

Holomorphic Structures on the Quantum Projective Line

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We show that much of the structure of the 2-sphere as a complex curve survives the q -deformation and has natural generalizations to the quantum 2-sphere—which, with additional structures, we identify with the quantum projective line. Notably among these is the identification of a quantum homogeneous coordinate ring with the coordinate ring of the quantum plane. In parallel with the fact that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information for complex structures on the surface, we formulate a notion of twisted positivity for twisted Hochschild and cyclic cocycles and exhibit an explicit twisted positive Hochschild cocycle for the complex structure on the sphere.

1 Introduction

Despite much progress in noncommutative geometry in the past 30 years, noncommutative complex geometry is not developed that much yet. To the best of our knowledge,

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the paper [9] is the first to outline a possible approach to the idea of a complex structure in noncommutative geometry, based on the notion of positive Hochschild cocycle on an involutive algebra. Other contributions include [14], where noncommutative complex structures motivated by supersymmetric quantum field theory were introduced, and [25], where a detailed study of holomorphic structure on noncommutative tori and of holomorphic vector bundles on them is carried out. In [8, Section VI.2] Connes explicitly shows that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. Although the corresponding problem of characterizing holomorphic structures on n -dimensional manifolds via positive Hochschild cocycles is still open, nevertheless, this result suggests positive Hochschild cocycles as a possible framework for holomorphic noncommutative structures. Indeed, this fits very well in the case of noncommutative tori, as complex structures defined in [25] can be shown to define a positive Hochschild 2-cocycle on the noncommutative torus [11, 12].

In the present paper, we study a natural complex structure on the Podleś quantum 2-sphere—which, with additional structure, we identify with the quantum projective line $\mathbb{C}P_q^1$ —that resembles in many aspects the analogous structure on the classical Riemann sphere. We shall concentrate on both algebraic and analytic aspects. While at the algebraic level, the complex structure we are using on the quantum projective line was already present in [19]; we move from this to the analytic level of holomorphic functions and sections. Indeed, it is well known that there are finitely generated projective modules over the quantum sphere that correspond to the canonical line bundles on the Riemann sphere in the $q \rightarrow 1$ limit. In this paper we study a holomorphic structure on these projective modules and give explicit bases for the space of corresponding holomorphic sections. Since these projective modules are in fact bimodules, we can define, in terms of their tensor products, a quantum homogeneous coordinate ring for $\mathbb{C}P_q^1$. We are able to identify this ring with the coordinate ring of the quantum plane.

In Section 2 we define the notion of a *complex structure* on an involutive algebra as a natural and minimal algebraic requirement on structures that ought to be present in any holomorphic structure on a noncommutative space; we also give several examples, some of which already present in the literature. We then define holomorphic structures on modules and bimodules and indicate, in special cases, a tensor product for bimodules. In Section 3 we look at the quantum projective line $\mathbb{C}P_q^1$ and its holomorphic structure defined via a differential calculus. This differential calculus, induced from the canonical left-covariant differential calculus on the quantum “group” $SU_q(2)$,

is the unique left-covariant one on $\mathbb{C}\mathbb{P}_q^1$. In Section 4 we compute explicit bases for the space of holomorphic sections of the canonical line bundles \mathcal{L}_n on $\mathbb{C}\mathbb{P}_q^1$, and we notice that they follow a pattern similar to the classical commutative case. This allows us to compute the quantum homogeneous coordinate ring of $\mathbb{C}\mathbb{P}_q^1$ and to show that it coincides with the coordinate ring of the quantum plane. In Section 5 we look for a possible positive Hochschild cocycle on the quantum sphere. Given that there are no nontrivial 2-dimensional cyclic cocycles on the quantum 2-sphere, we formulate a notion of twisted positivity for twisted Hochschild and cyclic cocycles and show that a natural twisted Hochschild cocycle is positive. The twist here is induced on $\mathbb{C}\mathbb{P}_q^1$ by the modular automorphism of the quantum $SU_q(2)$.

2 Holomorphic Structures in Noncommutative Geometry

We start with a general setup for complex and holomorphic structures in noncommutative geometry. In the next sections, we will apply this to the Podleś sphere seen as a quantum Riemann sphere. We try to introduce a scheme that applies also to other noncommutative holomorphic structures already present in the literature. In particular, we have in mind the holomorphic structure on the noncommutative torus that was introduced in [10] and further explored in [25]. Also, we require compatibility with the definitions in [14] of noncommutative complex and Kähler manifolds that originated from supersymmetric quantum theory.

2.1 Noncommutative complex structures

Suppose \mathcal{A} is an algebra over \mathbb{C} equipped with a *differential $*$ -calculus* $(\Omega^\bullet(\mathcal{A}), d)$. Recall that this is a graded differential $*$ -algebra $\Omega^\bullet(\mathcal{A}) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{A})$ with $\Omega^0(\mathcal{A}) = \mathcal{A}$; thus, $\Omega^\bullet(\mathcal{A})$ has the structure of an \mathcal{A} -bimodule. The differential $d : \Omega^\bullet(\mathcal{A}) \rightarrow \Omega^{\bullet+1}(\mathcal{A})$ satisfying a graded Leibniz rule, $d(\alpha\beta) = (d\alpha)\beta + (-1)^{\text{dg}(\alpha)}\alpha(d\beta)$, and $d^2 = 0$. Also the differential commutes with the $*$ -structure $d(a^*) = d(a)^*$.

Definition 2.1. A complex structure on \mathcal{A} for the differential calculus $(\Omega^\bullet(\mathcal{A}), d)$ is a bigraded differential $*$ -algebra $\Omega^{(\bullet, \bullet)}(\mathcal{A})$ with two differentials $\partial : \Omega^{(p, q)}(\mathcal{A}) \rightarrow \Omega^{(p+1, q)}(\mathcal{A})$ and $\bar{\partial} : \Omega^{(p, q)}(\mathcal{A}) \rightarrow \Omega^{(p, q+1)}(\mathcal{A})$ ($p, q \geq 0$) such that the following hold:

$$\Omega^n(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p, q)}(\mathcal{A}); \quad \partial(a)^* = \bar{\partial}(a^*); \quad \text{and} \quad d = \partial + \bar{\partial}.$$

Also, the involution $*$ maps $\Omega^{(p, q)}(\mathcal{A})$ to $\Omega^{(q, p)}(\mathcal{A})$. □

In the following, we will also abbreviate $(\mathcal{A}, \bar{\partial})$ for a complex structure on \mathcal{A} . Note that the complex $(\Omega^{(0,\bullet)}(\mathcal{A}), \bar{\partial})$ forms a differential calculus as well.

Remark 2.2. For the purpose of the present paper, the above definition suffices. In general one may need to add some kind of integrability condition. We postpone this to future work. These lines of research are also being pursued in [4]. \square

Definition 2.3. Let $(\mathcal{A}, \bar{\partial})$ be an algebra with a complex structure. The algebra of holomorphic elements in \mathcal{A} is defined as

$$\mathcal{O}(\mathcal{A}) := \ker \left\{ \bar{\partial} : \mathcal{A} \rightarrow \Omega^{(0,1)}(\mathcal{A}) \right\}. \quad \square$$

In the following, we will also loosely speak about $\mathcal{O}(\mathcal{A})$ as holomorphic functions. Note that by the Leibniz rule, this is indeed an algebra over \mathbb{C} .

Example 2.4. The motivating example is the de Rham complex (with complex coefficients) on a complex manifold. It is of course a complex structure in the above sense for the complexified de Rham differential calculus. \square

Example 2.5. Let L be a *real* Lie algebra with a *complex structure*

$$L^{\mathbb{C}} =: L_0 \oplus \overline{L_0}$$

Given $L \rightarrow \text{Der}(\mathcal{A}, \mathcal{A})$, an action of L by $*$ -derivations on an involutive algebra \mathcal{A} , then the Chevalley–Eilenberg complex

$$\Omega^{\bullet} \mathcal{A} := \text{Hom}_{\mathbb{C}}(\Lambda^{\bullet} L^{\mathbb{C}}, \mathcal{A}),$$

for the Lie algebra cohomology of $L^{\mathbb{C}}$ with coefficients in \mathcal{A} , is a differential calculus for \mathcal{A} . A complex structure on \mathcal{A} for this differential calculus is defined by setting

$$\Omega^{(p,q)} \mathcal{A} := \text{Hom}_{\mathbb{C}}(\Lambda^p L_0 \otimes \Lambda^q \overline{L_0}, \mathcal{A})$$

as the space of (p, q) -forms. \square

Example 2.6. The noncommutative torus \mathcal{A}_θ is defined as the involutive algebra generated by two unitaries U_1 and U_2 satisfying $U_1 U_2 = e^{2\pi i \theta} U_2 U_1$, for a fixed $\theta \in \mathbb{R}$. The two basic derivations on this torus are given by $\delta_j(U_k) = 2\pi i \delta_{jk} U_k$, for $j, k = 1, 2$, and generate an action of the abelian Lie algebra \mathbb{R}^2 on \mathcal{A}_θ . Any $\tau \in \mathbb{C} \setminus \mathbb{R}$ defines a complex structure on the Lie algebra \mathbb{R}^2 :

$$\mathbb{R}^2 \otimes \mathbb{C} = L_0 \oplus \overline{L_0},$$

where $L_0 := e_1 + \tau e_2$ with (e_1, e_2) the standard basis of \mathbb{R}^2 . The construction of Example 2.5 then yields a complex structure on the noncommutative torus which was already present in [10]. The only holomorphic functions are the constants.

