Principal Fibrations from Noncommutative Spheres

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Abstract: We construct noncommutative principal fibrations \( S_0^7 \rightarrow S_0^4 \) which are deformations of the classical \( SU(2) \) Hopf fibration over the four sphere. We realize the noncommutative vector bundles associated to the irreducible representations of \( SU(2) \) as modules of coequivariant maps and construct corresponding projections. The index of Dirac operators with coefficients in the associated bundles is computed with the Connes-Moscovici local index formula. “The algebra inclusion \( A(S_0^4) \hookrightarrow A(S_0^7) \) is an example of a not-trivial quantum principal bundle.”

1. Introduction

The ADHM construction [2, 3] of instantons in Yang-Mills theory has at its heart the theory of connections on principal and associated bundles. A central example is the basic \( SU(2) \)-instanton on \( S^4 \) which is described by the well-known Hopf \( SU(2) \)-principal bundle \( S^7 \rightarrow S^4 \) and connections thereon.

In this paper, we consider a noncommutative version of this Hopf fibration, in the framework of the isospectral deformations introduced in [11], while trying to understand the structure behind the noncommutative instanton bundle found there.

In Sect. 2 we will review the construction of \( \theta \)-deformed spheres where \( \theta \) is an anti-symmetric real-valued matrix. Apart from the noncommutative spheres \( S_0^m \), we also introduce differential calculi \( \Omega(S_0^m) \) as quotients of the universal differential calculi. On the sphere \( S_0^m \) one constructs a noncommutative Riemannian spin geometry \( (C^\infty(S_0^m), D, \mathcal{H}) \) in which the Dirac operator \( D \) is the classical one and \( \mathcal{H} = L^2(S^m, S) \) is the usual Hilbert space of spinors. Then the deformations are isospectral, as mentioned. Furthermore, one also constructs a Hodge star operator \(*_\theta \) acting on the differential calculus \( \Omega(S_0^m) \) which is most easily defined using the so-called splitting homomorphism [10].

In Sect. 3, we focus on two noncommutative spheres \( S_0^4 \) and \( S_0^7 \) starting from the algebras \( A(S_0^4) \) and \( A(S_0^7) \) of polynomial functions on them. The latter algebra carries
an action of the (classical) group $SU(2)$ by automorphisms in such a way that its invariant elements are exactly the polynomials on $S_4^\theta$. The anti-symmetric $2 \times 2$ matrix $\theta$ is given by a single real number also denoted by $\theta$. On the other hand, the requirements that $SU(2)$ acts by automorphisms and that $S_4^\theta$ makes the algebra of invariant functions, give the matrix $\theta'$ in terms of $\theta$. This yields a one-parameter family of noncommutative Hopf fibrations.

For each irreducible representation $V^{(n)} := \text{Sym}^n(C^2)$ of $SU(2)$ we construct the noncommutative vector bundles $E^{(n)}$ associated to the fibration $S_7^\theta' \to S_4^\theta$. By dualizing the classical construction, these bundles are described by the module of coequivariant maps from $C^2$ to $A(S_7^\theta')$. As expected, these modules are finitely generated projective and we construct explicitly the projections $p^{(n)} \in M_4^{S_4^\theta}(A(S_4^\theta))$ such that these modules are isomorphic to the image of $p^{(n)}$ in $A(S_4^\theta)$. Then, one defines connections $\nabla = p^{(n)}d$ as maps from $\Gamma(S_4^\theta, E^{(n)})$ to $\Gamma(S_0^4, E^{(n)}) \otimes A(S_0^4) \Omega^1(S_0^4)$, where $\Omega^*(S_0^4)$ is the quotient of the universal differential calculus mentioned above. The corresponding connection one-form $A$ turns out to be valued in a representation of the Lie algebra $su(2)$.

By using the projection $p^{(n)}$, the Dirac operator with coefficients in the noncommutative vector bundles $E^{(n)}$ is given by $D_{p^{(n)}} : = p^{(n)}D_{p^{(n)}}$. In order to compute its index, we first show that the local index theorem of Connes and Moscovici [12] takes a very simple form in the case of isospectral deformations. Indeed, for these deformations and with any projection $e$, one finds,

$$\text{Ind} D_e = \text{Res}_{z=0} z^{-1} \text{Tr} \left( \gamma (e - \frac{1}{2}) |D|^{-2z} \right)$$

$$+ \sum_{k \geq 1} c_k \text{Res}_{z=0} \text{Tr} \left( \gamma (e - \frac{1}{2}) |D|^{-2(k+z)} \right)$$

with some proper coefficients $c_k$. When applied to the projections $p^{(n)}$ on $S_4^\theta$, we obtain exactly as in the classical case,

$$\text{Ind} D_{p^{(n)}} = \frac{1}{6} n(n+1)(n+2).$$

Finally, in Sect. 5 we show that the fibration $S_7^\theta' \to S_4^\theta$ is a ‘not-trivial principal bundle with structure group $SU(2)$’. This means that the inclusion $\mathcal{A}(S_0^4) \hookrightarrow \mathcal{A}(S_0^4)$ is a not-cleft Hopf-Galois extension [21, 28]; in fact, it is a principal extension [6]. On this extension, we find an explicit form of the (strong) connection which induces connections on the associated bundles $E^{(n)}$ as maps from $\Gamma(S_0^4, E^{(n)})$ to $\Gamma(S_0^4, E^{(n)}) \otimes A(S_0^4) \Omega^1(\mathcal{A}(S_0^4))$, where $\Omega^*(\mathcal{A}(S_0^4))$ is the universal differential calculus on $\mathcal{A}(S_0^4)$. We show that these connections coincide with the Grassmannian connections $\nabla = p^{(n)}d$ on the quotient $\Omega(S_0^4)$ of the universal differential calculus alluded to before.

