Combinatorics and Feynman graphs for gauge theories

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ABSTRACT. We give an overview of the use of combinatorics in renormalization of gauge theories, using the Connes–Kreimer Hopf algebra. We prove some physical results in perturbative quantum gauge theories without relying on the formal manipulations involving path integrals. Instead, we take a perturbative series of Feynman graphs as a starting point. After a careful setup and counting of Feynman graphs, we study the structure of the renormalization Hopf algebra of gauge theories on the level of Green's functions. This involves Slavnov–Taylor identities, described by Hopf ideals, and Dyson–Schwinger equations, described by Hochschild cocycles [**7**]. As a new result, we prove the Kreimer's gauge theory theorem formulated in [**14**].

1. Introduction

These lecture notes are a compilation and extension of my previous work [**20, 18, 19, 21**] on renormalization Hopf algebras of gauge theories. It is supposed to serve as a self-contained treatment of the following themes:

(1) Feynman graphs in (non-abelian) gauge theories and their combinatorics
(2) Renormalization of gauge theories using Hopf algebras of Feynman graphs
(3) Dyson–Schwinger equations in terms of Hochschild cocycles.

The last point is based on the articles [**7, 2, 14**].

The description of (BPHZ)-renormalization in terms of Hopf algebras dates back to [**12**]. It involved rooted trees that encoded the combinatorial structure of renormalization. The description in terms of Feynman graphs – which are the objects of particular computational use in physics – was established in [**4, 5**]. Although a lot of progress has been made since then in several directions we will focus on the particular case of gauge theories.

Of course, their Hopf algebraic description was contained in the original papers *loc.cit.* – which after all applied to *any* quantum field theory – but the richer structure due to the presence of a gauge symmetry was explored in this context more recently. Non-abelian gauge theories were dissected in [**13**], with a prominent role played by the Slavnov–Taylor identities for the couplings and Dyson–Schwinger equations. The compatibility of these Slavnov–Taylor identities with the Hopf algebra was established in [**20**]. The Dyson–Schwinger equations were shown to be nicely captured by Hochschild cohomology of the Hopf algebra, see [**7, 2, 14**]. They also formed the starting point – via [**15**] – for a powerful method to obtain

solutions to these latter equations in [**1**]. The relation of the Hopf algebra with the gauge symmetry was discussed in [**16**].

In these lecture notes, we try to describe in a self-contained manner these steps towards an understanding of renormalization of perturbative gauge theories. In Section 2, we start with some background in perturbative quantum field theory. A precise definition of Feynman graphs is given in Section 3, where also the Connes–Kreimer Hopf algebra is defined. After proving some combinatorial identities, we derive at the end of that section formulas for the coproduct on full (1PI) Green's functions. This we apply in Section 4 to two physical theories, namely quantum electrodynamics and quantum chromodynamics.

Finally, in section 5 we review the approach taken in [**7, 2, 14**] on Dyson–Schwinger equations in terms of Hochschild cocycles of renormalization Hopf algebras. As a new result, we prove the gauge theory theorem [**14**, Theorem 5].

2. Crash course in perturbative quantum field theory

We start by giving some background from physics and try to explain the origin of Feynman graphs in the perturbative approach to quantum field theory.

Our starting point is the interpretation of the path integral as a formal series in Feynman graphs. It encodes the probability amplitudes for physical processes. Let us make this more explicit by some examples taken from physics.

EXAMPLE 1. *The interaction of the photon with the electron in quantum electrodynamics (QED) is described by the following formal expansion,*

Here all graphs appear that can be built from the vertex that connects a wiggly line (the photon) to two straight lines (the electron). The factor $\frac{1}{2}$ is a symmetry factor (cf. Definition 11 below).

EXAMPLE 2. *The quartic gluon self-interaction in quantum chromodynamics is given by*

This expansion involves the gluon vertex of valence 3 and 4 (wiggly lines), as well as the quark-gluon interaction (involving two straight lines)

We shall call these formal expansions *Green's functions*. Of course, this name originates from the theory of partial differential equations and the zeroth order terms in the above expansions are in fact Green's functions in the usual sense. The expansion is then treated as an asymptotic series for a Green's function for a perturbed differential operator. We use the notation $G^{\prec}$ and $G^{\times}$ for the Green's function, indicating the external structure of the graphs in the above two expansions, respectively.

From these expansions, physicists can actually derive numbers, giving the probability amplitudes mentioned above. The rules of this game are known as the Feynman rules; we briefly list them for the case of quantum electrodynamics. Feynman rules for non-abelian gauge theories can be found in most standard textbooks on

quantum field theory (see for instance [**3**, Section 2.14]).

Assigning momentum k to each edge of a graph, we have:

$$\text{~~~~~}_{k} \quad = \frac{1}{k^2 + i\epsilon}\left(-\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2 + i\epsilon}(1 - \xi)\right)$$

$$\underline{\quad\quad}_{k} \quad = \frac{1}{\gamma^\mu k_\mu + m}$$

$$\begin{array}{c} k_3 \\ k_1 \text{~~~}\!\!< \\ k_2 \end{array} \quad = -ie\gamma^\mu\delta(k_1 + k_2 + k_3)$$

Here, e is the electron charge, m the electron mass and γ^μ are 4×4 Dirac gamma matrices; they satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\delta^{\mu\nu}$. Also, ϵ is an infrared regulator and $\xi \in \mathbb{R}$ is the so-called gauge fixing parameter. In addition to the above assignments, one integrates the above internal momenta k (for each internal edge) over $\mathbb{R}^4$.

EXAMPLE 3. *Consider the following electron self-energy graph* Γ:
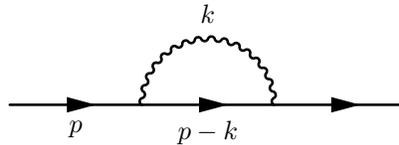
According to the Feynman rules, the amplitude for this graph is

$$(1) \quad U(\Gamma) = \int d^4k \; (e\gamma^\mu) \frac{1}{\gamma^\kappa(p_\kappa - k_\kappa) + m}(e\gamma^\nu)\left(-\frac{\delta_{\mu\nu}}{k^2 + i\epsilon} + \frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2}(1 - \xi)\right)$$

with summation over repeated indices understood.

The alert reader may have noted that the above improper integral is actually not well-defined. This is the typical situation – i.e. happening for most graphs – and are the famous divergences in perturbative quantum field theory. This apparent failure can be resolved, leading eventually to spectacularly accurate predictions in physics.

The theory that proposes a solution to these divergences is called *renormalization*. This process consists of two steps: *regularization* and *subtraction*. Let us give two examples of a regularization prescription.

The first we consider is a *momentum cut-off*. This means that we perform the integral above up to a real parameter Λ. More precisely, we make the replacement

$$\int d^4k \rightsquigarrow \int_{|k| \leq \Lambda} d^4k.$$

Let us consider the type of integrations we would like to perform, in a very simplified form. If the integrand is $(k^2 + m^2)^{-2}$ then

$$\int_{|k|\leq\Lambda} d^4k \frac{1}{(k^2+m^2)^2} \sim \log\Lambda.$$

This explain the divergent integrals we encountered above as $\Lambda \to \infty$, but we now have a control of the divergence. Although the momentum cut-off regularization is simple and physically natural, it is not the best regularization prescription for gauge theories since it breaks the gauge invariance. Nevertheless, it is the starting point for the powerful Wilsonian approach to renormalization, which has been studied in the Hopf algebraic setup as well in [**11**].

Another regularization prescription is *dimensional regularization*. Instead of integrating in 4 dimensions, one integrates in $4 - z$ dimensions, with z a complex number. Of course, this only makes sense after prescribing some rules for such an integration. The key rule is the following:

$$(2) \qquad\qquad \int d^D k \ e^{-\pi\lambda k^2} = \lambda^{-D/2} \qquad (D \in \mathbb{C}).$$

This formula clearly holds for D a positive integer, where it is just the Gaussian integral. However, if we *demand* it to hold for any complex D, it turns out to provide a very convenient regularization prescription. So, let us consider once more integration over $(k^2+m^2)^{-2}$, but now in $4-z$ dimensions. We write using so-called Schwinger parameters, or, equivalently the Laplace transform,

$$\frac{1}{k^2+m^2} = \int_{s>0} ds \ e^{-s(k^2+m^2)}.$$

Then, using the above Eq. (2) we find that

$$\int d^{4-z}k \frac{1}{(k^2+m^2)^2} = \int_{s>0} ds \int_{t>0} dt \int d^{4-z}k \ e^{-(s+t)(k^2+m^2)}$$

$$= \pi^{2-z/2} \int_{s>0} ds \int_{t>0} dt \ (s+t)^{-2+z/2} e^{-(s+t)m^2}$$

where we assumed that we could interchange the integrals. If we now change variables to $s = xy$ and $t = (1-x)y$ we obtain

$$\pi^{2-z/2} \int_0^1 dx \int_{y>0} dy y^{-1+z/2} e^{-ym^2} = \pi^{2-z/2} m^{-z} \Gamma(z/2),$$

with Γ the complex gamma function. It has a pole at $z = 0$, reflecting the divergence before regularization. Again, this gives us control on the divergence.

The second step in the process of renormalization is *subtraction*. For dimensional regularization, we let T be the projection onto the pole part of Laurent series in z, i.e.,

$$T\left[\sum_{n=-\infty}^{\infty} a_n z^n\right] = \sum_{n<0} a_n z^n$$

More generally, we have a projection on the divergent part in the regularizing parameter. This is the origin of the study of Rota-Baxter algebras in the setting of quantum field theories [**9**]. We will however restrict ourselves to dimensional regularization with this so-called *minimal substraction*, as it is a well suited renormalization scheme for gauge theories.

For the above graph Γ, we define the *renormalized amplitude* $R(\Gamma)$ by simply subtracting the divergent part, that is, $R(\Gamma) = U(\Gamma) - T[U(\Gamma)]$. Clearly, the result is finite for $z \to 0$. More generally, a graph Γ might have subgraphs $\gamma \subset \Gamma$ which lead to subdivergences in $U(\Gamma)$. The so-called *BPHZ-procedure* (after its inventors Bogoliubov, Parasiuk, Hepp and Zimmermann) provides a way to deal with those subdivergences in a recursive manner. It gives for the *renormalized amplitude*:

$$(3a) \qquad R(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

where C is the so-called *counterterm* defined recursively by

$$(3b) \qquad C(\Gamma) = -T \left[U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma) \right]$$

The two sums here are over all subgraphs in a certain class; we will make this more precise in the next section.

2.1. Gauge theories. We now focus on a special class of quantum field theories – quantum gauge theories – which are of particular interest for real physical processes. Without going too much into details of what classical gauge field theories are, we focus on the consequences on the quantum side of the presence of a classical gauge symmetry. Such a gauge symmetry acts (locally) on the classical fields, thereby leaving invariant the theory that is defined, say, by a Lagrangian. These so-called gauge transformations form a group, called the *gauge group*. This group is typically infinite dimensional, since it consists of functions on space-time taking values in a Lie group. For quantum electrodynamics this Lie group is abelian and just $U(1)$, for quantum chromodynamics – the theory of gluons and quarks – it is $SU(3)$.