Now, the conformal class of a general constant metric in two dimensions is parametrized by a complex number $\tau \in \mathbb{C}$, $\text{Im } \tau > 0$. Up to a conformal factor, the metric is given by

$$g = (g_{ij}) = \begin{pmatrix} 1 & \text{Re } \tau \\ \text{Re } \tau & |\tau|^2 \end{pmatrix}. \quad (2.1)$$

The complex structure on \mathcal{A}_θ is then given by

$$\partial_{(\tau)} = \frac{1}{(\tau - \bar{\tau})} (-\bar{\tau} \delta_1 + \delta_2), \quad \bar{\partial}_{(\tau)} = \frac{1}{(\tau - \bar{\tau})} (\tau \delta_1 - \delta_2). \quad (2.2)$$

The cyclic 2-cocycle which “integrates” 2-forms on the noncommutative torus is

$$\Psi(a_0, a_1, a_2) = \frac{i}{2\pi} \text{tr}_\theta (a_0 (\delta_1 a_1 \delta_2 a_2 - \delta_2 a_1 \delta_1 a_2)), \quad (2.3)$$

where tr_θ indicates the unique invariant normalized faithful trace on \mathcal{A}_θ . Its normalization ensures that for any hermitian idempotent $p \in \mathcal{A}_\theta$, the quantity $\Psi(p, p, p)$ is an integer: it is indeed the index of a Fredholm operator. By working with the metric (2.1), the positive Hochschild cocycle Φ associated with the cyclic one (2.3) is given by

$$\Phi(a_0, a_1, a_2) = \frac{2}{\pi} \text{tr}_\theta (a_0 \partial_{(\tau)} a_1 \bar{\partial}_{(\tau)} a_2). \quad (2.4)$$

The cocycle (2.4) seen as the conformal class of a general constant metric on the torus is in [8, Section VI.2]. We return to this later on in Section 5. \square

Example 2.7. The hermitian spectral data introduced in [14] in a noncommutative geometry approach to supersymmetric field theories are examples of a noncommutative complex structure. We refer in particular to Corollary 2.34 there for more details. \square

2.2 Holomorphic connections on modules

We will now lift the complex structures described in the previous section to noncommutative vector bundles. This requires the introduction of connections on (left or right) \mathcal{A} -modules. We will work with left module structures, although this choice is completely irrelevant. Recall that a *connection* on a left \mathcal{A} -module \mathcal{E} for the differential calculus $(\Omega^\bullet(\mathcal{A}), d)$ is a linear map $\nabla : \mathcal{E} \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ satisfying the (left) Leibniz rule

$$\nabla(a\eta) = a\nabla(\eta) + da \otimes_{\mathcal{A}} \eta, \quad \text{for } a \in \mathcal{A}, \eta \in \mathcal{E}.$$

We can extend the connection to a map $\nabla : \Omega^p(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \Omega^{p+1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ via the graded (left) Leibniz rule

$$\nabla(\omega\rho) = (-1)^p \omega \nabla(\rho) + (d\omega)\rho, \quad \text{for } \omega \in \Omega^p(\mathcal{A}), \rho \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}. \quad (2.5)$$

The *curvature* of the connection is defined as $F(\nabla) = \nabla^2$; one shows that it is left \mathcal{A} -linear, that is, it is an element in $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E})$.

Definition 2.8. Let $(\mathcal{A}, \bar{\partial})$ be an algebra with a complex structure. A holomorphic structure on a left \mathcal{A} -module \mathcal{E} with respect to $(\mathcal{A}, \bar{\partial})$ is a flat $\bar{\partial}$ -connection, that is, a linear map $\bar{\nabla} : \mathcal{E} \rightarrow \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ satisfying

$$\bar{\nabla}(a\eta) = a\bar{\nabla}(\eta) + \bar{\partial}a \otimes_{\mathcal{A}} \eta, \quad \text{for } a \in \mathcal{A}, \eta \in \mathcal{E}. \quad (2.6)$$

and such that $F(\bar{\nabla}) = \bar{\nabla}^2 = 0$.

If in addition \mathcal{E} is a finitely generated projective \mathcal{A} -module, we shall call the pair $(\mathcal{E}, \bar{\nabla})$ a holomorphic vector bundle. \square

The last part of the definition is motivated by the classical case: a vector bundle on a complex manifold is holomorphic if and only if it admits a flat $\bar{\partial}$ -connection. Indeed, there is a one-to-one correspondence between holomorphic structures on vector bundles over a complex manifold and (equivalence classes of) flat $\bar{\partial}$ -connections. The equivalence

is, with respect to gauge transformation, a concept which can be generalized to the noncommutative setup as well.

Definition 2.9. Two holomorphic structures $\bar{\nabla}_1$ and $\bar{\nabla}_2$ on an \mathcal{A} -module \mathcal{E} are gauge equivalent if there exists an invertible element $g \in \text{End}_{\mathcal{A}}(\mathcal{E})$ such that $\bar{\nabla}_2 = g^{-1} \circ \bar{\nabla}_1 \circ g$. \square

More generally, it follows from the Leibniz rule (2.6) that the difference of any two connections is \mathcal{A} -linear, that is, for $\bar{\nabla}_1$ and $\bar{\nabla}_2$ holomorphic structures on \mathcal{E} , it holds that

$$\nabla_1 - \nabla_2 \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}). \quad (2.7)$$

In particular, for finitely generated projective \mathcal{A} -modules, a $\bar{\nabla}$ -connection is given by an element $A \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^{(0,1)} \otimes_{\mathcal{A}} \mathcal{E})$. Indeed, such a module \mathcal{E} is a direct summand of a free \mathcal{A} -module and so inherits a connection ∇_0 from the trivial connection $\bar{\partial}$ on the free module (acting diagonally, after identifying the free module with N copies of \mathcal{A}). By Equation (2.7), any other connection on \mathcal{E} is then given by $\bar{\nabla}_0 + A$ with A as said. We call A the *connection (0,1)-form*.

Since $\bar{\nabla}$ is a flat connection, there is the following complex of vector spaces

$$0 \rightarrow \mathcal{E} \xrightarrow{\bar{\nabla}} \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\bar{\nabla}} \Omega^{(0,2)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\bar{\nabla}} \dots \quad (2.8)$$

where $\bar{\nabla}$ is extended to $\Omega^{(0,q)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ by a Leibniz rule similar to Equation (2.6).

Definition 2.10. The j -th cohomology group of the complex (2.8) is denoted by $H^j(\mathcal{E}, \bar{\nabla})$. In particular, the zeroth cohomology group $H^0(\mathcal{E}, \bar{\nabla})$ is called the space of holomorphic sections of \mathcal{E} . \square

As a consequence of the Leibniz rule (2.6), the space $H^0(\mathcal{E}, \bar{\nabla})$ is a left $\mathcal{O}(\mathcal{A})$ -module.

2.3 Holomorphic structures on bimodules and their tensor products

In the previous section, we focused on connections on modules carrying a left algebra action. Now, we would like to define tensor products of connections on bimodules, and for that we need some compatibility with the right action of the algebra. The following

definition was proposed in [22] for linear connections and [13] for the general case. See also [1, Section II.2].

Definition 2.11. A bimodule connection on an \mathcal{A} -bimodule \mathcal{E} for the calculus $(\Omega^\bullet(\mathcal{A}), d)$ is given by a connection $\nabla : \mathcal{E} \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ (for the left module structure) for which there is a bimodule isomorphism

$$\sigma(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E},$$

such that the following twisted right Leibniz rule holds

$$\nabla(\eta a) = \nabla(\eta)a + \sigma(\nabla)(\eta \otimes da), \quad \text{for } \eta \in \mathcal{E}, a \in \mathcal{A}. \quad (2.9)$$

In particular, this definition applies to the differential calculus $(\Omega^{(0,\bullet)}(\mathcal{A}), \bar{d})$, thus giving a notion of *holomorphic structure on bimodules*.

Next, suppose we are given two \mathcal{A} -bimodules \mathcal{E}_1 and \mathcal{E}_2 with two bimodule connections ∇_1, ∇_2 , respectively. Denote the corresponding bimodule isomorphisms by σ_1 and σ_2 . The following result establishes their *tensor product connection*.

Proposition 2.12. The map $\nabla : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \mapsto \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ defined by

$$\nabla := \nabla_1 \otimes 1 + (\sigma_1 \otimes \text{id})(1 \otimes \nabla_2)$$

yields a σ -compatible connection on the \mathcal{A} -bimodule $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$, with bimodule isomorphism $\sigma : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$, defined as the composition $(\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2)$. \square

Proof. Clearly, this map satisfies the left Leibniz rule, since ∇_1 does. Well-definedness of the map on $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ follows by an application of the twisted Leibniz rule (2.9) for ∇_1 and of the left Leibniz rule for ∇_2 . Also, σ -compatibility follows from the twisted Leibniz rule (2.9) and the fact that the σ_i are bimodule maps. \blacksquare

It would be desirable that the flatness condition on holomorphic structures survives upon taking the tensor product. Unfortunately, this is not the case for the above tensor product, as shown by the following example.

Example 2.13. Let $\mathcal{E} = \mathcal{A} \otimes V$ and $\mathcal{F} = \mathcal{A} \otimes W$ be two free \mathcal{A} -bimodules, with V and W vector spaces. Since a connection on a free module is determined by its action on basis vectors e_i of V and f_j of W , respectively, we define connections $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{F}}$ on \mathcal{E} and \mathcal{F} by two matrices of 1-forms A_{ik} and B_{jl} , respectively. The tensor product connection is then determined by

$$\nabla(e_i \otimes f_j) = \sum_k A_{ik} \otimes e_k \otimes f_j + \sum_l B_{jl} \otimes e_i \otimes f_l.$$

One then computes that its curvature is

$$\begin{aligned} \nabla^2(e_i \otimes f_j) &= \sum_k F(\nabla_{\mathcal{E}})_{ik} \otimes (e_k \otimes f_j) + \sum_l F(\nabla_{\mathcal{F}})_{jl} \otimes e_i \otimes f_l \\ &\quad - \sum_{kl} (A_{ik} B_{jl} + B_{jl} A_{ik}) \otimes (e_k \otimes f_l) \end{aligned}$$

with $F(\nabla_{\mathcal{E}})_{ik} = dA_{ik} + \sum_r A_{ir} A_{rk}$ and similarly for the curvature of $\nabla_{\mathcal{F}}$. Thus, the curvature of the tensor product connection is not the sum of the curvatures of the two connections, unless of course the differential calculus is (graded) commutative. Indeed, in the latter case, the term $A_{ik} B_{jl} + B_{jl} A_{ik}$ vanishes. \square

Possibly, a modification of the tensor product could overcome the problem. For the complex 1-dimensional case of interest for this paper, we can ignore the problem since then flatness of any holomorphic connection is automatic.

3 The Hopf Bundle on the Quantum Projective Line

The most natural way to define the quantum projective line $\mathbb{C}P_q^1$ is as a quotient of the sphere S_q^3 for an action of $U(1)$. It is the standard Podleś sphere S_q^2 with additional structure, and the construction we need is the well-known quantum principal $U(1)$ -bundle over the standard Podleś sphere S_q^2 and whose total space is the manifold of the quantum group $SU_q(2)$. This bundle is an example of a quantum homogeneous space [6]. In the following, without loss of generality, we will assume that $0 < q < 1$. We shall also use the “ q -number”

$$[s] = [s]_q := \frac{q^{-s} - q^s}{q^{-1} - q}, \quad (3.1)$$

defined for $q \neq 1$ and any $s \in \mathbb{R}$.