2. Noncommutative Spherical Manifolds

In this section, we will recall the construction of the noncommutative spheres $S_0^\theta$ as introduced in [11] and elaborated in [10]. Essentially, these $\theta$-deformations are a natural extension of the noncommutative torus (for a review see [29]) to (compact) Riemannian manifolds carrying an action of the $n$-torus $\mathbb{T}^n$. In this paper we will restrict only to the cases of planes and spheres.
For $\lambda^{\mu\nu} = e^{2\pi i \theta_{\mu\nu}}$, where $\theta_{\mu\nu}$ is an anti-symmetric real-valued matrix, the algebra $A(\mathbb{R}_{\theta}^{2n})$ of polynomial functions on the noncommutative $2n$-plane is defined to be the unital $\ast$-algebra generated by $2n$ elements $z^{\mu}$, $\bar{z}^{\mu}$ ($\mu = 1, \ldots, n$) with relations

$$z^{\mu} \bar{z}^{\nu} = \lambda^{\mu\nu} z^{\nu} \bar{z}^{\mu}; \quad \bar{z}^{\mu} \bar{z}^{\nu} = \lambda^{\mu\nu} z^{\nu} \bar{z}^{\mu}; \quad \bar{z}^{\mu} z^{\nu} = \lambda^{\mu\nu} \bar{z}^{\nu} z^{\mu}. \quad (3)$$

The involution $\ast$ is defined by putting $z^{\mu} \ast = \bar{z}^{\mu}$. For $\theta = 0$ one recovers the commutative $\ast$-algebra of complex polynomial functions on $\mathbb{R}^{2n}$.

Let $A(S_{n-1}^{2})$ be the $\ast$-quotient of $A(\mathbb{R}_{\theta}^{2n})$ by the two-sided ideal generated by the central element $\sum_{\mu} z^{\mu} \bar{z}^{\mu} - 1$. We will denote the images of $z^{\mu}$ under the quotient map again by $z^{\mu}$.

A key role in what follows is played by the action of the abelian group $\mathbb{T}^{n}$ on $A(\mathbb{R}_{\theta}^{2n})$ by automorphisms. For $s = (s_{\mu}) \in \mathbb{T}^{n}$, the $\ast$-automorphism $\sigma_{s}$ is defined on the generators by $\sigma_{s}(z^{\mu}) = e^{2\pi i s_{\mu}} z^{\mu}$. Clearly, $s \mapsto \sigma_{s}$ is a group-homomorphism from $\mathbb{T}^{n} \to \text{Aut}(A(\mathbb{R}_{\theta}^{2n}))$. In the special case that $\theta = 0$, we see that $\sigma$ is induced by a smooth action of $\mathbb{T}^{n}$ on the manifold $\mathbb{R}^{2n}$. Since the ideal generating $A(S_{n-1}^{2})$ is invariant under the action of $\mathbb{T}^{n}$, $\sigma$ induces a group-homomorphism from $\mathbb{T}^{n}$ into the group of automorphisms on the quotient $A(S_{n-1}^{2})$ as well.

We continue by defining the unital $\ast$-algebra $A(\mathbb{R}_{\theta}^{2n+1})$ of polynomial functions on the noncommutative $(2n + 1)$-plane which is given by adjoining a central self-adjoint generator $x$ to the algebra $A(\mathbb{R}_{\theta}^{2n})$, i.e. $x^{*} = x$ and $x z^{\mu} = z^{\mu} x$ ($\mu = 1, \ldots, n$). The action of the group $\mathbb{T}^{n}$ is extended trivially by $\sigma_{s}(x) = x$. Let $A(S_{n}^{2})$ be the $\ast$-quotient of $A(\mathbb{R}_{\theta}^{2n+1})$ by the ideal generated by the central element $\sum_{\mu} z^{\mu} \bar{z}^{\mu} + x^{2} - 1$. As before, we will denote the canonical images of $z^{\mu}$ and $x$ again by $z^{\mu}$ and $x$, respectively. Since $\mathbb{T}^{n}$ leaves this ideal invariant, it induces an action by $\ast$-automorphisms on the quotient $A(S_{n}^{2})$.

**Example 1.** For $n = 2$ we obtain the noncommutative sphere $S_{2}^{1}$, which was found in [11]. We adopt the notation used therein and let $A(S_{2}^{1})$ be generated by $a$, $b$ and a central $x$ with $x = x^{*}$ and relations

$$a b = \lambda b a, \quad a b^{*} = \bar{\lambda} b^{*} a, \quad a a^{*} = a^{*} a, \quad b b^{*} = b^{*} b, \quad (4)$$

together with the spherical relation $a a^{*} + b b^{*} + x^{2} = 1$. Here $\lambda = e^{2\pi i \theta}$ with $\theta$ a real number.

We will now construct a differential calculus on $\mathbb{R}_{\theta}^{2n}$. For $m = 2n$, the complex unital associative graded $\ast$-algebra $\Omega(\mathbb{R}_{\theta}^{2n})$ is generated by $2n$ elements $z^{\mu}$, $\bar{z}^{\mu}$ of degree 0 and $2n$ elements $d z^{\mu}$, $d \bar{z}^{\mu}$ of degree 1 with relations:

$$dz^{\mu} d\bar{z}^{\nu} + \lambda^{\mu\nu} dz^{\nu} d\bar{z}^{\mu} = 0; \quad d\bar{z}^{\mu} d\bar{z}^{\nu} + \bar{\lambda}^{\mu\nu} d\bar{z}^{\nu} d\bar{z}^{\mu} = 0; \quad dz^{\mu} d\bar{z}^{\nu} + \lambda^{\mu\nu} dz^{\nu} d\bar{z}^{\mu} = 0; \quad d\bar{z}^{\mu} d\bar{z}^{\nu} + \bar{\lambda}^{\mu\nu} d\bar{z}^{\nu} d\bar{z}^{\mu} = 0; \quad \bar{\lambda} = \lambda^{\ast}. \quad (5)$$