When (perturbatively) quantizing the gauge theory, one is confronted with this extra infinity, in addition to the divergences discussed in the previous subsection. A way to handle it is by *fixing the gauge*, in other words, by choosing an orbit under the action of the gauge group. All this can be made quite precise in the so-called *BRST-formalism*, after its inventors Becchi, Rouet, Stora and Tyutin. Although in this process the gauge symmetry completely disappears, certain identities between Green's functions appear. This is a purely 'quantum property' and therefore interesting to study. In addition, being potential identities between full Green's functions, it is interesting with a view towards nonperturbative quantum field theory.

For quantum electrodynamics, the identities are simple and linear in the Green's functions:

$$(4) \qquad U\left(G \, \overset{\prec}{} \right) = U\left(G \, \text{---} \right).$$

These are known as *Ward identities* as they were first derived by Ward in [**23**]. The apparent mismatch between the number of external lines on the left and right-hand-side is resolved because the vertex graphs are considered at *zero momentum transfer*. This means that the momentum on the photon line is evaluated at $p = 0$.

For non-abelian gauge theories such as quantum chromodynamics (QCD), the identities are quadratic in the Green's functions and read:

$$U\left(G^{\,\prec}\right)\,U\left(G^{\,\prec}\right)=U\left(G^{\,\times}\right)\,U\left(G^{\,-}\right);$$

(5)
$$U\left(G^{\,\prec}\right)\,U\left(G^{\,\prec}\right)=U\left(G^{\,\times}\right)\,U\left(G^{\,\cdots}\right);$$

$$U\left(G^{\,\prec}\right)\,U\left(G^{\,\prec}\right)=U\left(G^{\,\times}\right)\,U\left(G^{\,\cdots}\right).$$

The dotted and straight line here corresponds to the ghost and quark, respectively. After their inventors, they are called the *Slavnov–Taylor identities* for the couplings [**17, 22**].

The importance of these identities lie in the fact that they are compatible with renormalization under the condition that gauge invariance is compatible with the regularization procedure. In fact, it turns out that dimensional regularization satisfies this requirement, see for instance [**10**, Section 13.1]. As a consequence, the Slavnov–Taylor identities hold after replacing U by R or C in the above formula. For instance, in the case of quantum electrodynamics one obtains the identity $Z_1 = Z_2$ that was actually derived by Ward in [**23**], where $Z_1 = C(G^{\,\prec})$ and $Z_2 = C(G^{\,-})$. For quantum chromodynamics on the other hand, one derives the formulae

$$(6)\qquad \frac{Z^{\,\prec}}{Z^{\,-}\sqrt{Z^{\,\cdots}}}=\frac{Z^{\,\prec}}{Z^{\,\cdots}\sqrt{Z^{\,\cdots}}}=\frac{Z^{\,\prec}}{\left(Z^{\,\cdots}\right)^{3/2}}=\frac{\sqrt{Z^{\,\times}}}{Z^{\,\cdots}},$$

where the notation is as above: $Z^r := C(G^r)$. The above formula can be readily obtained from the above Slavnov–Taylor identities (5) after replacing U by C. They are the key to proving renormalizability of non-abelian gauge theories, let us try to sketch this argument.

First of all, the different interactions that are present in the theory can be weighted by a coupling constant. For example, in QCD there are four different interactions: gluon-quark, gluon-ghost, cubic and quartic gluon self-interaction. All of these come with their own coupling constants and gauge invariance (or rather, BRST-invariance) requires them to be identical. In the process of renormalization, the coupling constants are actually not constant and depend on the energy scale. This is the *running of the coupling constant* and is the origin of the renormalization group describing how they change. For QCD, the four coupling constants $g_{0,\,\prec}, g_{0,\,\prec}, g_{0,\,\prec}, g_{0,\,\times}$ are expressed in terms of the original coupling constant g as

$$(7)\qquad g_{0,\,\prec}=\frac{Z^{\,\prec}}{Z^{\,-}\sqrt{Z^{\,\cdots}}}g,\quad g_{0,\,\prec}=\frac{Z^{\,\prec}}{Z^{\,-}\sqrt{Z^{\,\cdots}}}g,$$

$$g_{0,\,\prec}=\frac{Z^{\,\prec}}{\left(Z^{\,\cdots}\right)^{3/2}}g,\quad g_{0,\,\times}=\frac{\sqrt{Z^{\,\times}}}{Z^{\,\cdots}}g.$$

We see that the Slavnov–Taylor identities guarantee that *the four coupling constants remain equal* after renormalization.

The above compatibility of renormalization with the Slavnov–Taylor identities is usually derived using the Zinn-Justin equation (or the more general BV-formalism) relying heavily on path integral techniques. Our goal in the next sections

is to derive this result taking the formal expansion of the Green's functions in Feynman graphs as a starting point. We will work in the setting of the Connes–Kreimer Hopf algebra of renormalization.

3. The Hopf algebra of Feynman graphs

We start with some definitions on Feynman graphs and their symmetries, thereby making precise several properties needed later.

3.1. Feynman graphs. The Feynman graphs we will consider are built from a certain set of edges and vertices R, and following [**14**] we write $R = R_V \cup R_E$. For example, in massive ϕ^3-theory, the set R_V contains the bi- and trivalent vertex and R_E the straight line, but more interesting theories such as gauge theories contain different types of edges and vertices (for example involving curly, dotted and straight lines) corresponding to different particles. More precisely, we have the following definition [**6**].

DEFINITION 4. *A* Feynman graph Γ *is given by a set* $\Gamma^{[0]}$ *of vertices each of which is an element in* R_V *and* $\Gamma^{[1]}$ *of edges in* R_E, *and maps*

$$\partial_j : \Gamma^{[1]} \to \Gamma^{[0]} \cup \{1, 2, \dots, N\}, \qquad j = 0, 1,$$

that are compatible with the type of vertex and edge as parametrized by R_V *and* R_E, *respectively. Moreover, we exclude the case that* ∂_0 *and* ∂_1 *are both in* $\{1, 2, \dots, N\}$. *The set* $\{1, 2, \dots, N\}$ *labels the external lines, so that* $\sum_j \operatorname{card} \partial_j^{-1}(v) = 1$ *for all* $v \in \{1, \dots, N\}$.

The set of external lines *is* $\Gamma_{\mathrm{ext}}^{[1]} = \cup_i \partial_i^{-1}\{1, \dots, N\}$ *and its complement* $\Gamma_{\mathrm{int}}^{[1]}$ *in* $\Gamma^{[1]}$ *is the set of* internal lines.

Note that the elements in $\Gamma_{\mathrm{ext}}^{[1]}$ can thus be labeled as $e_1, \dots, e_N$ where $e_k := \cup_i \partial_i^{-1}(k)$ and we understand this labeling as being fixed.

REMARK 5. *An equivalent way of looking at the above 'types' of edges and vertices is the following, staying more closely to the conventional graph theory. Instead of assigning a certain 'type' to each edge, we could just as well color them, with the color now corresponding to a propagating particle in our theory. A vertex then represents an interaction of the particles that correspond to the colors that meet at that vertex. It happened so that nature has chosen only certain interactions between certain particles, so we restrict ourselves to graphs that only involve the vertices with allowed coloring.*

If a Feynman graph Γ has two external lines, both corresponding to the same field, we would like to distinguish between propagators and mass terms. In more mathematical terms, since we have vertices of valence two, we would like to indicate whether a graph with two external lines corresponds to such a vertex, or to an edge. A graph Γ with two external lines is dressed by a bullet when it corresponds to a vertex, i.e. we write $\Gamma_\bullet$. The above correspondence between Feynman graphs and vertices/edges is given by the *residue* $\mathrm{res}(\Gamma)$. It is defined as the vertex or edge the graph corresponds to after collapsing all its internal points. For example, we have:

$$\mathrm{res}\left(\rule{0pt}{1.5em}\quad \right) = \quad \qquad \text{and} \qquad \mathrm{res}\left(\rule{0pt}{1.5em}\quad \right) = \underline{\quad\quad}$$

but if the last graph is dressed with a bullet it is understood to correspond to a valence 2 vertex:

$$\mathrm{res}\left(\text{\includegraphics{graph}}\bullet\right) = \text{\includegraphics{dot}}.$$

For the purpose of renormalization, one is mainly interested in one-particle irreducible Feynman graphs with residues in the set R.

DEFINITION 6. *A Feynman graph is called* one-particle irreducible (1PI) *if it is not a tree and can not be disconnected by removal of a single edge.*

For example, all graphs in this paper are one-particle irreducible, *except* the following which is one-particle reducible:

3.2. The Hopf algebra structure on graphs. We restrict to the class of 1PI Feynman graphs Γ for which $\mathrm{res}(\Gamma) \in R$ and will denote a generic graph with residue $r \in R$ by Γ^r. If it also has loop number L, we denote it by Γ^r_L.

DEFINITION 7 (Connes–Kreimer [**4**]). *The Hopf algebra H of Feynman graphs is the free commutative algebra (over $\mathbb{C}$) generated by all 1PI Feynman graphs with residue in R, with counit $\epsilon(\Gamma) = 0$ unless $\Gamma = \emptyset$, in which case $\epsilon(\emptyset) = 1$, coproduct,*

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over disjoint unions of 1PI subgraphs with residue in R. The quotient Γ/γ is defined to be the graph Γ with the connected components of the subgraph contracted to the corresponding vertex/edge. If a connected component γ' of γ has two external lines, then there are possibly two contributions corresponding to the valence two vertex and the edge; the sum involves the two terms $\gamma_\bullet \otimes \Gamma/(\gamma \to \bullet)$ and $\gamma \otimes \Gamma/\gamma$.
   The antipode is given recursively by,

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma} S(\gamma)\Gamma/\gamma.$$

The above Hopf algebra is an example of a connected graded Hopf algebra: it is graded by the *loop number* $L(\Gamma)$ of a graph Γ. Indeed, one checks that the coproduct (and obviously also the product) satisfy the grading by loop number and H^0 consists of complex multiples of the empty graph, which is the unit in H, so that $H^0 = \mathbb{C}1$. We denote by q_l the projection in H onto H^l.

In addition, there is another grading on this Hopf algebra. It is given by the number of vertices and already appeared in [**4**]. However, since we consider vertices and edges of different types (wiggly, dotted, straight, *et cetera*), we extend to a multigrading as follows. As in [**20**], we denote by $m_{\Gamma,r}$ the number of vertices/internal edges of type r appearing in Γ, for $r \in R$. Moreover, let $n_{\gamma,r}$ be the number of connected components of γ with residue r. For each $v \in R_V$ we define a degree d_v by

$$d_v(\Gamma) = m_{\Gamma,v} - n_{\Gamma,v}.$$

The multidegree indexed by R_V is compatible with the Hopf algebra structure as follows easily from the following relation:

$$m_{\Gamma/\gamma,v} = m_{\Gamma,v} - m_{\gamma,v} + n_{\gamma,v},$$

and the fact that $m_{\Gamma\Gamma',v} = m_{\Gamma,v} + m_{\Gamma',v}$, and $n_{\Gamma\Gamma',v} = n_{\Gamma,v} + n_{\Gamma',v}$. This gives a decomposition

$$H = \bigoplus_{n_1,\ldots,n_k \in \mathbb{Z}^k} H^{n_1,\ldots,n_k},$$

where $k = |R_V|$. We denote by $p_{n_1,\ldots,n_k}$ the projection onto $H^{n_1,\ldots,n_k}$.