3.1 The algebras of S_q^3 and CP_q^1

The manifold of S_q^3 is identified with the manifold of the quantum group $SU_q(2)$. Its coordinate algebra $\mathcal{A}(SU_q(2))$ is the $*$ -algebra generated by elements a and c , with relations

$$\begin{aligned} ac &= qca, & ac^* &= qc^*a, & cc^* &= c^*c, \\ a^*a + c^*c &= aa^* + q^2cc^* = 1. \end{aligned} \quad (3.2)$$

These are equivalent to requiring that the “defining” matrix

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix},$$

is unitary: $UU^* = U^*U = 1$. The Hopf algebra structure for $\mathcal{A}(SU_q(2))$ is given by coproduct, antipode, and counit:

$$\Delta U = U \otimes U, \quad S(U) = U, \quad \epsilon(U) = 1.$$

The quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is the Hopf $*$ -algebra generated as an algebra by four elements K, K^{-1}, E , and F with $KK^{-1} = 1$ and subject to relations

$$K^\pm E = q^\pm EK^\pm, \quad K^\pm F = q^\mp FK^\pm, \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}. \quad (3.3)$$

The $*$ -structure is simply

$$K^* = K, \quad E^* = F, \quad F^* = E,$$

and the Hopf algebra structure is provided by coproduct Δ , antipode S , and counit ϵ :

$$\Delta(K^\pm) = K^\pm \otimes K^\pm, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F,$$

$$S(K) = K^{-1}, \quad S(E) = -qE, \quad S(F) = -q^{-1}F,$$

$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0.$$

There is a bilinear pairing between $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}(\mathrm{SU}_q(2))$, given on generators by

$$\begin{aligned} \langle K, a \rangle &= q^{-1/2}, & \langle K^{-1}, a \rangle &= q^{1/2}, & \langle K, a^* \rangle &= q^{1/2}, & \langle K^{-1}, a^* \rangle &= q^{-1/2}, \\ \langle E, c \rangle &= 1, & \langle F, c^* \rangle &= -q^{-1}, \end{aligned}$$

and all other couples of generators pairing to 0. One regards $\mathcal{U}_q(\mathfrak{su}(2))$ as a subspace of the linear dual of $\mathcal{A}(\mathrm{SU}_q(2))$ via this pairing. Then there are [28] canonical left and right $\mathcal{U}_q(\mathfrak{su}(2))$ -module algebra structures on $\mathcal{A}(\mathrm{SU}_q(2))$ such that

$$\langle g, h \triangleright x \rangle := \langle gh, x \rangle, \quad \langle g, x \triangleleft h \rangle := \langle hg, x \rangle, \quad \forall g, h \in \mathcal{U}_q(\mathfrak{su}(2)), x \in \mathcal{A}(\mathrm{SU}_q(2)).$$

They are given by $h \triangleright x := \langle \mathrm{id} \otimes h, \Delta x \rangle$ and $x \triangleleft h := \langle (h \otimes \mathrm{id}), \Delta x \rangle$, or equivalently,

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle, \quad x \triangleleft h := \langle h, x_{(1)} \rangle x_{(2)},$$

in the Sweedler notation, $\Delta(x) = x_{(1)} \otimes x_{(2)}$, for the coproduct. These right and left actions are mutually commuting:

$$(h \triangleright a) \triangleleft g = (a_{(1)} \langle h, a_{(2)} \rangle) \triangleleft g = \langle g, a_{(1)} \rangle a_{(2)} \langle h, a_{(3)} \rangle = h \triangleright (\langle g, a_{(1)} \rangle a_{(2)}) = h \triangleright (a \triangleleft g),$$

and since the pairing satisfies

$$\langle (Sh)^*, x \rangle = \overline{\langle h, x^* \rangle}, \quad \forall h \in \mathcal{U}_q(\mathfrak{su}(2)), x \in \mathcal{A}(\mathrm{SU}_q(2)),$$

the $*$ -structure is compatible with both actions:

$$h \triangleright x^* = ((Sh)^* \triangleright x)^*, \quad x^* \triangleleft h = (x \triangleleft (Sh)^*)^*, \quad \forall h \in \mathcal{U}_q(\mathfrak{su}(2)), x \in \mathcal{A}(\mathrm{SU}_q(2)).$$

We list here the left action on powers of generators. For $s = 0, 1, \dots$, one finds

$$\begin{aligned} K^\pm \triangleright a^s &= q^{\mp \frac{s}{2}} a^s, & K^\pm \triangleright a^{*s} &= q^{\pm \frac{s}{2}} a^{*s}, & K^\pm \triangleright c^s &= q^{\mp \frac{s}{2}} c^s, & K^\pm \triangleright c^{*s} &= q^{\pm \frac{s}{2}} c^{*s}; \\ F \triangleright a^s &= 0, & F \triangleright a^{*s} &= q^{(1-s)/2} [s] c a^{*s-1}, & F \triangleright c^s &= 0, & F \triangleright c^{*s} &= -q^{-(1+s)/2} [s] a c^{*s-1}; \\ E \triangleright a^s &= -q^{(3-s)/2} [s] a^{s-1} c^*, & E \triangleright a^{*s} &= 0, & E \triangleright c^s &= q^{(1-s)/2} [s] c^{s-1} a^*, & E \triangleright c^{*s} &= 0. \end{aligned} \quad (3.4)$$

The principal bundle structure comes from an additional (right) action of the group $U(1)$ on $SU_q(2)$, given via a map $\alpha : U(1) \rightarrow \text{Aut}(\mathcal{A}(SU_q(2)))$, explicit on generators by

$$\alpha_u \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \text{for } u \in U(1). \quad (3.5)$$

The invariant elements for this action form a subalgebra of $\mathcal{A}(SU_q(2))$ which is by definition the coordinate algebra $\mathcal{A}(S_q^2)$ of the standard Podleś sphere S_q^2 of [23]. The inclusion $\mathcal{A}(S_q^2) \hookrightarrow \mathcal{A}(SU_q(2))$ is a quantum principal bundle [6] with classical structure group $U(1)$. Moreover, the sphere S_q^2 (or the projective line $\mathbb{C}P_q^1$ then) is a quantum homogeneous space of $SU_q(2)$, and the (left) coaction of $\mathcal{A}(SU_q(2))$ on to itself restricts a left coaction $\Delta_L : \mathcal{A}(S_q^2) \rightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2)$; dually, it survives a right action of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(S_q^2)$ as twisted derivations.

As a set of generators for $\mathcal{A}(S_q^2)$, we may take

$$B_- := ac^*, \quad B_+ := ca^*, \quad B_0 := cc^*, \quad (3.6)$$

for which one finds relations

$$\begin{aligned} B_- B_0 &= q^2 B_0 B_-, \\ B_- B_+ &= q^2 B_0 (1 - q^2 B_0), \quad B_+ B_- = B_0 (1 - B_0), \end{aligned}$$

and $*$ -structure $(B_0)^* = B_0$ and $(B_\pm)^* = B_\mp$.

Later on in Section 3.4, we shall describe a natural complex structure on the quantum 2-sphere S_q^2 for the unique 2-dimensional covariant calculus on it. This will transform the sphere S_q^2 into a quantum Riemannian sphere or quantum projective line $\mathbb{C}P_q^1$. Having this in mind, with a slight abuse of “language,” from now on we will speak of $\mathbb{C}P_q^1$ rather than S_q^2 .

3.2 The C^* -algebras $C(SU_q(2))$ and $C(\mathbb{C}P_q^1)$

We recall [17] that the algebra $\mathcal{A}(SU_q(2))$ has a vector space basis consisting of matrix elements of its irreducible corepresentations, $\{t_{mn}^l : 2l \in \mathbb{N}, m, n = -l, \dots, l-1, l\}$; in particular,

$$t_{00}^0 = 1, \quad t_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} = a, \quad t_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} = c. \quad (3.7)$$

The coproduct has the matricial form $\Delta t_{mn}^l = \sum_k t_{mk}^l \otimes t_{kn}^l$ while the product is

$$t_{rs}^j t_{mn}^l = \sum_{k=|j-l|}^{j+l} C_q \begin{pmatrix} j & l & k \\ r & m & r+m \end{pmatrix} C_q \begin{pmatrix} j & l & k \\ s & n & s+n \end{pmatrix} t_{r+m, s+n}^k, \quad (3.8)$$

where the $C_q(-)$ factors are q -Clebsch–Gordan coefficients [5, 16].

We denote by $C(\mathrm{SU}_q(2))$ the C^* -completion of $\mathcal{A}(\mathrm{SU}_q(2))$; it is the universal C^* -algebra generated by a, a^*, c, c^* subject to relations (3.2). One of the main features of this compact quantum group is the existence of a unique (left-)invariant Haar state [28] that we shall denote by h . This state is faithful and is determined by setting $h(1) := 1$ and $h(t_{mn}^l) := 0$ if $l > 0$. Let $L^2(\mathrm{SU}_q(2)) := L^2(\mathrm{SU}_q(2), h)$ be the Hilbert space of its Gelfand–Naimark–Segal (GNS) representation π ; then the GNS map $\eta : C(\mathrm{SU}_q(2)) \rightarrow L^2(\mathrm{SU}_q(2))$ is injective and satisfies

$$\|\eta(t_{mn}^l)\|_0^2 := h((t_{mn}^l)^* t_{mn}^l) = \frac{q^{-2m}}{[2l+1]}, \quad (3.9)$$

and the vectors $\eta(t_{mn}^l)$ are mutually orthogonal. From the formula

$$C_q \begin{pmatrix} l & l & 0 \\ -m & m & 0 \end{pmatrix} = (-1)^{l+m} \frac{q^{-m}}{[2l+1]^{\frac{1}{2}}},$$

we see that the involution in $C(\mathrm{SU}_q(2))$ is given by

$$(t_{mn}^l)^* = (-1)^{2l+m+n} q^{n-m} t_{-m, -n}^l. \quad (3.10)$$

In particular, $t_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} = -qc^*$ and $t_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} = a^*$, as expected.

An orthonormal basis of $L^2(\mathrm{SU}_q(2))$ is obtained by normalizing the matrix elements

$$|lmn\rangle := q^m [2l+1]^{\frac{1}{2}} \eta(t_{mn}^l), \quad (3.11)$$

for $2l \in \mathbb{N}$ and $m, n = -l, \dots, l-1, l$.

Recall [17] that the irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$ are labeled by non-negative half-integers l ; the corresponding representation spaces V_l are of dimension $2l+1$. By construction, the Hilbert space $L^2(\mathrm{SU}_q(2))$ is (the completion of) $\bigoplus_l V_l \otimes V_l$.

