Gauge theories on the circle

Author: Fokke de Haan

Supervisor: Dr.W.D. van Suijlekom

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Contents

Abstract

Given a Lie group G and a principal G-bundle $P \to \mathbb{T}$, let $\mathscr{A}$ denote the space of smooth connections on P. For any discrete subset Λ of the circle $\mathbb{T}$, this space of connections can be identified with G^n where n is the amount of points in Λ: any connection can be identified with the parallel transports between the points of Λ. The space $L^2(\mathscr{A}^*, d\mu_t)$ as described in [15],[27] can be written as the direct limit of the spaces $L^2(G^\Lambda, d\rho_t)$, where ρ_t is a heat kernel measure on G^Λ. This direct system of Hilbert spaces maintains several of the properties described for the direct system described in [1] of spaces $L^2(G^\Lambda, dx)$ where dx is the Haar measure on G^Λ.

CHAPTER 1

Introduction

The main topic of this thesis will be principal bundles and principal connections. If G is a Lie group, a principal G-bundle is a fiber bundle $P \to M$ that locally looks like $M \times G$. Furthermore, P should come equipped with a right action of G that generalizes the action $g(m, h) = (m, gh)$ on $M \times G$. A connection on such a principal bundle is essentially a way to identify different fibers by transporting elements between them along paths on the manifold. These principal bundles and connections were independently discovered by physicists as they can be used to formulate gauge theories - a type of field theory used to describe the interactions of elementary particles. For example, quantum electrodynamics is a gauge theory with as its symmetry group $U(1)$. The standard model of physics can also be described as a gauge theory, with a more complicated Lie group as its symmetry group. The connections on a principal bundle are known to physicists as gauge fields. The magnetic potential in electrodynamics would be an example of such a field. In this thesis, we will focus purely on principal bundles and connections on the circle.

If Λ is a discrete subset of the circle - so a finite set of points - the principal bundle on the circle induces a principal bundle on Λ. The notion of a connection on this principal bundle as is not very interesting, but connections on the circle do have a discrete analogue. If the finite subset Λ has - say - n points, any connection on the entire circle induces an element of G^n, because the parallel transport between any two points of Λ can be identified with a point of the Lie group. We will look at the question, whether we can in some sense consider the space of connections on the entire circle to be the limit of these spaces G^n of discrete connections on the finite subsets.

There is a great deal of preliminary results and definitions that will need to be treated, which we will do in the second chapter. In particular we will give a short recap of the theory of unbounded operators, Lie groups and Sobolev spaces. Furthermore, several basic results about the Fock space $\Gamma(\mathscr{H})$ associated to a Hilbert space $\mathscr{H}$ will be discussed. Lastly, we will look at the notion of a direct limit, particularly the direct limit of a system of Hilbert spaces. For a more detailed discussion of the material dealt with in any particular section, one can read the short introductions before each section.

The third chapter is a chapter containing several technical results about Gaussian measures on infinite dimensional spaces that will be useful throughout the rest of the thesis. We will also discuss a generalization of such measures called Gross measures, where the norm need not be associated to a Hilbert-Schmidt operator but has a more general form. We will come across several examples of such measures in the fourth and fifth chapter especially.

In the fourth chapter we will finally look at gauge theories on the circle. The first section will deal with the definition of principal bundles along with some basic results. The second section we will see that the space of smooth connections on a principal bundle on the circle - denoted $\mathscr{A}$ - can be completed to a Hilbert space $\mathscr{H}$ of square-integrable connections. We can further complete this space to the space of distributions $\mathscr{A}^*$, which we will show can be equipped with a Gaussian measure as defined in chapter 3. The main result will be showing that Fock space of $\mathscr{H}$ is naturally equivalent to the space of square-integrable functions on $\mathscr{A}^*$ with respect to this Gaussian measure.

The fifth chapter will deal with the discrete approximations of gauge theories on the circle. As discussed earlier, if we look at only a finite set of n points of the circle, the most logical generalization of the space of connections would be G^n - identified as the parallel transport between the points. We will see that the L^2-spaces $L^2(G^n, d\rho_t)$ with respect to a heat kernel measure forms a direct system of Hilbert spaces, with as its direct limit the space of square-integrable functions on $\mathscr{A}^*$ as introduced in the previous chapter.

The final chapter compares the results from above with the results found in [1], in which the direct limit of the Hilbert spaces $L^2(G^n)$ with respect to the Haar-measure is described. Several of the propositions from this article still hold in our new directed system, and one could argue the directed system from chapter 5 has the directed system from this article as a limiting case.

1.1 Acknowledgments

I especially want to thank the other students that were working in the masterkamer in this past year. Despite sometimes being a bit of a distraction a project like this would have been far more difficult to complete working alone. Of course I would also like to thank my advisor Walter for a great amount of advice and direction and an even greater amount of patience.

CHAPTER 2

Preliminaries

In this chapter we will briefly look at some preliminary results and definitions that will be useful throughout the rest of the text. In the first section some notions from functional analysis will be recalled, in particular some of the theory of unbounded operators. Furthermore we will look at Sobolev spaces and at Fock spaces. The second section will deal with direct limits. The final section will deal with Lie groups and lie algebras.

2.1 Functional analysis

In this section, we will look at some of the basics of the theory of unbounded operators on Hilbert spaces. In particular, we will define the adjoint of such an operator and discuss what it means for such an unbounded operator to be self adjoint. Furthermore, we will prove several lemmas that will be convenient throughout the rest of the text. In this thesis, we will come across several unbounded operators, mostly either differential operators on multiplication operators. In particular we will come across the Laplace operator Δ_G on a Lie group G.

Next, we will look at Sobolev spaces. As we will only need Sobolev spaces of functions on one-dimensional spaces - either an interval or the circle - it will not be as involved as the general theory.

The last subsection will deal with Fock spaces.

2.1.1 Unbounded operators

This section will be a quick recap of the theory of unbounded operators. We will not look at proofs for most of the statements in this section. For all the proofs and required background one can look at the eighth chapter of [26]. Throughout this chapter let $\mathscr{H}$ denote a Hilbert space.

Definition 2.1. *A (densely defined) unbounded operator D on $\mathscr{H}$ is a linear map $D : \operatorname{dom}(D) \to \mathscr{H}$. where $\operatorname{dom}(D)$ is a dense subspace of $\mathscr{H}$ we call the domain of D.*

Definition 2.2. *Let D be an unbounded operator. Define*

$$\operatorname{dom}(D^*) = \{ x \in \mathscr{H} \mid y \mapsto \langle x, Dy \rangle \text{ is continuous} \},$$

where the functional is defined on $\operatorname{dom}(D)$. If $x \in \operatorname{dom}(D^*)$, the functional $y \mapsto \langle x, Dy \rangle$ can be extended to an element of $\mathscr{H}^*$. By the Riesz representation theorem, this functional corresponds *uniquely to an element of $\mathscr{H}$. This element will be D^*x. Therefore the operator $D^* : \mathrm{dom}\,(D^*) \to \mathscr{H}$ is defined entirely by demanding that*

$$\langle x, Dy\rangle = \langle D^*x, y\rangle$$

for all $y \in \mathrm{dom}\,(D)$.

It is not necessary for $\mathrm{dom}\,(D^*)$ to once again be dense in $\mathscr{H}$. As such D^* need not be a densely defined operator. For example, if $\mathscr{H} = l^2(\mathbb{N})$, and $\mathrm{dom}\,(D) = l^0(\mathbb{N})$ where $Dx = \left(\sum_{i=1}^{\infty} x(i)\right) e_1$, the functional

$$y \mapsto \langle x, Dy\rangle = x_1 \left(\sum_{i=1}^{\infty} y(i)\right)$$

is only continuous if $x_1 = 0$. Such functions are not dense in $\mathscr{H}$.

However, if D^* is again densely defined we will say the operator D is closable. The closure of D is exactly $(D^*)^*$.

Definition 2.3. *Suppose D is an unbounded operator such that $D^* = D$. We call such a D self-adjoint.*

Note that this is stronger than simply requiring that $\langle Dx, y\rangle = \langle x, Dy\rangle$ for all $x, y \in \mathrm{dom}\,(D)$. Such a D is called symmetric.

Definition 2.4. *An unbounded operator D is called essentially self-adjoint if its closure $(D^*)^*$ is self-adjoint.*

In this case $D^* = (D^*)^*$ so D^* is already the closure of D.

The following proposition is theorem 8.3 in [26], along with its corollary.

Proposition 2.5. *Suppose D is a symmetric unbounded operator. Then D is essentially self-adjoint if and only if the operators $(D \pm i)$ have dense range.*

Definition 2.6. *Let D be a closed unbounded operator. That is to say, $D = (D^*)^*$. The spectrum of D, denoted $\sigma(D)$, is the set of $\lambda \in \mathbb{C}$ such that $D - \lambda$ is not bijective.*

If $\lambda \notin \sigma(D)$, then the inverse $(D - \lambda)^{-1}$ exists as a bounded linear operator.

Definition 2.7. *Let D be an unbounded operator and let $\mu \notin \sigma(D)$. The resolvent of D is given by $R(D, \mu) = (D - \mu)^{-1}$.*

Lemma 2.8. *Let D be a self-adjoint operator on $\mathscr{H}$ and let $\mu \notin \sigma(D)$. Then*

$$\|R(D, \mu)\| \leq \frac{1}{dist(\sigma(D), \mu)}.$$

Proof. Let $\mu \in \rho(D)$. Then, for any $\lambda \neq \mu$,

$$D - \lambda = \left(R(D, \mu) - (\lambda - \mu)^{-1}\right)(D - \mu)(\mu - \lambda),$$

so $\lambda \in \rho(D)$ if and only if $(\lambda - \mu)^{-1} \in \rho(R(D, \mu))$. Hence,

$$\sigma(R(D, \mu)) \setminus \{0\} = \left\{ \frac{1}{\lambda - \mu} \middle| \lambda \in \sigma(D) \right\}.$$

We get the desired result by using that the spectral radius and norm coincide for bounded normal operators. □

Proposition 2.9. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let D_1, D_2 be self-adjoint operators on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. The operator given by*

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

with domain $\operatorname{dom}(D) = \operatorname{dom}(D_1) \oplus \operatorname{dom}(D_2)$ *is self-adjoint also.*

The following proposition is dealt with in section 8.10 in [26].

Proposition 2.10. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let D_1, D_2 be self-adjoint operators on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Then the operator D_0 on $\mathscr{H}_1 \otimes \mathscr{H}_2$ with domain* $\operatorname{span}\{x \otimes y | x \in \operatorname{dom}(D_1), y \in \operatorname{dom}(D_2)\}$ *given by*

$$D_0(x \otimes y) = (D_1 x \otimes y) + (x \otimes D_2 y)$$

is essentially self adjoint.

We will denote its closure by $D_1 \otimes 1 + 1 \otimes D_2$.

2.1.2 Sobolev spaces

Let us denote by I the unit interval $[0, 1]$. This section will focus mostly on describing the Sobolev spaces $H^k(I, \mathbb{R})$ and $H^k(\mathbb{T}, \mathbb{R})$, where $\mathbb{T}$ denotes the circle. These are Hilbert spaces of functions - or in the case that k is negative, distributions - that are in some sense differentiable.

Definition 2.11. *Let $k \in \mathbb{N}_{>0}$. The Sobolev space $H^k(I, \mathbb{R})$ of order k is the space*

$$H^k(I, \mathbb{R}) = \{ \varphi \in L^2(I, \mathbb{R}) \mid \varphi^{(k)} \in L^2(I, \mathbb{R}) \}$$

Here $\varphi^{(k)}$ is the k-th derivative of φ as a distribution. The full definition can be found in the third chapter of [13] - in particular definition 3.5 - but will not be needed in all its generality in the rest of this thesis.

It can be shown that any function on an interval that is an element of $H^k(I, \mathbb{R})$ for some $k \geq 1$ is necessarily continuous.

These Sobolev spaces are Hilbert spaces with respect to the inner product

$$\langle \varphi, \psi \rangle_k = \sum_{j=1}^{k} \langle \varphi^{(k)}, \psi^{(k)} \rangle.$$

Definition 2.12. *Let $k \in \mathbb{N}_{>0}$. The dual Sobolev space $H^{-k}(I, \mathbb{R})$ is the dual space of $H^k(I, \mathbb{R})$.*

We can similarly consider the space $H^k(\mathbb{T}, \mathbb{R})$ of real-valued functions that are k-times weakly differentiable on the circle. It is the closed subspace of $H^k(I, \mathbb{R})$ of functions f such that $f(0) = f(1)$.

Proposition 2.13. *Let G be a Lie group and let $\mathfrak{g}$ be a lie algebra. Then $\sigma \in H^1(I, \mathfrak{g})$ if and only if*

$$\int_I |\dot{\sigma}\sigma^{-1}|^2 d\tau < \infty.$$

The right-hand expression in the above proposition is known as the energy of the path σ. As such, $H^1(I, \mathfrak{g})$ is also known as finite energy paths.

Definition 2.14. *Let $f_n \in C^\infty(\mathbb{T})$ be a sequence of smooth functions. Then we will say $f_n \to f$ if and only if $f_n \to f$ in $H^k(\mathbb{T}, \mathbb{R})$ for every non-negative integer k.*

2.1.3 Fock spaces

In this section the notion of a Fock space is introduced.

Definition 2.15. *Let $\mathscr{H}$ be a Hilbert space. The n-fold tensor product of $\mathscr{H}$ will be denoted by*

$$\mathscr{H}^{\otimes n} = \underbrace{\mathscr{H} \otimes \ldots \otimes \mathscr{H}}_{n \text{ copies}}.$$

For $v_1, \ldots, v_n \in \mathscr{H}$, the symmetric tensor product is defined as

$$v_1 \odot \ldots \odot v_n = \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

The closure of the subspace of $\mathscr{H}^{\otimes n}$ spanned by such symmetric tensor products is called the n-fold symmetric tensor product of $\mathscr{H}$ and will be denoted by $\mathscr{H}^{\odot n}$.

More generally, if $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, we can define the symmetric tensor product $\mathscr{H}_1 \odot \mathscr{H}_2$ of the two to be the Hilbert subspace of $(\mathscr{H}_1 \otimes \mathscr{H}_2) \oplus (\mathscr{H}_2 \otimes \mathscr{H}_1)$ generated by elements of the form $v \otimes w + w \otimes v$, where $v \in \mathscr{H}_1$ and $w \in \mathscr{H}_2$. There is also a notion of asymmetric tensor products of Hilbert spaces, but this notion will not be needed in the rest of this text.

Definition 2.16. *Let $\mathscr{H}$ be a Hilbert space. The symmetric (or bosonic) Fock space of $\mathscr{H}$ is the Hilbert space*

$$\Gamma(\mathscr{H}) = \widehat{\bigoplus_{n \geq 0}} (\mathscr{H})^{\odot n}.$$

Suppose a Hilbert space $\mathscr{H}$ can be decomposed as a direct sum $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. If $v_1 \odot \ldots \odot v_k \odot w_1 \odot \ldots w_l \in \mathscr{H}^{\odot k+l}$, where the $v_i \in \mathscr{H}_1$ and $w_j \in \mathscr{H}_2$, it his could also be interpreted as an element of $\mathscr{H}_1^{\odot k} \odot \mathscr{H}_2^{\odot l}$. The induced map $U : \Gamma(\mathscr{H}) \to \Gamma(\mathscr{H}_1) \odot \Gamma(\mathscr{H}_2)$ is unitary.

Definition 2.17. *Let $n \in \mathbb{N}$, let $\mathscr{H}$ be a Hilbert space and let $u \in \mathscr{H}$. Define the operators $a_n(u) : \mathscr{H}^{\odot n+1} \to \mathscr{H}^{\odot n}$ as*

$$a_n(u_1 \odot \ldots \odot u_{n+1}) = \sum_{i=1}^{n+1} \left(\langle u, u_i \rangle \odot u_1 \odot \ldots \odot \widehat{u_i} \odot \ldots \odot u_{n+1} \right).$$

Furthermore, define $a_{-1}u : \mathbb{C} \to \mathbb{C}$ to be $a_{-1}(u)(\lambda) = 0$.

The adjoint of $a_n(u)$ is given by

$$a_n^*(u)(u_1 \odot \ldots \odot u_n) = u \odot u_1 \odot \ldots \odot u_n.$$

Consider the unbounded operator $a(u) : \Gamma(\mathscr{H}) \to \Gamma(\mathscr{H})$ with as a domain all the finite combinations of elements of $\mathscr{H}^{\odot n}$ for some n, that equals a_n on an element of the $n+1$-fold tensor product. We can define $a^*(u)$ in a similar way. These operators $a(u)$ and $a^*(u)$ certainly have densely defined adjoints, because the adjoint $(a(u))^*$ contains $a^*(u)$ and vice versa. Hence, they are both closable.

These closures will still be denoted $a^*(u)$ and $a(u)$ respectively. They are called the creation and annihilation operators.

Proposition 2.18. *$a^*(u)$ is indeed the adjoint of $a(u)$, justifying the notation.*

Proof. As had already been established, $a(u)^* \supseteq a^*(u)$. As for the converse, suppose $f = (f_n)_{n\in\mathbb{N}} \in \operatorname{dom}(a(u)^*)$ and let $g \in \mathscr{H}^{\odot n+1}$. Then

$$\begin{aligned}\langle a(u)^* f, g\rangle &= \langle f, a(u)g\rangle \\ &= \langle f_n, a_n(u)g\rangle \\ &= \langle a_n^*(u)f_n, g\rangle \\ &= \langle a^*(u)f, g\rangle.\end{aligned}$$

Hence, if $a(u)^* f := k = (k_n)_{n\in\mathbb{N}}$, then $k_{n+1} = a_n^*(u)f_n = (a^*(u)f)_{n+1}$. It follows that $f \in \operatorname{dom}(a^*(u))$ and that $a^*(u)f = a(u)^*f$ as required. □

One very convenient set of vectors lying dense in the Fock space $\Gamma(\mathscr{H})$ of some Hilbert space $\mathscr{H}$ is given by the exponential vectors, sometimes also called coherent vectors.

Definition 2.19. *Let $v \in \mathscr{H}$. The exponential vector of v is the element in $\Gamma(\mathscr{H})$ given by*

$$\exp(v) = \sum_{n\in\mathbb{N}} \frac{1}{n!} v^{\odot n}.$$

The normalization is exactly such that given $v, w \in \mathscr{H}$, we have that $\langle \exp(v), \exp(w)\rangle = \exp(\langle v, w\rangle)$. Such vectors are also occasionally called coherent vectors. Proofs of the following propositions can be found in chapter 2 of [23].