LEMMA 8. *There is the following relation between the grading by loop number and the multigrading by number of vertices:*

$$\sum_{v \in R_V} (N(v) - 2)d_v = 2L$$

where $N(v)$ is the number of lines attached to v, i.e., $N(v) := |\partial(v)|$.

PROOF. This can be easily proven by induction on the number of internal edges using invariance of the quantity $\sum_v (N(v) - 2)d_v - 2L$ under the adjoint of an edge. ☐

3.2.1. *Renormalization as a Birkhoff decomposition.* We now demonstrate how to obtain Equation (3) for the renormalized amplitude and the counterterm for a graph as a Birkhoff decomposition in the group of characters of H. Let us first recall the definition of a Birkhoff decomposition.

We let $l : C \to G$ be a loop with values in an arbitrary complex Lie group G, defined on a smooth simple curve $C \subset \mathbb{P}_1(\mathbb{C})$. Let $C_\pm$ be the two complements of C in $\mathbb{P}_1(\mathbb{C})$, with $\infty \in C_-$. A *Birkhoff decomposition* of l is a factorization of the form

$$l(z) = l_-(z)^{-1}l_+(z); \qquad (z \in C),$$

where $l_\pm$ are (boundary values of) two holomorphic maps on $C_\pm$, respectively, with values in G. This decomposition gives *a natural way to extract finite values from a divergent expression.* Indeed, although $l(z)$ might not holomorphically extend to C_+, $l_+(z)$ is clearly finite as $z \to 0$.

We now look at the group $G(K) = \mathrm{Hom}_\mathbb{Q}(H, K)$ of K-valued characters of a connected graded commutative Hopf algebra H, where K is the field of convergent Laurent series in z.[1] The product, inverse and unit in the group $G(K)$ are defined by the respective equations:

$$\phi * \psi(X) = \langle \phi \otimes \psi, \Delta(X) \rangle,$$
$$\phi^{-1}(X) = \phi(S(X)),$$
$$e(X) = \epsilon(X),$$

for $\phi, \psi \in G(K)$. We claim that a map $\phi \in G(K)$ is in one-to-one correspondence with loops l on an infinitesimal circle around $z = 0$ and values in $G(\mathbb{Q}) = \mathrm{Hom}_\mathbb{Q}(H, \mathbb{Q})$. Indeed, the correspondence is given by

$$\phi(X)(z) = l(z)(X),$$

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[1]In the language of algebraic geometry, there is an affine group scheme G represented by H in the category of commutative algebras. In other words, $G = \mathrm{Hom}_\mathbb{Q}(H, \cdot)$ and $G(K)$ are the K-points of the group scheme.

and to give a Birkhoff decomposition for l is thus equivalent to giving a factorization $\phi = \phi_-^{-1} * \phi_+$ in $G(K)$. It turns out that for graded connected commutative Hopf algebras such a factorization exists.

THEOREM 9 (Connes–Kreimer [4]). *Let H be a graded connected commutative Hopf algebra. The Birkhoff decomposition of $l : C \to G$ (given by an algebra map $\phi : H \to K$) exists and is given dually by*

$$\phi_-(X) = \epsilon(X) - T\left[m(\phi_- \otimes \phi)(1 \otimes (1 - \epsilon)\Delta(X)\right]$$

and $\phi_+ = \phi_- * \phi$.

The graded connected property of H assures that the recursive definition of ϕ_- actually makes sense. In the case of the Hopf algebra of Feynman graphs defined above, the factorization takes the following form:

$$\phi_-(\Gamma) = -T\left[\phi(\Gamma) + \sum_{\gamma \subsetneq \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma)\right]$$
$$\phi_+(\Gamma) = \phi(\Gamma) + \phi_-(\Gamma) + \sum_{\gamma \subsetneq \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma)$$

The key point is now that the Feynman rules actually define an algebra map $U : H \to K$ by assigning to each graph Γ the regularized Feynman rules $U(\Gamma)$, which are Laurent series in z. When compared with Equations (3) one concludes that the algebra maps U_+ and U_- in the Birkhoff factorization of U are precisely the renormalized amplitude R and the counterterm C, respectively. Summarizing, we can write the BPHZ-renormalization as the Birkhoff decomposition $U = C^{-1} * R$ of the map $U : H \to K$ dictated by the Feynman rules.

Although the above construction gives a very nice geometrical description of the process of renormalization, it is a bit unphysical in that the Hopf algebra relies on individual graphs. Rather, as mentioned before, in physics the probability amplitudes are computed from the full expansion of Green's functions. Individual graphs do not correspond to physical processes and therefore a natural question to pose is how the Hopf algebra structure behaves at the level of the Green's functions. Let us make the following definition.

DEFINITION 10. *For a vertex or edge $r \in R$ we define the 1PI Green's function by*

$$(8) \qquad G^r = 1 \pm \sum_{\text{res}(\Gamma)=r} \frac{\Gamma}{\text{Sym}(\Gamma)}$$

where the sign is $+$ if r is a vertex and $-$ if it is an edge. Finally, we denote the restriction of the sum to graphs Γ at loop order $L(\Gamma) = L$ by G_L^r.

We are particularly interested in the form of the coproduct on 1PI Green's functions, and more generally, the Hopf algebra structure of Green's functions. Before we derive the coproduct on these Green's functions, we need a significant amount of combinatorial relations and identities.

$$\mathrm{Sym}\big(\ \sim\!\!\bigcirc\!\!\sim\ \big) = 2 \qquad\qquad \mathrm{Sym}\big(\ \underline{\ \ \overset{\frown}{\ \ }\ \ }\ \big) = 1$$

FIGURE 1. Automorphisms of Feynman graphs respect the type of vertex/edge in R.

3.3. Intermezzo: Counting Feynman graphs. In this subsection we recall some of the combinatorial properties we have derived in [**19**]. Let us start with the natural notion of graph automorphisms, taking Definition 4 of Feynman graphs above as a starting point.

DEFINITION 11. *An* automorphism *of a Feynman graph Γ is given by an iso-morphism $g^{[0]}$ from $\Gamma^{[0]}$ to itself, and an isomorphism $g^{[1]}$ from $\Gamma^{[1]}$ to itself that is the identity on $\Gamma_{\mathrm{ext}}^{[1]}$ and such that for all $e \in \Gamma^{[1]}$,*

$$(9) \qquad\qquad \cup_j g^{[0]}(\partial_j(e)) = \cup_j \partial_j(g^{[1]}(e)).$$

Moreover, we require $g^{[0]}$ and $g^{[1]}$ to respect the type of vertex/edge in the set R.

The automorphism group $\mathrm{Aut}(\Gamma)$ *of Γ consists of all such automorphisms; its order is called the* symmetry factor *of Γ and is denoted by $\mathrm{Sym}(\Gamma)$.*

Similarly, there is a notion of an isomorphism of two graphs Γ and Γ' as a pair of maps that intertwines the maps ∂_i as in Eq. (9). We remark that we correct in this way for the apparent orientation given by the two maps ∂_0 and ∂_1 and we stress that the fermionic lines are unoriented. We take the complex character of the fermionic fields into account by summing over all possible orientations once we apply the Feynman rules.

Note that for $\Gamma = \prod_i \Gamma_i$ the disjoint union of n graphs, the symmetry factor is given by $\mathrm{Sym}(\Gamma) = n_1! \cdots n_k! \, \mathrm{Sym}(\Gamma_1) \cdots \mathrm{Sym}(\Gamma_n)$ where n_i are the numbers of isomorphic (with fixed external lines) connected components of Γ'. Equivalently, one has for a 1PI graph Γ',

$$(10) \qquad\qquad \mathrm{Sym}(\Gamma\ \Gamma') = n_c(\Gamma, \Gamma')\mathrm{Sym}(\Gamma)\mathrm{Sym}(\Gamma'),$$

with $n_c(\Gamma, \Gamma')$ the number of connected components of $\Gamma\Gamma'$ that are isomorphic to Γ'.

The above definition of automorphism differs from the usual notion of graph automorphism (cf. for instance [**8**]) in that the latter might also permute the elements in $\{1, \ldots, N\}$ when understood as external vertices. In the above notation, such an automorphism of Γ would be given by an isomorphism $g^{[0]}$ from $\Gamma^{[0]} \cup \{1, \ldots, N\}$ to itself, and an isomorphism $g^{[1]}$ from $\Gamma^{[1]}$ to itself such that Equation (9) holds.

If Γ is a connected Feynman graph with external lines labeled by $\{1, \ldots, N\}$, we can construct another graph Γ^σ, by permuting the external lines by an element $\sigma \in S_N$, respecting the type of external lines. The graph Γ^σ is given by the same sets $\Gamma^{[0]}$ and $\Gamma^{[1]}$ but with maps

$$\partial_j^\sigma := \sigma \circ \partial_j : \Gamma^{[1]} \to \Gamma^{[0]} \cup \{1, \cdots, N\}.$$

This permutation affects the labeling of the external lines by $\{1, \ldots N\}$, which explains the terminology permutation of external lines; we write e^σ for the edge in Γ^σ corresponding to an edge $e \in \Gamma^{[1]}$ under the permutation σ.

DEFINITION 12. *A permutation σ of the external lines of Γ is called* trivial *if there exists an isomorphism between Γ^σ and Γ, leaving the labeling of the external lines fixed.*
The number of non-isomorphic graphs Γ^σ obtained by a permutation σ of the external lines of Γ, is denoted by $|\Gamma|_\vee$ and extended to disconnected graphs by $|\Gamma\Gamma'|_\vee = |\Gamma|_\vee |\Gamma'|_\vee$.