This gives two commuting representations of $\mathcal{U}_q(\mathfrak{su}(2))$ on $L^2(\mathrm{SU}_q(2))$ of which we need only one:

$$\begin{aligned}\sigma(K) |lmn\rangle &= q^n |lmn\rangle, \\ \sigma(E) |lmn\rangle &= \sqrt{[l-n][l+n+1]} |lm, n+1\rangle, \\ \sigma(F) |lmn\rangle &= \sqrt{[l-n+1][l+n]} |lm, n-1\rangle.\end{aligned}\tag{3.12}$$

The second one would move the label m while not changing n . The representation is such that the left regular representation π of $C(\mathrm{SU}_q(2))$ is equivariant with respect to the left $\mathcal{U}_q(\mathfrak{su}(2))$ -action

$$\sigma(g)\pi(f) = \pi(g \triangleright f), \quad \text{for } g \in \mathcal{U}_q(\mathfrak{su}(2)), f \in C(\mathrm{SU}_q(2)).$$

There is a similar statement with respect to the right action of $\mathcal{U}_q(\mathfrak{su}(2))$ on $C(\mathrm{SU}_q(2))$ for which the representation of the former involves the spin index m rather than n , that is, in their action on $L^2(\mathrm{SU}_q(2))$. As before on the polynomial level, the action of the element $K \in \mathcal{U}_q(\mathfrak{su}(2))$ on $C(\mathrm{SU}_q(2))$ is closely related to a right $U(1)$ -action. Indeed, the formula (3.5) extends to an action of $U(1)$ on $C(\mathrm{SU}_q(2))$ by automorphisms. The invariant subalgebra in $C(\mathrm{SU}_q(2))$ under this $U(1)$ -action is by definition the C^* -algebra of the Podleś sphere or, in our notation, $C(\mathbb{CP}_q^1) = C(\mathrm{SU}_q(2))^{U(1)}$.

The corresponding action of $U(1)$ on the Hilbert space $L^2(\mathrm{SU}_q(2))$ reads (compare with the first equation in (3.12))

$$\rho(u) |lmn\rangle = u^n |lmn\rangle \quad \text{for } u \in U(1),\tag{3.13}$$

making the GNS representation $U(1)$ -equivariant:

$$\rho(u)\pi(f) = \pi(\alpha_u(f)).$$

Moreover, $U(1)$ -invariance of the Haar state implies that the restriction of the GNS map $C(\mathbb{CP}_q^1) \rightarrow L^2(\mathrm{SU}_q(2))^{U(1)}$ is injective; we will accordingly denote this Hilbert space by $L^2(\mathbb{CP}_q^1) := L^2(\mathrm{SU}_q(2))^{U(1)}$. More explicitly, we derive from (3.13) that

$$L^2(\mathbb{CP}_q^1) = \mathrm{Span}\{|lm0\rangle : l \in \mathbb{N}, m = -l, -l+1, \dots, l\}^{\mathrm{clos}}.$$

3.3 The line bundles over $\mathbb{C}P_q^1$

The right action of the group $U(1)$ on the algebra $\mathcal{A}(SU_q(2))$ allows one to give a vector space decomposition $\mathcal{A}(SU_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$, where

$$\mathcal{L}_n := \{x \in \mathcal{A}(SU_q(2)) : \alpha_u(x) = u^{n/2}x, \quad \forall u \in U(1)\}. \quad (3.14)$$

Equivalently [20, Eq. (1.10)], these could be defined via the action of $K \in \mathcal{U}_q(\mathfrak{su}(2))$. Indeed, if H denotes the infinitesimal generator of the action α , the group-like element K can be written as $K = q^{-H}$. In particular, $\mathcal{L}_0 = \mathcal{A}(\mathbb{C}P_q^1)$. Also, $\mathcal{L}_n^* = \mathcal{L}_{-n}$ and $\mathcal{L}_n \mathcal{L}_m = \mathcal{L}_{n+m}$. Each \mathcal{L}_n is clearly a bimodule over $\mathcal{A}(\mathbb{C}P_q^1)$. It was shown in [27, Prop. 6.4] that each \mathcal{L}_n is isomorphic to a projective left $\mathcal{A}(\mathbb{C}P_q^1)$ -module of rank 1 and winding number $-n$. We have indeed the following proposition.

Proposition 3.1. The natural map $\mathcal{L}_n \otimes \mathcal{L}_m \rightarrow \mathcal{L}_{n+m}$ defined by multiplication induces an isomorphism of $\mathcal{A}(\mathbb{C}P_q^1)$ -bimodules

$$\mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \mathcal{L}_m \simeq \mathcal{L}_{n+m}. \quad \square$$

Proof. This follows from the representation theory of $U(1)$ and the relations

$$\begin{aligned} a \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} c - qc \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} a &= 0, \\ a \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} c^* - qc^* \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} a &= 0, \\ c \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} c^* - c^* \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} c &= 0, \end{aligned}$$

and so on, that can be easily established. ■

Using the explicit action of K in the first line of (3.4), a generating set for the \mathcal{L}_n 's as $\mathcal{A}(\mathbb{C}P_q^1)$ -modules is found to be given by elements

$$\begin{aligned} \{c^{*\mu} a^{*n-\mu}, \mu = 0, \dots, n\} & \quad \text{for } n \geq 0, \\ \{a^\mu c^{|\mathfrak{n}|-\mu}, \mu = 0, \dots, |\mathfrak{n}|\} & \quad \text{for } n \leq 0, \end{aligned} \quad (3.15)$$

and any $\phi \in \mathcal{L}_n$ is written as

$$\phi_f = \begin{cases} \sum_{\mu=0}^n f_\mu c^{*\mu} a^{*n-\mu} = \sum_{\mu=0}^n c^{*\mu} a^{*n-\mu} \tilde{f}_\mu & \text{for } n \geq 0, \\ \sum_{\mu=0}^{|n|} f_\mu a^\mu c^{|n|-\mu} = \sum_{\mu=0}^{|n|} a^\mu c^{|n|-\mu} \tilde{f}_\mu & \text{for } n \leq 0, \end{cases}$$

with f_μ and \tilde{f}_μ generic elements in $\mathcal{A}(\mathbb{CP}_q^1)$. It is worth stressing that the elements in (3.15) are not independent over $\mathcal{A}(\mathbb{CP}_q^1)$ since the \mathcal{L}_n are not free modules.

From the definition of the \mathcal{L}_n 's in (3.14) and the relations (3.3) of $\mathcal{U}_q(\mathfrak{su}(2))$, one gets that

$$E \triangleright \mathcal{L}_n \subset \mathcal{L}_{n+2}, \quad F \triangleright \mathcal{L}_n \subset \mathcal{L}_{n-2}. \tag{3.16}$$

On the other hand, commutativity of the left and right actions of $\mathcal{U}_q(\mathfrak{su}(2))$ yields that

$$\mathcal{L}_n \triangleleft g \subset \mathcal{L}_n, \quad \forall g \in \mathcal{U}_q(\mathfrak{su}(2)).$$

A Poincaré–Birkhoff–Witt (PBW) basis for $\mathcal{A}(\text{SU}_q(2))$ is given by monomials $\{a^m c^k c^{*l}\}$ for $k, l = 0, 1, \dots$, and $m \in \mathbb{Z}$ with the convention that a^{-m} is a short-hand notation for a^{*m} ($m > 0$). Also, a similar basis for \mathcal{L}_n is given by the monomials $a^{l-k} c^k c^{*l+n}$; indeed, from (3.4) it follows that $K \triangleright (a^m c^k c^{*l}) = q^{(-m-k+l)/2} a^m c^k c^{*l}$; then the requirement that $-m - k + l = n$ is met by renaming $l \rightarrow l + n$ forcing in turn $m = l - k$. In particular, the monomials $a^{l-k} c^k c^{*l}$ are the only K -invariant elements, thus providing a PBW basis for $\mathcal{L}_0 = \mathcal{A}(\mathbb{CP}_q^1)$.

At the C^* -algebraic level, we can use the $U(1)$ -action to decompose the $C(\text{SU}_q(2))$ in $C(\mathbb{CP}_q^1)$ -modules. Thus, we consider the $C(\mathbb{CP}_q^1)$ -modules

$$\Gamma(\mathcal{L}_n) := \{f \in C(\text{SU}_q(2)) : \alpha_u(f) = u^{n/2} f, \quad \forall u \in U(1)\},$$

as spaces of continuous sections on the line bundles \mathcal{L}_n . Accordingly, the space of L^2 -sections are defined as

$$\begin{aligned} L^2(\mathcal{L}_n) &:= \{\psi \in L^2(\text{SU}_q(2)) : \rho(u)\psi = u^{n/2}\psi, \quad \forall u \in U(1)\} \\ &= \text{Span}\{|l, m, n/2\rangle : l = |n|, |n| + 1, \dots; m = -l, \dots, l\}^{d_{\text{os}}}, \end{aligned}$$

the second line following at once from (3.13).

3.4 The calculi on the principal bundle

The principal $U(1)$ -bundle $\mathcal{A}(\mathbb{C}P_q^1) \hookrightarrow \mathcal{A}(SU_q(2))$ is endowed [6, 7] with compatible nonuniversal calculi obtained from the 3-dimensional left-covariant calculus [28] on $SU_q(2)$ we describe first. We then give the unique left-covariant 2-dimensional calculus [24] on the projective line $\mathbb{C}P_q^1$ obtained by restriction.

The differential calculus we take on the quantum group $SU_q(2)$ is the 3-dimensional left-covariant one already developed in [28]. Its quantum tangent space $\mathcal{X}(SU_q(2))$ is generated by the three elements:

$$X_z = \frac{1 - K^4}{1 - q^{-2}} = (X_z)^*, \quad X_- = q^{-1/2}FK, \quad X_+ = q^{1/2}EK = (X_-)^*, \quad (3.17)$$

whose coproducts are easily found:

$$\Delta X_z = 1 \otimes X_z + X_z \otimes K^4, \quad \Delta X_{\pm} = 1 \otimes X_{\pm} + X_{\pm} \otimes K^2.$$

From these, one also infers that

$$S(X_z) = -X_z K^{-4}, \quad S(X_{\pm}) = -X_{\pm} K^{-2}.$$

The dual space of 1-forms $\Omega^1(SU_q(2))$ has a basis

$$\omega_z = a^* da + c^* dc, \quad \omega_- = c^* da^* - qa^* dc^*, \quad \omega_+ = adc - qcda, \quad (3.18)$$

of left-invariant forms, that is,

$$\Delta_L^{(1)}(\omega_s) = 1 \otimes \omega_s, \quad s = z, \pm, \quad (3.19)$$

with $\Delta_L^{(1)}$ the (left) coaction of $\mathcal{A}(SU_q(2))$ on to itself extended forms. The differential $d : \mathcal{A}(SU_q(2)) \rightarrow \Omega^1(SU_q(2))$ is then given by

$$df = (X_+ \triangleright f) \omega_+ + (X_- \triangleright f) \omega_- + (X_z \triangleright f) \omega_z, \quad (3.20)$$

for any $f \in \mathcal{A}(\mathrm{SU}_q(2))$. By taking the conjugate of (3.18), the requirement of a differential $*$ -calculus, $d f^* = (d f)^*$ (see Section 2.1), yields $\omega_-^* = -\omega_+$ and $\omega_z^* = -\omega_z$. The bimodule structure is:

$$\begin{aligned} \omega_z a &= q^{-2} a \omega_z, & \omega_z a^* &= q^2 a^* \omega_z, & \omega_\pm a &= q^{-1} a \omega_\pm, & \omega_\pm a^* &= q a^* \omega_\pm \\ \omega_z c &= q^{-2} c \omega_z, & \omega_z c^* &= q^2 c^* \omega_z, & \omega_\pm c &= q^{-1} c \omega_\pm, & \omega_\pm c^* &= q c^* \omega_\pm. \end{aligned} \quad (3.21)$$

Higher dimensional forms can be defined in a natural way by requiring compatibility for commutation relations and that $d^2 = 0$. One has:

$$d\omega_z = -\omega_- \wedge \omega_+, \quad d\omega_+ = q^2(1 + q^2) \omega_z \wedge \omega_+, \quad d\omega_- = -(1 + q^{-2}) \omega_z \wedge \omega_-, \quad (3.22)$$

together with commutation relations

$$\omega_\pm \wedge \omega_\pm = \omega_z \wedge \omega_z = 0, \quad \omega_- \wedge \omega_+ + q^{-2} \omega_+ \wedge \omega_- = 0, \quad \omega_z \wedge \omega_\mp + q^{\pm 4} \omega_\mp \wedge \omega_z = 0. \quad (3.23)$$

Finally, there is a unique top form $\omega_- \wedge \omega_+ \wedge \omega_z$.