There is a unique differential $d$ on $\Omega(\mathbb{R}_{\theta}^{2n})$ such that $d: z^{\mu} \mapsto dz^{\mu}$. The involution $\omega \mapsto \omega^{*}$ for $\omega \in \Omega(\mathbb{R}_{\theta}^{2n})$ is the graded extension of $z^{\mu} \mapsto \bar{z}^{\mu}$, i.e. it is such that $(d\omega)^{*} = d\omega^{*}$ and $(\omega_{1} \omega_{2})^{*} = (-1)^{m_{1} m_{2}} \omega_{2}^{*} \omega_{1}^{*}$ for $\omega_{i} \in \Omega^{m_{i}}(\mathbb{R}_{\theta}^{2n})$. For $m = 2n + 1$, we adjoin to $\Omega(\mathbb{R}_{\theta}^{2n})$ one generator $x$ of degree 0 and one generator $d x$ of degree 1 such that

$$x d x = d x x; \quad x \omega = \omega x; \quad d x \omega = (-1)^{m_{\omega}} \omega d x. \quad (6)$$
We extend the differential d and the graded involution \(\omega \mapsto \omega^*\) of \(\Omega(\mathbb{R}^{2n})\) to \(\Omega(\mathbb{R}^{2n+1})\) by setting \(x^* = x\) and \((dx)^* = dx\), so that \((dx)^* = dx\).

The differential calculi \(\Omega(S^m_\theta)\) on the noncommutative spheres \(S^m_\theta\) are defined to be the quotients of \(\Omega(\mathbb{R}^{m+1})\) by the differential ideals generated by the central elements \(\sum z^\mu \overline{z}^\nu - 1\) and \(\sum z^\mu \overline{z}^\nu + x^2 - 1\), for \(m = 2n - 1\) and \(m = 2n\) respectively.

The action of \(\mathbb{R}\) by \(*\)-automorphisms on \(\mathcal{A}(M_\theta)\) can be easily extended to the differential calculi \(\Omega(M_\theta)\), for \(M = \mathbb{R}_\theta^m\) and \(M = S^m_\theta\), by imposing \(\sigma \circ d = d \circ \sigma\).

In [10], the so-called splitting homomorphism was introduced. For the cases \(M = \mathbb{R}_\theta^m\) or \(M = S^m_\theta\), this map identifies \(\mathcal{A}(M_\theta)\) with a subalgebra of \(\mathcal{A}(M) \otimes \mathcal{A}(T^m_\theta)\), and this identification allows one to use techniques from commutative differential geometry on \(\mathcal{A}(M)\) and extend it to \(\mathcal{A}(M_\theta)\). Let us recall the definition of the noncommutative \(n\)-torus \(T^n_\theta\). The unital \(*\)-algebra \(\mathcal{A}(T^n_\theta)\) of polynomial functions is generated by \(n\) unitary elements \(U^\mu\) with relations

\[
U^\mu U^\nu = \lambda^{\mu\nu} U^\nu U^\mu, \quad (\mu, \nu = 1, \ldots, n) \tag{7}
\]

with \(\lambda^{\mu\nu} = e^{2\pi i \theta \nu - \theta \mu}\) as before. There is a natural action of \(\mathbb{T}^n\) on \(\mathcal{A}(T^n_\theta)\) by \(*\)-automorphisms given by \(\tau_s(U^\mu) = e^{2\pi i \theta \nu} U^\mu\) with \(s = (s_\mu) \in \mathbb{T}^n\). This allows one to define a diagonal action \(\sigma \times \tau^{-1}\) of \(\mathbb{T}^n\) on \(\mathcal{A}(R^n \times T^n_\theta) := \mathcal{A}(R^n) \otimes \mathcal{A}(T^n_\theta)\) by \(s \mapsto \sigma_s \otimes \tau^{-s}\). That is, \(s \mapsto (\sigma \times \tau^{-1})_s\) is a group-homomorphism of \(\mathbb{T}^n\) into \(\text{Aut} \mathcal{A}(R^n \times T^n_\theta)\).

If \(z^\mu(0)\) denote the classical coordinates of \(R^n\) corresponding to \(z^\mu\) for \(\theta = 0\), one defines the splitting homomorphism on the generators of \(\mathcal{A}(R^{2n}_\theta)\) by

\[
st : \mathcal{A}(R^{2n}_\theta) \rightarrow \mathcal{A}(R^{2n}) \otimes \mathcal{A}(T^n_\theta); \quad z^\mu \mapsto z^\mu(0) \otimes U^\mu. \tag{8}
\]

One checks that \(st\) induces an isomorphism between the algebra \(\mathcal{A}(R^{2n}_\theta)\) and the subalgebra \(\mathcal{A}(R^{2n} \times T^n_\theta)^{\sigma \times \tau^{-1}}\) of \(\mathcal{A}(R^{2n} \times T^n_\theta)\) consisting of fixed points of the previous diagonal action of \(\mathbb{T}^n\). By setting \(st(x) = x(0) \otimes 1\) the splitting homomorphism extends trivially to a map from \(\mathcal{A}(R^{2n+1}_\theta)\) to \(\mathcal{A}(R^{2n+1}) \otimes \mathcal{A}(T^n_\theta)\), giving an algebra isomorphism \(\mathcal{A}(R^{2n+1}) \simeq \mathcal{A}(R^{2n} \times T^n_\theta)^{\sigma \times \tau^{-1}}\).