Proposition 2.20. *For any finite sequence $v_i \in \mathscr{H}$ the sequence $\exp(v_i)$ is a sequence of linearly independent vectors.*

Proposition 2.21. *Exponential vectors are dense in $\Gamma(\mathscr{H})$.*

Example 2.22. Let us briefly discuss the simplest nontrivial example of a Fock space, $\Gamma(\mathbb{C})$. This arises naturally as the Hilbert space associated with the one-dimensional harmonic oscillator. It follows that there is an unitary map

$$U : \Gamma(\mathbb{C}) \to L^2(\mathbb{R}, d\rho_t)$$

where ρ_t denotes the heat kernel measure of variance $t > 0$ on the real line. That is to say

$$\int_{x\in\mathbb{R}} f(x) d\rho_t = \int f(x) \exp\left(\frac{x^2}{2t}\right) (2\pi t)^{-\frac{1}{2}} dx.$$

for any measurable function f on $\mathbb{R}$. Here dx denotes the standard Lebesgue measure. This measure is also known as the Gaussian measure on $\mathbb{R}$. The map U is given by diagonalizing the one-dimensional harmonic oscillator. It maps the ground state in $\Gamma(\mathbb{C})$ to the constant function 1 on $\mathbb{R}$.

Lemma 2.23. *The map U is given explicitly on exponential vectors by*

$$U(\exp(v)) = \exp(-v^2/2t - xv/t).$$

2.2 Direct limits

In this section, the notion of a direct limit is introduced. Most importantly, we will look at the direct limit of a directed system of Hilbert spaces. Furthermore, we will show that we can take limits of special sequences of self-adjoint unbounded operators, which will allow us to define operators on larger spaces later on in the thesis. Lastly, we will discuss the Kolmogorov extension theorem, allowing us to take limits of measure spaces.

2.2.1 General Definitions

Throughout the thesis occasionally some notions from category theory will come in handy. In particular, we will work with direct limits - which most of this section is about - and we will come across several functors. We will not use any non-trivial results from category theory at all, so no knowledge beyond the basic definitions is necessary.

Definition 2.24. *A (small) category consists of the following:*

1. *a set of objects C;*
2. *a set of morphisms $f : c \to d$ where $c, d \in C$. The identity on any object c is always assumed to be in this set.;*
3. *a composition of morphisms; if $f : c \to d$ and $g : d \to e$, then there is a morphism $g \circ f : c \to e$. Composition with the identity on an object c does nothing.*

Definition 2.25. *Let F be a map from two categories C_1, C_2, i.e. $F : C_1 \to C_2$ such that given a morphism $f : c \to d$ there is a Such a map F is called a functor is it respects composition, so if $F(g \circ f) = F(g) \circ F(f)$, and if $F(id) = id$.*

Definition 2.26. *Let $\mathbb{F}$ be a directed set. Assume that for every $f \in \mathbb{F}$ there is an object A_f, and that for every pair $f < h$ there is a morphism $u_{hf} : A_f \to A_h$ such that $u_{ff} = id$ for all f and such that if $f < h < g$, that $u_{gh} \circ u_{hf} = u_{fg}$. The family $\{A_f, u_{fh}\}$ is called a directed system.*

Definition 2.27. *Let $\{A_f, u_{fh}\}$ be a directed system and let A be an object such that for all $f \in \mathbb{F}$ there exists a morphism $u_f : A_f \to U$ such that $u_f u_{fh} = u_h$ for all $h < f$. We call such an object A a target.*

Definition 2.28. *Let $\{A_f, u_{fh}\}$ be a directed system. A target A_∞ is called the direct limit if it is the smallest target of the directed system.*

That is to say, A_∞ has the following universal property: If A is a target of the directed system with associated morphisms v_f, then there is a unique morphism $v_\infty : A_\infty \to A$ such that $v_f = v_\infty u_f$ for all $f \in (F)$.

By the uniqueness of the morphism, if the direct limit exists, it is unique up to isomorphism. We will also write $\lim A_f$ for the direct limit.

Example 2.29. Examples of categories that admit direct limits that we will come across are Hilbert spaces and C^*-algebras. We will deal with Hilbert spaces in the next section. For a construction of direct limits of C^*-algebras confer [18], where it is treated in chapter 11.

Let $\{\, A_f, \Sigma_f \,\}_{f \in \mathbb{F}}$ be net of measure spaces, where the Σ_f are the associated Σ-algebras. For any finite subset E of $\mathbb{F}$ let μ_E be a probability measure on $A_E = \prod_{f \in E} A_f$ equipped with the product Σ-algebra. Suppose that if $E, \tilde{E}$ are finite subsets of $\mathbb{F}$ such that $E \subseteq \tilde{E}$. Denote by $\pi_{E,\tilde{E}}$ the projection map on $A_{\tilde{E}}$ onto A_E. Suppose that for all such pairs, $\mu_{\tilde{E}} = \mu_E \pi_{E,\tilde{E}}$.

Theorem 2.30. *Under the assumptions above, there is a probability measure μ on $A = \prod_{f \in \mathbb{F}} A_f$ such that $\mu_E = \mu \pi_E$ where π_E denoted the projection onto A_E.*

2.2.2 Hilbert Spaces

The goal of this subsection will be to show that the category of Hilbert spaces, where morphisms are isometries, admits direct limits.

Recall that an operator p on a Hilbert space $\mathscr{H}$ is called an ortogonal projection precisely if $p^2 = p^* = p$. There is a one to one correspondence between projections and closed subspaces of $\mathscr{H}$, given by mapping a projection p to the closed subspace $p\mathscr{H}$. For two projections p and q we will say $p \leq q$ exactly if $p\mathscr{H}$ is contained in $q\mathscr{H}$, which is equivalent to saying that $pq = qp = p$. It is easy to show that $p \leq q$ if and only if $\|p(x)\| \leq \|q(x)\|$ for all $x \in \mathscr{H}$.

To show that the category of Hilbert spaces admits direct limits, let us first prove the following lemma.

Lemma 2.31. *An increasing net of projections (p_f) on a Hilbert space $\mathscr{H}$ converges strongly to the projection onto the closure of $\bigcup_f p_f(\mathscr{H})$.*

Proof. for any $x \in \mathscr{H}$, the net $\langle x, p_f x\rangle$ is increasing and bounded from above by $\|x\|^2$ and therefore convergent. The polarization identity then tells us that $\langle x, p_f y\rangle$ is convergent for any pair $x, y \in \mathscr{H}$. The assignment

$$(x, y) \mapsto \lim_f \langle x, p_f y\rangle$$

therefore defines a bounded sesquilinear form, so there is an operator p such that $\lim_f \langle x, p_f y\rangle = \langle x, py\rangle$. for all $x, y \in \mathscr{H}$. This operator is clearly a projection bounding the net (p_f) from above. Furthermore,

$$\|px - p_f x\|^2 = \langle (p - p_f)x, x\rangle \to 0$$

using that $p - p_f$ is a projection, so indeed $p_f \to p$ strongly. If $x \in (p_f(\mathscr{H}))^\perp$ for all f, then $p_f x = 0$ for all f and hence $px = 0$ as well. Conversely, if $x \in p_f(\mathscr{H})$ for some f, then $px = x$. It follows that p indeed projects onto $\overline{\bigcup_f p_f(\mathscr{H})}$. □

Remark. Note that the projection p obtained in the lemma from an increasing net p_f is minimal. Indeed, if q is another upper bound of the net p_f, then

$$\|p(x)\| = \lim_f \|p_f(x)\| \leq \|q(x)\|$$

because $\|p_f(x)\| \leq \|q(x)\|$ for all $f \in F$. So $p \leq q$ as desired. Therefore by this lemma, if we can find even a single target $\mathscr{H}$ for a directed system of Hilbert spaces $\{\mathscr{H}_f | f \in F\}$, an appropriate direct limit is given by $\mathscr{H}_\infty = (\lim_f p_f)\mathscr{H}$, where p_f is the projection onto $U_f(\mathscr{H}_f)$.

Proposition 2.32. *The category of Hilbert spaces admits direct limits.*

Proof. By the remark above, it suffices to create a single target for a given directed system $\{\mathscr{H}_f)_{f\in F}$ of Hilbert spaces. Consider $\Phi = \{(x_f)_{f\in F} | x_f \in \mathscr{H}_F\}$. Denote by $\mathscr{K}_0$ the space of sequences in Φ that are eventually constant, meaning that eventually $x_f = U_{fh}(x_h)$ for all $h \geq f$. Define a pseudo-inner product on this space by putting $\langle (x_f), (y_f)\rangle = \langle x_h, y_h\rangle_{\mathscr{H}_h}$ where h is an index where x_f and y_f are both already constant. Let us denote the Hilbert space obtained from completing the quotient by the null-space of this pre-inner product $\mathscr{H}_\infty$. Define the maps $U_h : \mathscr{H}_h \to \mathscr{H}_\infty$, $x_h \mapsto (y_f)$, where $(y_f) = U_{fh}(x_h)$ for all $f \geq h$, and the choice for all other f do not matter as the difference would be eventually zero and hence in the kernel of our pre-inner product. These are clearly isometries. □

The requirement that $U_{ff} = id$ for all $f \in F$ that is usually set for such directed families of objects is not needed if we are considering Hilbert spaces, as any isometric projection is necessarily the identity.

If $\{\mathscr{H}_f | f \in F\}$ is a directed family of Hilbert spaces, we can identify each $\mathscr{H}_f$ with its image $U_f(\mathscr{H}_f) \subseteq \mathscr{H}_\infty$; any directed system is unitarily equivalent to a net of subspaces with dense union in $\mathscr{H}_\infty$. In what follows we will therefore omit the isometries U_{hf} and U_f.

Let us now return to the matter of Fock spaces, as introduced in subsection 2.1.3.

Proposition 2.33. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let $u : \mathscr{H}_1 \to \mathscr{H}_2$ be an isometry. The induced map $\Gamma(u) : \Gamma(\mathscr{H}_1) \to \Gamma(\mathscr{H}_2)$ is again an isometry.*

Furthermore, it is obvious that given two isometries $V \xrightarrow{u} W \xrightarrow{v} X$, that $\Gamma(v)\Gamma(u) = \Gamma(vu)$. Furthermore, $\Gamma(id_V)$ is the identity on $\Gamma(V)$. In the language of category theory, taking the Fock space is a functor.

Proposition 2.34. *Let $\{\mathscr{H}_f\}$ be a directed system of Hilbert spaces. By the proposition above $\Gamma(\mathscr{H}_f)$ is also a directed system of Hilbert spaces. Let $\mathscr{H}_\infty$ be the direct limit of $\mathscr{H}_f$. Then there is a natural isometric isomorphism $\lim \Gamma(\mathscr{H}_f) \to \Gamma(\mathscr{H}_\infty)$.*

Proof. Because $\mathscr{H}_f$ is a subspace of $\mathscr{H}_\infty$ for all f, the spaces $\Gamma(\mathscr{H}_f)$ are subspaces of $\Gamma(\mathscr{H}_\infty)$. It is therefore sufficient to show that the union of these subspaces is dense. If $\psi = \psi_1 \odot \cdots \odot \psi_n \in \mathscr{H}_\infty^{\odot n}$ is nonzero, there exists an index f and elements $\varphi_i \in \mathscr{H}_f$ such that $\langle \psi_i, \varphi_i \rangle \neq 0$ for all i. Hence, there is an element φ of $\Gamma(\mathscr{H}_f)$ such that $\langle \psi, \varphi \rangle \neq 0$. That is to say, $\left(\bigcup_{f \in F} \Gamma(\mathscr{H}_f)\right)^\perp \cap \bigoplus_{n \in \mathbb{N}} \mathscr{H}_\infty^{\odot n} = \{0\}$. As $\Gamma(\mathscr{H}_\infty)$ is the direct sum of $\left(\bigcup_{f \in F} \Gamma(\mathscr{H}_f)\right)^\perp$ and the closure of $\bigcup_{f \in F} \Gamma(\mathscr{H}_f)$, the dense subspace $\bigoplus_{n \in \mathbb{N}} \mathscr{H}_\infty^{\odot n}$ must be contained entirely in the latter. In other words, $\bigcup_{f \in F} \Gamma(\mathscr{H}_f)$ is dense. □

2.2.3 Self adjoint Operators

Let us now consider a different category, namely the one where objects are pairs $(\mathscr{H}, D)$ of a Hilbert space along with a self adjoint operator. Morphisms in this category are defined to be isometries $u : \mathscr{H}_1 \to \mathscr{H}_2$ such that if $x \in \operatorname{dom}(D_1)$, then $ux \in \operatorname{dom}(D_2)$ and $D_2(ux) = u(D_1 x)$. We will say that the inclusion is compatible with the operators and that the operators are compatible with the inclusion.

Remark. Given a directed system of Hilbert spaces with a compatible family of operators -not necessarily self adjoint-, we can always define a densely defined operator D on the direct limit of the Hilbert spaces by considering the domain $\operatorname{dom}(D) = \bigcup_{f \in F} \operatorname{dom}(D_f)$ and putting $Dx = D_f x$ if $x \in \operatorname{dom}(D_f)$. However, even if all the D_f are bounded this limit need not be closable. As a counter example let us consider the direct system of simple functions F_Λ on the unit interval, with the L^2-norm. The index set in this case is partitions Λ of $[0, 1]$ where every segment is assumed to have positive measure to ensure every F_Λ is indeed a Hilbert space. The direct limit of this direct system is $L^2(0, 1)$. As operators on F_Λ choose D_Λ mapping $\varphi \in F_\Lambda$ to the constant function $\varphi(0)$. This is certainly a compatible family of closed (because bounded) operators. There is however certainly no compatible closable operator in the limit. Hence, if we allow operators that are not self adjoint, there might not be a closed operator in the limit; some directed systems in fact have no targets at all. It is in fact sufficient for the operator to be normal, although we will not need this.

Perhaps the most obvious non trivial example of such an isometry is given by the following example,

Example 2.35. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let D_1 and D_2 be self-adjoint operators. Then the operator

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

on the space $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ is also self-adjoint as per proposition 2.9. If $u : \mathscr{H}_1 \to \mathscr{H}_1 \oplus \mathscr{H}_2$ is the isometry given by mapping x to $(x, 0)$, then u is a compatible isometry between $(\mathscr{H}_1, D_1)$ and $(\mathscr{H}, D)$ and the same can be said for the isometric inclusion of $\mathscr{H}_2$ into $\mathscr{H}$.

We will see this example is in fact the only example, similar to how, if $u : \mathscr{H}_1 \to \mathscr{H}$ is an isometry between two Hilbert spaces, there is a Hilbert space $\mathscr{H}_2 = \mathscr{H}_1^\perp$ such that $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and such that u is the inclusion.

Proposition 2.36. *Suppose $(\mathscr{H}_1, D_1)$ and $(\mathscr{H}_2, D_2)$ are pairs of Hilbert spaces and self-adjoint operators and suppose $u : \mathscr{H}_1 \to \mathscr{H}_2$ is a compatible isometry. Then $D_1 p = p D_2 = D_2 p$, where p denotes the projection on $\mathscr{H}_2$ onto the closed subspace $u(\mathscr{H}_1)$.*

Proof. Let $x \in \operatorname{dom}(D_2)$ and let $y \in \operatorname{dom}(D_1)$. Then

$$|\langle px, D_1 y \rangle| = |\langle x, D_2 y \rangle| = |\langle D_2 x, y \rangle| \leq \|D_2 x\| \, \|y\|,$$

using in the first equality that D_1 maps its domain into $\mathscr{H}_1$. Therefore,

$$px \in \operatorname{dom}((D_1)^*) = \operatorname{dom}(D_1)$$

for all $x \in \operatorname{dom}(D_2)$. The operator D_2 maps $(1-p)\operatorname{dom}(D_2)$ into $\mathscr{H}_1^\perp$, because if $z \in \mathscr{H}_1^\perp$, we find $\langle px, D_2 z \rangle = \langle D_2 px, z \rangle = 0$. Therefore,

$$0 = pD(1-p)x = pD_2 x - pD_f px = pD_2 x - D_2 px.$$

But $D_2 px = D_1 px$ for all $x \in \operatorname{dom}(D_2)$ In other words, $D_1 px = pD_2 x$. ☐

Corollary 2.37. *With the assumptions as above, then $D_2 = D_1 \oplus (D_2)\big|_{\mathscr{H}_1^\perp}$. Here, the domain of $(D_2)\big|_{\mathscr{H}_1^\perp}$ is given by*

$$\operatorname{dom}\left((D_2)\big|_{\mathscr{H}_1^\perp}\right) = \mathscr{H}_1^\perp \cap \operatorname{dom}(D_2) = (1-p)\operatorname{dom}(D_2).$$

Definition 2.38. *Let $\mathbb{F}$ be a directed stet and let $\{H_f, D_f, u_{fh}\}_{f \in \mathbb{F}}$ be a directed family. Define the operator D on $\mathscr{H}_\infty$, with domain $\operatorname{dom}(D) = \bigcup_{f \in F} \operatorname{dom}(D_f)$, by putting $Dx = D_f x$ for $x \in \operatorname{dom}(D_f)$.*

Proposition 2.39. *The operator D as defined above is essentially self-adjoint.*

Proof. Let $x \in \operatorname{dom}(D_f), y \in \operatorname{dom}(D_h)$. Without loss of generality, we can assume $f = h$. Then $\langle x, Dy \rangle = \langle x, D_f y \rangle = \langle Dx, y \rangle$, so D is symmetric. We therefore only have to check whether the images of $(D \pm i)$ are indeed dense. Because all the D_f are self-adjoint, the images $D_f \pm i$ are exactly $\mathscr{H}_f$. As such, the images of $D \pm i$ are both $\bigcup \mathscr{H}_f$, which is dense in $\mathscr{H}_\infty$ by construction. ☐

Corollary 2.40. *There exists a unique self-adjoint operator D_∞ extending D. The domain of this operator is the subspace of $x \in \mathscr{H}_\infty$ such that $p_f x \in \mathrm{dom}\,(D_f)$ for all $f \in F$ and such that $\|D_f(p_f x)\|$ has a uniform bound. Furthermore*

$$\sigma\,(D_\infty) = \overline{\left(\bigcup_{f\in F} \sigma(D_f)\right)}$$

.

Proof. Let $x \in \mathrm{dom}\,(D_\infty)$. Then, by proposition 2.36 $p_f x \in \mathrm{dom}\,(D_f)$ for all f. Also, $\|D_f(p_f x)\| = \|p_f(D_\infty x)\| \leq \|D_\infty x\|$ is a uniform bound.

For the other direction let $x \in \mathscr{H}$ be such that $x_f \in \mathrm{dom}\,(D_f)$ and such that $\|D_f x_f\|$ is bounded by $M > 0$. Then

$$\langle x, Dy\rangle = \lim\langle x_f, D_f y_f\rangle = \lim\langle D_f x_f, y_f\rangle$$

which in absolute value is bounded by $M\,\|y\|$. It follows that $x \in \mathrm{dom}\,(D^*) = \mathrm{dom}\,(D_\infty)$ as required.

To prove the statement about the spectrum, let $f \in F$ be arbitrary. Then $p_f(D_\infty - \lambda) = (D_f - \lambda)p_f$ for all f and all scalars λ. Suppose $\lambda \in \sigma(D_\infty)$ and let $x \in \mathscr{H}_f$. There exists a $y \in \mathscr{H}$ such that $(D_\infty - \lambda)y = x$. But then also $(D_f - \lambda)(p_f y) = x$. Furthermore, the restriction of an injective map is again injective. Hence, $\sigma(D_f) \subseteq \sigma(D_\infty)$.