We summarize the above results in the following proposition.

Proposition 3.2. For the 3-dimensional left-covariant differential calculus on $\mathrm{SU}_q(2)$, the bimodules of forms are all (left, say) trivial $\mathcal{A}(\mathrm{SU}_q(2))$ -modules given explicitly as follows:

$$\begin{aligned} \Omega^0(\mathrm{SU}_q(2)) &= \mathcal{A}(\mathrm{SU}_q(2)), \\ \Omega^1(\mathrm{SU}_q(2)) &= \mathcal{A}(\mathrm{SU}_q(2)) \langle \omega_-, \omega_+, \omega_z \rangle, \\ \Omega^2(\mathrm{SU}_q(2)) &= \mathcal{A}(\mathrm{SU}_q(2)) \langle \omega_- \wedge \omega_+, \omega_- \wedge \omega_z, \omega_+ \wedge \omega_z \rangle, \\ \Omega^3(\mathrm{SU}_q(2)) &= \mathcal{A}(\mathrm{SU}_q(2)) \omega_- \wedge \omega_+ \wedge \omega_z; \end{aligned}$$

the exterior differential and commutation relations are obtained from (3.22) and (3.23), whereas the bimodule structure is obtained from (3.21). \square

The restriction of the above 3-dimensional calculus to the projective line $\mathbb{C}\mathbb{P}_q^1$ yields the unique left-covariant 2-dimensional calculus on the latter [19]. This unique calculus was realized in [26] via a Dirac operator.

The calculus on $\mathbb{C}\mathbb{P}_q^1$ is broken into a holomorphic and anti-holomorphic part in a natural way. The module of 1-forms $\Omega^1(\mathbb{C}\mathbb{P}_q^1)$ is shown to be isomorphic to the direct

sum $\mathcal{L}_{-2} \oplus \mathcal{L}_2$, that is, the line bundles with winding number ± 2 . Since the element K acts as the identity on $\mathcal{A}(\mathbb{CP}_q^1)$, the differential (3.20) becomes, when restricted to the latter,

$$df = (X_{-\triangleright} f) \omega_- + (X_{+\triangleright} f) \omega_+ \quad \text{for } f \in \mathcal{A}(\mathbb{CP}_q^1).$$

These lead one to break the exterior differential into a holomorphic and an anti-holomorphic part, $d = \bar{\partial} + \partial$, with

$$\bar{\partial} f = (X_{-\triangleright} f) \omega_-, \quad \partial f = (X_{+\triangleright} f) \omega_+, \quad \text{for } f \in \mathcal{A}(\mathbb{CP}_q^1). \quad (3.24)$$

Lemma 3.3. The two differentials ∂ and $\bar{\partial}$ satisfy

$$(\partial f)^* = \bar{\partial} f^*. \quad \square$$

Proof. This follows by direct computation:

$$(\partial f)^* = -\omega_-(X_{+\triangleright} f)^* = -\omega_-(S(X_+)^* \triangleright f^*) = \omega_-(K^{-2} X_{-\triangleright} f^*) = (X_{-\triangleright} f^*) \omega_- = \bar{\partial} f^*,$$

using the compatibility of the $*$ -structure with the left action. ■

The decomposition of the calculus shows that

$$\Omega^1(\mathbb{CP}_q^1) = \Omega^{(0,1)}(\mathbb{CP}_q^1) \oplus \Omega^{(1,0)}(\mathbb{CP}_q^1),$$

where $\Omega^{(0,1)}(\mathbb{CP}_q^1) \simeq \mathcal{L}_{-2} \omega_-$ is the $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodule generated by

$$\{\bar{\partial} B_-, \bar{\partial} B_0, \bar{\partial} B_+\} = \{a^2, ca, c^2\} \omega_- = q^2 \omega_- \{a^2, ca, c^2\}, \quad (3.25)$$

and $\Omega^{(1,0)}(\mathbb{CP}_q^1) \simeq \mathcal{L}_{+2} \omega_+$ is the $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodule generated by

$$\{\partial B_+, \partial B_0, \partial B_-\} = \{a^{*2}, c^* a^*, c^{*2}\} \omega_+ = q^{-2} \omega_+ \{a^{*2}, c^* a^*, c^{*2}\}. \quad (3.26)$$

That these two modules of forms are not free is also expressed by the existence of relations among the differential

$$\partial B_0 - q^{-2} B_- \partial B_+ + q^2 B_+ \partial B_- = 0, \quad \bar{\partial} B_0 - B_+ \bar{\partial} B_- + q^{-4} B_- \bar{\partial} B_+ = 0.$$

The 2-dimensional calculus on \mathbb{CP}_q^1 has a unique top 2-form ω with $\omega f = f\omega$, for all $f \in \mathcal{A}(\mathbb{CP}_q^1)$, and $\Omega^2(\mathbb{CP}_q^1)$ is the free $\mathcal{A}(\mathbb{CP}_q^1)$ -module generated by ω , that is,

$$\Omega^2(\mathbb{CP}_q^1) = \omega \mathcal{A}(\mathbb{CP}_q^1) = \mathcal{A}(\mathbb{CP}_q^1) \omega.$$

Now, both ω_{\pm} commutes with elements of $\mathcal{A}(\mathbb{CP}_q^1)$ and so does $\omega_- \wedge \omega_+$, which is taken as the natural generator $\omega = \omega_- \wedge \omega_+$ of $\Omega^2(\mathbb{CP}_q^1)$. The exterior derivative of any 1-form $\alpha = x\omega_- + y\omega_+ \in \mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_{+2}\omega_+$ is given by

$$d\alpha = d(x\omega_- + y\omega_+) = \partial x \wedge \omega_- + \bar{\partial} y \wedge \omega_+ = (X_{-} \triangleright y - q^2 X_{+} \triangleright x) \omega_- \wedge \omega_+. \quad (3.27)$$

We summarize the above results in the following proposition.

Proposition 3.4. The 2-dimensional differential calculus on the projective line \mathbb{CP}_q^1 is

$$\Omega^*(\mathbb{CP}_q^1) = \mathcal{A}(\mathbb{CP}_q^1) \oplus (\mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_{+2}\omega_+) \oplus \mathcal{A}(\mathbb{CP}_q^1) \omega_- \wedge \omega_+.$$

Moreover, the splitting $\Omega^1(\mathbb{CP}_q^1) = \Omega^{(1,0)}(\mathbb{CP}_q^1) \oplus \Omega^{(0,1)}(\mathbb{CP}_q^1)$ together with the two maps ∂ and $\bar{\partial}$ given in (3.24) constitute a complex structure (in the sense of Definition 2.1) for the differential calculus. \square

3.5 The holomorphic connection

The next ingredient is a connection on the quantum principal bundle with respect to the left-covariant calculus $\Omega(\mathbb{CP}_q^1)$. The connection will in turn determine a covariant derivative on any $\mathcal{A}(\mathbb{CP}_q^1)$ -module \mathcal{E} , in particular on the modules of sections of line bundles \mathcal{L}_n given in (3.14).

The most natural way to define a connection on a quantum principal bundle (with given calculi) is by splitting the 1-forms on the total space into horizontal and vertical ones [6, 7]. Since horizontal 1-forms are given in the structure of the principal bundle, one needs a projection on forms whose range is the subspace of vertical ones. The

projection is required to be covariant with respect to the right coaction of the structure Hopf algebra.

For the principal bundle over the quantum projective line $\mathbb{C}P_q^1$ that we are considering, a principal connection is a covariant left module projection $\Pi : \Omega^1(\mathrm{SU}_q(2)) \rightarrow \Omega_{\mathrm{ver}}^1(\mathrm{SU}_q(2))$. That is, $\Pi^2 = \Pi$ and $\Pi(x\omega) = x\Pi(\omega)$, for $\omega \in \Omega^1(\mathrm{SU}_q(2))$ and $x \in \mathcal{A}(\mathrm{SU}_q(2))$, and $\alpha \circ \Pi = \Pi \circ \alpha$, with a natural extension of the $U(1)$ -action α to 1-forms. Equivalently, it is a covariant splitting $\Omega^1(\mathrm{SU}_q(2)) = \Omega_{\mathrm{ver}}^1(\mathrm{SU}_q(2)) \oplus \Omega_{\mathrm{hor}}^1(\mathrm{SU}_q(2))$. It is not difficult to realize that with the left-covariant 3-dimensional calculus on $\mathcal{A}(\mathrm{SU}_q(2))$, a basis for $\Omega_{\mathrm{hor}}^1(\mathrm{SU}_q(2))$ is given by $\{\omega_-, \omega_+\}$. Furthermore,

$$\alpha_u(\omega_z) = \omega_z, \quad \alpha_u(\omega_-) = \omega_- u^{*2}, \quad \alpha_u(\omega_+) = \omega_+ u^2 \quad \text{for } u \in U(1).$$

Thus, a natural choice of a connection [6, 19] is to define ω_z to be vertical:

$$\Pi(\omega_z) := \omega_z, \quad \Pi(\omega_{\pm}) := 0.$$

With a connection, one has a covariant derivative for any $\mathcal{A}(\mathbb{C}P_q^1)$ -module \mathcal{E} :

$$\nabla : \mathcal{E} \rightarrow \Omega^1(\mathbb{C}P_q^1) \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \mathcal{E}, \quad \nabla := (\mathrm{id} - \Pi_z) \circ \mathrm{d},$$

and one readily shows the Leibniz rule property $\nabla(f\phi) = f\nabla(\phi) + (\mathrm{d}f) \otimes \phi$, for all $\phi \in \mathcal{E}$ and $f \in \mathcal{A}(\mathbb{C}P_q^1)$. We shall concentrate on \mathcal{E} being the line bundles \mathcal{L}_n given in (3.14). Then, with the left-covariant 2-dimensional calculus on $\mathcal{A}(\mathbb{C}P_q^1)$ (coming from the left covariant 3-dimensional calculus on $\mathcal{A}(\mathrm{SU}_q(2))$ as explained before), we have

$$\nabla\phi = (X_{+\triangleright}\phi)\omega_+ + (X_{-\triangleright}\phi)\omega_- = q^{-n-2}\omega_+(X_{+\triangleright}\phi) + q^{-n+2}\omega_-(X_{-\triangleright}\phi), \quad (3.28)$$

since $X_{\pm\triangleright}\phi \in \mathcal{L}_{n\pm 2}$. Using Proposition 3.1, we see that

$$\nabla\phi \in \omega_+\mathcal{L}_{n+2} \oplus \omega_-\mathcal{L}_{n-2} \simeq \Omega^1(\mathbb{C}P_q^1) \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \mathcal{E},$$

as required.