Furthermore, the map \(st\) will pass to the quotient, for \(m = 2n, 2n + 1\),

\[
st : \mathcal{A}(S^m_\theta) \rightarrow \mathcal{A}(S^m) \otimes \mathcal{A}(T^n_\theta) =: \mathcal{A}(S^m \times T^n_\theta), \tag{9}
\]

giving isomorphisms \(\mathcal{A}(S^m_\theta) \simeq \mathcal{A}(S^m \times T^n_\theta)^{\sigma \times \tau^{-1}}\).

The splitting homomorphism allows one to introduce algebras of smooth functions \(C^\infty(M_\theta)\), for \(M = \mathbb{R}^m\) or \(M = S^n\). They are defined to be the fixed point subalgebras of the diagonal action of \(\mathbb{T}^n\) on \(C^\infty(M) \otimes C^\infty(T^n_\theta)\). Here \(C^\infty(T^n_\theta)\) is the nuclear Fréchet algebra of smooth functions on \(T^n_\theta\) and \(\otimes\) denotes the completion of the tensor product in the projective tensor product topology (see [10] for more details).

The extension of the splitting homomorphism to the differential calculi yields isomorphisms \(\Omega(M_\theta) \simeq (\Omega(M) \otimes \mathcal{A}(T^n_\theta))^{\sigma \otimes \tau^{-1}}\), with \(M\) as above. This allows one to introduce a Hodge star operator on \(\Omega(M_\theta)\). Let \(\ast\) be the Hodge star operator on \(\Omega(M)\) defined with a \(\sigma\)-invariant metric on \(M\). The operator \(\ast \otimes \text{id}\) on \(\Omega(M) \otimes \mathcal{A}(T^n_\theta)\) restricted to the fixed point subalgebra of the diagonal action, defines the Hodge star operator \(\ast_\theta\) on
\( \Omega(M_0) \). Using this operator, one defines a hermitian structure on \( \Omega(M_0) \) in the following way. If \( \omega, \eta \in \Omega^p(M_0) \), then

\[
\langle \omega, \eta \rangle := *_\theta (\bar{\omega} \ast \eta)
\]

(10)
takes values in \( \mathcal{A}(M_0) \) and fulfills all properties of a hermitian structure on \( \Omega^p(M_0) \).

Finally, for the Dirac operator one has the following construction. Suppose for convenience that \( M = S^m \) and equip \( S^m \) with a Riemannian metric such that \( \mathbb{T}^m \) acts isometrically (this is always possible, for instance by averaging). Let \( S \) be a spin bundle over the spin manifold \( M \) and \( D \) the Dirac operator on \( \Gamma^{\infty}(S^m, S) \). The action of the group \( \mathbb{T}^m \) on \( S^m \) does not lift directly to the spinor bundle. Rather, there is a double cover \( \tilde{\mathbb{T}}^m \rightarrow \mathbb{T}^m \) and a group-homomorphism \( \tilde{s} \rightarrow V_\tilde{s} \) of \( \tilde{\mathbb{T}}^m \) into \( \text{Aut}(S) \) covering the action of \( \mathbb{T}^m \) on \( M \):

\[
V_\tilde{s}(f \psi) = \sigma_{\pi(s)}(f) V_\tilde{s}(\psi),
\]

(11)
for \( f \in C^\infty(S^m) \) and \( \psi \in \Gamma^{\infty}(M, S) \). It turns out that the proper notion of smooth sections \( \Gamma^{\infty}(S^m_n, S) \) of a spinor bundle on \( S^m_n \) is given by the subalgebra of \( \Gamma^{\infty}(S^m, S) \otimes C^\infty(\mathbb{T}_{\theta/2}^m) \) made of elements which are invariant under the diagonal action \( \mathbb{V} \times \tilde{\tau}^{-1} \) of \( \tilde{\mathbb{T}}^m \). Here \( \tilde{s} \mapsto \tilde{\tau}_\tilde{s} \) is the canonical action of \( \tilde{\mathbb{T}}^m \) on \( \mathcal{A}(\mathbb{T}_{\theta/2}^m) \). Since the Dirac operator \( D \) will commute with \( V_\tilde{s} \) one can restrict \( D \otimes \text{id} \) to the fixed point subalgebra \( \Gamma^{\infty}(S^m_n, S) \).

Next, let \( L^2(S^m, S) \) be the space of square integrable spinors on \( S^m \) and let \( L^2(\mathbb{T}_{\theta/2}^m) \) be the completion of \( C^\infty(\mathbb{T}_{\theta/2}^m) \) in the norm \( f \mapsto \| f \| = \tau(f^* f)^{1/2} \), with \( \tau \) the usual trace on \( C^\infty(\mathbb{T}_{\theta/2}^m) \). The diagonal action \( \mathbb{V} \times \tilde{\tau}^{-1} \) of \( \tilde{\mathbb{T}}^m \) extends to \( L^2(S^m, S) \otimes L^2(\mathbb{T}_{\theta/2}^m) \) and defines \( L^2(S^m_n, S) \) to be the fixed point Hilbert subspace. If \( D \) also denotes the closure of the Dirac operator on \( L^2(S^m, S) \), we denote the operator \( D \otimes \text{id} \) on \( L^2(S^m, S) \otimes L^2(\mathbb{T}_{\theta/2}^m) \) when restricted to \( L^2(S^m_n, S) \) by \( D \).

The triple \((C^\infty(S^m_n), L^2(S^m_n, S), D)\) satisfies all axioms of a noncommutative spin geometry (there is also a real structure \( J \)). In fact, this construction on \( S^m_n \) can be generalized to any compact Riemannian spin manifold, carrying an isometrical action of \( \mathbb{T}^m \). For more details, we refer to [11, 10].