Conversely, suppose $\lambda \in \mathbb{C} \setminus \overline{\left(\bigcup_{f\in\mathbb{N}} \sigma(D_f)\right)}$. Suppose $x \in \ker(D_\infty - \lambda)$. Then $p_f x \in \ker(D_f - \lambda)$, so $p_f x = 0$ for all $f \in F$. It follows that $x = 0$. As for the surjectivity, let $y \in \mathscr{H}$. By assumption there exists an $\varepsilon > 0$ such that $dist(\sigma(D_f), \lambda) > \varepsilon$ for all $f \in F$. By lemma 2.8, we have a uniform bound on the resolvents $R(D_n, \lambda)$. Consider the sequence $x_n = R(D_n, \lambda)(p_n y)$ of elements of $\mathrm{dom}\,(D_\infty)$. Clearly, $(D_\infty - \lambda)x_f = p_f y$ which converges to y. We conclude that $x \in \mathrm{dom}\,(D_\infty)$ and $(D_\infty - \lambda)x = y$, as desired. □

Proposition 2.41. *The operator D_∞ is minimal.*

Proof. If $x \in \mathrm{dom}\,(D_\infty)$, by lemma 2.31 the net $D_f x_f = p_f D_\infty x$ converges to $D_\infty x$. This gives a second characterization of the domain of D_∞ as the $x \in \mathscr{H}_\infty$ such that the net $D_f x_f$ converges.

Assume that $(\mathscr{K}, \Delta)$ is another target, with isometries v_f. Because $\mathscr{H}_\infty$ is the direct limit of the directed family of Hilbert spaces, there is a unique isometry $v : \mathscr{H}_\infty \to \mathscr{K}$. It now suffices to show that this isometry is compatible, so let $x \in \mathrm{dom}\,(()\,D_\infty)$. By the above remark, the net $D_f x_f$ converges to $D_\infty x$. By assumption $v_f(x_f) \in \mathrm{dom}\,(\Delta))$ and $\Delta(v_f(x_f)) = v_f D_f(x_f)$. Because Δ is closed, it follows that

$$\lim_{f\in\mathbb{F}}\,(v_f(x_f)) = vx \in \mathrm{dom}\,(\Delta)\,,$$

which means that v is compatible with the operators. Because v is unique as a map between Hilbert spaces, it is still unique in our case as well, meaning the pair $\mathscr{H}_\infty, D_\infty$ is minimal as desired. □

2.3 Lie groups

In this section we will recall the basic definitions regarding Lie groups and Lie algebras. We will show that if G is compact, we can find an Ad-invariant inner product on its Lie algebra $\mathfrak{g}$. This inner product gives rise to a differential operator Δ_G called the Casimir operator, which acts as a self-adjoint operator on $L^2(G, dg)$ where dg denotes the Haar measure. The differential equation $\Delta_G f = 1/2d/dtf$ on G has a fundamental solution p_t - the heat kernel. We will prove several statements about these three objects associated to G and $\mathfrak{g}$ that will be used throughout the text. For a more complete treatment of Lie groups or for proofs of the theorems below, one can read [14]. Throughout the text we will only consider Lie groups G that are compact and connected. Any such a group has a faithful finite-dimensional representation and as such is in fact a group of matrices without loss of generality.

Definition 2.42. *A Lie group is a manifold endowed with a group structure such that both the multiplication map and the inverse map are smooth.*

Definition 2.43. *The Lie algebra of a Lie group G is the tangent space T_eG at the identity. We will generally denote the Lie algebra of G by $\mathfrak{g}$.*

For every element X of the Lie algebra there is a one-parameter subgroup $t \mapsto \exp(tX)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp(tX) = X$.

We define the adjoint action of G on itself by $Ad_g(h) = ghg^{-1}$. Taking the derivative yields an action ad of G on $\mathfrak{g}$, so this action is given by

$$ad_g(X) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} Ad(g)\exp(tX)$$

where $g \in G$ and $X \in \mathfrak{g}$. Taking the derivative again gives an action $\mathfrak{ad}$ of $\mathfrak{g}$ on itself. As we recall this action gives rise to the Lie bracket on $\mathfrak{g}$.

A useful theorem will be the Peter Weyl theorem, which generalizes what we know as the Fourier-decomposition of an $L^2(U(1))$-function to a decomposition of elements of $L^2(G)$ for a more general Lie group G. Let G be a Lie group. We let $\widehat{G}$ denote the isomorphism classes of irreducible representations of G and let $HS(V)$ denote the Hilbert-Schmidt operators on the finite dimensional space V, which is a Hilbert space when equipped with the inner product $\langle \varphi, \psi \rangle = tr(\varphi^* \psi)\frac{1}{\dim(V)}$. For any $\pi \in \widehat{G}$, we can consider a map $\varphi \in HS(V_\pi)$ to be an element of $L^2(G)$ via $\varphi(g) = tr(\varphi \circ \pi(g)) = tr(\pi(g) \circ \varphi)$.

Theorem 2.44. *Let G be a compact Lie group. Then*

$$L^2(G) = \widehat{\bigoplus}_{\pi \in \widehat{G}} HS(V_\pi)$$

as Hilbert spaces.

Proposition 2.45. *Let G be a compact Lie group. Then there exists a probability measure dg - called the Haar measure - which is bi-invariant.*

Definition 2.46. *A metric g on G is called* **left-invariant if $g_b(X, Y) = g_{ab}(aX, aY)$ for all $a, b \in G$ and all $X, Y \in T_bG$.**

Here aX is a shorthand for $(dL_a)_bX$, as one would write for matrix groups.

Proposition 2.47. *There is a one-to-one correspondence between left-invariant metrics on G and inner products on $\mathfrak{g}$, given by*

$$g_a(X, Y) = \langle (dL_{a^{-1}})_aX, (dL_{a^{-1}})_aY \rangle.$$

where $a \in G$ and $X, Y \in T_aG$. Furthermore, such a left-invariant metric is bi-invariant exactly if $\langle \cdots, \cdots \rangle$ is ad-invariant, in the sense that ad_a is an isometry for all $a \in G$.

Proof. It is clear that a metric on G induces an inner product on $\mathfrak{g}$. As for the converse, we calculate

$$g_{ab}\left((dL_a)_b X, (dL_a)_b Y\right) = \langle (dL_{(ab)^{-1}})_a b (dL_a)_b X, (dL_{(ab)^{-1}})_a b (dL_a)_b Y\rangle = g_b(X, Y)$$

by the chain rule. Furthermore, recall that $ad_g = (dL_g)_{g^{-1}} \circ (dR_g^{-1})_e = (dR_g^{-1})_g \circ (dL_g)_e$, so clearly a bi-invariant metric induces an Ad-invariant inner product and vice versa. □

Remark. if $\langle \cdot, \cdot \rangle$ is an ad-invariant inner product, then $\mathfrak{ad}$ is skew-adjoint: if we suppose that

$$\langle ad_h(X), ad_h(Y)\rangle = \langle X, Y\rangle$$

for all $X, Y \in \mathfrak{g}$ and $h \in G$, inserting $h = \exp(tZ)$, differentiating and evaluating in $t = 0$ gives us the desired result.

As for the converse; if we also suppose G is connected, let $\langle \cdot, \cdot \rangle$ be an invariant inner product on $\mathfrak{g}$. Let $g \in G$ be arbitrary, say $g = \exp tZ$ for $Z \in \mathfrak{g}$. Then

$$\langle ad_g(X), ad_g(Y)\rangle = \langle \exp \mathfrak{ad}(Z)X, \exp \mathfrak{ad}(Z)Y\rangle = \langle X, \exp \mathfrak{ad}(-Z) \exp \mathfrak{ad}(Z)\rangle = \langle X, Y\rangle$$

as desired.

Proposition 2.48. *Let G be a compact Lie group. Then G has a bi-invariant metric.*

Proof. Let $(\cdot, \cdot)$ be some (real) inner product on $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$. Define $\varphi_{X,Y} : G \to \mathbb{R}$ as

$$\varphi_{X,Y}(g) = (ad_g(X), ad_g(Y)).$$

Define $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by putting

$$\langle X, Y\rangle = \int_{g\in G} \varphi_{X,Y}(g) dg,$$

It is clear from the properties of the Haar-measure that $\langle \cdot, \cdot \rangle$ defines an ad-invariant inner product. □

Definition 2.49. *Let $\{ X_i \mid 1 \le i \le n \}$ be an orthonormal basis of $\mathfrak{g}$ (with respect to a fixed ad-invariant inner product $\langle \cdot, \cdot \rangle$). The Casimir operator is the operator given by*

$$\Delta_G = -\sum_{i=1}^{n} X_i X_i.$$

It is an element of the universal enveloping algebra of $\mathfrak{g}$ independent of the choice of basis. In fact it is an element of the center of the universal enveloping algebra, as per the following lemma.

Lemma 2.50. *The commutator $[\Delta_G, X] = 0$ for all $X \in \mathfrak{g}$.*

Theorem 2.51. *The Casimir element Δ_G with domain the finite span of $HS(V_\pi)$ - with $\pi \in \hat{G}$ - defines an essentially self-adjoint operator on $L^2(G)$.*

Proof. Recall that $\mathfrak{g}$ works on $C^\infty(G) \subseteq L^2(G)$ as differential operators, where an $X \in \mathfrak{g}$ acts on an $f \in C^\infty(G)$ by

$$X(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\, |_{t=0}\, (f(g \exp(tX))).$$

It follows that for an $X \in \mathfrak{g}$, we have $X(\varphi \otimes v) = \varphi \otimes (\pi_*(X)v)$. As (π, V_π) is an irreducible representation of G, the derivative (π_*, V_π) is an irreducible representation of $\mathfrak{g}$. By Schur's lemma, Δ_G acts as a scalar on V_π, which we will call λ_π. As we have already seen, Δ is symmetric and negative and hence $\lambda_\pi \le 0$ for all π. The fact that Δ is essentially self-adjoint operator on $L^2(G)$ now follows from the previous chapter. □

Proposition 2.52. *The Casimir element is bi-invariant, in the sense that if* $\Psi \in \operatorname{dom}(\Delta_G)$ *and* Φ *is such that* $\Phi(g) = \Psi(gh)$ *then* $\Psi \in \operatorname{dom}(\Delta_G)$ *and* $\Delta_G\Phi(h) = \Delta_G\Psi(gh)$*. The same holds for left multiplication.*

Definition 2.53. *Let* G *be a compact connected Lie group. The heat kernel is the function* $p_t = \sum_{\pi\in\hat{G}} \exp(\lambda_p it/2) d_\pi \chi_\pi$*.*

Here d_π denotes the dimension of V_π. It is a smooth function which is strictly positive everywhere. The following lemma is a direct result of the orthogonality of matrix elements, c.f. theorem 2.44.

Lemma 2.54. *Let* G *be a compact Lie group and let* π, ρ *be irreducible representations. Then*

$$\int_G \chi_\pi(g^{-1}h)\chi_\rho(g)dg = \delta_{\pi,\rho}\frac{1}{\dim(V_\pi)}\chi_\pi(h).$$

Because p_t is positive everywhere, if G is a compact and connected Lie group it is immediate that $L^2(G, d\rho_t)$ and $L^2(G)$ are in fact the same vector space with the same topology, but with different inner products. As such we can formulate the lemma below about the following operator.

Definition 2.55. *Let* G *be a compact connected Lie group and let* $\langle\cdot,\cdot\rangle$ *be an ad-invariant inner product on* $\mathfrak{g}$*. Let* Δ_G *be the associated Casimir operator and let* p_t *be the associated heat kernel. Let* $v_t : L^2(G, d\rho_t) \to L^2(G)$ *be the canonical unitary map. Define the operator*

$$\Delta_G^t = v_t\Delta_G v_t^*.$$

We wish to show that in some sense $\lim_{t\to\infty} \Delta_G^t = \Delta_G$.

Lemma 2.56. *The domains coincide, i.e.* $\operatorname{dom}\left(\Delta_G^t\right) = \operatorname{dom}(\Delta_G)$ *for all* $t > 0$ *as linear subspaces. Furthermore, for any fixed* $\varphi \in \operatorname{dom}(\Delta)$*, we have that* $\Delta_G^t\varphi \to \Delta_G\varphi$ *as* t *goes to* ∞*.*

Proof. It is well known that the domain of Δ_G is $H^2(G)$, the second Sobolev space, which is closed under multiplication by smooth functions. Hence, the smoothness of p_t means the domain of Δ_G^t must also be $H^2(G)$. Furthermore, the product rule for the Laplacian tells us that,

$$\Delta_G((p_t)^{-\frac{1}{2}}f) = (p_t)^{-\frac{1}{2}}\Delta_G(f) + (\nabla(p_t)^{-\frac{1}{2}})(\nabla f) + f\Delta_G((p_t)^{-\frac{1}{2}}).$$

Now, because p_t - which is postive everywhere - converges to the constant function 1 as a smooth function, so does $\frac{1}{\sqrt{p_t}}$. As such, the gradient $\nabla(p_t)^{-\frac{1}{2}}$ and also $\Delta_G(p_t)^{-\frac{1}{2}}$ converge to zero as t gets large. Hence, what remains on the right-hand side is simply $\Delta_G(f)$, as desired. $\square$

CHAPTER 3

Gross measures on Hilbert spaces

Throughout this chapter let $\mathscr{H}$ be a fixed real separable Hilbert space. The goal of this chapter will be to define a measure on $\mathscr{H}$ - or rather, on $\overline{\mathscr{H}}^*$ which is our original Hilbert space completed with respect to a certain norm - that behaves as a Gaussian measure on a finite dimensional real Hilbert space would. Let us begin by defining what we mean by a Gaussian measure in the finite dimensional case.

Definition 3.1. *Let V be a finite-dimensional real Hilbert space of dimension d. We define the Gaussian measure of variance $t > 0$ to be the Borel measure on the real dual V^* given by multiplying the Lebesgue measure dx with the weight*

$$p_t(x) = (2\pi t)^{-d/2} \exp\left(\frac{-\|x\|^2}{2t}\right).$$

We will denote this measure by ρ_t.

We know from our knowledge of Gaussian integrals that this indeed defines a probability measure on V^*. Of course we could define such a measure when provided with any norm on V^*, not just one arising from an inner product. However, an important property of this measure that we will use in the rest of this thesis is that this measure - when V is equipped with a Hilbert space structure - is well behaved with regard to orthogonal subspaces.

As a further remark, of course taking the dual space is not necessary in the finite dimensional case. However, because in the infinite dimensional case we have to enlarge the space on which we define the measure, there will be a difference between the space itself and the dual space. Hence, defining the measure on the dual instead of on V itself is done for the purpose of consistency.

Defining a Gaussian measure on the infinite-dimensional space $\mathscr{H} = \mathscr{H}^*$ is slightly more tricky. In particular, there is no Σ-additive Borel measure μ_t on $\mathscr{H}$ that generalizes ρ_t on a finite dimensional space. This measure would have to correspond - heuristically - to the measure $\exp\left(-\frac{\|x\|^2}{2t}\right) dx$, where dx denotes what would be the equivalent of the Lebesgue measure. Furthermore, one would have to multiply by a vanishing normalization factor. Accepting that this is well defined for a moment, suppose p is a finite dimensional projection acting on $\mathscr{H}$, and suppose E is a so called cylindrical subset of $\mathscr{H}$, meaning that

$$E = \{x \in \mathscr{H} | px \in \tilde{E}\}$$

where $\tilde{E}$ is a Borel measurable subset of the finite dimensional real Hilbert space $p\mathscr{H}$. The measure is a probability measure on the orthogonal complement of $p\mathscr{H}$ and hence we find that $\tilde{\mu}(E)$ must be equal to $\rho_t(\tilde{E})$ where the latter measure is just the finite dimensional Gaussian measure on $p\mathscr{H}$. This is in analogy with Gaussian measures being well-behaved with respect to orthogonal subspaces. We could use this as a definition of a measure on $\mathscr{H}$, where the Σ-algebra is the smallest one containing all cylindrical sets. As we will see, this "measure" is not Σ-additive, nor is it clear that the Σ-algebra generated by cylindrical subsets is the entire Borel Σ-algebra. Let us summarize this in the following definitions. Furthermore, in the rest of this chapter we will neglect the parameter and put $t = 1$ for simplicity which is of course without loss of generality because the norm on a Hilbert space $\mathscr{H}$ can be rescaled.

Definition 3.2. *Let $\mathbb{F}$ be the partially ordered system of finite dimensional projections on $\mathscr{H}$. A function $f : \mathscr{H} \to \mathbb{C}$ is called cylindrical if $f(\psi) = F(p\psi)$ for some $p \in \mathbb{F}$ and some Borel-measurable $F : p\mathscr{H} \to \mathbb{C}$.*

Definition 3.3. *Let $\tilde{\mu}_t$ be the measure on the Σ-algebra generated by cylindrical subsets E of $\mathscr{H}$ defined by*

$$\tilde{\mu}_t(E) = \rho_t(\tilde{E})$$

where the latter measure is the Gaussian measure on $p\mathscr{H}$.

That this measure is not Σ-additive is proven in for example [3]. It does however define a Σ-additive measure on a slightly larger space, which we will define now. The slightly larger space $\overline{\mathscr{H}}^*$ will be the closure of $\mathscr{H}$ with respect to an operator T.

Let T be a strictly positive Hilbert-Schmidt operator on $\mathscr{H}$, by which we mean that it is both positive and injective. Denote by $\|\cdot\|_T$ the norm on $\mathscr{H}$ given by

$$\|x\|_T = \|Tx\|$$

for $x \in \mathscr{H}$. We still have to check that this defines a norm. Obviously homogeneity of $\|\cdot\|_T$ and its triangle inequality follow directly from the linearity of T. As for the non-degeneracy, this follows from the demand that T be strictly positive. Because T is bounded but not invertible $\|\cdot\|_T$ is strictly weaker than the ordinary norm on $\mathscr{H}$. There are sequences that are Cauchy with respect to $\|\cdot\|_T$ that are not Cauchy in $\mathscr{H}$. We can therefore consider the closure of $\mathscr{H}$ with respect to $\|\cdot\|_T$, which will be denoted by $\overline{\mathscr{H}}^*$.

The demand for T to be strictly positive is not entirely necessary but it will make our life slightly easier. The removal of the strictness demand would make $\|x\|_T$ into a semi-norm. This would still give us a closure bigger than the original Hilbert space and in fact yield the same theorems. However, throughout this thesis we will only come across norms - no semi-norms. Furthermore, some of proofs in this chapter will be slightly simpler if we neglect T that are not injective. The demand for positivity means we can work with a decreasing sequence of eigenvalues which will also make matters easier, but it is similarly inessential.

Because T is injective it has a measure theoretical inverse with as its domain the range of T.

Proposition 3.4. *The injective self-adjoint operator T has dense range.*

Proof. Suppose $x \in (T\mathscr{H})^\perp$. Then

$$0 = \langle x, T(Tx)\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$$

so it follows that x lies in the kernel of T. Because the operator was assumed to be injective, $x = 0$. In other words, T has dense range. □

Hence we can define $T^{-1} : \operatorname{dom}\left(T^{-1}\right) = T\mathscr{H} \to \mathscr{H}$ which is a densely defined unbounded operator. Let us denote $\operatorname{dom}\left(T^{-1}\right)$ by $\overline{\mathscr{H}}$ for reasons that will be obvious later. It is a self-adjoint operator by the positivity of T. Indeed, if we apply the spectral theorem for compact self adjoint operators to T, we get a sequence $\lambda_n > 0$ of eigenvalues with orthonormal basis of eigenvectors e_n. It follows that T^{-1} is diagonal with sequence λ_n^{-1}, so the self-adjointness follows from corollary 2.40.