LEMMA 13. *A permutation σ of the external lines of Γ is trivial if and only if there exists an automorphism g of the graph Γ not necessarily leaving the external lines fixed, such that $g^{[0]}|_{\{1,\ldots,N\}} = \sigma$.*

PROOF. Firstly, if σ is trivial, there exists an isomorphism $f : \Gamma^\sigma \to \Gamma$ and the pair $(f^{[0]} \circ \sigma,\ f^{[1]} \circ \sigma)$ is an automorphism g of Γ (without fixed external vertices), since,

$$\cup_j g^{[0]}(\partial_j(e)) = \cup_j f^{[0]}(\partial_j^\sigma(e^\sigma)) = \cup_j \partial_j(f^{[1]}(e^\sigma)) = \cup_j \partial(g^{[1]}(e)).$$

On the other hand, such an automorphism g is given by two maps $g^{[0]}$ and $g^{[1]}$, where $g^{[0]}$ is the product of two permutations of the disjoint sets $\Gamma^{[0]}$ and $\{1, \ldots, N\}$, say $f^{[0]}$ and σ, respectively. Correspondingly, σ acts on $\Gamma_{\mathrm{ext}}^{[1]}$ by permutation, so that also $g^{[1]} = f^{[1]} \circ \sigma$. This factorization gives rise to an isomorphism f from Γ^σ to Γ, which leaves external lines fixed.   □
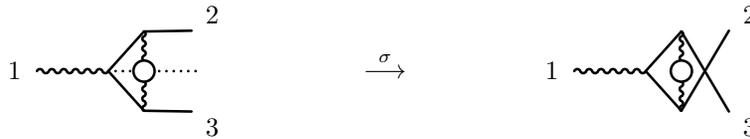

FIGURE 2. The permutation $\sigma = (23)$ of the external lines of the graph Γ is trivial since reflection in the dotted line induces an automorphism g of Γ such that $g^{[0]}|_{\{1,2,3\}} = \sigma$. Moreover, this is the only trivial permutation so that $|\Gamma|_\vee = 3!/2 = 3$

DEFINITION 14. *An* insertion place *for a (connected) graph γ in Γ is the subset of $\Gamma^{[0]} \cup \Gamma_{\mathrm{int}}^{[1]}$ consisting of vertices/internal edges of the form $r = \mathrm{res}(\gamma)$. It can be extended to disconnected graphs $\gamma = \prod_{i=1}^n \gamma_i$ by giving n-tuples of insertion places for $\gamma_1, \ldots, \gamma_n$, thereby allowing several insertions of the connected components with residue r in R_E on the same internal edge in Γ of the form r. The number of insertion places for γ in Γ is denoted by $\Gamma \mid \gamma$.*

An explicit expression for $\Gamma \mid \gamma$ can be obtained as follows [14]. Recall the notation $m_{\Gamma,r}$ for the number of vertices/edges r in $\Gamma^{[0]} \cup \Gamma^{[1]}$, for $r \in R$, and $n_{\gamma,r}$ for the number of connected components of γ with residue r. Since insertion of a vertex graph (i.e. with residue in R_V) on a $v \in \Gamma^{[0]}$ prevents a subsequent insertion at v of a vertex graph with the same residue, whereas insertion of an edge graph

(i.e. with residue in R_E) creates two new edges and hence two insertion places for a subsequent edge graph, we find the following expression,

$$
(11) \qquad \Gamma \mid \gamma = \prod_{v \in R_V} n_{\gamma,v}! \binom{m_{\Gamma,v}}{n_{\gamma,v}} \prod_{e \in R_E} n_{\gamma,e}! \binom{m_{\Gamma,e} + n_{\gamma,e} - 1}{n_{\gamma,e}}.
$$

Indeed, the binomial coefficients arises for each vertex v since we are choosing $n_{\gamma,v}$ out of $m_{\Gamma,v}$ whereas for an edge e we choose $n_{\gamma,e}$ out of $m_{\Gamma,e}$ *with repetition*. We extend this definition to empty graphs by defining $\Gamma \mid \emptyset = \emptyset \mid \gamma = \emptyset \mid \emptyset = 1$ for a 1PI graph γ, and $\emptyset \mid \gamma = 0$ for a disconnected graph γ.

REMARK 15. *Our expression for $\Gamma \mid \gamma$ differs slightly from the one given in* [**14**] *where additional factors of $1/n_{\gamma,r}!$ are present for $r \in R$. It turns out that the above expression appears naturally in the coproduct on 1PI Green's functions (see below).*

A few examples are in place:

$$
\text{\raisebox{-1em}{\image}} \bigg| \text{\raisebox{-1em}{\image}} = \binom{2}{1} = 2 \quad \text{whereas} \quad \text{\raisebox{-1em}{\image}} \bigg| \text{\raisebox{-1em}{\image}} = 2! \binom{3}{2} = 6.
$$

DEFINITION 16. *An* insertion *of a connected graph γ at the insertion place x in Γ, is given by a bijection between the set $\gamma_{\mathrm{ext}}^{[1]}$ of external lines of γ and the set $\partial^{-1}(x)$. If $x \in \Gamma^{[0]}$, $\partial^{-1}(x)$ denotes the set of lines attached to the vertex x, and if $x \in \Gamma_{\mathrm{ext}}^{[1]}$, $\partial^{-1}(x)$ denotes the set of adjacent edges to any internal point of x. The graph obtained in this way is denoted by $\Gamma \circ_{(x,\phi)} \gamma$.*

Two insertions (x,ϕ) and (x',ϕ') are called equivalent if $x = x'$ and $\phi' = \phi \circ \sigma$ for some trivial permutation σ of the external lines of γ. The set of all insertions of γ in Γ up to equivalence is denoted by $X(\Gamma,\gamma)$; it consists of equivalence classes $[x,\phi]$.

This definition of equivalence relation on insertions is motivated by the fact that $\Gamma \circ_{(x,\phi)} \gamma \simeq \Gamma \circ_{(x',\phi')} \gamma$ whenever $(x,\phi) \sim (x',\phi')$. We extend $X(\Gamma,\gamma)$ to disconnected graphs γ as follows. If $\gamma = \prod_{i=1}^{n} \gamma_i$ is the disjoint union of n graphs, the set $X(\Gamma,\gamma)$ of insertions of γ in Γ is defined as the set of n−tuples of pairs $([x_1,\phi_1], \ldots, [x_n,\phi_n])$, where $[x_1,\phi_1] \in X(\Gamma,\gamma_1)$ and $[x_{k+1},\phi_{k+1}]$ is an element in $X(\Gamma \circ_{(x_1,\phi_1)\ldots(x_k,\phi_k)} \prod_{i=1}^{k} \gamma_i, \gamma_{k+1})$ which is not part of any of the inserted graphs $\gamma_1, \ldots, \gamma_{k-1}$ for $k = 1, \ldots, n-1$. The cardinality of $X(\Gamma,\gamma)$ is the number $\Gamma \mid \gamma$ of insertion places for γ in Γ times the number $|\gamma|_\vee$ of non-trivial permutations of the external lines of γ.

We also need the following generalization for the number of insertion places.

DEFINITION 17. *Let Γ, γ, γ' be three (disjoint unions of) 1PI graphs. We define $\Gamma \mid \gamma \mid \gamma'$ to be the number of places to insert γ into Γ (say, at x using ϕ) and then subsequently insert γ' in $\Gamma \circ_{(x,\phi)} \gamma$. In other words,*

$$
\Gamma \mid \gamma \mid \gamma' := \frac{1}{|\gamma|_\vee} \sum_{[x,\phi] \in X(\Gamma,\gamma)} \Gamma \circ_{(x,\phi)} \gamma \mid \gamma'.
$$

Moreover, we set $\Gamma \mid \emptyset \mid \gamma' = \Gamma \mid \gamma'$ and $\emptyset \mid \gamma \mid \gamma' = \gamma \mid \gamma'$ if γ is 1PI and $\emptyset \mid \gamma \mid \gamma' = 0$ if γ is disconnected.

The factor $1/|\gamma|_\vee$ corrects for the overcounting due to the several (not equivalent) ways to insert γ into Γ at a particular place. Note that automatically $\Gamma \mid \gamma \mid \emptyset = \Gamma \mid \gamma$ and if $\Gamma, \gamma, \gamma' \neq \emptyset$, we have

$$(12) \qquad\qquad \Gamma \mid \gamma \mid \gamma' = \Gamma \mid \gamma\gamma' + (\Gamma \mid \gamma)(\gamma \mid \gamma').$$

Suppose γ is a 1PI graph. There is a natural action of $\mathrm{Aut}(\Gamma)$ on $X(\Gamma, \gamma)$ given by

$$g \cdot (x, \phi) = (gx, g \circ \phi).$$

One readily checks that this action respects the equivalence relation on insertions, and therefore acts on the equivalence classes $[x, \phi]$. Moreover, an element $g \in \mathrm{Aut}(\Gamma)$ naturally induces an isomorphism $\Gamma \circ_{(x,\phi)} \gamma$ to $\Gamma \circ_{g(x,\phi)} \gamma$.

For an element $[x, \phi]$ in $X(\Gamma, \gamma)$, we denote by $M(x, \phi)$ the number of graphs γ' in $\Gamma \circ_{(\phi,x)} \gamma$ that are images of γ under some element in $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)$. Moreover, $N(x, \phi)$ denotes the number of orbits $\mathrm{Aut}(\Gamma)[x', \phi']$ such that $\Gamma \circ_{(x',\phi')} \gamma \simeq \Gamma \circ_{(x,\phi)} \gamma$. Both definitions are independent of the choice of a representative (x, ϕ) as well as the choice of the element $[x', \phi']$ in the orbit. Indeed, an element g in $\mathrm{Aut}(\Gamma)$ will induce a natural isomorphism $\Gamma \circ_{(x',\phi)} \gamma \simeq \Gamma \circ_{g(x',\phi')} \gamma$.

LEMMA 18. *Suppose γ is a 1PI graph and let $[x, \phi] \in X(\Gamma, \gamma)$. The length of the orbit $\mathrm{Aut}(\Gamma)[x, \phi]$ is given by*

$$|\mathrm{Aut}(\Gamma)[x, \phi]| = \frac{\mathrm{Sym}(\gamma)\mathrm{Sym}(\Gamma)M(x, \phi)}{\mathrm{Sym}(\Gamma \circ_{(x,\phi)} \gamma)}.$$

PROOF. We use the orbit-stabilizer theorem, stating in this case that the orbit $\mathrm{Aut}(\Gamma)[x, \phi]$ is isomorphic to the left cosets of the stabilizer $\mathrm{Aut}(\Gamma)_{[x,\phi]}$ in $\mathrm{Aut}(\Gamma)$. In particular, we have for its length,

$$|\mathrm{Aut}(\Gamma)[x, \phi]| = \left[\mathrm{Aut}(\Gamma) : \mathrm{Aut}(\Gamma)_{[x,\phi]}\right] = \frac{|\mathrm{Aut}(\Gamma)|}{|\mathrm{Aut}(\Gamma)_{[x,\phi]}|}.$$

The order of $\mathrm{Aut}(\Gamma)_{[x,\phi]}$ can be computed as follows. Let $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma$ be the subgroup of $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)$ consisting of automorphisms that map γ to itself (but possibly permuting the external lines of γ). There is a short exact sequence of groups

$$1 \to \mathrm{Aut}(\gamma) \to \mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma \to \mathrm{Aut}(\Gamma)_{[x,\phi]} \to 1.$$

Indeed, the image $\widetilde{g}$ inside $\mathrm{Aut}(\Gamma)$ of an element g in $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma$ is defined by restricting g to $\Gamma - \{x\}$ and by the identity map on the vertex x. Then, by Lemma 13, $\widetilde{g}$ might permute the edges connected to the vertex x but always in a trivial way, since g induces an automorphism of γ not necessarily leaving its external lines fixed. Therefore, $\widetilde{g}(x, \phi) = (x, \phi \circ \sigma)$ for some trivial permutation σ of $\gamma_{\mathrm{ext}}^{[1]}$, so that it is an element in the fixed point subgroup $\mathrm{Aut}(\Gamma)_{[x,\phi]}$. Moreover, the kernel of the map that sends such a g to $\widetilde{g}$ consists precisely of those elements in $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma$ that correspond to the identity on Γ; in other words, these are automorphisms of γ that leave external lines fixed.