The curvature of the connection ∇ is the $\mathcal{A}(\mathbb{C}P_q^1)$ -linear map

$$\nabla^2 : \mathcal{E} \rightarrow \Omega^2(\mathbb{C}P_q^1) \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \mathcal{E}.$$

On $\phi \in \mathcal{L}_n$ one finds

$$\nabla^2 \phi = -q^{-2n-2} \omega_+ \wedge \omega_- (X_z \triangleright \phi),$$

with X_z the (vertical) vector field in (3.17), or

$$\nabla^2 = -q^{-n-1} [n] \omega_+ \wedge \omega_- \tag{3.29}$$

as an element in $\text{Hom}_{\mathcal{A}(\mathbb{CP}_q^1)}(\mathcal{L}_n, \Omega(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_n)$.

In Section 4.2 we shall study at length holomorphic vector bundles on \mathbb{CP}_q^1 coming from the natural splitting of the connection ∇ defined in (3.28) into a holomorphic and anti-holomorphic part: $\nabla = \nabla^\partial + \nabla^{\bar{\partial}}$, with

$$\nabla^\partial \phi = q^{-n-2} \omega_+ (X_+ \triangleright \phi), \quad \nabla^{\bar{\partial}} \phi = q^{-n+2} \omega_- (X_- \triangleright \phi), \tag{3.30}$$

for which there are corresponding Leibniz rules: $\nabla^\partial(f\phi) = f\nabla^\partial(\phi) + (\partial f) \otimes \phi$ and $\nabla^{\bar{\partial}}(f\phi) = f\nabla^{\bar{\partial}}(\phi) + (\bar{\partial} f) \otimes \phi$, for all $\phi \in \mathcal{E}$ and $f \in \mathcal{A}(\mathbb{CP}_q^1)$. The operator $\nabla^{\bar{\partial}}$ clearly satisfies the conditions of a holomorphic structure as given in Definition 2.8. In particular, these connections are automatically flat as it is evident from the curvature in (3.29) being a (1, 1)-form. Thus, we have the following:

Definition 3.5. The operator $\nabla^{\bar{\partial}}$ in (3.30) will be called the *standard holomorphic structure* on \mathcal{L}_n ; we denote $\bar{\nabla}_{(n)} = \nabla^{\bar{\partial}}$ or even simply $\bar{\nabla}$ when there is no room for confusion. □

It is natural to expect that modulo gauge equivalence, as given in Definition 2.9, the holomorphic structure on \mathcal{L}_n , is unique; as of now we are unable to prove the uniqueness.

3.6 Tensor products

We next study the tensor product of two such noncommutative line bundles with connections. Similar to [18, Lemma 18], we have the following:

Lemma 3.6. For any integer n , there is a “twisted flip” isomorphism

$$\Phi_{(n)} : \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \Omega^{(0,1)}(\mathbb{CP}_q^1) \xrightarrow{\sim} \Omega^{(0,1)}(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_n$$

as $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules. There is an analogous map for $\Omega^{(1,0)}(\mathbb{CP}_q^1)$ replacing $\Omega^{(0,1)}(\mathbb{CP}_q^1)$. □

Proof. First, note that due to relations (3.21), not only $\Omega^{(0,1)}(\mathbb{CP}_q^1) \simeq \mathcal{L}_{-2}\omega_-$, but also $\Omega^{(0,1)}(\mathbb{CP}_q^1) \simeq \omega_- \mathcal{L}_{-2}$ as $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules (this was indeed given in (3.25)). It follows, using Proposition 3.1, that we have the following diagram of $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules:

$$\begin{array}{ccc} \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \Omega^{(0,1)}(\mathbb{CP}_q^1) & \dashrightarrow & \Omega^{(0,1)}(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_n \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{L}_{n-2}\omega_- & \xrightarrow{\sim} & \omega_- \mathcal{L}_{n-2} \end{array}$$

where for the bottom row we have used again the commutation relations (3.21) that hold inside $\Omega^1(\mathrm{SU}_q(2))$. The dashed arrow is the desired bimodule isomorphism $\Phi_{(n)}$. ■

Given the twisted flip, the following propositions can be easily verified.

Proposition 3.7. The holomorphic structure $\bar{\nabla}$ on \mathcal{L}_n is a bimodule connection with $\sigma(\bar{\nabla}) = \Phi_{(n)}$, that is, it satisfies the twisted right Leibniz rule

$$\bar{\nabla}(\eta f) = \bar{\nabla}(\eta) f + \Phi_{(n)}(\eta \otimes \bar{\partial} f), \quad \text{for } \eta \in \mathcal{L}_n, \quad f \in \mathcal{A}(\mathbb{CP}_q^1). \quad \square$$

Proposition 3.8. Let $(\mathcal{L}_{n_i}, \bar{\nabla}_{n_i})$, $i = 1, 2$, be two line bundles with standard holomorphic structure. Then the tensor product connection $\bar{\nabla}_{n_1} \otimes 1 + (\Phi_{(n_1)} \otimes 1)(1 \otimes \bar{\nabla}_{n_2})$ coincides with the standard holomorphic structure on $\mathcal{L}_{n_1} \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_{n_2}$ when identified with $\mathcal{L}_{n_1+n_2}$. □

4 Holomorphic Structures on the Quantum Projective Line

We will study more closely the holomorphic structure on the quantum projective line given in terms of the ∂ and $\bar{\partial}$ -operator in (3.24) and of the ∇^∂ and $\nabla^{\bar{\partial}}$ -connection in (3.30). We start with holomorphic functions before moving to holomorphic sections.

4.1 Holomorphic functions

Recall Definition 2.3 above for holomorphic (polynomial) functions. On \mathbb{CP}_q^1 these are elements in the kernel of the operator $\bar{\partial} : \mathcal{A}(\mathbb{CP}_q^1) \rightarrow \Omega^{(0,1)}(\mathbb{CP}_q^1)$. Equivalently (cf. Equation (3.24)), these are elements in the kernel of $F = q^{1/2} X_- : \mathcal{A}(\mathbb{CP}_q^1) \rightarrow \mathcal{A}(\mathrm{SU}_q(2))$, acting as in (3.4). We will derive the triviality of this kernel from the analogous, but more general, result in the Hilbert space $L^2(\mathbb{CP}_q^1)$.

Recall from Section 3.2 that there is an injective map $\mathcal{A}(\mathbb{C}P_q^1) \rightarrow L^2(\mathbb{C}P_q^1)$; it is the composition of the map from $\mathcal{A}(\mathbb{C}P_q^1)$ into its C^* -algebraic completion with the (restriction of the) GNS-map $\eta : C(\mathbb{C}P_q^1) \rightarrow L^2(\mathbb{C}P_q^1)$. These maps are equivariant with respect to the left $\mathcal{U}_q(\mathfrak{su}(2))$ action so that triviality of the kernel of F in $\mathcal{A}(\mathbb{C}P_q^1)$ would follow from triviality of the kernel of $\sigma(F)$ in $L^2(\mathbb{C}P_q^1)$, acting in the representation (3.12). Since the operator $\sigma(F)$ on $L^2(\mathrm{SU}_q(2))$ is unbounded, we need to specify its domain, and we choose

$$\mathrm{Dom}(\sigma(F)) := \{\psi \in L^2(\mathrm{SU}_q(2)) : \sigma(F)\psi \in L^2(\mathrm{SU}_q(2))\}. \tag{4.1}$$

It clearly contains the image of $\mathcal{A}(\mathbb{C}P_q^1)$ inside $L^2(\mathbb{C}P_q^1) \subset L^2(\mathrm{SU}_q(2))$ under the map η above. Indeed, any polynomial in the B_0 and B_\pm when mapped in $L^2(\mathbb{C}P_q^1)$ can be written as a finite linear combination of the basis vectors $|lmm\rangle$ through the relations (3.6) and (3.11). Let us now consider the restriction of $\sigma(F)$ to an operator $L^2(\mathbb{C}P_q^1) \rightarrow L^2(\mathrm{SU}_q(2))$.

Proposition 4.1. The kernel of $\sigma(F)$ restricted to $L^2(\mathbb{C}P_q^1)$ is \mathbb{C} . □

Proof. Let $\psi = \sum_{l,m} \psi_{lm} |lm\mathbf{0}\rangle \in L^2(\mathbb{C}P_q^1)$ be in $\ker \sigma(F)$. Then

$$0 = \sigma(F)\psi = \sum_{l,m} \psi_{lm} \sqrt{[l+1][l]} |l, m, -1\rangle.$$

Since the $|l, m, -1\rangle$ are linearly independent and $[l+1][l] \neq 0$ as long as $l \neq 0$, we conclude that the $\psi_{lm} = 0$ for all l, m , unless $l = 0$ whence $\psi = \psi_{00} \in \mathbb{C}$. ■

Corollary 4.2. There are no nontrivial holomorphic polynomial functions on $\mathbb{C}P_q^1$. □

We next turn to the question of the existence of nontrivial holomorphic *continuous* functions on $\mathbb{C}P_q^1$. Thus, we consider the kernel of $\bar{\partial}$ in $C(\mathrm{SU}_q(2))$; since $\bar{\partial}$ is an unbounded derivation, we define its domain in $C(\mathrm{SU}_q(2))$ as

$$\mathrm{Dom}(\bar{\partial}) := \{f \in C(\mathrm{SU}_q(2)) : \|F \triangleright f\| < \infty\}. \tag{4.2}$$

We again have as a corollary to Proposition 4.1

Corollary 4.3. There are no nontrivial holomorphic functions in $\mathrm{Dom}(\bar{\partial}) \cap C(\mathbb{C}P_q^1)$. □

Proof. If $f \in \ker(\bar{\partial})|_{\mathcal{C}(\mathbb{CP}_q^1)}$, by the equivariance of the GNS representation, the corresponding elements $\eta(f) \in L^2(\mathbb{CP}_q^1)$ under the continuous GNS map should be in the kernel of $\sigma(F)$. By Proposition 4.1, the only possibility is that f be a constant. ■

Consequently, $\mathcal{O}(\mathbb{CP}_q^1) \simeq \mathbb{C}$ (with a slight abuse of notation), a result which completely parallels the classical case ($q = 1$) of holomorphic functions on the Riemann sphere.