Definition 3.5. *Let $\overline{\mathscr{H}}^*$ denote the completion of $\mathscr{H}$ with respect to $\|\cdot\|_T$. We call a subset E of $\overline{\mathscr{H}}^*$ cylindrical if it is of the form*

$$E = \{\, x \in \mathscr{H} \mid px \in \tilde{E} \,\}$$

where p is now a finite dimensional projection onto a finite-dimensional subspace of $\overline{\mathscr{H}} = \operatorname{dom}\left(T^{-1}\right)$.

Because $\overline{\mathscr{H}}$ is a subset of $\mathscr{H}$, if E is a cylindrical subset of $\overline{\mathscr{H}}$ then $E \cap \mathscr{H}$ is a cylindrical subset of $\mathscr{H}$.

The main focus of this chapter will be proving the following theorem:

Theorem 3.6. *Let $\mathscr{H}$ be a real Hilbert space and let T be a Hilbert-Schmidt operator. Let $\overline{\mathscr{H}}^*$ be the completion of $\mathscr{H}$ with respect to the norm $\|x\|_T = \|Tx\|$. Then there is a Borel measure μ on $\overline{\mathscr{H}}^*$ such that for any cylindrical set E*

$$\mu_t(E) = \tilde{\mu}(E \cap \mathscr{H})$$

In fact, the only thing we will use in the proof is that the norm $\|\cdot\|_T$ has the following property: For every $\varepsilon > 0$ there is a finite dimensional projection $p_0 \in \mathbb{F}$ such that

$$\tilde{\mu}\{\|pA\|_T > \varepsilon\} < \varepsilon.$$

for all $p \in \mathbb{F}$ such that $p \perp p_0$. If a norm $\|\cdot\|'$ is of the form $\|\cdot\|_T$ for some linear operator T, the norm has the above property if and only if T is a Hilbert-Schmidt operator. However, certainly not every norm with the above property is of such a form - we will see an example in the next chapters. Such measures are known as Gross-measurable norms and as such are a generalization of Hilbert-Schmidt operators. We will also occasionally call such norms measurable (sans the Gross) even though in some of the the literature this sometimes means something else. The definition of these measures was first given by Leonard Gross in [11], whence the name.

We will prove the theorem in several steps. The proof is from [20], slightly adapted to suit our particular situation.

Lemma 3.7. *For every $\varepsilon > 0$ there is a finite dimensional projection $p_0 \in \mathbb{F}$ such that*

$$\tilde{\mu}\,\{\,\|pA\|_T > \varepsilon\,\} < \varepsilon$$

for all $p \in \mathbb{F}$ such that $p \perp p_0$.

Proof. By the spectral theorem for self-adjoint compact operators, let λ_n be the sequence of eigenvalues of T with respective eigenvectors e_n. Because T is Hilbert-Schmidt, the sequence λ_n is square summable. Let $1 > \varepsilon > 0$ be arbitrary. Because the measure is a probability measure, we can do this without loss of generality. By the definition of $\tilde{\mu}$, given a $p \in \mathbb{F}$,

$$\tilde{\mu}\,\{\,\|pA\|_T > \varepsilon\,\} = \rho\,\{\,v \in p\mathscr{H} \mid \|Tv\| > \varepsilon\,\}.$$

Hence, let N be a natural number such that $\sum_{m>N} \lambda_m^2 < \delta$, where $\delta > 0$ will be specified later. Let p_0 be the projection onto the finite-dimensional subspace spanned by $\{e_m | m \leq N\}$. Suppose that $p \in \mathbb{F}$ is orthogonal to p_0. If $v \in p\mathscr{H}$, say $v = \sum v_m e_m$

$$\|Tv\|^2 = \sum_{m>N} \lambda_n^2 |v_m|^2 \leq \delta \sup(|v_m|).$$

It follows that

$$\rho\{\, v \in p\mathscr{H} \mid \|Tv\| > \varepsilon \,\} \geq \rho\{\, v \in p\mathscr{H} \mid \sup_{m>N} \}\, (|v_m|) > \delta^{-1}\varepsilon.$$

Because Gaussian measures are rotation invariant, we can assume without loss of generality that $p\mathscr{H}$ has elements of the form $e_m \xi_j$ as a basis. Then

$$\begin{aligned} \rho\{\, v \in p\mathscr{H} \mid \sup_{m>N} (|v_m|) > \delta^{-1}\varepsilon \,\} &= \int_{v_m > \delta^{-1}\varepsilon} d\rho \\ &= \left(\int_{v > \delta^{-1}\varepsilon} d\rho \right)^d. \end{aligned}$$

If we set δ such that $\int_{v>\delta^{-1}\varepsilon} d\rho$ is bounded by ε, we are done because $\varepsilon^d < \varepsilon$. Because the one-dimensional integral is finite, such a δ certainly exists. □

Firstly, because the net of measures indexed by finite dimensional subspaces of $\mathscr{H}$ is a compatible net of probability measures, by Kolmogorov's extension theorem - confer proposition 2.30 - there is a probability space in the limit, which we will denote by $(\Omega, \mathbb{P})$. In other words, there is a Σ-additive probability measure $\mathbb{P}$ on a space Ω and random variables $n : \mathscr{H} \to L^2(\Omega)$ such that

$$\tilde{\mu}\{\, x \in \overline{\mathscr{H}}^* \mid (\langle e_1, x\rangle, \ldots, \langle e_d, x\rangle) \in E \,\} = \mathbb{P}[(n(e_1), \ldots, n(e_d)) \in E]$$

for every finite orthonormal sequence e_i.

Proposition 3.8. *Let $x \in \mathscr{H}$. The net $\|px\|_T$ indexed by finite dimensional projections p converges in probability to a random variable $\|x\|_T$ on Ω.*

By this we mean, that there is a random variable $\|x\|_T$ such that for all $\varepsilon > 0$, the probability $\mathbb{P}(|\, \|px\|_T - \|x\|_T \,| > \varepsilon)$ converges to zero as p approaches the identity. For a proof, again see [20].

Lemma 3.9. *There is a strictly positive Hilbert Schmidt operator $S > T$ on $\mathscr{H}$ such that the closure in the norm $\|\cdot\|_S$ is compactly contained in $\overline{\mathscr{H}}^*$.*

Proof. Let a_n be the non-increasing square-summable positive sequence associated to T via the spectral theorem. It is sufficient to show that there is non-increasing square-summable positive sequence b_n such that $a_n b_n^{-1}$ converges to zero. Suppose w.l.o.g. that $\sum_{n\in\mathbb{N}} a_n^2 = 1$. For every integer k there is an integer N_k, such that

$$\sum_{n>N_k} a_n^2 < \frac{1}{2^k}.$$

Without loss of generality we can choose these integers such that $N_{k+1} > N_k$. Now define $f(n)$ to be the smallest integer such that $\sum_{n\in\mathbb{N}} a_n^2 < 2^{f(m)}$. Let $b_n = 2^{f(n)/4} a_n \geq a_n$. If n is such that $N_{k+1} > n > N_k$, then certainly $f(n) < k$, so $2^{f(n)} < 2^k$. Hence

$$\begin{aligned}\sum_{n\in\mathbb{N}} b_n^2 &= \sum_{n\in\mathbb{N}} 2^{f(m)/2} a_n^2 \\ &\leq \sum_{k=0}^{\infty} 2^{k/2} \sum_{n=N_k}^{N_{k+1}} a_n^2 \\ &\leq \sum_{k=0}^{\infty} 2^{k/2} \sum_{n=N_k}^{\infty} a_n^2 \\ &\leq \sum_{k=0}^{\infty} 2^{-k/2}\end{aligned}$$

which is certainly finite. Furthermore, $a_n b_n^{-1}$ is a sequence converging to zero, so the inclusion is compact as desired. □

Now we are at the point where we can prove that the measure μ is Σ-additive on $\overline{\mathscr{H}}^*$. Let $\varepsilon > 0$ be arbitrary. For any random variable X, we have that $\mathbb{P}[|X| > M]$ converges to zero as M goes to infinity. Hence there is a radius R such that $\mathbb{P}[\|x\|_S > R] < \varepsilon$. By lemma 3.9, the set $C = cl\{x \in \mathscr{H} | \, \|x\|_S \leq R\}$ is compact in H^{-1}. Furthermore if E is cylindrical, say with respect to a projection p, and $E \cap C = \emptyset$, then

$$\begin{aligned}\mu(E) &\leq 1 - \rho(C \cap p\mathscr{H}) \\ &\leq 1 - \rho\{\, x \in p\mathscr{H} \mid \|x\|_S \leq R \,\} \\ &= 1 - \mu\{\, x \in \mathscr{H} \mid \|px\|_S \leq R \,\} \\ &= 1 - \mathbb{P}[\|px\|_S \leq R] \\ &\leq 1 - \mathbb{P}[\|x\|_S \leq R] \\ &< \varepsilon\end{aligned}$$

by the definition of R, using in the second to last line that $\|px\|_S \leq \|x\|_S$ for all $x \in \mathscr{H}$.

Therefore, for every ε there is a compact C such that whenever $E \cap C$ is empty, $\mu(E) < \varepsilon$. Let $\varepsilon > 0$ be arbitrary and let C be as defined above. If we write $H^{-1} = \bigcup_{n\in\mathbb{N}} E_n$ where all the E_n are both cylindrical and open, C is contained in finitely many of them. Therefore there is some N such that

$$\sum_{n\in\mathbb{N}} \mu(E_n) \geq 1 - \mu\left(B - \bigcup_{n<NE_n}\right) \geq 1 - \varepsilon$$

using that μ is finitely additive. The Σ-algebras on the finite dimensional subspaces are generated by such open E, so cylindrical and open E still generate the cylindrical Σ-algebra. Hence, μ is a Σ-additive measure.

Theorem 3.10. *The Σ-algebra generated by cylinder sets in $\overline{\mathscr{H}}^*$ is the Borel Σ-algebra.*

This theorem is theorem 4.2 in [20].

In conclusion, given a Hilbert-Schmidt operator on a Hilbert space $\mathscr{H}$ there exists a Borel measure μ on $\overline{\mathscr{H}}^*$ such that if E is cylindrical, say $E = \{x \in \mathscr{H} | px \in \tilde{E}\}$, for some finite dimensional p

and some Borel-measurable $\tilde{E}$ in $p\mathscr{H}$, that $\mu(E) = \rho(\tilde{E})$, where ρ denotes the standard Gaussian measure. Note that because of the fact that $\mathscr{H}$ can not be equipped with such a measure, it must be a measure zero subspace of $\overline{\mathscr{H}}$.

3.1 Gross-measurable norms

If $|\cdot|$ is a general Gross-measurable norm, the constructed measure is still Σ-finite and all the statements made in this chapter can be generalized - the only statements made that do not readily generalize being lemma 3.9, and the claim that $\|\cdot\|_T$ is weaker than the ordinary norm on $\mathscr{H}$. For a proof of lemma 3.9 in greater generality, see lemmas 4.4 and 4.5 in [20]. For a proof of the claim that any Gross-measurable norm is weaker than the original norm, see lemma 4.2 in the same book. Instead of letting $\overline{\mathscr{H}}$ be the domain of the inverse of the Hilbert-Schmidt operator, we can define it as the dual space of $\overline{\mathscr{H}}^*$ interpreted as a subspace of $\mathscr{H}$.

Let us prove several lemmas about Gross-measurable norms that will come in useful in the next section and also in the next chapter, where we will use the above construction again. The first lemma is straightforward.

Lemma 3.11. *Let $\mathscr{H}$ be a real Hilbert space with measurable norm $|\cdot|_1$. Let $|\cdot|_2$ be a norm weaker than $|\cdot|_1$. Then $|\cdot|_2$ is also measurable.*

Proof. Let $\varepsilon > 0$. There exists a positive constant c such that $c|x|_1 \geq |x|_2$ for all $x \in \mathscr{H}$ by assumption. Note that therefore for any finite dimensional projection p

$$\mu_0 \{ x \in \mathscr{H} \mid |px|_2 > \varepsilon \} \leq \tilde{\mu} \{ x \in \mathscr{H} \mid |px|_1 > \frac{\varepsilon}{c} \}.$$

Suppose that $c \leq 1$. There exists a p_0 such that for all p orthonogal to p_0,

$$\tilde{\mu} \{ x \in \mathscr{H} \mid |px|_1 > \varepsilon < \varepsilon \}.$$

It follows that $\tilde{\mu} \{ x \in \mathscr{H} \mid |px|_2 > \varepsilon \} < \varepsilon$ for all p orthogonal to p_0.

Suppose that $c > 1$. There exists a p_0 such that for all p orthonogal to p_0, we have

$$\tilde{\mu} \{ x \in \mathscr{H} \mid |px|_2 > \varepsilon \} \leq \tilde{\mu} \{ x \in \mathscr{H} \mid |px|_1 > \frac{\varepsilon}{c} \} < \frac{\varepsilon}{c} < \varepsilon.$$ □

The following two statements are direct corollaries:

Corollary 3.12. *Let $\mathscr{H}$ be a real Hilbert space with equivalent norms $|\cdot|_1$ and $|\cdot|_2$. Then $|\cdot|_1$ is Gross-measurable if and only if $|\cdot|_2$ is.*

Corollary 3.13. *Let $\mathscr{H}$ be a Hilbert space and let $|\cdot|$ be a measurable norm. Let A be a positive bounded linear operator on $\mathscr{H}$ such that $Ax = 0$ only for $x = 0$. Define the norm $|x|_A = |Ax|$. This norm is again measurable.*

If $|\cdot|$ is of the form $\|\cdot\|_T$ for some Hilbert-Schmidt operator, this statement is of course obvious because Hilbert-Schmidt operators are an ideal in $B(\mathscr{H})$.

The next lemma is fairly obvious: the measure constructed from a Gross-measurable norm does not depend on the norm explicitly, only on the equivalence class thereof. Indeed, for two equivalent measurable norms, the notions of a set being cylindrical are obviously the same, and their measures agree by definition.

Lemma 3.14. *Let $\mathscr{H}$ be a real Hilbert space with equivalent Gross-measurable norms $\|\cdot\|_0$ and $|\cdot|$. Then the measures μ and $\tilde{\mu}$ constructed from these norms on $\overline{\mathscr{H}}^*$ coincide.*

If moreover one of the norms is weaker than the other, we can identify the smallest closure as a full-measure subspace of the larger one, by a similar argument.

Lemma 3.15. *Let $\mathscr{H}_1, \mathscr{H}_2$ be real Hilbert spaces and let $\|\cdot\|_1, \|\cdot\|_2$ be Gross-measurable norms on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Suppose there is a unitary map $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that for all $x \in \mathscr{H}_1$*

$$\|x\|_1 = \|Ux\|_2$$

Then U extends to a unitary map $\overline{U} : \overline{\mathscr{H}_1}^* \to \overline{\mathscr{H}_2}^*$ that is an isometry of measure spaces.

Proof. Denote by μ_1 and μ_2 the measures on $\overline{\mathscr{H}_1}^*$ and $\overline{\mathscr{H}_2}^*$ respectively. It is clear that U extends to a unitary map $\overline{U} : \overline{\mathscr{H}_1}^* \to \overline{\mathscr{H}_2}^*$, which is a homeomorphism and as such measurable in both directions. Because we can exchange $\mathscr{H}_1$ and $\mathscr{H}_2$ it suffices to show that if $E \subseteq \mathscr{H}_1$ is cylindrical, that then $\overline{U}(E)$ has the same measure as E. Suppose $E = \{ x \in \mathscr{H}_1 \mid px \in \tilde{E} \}$ for p a finite dimensional projection and $\tilde{E}$ Borel-measurable. Then $\overline{U}(E) = \{ x \in \mathscr{H}_2 \mid U^{-1}pUx \in \tilde{E} \}$, so $\overline{U}(E)$ is also cylindrical. Because U preserves the norm on $\mathscr{H}$, the measures agree. □

Of course it is sufficient for the norms to merely be equivalent, as per what we have previously proven.

CHAPTER 4

Gauge theories on the circle

This chapter will first consider principal bundles on the circle, which is a fiber bundle P where the fibers are a Lie group G. Furthermore G acts on P from the right. A connection on such a principal bundle can be described to be an element of $\Omega^1(\mathbb{T}, \mathfrak{g})$. The most important property of a connection for us will be that one can take parallel transport with respect to a connection. By this we will mean, that given a path γ on the manifold we can transport an element p such that $p \in P_{\gamma(0)}$ along the path to an element $p' \in P_{\gamma(1)}$, allowing us to identify fibers. We will see that we can identify connections on the circle with functions $f \in C^\infty(\mathbb{T}, \mathfrak{g})$. We will denote space of smooth connections by $\mathscr{A}$.

In order to use methods from functional analysis we consider the larger space of square integrable connections $\mathscr{H} = L^2(\mathbb{T}, \mathfrak{g})$. In fact this choice of configuration space is motivated by how we will define the gauge group $\mathscr{G}$ which is the group of maps $P \to P$ that do not affect the structure of the principal bundle.

For any postive t, there is a natural measure μ_t on this space of generalized connections $\mathscr{A}^*$ that makes $L^2(\mathscr{A}^*, d\mu_t)$ naturally isomorphic to the Fock space $\Gamma(\mathscr{H})$ over the space of square integrable connections.

Throughout this chapter let G be a compact and connected Lie group with lie algebra that we will denote by $\mathfrak{g}$. Recall that because G is compact, we can equip $\mathfrak{g}$ with an ad-invariant inner product $\langle \cdot, \cdot \rangle$.

4.1 Principal bundles on the circle

Definition 4.1. *Suppose G acts freely from the left on a manifold P. Let M be a manifold and let $\pi : P \to M$ be a smooth surjective mapping. We call $P \xrightarrow{\pi} M$ a left principal G-bundle if*

- *if $p, q \in P$, then $\pi(p) = \pi(q)$ precisely if $p = gq$ for some $g \in G$;*
- *the bundle admits local sections, that is to say: for any $m \in M$ there exists an open neighborhood U and a map $s_U : U \to P$ such that $\pi \circ s_U = id$.*

Remark. In the literature G is generally assumed to be acting from the right. However, we will let G act from the left. Obviously these definitions are equivalent, by switching out G for the opposite group, i.e. by replacing any group element g by g^{-1}.

The simplest example of such a principal G-bundle on a given manifold M would be the trivial principal bundle $P = M \times G$, with action $g(m, h) = (m, gh)$ and the surjective map being projection onto the first argument. The section can then be defined globally by putting $s(m) = (m, e)$ for all points $m \in M$. Conversely, the existence of local sections is equivalent to the existence of a local trivialization by identifying $s_U(m) \in P_m$ with the identity element.