We conclude that the quotient group $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma / \mathrm{Aut}(\gamma)$ is isomorphic to $\mathrm{Aut}(\Gamma)_{[x,\phi]}$. Since $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)$ is generated by the elements in $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma$ and automorphisms that map γ isomorphically to a subgraph γ' of Γ, we see that

$$|\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma| = \frac{|\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)|}{M(x, \phi)}.$$

Combining these results, we conclude that

$$|\mathrm{Aut}(\Gamma)[x,\phi]| = \frac{|\mathrm{Aut}(\gamma)|\,|\mathrm{Aut}(\Gamma)|}{|\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)_\gamma|} = \frac{\mathrm{Sym}(\gamma)\mathrm{Sym}(\Gamma)M(x,\phi)}{\mathrm{Sym}(\Gamma \circ_{(x,\phi)} \gamma)}.$$

$\square$

As a final preparation to the next section, we will write the coproduct as a sum of maps Δ_γ, with γ a disjoint union of 1PI graphs (with fixed external lines). They are given by

$$\Delta_\gamma(\Gamma) = \sum_{\gamma' \subset \Gamma, \gamma' \simeq \gamma} \Gamma/\gamma',$$

and defined to be zero if Γ contains no subgraphs isomorphic to γ. In particular, $\Delta_\emptyset$ is the identity map, $\Delta_\Gamma(\Gamma) = \emptyset$ and $\Delta_\gamma(\emptyset) = 0$ if $\gamma \neq \emptyset$. However, since only subgraphs isomorphic to γ enter in this formula – hence no reference is made to a particular labeling of the external lines of γ – we have to correct by a factor of $|\gamma|_\vee$ if we are to sum over all disjoint unions of 1PI graphs *with fixed external lines*,

$$\Delta = \sum_\gamma \frac{1}{|\gamma|_\vee} \, \gamma \otimes \Delta_\gamma.$$

We recall the following combinatorial factor from [4]; for a given Γ, γ, Γ', we denote by $n(\Gamma, \gamma, \Gamma')$ the number of subgraphs $\gamma' \simeq \gamma$ in Γ such that $\Gamma/\gamma \simeq \Gamma'$. With this definition, we can write

(13) $$\Delta_\gamma(\Gamma) = \sum_{\Gamma'} n(\Gamma, \gamma, \Gamma') \, \Gamma',$$

which also yields the following formula for the coproduct,

$$\Delta(\Gamma) = \sum_{\gamma, \Gamma'} \frac{n(\Gamma, \gamma, \Gamma')}{|\gamma|_\vee} \, \gamma \otimes \Gamma'.$$

REMARK 19. *From this last formula, one easily derives the Lie bracket on Feynman graphs as defined in [4]. Indeed, one can define a pre-Lie product between 1PI graphs Γ_1, Γ_2 by duality*

$$\langle \Gamma_1 * \Gamma_2, \Gamma \rangle := \langle \Gamma_1 \otimes \Gamma_2, \Delta(\Gamma) \rangle,$$

with the pairing given by $\langle \Gamma_1, \Gamma_2 \rangle = 1$ if $\Gamma_1 \simeq \Gamma_2$ and zero otherwise. This pre-Lie product defines a Lie bracket by $[\Gamma_1, \Gamma_2] = \Gamma_1 * \Gamma_2 - \Gamma_2 * \Gamma_1$ with $*$ given explicitly by

$$\Gamma_1 * \Gamma_2 = \sum_\Gamma \frac{n(\Gamma, \Gamma_1, \Gamma_2)}{|\Gamma_1|_\vee} \, \Gamma.$$

LEMMA 20. *If Γ and γ are nonempty (connected) 1PI graphs, then*

$$n(\Gamma \circ_{(x,\phi)} \gamma, \gamma, \Gamma) = M(x,\phi)N(x,\phi).$$

PROOF. We have to count the number of subgraphs $\gamma' \simeq \gamma$ of $\Gamma \circ_{(x,\phi)} \gamma$ such that there is an isomorphism $(\Gamma \circ_{(x,\phi)} \gamma)/\gamma' \simeq \Gamma$.

This isomorphism can be trivial in the sense that there exists an element in $\mathrm{Aut}(\Gamma \circ_{(x,\phi)} \gamma)$ mapping γ' to γ. Otherwise, the existence of such an isomorphism implies that there is an isomorphism $\Gamma \circ_{(x,\phi)} \gamma \simeq \Gamma \circ_{(x',\phi')} \gamma$, with (x',ϕ') the image in Γ of $\mathrm{res}(\gamma')$ in the quotient $(\Gamma \circ_{(x,\phi)} \gamma)/\gamma'$; such an isomorphism maps γ in $\Gamma \circ_{(x,\phi)} \gamma$ to a certain subgraph γ' of Γ. Moreover, $[x,\phi]$ and $[x',\phi']$ are in disjoint

Aut(Γ)-orbits, since if $(x', \phi') = g(x, \phi)$, the isomorphism would be the composition of an element in Aut($\Gamma \circ_{(x,\phi)} \gamma$) and an element in Aut($\Gamma$).

We claim that all subgraphs γ' obtained in this way (for disjoint orbits) are all different subgraphs of Γ, and cannot be the image of γ under the action of an element in Aut($\Gamma \circ_{(x,\phi)} \gamma$). This would then lead to $M(x, \phi)N(x, \phi)$ many subgraphs γ' of $\Gamma \circ_{(x,\phi)} \gamma$ satisfying $(\Gamma \circ_{(x,\phi)} \gamma)/\gamma' \simeq \Gamma$.

Let $[x, \phi], [x', \phi'], [x'', \phi''] \in X(\Gamma, \gamma)$ be in disjoint orbits and suppose that there are isomorphisms

$$g' : \Gamma \circ_{(x',\phi')} \gamma \to \Gamma \circ_{(x,\phi)} \gamma,$$

$$g'' : \Gamma \circ_{(x'',\phi'')} \gamma \to \Gamma \circ_{(x,\phi)} \gamma,$$

mapping γ to subgraphs γ' and γ'' in Γ, respectively. If γ' and γ'' coincide (up to an isomorphism h), then the composition $(g'')^{-1} \circ h \circ g'$ gives an isomorphism from $\Gamma \circ_{(x',\phi')} \gamma$ to $\Gamma \circ_{(x'',\phi'')} \gamma$ mapping γ to itself. It therefore induces an element in Aut(Γ) that sends $[x', \phi']$ to $[x'', \phi'']$, which cannot be true. We conclude that γ' and γ'' are different subgraphs of Γ.

On the other hand, if there is an element ϕ in Aut($\Gamma \circ_{(x,\phi)} \gamma$) that maps γ to such a subgraph $\gamma' \in \Gamma$, the composition $\phi^{-1} \circ g'$ would map $\Gamma \circ_{(x',\phi')} \gamma$ to $\Gamma \circ_{(x,\phi)} \gamma$ isomorphically, sending γ to itself. Again, such a map must be induced by an element in Aut(Γ) mapping $[x, \phi]$ to $[x', \phi']$, contradicting our assumptions. $\square$

LEMMA 21. *Let γ, γ' be as above. Then,*

$$\Delta_{\gamma\gamma'} = \frac{1}{n_c(\gamma, \gamma')} \, \Delta_\gamma \Delta_{\gamma'} - \rho_{\gamma\gamma'},$$

where $n_c(\gamma, \gamma')$ is the number of connected components of $\gamma\gamma'$ (cf. Eq. (10)) and $\rho_{\gamma\gamma'}$ is defined by

$$\rho_{\gamma\gamma'} = \sum_{[y,\psi] \in X(\gamma,\gamma')} \frac{\text{Sym}(\gamma \circ_{(y,\psi)} \gamma')}{\text{Sym}(\gamma\gamma')} \Delta_{\gamma \circ_{(y,\psi)} \gamma'}.$$

PROOF. Consider $\Delta_{\gamma\gamma'}(\Gamma)$ on a 1PI graph Γ; if γ and γ' appear as disjoint subgraphs of Γ, this expression is given by $\Delta_\gamma \Delta_{\gamma'}(\Gamma)$, up to a factor of $n_c(\gamma, \gamma')$ which corrects for the overcounting. Indeed, let $\gamma_1, \ldots, \gamma_m$ denote all subgraphs of Γ that are isomorphic to γ. If $m \geq n + 1$, then

$$\Delta_{\gamma^{n+1}}(\Gamma) = \sum_{\substack{\{i_1,\ldots,i_{n+1}\} \\ \subset \{1,\ldots,m\}}} \frac{1}{(n+1)!} \, \Gamma/\gamma_{i_1} \cdots \gamma_{i_{n+1}};$$

$$\Delta_{\gamma^n} \Delta_\gamma(\Gamma) = \sum_{i=1}^m \sum_{\substack{\{i_1,\ldots,i_n\} \\ \subset \{1,\ldots,\hat{i},\ldots,m\}}} \frac{1}{n!} \, \Gamma/\gamma_i \gamma_{i_1} \cdots \gamma_{i_n},$$

leading precisely to the factor $n_c(\gamma^n, \gamma) = n + 1$. On the other hand, if $m < n + 1$, then both terms vanish.

In the case that Γ contains a subgraph $\widetilde{\gamma}$ such that $\widetilde{\gamma}/\gamma' \simeq \gamma$, we find a discrepancy between the two terms which is given by the following sum,

$$\rho_{\gamma\gamma'}(\Gamma) = \frac{1}{n_c(\gamma, \gamma')} \sum_{\widetilde{\gamma} \subset \Gamma, \widetilde{\gamma}/\gamma' \simeq \gamma} n(\widetilde{\gamma}, \gamma', \gamma) \, \Gamma/\widetilde{\gamma}.$$

Here $n(\widetilde{\gamma}, \gamma', \gamma)$ is by definition the number of disjoint subgraphs of $\widetilde{\gamma}$ that are isomorphic to γ' and such that $\widetilde{\gamma}/\gamma' \simeq \gamma$, which do indeed all contribute to $\Delta_\gamma \Delta_{\gamma'}(\Gamma)$. We replace the above sum by a sum over insertion places of γ' in γ, while correcting for the equivalent insertions. The latter correcting factor is given as the number of elements $[y', \phi'] \in X(\gamma, \gamma')$ such that $\gamma \circ_{(y',\phi')} \simeq \gamma \circ_{(y,\phi)}$. Such an isomorphism can be induced by an element $g \in \mathrm{Aut}(\gamma)$, with $[y', \phi'] = g[y, \phi]$ but leaving γ' untouched, leading to a factor of $|\mathrm{Aut}(\gamma)[y, \phi]|$. The number of isomorphisms $\gamma \circ_{(y',\phi')} \simeq \gamma \circ_{(y,\phi)}$ that are not induced by such an element, is given precisely by the factor $N(y, \phi)$. Thus, on inserting the expression for $n(\gamma \circ_{(y,\psi)} \gamma', \gamma', \gamma)$ derived in Lemma 20, we infer that,

$$\rho_{\gamma\gamma'}(\Gamma) = \frac{1}{n_c(\gamma, \gamma')} \sum_{[y,\psi] \in X(\gamma,\gamma')} \frac{M(y,\psi)N(y,\psi)}{N(y,\psi)|\mathrm{Aut}(\gamma)[y,\psi]|} \, \Delta_{\gamma \circ_{(y,\psi)} \gamma'}(\Gamma)$$

$$= \sum_{[y,\psi] \in X(\gamma,\gamma')} \frac{\mathrm{Sym}(\gamma \circ_{(y,\psi)} \gamma')}{\mathrm{Sym}(\gamma\gamma')} \, \Delta_{\gamma \circ_{(y,\psi)} \gamma'}(\Gamma),$$

where we have applied Lemma 18 in going to the second line. We have also used Equation (10) to replace $n_c(\gamma, \gamma')\mathrm{Sym}(\gamma)\mathrm{Sym}(\gamma')$ by $\mathrm{Sym}(\gamma\gamma')$. $\square$

3.4. The coproduct on Green's functions. In this section we derive a key formula for the coproduct on the Green's function defined above. We start with a technical lemma; recall that $\sum_{\Gamma_L^r}$ denotes the sum over all 1PI graphs with residue r and loop number L.