4.2 Holomorphic vector bundles

In this section we shall study holomorphic vector bundles on \mathbb{CP}_q^1 coming from the natural splitting of the connection ∇ defined in (3.28) into a holomorphic and anti-holomorphic part. The anti-holomorphic connection on the modules \mathcal{L}_n is given by (cf. (3.30))

$$\bar{\nabla}_{(n)} = q^{-n+2}\omega_-(X_{-\triangleright}\phi),$$

which we simply denote by $\bar{\nabla}$ when no confusion arises.

Theorem 4.4. Let n be a positive integer. Then

1. $H^0(\mathcal{L}_n, \bar{\nabla}) = 0$,
2. $H^0(\mathcal{L}_{-n}, \bar{\nabla}) \simeq \mathbb{C}^{n+1}$.

These results continue to hold when considering continuous sections $\Gamma(\mathcal{L}_n)$ as modules over the C^* -algebra $\mathcal{C}(\mathbb{CP}_q^1)$. □

Proof. We derive this from the more general result in the Hilbert spaces $L^2(\mathcal{L}_n)$. There, an element $\phi \in L^2(\mathcal{L}_n)$ is in the kernel of $\bar{\nabla}$ if and only if it is in $\text{Dom}(\sigma(F))$ defined in Equation (4.1) (intersected with $L^2(\mathcal{L}_n)$) and such that $\sigma(F)\phi = 0$. This follows easily from the definition of the anti-holomorphic connection in (3.30).

For $n > 0$ let $\phi = \sum_{l,m} \phi_{lm} |l, m, n/2\rangle \in L^2(\mathcal{L}_n)$ be in $\ker \sigma(F)$. Then

$$0 = \sigma(F)\phi = \sum_{l,m} \phi_{lm} \sqrt{[l - n/2 + 1][l + n/2]} |l, m, n/2 - 1\rangle.$$

Since the $|l, m, n/2 - 1\rangle$ are linearly independent and $[l - n/2 + 1][l + n/2] \neq 0$ as long as $l + n/2 \neq 0$, we conclude that the $\phi_{lm} = 0$ for all l, m (since $l \geq n/2 > 0$).

For the second statement, let $\phi = \sum_{l,m} \phi_{lm} |lm, -n/2\rangle \in L^2(\mathcal{L}_{-n})$ be in $\ker \sigma(F)$. Then

$$0 = \sigma(F)\phi = \sum_{l,m} \phi_{lm} \sqrt{[l + n/2 + 1][l - n/2]} |lm, -n/2 - 1\rangle.$$

Now $[l + n/2 + 1][l - n/2]$ vanishes for $l = n/2$ so that $\phi_{lm} = 0$ unless $l = n/2$. With this restriction the integer label m in ϕ_{lm} runs from $-n/2$ to $n/2$, thus giving $n + 1$ complex degrees of freedom. ■

We finally have

Theorem 4.5. The space $R = \bigoplus_{n \geq 0} H^0(\mathcal{L}_{-n}, \bar{\nabla})$ carries a ring structure and is isomorphic to the quantum plane

$$R \simeq \mathbb{C}\langle a, c \rangle / (ac - qca). \quad \square$$

Proof. The ring structure is induced from the tensor product $\mathcal{L}_{-n} \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_{-m} \simeq \mathcal{L}_{-n-m}$. From the proof of Theorem 4.4, we know that $H^0(\mathcal{L}_{-1}, \bar{\nabla})$ is spanned by $|1/2, m, -1/2\rangle$ with $m = \pm 1/2$. According to Equations (3.11) and (3.7), they correspond to the elements $a, c \in \mathcal{A}(\text{SU}_q(2))$. The result then follows from the identity $a \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} c - qc \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} a = 0$ which, as already mentioned, can be easily established. ■

Note that this quantum homogeneous coordinate ring R coincides precisely with the twisted homogeneous coordinate rings of [2, 3] associated to the line bundle $\mathcal{O}(1)$ on \mathbb{CP}^1 and a suitable twist.

5 Twisted Positive Hochschild Cocycle

In [8, Section VI.2] it is shown that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a complex structure on the surface. The relevant positive cocycle is in the same Hochschild cohomology class of the cyclic cocycle giving the fundamental class of the manifold. Although the problem of characterizing complex structures on n -dimensional manifolds via positive Hochschild cocycles is still open, nevertheless, Connes' result suggests positive cyclic and Hochschild cocycles as a starting point in defining complex noncommutative structures.

It is well known that there are no nontrivial 2-dimensional cyclic cocycles on the quantum 2-sphere [21]. Thus, we shall try and formulate a notion of *twisted positivity* for twisted Hochschild and cyclic cocycles and exhibit an example of it in the case of our complex structure on the quantum 2-sphere.

Recall that a Hochschild $2n$ -cocycle φ on an $*$ -algebra A is called *positive* [9] if the following pairing defines a positive sesquilinear form on the vector spaces $A^{\otimes(n+1)}$

$$\langle a_0 \otimes a_1 \otimes \cdots \otimes a_n, b_0 \otimes b_1 \otimes \cdots \otimes b_n \rangle = \varphi(b_0^* a_0, a_1, \dots, a_n, b_n^*, \dots, b_1^*).$$

For $n=0$ one recovers the standard notion of a positive trace on an $*$ -algebra. Given a differential graded $*$ -algebra of noncommutative differential forms $(\Omega A, d)$ on A , a Hochschild $2n$ -cocycle on A defines a sesquilinear pairing on the space $\Omega^n A$ of n -forms (typically, these would be middle-degree forms). For $\omega = a_0 da_1 \cdots da_n$ and $\eta = b_0 db_1 \cdots db_n$, one defines

$$\langle \omega, \eta \rangle := \varphi(b_0^* a_0, a_1, \dots, a_n, b_n^*, \dots, b_1^*),$$

extended by linearity. One has that $\langle a\omega, \eta \rangle = \langle \omega, a^*\eta \rangle$ for all $a \in A$. Positivity of φ is equivalent to positivity of this sesquilinear form on $\Omega^n A$.

Before we introduce a twisted analog of the notion of positivity, we need to briefly recall twisted Hochschild and cyclic cohomologies.

Then, let A be an algebra and $\sigma : A \rightarrow A$ be an automorphism of A . For $n \geq 0$ let $C^n(A) = \text{Hom}_{\mathbb{C}}(A^{\otimes(n+1)}, \mathbb{C})$ be the space of $(n+1)$ -linear functionals (the n -cochains) on A . Define the twisted cyclic operator $\lambda_\sigma : C^n(A) \rightarrow C^n(A)$ by

$$(\lambda_\sigma \varphi)(a_0, \dots, a_n) = (-1)^n \varphi(\sigma(a_n), a_0, a_1, \dots, a_{n-1}).$$

Clearly, $(\lambda_\sigma^{n+1} \varphi)(a_0, \dots, a_n) = \varphi(\sigma(a_0), \dots, \sigma(a_n))$. Let

$$C_\sigma^n(A) = \ker((1 - \lambda_\sigma^{n+1}) : C^n(A) \rightarrow C^n(A))$$

denote the space of *twisted Hochschild n -cochains* on A . Define the *twisted Hochschild coboundary operator* $b_\sigma : C_\sigma^n(A) \rightarrow C_\sigma^{n+1}(A)$ and the operator $b'_\sigma : C_\sigma^n(A) \rightarrow C_\sigma^{n+1}(A)$ by

$$\begin{aligned} b_\sigma \varphi(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} \varphi(\sigma(a_{n+1}) a_0, a_1, \dots, a_n), \\ b'_\sigma \varphi(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}). \end{aligned}$$

One checks that b_σ sends twisted cochains to twisted cochains. The cohomology of the complex $(C_\sigma^*(A), b_\sigma)$ is the *twisted Hochschild cohomology* of A . We also need the twisted cyclic cohomology. An n -cochain $\varphi \in C^n(A)$ is called *twisted cyclic* if $(1 - \lambda_\sigma)\varphi = 0$, or equivalently,

$$\varphi(\sigma(a_n), a_0, \dots, a_{n-1}) = (-1)^n \varphi(a_0, a_1, \dots, a_n),$$

for all a_0, a_1, \dots, a_n elements in A . Denote the space of cyclic n -cochains by $C_{\lambda, \sigma}^n(A)$. Obviously, $C_{\lambda, \sigma}^n(A) \subset C_\sigma^n(A)$. The relation $(1 - \lambda_\sigma)b_\sigma = b'_\sigma(1 - \lambda_\sigma)$ shows that the operator b_σ preserves the space of cyclic cochains, and we obtain the twisted cyclic complex of the pair (A, σ) , denoted $(C_{\lambda, \sigma}^n(A), b_\sigma)$. The cohomology of the twisted cyclic complex is called the *twisted cyclic cohomology* of A and denoted $\text{HC}_\sigma^\bullet(A)$.

Definition 5.1. A twisted Hochschild $2n$ -cocycle φ on an $*$ -algebra A is called *twisted positive* if the pairing

$$\langle a_0 \otimes a_1 \otimes \dots \otimes a_n, b_0 \otimes b_1 \otimes \dots \otimes b_n \rangle := \varphi(\sigma(b_0^*) a_0, a_1, \dots, a_n, b_n^*, \dots, b_1^*)$$

defines a positive sesquilinear form on the vector space $A^{\otimes(n+1)}$. □

Let us now go back to the quantum projective line $\mathbb{C}P_q^1$. Let $h : \mathcal{A}(\text{SU}_q(2)) \rightarrow \mathbb{C}$ denote the normalized Haar state of $\text{SU}_q(2)$. It is a positive twisted trace obeying

$$h(xy) = h(\sigma(y)x), \quad \text{for } x, y \in \mathcal{A}(\text{SU}_q(2)),$$

with (modular) automorphism $\sigma : \mathcal{A}(\text{SU}_q(2)) \rightarrow \mathcal{A}(\text{SU}_q(2))$ given by

$$\sigma(x) = K^2 \triangleright x \triangleleft K^2.$$

(cf. [17, Prop. 4.15]). When restricted to $\mathcal{A}(\mathbb{C}P_q^1)$, it induces the automorphism

$$\sigma : \mathbb{C}P_q^1 \rightarrow \mathbb{C}P_q^1, \quad \sigma(x) = x \triangleleft K^2, \quad \text{for } x \in \mathcal{A}(\mathbb{C}P_q^1).$$