The only base manifold M of the principal bundles P we will consider throughout this thesis will be the circle

$$\mathbb{T} = \{ \exp(2\pi i s) \mid s \in [0,1) \}$$

which as we know is a compact connected manifold. The circle can also be equipped with a Lie group structure - it is an Abelian Lie group when you equip it with the multiplication it inherits from $\mathbb{C}$ - and as such it can act as a symmetry group G as well; if we are using the circle as a base manifold we will write $\mathbb{T}$ and in the case that the circle acts as the symmetry group G of the principal bundle we will write $U(1)$ as to prevent confusion. Furthermore we will limit ourselves to compact and connected Lie groups G. As such, we will only need to discuss trivial principal bundles, because of the following fact which will be stated without proof:

Fact 4.2. *Any principal bundle on the circle with a compact and connected group G is trivializable.*

However, there is of course no canonical trivialization so even though in the rest of this section we will only need consider principal bundles of the form $P = \mathbb{T} \times G$, we will not choose a fixed trivialization whenever it is not necessary. For a more general treatment of principal bundles, one can refer to chapter II of [19]. Another argument against picking a fixed trivialization is that we will see that not all of the structure we will consider throughout the chapter is independent of our choice of trivialization.

Definition 4.3. *The fundamental vector field of $X \in \mathfrak{g}$ is the smooth vector field defined in a point $p \in \mathbb{T}$ by*

$$a_p(X) := \frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} \left(\exp(tX)p\right).$$

If we fix a trivialization and as such write $P = \mathbb{T} \times G$ and $p = (s, g)$, this vector field is given by $a_{s,g}(X) = (0, Xg)$. The following proposition follows immediately.

Proposition 4.4. *Let P be a principal bundle and let $X \in T_pP$. Then $(d\pi)_p(X) = 0$ if and only if $X = a_p(Y)$ for some $Y \in \mathfrak{g}$.*

Proof. It is immediate that $d\pi(a) = 0$, because $\pi\left(\exp(tX)p\right) = \pi(p)$ has vanishing derivative for any $p \in P$ and $X \in \mathfrak{g}$. The converse is a straightforward application of the rank-nullity theorem; recall that M has dimension $dim(P) - dim(G)$, so the kernel of $(d\pi)_p$ must be of the same dimension as G. Furthermore, $a_p : \mathfrak{g} \to T_pP$ is injective, because $A_p : G \to P_{\pi(p)}$ is a diffeomorphism. □

We will call an $X \in T_pP$ such that $(d\pi)_p(X) = 0$ vertical. A tangent vector X is vertical precisely if the associated path lies tangent to the fiber $P_{\pi(p)}$.

A connection is a family of subspaces H_p of T_pP varying smoothly in p such that $T_pP = H_p \oplus V_p$, where V_p denotes the vertical tangent vectors, varying smoothly in p. Furthermore, we would like these horizontal subspaces to be compatible with the action of G, i.e. that $gH_p = H_{gp}$. A convenient way to capture the notion of such horizontal subspaces compatible with the action of G is given by the following definition. A convenient way to capture this notion is the following:

Definition 4.5. *A 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ is called a connection 1-form if $\omega(a(Y)) = Y$ for all $X \in \mathfrak{g}$ and if $\omega(gX) = Ad_g\omega(X)$ for all $g \in G$ and $X \in \mathfrak{g}$.*

The horizontal subspaces H_p associated to one of these connection 1-forms ω are the kernels of ω_p. Because ω has full rank this is indeed a decomposition of T_pP into vertical vectors and horizontal vectors as desired. The converse also holds.

Proposition 4.6. *Suppose $P = \mathbb{T} \times G$. If ω is a connection on P it is determined entirely by the 1-form $A \in \Omega^1(M; \mathfrak{g})$ given by*

$$A_m(X) = \omega_{(m,e)}(X, 0)$$

for $X \in T_mM$.

Proof. Let $p = (m, g) \in P$ and let $X \in T_mM$, $Y \in T_gG$. Then

$$\begin{aligned}\omega_{(m,g)}(X, Y) = \omega_{g(m,e)}(X, Y) &= \omega_{m,e}\left((dL_g)_p(X, (dR_g^{-1})_g Y)\right) \\ &= Ad_g\omega_{(m,e)}\left(X, (dR_g^{-1})_g Y\right).\end{aligned}$$

Furthermore, the demand that $\omega(a(Z)) = \omega(0, Z) = Z$ for all $Z \in \mathfrak{g}$ means that

$$\omega_{(m,g)}(X, Y) = ad_g(A_m(X)) + (dR_g^{-1})_g Y).$$

Hence, on a trivial bundle the values of a connection 1-form ω are determined entirely by η as defined above. Conversely, any such $A \in \Omega^1(M; \mathfrak{g})$ defines a connection 1-form on the trivial bundle, so there is one-to-one correspondence between connections and $A \in \Omega^1(M; \mathfrak{g})$. The connection 1-form coming from a element of $A \in \Omega^1(M; \mathfrak{g})$ is given explicitly by

$$\omega_{(m,g)}(X, Y) = ad_g(A_m(X)) + Yg^{-1}. \qquad \square$$

The 1-form A given above is however not independent of the trivialization, which will be proven later by showing that A is not gauge invariant.

A curve on P is called horizontal if all the tangent vectors are horizontal, i.e. if $\omega(\frac{d(}{dt}u)) = 0$. A connection on a principal bundle allows us to lift paths on the original manifold in a canonical way; given a path on M and an initial point in the fiber lying above its starting point the path can be lifted to a horizontal path through M, meaning that it is perpendicular to the fibers everywhere . This allows us to identify fibers with each other along a path, by *parallel transporting* an element of one fiber along it.

Proposition 4.7. *Let ω be a connection on P and let γ be a curve on $\mathbb{T}$. Let $u_0 \in P_{\gamma(t_0)}$. Then there is a unique horizontal curve u on P such that $\pi \circ u = \gamma$ and such that $u(t_0) = u_0$. Furthermore, the path lifting is G-equivariant.*

By G-equivariance we mean that lifting a path γ starting in gu_0 is the same as lifting the path but choosing u_0 as a starting point, and then acting by g on u pointwise.

A proof of this proposition can be found in [19] where it is listed as proposition 3.2 of the second chapter. Essentially, because any curve is compact without loss of generality we can concentrate on the case where P is trivial. Finding the path lifting then corresponds to solving the differential equation

$$\dot{u} = -uA(\dot{\gamma})$$

with the starting condition as given which can be shown to have a unique solution. Here A is the connection on the trivialization associated to ω.

Definition 4.8. *The parallel transport maps* $F_\gamma^{s,t} : P_{\gamma(s)} \to P_{\gamma(t)}$ *are the* G*-equivariant diffeomorphisms that map* $p = u(s)$ *to* $u(t)$ *above the curve* γ*.*

We can recover the connection 1-form from the parallel transport. Explicitly, suppose γ is a curve lying entirely in a trivialization U. If γ is such that $\gamma(0) = m$ and such that $\dot\gamma(0) = X$, we can lift γ to a curve (γ, α) on $U \times G$, where $\alpha(0) = e$. Then the assignment $X \to \dot\alpha(0)$ defines a $\mathfrak{g}$-valued one-form on U, which is equal to $s_U^*\omega$ by construction.

Definition 4.9. *A gauge transformation of* P *is a* G*-equivariant diffeomorphism* $\Phi : P \to P$ *such that* $\pi \circ \Phi = \pi$*. The space of gauge transformations is denoted by* $\mathscr{G}$*.*

A gauge transformation Φ preserves the fibers by definition and hence we can define the map $f : P \to G$ where $f(p)$ is the unique element of G such that $f(p)p = \Phi(p)$. Because Φ is G-equivariant, this expression is independent of the choice of p and this map is clearly smooth. Furthermore $f(gp)$ is the element of G such that $f(gp)gp = \Phi(gp) = g\Phi(p)$. Hence $f(gp) = Ad_g f(p)$, because $Ad_g f(p)gp = gf(p)p = g\Phi(p)$ as desired. It follows that f is Ad-invariant. Hence we get a useful way to think about such gauge transformations - as an Ad-invariant map $P \to G$, acting from the right.

If Φ is a gauge transformation, the associated connection 1-form is given by $\omega^\Phi = (\Phi^{-1})^*\omega$.

On a trivialization U, the gauge group can be identified with $C^\infty(U, G)$. In this case, for an $f \in C^\infty(U, G)$ and $(m, g) \in P$,

$$f \cdot (m, g) = (m, gf(m)).$$

The action of such an $f \in C^\infty(U, G)$ on the one-form $\omega_U \in \Omega^1(U, \mathfrak{g})$ associated to a connection one-form ω is given by

$$f \cdot A = f^{-1}Af - f^{-1}df$$

Proof. Consider the connection ω on the trivial principal G-bundle $P = U \times G$. Recall that in a point $p = (m, g) \in P$ we can write

$$\omega_{(m,g)}(X, Y) = ad_g A(X) + Yg^{-1}$$

where $X \in T_m U$ and $Y \in T_g G$. Therefore

$$(f \cdot \omega)_{(m,g)}(X, Y) = ad_g(f \cdot A)(X) + Yg^{-1}).$$

But also, by definition $f \cdot \omega = (\Phi^{-1})^*\omega$ where Φ is given by $\Phi(m, g) = (m, gf(m))$, so for any $(X, Y) \in T_m M \times T_g G$ we have

$$\begin{aligned}(f \cdot \omega)_{(m,g)}(X, Y) &= \left((\Phi^{-1})^*\omega\right)_{(m,g)}(X, Y) \\ &= \omega_(m, gf(m)^{-1})(X, Yf^{-1}(m) - gf(m)^{-1}df(X)f(m)^{-1}) \\ &= ad_{gf(m)^{-1}}(A)(X) + Yg^{-1} - gf(m)^{-1}df(X)g^{-1}.\end{aligned}$$

Therefore, putting the two equations together,

$$Ad_g(f \cdot A)(X) = ad_{gf(m)^{-1}}(A)(X) - gf(m)^{-1}df(X)g^{-1},$$

and hence

$$(f \cdot A)(X) = Ad_{f^{-1}(m)}A(X) - f(m)^{-1}df(X),$$

as desired. ∎

Throughout the rest of this thesis, we will identify points on the circle with points in $[0,1)$ in the obvious way, by defining $\theta : \mathbb{T} \to [0,1) \subseteq \mathbb{R}$ to be the map sending an element $\exp(2\pi i s)$ to s (abusing notation slightly). This allows us to identify the space of smooth connections $\mathscr{A} = \Omega^1(\mathbb{T}, \mathfrak{g})$ with $C^\infty(\mathbb{T}, \mathfrak{g})$. Explicitly, a smooth map $A \in C^\infty(\mathbb{T}, \mathfrak{g})$ can be interpreted as a connection 1-form via

$$A(X) = A(s)d\theta(X) \tag{4.1}$$

where $X \in T_x\mathbb{T}$ for some $s \in \mathbb{T}$. The measure on the circle that we will use will be the standard Lebesgue measure on $[0,1)$ under this identification, i.e. the standard Lebesgue measure on the circle. Note that this identification is still dependent on the trivialization because the identification of the space of connections with $\Omega^1(\mathbb{T}, \mathfrak{g})$ was dependent on the trivialization to begin with.

Because parallel transport is independent of the parameterization, instead of choosing some path γ on the circle we could always choose an arc between the points parameterized as above. The path lifting of this curve with respect to some connection A starting in the identity is then always given by $\theta(A)$, the unique smooth path such that

$$\dot{\theta}(A)\theta(A)^{-1} = A$$

The space of smooth connections is too small to be used to define any kind of quantum theory, as discussed in [21]. Instead we consider the larger real Hilbert space $\mathscr{H} = L^2(\mathbb{T}, \mathfrak{g}) = L^2(\mathbb{T}) \otimes \mathfrak{g}$. Its complexification we will denote by $\mathscr{H}_\mathbb{C} = L^2(\mathbb{T}, \mathfrak{g}_\mathbb{C})$. This will later allow us to also choose a larger gauge group. In fact, the gauge group will be the largest subgroup of all continuous G-valued loops for which the action is still well defined (confer [21]). A vector $A \in \mathscr{H}$ can be considered to be a connection on the circle by the same formula as formula (4.1) above. We will construct a measure on an even larger space of connections $\mathscr{A}^* = C^\infty(\mathbb{T}, \mathfrak{g})^*$. A goal of the rest this chapter will be to show that there exists a unitary equivalence between the Fock space over the Hilbert space $\mathscr{H}_\mathbb{C} = L^2(\mathbb{T}, \mathfrak{g}_\mathbb{C})$ and the space of square-integrable (complex) functions on the set of distributions on the circle $\mathscr{A}^*$, equipped with this Gaussian measure.

We will see in the next chapter that the notion of parallel transport generalizes to this space in a continuous way. In fact the notion of parallel transport generalizes to an even larger class of connections.

4.2 Fock spaces and Gaussian measures

Recall from chapter 3 that on the real dual V^* of a finite dimensional real Hilbert space V, the Gaussian measure of variance $t > 0$ is the Borel probability measure ρ_t given by

$$\int_V f d\rho_t = \int_{v \in V^*} f(v) \exp\left(\frac{-\|v\|^2}{2t}\right) (2\pi t)^{-\frac{d}{2}} dv$$

for any Borel-measurable f on V^*. Here d is the dimension of V and dv is the standard Lebesgue measure on V^*.

For different spaces V and W we denote both the measures by ρ_t, to suppress the amount of indices and suffices we need to write. Because these measures are only used in the context of L^2-spaces, and the space the measure is defined on is also always given in this expression, this will hopefully not lead to ambiguity at any point.

In the preliminary section on Fock spaces we looked at the one dimensional case in example 2.22 and saw that the space $L^2(\mathbb{R}, d\rho_t)$ is unitarily equivalent to $\Gamma(\mathbb{C})$. The measure on the real line in this example being exactly the Gaussian measure as defined above. We will see below that this correspondence generalizes to the d-dimensional case.

Definition 4.10. *Let V, W be finite-dimensional real Hilbert spaces and let $u : V \to W$ be an isometry. Define $L^2(u) : L^2(V^*, d\rho_t) \to L^2(W^*, d\rho_t)$ given by mapping a square-integrable function Ψ on V^* to*

$$\left(L^2(u)\Psi\right)(w^*) = \Psi(w^* \circ u),$$

where $w^* \in W^*$.

Proposition 4.11. *If u is an isometry, the mapping $L^2(u)$ as defined above is an isometry as well.*

Proof. Without loss of generality let us assume that V is a subspace of W and that u is the identity, by identifying V with its image $u(V)$. Denote by $V^\perp$ the orthogonal complement of V in W. Because $W = V \oplus V^\perp$, we can decompose any element $\Psi \in L^2(W^*, d\mu_t)$ as $\Psi = \Psi_V \otimes \Psi_{V^\perp}$ where $\Psi_V \in L^2(V^*, d\mu_t)$ and the same for $V^\perp$. Let $\Phi \in L^2(V^*, d\mu_t)$. To thusly decompose $U_V(\Phi)$, note that

$$U_V(\Phi)(w^*) = \Phi(w^* \circ id_V) = \Phi(w^* \circ id_V) \cdot 1(w^* \circ id_{V^\perp})$$

for any $w^* \in W^*$. Here 1 denotes the constant function 1 on $V^\perp$, which is of unit norm because the measure is a probability measure. Hence, $U_V(\Phi) = \Phi \otimes 1$, which has the same norm as Φ. □

It follows immediately from the definition that $L^2(id_V)$ is the identity on $L^2(V^*, d\rho_t)$. Furthermore it is easy to see that this assignment $L^2(u)$ respects composition: given a pair of isometries $V \xrightarrow{u} W \xrightarrow{v} X$, again $L^2(v)L^2(u) = L^2(vu)$. In other words, the assignment L^2 of finite dimensional real Hilbert spaces to complex Hilbert spaces and of isometries to isometries is a functor. As an immediate corollary, the functor L^2 maps a directed system of finite dimensional real Hilbert spaces to a directed system of Hilbert spaces. We will see later that the direct limit of $L^2(V^*, d\mu_t)$ where the V are finite dimensional subspaces of $\mathscr{A}$ can be identified with the space $L^2(\mathscr{A}^*, d\mu_t)$ that we are looking to define.

We have now defined two ways of simultaneously mapping finite dimensional Hilbert spaces to complex Hilbert spaces and isometries to isometries: we have $V \mapsto \Gamma(V_\mathbb{C})$, taking the Fock space, and $V \mapsto L^2(V^*, d\mu_t)$. The following theorem essentially states that these two assignments are equivalent.

Theorem 4.12. *Let V be a finite dimensional real Hilbert space and let $V_\mathbb{C}$ denote its complexification. There is a unique natural unitary map $U_V : \Gamma(V_\mathbb{C}) \to L^2(V^*, d\mu_t)$ such that in the one dimensional case $U_\mathbb{R}$ is the map from proposition 2.22.*

Here by naturality the following is meant: if $u : V \to W$ is an isometry of two real finite dimensional Hilbert spaces, then $\Gamma(u)U_V = U_W L^2(u)$. In the language of category theory, U_V would be referred to as an equivalence of functors.

Proof. We will prove this theorem in several steps. Let us begin with the following lemma, which essentially says that both Γ and L^2 as functors respect orthogonal complements.

Lemma 4.13. *Let V be a finite dimensional real Hilbert space that decomposes as $V = W \oplus W^\perp$. Let $u : W \to V$ be the isometric inclusion, and similarly define $u^\perp : W^\perp \to V$. Then*

- $\Gamma(id_V) = \Gamma(u) \otimes \Gamma(u^\perp)$

- $L^2(id_V) = L^2(u) \otimes L^2(u^{\perp})$

where we implicitly make the identifications $\Gamma(W \oplus W^{\perp}) = \Gamma(W) \odot \Gamma(W^{\perp})$ *and do the same for the* L^2*-spaces.*

Hence the demand for naturality then fixes U_V for any V. Indeed if V is of dimension n, choosing a single unit-vector $v \in V$ gives us a unitary map $L^2(V^*, d\mu_t) \to L^2(\mathbb{R}v^*, d\mu_t) \otimes L^2(v^{\perp}, d\mu_t)$. The naturality therefore forces us to put $U_V = U_{\mathbb{R}} \otimes U_{v^{\perp}}$, by the lemma above. Proceeding inductively until we have constructed an orthonormal basis for V, we see that the map decomposes as $U_V = U_{\mathbb{R}}^{\otimes n}$. Hence, the map - if it exists - is indeed unique as $U_{\mathbb{R}}$ was fixed.