LEMMA 22 ([**20**]). *If γ is a 1PI graph of loop order K, then the following holds:*

(14)
$$\sum_{\Gamma_L^r} \frac{\Gamma \mid \gamma_0}{|\gamma|_\vee \mathrm{Sym}(\Gamma)} \, \Delta_\gamma(\Gamma) = \sum_{\widetilde{\Gamma}_{L-K}^r} \frac{\widetilde{\Gamma} \mid \gamma \mid \gamma_0}{\mathrm{Sym}(\gamma)\mathrm{Sym}(\widetilde{\Gamma})} \, \widetilde{\Gamma}.$$

PROOF. If $\gamma = \emptyset$, there is nothing to prove, since $\Delta_\gamma(\Gamma) = \Gamma$, $\mathrm{Sym}(\emptyset) = 1$ and $\Gamma \mid \emptyset \mid \gamma_0 \equiv \Gamma \mid \gamma_0$. We claim that the following equality holds for $\gamma, \widetilde{\Gamma} \neq \emptyset$,

$$\sum_\Gamma \frac{(\Gamma \mid \gamma_0) \, n(\Gamma, \gamma, \widetilde{\Gamma})}{|\gamma|_\vee \mathrm{Sym}(\Gamma)} = \sum_{[x,\phi] \in X(\widetilde{\Gamma}, \gamma)} \frac{(\widetilde{\Gamma} \circ_{(x,\phi)} \gamma \mid \gamma_0) \, n(\widetilde{\Gamma} \circ_{(x,\phi)} \gamma, \gamma, \widetilde{\Gamma})}{|\gamma|_\vee N(x,\phi)|\mathrm{Aut}(\widetilde{\Gamma})[x,\phi]|\mathrm{Sym}(\widetilde{\Gamma} \circ_{(x,\phi)} \gamma)}$$

Indeed, one can replace the sum on the left-hand-side over Γ by a sum over insertion places of γ in $\widetilde{\Gamma}$ (so that $\Gamma \simeq \widetilde{\Gamma} \circ_{(x,\phi)} \gamma$ for some $[x, \phi] \in X(\Gamma, \gamma)$, and also $\mathrm{res}(\widetilde{\Gamma}) = \mathrm{res}(\Gamma)$), provided one divides by a combinatorial factor counting the number of equivalent insertions. This factor is given as the number of elements $[x', \phi'] \in X(\Gamma, \gamma)$ such that $\widetilde{\Gamma} \circ_{(x',\phi')} \gamma \simeq \widetilde{\Gamma} \circ_{(x,\phi)} \gamma$, in which case $\mathrm{Sym}(\widetilde{\Gamma} \circ_{(x,\phi)} \gamma) = \mathrm{Sym}(\widetilde{\Gamma} \circ_{(x',\phi')} \gamma)$ and also $\widetilde{\Gamma} \circ_{(x,\phi)} \gamma \mid \gamma_0 = \widetilde{\Gamma} \circ_{(x',\phi')} \gamma \mid \gamma_0$.

Such an isomorphism $\widetilde{\Gamma} \circ_{(x',\phi')} \gamma \simeq \widetilde{\Gamma} \circ_{(x,\phi)} \gamma$ can be induced by an element in $g \in \mathrm{Aut}(\widetilde{\Gamma})$ with $[x', \phi'] = g[x, \phi]$ but leaving γ untouched. This leads to division by the length of the orbit $\mathrm{Aut}(\widetilde{\Gamma})[x, \phi]$. Otherwise, an isomorphism from $\widetilde{\Gamma} \circ_{(x,\phi)} \gamma$ to $\widetilde{\Gamma} \circ_{(x',\phi')} \gamma$ has to map γ to an isomorphic subgraph $\gamma' \subset \widetilde{\Gamma}$. In that case, it can not be induced by an element in $\mathrm{Aut}(\widetilde{\Gamma})$, leading precisely to the additional factor of $N(x, \phi)$.

Equation (14) now follows directly by inserting the expressions obtained in Lemma 18 and 20 in the above equation and summing over all 1PI graphs $\widetilde{\Gamma}$, as in Equation (13). We also noted on the way that by definition

$$\frac{1}{|\gamma|_{\vee}} \sum_{[x,\phi] \in X(\widetilde{\Gamma},\gamma)} \widetilde{\Gamma} \circ_{(x,\phi)} \gamma \mid \gamma_0 = \widetilde{\Gamma} \mid \gamma \mid \gamma_0.$$

The case $\widetilde{\Gamma} = \emptyset$ arises whenever $K = L$ and $\gamma \simeq \Gamma$, in which case the combinatorial factors $\Gamma \mid \gamma_0$ and $\emptyset \mid \gamma \mid \gamma_0$ coincide. $\square$

PROPOSITION 23 ([**20**]). *The coproduct takes the following form on the 1PI Green's functions:*

$$\Delta(G_L^r) \equiv \sum_{\Gamma_L^r} \frac{1}{\mathrm{Sym}(\Gamma)} \, \Delta(\Gamma) = \sum_{K=0}^{L} \sum_{\gamma_K, \widetilde{\Gamma}_{L-K}^r} \frac{\widetilde{\Gamma} \mid \gamma}{\mathrm{Sym}(\gamma)\mathrm{Sym}(\widetilde{\Gamma})} \, \gamma \otimes \widetilde{\Gamma},$$

where the sums are over all 1PI graphs $\Gamma, \widetilde{\Gamma}$ with the indicated residue and loop number, and graphs γ at the indicated loop order that are disjoint unions of 1PI graphs.

PROOF. Since $\Delta = \frac{1}{|\gamma|_{\vee}} \sum_{\gamma} \gamma \otimes \Delta_{\gamma}$, this would follow from the validity of Eq. (14) *for γ any disjoint union of 1PI graphs* at loop order $K < L$ and an auxiliary graph γ_0. Indeed, putting $\gamma_0 = \emptyset$ and summing over γ then gives the desired result. We show that this analogue of Equation (14) holds by induction on the number of connected components of γ.

Assume that Equation (14) holds for γ a (non-empty) disjoint union of 1PI graphs of loop order K. We will prove that it also holds for the disjoint union $\gamma\gamma' = \gamma \cup \gamma'$ of it with a non-empty 1PI graph γ' of loop order K'. An application of Lemma 21 yields,

$$\frac{\Gamma \mid \gamma_0}{|\gamma\gamma'|_{\vee}\mathrm{Sym}(\Gamma)} \, \Delta_{\gamma\gamma'}(\Gamma) = \frac{\Gamma \mid \gamma_0}{n_c(\gamma,\gamma')|\gamma\gamma'|_{\vee}\mathrm{Sym}(\Gamma)} \, \Delta_{\gamma}\Delta_{\gamma'}(\Gamma)$$

(15)
$$- \frac{\Gamma \mid \gamma_0}{|\gamma\gamma'|_{\vee}\mathrm{Sym}(\Gamma)} \, \rho_{\gamma\gamma'}(\Gamma).$$

Since γ' is a 1PI graph, we can apply Lemma 22 to the first term, which gives for the sum over all graphs Γ_L^r,

$$\frac{1}{n_c(\gamma,\gamma')} \sum_{\Gamma_L^r} \frac{\Gamma \mid \gamma_0}{|\gamma\gamma'|_{\vee}\mathrm{Sym}(\Gamma)} \, \Delta_{\gamma}\Delta_{\gamma'}(\Gamma)$$

$$= \frac{1}{n_c(\gamma,\gamma')} \sum_{\Gamma_{L-K'}^r} \frac{\Gamma \mid \gamma' \mid \gamma_0}{|\gamma|_{\vee}\mathrm{Sym}(\gamma')\mathrm{Sym}(\Gamma)} \, \Delta_{\gamma}(\Gamma)$$

$$= \frac{1}{n_c(\gamma,\gamma')} \sum_{\Gamma_{L-K'}^r} \frac{\Gamma \mid \gamma'\gamma_0 + (\Gamma \mid \gamma')(\gamma' \mid \gamma_0)}{\mathrm{Sym}(\gamma')|\gamma|_{\vee}\mathrm{Sym}(\Gamma)} \, \Delta_{\gamma}(\Gamma),$$

using also Equation (12). The induction hypothesis – that is, validity of Eq. (14) in the case of γ – now yields,

$$\frac{1}{n_c(\gamma,\gamma')} \sum_{\Gamma_L^r} \frac{\Gamma \mid \gamma_0}{|\gamma\gamma'|_{\vee}\mathrm{Sym}(\Gamma)} \, \Delta_{\gamma}\Delta_{\gamma'}(\Gamma) = \sum_{\widetilde{\Gamma}_{L-K-K'}^r} \frac{\widetilde{\Gamma} \mid \gamma \mid \gamma'\gamma_0 + (\widetilde{\Gamma} \mid \gamma \mid \gamma')(\gamma' \mid \gamma_0)}{\mathrm{Sym}(\gamma\gamma')\mathrm{Sym}(\widetilde{\Gamma})} \, \widetilde{\Gamma}$$

combining once more the symmetry factors $\mathrm{Sym}(\gamma)$ and $\mathrm{Sym}(\gamma')$ with the help of $n_c(\gamma, \gamma')$. For the second term in Equation (15), we can use the induction hypothesis on $\Delta_{\gamma \circ_{(y,\psi)} \gamma'}$ to show that

$$\sum_{\Gamma_L^r} \frac{\Gamma \mid \gamma_0}{|\gamma\gamma'|_\vee \mathrm{Sym}(\Gamma)} \, \rho_{\gamma\gamma'}(\Gamma) = \sum_{\widetilde{\Gamma}_{L-K-K'}^r} \sum_{[y,\psi] \in X(\gamma,\gamma')} \frac{\widetilde{\Gamma} \mid \gamma \circ_{(y,\psi)} \gamma' \mid \gamma_0}{|\gamma'|_\vee \mathrm{Sym}(\gamma\gamma') \mathrm{Sym}(\widetilde{\Gamma})} \, \widetilde{\Gamma},$$

since $|\gamma \circ_{(y,\psi)} \gamma'|_\vee = |\gamma|_\vee$. We conclude with the following equality,

$$\widetilde{\Gamma} \mid \gamma\gamma' \mid \gamma_0 = \widetilde{\Gamma} \mid \gamma \mid \gamma'\gamma_0 + (\widetilde{\Gamma} \mid \gamma \mid \gamma')(\gamma' \mid \gamma_0) - \frac{1}{|\gamma'|_\vee} \sum_{[y,\psi] \in X(\gamma,\gamma')} \widetilde{\Gamma} \mid \gamma \circ_{(y,\psi)} \gamma' \mid \gamma_0,$$

which follows easily from Definition 17. Indeed, by definition

$$\begin{aligned}
\widetilde{\Gamma} \mid \gamma \mid \gamma'\gamma_0 &+ (\widetilde{\Gamma} \mid \gamma \mid \gamma')(\gamma' \mid \gamma_0) \\
&= \frac{1}{|\gamma\gamma'|_\vee} \sum_{[x,\phi] \in X(\widetilde{\Gamma},\gamma)} \sum_{[x',\phi'] \in X(\widetilde{\Gamma}\circ_{(x,\phi)}\gamma,\gamma')} (\widetilde{\Gamma} \circ_{(x,\phi)} \gamma) \circ_{(x',\phi')} \gamma' \mid \gamma_0,
\end{aligned}$$

which counts the number of places to insert $\gamma\gamma'$ and then γ_0 in $\widetilde{\Gamma}$. Subtraction of the number of such places with γ' sitting inside γ, leads precisely to the number of places to subsequently insert $\gamma\gamma'$ and γ_0 in $\widetilde{\Gamma}$.   □