The bi-invariance of h on $\mathcal{A}(\text{SU}_q(2))$ reduces to left invariance on $\mathcal{A}(\mathbb{C}P_q^1)$, that is to say,

$$(\text{id} \otimes h) \circ \Delta_L(x) = h(x) 1_{\mathcal{A}(\mathbb{C}P_q^1)}, \quad \text{for } x \in \mathcal{A}(\mathbb{C}P_q^1),$$

where Δ_L is the coaction of $\mathcal{A}(\text{SU}_q(2))$ on $\mathcal{A}(\mathbb{C}P_q^1)$ alluded to at the end of Section 3.1. Dually, there is invariance for the right action of $\mathcal{U}_q(\text{su}(2))$ on $\mathcal{A}(\mathbb{C}P_q^1)$:

$$h(x \triangleleft v) = \varepsilon(v)h(x), \quad \text{for } x \in \mathcal{A}(\mathbb{C}P_q^1), v \in \mathcal{U}_q(\text{su}(2)).$$

With $\omega_- \wedge \omega_+$ the central generator of $\Omega^2(\mathbb{C}P_q^1)$, h the Haar state on $\mathcal{A}(\mathbb{C}P_q^1)$, and σ its above modular automorphism, the linear functional

$$\int_h : \Omega^2(\mathbb{C}P_q^1) \rightarrow \mathbb{C}, \quad \int_h a \omega_- \wedge \omega_+ := h(a),$$

defines [26] a nontrivial twisted cyclic 2-cocycle τ on $\mathcal{A}(\mathbb{C}P_q^1)$ by

$$\tau(a_0, a_1, a_2) := \frac{1}{2} \int_h a_0 da_1 \wedge da_2.$$

The non-triviality means that there is no twisted cyclic 1-cochain α on $\mathcal{A}(\mathbb{C}P_q^1)$ such that $b_\sigma \alpha = \tau$ and $\lambda_\sigma \alpha = \alpha$. Thus, τ is a nontrivial class in $\text{HC}_\sigma^2(\mathbb{C}P_q^1)$.

Proposition 5.2. The cochain $\varphi \in \mathcal{C}^2(\mathcal{A}(\mathbb{C}P_q^1))$ defined by

$$\varphi(a_0, a_1, a_2) = \int_h a_0 \partial a_1 \bar{\partial} a_2$$

is a twisted Hochschild 2-cocycle on $\mathcal{A}(\mathbb{C}P_q^1)$, that is to say, $b_\sigma \varphi = 0$ and $\lambda_\sigma^3 \varphi = \varphi$; it is also positive, with positivity expressed as

$$\int_h a_0 \partial a_1 (a_0 \partial a_1)^* \geq 0$$

for all $a_0, a_1 \in \mathcal{A}(\mathbb{C}P_q^1)$. □

Before giving the proof, we prove a preliminary result.

Lemma 5.3. For any $a_0, a_1, a_2, a_3 \in \mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$, it holds that

$$\int_h a_0(\partial a_1 \bar{\partial} a_2) a_3 = \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2. \quad \square$$

Proof. Write $\partial a_1 \bar{\partial} a_2 = \gamma \omega_- \wedge \omega_+$, for some $\gamma \in \mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$. Using the fact that $\omega_- \wedge \omega_+$ commutes with elements in $\mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$, we have that

$$\begin{aligned} \int_h a_0(\partial a_1 \bar{\partial} a_2) a_3 - \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2 &= \int_h a_0 \gamma \omega_- \wedge \omega_+ a_3 - \int_h \sigma(a_3) a_0 \gamma \omega_- \wedge \omega_+ \\ &= \int_h a_0 \gamma a_3 \omega_- \wedge \omega_+ - \int_h \sigma(a_3) a_0 \gamma \omega_- \wedge \omega_+ \\ &= h(a_0 \gamma a_3) - h(\sigma(a_3) a_0 \gamma) = 0 \end{aligned}$$

from the twisted property of the Haar state. ■

Proof. of Proposition 5.2

Using the derivation property of ∂ and $\bar{\partial}$, we have that

$$\begin{aligned} (b_\sigma \varphi)(a_0, a_1, a_2, a_3) &= \int_h a_0 a_1 \partial a_2 \bar{\partial} a_3 - \int_h a_0 \partial(a_1 a_2) \bar{\partial} a_3 \\ &\quad + \int_h a_0 \partial a_1 \bar{\partial}(a_2 a_3) - \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2 = \int_h a_0(\partial a_1 \bar{\partial} a_2) a_3 - \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2 = 0, \end{aligned}$$

from the previous lemma.

Next, the cyclic condition follows from the invariance of the Haar state and of the calculus. Indeed, from the commutativity of the left and right $\mathcal{U}_q(\mathfrak{su}(2))$ -actions, it holds that

$$\varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2)) = \int_h \sigma(a_0) \partial \sigma(a_1) \bar{\partial} \sigma(a_2) = \int_h \sigma(a_0 \partial a_1 \bar{\partial} a_2);$$

writing $a_0 \partial a_1 \bar{\partial} a_2 = \gamma \omega_- \wedge \omega_+$, for some $\gamma \in \mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$, left $\mathcal{U}_q(\mathfrak{su}(2))$ invariance of the forms ω_\pm , that is, $\omega_\pm \triangleleft K = \omega_\pm$ (part of the dual invariance statement of (3.19)), yields $\sigma(a_0 \partial a_1 \bar{\partial} a_2) = \sigma(\gamma) \omega_- \wedge \omega_+$ and in turn,

$$\begin{aligned} \varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2)) &= \int_h \sigma(\gamma) \omega_- \wedge \omega_+ = h(\sigma(\gamma)) = h(\gamma \triangleleft K^2) = h(\gamma) = \int_h \gamma \omega_- \wedge \omega_+ \\ &= \int_h a_0 \partial a_1 \bar{\partial} a_2 = \varphi(a_0, a_1, a_2). \end{aligned}$$

Finally, for the twisted positivity of φ , the hermitian scalar product on $\Omega^{(1,0)}(\mathbb{C}P_q^1)$,

$$\langle a_0 \partial a_1, b^0 \partial b^1 \rangle := \varphi(\sigma(b_0^*) a_0, a_1, b_1^*) = \int_h \sigma(b_0^*) a_0 \partial a_1 \bar{\partial} b_1^*,$$

determines a positive sesquilinear form if for all $a_0, a_1 \in A(\mathbb{C}P_q^1)$, it holds that

$$\int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^* = \int_h a_0 \partial a_1 (a_0 \partial a_1)^* \geq 0.$$

The first equality follows again from the lemma. Indeed,

$$\int_h a_0 \partial a_1 (a_0 \partial a_1)^* = \int_h a_0 \partial a_1 (\partial a_1)^* a_0^* = \int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^*.$$

Then, if $\partial a_1 = \gamma \omega_+$, it follows that $\bar{\partial} a_1^* = (\partial a_1)^* = -\omega_- \gamma^*$; then

$$\begin{aligned} \int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^* &= - \int_h \sigma(a_0^*) a_0 \gamma \omega_+ \wedge \omega_- \gamma^* = q^2 \int_h \sigma(a_0^*) a_0 \gamma \gamma^* \omega_- \wedge \omega_+ \\ &= q^2 h(\sigma(a_0^*) a_0 \gamma \gamma^*) = q^2 h(a_0 \gamma \gamma^* (a_0)^*) = q^2 h(a_0 \gamma (a_0 \gamma^*)^*) \geq 0, \end{aligned}$$

the positivity being evident. ■

Proposition 5.4. The twisted Hochschild cocycles τ and φ are cohomologous. □

Proof. Let us define a twisted Hochschild 1-cochain ψ on $\mathcal{A}(\mathbb{C}P_q^1)$ by

$$\psi(a, b) = \frac{1}{2} \int_h a \partial \bar{\partial}(b),$$

for $a, b \in \mathcal{A}(\mathbb{C}P_q^1)$. It is a twisted cochain since

$$\begin{aligned} 2 \psi(\sigma(a), \sigma(b)) &= \int_h \sigma(a) \partial \bar{\partial}(\sigma(b)) = \int_h \sigma(a) \sigma(\partial \bar{\partial}(b)) \\ &= \int_h \sigma(a \partial \bar{\partial}(b)) = \int_h a \partial \bar{\partial}(b) = 2 \psi(a, b), \end{aligned}$$

for the invariance of the integral as seen before. We have

$$\begin{aligned} (b_\sigma \psi)(a_0, a_1, a_2) &= \psi(a_0 a_1, a_2) - \psi(a_0, a_1 a_2) + \psi(\sigma(a_2) a_0, a_1) \\ &= \frac{1}{2} \int_h (a_0 a_1 \partial \bar{\partial}(a_2) - a_0 \partial \bar{\partial}(a_1 a_2) + \sigma(a_2) a_0 \partial \bar{\partial}(a_1)) \\ &= \frac{1}{2} \int_h (a_0 a_1 \partial \bar{\partial}(a_2) - a_0 \partial \bar{\partial}(a_1 a_2) + a_0 (\partial \bar{\partial}(a_1)) a_2). \end{aligned}$$

On the other hand,

$$\frac{1}{2} a_0 da_1 \wedge da_2 = a_0 \partial a_1 \bar{\partial} a_2 + \frac{1}{2} a_0 (-\partial \bar{\partial}(a_1 a_2) + (\partial \bar{\partial} a_1) a_2 + a_1 \partial \bar{\partial} a_2).$$

Comparing these last two relations, we find that

$$\frac{1}{2} \int_h a_0 da_1 \wedge da_2 = \int_h a_0 \partial a_1 \bar{\partial} a_2 + (b_\sigma \psi)(a_0, a_1, a_2)$$

or $\varphi - \tau = b_\sigma \psi$, as stated. ■

It is worth stressing that φ is not a twisted cyclic cocycle, only a Hochschild one. In fact, it is trivial as a twisted Hochschild cocycle since it is known that for the modular automorphism σ , the twisted Hochschild cohomology of the algebra $\mathcal{A}(\mathbb{C}P^1_q)$ is trivial [15]. To get nontrivial twisted Hochschild cohomology, one needs twisting with the inverse modular automorphism.

6 Final Remarks

We have shown that much of the structure of the 2-sphere as a complex curve, or equivalently as a conformal manifold, actually survives the q -deformation and has natural generalizations on the quantum 2-sphere. Chiefly among these is the identification of a quantum homogeneous coordinate ring with the coordinate ring of the quantum plane. Also, in parallel with the fact that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information for complex structures on the surface [8, Section VI.2], we have formulated a notion of twisted positivity for twisted Hochschild and cyclic cocycles—given that there are no nontrivial 2-dimensional cyclic cocycles on the quantum 2-sphere—and exhibited an example of a twisted positive Hochschild cocycle in the case of our complex structure on this sphere. Now, additional tools of noncommutative geometry are available there,

including the abstract perturbation of conformal structures by Beltrami differentials as explained in [8, Example 8, Section VI.4]. A great challenge is to prove an analog of the measurable Riemann mapping theorem in the q -deformed case. The formalism of q -groups allows one to set up a simple algebraic framework but the real challenge resides in the analysis. An attack of these problems should await a future time.

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