Definition 4.14. *Let V be a finite dimensional real Hilbert space and let $v \in V$. Consider* $f = \exp(v) = \sum_{n\in\mathbb{N}} \frac{1}{n!} v^{\odot n} \in \Gamma(V)$. *For $x \in V^*$, define*

$$U_V(f)(x) = \exp(-\|v\|^2/2t - \langle x, v\rangle/t).$$

Because all exponential vectors are linearly independent, this expression is well defined. Furthermore, because vectors of the form $\exp(v)$ lie densely in $\Gamma(V_{\mathbb{C}})$, if we can show the linear extension of this assignment to be unitary on the subspace generated by such vectors, it extends to a unitary map on the entire space. Let us put $t = 1$ to simplify notation. Clearly the case for a general $t > 0$ can be proven in the same way. For $v, w \in V$ we calculate

$$\begin{aligned}\langle U_V(\exp(v)), U_V(\exp(w))\rangle &= \int\limits_{x\in A^*} \left(\exp\left(-\langle x, v\rangle - \langle x, w\rangle\right)\exp\left(-\|v\|^2/2 - \|w\|^2/2\right)\right) d\mu \\ &= \int\limits_{x\in A^*} \left(\exp\left(-\langle x, v+w\rangle\right)\exp\left(\langle v+w, v+w\rangle/2 - \langle v, w\rangle\right)\right) d\mu \\ &= \exp(\langle v, w\rangle) \int\limits_{x\in A^*} \left(\exp\left(-\langle x, v+w\rangle\right)\exp\left(\|v+w\|^2/2\right)\right) d\mu.\end{aligned}$$

Furthermore, using our knowledge of Gaussian integrals we calculate

$$\begin{aligned}\int\limits_{x\in A^*} \exp\left(-\langle x, v\rangle\right)\exp\left(-\|v\|^2/2\right) d\mu &= \int\limits_{\lambda\in\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\langle \lambda v, v\rangle - \|v\|^2 \frac{1}{2} - \|v\|^2 \lambda^2 \frac{1}{2}\right) \|v\| \, d\lambda \\ &= 1\end{aligned}$$

for any arbitrary v. Hence, $\langle U_V(\exp(v)), U_V(\exp(w))\rangle = \exp(\langle v, w\rangle) = \langle \exp(v), \exp(w)\rangle$ as desired.

In the one dimensional case, $U_{\mathbb{R}}$ as defined above agrees with the map from proposition 2.22. This follows directly from lemma 2.23. The only thing left to show is therefore that the maps U_V are natural in V.

Proof. Let $u : V \to W$ be an isometry between finite dimensional real Hilbert spaces. Then, for any $v \in V$ we have that $\Gamma(u)$
exp$(v) = \exp(uv)$, and hence for any arbitrary $x \in W^*$

$$\begin{aligned}(U_W(\Gamma(u)\exp(v)))(x) &= (U_W(\exp(uv))) = \exp(-\|uv\|^2/2t - \langle x, uv\rangle/t) \\ &= \exp(-\|v\|^2/2t - \langle x \circ u, v\rangle/t) = \left(L^2(u)U_V(\exp(v))\right)(x)\end{aligned}$$

using in the second to last equality that u is an isometry, so $L^2(u)U_V = U_W\Gamma(u)$ as desired. □

This completes the proof of theorem 4.12. □

There are very many ways to describe U_V. For a description that is perhaps easier to calculate for a general element of the Fock space, see chapter 2 of [23] where it is shown that there is a sequence of symmetric elements $: x :^{\otimes n} \in (V^*)^{\otimes n}$ such that for any $f = (f_n) \in \Gamma(V)$ we can express $U_V(f)(x) = \sum_{n\in\mathbb{N}} \langle : x :^{\otimes n}, f_n\rangle$. The proof that this approach is equivalent to ours is proposition 3.3.10 in the same book.

It would make sense to think that this natural equivalence would in some way carry over to the infinite dimensional case. If we were to construct a Gaussian measure on some closure $\overline{\mathscr{H}}^*$ of $\mathscr{H}$ using the construction from chapter 2, we would hope that $L^2(\overline{\mathscr{H}}^*, d\mu_t) \cong \Gamma(\mathscr{H})$ still and that this equivalence is still natural.

Proposition 4.15. *There exists a family of Borel-measures $d\mu_t$ of variance $t > 0$ on $\mathscr{A}^*$ specified by the following property: If E is a subset of $\mathscr{A}^*$ such that*

$$E = \{x \in \mathscr{A}^* | \langle x, y_i\rangle \in \tilde{E} \text{ for all } i\}$$

for some orthonormal sequence $y_i \in \mathscr{A}$ in $\mathscr{H}$, then $\mu_t(E) = \rho_t^{p\mathscr{H}}(\tilde{E})$ where p is the projection onto the subspace of $\mathscr{H}$ spanned by the y_i.

This measure is not independent of the trivialization of the principal bundle. We will see however that of course the Σ-algebra does not change but also that the notions of a set being a measure zero subset coincide. That is to say, the measure is quasi-invariant under changing trivialization. The implication will be that the action of the Gauge group will not be unitary, but the action will at least induce a well defined action on $L^2(\mathscr{A}^*, d\mu_t)$. The demand for the measure to coincide with the finite dimensional measure on cylindrical subspaces can also be formulated in terms of integrals as follows: if $f \in L^2(\mathscr{A}*, d\mu_t)$ only depends on a finite dimensional subspace, say $f(A) = F(\langle A, \varphi_1\rangle, \dots, \langle A, \varphi_n\rangle)$ where the $\varphi_i \in \mathscr{A}$ are of unit norm in $\mathscr{H}$, then

$$\int_{A\in\mathscr{A}^*} f d\mu_t = \int_{x\in\mathbb{R}^n} F(x) d\rho_t$$

where the latter measure is the finite dimensional Gaussian measure.

Proof. Let $\xi_1, \dots, \xi_d$ be an orthonormal basis for $\mathfrak{g}$, because $L^2(\mathbb{T}, \mathfrak{g}) = L^2(\mathbb{T}, \mathbb{R}) \otimes \mathfrak{g}$, products of the form $e_m x_j \in L^2(\mathbb{T}, \mathfrak{g})$ given by $(e_m\xi_j)(s) = \exp(2\pi m s)\xi_j$ form an orthonormal basis for $L^2(\mathbb{T}, \mathfrak{g})$. Let $T : L^2(\mathbb{T}, \mathfrak{g}) \to L^2(\mathbb{T}, \mathfrak{g})$ be given by

$$T(e_m\xi_j) = \frac{1}{\sqrt{(1+m^2)}} e_m\xi_j$$

This operator T is a Hilbert-Schmidt operator because the sequence $\lambda_m = \frac{1}{1+m^2}$ is summable: $0 < \frac{1}{1+n^2} < \frac{1}{n^2}$, for all n but $n = 0$, which is a sequence well known to be summable. Indeed it can be shown that

$$\sum_{m\in\mathbb{Z}} \frac{1}{1+m^2} = \pi \coth(\pi)$$

which is the Hilbert-Schmidt norm of T. Let k be an integer larger than 1 and let us now define $\|\cdot\|_{T^k}$ to be the norm defined on $\mathscr{H}$ given by $\|A\|_{T^k} = \left\|T^k(A)\right\|$. The Sobolev space $H^{-k}(\mathbb{T}, \mathfrak{g})$ is the closure of $\mathscr{H}$ with respect to $\|\cdot\|_{T^k}$. Furthermore, the Sobolev space $H^k(\mathbb{T}, \mathfrak{g})$ is exactly the domain of T^{-k}. Both of these claims can be found as lemma 6.8 in [13].

As per lemma 5.2, the Sobolev dual spaces $H^{-k}(\mathbb{T}, \mathfrak{g})$ carry Gaussian measures which will be denoted μ_t^k. Furthermore, by lemma 3.14 and 3.15 if $k > 0$ is an integer $H^{-k}(\mathbb{T}, \mathfrak{g})$ is contained in $H^{-k-1}(\mathbb{T}, \mathfrak{g})$ as a measure one subspace.

We can now define a measure on $\mathscr{A}^*$ as follows: Let μ_t be the measure on $\mathscr{A}* = \bigcup_{k\in\mathbb{N}} H^{-k}(\mathbb{T}, \mathfrak{g})$ by considering the Σ-algebra generated by sets E such that $E \cap H^{-k}(\mathbb{T}, \mathfrak{g})$ is Borel-measurable for some k (and hence for all $m > k$ as well) and putting $\mu_t(E) = \mu_t^k(E \cap H^{-k}(\mathbb{T}, \mathfrak{g}))$.

This defines a Borel measure because the topology on $\mathscr{A}^*$ is exactly the final topology with respect to the maps $H^{-k}(\mathbb{T}, \mathfrak{g}) \to \mathscr{A}^*$, so sets that are open in $H^{-k}(\mathbb{T}, \mathfrak{g})$ form a basis for the topology of $\mathscr{A}^*$, and they are all measurable. □

There are several ways to construct this measure. The reason for constructing the measure as we did above is because we will use its form explicitly in what follows. For perhaps an easier construction that does not depend on the construction we discussed in chapter 3, one can consider [3] where this measure and measures like it are the focus of the entire second chapter, or [23] where the existence of this measure is listed as theorem 1.5.2.

Lemma 4.16. *Let F be the directed set of finite dimensional subspaces of $\mathscr{A}$. Then*

$$\lim_{V\in F} L^2(V^*, d\mu_t) = L^2(\mathscr{A}^*, d\mu_t).$$

Proof. Note that the indicator maps 1_A where A is a cylindrical subset of $\mathscr{A}^*$ are cylindrical, because given $y_1, \ldots, y_n \in A$ such that $A = \{x \in \mathscr{A}^* | (x(y_1), \ldots, x(y_n)) \in E\}$ for some Borel-measurable E, we have that $1_A(x) = 1_E((x(y_1), \ldots, x(y_n)))$. hence

$$\int_A \Psi d\mu_t = 0$$

for all cylinder sets A. Because such sets generate the Σ-algebra, Ψ must be identically zero almost everywhere with respect to μ_t. In other words: cylindrical functions are dense in $L^2(\mathscr{A}^*, d\mu_t)$. □

Furthermore, as per the following theorem $L^2(\mathscr{A}^*, d\mu_t)$ is a proper infinite dimensional generalization of $L^2(V^*, d\mu_t)$ in the finite dimensional case.

Theorem 4.17. *There is a natural unitary map $U_{\mathscr{H}} : \Gamma(\mathscr{H}_{\mathbb{C}}) \to L^2(\mathscr{A}^*, d\mu_t)$.*

Proof. Let F be the partially ordered system of finite dimensional subsets of $\mathscr{A}$. Because $\mathscr{A}$ is dense in $\mathscr{H}$, the direct limit is $\mathscr{H}$. Hence, by lemma 2.34, $\lim_{V\in F} \Gamma(V_{\mathbb{C}}) = \Gamma(\mathscr{H}_{\mathbb{C}})$. Furthermore, by theorem 4.12, there is a natural unitary equivalence $U_V : \Gamma(V_{\mathbb{C}}) \to L^2(V^*, d\mu_t)$. That is to say, there are isometries $u_V = U_V \circ \Gamma(id_V) : \Gamma(V_{\mathbb{C}}) \to \Gamma(\mathscr{H}) \to L^2(\mathscr{A}^*, d\mu_t)$ such that $u_W \circ \Gamma(id_{V,W}) = u_W$ for any pair $V \subseteq W$. It therefore follows from the uniqueness of the direct limit that there is a unitary map $U_{\mathscr{H}} : \Gamma(\mathscr{H}_{\mathbb{C}}) \to \lim L^2(V^*, d\mu_t)$. □

What we mean by that this unitary map is still natural is the following: if V is a finite dimensional subspace of $\mathscr{A}$, then

$$L^2(u_{V^*})U_V = U_{\mathscr{H}}\Gamma(u_V)$$

where $L^2(u_{V^*})$ denotes the embedding of $L^2(V^*, d\mu_t)$ into $L^2(\mathscr{A}^*, d\mu_t)$ and where by u_V we mean the inclusion of V into $\mathscr{H}$.

Note that we said *a* generalization. There is no canonical general construction to make an L^2-space out of an arbitrary real Hilbert space $\mathscr{H}$. One could in general choose a (real-valued) strictly positive injective Hilbert-Schmidt operator T on $\mathscr{H}$, construct a measure μ using lemma 3.6 on the closure $\overline{\mathscr{H}}$ of the space with respect to the norm $\|\cdot\|_T$ and consider $L^2(\mathscr{H}) = L^2(\overline{\mathscr{H}}, d\mu)$. This does make all the previous theorems and propositions true, when replacing $\mathscr{A}$ by $\operatorname{dom}\left(T^{-1}\right)$. However, there is no canonical way to choose a Hilbert-Schmidt operator or of choosing a Gross-measurable norm, which of course would also work. However, because on any finite dimensional space all norms are equivalent all norms are Gross-measurable (this is of course trivial) and they all produce the same measure space, namely just the finite dimensional space itself in the ordinary norm.

CHAPTER 5

Discrete approximations

In this chapter we will look at discrete subsets Λ of the circle and discuss principal bundles on such discrete subsets. The first main result will be that we can extend the parallel transport map to almost all of $\mathscr{A}^*$ - where originally we had only defined how to take the parallel transport with respect to smooth connections. This extension of the parallel transport map will allow us to write $L^2(\mathscr{A}^*, d\mu_t)$ as the direct limit of spaces of the form $L^2(G^\Lambda, d\rho_t)$, where $d\rho_t$ is a heat kernel measure and where G^Λ is the space of connections on the discrete subset. The notion of a discrete principal bundle for a general manifold can also be found in [2], which deals with gauge theories on graphs. Our discrete subsets Λ of the circle can be considered to be graphs by connecting all adjacent points with a single edge.

One can think of some graphs being finer than others - for example if a graph Λ_1 is obtained from another graph Λ_2 by adding a single point along one of the edges. For a more general definition of refining a graph, see for example the third section of [1]. In the chapter to follow we not only restrict ourselves to just the circle but we also only allow only a specific way of refining a graph. In particular, a graph Λ_1 is only finer than another graph Λ_2 if the second is a subset of the first. That is to say, we only allow for the method of refining described above, by adding points along an edge. This ordering will make the spaces $L^2(G^\Lambda, d\rho_t)$ into a directed set of Hilbert spaces.

5.1 Parallel transport

Recall from the previous chapter that there exists a measure on the space of connections $\mathscr{A}^*$ which we called μ_t. Furthermore, we saw that there was a natural unitary equivalence $L^2(\mathscr{A}^*, d\mu_t) \cong \Gamma(L^2(\mathbb{T}, \mathfrak{g}_\mathbb{C}))$. In this section we will show that given a graph Λ lying in the circle, there exists a measure ρ_t on G^Λ called the heat kernel measure such that $L^2(G^\Lambda, d\rho_t)$ can be isometrically embedded into $L^2(\mathscr{A}^*, d\mu_t)$ by identifying them with the elements of the latter space invariant under action of some normal subgroup of the gauge group $\mathscr{G}$. Furthermore, we will see that the direct limit of $L^2(G^\Lambda, d\rho_t)$ is exactly $L^2(\mathscr{A}^*, d\mu_t)$.

The goal of this section will be showing that the notion of taking the parallel transport around the circle can be extended from the smooth connections to almost all of the larger space of distributions on the circle. If we choose an appropriate measure called the Wiener measure $d\nu_t$ on the space $W(G)$ of continuous G-valued paths starting in the identity, this extended parallel transport map is in fact an almost isometry of measure spaces between $\mathscr{A}^*$ and $W(G)$. We will use this to prove our theorem.

Definition 5.1. *Let $W(G)$ be the space of continuous paths $\sigma : I \to G$ such that $\sigma(0) = e$.*

Proposition 5.2. *There exists a measure on $W(G)$ specified entirely by the following property: if f is an integrable function on $W(G)$ only depending on finitely many points, say $f(\sigma) = F(g_1, \ldots, g_n)$ for some Haar-measurable F on G^n, then*

$$\int_{W(G)} f(\sigma) d\nu_t = \int_{G^n} F(g_1, \ldots, g_n) \prod_j d\rho_{tl_j}(g_{j-1}^{-1} g_j).$$

That is to say, if f is a measurable function on $W(G)$ that only depends on a finite set of points, say on $\Lambda = (x_1, \ldots, x_n)$, then the integral can be expressed as above. By an argument akin to the one provided for lemma 4.16, the Hilbert space $L^2(W(G), d\nu_t)$ can be written as the direct limit of cylindrical functions. That is to say, $L^2(W(G), d\nu_t) = \lim_{\Lambda \in F} L^2(G^\Lambda, d\nu_t)$, where $d\nu_t^\Lambda$ is the measure on G^n from definition 5.2 given above.

This measure is known as the Wiener measure on $W(G)$. For more details about this measure, see for example [7].

Proposition 5.3. *The holonomy-map $\theta : \Omega^1(\mathbb{T}, \mathfrak{g}) \to C^\infty(\mathbb{T}, G)$ can be extended to a map*

$$\theta : \mathscr{A}^* \to W(G)$$

defined almost everywhere with respect to μ_t.

Theorem 5.4. *The map $U_\theta : L^2(W(G), d\nu_t) \to L^2(\mathscr{A}^*, d\mu_t)$ given by mapping Ψ to*

$$U_\theta(\Psi)(A) = \Psi(\theta(A))$$

is well defined and an isometry.

We will prove these theorems simultaneously in several steps. We will furthermore once again neglect the parameter t by putting $t = 1$ throughout the proof which is without loss of generality because we can rescale the inner product on $\mathfrak{g}$ by any positive number.

Definition 5.5. *Let $H_0^1(I, \mathfrak{g})$ be the subspace of $H^1(I, \mathfrak{g})$ of functions σ such that $\sigma(0) = 0$*

It is a closed subspace of $H^1(I, \mathfrak{g})$. We can therefore equip $H_0^1(I, \mathfrak{g})$ with two equivalent norms: the ordinary Sobolev norm that it inherits from $H^1(I, \mathfrak{g})$, but also the norm given by $\|\sigma\|_{H_0^1} = \|\dot{\sigma}\|$ where the latter norm is the L^2-norm. These norms are equivalent, see for example [8]. The main advantage of the latter norm being, that it makes the mapping $\int : L^2(I, \mathfrak{g}) \to H_0^1(I, \mathfrak{g})$ defined by

$$\left(\int \varphi \right)(s) = \int_{\tau=0}^{s} \varphi(\tau)$$

into a unitary map.

We will once again construct a Gaussian measure on the slightly larger space $W(\mathfrak{g})$ of continuous paths σ through $\mathfrak{g}$ such that $\sigma(0) = 0$, using the same procedure as we did in the last section of the previous chapter.

Proposition 5.6. *The norm $|\cdot|$ on $H_0^1(I, \mathfrak{g})$ given by*

$$|\sigma| = \sup_{s \in \mathbb{T}} \{ \langle \sigma(s), \sigma(s) \rangle_{\mathfrak{g}}^{\frac{1}{2}} \} .$$

is Gross-measurable.

It is significantly more difficult to establish that this norm is Gross-measurable now that there is no associated Hilbert-Schmidt operator. A direct but lengthy proof can be found in [12], a shorter one in [9].