### 3. Hopf Fibration and Associated Bundles on \( S^4_\theta \)

We will now construct a \( \theta \)-deformation of the Hopf fibration \( SU(2) \rightarrow S^7 \rightarrow S^4 \). For convenience, the classical fibration is described in some detail in App. A. Firstly, we recall that while there is a \( \theta \)-deformation of the manifold \( S^3 \simeq SU(2) \), to a sphere \( S^3_\theta \), on the latter there is no compatible group structure so that there is no \( \theta \)-deformation of the group \( SU(2) \) [10]. Therefore, we must choose the matrix \( \theta'_{\mu \nu} \) in such a way that the non-commutative 7-sphere \( S^3_\theta \) carries a classical \( SU(2) \) action, which in addition is such that the subalgebra of \( \mathcal{A}(S^3_\theta) \) consisting of \( SU(2) \)-invariant polynomials is exactly \( \mathcal{A}(S^3_\theta) \). As expected, we will find that \( \theta' \) is expressed in terms of \( \theta \). Then we construct the finitely generated projective modules \( \Gamma(S^3_\theta, E^{(n)}) \), associated to the irreducible representations \( V^{(n)} \) of \( SU(2) \) as the space of \( SU(2) \)-coequivariant maps from \( V^{(n)} \) to \( \mathcal{A}(S^3_\theta) \). We will construct projections \( p^{(n)} \in \text{Mat}_{p^{(n)}}(\mathcal{A}(S^3_\theta)) \) such that \( \Gamma(S^3_\theta, E^{(n)}) \simeq p^{(n)}(\mathcal{A}(S^3_\theta))^\mathbb{C}^{p^{(n)}} \). In the special case of the defining representation, we recover the basic instanton projection on the sphere \( S^4_\theta \) constructed in [11].
As mentioned, the interplay of the noncommutative spheres $S^7_0$ and $S^7_\mu$ is in that $\mathcal{A}(S^7_\mu)$ will be required to carry an action of $SU(2)$ by automorphisms and this action is such that

$$\mathcal{A}(S^7_\mu) = \text{Inv}_{SU(2)}(\mathcal{A}(S^7_\mu)).$$

(12)

These requirements will restrict the values of $\lambda^{ij} = e^{2\pi i \theta_{ij}}$ is such a manner that there is essentially only 'one' noncommutative 7-sphere such that the invariance condition (12) is satisfied, with a compatible right $SU(2)$ action on $S^7_\mu$. This action on the generators of $\mathcal{A}(S^7_\mu)$ is simply defined by

$$\alpha_w : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \left( \begin{array}{cc} w & 0 \\ 0 & w \end{array} \right), \quad w = \left( \begin{array}{cc} w^1 & w^2 \\ -w^2 & w^1 \end{array} \right).$$

(13)

Here $w^1$ and $w^2$, satisfying $w^1 \overline{w}^1 + w^2 \overline{w}^2 = 1$, are the coordinates on $SU(2)$. By imposing that the map $w \mapsto \alpha_w$ embeds $SU(2)$ in $\text{Aut}(\mathcal{A}(S^7_\mu))$ we find that $\lambda^{12} = \lambda^{34} = 1$ and $\lambda^{14} = \lambda^{23} = \lambda^{24} = \lambda^{13} =: \lambda'$.

In terms of the splitting homomorphism, this means that we can identify $\mathcal{A}(S^7_\mu)$ with a certain subalgebra of $\mathcal{A}(S^7 \times \mathbb{T}^2_\mu)$ instead of $\mathcal{A}(S^7 \times \mathbb{T}^2_\mu)$. In fact, we can write

$$z^1 = z^1_{(0)} \otimes u, \quad z^3 = z^3_{(0)} \otimes v,$$

$$z^2 = z^2_{(0)} \otimes u, \quad z^4 = z^4_{(0)} \otimes v,$$

(14)

for two unitaries $u, v$ satisfying $uv = \lambda' vu$, i.e. the generators of $\mathcal{A}(\mathbb{T}^2_\mu)$.

The subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7_\mu)$ can be found in the following way. By using the splitting homomorphism, a general element $a \in \mathcal{A}(S^7_\mu)$ can be written as a finite sum: $a = \sum a^i_{(0)} \otimes u^i$, where $a^i_{(0)} \in \mathcal{A}(S^7)$ and $u^i \in \mathcal{A}(\mathbb{T}^2_\mu)$. Then, from the diagonal nature of the action of $SU(2)$ on $\mathcal{A}(S^7_\mu)$ and the above formulae for $z^1, \ldots , z^4$ we have that $\alpha_w(a) = \sum \alpha_w(a^i_{(0)}) \otimes u^i$, encoding the fact that $SU(2)$ essentially acts classically. But this means that any invariant polynomial $a = \alpha_w(a)$ induces a classical invariant polynomial $a_{\mu}(a)$. Hence, the subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7_\mu)$ is completely determined by the classical subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7)$. From App. A we can conclude that

$$\text{Inv}_{SU(2)}(\mathcal{A}(S^7_\mu)) = \mathbb{C}[1, z^1 z^3 + z^2 z^4, -z^1 z^4 + z^2 z^3, z^1 z^3 + z^2 z^4]$$

(15)

modulo the relations in the algebra $\mathcal{A}(S^7_\mu)$. We identify

$$\alpha = 2(z^1 z^3 + z^2 z^4), \quad \beta = 2(-z^1 z^4 + z^2 z^3),$$

$$x = z^1 z^4 + z^2 z^3 - z^3 z^4 - z^4 z^3,$$

(16)

and compute that $\alpha \alpha^* + \beta \beta^* + x^2 = 1$. By imposing commutation rules $\alpha \beta = \lambda \beta \alpha$ and $\alpha \beta^* = \lambda^* \beta^* \alpha$, as in Example 1, we infer that $\lambda^{14} = \lambda^{23} = \lambda^{24} = \lambda^{13} = \pm \sqrt{\lambda}$.