With the explicit expressions (11) for the number of insertions $\widetilde{\Gamma} \mid \gamma$ we can further simplify this expression. Recall the grading d_v by the number of vertices, and the projection $p_{n_1,\ldots,n_k}$ onto the vector subspace of H consisting of graphs with n_1 vertices of type v_1, n_2 vertices of type v_2, *et cetera*.

LEMMA 24 ([**19**]).

$$\Delta(G^r) = \sum_{\mathrm{res}(\Gamma)=r} \prod_{v \in R_V} (G^v)^{m_{\Gamma,v}} \prod_{e \in R_E} (G^e)^{-m_{\Gamma,e}} \otimes \frac{\Gamma}{\mathrm{Sym}(\Gamma)}.$$

PROOF. Let us simplify a little and consider a scalar field theory with just one type of vertex and edge, i.e. $R = \{ \prec, - \}$. We consider the sum

$$\sum_\gamma \frac{\Gamma \mid \gamma}{\mathrm{Sym}(\gamma)} \gamma = \sum_{\gamma_v} \frac{n_{\gamma,v}!}{\mathrm{Sym}(\gamma_v)} \binom{m_{\Gamma,v}}{n_{\gamma,v}} \gamma_v \sum_{\gamma_e} \frac{n_{\gamma,e}!}{\mathrm{Sym}(\gamma_e)} \binom{m_{\Gamma,e} + n_{\gamma,v} - 1}{n_{\gamma,v}} \gamma_e,$$

naturally split into a sum over vertex and edge graphs. We have also inserted the above combinatorial expression for the number of insertion places. Next, we write $\gamma_v = \gamma'_v \gamma''_v$ and try factorize the sum over γ_v into a sum over γ'_v (connected) and γ''_v. Some care should be taken here regarding the combinatorial factors but let us ignore them for the moment. In fact, if we fix the number of connected components $h^0(\gamma_v)$ of γ_v in the sum to be n_V we can write

$$\sum_{h^0(\gamma_v)=n_V} n_V! \frac{\gamma_v}{\mathrm{Sym}(\gamma_v)} = \sum_{h^0(\gamma_v)=n_V} \left(\sum_{\substack{\gamma'_v, \gamma''_v \\ \gamma'_v \gamma''_v \simeq \gamma_v}} \frac{n_c(\gamma''_v, \gamma'_v) + 1}{n_V} \right) n_V! \frac{\gamma_v}{\mathrm{Sym}(\gamma_v)},$$

with γ_v' a connected graph. Here, we have simply inserted 1,

$$\sum_{\substack{\gamma_v',\gamma_v'' \\ \gamma_v'\gamma_v''\simeq\gamma_v}} \frac{n_c(\gamma_v'',\gamma_v')+1}{n_V} = \sum_{\gamma_v'} \frac{n_c(\gamma_v,\gamma_v')}{n_V} = 1,$$

which follows directly from the definition of $n_c(\gamma_v,\gamma_v')$ as the number of connected components of γ_v isomorphic to γ_v'. Now, by definition $\mathrm{Sym}(\gamma_v'\gamma_v'') = (n_c(\gamma_v'',\gamma_v')+ 1)\mathrm{Sym}(\gamma_v')\mathrm{Sym}(\gamma_v'')$ for a connected graph γ_v' so that we obtain for the above sum

$$\sum_{\gamma_v'} \frac{\gamma_v'}{\mathrm{Sym}(\gamma_v')} \sum_{h^0(\gamma_v'')=n_V-1} (n_V-1)!\frac{\gamma_v''}{\mathrm{Sym}(\gamma_v'')} = \cdots = (G^v-1)^{n_V},$$

by applying the same argument n_V times. Recall also the definition of the Green's function G^v from Eq. (8). A similar argument applies to the edge graphs, leading to a contribution $(1-G^e)^{n_E}$, with n_E the number of connected components of γ_e. When summing over n_V and n_E, taking also into account the combinatorial factors, we obtain:

$$\sum_{n_V=0}^{\infty} \binom{m_{\Gamma,v}}{n_V}(G^v-1)^{n_V} \sum_{n_E=0}^{\infty} \binom{m_{\Gamma,e}+n_E-1}{n_E}(1-G^e)^{n_E} = (G^v)^{m_{\Gamma,v}}(G^e)^{-m_{\Gamma,e}}$$

The extension to the general setting where the set R contains different types of vertices and edges is straightforward. $\square$

THEOREM 25 ([**19**]). *If we label our vertices as $R_V = \{v_1,\cdots,v_k\}$, then*

$$\Delta(G^r) = \sum_{n_1,\ldots,n_k\in\mathbb{Z}} G^r \prod_{i=1}^{k}\left(\frac{G^{v_i}}{\prod_e (G^e)^{N_e(v_i)/2}}\right)^{n_i} \otimes p_{n_1,\ldots,n_k}(G^r).$$

PROOF. An additional counting of the number of edges and numbers of vertices in Γ gives the following relations:

$$2m_{\Gamma,e} + N_e(\mathrm{res}(\Gamma)) = \sum_{v\in R_V} N_e(v)m_{\Gamma,v}$$

where $N_e(r)$ is the number of lines (of type e) attached to $r \in R$. For instance $N_e(\prec)$ equals 2 if e is an electron line and 1 if e is a photon line. One checks the above equality by noting that the left-hand-side counts the number of internal half lines plus the external lines which are connected to the vertices that appear at the right-hand-side, taken into account their valence.

With this formula, we can write the expression of Lemma 24 as

$$(16)\qquad \Delta(G^r) = \prod_e (G^e)^{N_e(r)/2} \sum_{\mathrm{res}(\Gamma)=r} \prod_v \left(\frac{G^v}{\prod_e (G^e)^{N_e(v)/2}}\right)^{m_{\Gamma,v}} \otimes \frac{\Gamma}{\mathrm{Sym}(\Gamma)}.$$

It now remains to observe that $m_{\Gamma,v} = d_v(\Gamma)$ unless $v = r$ (the residue of Γ) in which case $m_{\Gamma,v} = d_v(\Gamma) + 1$. This yields the extra factor G^r. $\square$

4. Examples of the Hopf algebra on Green's functions

4.1. Quantum electrodynamics. Let us now apply the above formula to the case of quantum electrodynamics (QED). In (massless) quantum electrodynamics, there is only the vertex of valence three, describing the interaction of the photon with a pair of electrons. There are two types of edges corresponding to the photon (wiggly edge) and the electron (straight edge). Summarizing, we have in the notation of the previous section: $R = R_V \cup R_E$ with

$$R_V = \{\; \prec\!\!\!\!\prec \;\};$$
$$R_E = \{\; -\!\!\!- \;,\; \sim\!\!\!\sim \;\}.$$

In particular, this means that in the process of renormalization, only three types of graphs are of importance: the vertex graph, the electron self-energy graph and the vacuum polarization. Correspondingly, we have the three 1PI Green's functions,

$$G^{\prec} = 1 + \sum_{\Gamma^{\prec}} \frac{\Gamma}{\mathrm{Sym}(\Gamma)};$$

$$G^e = 1 - \sum_{\Gamma^e} \frac{\Gamma}{\mathrm{Sym}(\Gamma)},$$

with $e = -\!\!\!-,\; \sim\!\!\!\sim$.

Since there is only one vertex in QED, we can use Lemma 8 to simplify Theorem 25 above.

PROPOSITION 26 ([**18**]). *For $r = \prec,\; -\!\!\!-$ or $\sim\!\!\!\sim$ the following holds*

$$\Delta(G^r) = \sum_{l=0}^{\infty} G^r \left(\frac{G^{\prec}}{G^e \sqrt{G^{\sim}}} \right)^{2l} \otimes q_l(G^r)$$

with q_l the projection onto graphs of loop order l. $\square$

COROLLARY 27. *The elements $q_l(G^{\prec}) - q_l(G^{-}) \in H$ for $l = 1, 2, \ldots$ generate a Hopf ideal I, i.e.*

$$\Delta(I) \subseteq I \otimes H + H \otimes I, \qquad \epsilon(I) = 0, \qquad S(I) \subseteq I.$$

PROOF. This follows easily by applying Proposition 26 to the coproduct evaluated on the difference $G^{\prec} - G^{-}$, in combination with the recursive definition of the antipode. $\square$

The identities $G^{\prec} = G^{-}$ which hold in the corresponding quotient Hopf algebra H/I have a physical meaning: they are the famous *Ward identities of quantum electrodynamics* [**23**]. The above claim that they can be implemented on the Hopf algebra of Feynman graphs corresponds to the physical statement that the Ward identities are compatible with renormalization. In fact, we have the following.

PROPOSITION 28. *Suppose the regularized (but unrenormalized) Feynman rules $U : H \to K$ satisfy the Ward identities. Then the counterterms C and the renormalized Feynman rules R satisfy the Ward identities:*

$$C(G^{\prec}) = C(G^{-}); \qquad R(G^{\prec}) = R(G^{-})$$

Note that the first equation is usually written as $Z_1 = Z_2$ [**23**].

PROOF. This follows directly from the Birkhoff decomposition (cf. Theorem 9 above) applied to the character group of the graded connected commutative Hopf algebra H/I. □

4.2. Quantum chromodynamics. We work in the setting of the non-abelian gauge theory *quantum chromodynamics* (QCD). It describes the interaction between quarks (the fermions) via gluons (the gauge bosons).

In contrast with quantum electrodynamics described previously, there are now three vertices of valence three, describing the interaction of the fermion and ghost with the gluon, as well as the cubic gluon self-interaction. In addition, there is the quartic gluon self-interaction. This means that the Feynman graphs are built from the following two sets of vertices and edges:

$$R_V = \{ \text{⚛}, \ \text{⚛}, \ \text{⚛}, \ \text{⚛} \};$$

$$R_E = \{ \ \text{——}, \ \text{········}, \ \text{⚛⚛⚛⚛} \},$$

where the plain, dotted and curly lines represent the quark, ghost and gluon, respectively.