Proposition 5.7. *The completion of $H_0^1(I,\mathfrak{g})$ in this norm is exactly $W(\mathfrak{g})$.*

Proof. Let σ be a continuous path through $\mathfrak{g}$ starting in zero. For any partition Λ we can consider the function σ_Λ that takes value $\sigma(x_i)$ in points of the partition and is a straight line in between points of the partition. Because $H^1(I,\mathfrak{g})$ is well-behaved under concatenation, all these functions lie in $H_0^1(I,\mathfrak{g})$ and as such $H^1(I,\mathfrak{g})$ is dense in $W(\mathfrak{g})$, as the net converges to σ in the norm. Furthermore, $W(\mathfrak{g})$ is closed with respect to this norm as is well known. □

Hence, by the same construction we used in the proof of lemma 5.2, we get measures $d\nu_t$ on $W(\mathfrak{g})$ of continuous paths through $\mathfrak{g}$ such that $\sigma(0) = 0$.

Let $S : H_0^1(I,\mathfrak{g}) \to L^2(\mathbb{T},\mathfrak{g})$ be the unitary map given by $S(\sigma) = \dot{\sigma}$. It is unitary. To see that it is surjective, note that if $A \in L^2(\mathbb{T},\mathfrak{g})$, that $\sigma(s) = \int_{\tau=0}^{s} A(\tau)d\tau$ is in $H_0^1(I,\mathfrak{g})$ and has derivative A. Therefore, S extends to a unitary map $\overline{S} : L^2(I,\mathfrak{g}) \to H^{-1}(\mathbb{T},\mathfrak{g})$.

Recall that if $\xi_1,\ldots,\xi_d$ is an orthonormal basis for $\mathfrak{g}$, we had defined $T : L^2(\mathbb{T},\mathfrak{g}) \to L^2(\mathbb{T},\mathfrak{g})$ as

$$T(e_m\xi_j) = \frac{1}{\sqrt{(1+m^2)}} e_m\xi_j$$

where $e_n(s) = \exp(ins)$.

Proposition 5.8. *The norms on $H_0^1(I,\mathfrak{g})$ given by $\|S\sigma\|_T$ and $\|A\|_{L^2}$ where the latter norm is taken in $L^2(I,\mathfrak{g}) = L^2(\mathbb{T},\mathfrak{g})$ are equivalent.*

Proof. The norms $\|\cdot\|_T$ and $\|\varphi\|_S = \|\dot{\varphi}\|$ are equivalent on $H_0^1(\mathbb{T},\mathfrak{g})$. Note that T maps a constant function $X \in \mathfrak{g}$ to itself and as such the norms are also equivalent on the subspace of maps of the form $s \mapsto sX$. because $H_0^1(I,\mathfrak{g}) = H_0^1(\mathbb{T},\mathfrak{g}) \oplus \mathfrak{g}$ where by $\mathfrak{g}$ we mean $s \mapsto sX$, for $X \in \mathfrak{g}$, the norms are therefore equivalent on the larger space. □

Proposition 5.9. *The L^2-norm on $H_0^1(I,\mathfrak{g})$ is Gross-measurable.*

This follows immediately from the fact that on a finite measure space, the L^2-norm is weaker than the sup-norm which we already knew to be measurable. There are therefore two extensions of $H_0^1(I,\mathfrak{g})$ carrying measures. with the extension of S being an isometry of measure spaces by lemma 3.15. Furthermore, by by lemma 3.15 we have an isometry of measure spaces $S : L^2(I,\mathfrak{g}) \to H^{-1}(\mathbb{T},\mathfrak{g})$ given on $H_0^1(I,\mathfrak{g})$ by taking the derivative. Because $C(I,\mathfrak{g})$ is a measure one subspace of $L^2(I,\mathfrak{g})$ and because $H^{-1}(\mathbb{T},\mathfrak{g})$ is a measure one subspace of $\mathscr{A}^*$ We have an associated almost isometry of measure spaces $S : W(\mathfrak{g}) \to \mathscr{A}^*$

We can define this map explicitly on all of $W(\mathfrak{g})$: every $\mathfrak{g}$-valued path $\eta \in W(\mathfrak{g})$ can be considered to be a functional on $\mathscr{A}$, i.e. as an element of $\mathscr{A}^*$, by considering the Stieltjes integral

$$S(\eta)(\sigma) = \int_I \langle \sigma(\tau), d\eta(\tau) \rangle d\tau$$

where $\sigma \in E$. Explicitly, $S(\eta)(\sigma)$ is the limit of the net

$$S_\Lambda(\eta,\sigma) = \sum_{i=1}^{\#\Lambda} \langle \sigma(x_i), \eta(x_{i+1}) - \eta(x_i) \rangle.$$

indexed by partitions of the unit interval. Recall that because such Stieltjes integrals still admit integration by parts and any smooth map is of bounded variation, this is indeed a well defined expression.

The following theorem is due to Itô [17]

Theorem 5.10. *There is an almost isometry of measure spaces $\iota : W(\mathfrak{g}) \to W(G)$ both equipped with their Wiener measures of the same variance.*

Explicitely, if $\Lambda = (x_1, \ldots, x_n)$ is a partition of the unit interval and $s \in I$ lies between x_i and x_{i+1}, we define inductively

$$\begin{aligned} \iota_\Lambda(\sigma)(0) &= e \\ \iota_\Lambda(\sigma)(s) &= \iota_\Lambda(x_i)\exp(\sigma(s) - \sigma(x_i)), \end{aligned}$$

where σ is a path through $\mathfrak{g}$ starting in zero. The limit exists almost everywhere in $W(\mathfrak{g})$ with respect to the Wiener measure.

Let us now concentrate on proving theorem 5.4, i.e. on proving that the map θ that we have defined meaningfully extends the notion of holonomy.

Lemma 5.11. *If $A \in L^2(I, gl_n(\mathbb{R}))$, there is a unique path $\theta(A) \in H^1(I, GL_n(\mathbb{R}))$ such that $\theta_0(A) = I_n$ and such that $\dot{\theta}(A)\theta(A)^{-1} = A$. Furthermore, the map $\theta : L^2(I, gl_n(\mathbb{R})) \to H^1(I, GL_n(\mathbb{R}))$ is continuous.*

A proof can be found in [24] or as lemma 4.1 in [6]. In fact, it is shown in these references that the map is smooth as a map between infinite dimensional manifolds that are locally homeomorphic to a Hilbert space, although we will not need the lemma in this generality.

The desired result now follows almost directly from the previous proposition: if $A \in L^2(\mathbb{T}, \mathfrak{g})$, there is a sequence A_{Λ_n} of functions that are simple with respect to partitions Λ_n such that the union of all Λ_n lies dense in $\mathbb{T}$. The holonomy of a simple function A_{Λ_n}, say that

$$A_{\Lambda_n} = \sum X_i 1_{[x_{i-1}, x_i]}$$

where $X_i \in \mathfrak{g}$, is easily calculated to be

$$\theta\left(A_{\Lambda_n}\right) = \exp(l_1 X_1)\ldots\exp(l_{i-1}X_{i-1}) \cdot \exp((s - x_i)X_i)$$

by evaluating the parallel transport segment by segment. This is precisely equal to $\iota \circ \int (A_{\Lambda_n}) = \iota(S^{-1}(A_{\Lambda_n}))$. By continuity of taking the holonomy, we obtain that $\iota \circ \int = \theta$ for all $A \in L^2(\mathbb{T}, \mathfrak{g})$, and hence for all smooth connections as well.

In conclusion, we showed that he expression for the parallel transport

$$\theta(\sigma)(s) = \lim_{\Lambda \in \mathbb{F}} \prod_{x_i \in \Lambda} \exp\left(\int_{x_{i-1}}^{x_i} \sigma(\tau) d\tau\right)$$

that holds for smooth σ can be extended to almost all of $\mathscr{A}^*$.

5.2 Partitions of the circle

If instead of a principal bundle on the entirety of the circle we consider a principal bundle on a discrete subset consisting of only a finite number of points on the circle, the notion of a connection does not have an obvious analogy. After all, there are no smooth paths between the different points anymore, so no paths to transport along. However, as we recall, the parallel transport maps are smooth G-equivariant maps between the fibers above the initial and final points. Furthermore we will prove further along in the text that any smooth G-equivariant map between fibers above distinct points on the circle comes from the parallel transport along the arc between the points with respect to some connection. It therefore makes sense to generalize the notion of a connection to the discretized case by considering the parallel transport maps between the points.

As any two points on the circle are essentially connected by two paths, one clockwise, one counterclockwise, it makes sense to consider graphs where only such paths are allowed. If we always assume the orientation to be counterclockwise, there is a unique path. Hence, instead of considering graphs $\varphi = (E, V)$ on the circle, we will consider only a set of vertices V. The edge associated to a pair x, y of points on the circle is the counterclockwise segment of the circle lying in between the two points. For simplicity we will take the last point of such a graph to always be 1.

Definition 5.12. *A partition of the circle is a sequence $\Lambda = (x_1, \ldots, x_n)$ of distinct points lying in the circle, such that $x_1 > 0$, $x_n = 1$ and such that the points are in counterclockwise order.*

By convention x_0 will mean 0. That is to say, $x_0 = x_n$, as one would expect when discussing the circle. Such a choice of n distinct points partitions the circle into n segments. We will use the two notions - a partition as a set of points and a partition as a set of segments - interchangeably; the set of edges is no longer needed because it is specified entirely by the set of vertices. Hence a partition Λ_1 is coarser than Λ_2 if Λ_1 is a subset of Λ_2. The inclusion of the point $x_n = 1$ is not strictly necessary, but will allow us to continue to make the identification of $\mathbb{T}$ with $[0, 1)$.

As we know any connection on M induces a smooth map $F : P_{s(e)} \to P_{t(e)}$.

Proposition 5.13. *Let P be an arbitrary principal G-bundle over $\mathbb{T}$. Consider a distinct pair of points s_1, s_2 on the circle and a path γ going from s_1 to s_2. If $p \in P_{s_2}$ is arbitrary there is a connection one form such that the path-lifting α of γ is such that $\alpha(t) = p$.*

Proof. It is sufficient to show the existence of a $h \in \mathscr{G}$ such that $h(t) = g$ and such that $h(s) = e$, which follows from the path-connectedness of G. □

By the same argument, given a finite amount of points s_i on the circle, a piecewise smooth path σ visiting these points, and $p_i \in P_{s_i}$, there is a connection such that the path lifting of σ takes the values p_i above s_i. Hence given a segment of the circle $e = [x_i, x_{i+1}]$, it makes sense to define the space of connections on e, denoted A_e, to be the space of smooth maps $F : P_{x_i} \to P_{x_{i+1}}$ compatible with the action of G. The space of all connections on ϕ is then $A = \prod_{e \in E} A_e$ where E denotes the set of segments of our partition.

The following proposition is a direct corollary of the product rule on Lie groups:

Proposition 5.14. *Let ω be a connection, let $g \in \mathscr{G}$ and let γ be a curve. Denote by T the parallel transport with respect to ω and by S the one with respect to $g\omega$. Then $S_\gamma^{s,t} = g(\gamma(t)) T_\gamma^{s,t} g(\gamma(s))^{-1}$.*

So in the discretized version of the principal bundle by analogy the gauge group $\mathscr{G}$ acts on the space on connections A by conjugation.

Definition 5.15. *Let Λ be a partition of the circle, say $\Lambda = (x_1, \ldots, x_n)$. Define $u_\Lambda : L^2(G^\Lambda, d\rho_t^\Lambda) \to L^2(\mathscr{A}^*, d\mu_t)$ by*

$$(u_\Lambda \Psi)(A) = \Psi(\theta_{x_1}(A), \ldots, \theta_{x_{n-1}}(A)^{-1} \theta_{x_n}(A))$$

for $\Psi \in L^2(G^\Lambda, d\rho_t)$ and $A \in \mathscr{A}^*$ a connection.

The elements $\theta_{x_{i-1}}(A)^{-1}\theta_{x_i}(A)$ of G are exactly the parallel transports of the identity element e from x_{i-1} to x_i with respect to the connection A.

Theorem 5.16. *Let Λ be a partition. Then u_Λ as defined above is an isometry.*

Proof. Let us denote by κ the map

$$\kappa(g_1, \dots, g_n) = (g_1, g_1 g_2, \dots, \prod_{i=1}^{n} g_i).$$

Recall ν_t was the measure introduced in definition 5.2. Then composing with κ is an isometry by the definition of the measures. Indeed, using the properties of the Haar measure

$$\begin{aligned}\int_{G^n} f(g_1, \dots, g_n) d\nu_t &= \int_{G^n} f(g_1, \dots, g_n) \prod_j d\rho_{tl_j}(g_{j-1}^{-1} g_j) \\ &= \int_{G^n} f(g_1, g_1 g_2 \dots, g_1 \cdots g_n) \prod_j d\rho_{tl_j}(g_j) \\ &= \int_{G^n} (f \circ \kappa)(g_1 \dots, g_n) d\rho_t(g_1, \dots, g_n).\end{aligned}$$

for any $f \in L^2(G^n, d\nu_t)$. Furthermore, the inclusion of $L^2(G^n, d\nu_t)$ is isometric by the definition of the Wiener measure. It follows that u_Λ is an isometric inclusion as desired. □

Any function of the parallel transports between points of a partition is of course also a function of the parallel transports between the points of any finer partition. For example, let Λ_1 be the circle entirely - so one vertex and one edge -, and let $\Psi \in L^2(G^{\Lambda_1}, d\rho_t)$. If Λ_2 is the circle split up into two segments - let's say we have added a point x to the partition. Then $\Psi \in L^2(G^{\Lambda_2}, d\rho_t)$ as well, given by $\Psi(g, h) = \Psi(gh)$, because the parallel transport between 0 and 1 is exactly the product of the parallel transports between 0 and x and x and 1 respectively. It follows that the system $\{ L^2(G^\Lambda, d\rho_t), u_\Lambda \}$ is a direct system of Hilbert spaces.

We can of course also show this directly, which is equivalent to showing that the Wiener measure is well defined which we neglected to show earlier. Now, any finer partition can be obtained by splitting up segments in two - i.e. by adding a single point. Furthermore, if we split up - say - the i-th segment, the only elements that change is the parallel transport along this segment. After all, the other segments don't change and hence neither does the parallel transport along these segments. Because the measure splits up, we need not consider any of the other elements of the partition. Without loss of generality we can therefore consider Λ_1 to be the whole circle - but now of length l, and Λ_2 to be the circle split into two segments of lengths l_1 and $l_2 = l - l_1$ respectively.

Hence let $\Psi, \Phi \in L^2(G^{\Lambda_1}, d\rho_t)$ and let us shorten the map $u_{2,1}$ to just u. Recall that we can write the heat kernel on G as $p_t = \sum_{\pi\in\hat{G}} \exp(\lambda_p it/2) d_\pi \chi_\pi$. We calculate

$$\begin{aligned}
\langle u\Phi, u\Psi\rangle &= \int_{G\times G} \overline{\Phi}(gh)\Psi(gh)d\rho_t(g,h) \\
&= \int_G\int_G \overline{\Phi}(gh)\Psi(gh)d\rho_{tl_1}(g)d\rho_{tl_2}(h)dgdh \\
&= \int_G\int_G \overline{\Phi}(h)\Psi(h)\left(p_{tl_1}(g)\left(p_{tl_2}(g^{-1}h)\right)\right)dgdh \\
&= \int_G\int_G \overline{\Phi}(h)\Psi(h)\left(\sum_{\pi,\nu\in\hat{G}} d_\pi d_\nu \exp((l_1\lambda_\pi + l_2\lambda_\nu)t/2)\chi_\pi(g)\chi_\nu(g^{-1}h)\right)dgdh \\
&= \int_G \overline{\Phi}(h)\Psi(h)\left(\sum_{\pi\in\hat{G}} d_\pi \exp(l\lambda_\pi t/2)\chi_\pi(h)\right)dh \\
&= \langle\Phi,\Psi\rangle,
\end{aligned}$$

using the properties of the Haar measure and using the Schur orthogonality of the characters which is lemma 2.54.

Let us return to the gauge group $\mathscr{G}$ which as we recall we defined as the space $\mathscr{G} = H^1(\mathbb{T}, G)$ of finite energy paths through G. Because we interpret any path σ in $W(G)$ as the parallel transport of some generalized connection, the action is defined to be consistent with this interpretation Hence the action of $\mathscr{G}$ is given by

$$f\cdot\sigma(s) = f(0)\sigma(s)f(s)^{-1}$$

It is of course not at all obvious that this action is even well-defined. This is proven in [22].

Let Λ be a partition of the circle. Consider $\mathscr{G}_\Lambda$ to be the subgroup

$$\mathscr{G}_\Lambda = \{f\in\mathscr{G} | f(x_i) = e \text{ for all } x_i\in\Lambda\}.$$

Then clearly $\mathscr{G}_\Lambda$ is a normal subgroup of $\mathscr{G}$.

Proposition 5.17. *The space of square-integrable functions that are invariant under the action of $\mathscr{G}_\Lambda$, denoted $L^2(\mathscr{A}^*, d\mu_t)^{\mathscr{G}_\Lambda}$ is precisely $L^2(G^\Lambda, d\rho_t)$. More explicitly, if $\Psi\in L^2(\mathscr{A}^*, d\mu_t)$ is invariant under the action of $\mathscr{G}_\Lambda$, then there is a square-integrable function F on G^Λ such that*

$$\Psi(A) = F = \left(\theta_{x_1}(A), \ldots, \theta_{x_{n-1}}(A)^{-1}\theta_{x_n}(A)\right)$$

almost everywhere with respect to $d\mu_t$.

Proof. Firstly, note that indeed any element of $L^2(G^\Lambda, d\rho_t)$ is invariant under the action of $\mathscr{G}_\Lambda$, because the action of $f\in\mathscr{G}$ on $L^2(G^\Lambda, d\rho_t)$ only depends on the values f takes in the points of the partition.

To see that these are all elements of $L^2(\mathscr{A}^*, d\mu_t)$ that are invariant under the action of this normal subgroup, let us first note that all the structure is well behaved under concatenation. If Λ is a partition, then there is a one to one correspondence between $\mathscr{G}_\Lambda$ and tuples of n elements of $H^1_0(\mathbb{T}, \mathfrak{g})$, by sending an element $f\in\mathscr{G}_\Lambda$ to $(\overline{f}_i)$, given by

$$\overline{f}_i(s) = f\left(\frac{s}{l_i} - x_i\right).$$

Similarly, for any two paths $\sigma, \tau \in W(G)$ we can create a path $\sigma * \tau \in W(G)$ given by

$$\begin{aligned} \sigma * \tau(s) &= \sigma(2s) \text{ if } s < \frac{1}{2} \\ &= \sigma(1)\tau(2s-1) \text{ else,} \end{aligned}$$

If Ψ is a measurable map on $W(G)$, then

$$\int_{\sigma \in W(G)} \Psi(\sigma) d\nu_t = \int_{\tau \in W(G)} \int_{\sigma \in W(G)} \Psi(\sigma * \tau) d\nu_{\frac{t}{2}} d\nu_{\frac{t}{2}}$$

by the construction of the Wiener measure; any partition consists of points that lie either before after after the half-way point. We can generalize this to a general partition in the obvious way, by concatenating tuples of paths and the action is compatible with this decomposition. Hence, by induction to the number of elements in the partition Λ, if we assume we know the theorem for all partitions of less than n elements: suppose Λ has n elements. Suppose $\Psi \in L^2(\mathscr{A}^*, d\mu_t)$ is invariant under the group action of $\mathscr{G}_\Lambda$. By splitting the interval in half between the first and second element and integrating over the first segment with respect to the Wiener-measure, we obtain a function that is invariant under the action of $\mathscr{G}_{\Lambda_1}$, i.e. and element of $L^2(G^{\Lambda_1}, d\rho_t)$. The same holds for splitting it up in between the second last and last elements. Hence the element must be cylindrical with respect to the entire partition, exactly as we wanted.