On $S^7_\mu$. We conclude that $\text{Inv}_{SU(2)}(\mathcal{A}(S^7_\mu)) = \mathcal{A}(S^4_\mu)$ for $\lambda^{ij} = e^{2\pi i \theta_{ij}}$ of the following form:

$$\lambda_{ij} = \left( \begin{array}{cc} 1 & \mu \\ \overline{\mu} & 1 \end{array} \right), \quad \mu = \sqrt{\lambda},$$

(17)
or equivalently
\[
\theta_{ij}' = \theta \left( \begin{array}{cccc}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
-1 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
\end{array} \right).
\] (18)

There is a nice description of the instanton projection constructed in [11] in terms of ket-valued polynomials on \( S_7^\theta \). The latter are elements in the right \( A(S_7^\theta) \)-module \( E := \mathbb{C}^4 \otimes A(S_7^\theta) =: A(S_7^\theta)^4 \), with a hermitian structure given by \( \langle \xi, \eta \rangle = \sum_j \xi_j^* \eta_j \). To any \( \xi \in E \) one associates its dual \( \langle \xi \rangle \in E^* \) by setting \( \langle \xi \rangle(\eta) = \langle \xi, \eta \rangle \), for all \( \eta \in E \).

Similarly to the classical case (see App. A), we define \( |\psi_1\rangle, |\psi_2\rangle \in A(S_7^\theta)^4 \) by
\[
|\psi_1\rangle = (z^1, -\bar{z}^2, z^3, -\bar{z}^4)^t, \quad |\psi_2\rangle = (\bar{z}^2, z^1, z^4, \bar{z}^3)^t.
\] (19)

with \( t \) denoting transposition. They satisfy \( \langle \psi_k | \psi_l \rangle = \delta_{kl} \), so that the \( 4 \times 4 \)-matrix \( p = |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2| \) is a projection, \( p^2 = p = p^* \), with entries in \( A(S_4^\theta) \). Indeed, let us introduce the matrix
\[
u = (|\psi_1\rangle, |\psi_2\rangle) = \begin{pmatrix}
z^1 & \bar{z}^2 \\
\bar{z}^3 & \bar{z}^4 \\
z^3 & z^4 \\
\bar{z}^1 & \bar{z}^3
\end{pmatrix}.
\] (20)

Then \( u^* u = 1_2 \) and \( p = uu^* \). The action (13) becomes
\[
\alpha_w(u) = uw,
\] (21)

from which the invariance of the entries of \( p \) follows at once. Explicitly one finds
\[
p = \frac{1}{2} \begin{pmatrix}
1 + x & 0 & \alpha & \beta \\
0 & 1 + x - \mu \beta^* & \mu \alpha^* \\
\alpha^* & -\mu \beta & 1 - x & 0 \\
\beta^* & \mu \alpha & 0 & 1 - x
\end{pmatrix}.
\] (22)

The projection \( p \) is easily seen to be equivalent to the projection describing the instanton on \( S_4^\theta \) constructed in [11]. Indeed, if one defines
\[
|\tilde{\psi}_1\rangle = (z^1, -\mu \bar{z}^2, z^3, -\bar{z}^4)^t, \quad |\tilde{\psi}_2\rangle = (\bar{z}^2, \mu z^1, z^4, \bar{z}^3)^t,
\] (23)

one obtains exactly the projection obtained therein, that is,
\[
\tilde{p} = \frac{1}{2} \begin{pmatrix}
1 + x & 0 & \alpha & \beta \\
0 & 1 + x - \lambda \beta^* & \lambda \alpha^* \\
\alpha^* & -\lambda \beta & 1 - x & 0 \\
\beta^* & \lambda \alpha & 0 & 1 - x
\end{pmatrix}.
\] (24)

We will denote the image of \( p \) in \( A(S_4^\theta)^4 \) by \( \Gamma(S_4^\theta, E) = p A(S_4^\theta)^4 \) which is clearly a right \( A(S_4^\theta) \)-module. Another description of the module \( \Gamma(S_4^\theta, E) \) comes from considering coequivariant maps from \( \mathbb{C}^2 \) to \( A(S_4^\theta) \) [16]. The defining left representation
of $SU(2)$ on $\mathbb{C}^2$ is given by $SU(2) \times \mathbb{C}^2 \to \mathbb{C}^2; (w, v) \mapsto w \cdot v$. The collection $\text{Hom}_{SU(2)}(\mathbb{C}^2, A(S_0'))$ of coequivariant maps, i.e. of maps $\phi : \mathbb{C}^2 \to A(S_0')$, such that

$$\phi(w^{-1} \cdot v) = \alpha_w(\phi(v)), \quad (25)$$

is a right $A(S_0')$-module (it is in fact also a left $A(S_0')$-module).