The relation between the number of vertices and loop order is not as simple as in QED, due to the presence of 4 different vertices. Nevertheless, we can explore the structure of the Hopf algebra by introducing the following formal elements corresponding to each vertex $v \in R_V$:

$$X_v := \left(\frac{G^v}{\prod_e (G^e)^{N_e(v)/2}} \right)^{1/N(v)-2}.$$

Theorem 25 in combination with the fact that Δ is an algebra map then yields the following result.

PROPOSITION 29. *The coproduct on X_v ($v \in R_V$) is given by*

$$\Delta(X_v) = \sum_{n_1,\ldots,n_4} X_v (X_{\text{⚛}})^{n_1} (X_{\text{⚛}})^{n_2} (X_{\text{⚛}})^{n_3} (X_{\text{⚛}})^{2n_4} \otimes p_{n_1,\ldots,n_4}(X_v).$$

COROLLARY 30. *The elements $q_l(X_v) - q_l(X_{v'})$ for any $v, v' \in R_V$ and $l = 1, 2, \ldots$ generate a Hopf ideal I in H.*

PROOF. Let I be the algebra ideal in H generated by $q_l(X_v) - q_l(X_{v'})$. In the expression of Proposition (29) for the coproduct on X_v, we can replace each X_v that appears on the first leg of the tensor product by $X_{\widetilde{v}}$ for some fixed but arbitrary $\widetilde{v} \in R_V$, as long as we add terms in $I \otimes H$ to it. Thus, for all $v \in R_V$ we have

$$\Delta(X_v) = \sum_{n_1,\ldots,n_4} (X_{\widetilde{v}})^{n_1+\cdots 2n_4+1} \otimes p_{n_1,\ldots,n_4}(X_v) + I \otimes H$$

$$= \sum_l (X_{\widetilde{v}})^{2l+1} \otimes q_l(X_v) + I \otimes H,$$

where in going to the second line, we have applied Lemma 8 to write $n_1 + \cdots 2n_4 = 2l$. With the first leg of $\Delta(X_v)$ independent of v, we easily obtain that

$$\Delta(X_v) - \Delta(X_{v'}) = \sum_l (X_{\widetilde{v}})^{2l+1} \otimes q_l(X_v - X_{v'}) + I \otimes H$$

thus establishing that $\Delta(I) \subset I \otimes H + H \otimes I$. The recursive definition of the antipode allows one to conclude directly that $S(I) \subset I$. □

Again these identities have a physical meaning, they are the *Slavnov–Taylor identities for the couplings* in quantum chromodynamics. Again, the above fact that they can be imposed on the Hopf algebra is a rephrasing of the physical fact that these identities are compatible with renormalization. More precisely, we have the following

PROPOSITION 31. *Suppose the regularized (but unrenormalized) Feynman rules $U : H \to K$ satisfy the Slavnov–Taylor identities for the couplings. Then the counterterms C and the renormalized Feynman rules R satisfy the Slavnov–Taylor identities:*

$$C(X \underset{\prec}{\smile}) = C(X \underset{\prec}{\smile}) = C(\underset{\prec}{\smile}) = C(\sqrt{\times})$$
$$R(X \underset{\prec}{\smile}) = R(X \underset{\prec}{\smile}) = R(\underset{\prec}{\smile}) = R(\sqrt{\times})$$

Again, the first equation is typically written in terms of Z-factors. This would lead precisely to Equation (6) above.

PROOF. As in the case of QED, this follows from the Birkhoff decomposition in the character group of the quotient Hopf algebra H/I. □

Let us end this section with a formula for the coproduct on the element $X := X_v^{1/N(v)-2}$ in H/I:

$$\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X). \tag{17}$$

5. Dyson–Schwinger equations and Hochschild cohomology

In this section, we will review how Hochschild cohomology fits nicely in the context of renormalization Hopf algebras, following [**7, 2**] and [**14**]. In particular, we will relate it to the Dyson–Schwinger equations and prove the so-called gauge theory theorem that was announced in [**14**].

Let us first recall the definition of Hochschild cohomology for Hopf algebras, – or, more generally, for bialgebras – with values in a bicomodule. This dualizes the definition of Hochschild cohomology for algebras to bialgebras. Let H be a bialgebra and M an H-bicomodule, i.e. there are two cocommuting left and right coactions $\rho_L : M \to H \otimes M$ and $\rho_R : M \to M \otimes H$. We denote by $C^n(H, M)$ the space of linear maps $\phi : M \to H^{\otimes n}$ and define the *Hochschild coboundary map* $b : C^n(H, M) \to C^{n+1}(H, M)$ by

$$b\phi = (\mathrm{id} \otimes \phi)\rho_L + \sum_{i=1}^{n}(-1)^n \Delta_i \phi + (-1)^{n+1}(\phi \otimes \mathrm{id})\rho_R.$$

where Δ_i denotes the application of the coproduct on the i'th factor in $H^{\otimes n}$. Coassociativity implies that b is a differential, i.e. that $b^2 = 0$.

DEFINITION 32. *The Hochschild cohomology $HH^{\bullet}(H, M)$ of the bialgebra H with values in the H-comodule M is defined as the cohomology of the complex $(C^{\bullet}(H, M), b)$ defined above.*

We are interested in the particular case that $M = H$ is a comodule over itself, with $\rho_L = \Delta$ but with $\rho_R = (\text{id} \otimes \epsilon)\Delta$. We denote the Hochschild cohomology groups in this case by $HH^\bullet(H, H_\epsilon)$ or simply $HH^\bullet_\epsilon(H)$ as in [**7**]. Let us consider the case $n = 1$, then $\phi \in HH^1_\epsilon(H)$ means simply that

$$\Delta\phi = (\text{id} \otimes \phi)\Delta + (\phi \otimes \mathbb{I}).$$

where $(\phi \otimes \mathbb{I})(h) \equiv \phi(h) \otimes 1$ for $h \in H$. As was observed in [**7**] the grafting operator on rooted trees is an example of such a 1-cocycle. We will give an example in the case of the Hopf algebra of Feynman graphs (cf. (2) of Theorem 33 below), following [**14**].

This starts with the observation that the Green's functions can be dissected as follows [**14**, Theorem 4]:

$$(18) \qquad G^r = \sum_{\gamma \text{ prim}} B_+^\gamma \left(\frac{\prod_{v \in R_V} (G^v)^{m_{\gamma,v}}}{\prod_{e \in R_E} (G^e)^{m_{\gamma,e}}} \right) = \sum_{\gamma \text{ prim}} B_+^\gamma(G^{\text{res}(\gamma)} X^{2l(\gamma)})$$

where B_+^γ is the (normalized) grafting operator that inserts in γ the graphs given as its argument on the appropriate insertion places. The sum is over all primitive graphs γ, i.e. satisfying $\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$. It is clear that any graph in G^r is of the form $B_+^\gamma(\Gamma_1 \cdots \Gamma_N)$ for some 1PI graphs $\Gamma_1, \ldots, \Gamma_N$ but this decomposition is highly non-unique. In order to correct for the overcounting, the grafting operators have to be normalized appropriately as was done in [**14**]. We will instead take Eq. (18) as a *definition* of the normalized maps B_+^γ, without explicitly describing this normalization. The sum of the B_+^γ over all primitive 1PI Feynman graphs at a given loop order and with given residue will be denoted by $B_+^{k;r}$, as in *loc. cit.*. More precisely,

$$B_+^{k;r} = \sum_{\substack{\gamma \text{ prim} \\ l(\gamma)=k \\ \text{res}(\gamma)=r}} \frac{1}{\text{Sym}(\gamma)} B_+^\gamma$$

and, of course, $G^r = \sum_{k,r} B_+^{k;r}(X_{k,r})$, where we have denoted $X_{k,r} = G^r X^{2k}$. With this and the formulas of the previous section on QCD, we can prove the *gauge theory theorem* as formulated in [**14**, Theorem 5]:

THEOREM 33. *Let $\widetilde{H} = H/I$ be the Hopf algebra of QCD Feynman graphs (cf. Sect. 4.2) with the Slavnov–Taylor identities for the couplings imposed.*

(1) $G^r = \sum_{k=0}^\infty B_+^{k;r}(X_{k,r})$
(2) $\Delta(B_+^{k;r}(X_{k,r})) = B_+^{k;r}(X_{k,r}) \otimes \mathbb{I} + (\text{id} \otimes B_+^{k;r})\Delta(X_{k,r}).$
(3) $\Delta(G_k^r) = \sum_{j=0}^k \text{Pol}_j^r(G) \otimes G_{k-j}^r.$

where $\text{Pol}_j^r(G)$ is a polynomial in the G_m^r of degree j, determined as the order j term in the loop expansion of $G^r X^{2k-2j}$.

PROOF. The first claim is just the definition of the $B_+^{k;r}$. For (2), we first enhance the result of Proposition 29 to partial sums in G^r over graphs that have 'primitive residue' isomorphic to a fixed primitive graph γ. In other words, if $G^{r,\gamma}$ is the part of G^r that sums only over graphs that are obtained by inserting graphs into the primitive graph γ, then

$$\Delta(G^{r,\gamma}) = G^{r,\gamma} \otimes 1 + \sum_{l=1}^\infty G^r X^{2l} \otimes q_l(G^{r,\gamma}).$$

Here we have imposed the Slavnov–Taylor identities for the couplings to write this in terms of a single coupling, X. Combing Proposition 29 and Eq. (17) we obtain for the coproduct of $X_{k,r} = G^r X^{2k}$:

$$\Delta(X_{k,r}) = \sum_{l=0}^{\infty} G^r X^{2l+2k} \otimes q_l X_{k,r}.$$

Since $G^{r,\gamma} = B_+^\gamma(X_{k,r})$, it follows by a combination of the above two formula that

$$\Delta(B_+^\gamma(X_{k,r})) = B_+^\gamma(X_{k,r}) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^\gamma)\Delta(X_{k,r}).$$

and summing over all primitive graphs with residue r at loop order k gives the desired result.

Finally, (2) follows by combining Theorem 25 with Proposition 29, thereby taking into account the Slavnov–Taylor identities. □

REMARK 34. *We have corrected for the apparent misprint in* [**14**, Eq. (83)].

In fact, this proves the slightly stronger result that actually every B_+^γ defines a Hochschild 1-cocycle:

PROPOSITION 35. *For γ a primitive graph at loop order k and residue r, we have*

$$\Delta(B_+^\gamma(X_{k,r})) = B_+^\gamma(X_{k,r}) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^\gamma)\Delta(X_{k,r}).$$

Actually, the above results apply in full generality for any Hopf algebra as defined in Definition 7. However, the meaning of the Hopf ideals as imposing Slavnov–Taylor identities can only be given in the context of a non-abelian gauge theory. Moreover, the above Hochschild cocycles $B_+^{k;r}$ play an important role in that they give quantum equation of motions. These *Dyson–Schwinger equations* are the recursive construction of the 1PI Green's functions G^r from the lower order Green's functions in $X_{k,r}$ forming the argument of the B_+-operations. In fact, Equation (18) are precisely the Dyson–Schwinger equations for quantum chromodynamics.

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