As such, we only have yet to establish the base case $n = 1$, i.e. the trivial partition containing only one point. This theorem is known as the Gross ergodicity theorem, and can be found as Theorem 2.2 in [16]. □

Hence, the space of functions invariant under the action of $\mathscr{G}_\Lambda$ is precisely the subspace of functions that depend only on the parallel transport along the segment of Λ.

If Λ_1 is coarser than Λ_2, then $\mathscr{G}_{\Lambda_1}$ is contained in $\mathscr{G}_{\Lambda_2}$. Hence any function that is invariant with respect to $\mathscr{G}_{\Lambda_1}$ is also invariant with respect with respect to $\mathscr{G}_{\Lambda_2}$. The inclusion map is given by what amounts to a coproduct: if for simplicity of notation we assume for now that Λ_2 was obtained by adding one point y between x_i and x_{i+1} to Λ_1, then $F \in L^2(G^{\Lambda_1}, d\rho_t)$ maps to $\overline{F} \in L^2(G^{\Lambda_2}, d\rho_t)$ given by

$$\overline{F}(g_1, \ldots, g_i, h, g_{i+1}, \ldots, g_n) = F(g_1, \ldots, g_i, hg_{i+1}, \ldots, g_n).$$

Any finer partition can be obtained by a finite amount of one-point additions so the general form of the inclusion can be derived from this. This inclusion is exactly the one from theorem 5.16.

On any of these spaces we can consider the action of the remainder of the gauge group $\mathscr{G}^\Lambda = \mathscr{G}/\mathscr{G}_\Lambda$. Now, if $f_1, f_2 \in \mathscr{G}$ are such that $f_1(x) = f_2(x)$ for all $x \in \Lambda$, then $f_1 f_2^{-1} \in \mathscr{G}_\Lambda$. Hence this quotient can be identified with G^n by evaluating the points of the partition.

The action of $\mathscr{G}^\Lambda$ is given by conjugation. That is to say, if $f \in \mathscr{G}$ and $\Psi \in L^2(G^\Lambda, d\rho_t)$, then

$$f\Psi(g_1, \ldots, g_n) = \Psi(f(0)g_1 f(x_1)^{-1}, \ldots, f(x_{n-1})g_n f(x_n)^{-1}),$$

as we would expect as per proposition 5.14.

To summarize the previous two chapters, we have seen that our configuration space $L^2(\mathscr{A}^*, d\mu_t) \cong \Gamma(\mathscr{H}_\mathbb{C})$ can be written as the direct limit $\lim_{\Lambda \in \mathbb{F}} L^2(G^\Lambda, d\rho_t)$.

CHAPTER 6

Algebras of Observables

6.1 Preliminary definitions

Definition 6.1. *A Banach *-algebra is a Banach space B with an algebra structure and involution $a \mapsto a^*$ such that $\|ab\| \leq \|a\| \|b\|$ and such that $\|a^*\| = \|a\|$ for all $a, b \in B$.*

Definition 6.2. *A C^*-algebra is a Banach *-algebra A with the C^*-property, to wit*

$$\|a\|^2 = \|a^* a\|$$

for all $a \in A$.

Example 6.3. For any given Hilbert space $\mathscr{H}$, the space of bounded operators $B(\mathscr{H})$ on the space is a C^*-algebra. The compact operators $K(\mathscr{H})$ are a C^*-ideal in $B(\mathscr{H})$.

Definition 6.4. *A groupoid is a (small) category $\mathbb{G}$ in which every morphism is invertible.*

For example, a group is a groupoid with only one object. As such we could think about a groupoid as a set $\mathbb{G}$ with a partially defined multiplication map; not all elements can be multiplied, only ones that have matching source and target.

Example 6.5. Given a space X, the pair groupoid of X has as its space of objects X. Between any two objects of the pair groupoid $x, y \in X$ there is exactly one morphism which we will denote by (x, y).

The space $C(\mathbb{G})$ on the pair groupoid of a compact Lie group G acts on $L^2(G, d\rho_t)$ by

$$\pi(h)\Psi(g) = \int_G h(x, y)\Psi(y) d\rho_t(y).$$

Definition 6.6. *Let $C^*(\mathbb{G})$ be the norm-closure of the image $\pi(C(\mathbb{G}))$ in $B(L^2(G))$.*

It does in fact hold that $C^*(\mathbb{G}) \cong K(L^2(G))$ - the compact operators. Confer definition 3 of [1].

6.2 The circle

Let us return to the circle and connections on a principal bundle P over $\mathbb{T}$.

Definition 6.7. *Given a partition Λ of the circle, define $\mathbb{G}^\Lambda$ to be the pair groupoid of G^Λ.*

We can make $C^*(\mathbb{G}^\Lambda)$ into a direct system of C^*-algebras as follows: if Λ_1, Λ_2 are partitions such that Λ_2 is Λ_1 with one single point added, th As always the general map is given by composing.

Because $\{ L^2(G^\Lambda, d\rho_t), u_\Lambda \}$ is a direct system of Hilbert spaces, we also have a direct system of C^*-algebras $K(L^2(G^\Lambda, d\rho_t))$ where the map $v_{j,i} : K\left(L^2(G^{\Lambda_i}, d\rho_t)\right) \to K\left(L^2(G^{\Lambda_j}, d\rho_t)\right)$ is given by $v_{j,i}(a) = uau^*$.

The indexes were suppressed to avoid notational clutter. We know that u - if we add only one point to the partition - is simply the coproduct. We can also express u^* in a more convenient form.

As usual we can limit ourselves to simply splitting a segment in two; consider splitting the circle up into two segments of lengths l_1 and $l_2 = l - l_1$ respectively.

Lemma 6.8. *Suppose we have for now set the parameter to tl instead of t. Suppose $\Lambda = \{ 0, l_1 \}$. Let $\Psi \in L^2(G^\Lambda, d\rho_t)$ and let u denote the coproduct. Then*

$$u^*\Psi(g) = \int_{b\in G} \Psi(b, b^{-1}g)\frac{p_{tl}(b, b^{-1}a)}{p_{tl}(a)p_{tl}(b)}d\rho_{tl}(b).$$

Proof. Let $\Phi \in L^2(G, d\rho_t)$. Let $l_1 = l_2 = l$. We calculate

$$\begin{aligned}
\langle \Phi, u^*\Psi \rangle &= \langle u\Phi, \Psi \rangle \\
&= \int_{G\times G} \overline{\Phi}(gh)\Psi(g,h)p_{tl_1}(g)p_{tl_2}(h)dgdh \\
&= \int_{G\times G} \overline{\Phi}(h)\Psi(g, g^{-1}h)p_{tl}(g, g^{-1}h)dgdh \\
&= \int_{h\in G} \overline{\Phi}(h)\left(\int_{g\in G} \Psi(g, g^{-1}h)p_{tl}(g, g^{-1}h)\frac{1}{p_{tl}(h)}dg\right)p_{tl}(h)dh \\
&= \langle \Phi, \tilde{\Psi} \rangle,
\end{aligned}$$

where $\tilde{\Psi}$ is given by

$$h \mapsto \int_{g\in G} \Psi(g, g^{-1}h)\frac{p_{tl}(g, g^{-1}h)}{p_{tl}(h)p_{tl}(g)}d\rho_{tl}(g),$$

exactly as desired. □

It follows that if Λ_1, Λ_2 are partitions, where $\Lambda_2 = \{ 0, x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_n \}$ - i.e. Λ_1 but with one point added in the i-th segment - then

$$u^*\Psi(g_1, \ldots, g_n) = \int_{b\in G} \Psi(g_1, \ldots, g_{i-1}, b, b^{-1}g_i, \ldots, g_n)\frac{p_{l_it}(b, b^{-1}a)}{p_{tl_i}(a)p_{tl_i}(b)}d\rho_t(b)$$

for any $\Phi \in L^2(G_1^\Lambda, d\rho_t)$, because we can essentially ignore the elements g_j not adjoining the segment.

Definition 6.9. *Let Λ_1 be a partition and let Λ_2 be the partition obtained by splitting the i-th segment in two segments - i.e. by adding a single point. Define the map $T_{2,1} : C(\mathbb{G}_{\Lambda_1}) \to C(\mathbb{G}_{\Lambda_2})$ by putting*

$$\begin{aligned}
&Tf(g_1, \ldots, g_{i-1}, g, g_i, \ldots, g_n, h_1, \ldots, h_{i-1}, h, h_i, \ldots, h_n) \\
&= f(g_1, \ldots, g_{i-1}, gg_i, \ldots, g_n, h_1, \ldots, h_{i-1}, hh_i, \ldots, h_n).
\end{aligned}$$

As usual the map between two general partitions is given by composing.

Theorem 6.10. *The two systems are equivalent. That is to say, if $f \in C(\mathbb{G}_{\Lambda_1})$ then $\pi(T_{2,1}f) = u\pi(f)u^*$.*

Proof. Let us first consider the case where $\Lambda_1 = \{0\}$ and where $\Lambda_2 = \{0, x_i\}$. We calculate for $f \in C(\mathbb{G}_\Lambda)$ and $\Psi \in L^2(G, d\rho_t)$

$$\begin{aligned}(u\pi(f)u^*)\,\Psi(g,h) &= ((\pi(f)u^*)\Psi)(gh)\\ &= \int_G f(b,gh)u^*\Psi(b)d\rho_t(b)\\ &= \int_G\int_G f(b,gh)\Psi(aa^{-1}b)\frac{p_t(a,a^{-1}b)}{p_t(b)p_t(a)}d\rho_t(a)d\rho_t(b)\\ &= \int_G\int_G f(b,gh)\Psi(b)p_t(a,a^{-1}b)dadb\\ &= \int_G\int_G f(ab,gh)\Psi(ab)d\rho_t(a,b)\\ &= (\pi(Tf)\Psi)(g,h).\end{aligned}$$

where of course by T we mean $T_{\Lambda,\{0\}}$. As such, if Λ_2 is Λ_1 with one point added in the i-th segment, then

$$(u\pi(f)u^*)\,\Psi(g_1,\ldots,g_n) = (T_{2,1}f\Psi)$$

because again we can ignore the elements g_j belonging to segments of Λ_1 that are not the i-th segment. □

Hence - perhaps disappointingly - we obtain the compact operators in the limit as well, similar to [1] where it is proposition 21.

Definition 6.11. *Let Λ be a partition of the circle. Define the operator $Delta_\Lambda$ as*

$$\Delta_\Lambda = \sum_{i=1}^{n} l_i 1\otimes\ldots\otimes 1\otimes\Delta_G\otimes 1\otimes\ldots\otimes 1$$

on $L^2(G^\Lambda)$.

This is a self-adjoint operator, because it is the Casimir operator on G^n associated to the *ad*-invariant inner product

$$\langle (X_1,\ldots,X_n),(Y_1,\ldots,Y_n)\rangle = \sum_{i=1}^{n}\langle X_i, Y_i\rangle\frac{1}{l_i}$$

where the X_i and Y_i are elements of $\mathfrak{g}$.

Let us denote by $y : L^2(G^\Lambda) \to L^2(G^\Lambda, d\rho_t)$ the obvious unitary map given on $\Psi \in L^2(G^\Lambda)$ by

$$y(\Psi) = \frac{\Psi}{\sqrt{p_t}}.$$

Again we suppress the indexes whenever possible.

Definition 6.12. *Define the operator Δ_Λ^t on $L^2(G^\Lambda, d\rho_t)$ by*

$$\Delta_\Lambda^t = y\Delta_\Lambda y^*.$$

The sequence of operators Δ_Λ^t is not necessarily a compatible sequence of operators. For this to hold, after all, we would require that $p_t(ab) = p_{l_1 t}(a)p_{l_2 t}(b)$ which does not generally hold. However, because in the limit as $t \to \infty$ both sides of this equation converge to the constant function 1 uniformly we expect the direct system to become compatible with Δ_Λ in this limit.

6.3 The limit $t \to \infty$

Both in [5] and [4] the space $L^2(\mathscr{A}^*, d\mu_t)$ is considered in order to take the limit of $t \to \infty$. In this section we will consider the same limit. In particular we will discuss shortly the fact that if we let t approach infinity, we again obtain a direct system of Hilbert spaces. In fact, it is the exact direct system described in [1] where the manifold is taken to be the circle $\mathbb{T}$ and the restriction of only allowing the splitting of segments and not any other refinement of graphs is still observed. In this limit, the spaces $L^2(V^*, d\rho_t)$ should go to $L^2(V^*)$, with the measure being the standard Lebesgue measure, because p_t (on V^*) converges to 1 - at least pointwise. However, even if we could rigourously establish this convergence, the Hilbert spaces $L^2(V^*)$ no longer form a directed system because the measure in no longer a probability measure and hence the maps between them are no longer isometries. However, if we write $L^2(\mathscr{A}^*, d\mu_t)$ as the direct limit of the spaces $L^2(G^\Lambda, d\rho_t)$ we *can* rigourously take the limit of $t \to \infty$ and it still forms a directed system because the normalized Haar-measure on G^Λ is of course still a probability measure. More precisely formulated, the measures ρ_t on any space G^Λ converge uniformly to the normalized Haar measure as $t \to \infty$. Because the maps between $L^2(G^{\Lambda_1}, d\rho_t) \to L^2(G^{\Lambda_2}, d\rho_t)$ do not depend on t - they are given by composition of the coproduct - and because the measures in this limit are probability measures still - we again obtain a direct system of Hilbert spaces. Furthermore, even though $\Delta^t_{G^\Lambda}$ is not a compatible sequence of operators - as discussed in the previous section - we will see that in the limit $t \to \infty$, we do obtain a compatible system. In [5] an operator is carefully defined on $L^2(\mathscr{A}^*, d\mu_t)$. This operator has the property that if $\Psi \in L^2(G) = \lim_{t\to\infty} L^2(G, d\rho_t)$ lies in the domain of Δ_G, that $\Delta(\Psi) = \Delta_G(\Psi)$. The operator we find in the limit will have the same property. In fact, we will take this property as a starting principal for finding our operator.

The following is a corollary of proposition 2.56.

Proposition 6.13. *In the limit $t \to \infty$, the operators Δ^t_Λ "converge" to Δ_Λ.*

Theorem 6.14. *The family $\{ L^2(G^\Lambda), \Delta_\Lambda \}$ indexed by partitions of the circle is a directed system of Hilbert spaces with compatible operators, in the sense of subsection 2.2.3.*

Proof. Of course we have made the claim that the spaces $L^2(G^\Lambda)$ with maps U_{Λ_2,Λ_1} given by the coproduct is a directed system of Hilbert spaces. Indeed, it is sufficient to show that if Λ_2 is obtained from Λ_1 by adding one point in the i-th segment, the map is unitary. That it respects composition follows from the fact that is does for the $t > 0$ system. However, this is immediate from the properties of the Haar measure. Indeed,

$$\int_{G\times G} \Psi(gh)dgdh = \int_G \int_G \Psi(g)dgdh = \int_G \Psi(g)dg$$

for any integrable Ψ because the measure in the limit is a probability measure.

Furthermore, to see that the operators are compatible, recall that the Casimir element is bi-invariant. Hence, if $\Psi \in \operatorname{dom}(\Delta_G)$ and Φ is such that $\Phi(g) = \Psi(gh)$, then $\Psi \in \operatorname{dom}(\Delta_G)$ and $\Delta_G\Phi(h) = \Delta_G\Psi(gh)$ the same holds for left multiplication. Let $\Psi \in \operatorname{dom}(\Delta_1)$. Then $u\Psi \in \operatorname{dom}(\Delta_2)$. If we assume that we split up the i-th segment in to segments of lengths L_1 and L_2 respectively. Then

$$\Delta_2 u\Psi(g_1, \ldots, h, \ldots, g_n) = \left(\sum_{j\neq i} l_i 1 \otimes \ldots \otimes \Delta_G \otimes \ldots 1 \right.$$
$$\left. + L_1 1 \otimes \ldots \otimes \Delta_G \otimes \ldots \otimes 1 + L_2 1 \otimes \ldots \otimes \Delta_G \otimes \ldots \otimes 1 \right) u\Psi = \Delta_1\Psi$$

using the bi-invariance of the Casimir element and using that $L_1 + L_2 = l_i$. □

The following follows directly from the properties of the Haar measure:

Proposition 6.15. *The action of* $\mathscr{G}^\Lambda = \mathscr{G}/\mathscr{G}_\Lambda$ *is unitary.*

As a direct corollary the action of $\mathscr{G}$ on the direct limit $L^2(\mathscr{A}^*)$ is unitary.

In conclusion, $\lim L^2(G^\Lambda)$ is a possible candidate for defining $L^2(\mathscr{A}^*)$ with respect to a non-existent infinite dimensional Lebesgue measure, with closed (self-adjoint) Hamiltonian Δ_∞. This operator has precisely the property desired that is corresponds to Δ_G on elements of the form $f(A) = F(\theta(A))$ for some $F : G \to \mathbb{C}$ smooth enough to be in the domain of the Laplacian.

Example 6.16. As a concrete example, let us consider briefly as a Lie group $G = U(1)$. We will show that if we choose this as our symmetry group, the spectrum of the operator Δ_∞ in the limit is all of R_+.

Let us consider as a convenient subset of the directed set of partitions of the circle, the partitions $\Lambda_n = \{ \frac{k}{2^n} \mid k = 1, \cdots, 2^n \}$. The spectra of the operators Δ_n are given exactly by

$$\sigma(\Delta_n) = \{ \frac{1}{2^n} \sum_{k=1}^{2^n} m_k^2 \mid m_k \in \mathbb{Z} \} .$$

Because the union of these partitions are also dense in $\mathbb{T}$, we have that

$$\lim_{\Lambda \in F} L^2(G^\Lambda) = \lim_{n \to \infty} L^2(G^{\Lambda_n}).$$

As such, it suffices to show that any positive real number can be written as the limit of numbers of the form $\frac{1}{2^n} \sum_{k=1}^{2^n} m_k^2$.

Let $\lambda \in \mathbb{R}_+$ be arbitrary. Let us consider the binary expansion of λ, so write

$$\lambda = \sum_{j=1}^{\infty} 2^{-j} \lambda_j$$

where $\lambda_j = 0$ or 1. Now, any positive integer can be written as the sum of 4 or more squares. Hence, the integer $\sum_{j=1}^{N} 2^{N-j} \lambda_j$ can be written as the sum of N squares, say

$$\sum_{j=1}^{N} 2^{N-j} \lambda_j = m_1^2 + \cdots + m_N^2.$$

It follows that any partial sum of the binary expansion of λ can be written as an element of the spectrum of some Δ_N. Recall from corollary 2.40 that the spectrum of Δ_∞ is the closure of the unions of all spectra $\sigma(\Delta_N)$. Hence, $\lambda \in \sigma(\Delta_\infty)$ as desired.

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