Since $SU(2)$ acts classically on $A(S_0')$, one sees that the coequivariant maps are given on the canonical basis $|e_1, e_2\rangle$ of $\mathbb{C}^2$ by $\phi(e_k) = \langle \psi_k | f \rangle$ for $|f\rangle = |f_1, f_2, f_3, f_4\rangle$, with $f_i \in A(S_0')$ (cf. App. A). We then have the following isomorphism

$$\Gamma(S_0', E) \cong \text{Hom}_{SU(2)}(\mathbb{C}^2, A(S_0')), \quad \sigma = p|f\rangle \leftrightarrow \phi : e_k \mapsto \langle \psi_k | f \rangle. \quad (26)$$

More generally, one can define the right $A(S_0')$-module $\Gamma(S_0', E^{(n)})$ associated with any irreducible representation $\rho_n : SU(2) \to GL(V^{(n)})$, with $V^{(n)} = \text{Sym}^n(\mathbb{C}^2)$, for a positive integer $n$. The module of coequivariant maps $\text{Hom}_{\rho_n}(V^{(n)}, A(S_0'))$ consists of maps $\phi : V^{(n)} \to A(S_0')$ satisfying

$$\phi(\rho_n^{-1}(w) \cdot v) = \alpha_w(\phi(v)). \quad (27)$$

It is easy to see that these maps are of the form $\phi_{(n)}(e_k) = \langle \phi_k | f \rangle$ on the basis $|e_1, \ldots, e_{n+1}\rangle$ of $V^{(n)}$, where now $|f\rangle \in A(S_0')^{(n)}$ and

$$|\phi_k\rangle = \frac{1}{a_k} |\psi_1\rangle^{\otimes(n-k+1)} \otimes |\psi_2\rangle^{\otimes(k-1)} \quad (k = 1, \ldots, n+1), \quad (28)$$

with $\otimes_S$ denoting symmetrization and $a_k$ are suitable normalization constants. These vectors $|\phi_k\rangle \in \mathbb{C}^{4^n} \otimes A(S_0') =: A(S_0')^{4^n}$ are orthogonal (with the natural hermitian structure), and with $a_k^2 = \binom{n}{k-1}$ they are also normalized. Then

$$p_{(n)} := |\phi_1\rangle \langle \phi_1| + |\phi_2\rangle \langle \phi_2| + \cdots + |\phi_{n+1}\rangle \langle \phi_{n+1}| \in \text{Mat}_{4^n}(A(S_0')) \quad (29)$$

defines a projection $p^2 = p = p^*$. That its entries are in $A(S_0')$ and not in $A(S_0')^{4^n}$ is easily seen. Indeed, much as it happens for the vector $u$ in (21), for every $i = 1, \ldots, 4^n$, the vector $u_{(i)} = (|\phi_1\rangle_i, |\phi_2\rangle_i, \ldots, |\phi_{n+1}\rangle_i)$ transforms under the action of $SU(2)$ to the vector $(|\phi_1\rangle_i, \ldots, |\phi_{n+1}\rangle_i) \cdot \rho_n(\nu)$ so that each entry $\sum_k |\phi_k\rangle_i \langle \phi_k|_j$ of $p_{(n)}$ is $SU(2)$-invariant and hence an element in $A(S_0')$. With this we proved the following.

**Proposition 2.** The module of coequivariant maps $\text{Hom}_{\rho_n}(V^{(n)}, A(S_0'))$ is isomorphic to $\Gamma(S_0', E^{(n)}) := p_{(n)}(A(S_0')^{4^n})$ (as right-$A(S_0')$ modules) with the isomorphism given explicitly by:

$$\Gamma(S_0', E^{(n)}) \cong \text{Hom}_{\rho_n}(V^{(n)}, A(S_0')), \quad \sigma_{(n)} = p_{(n)}|f\rangle \equiv \phi_{(n)} : e_k \mapsto \langle \phi_k | f \rangle.$$
Using the splitting homomorphisms of the previous section, one can lift this whole construction to the smooth level. One proves that the $C^\infty(S^4_\theta)$-module $\Gamma^\infty(S^4_\theta, E^{(n)})$ defined by $p_{(n)}(C^\infty(S^4_\theta))^{d\theta}$ is isomorphic to $\text{Hom}_{\mathbb{C}_\theta}(V^{(n)}, C^\infty(S^7_\theta))$.

With the projections $p_{(n)}$, one associates (Grassmannian) connections on the modules $\Gamma(S^4_\theta, E^{(n)})$ in a canonical way:

$$\nabla = p_{(n)} \circ d : \Gamma(S^4_\theta, E^{(n)}) \to \Gamma(S^4_\theta, E^{(n)} \otimes_{A(S^4_\theta)} \Omega^1(S^4_\theta)), \tag{30}$$

where $(\Omega^*(S^4_\theta), d)$ is the differential calculus defined in the previous section. An expression for these connections as acting on coequivariant maps can be obtained using the above isomorphism and results in:

$$\nabla(\phi)(e_k) = d(\phi(e_k)) + A_{kl} \phi(e_l), \tag{31}$$

where $A_{kl} = (\delta_{kl}|d\phi_l) \in \Omega^1(S^4_\theta)$. The corresponding matrix $A$ is called the connection one-form; it is clearly anti-hermitian, and it is valued in the derived representation space, $\rho_n : su(2) \to \text{End}(V^{(n)})$, of the Lie algebra $su(2)$.

The case $n = 1$ describes classically (i.e. $\theta = 0$) the charge $-1$ instanton [2]. An instanton is defined as a connection on $\Gamma(S^4, E)$ with (anti-)selfdual curvature $F$, i.e. $*F = \pm F$ with $*$ the Hodge star operator. In physics, instantons are of importance since they are extrema of the Yang-Mills action. The equation of motion obtained from this action by a variational method is called the Yang-Mills equation $[\nabla, F = 0]$. In the case that $F$ is (anti-)selfdual, this equation of motion follows directly from the Bianchi identity $[\nabla, F] = 0$. There is no need to stress the huge importance of instantons (and in general Yang-Mills gauge theory) both in physics and in mathematics.

On the noncommutative sphere $S^4_{\theta}$, the curvature of the connection $p \circ d$ constructed above satisfies the following anti-selfdual equation [10] (see also [1, 25])$^1$:

$$*_{\theta} p(dp)^2 = -p(dp)^2. \tag{32}$$

In order to fully justify the name instanton, one should find a noncommutative analogue of the Yang-Mills action such that connections with an (anti-)selfdual curvature are its extrema. This will be discussed elsewhere [26].

4. Index of Dirac Operators

We know from Sect. 2 that there is a structure of noncommutative spin geometry on the sphere $S^4_{\theta}$ given by a ‘triple’ $(C^\infty(S^4_{\theta}), L^2(S^4_{\theta}, S), D, \gamma)$ with $\gamma = \gamma_{\theta}$ the grading. In this section we shall compute explicitly the index of the Dirac operator with coefficients in the bundles $E^{(n)}$, that is the index of the operator of $D_{p_{(n)}} := p_{(n)}(D \otimes I_{\gamma}) p_{(n)}$. In order to do that, we will use the (‘even dimensional’ version of the) local index formula of Connes and Moscovici [12] which we shall briefly describe.

Suppose in general that $(A, H, D, \gamma)$ is an even $p$-summable spectral triple with discrete simple dimension spectrum. Let $C_s(A)$ be the complex consisting of cycles over the algebra $A$, that is in degree $n$, $C_s(A) := A^{\otimes (n+1)}$. On this complex there are defined the Hochschild operator $b : C_n(A) \to C_{n-1}(A)$ and the boundary operator $B : C_n(A) \to C_{n+1}(A)$, satisfying $b^2 = 0$, $B^2 = 0$, $bB + Bb = 0$; thus $(b + B)^2 = 0$. From general homological theory, one defines a bicomplex $CC_s(A)$ by $CC_{(s,m)}(A) :=$

---

$^1$ An early attempt to write self-duality equations in terms of projections was done in [15].
Theorem 3 (Connes–Moscovici [12]).

(a) An even cocycle \( \phi^* = \sum_{k \geq 0} \phi^k \) in \( CC^*(\mathcal{A}) \), \( (b + B)\phi^* = 0 \), is defined by the following formula. For \( k = 0 \),
\[
\phi^0(a) := \text{Res}_{z=0} \frac{1}{z} \text{Tr}(\gamma a[D]^{-2z});
\]

whereas for \( k \neq 0 \),
\[
\phi^k(a_0, \ldots, a^{2k}) := \sum_a c_{k,a} \\
\times \text{Res}_{z=0} \text{Tr}(\gamma a^0[D, a^1]^{|a_1|} \cdots [D, a^{2k}]^{|a_{2k}|} |D|^{-2(|a|+k+z)}),
\]

where
\[
c_{k,a} = (-1)^{|a|} \Gamma(k + |a|)(a!(a_1 + 1)(a_1 + a_2 + 2) \cdots (a_1 + \cdots + a_{2k} + 2k))^{-1}
\]

and \( T^{(j)} \) denotes the \( j \)-th iteration of the derivation \( T \mapsto [D^2, T] \).

(b) For \( e \in K_0(\mathcal{A}) \), the Chern character \( \text{ch}_e(e) = \sum_{k \geq 0} \text{ch}_k(e) \) is the even cycle in \( CC^*(\mathcal{A}) \), \( (b + B)\text{ch}_e(e) = 0 \), defined by the following formula. For \( k = 0 \),
\[
\text{ch}_0(e) := \text{Tr}(e);
\]

whereas for \( k \neq 0 \)
\[
\text{ch}_k(e) := (-1)^k \frac{(2k)!}{k!} \sum_{\alpha} (\epsilon_{ii1i2} - \frac{1}{2} \delta_{ii1i2}) \otimes \epsilon_{ii2i3} \otimes \cdots \otimes \epsilon_{i2k0}. \]

(c) The index is given by the natural pairing between cycles and cocycles
\[
\text{Ind} \ D_e = \langle \phi^*, \text{ch}_e(e) \rangle.
\]

We concentrate on a compact Riemannian spin manifold \( M \) of even dimension carrying an isometric action of an \( n \)-torus. Set \( \mathcal{H} := L^2(M, S) \) and recall the grading on \( B(\mathcal{H}) \) with respect to the action of \( \mathbb{T}^n \) [11]. An element \( T \in B(\mathcal{H}) \) that is smooth for the action of \( \mathbb{T}^n \), i.e. such that the map \( \mathbb{T}^n \ni s \mapsto \alpha_s(T) \) with \( \alpha_s \), defined by \( \alpha_s(T) := U(s)TU(s)^{-1} \) is smooth for the norm topology, can be expanded as \( T = \sum T_r \) with \( r = (r_1, r_2, \ldots, r_n) \) a multi-index, and with each \( T_r \) of homogeneous degree \( r \) under the action of \( \mathbb{T}^n \), i.e.
\[
\alpha_s(T_r) = e^{2\pi i \sum_{\mu=1}^{\infty} r_\mu s_\mu} T_r \quad (s \in \mathbb{T}^n).
\]

Note that \( \alpha_s \) coincides on \( \pi(C^\infty(M)) \subset B(\mathcal{H}) \) with the automorphism \( \sigma_s \) defined in Sect. 2. Then, let \( (p_1, p_2, \ldots, p_n) \) be the infinitesimal generators of the action of \( \mathbb{T}^n \) so that \( U(s) = \exp 2\pi i \sum_{\mu=1}^{\infty} s_\mu p_\mu \). For \( T \in B(\mathcal{H}) \) we define a twisted representation on \( \mathcal{H} \) by
\[
L_\theta(T) := \sum_r T_r U(r_\mu \theta_{\mu 1}, \ldots, r_\mu \theta_{\mu n}) = \sum_r T_r \exp \left[ 2\pi i \sum_{\mu} r_\mu \theta_{\mu v} p_v \right] \]
with $\theta$ an $n \times n$ anti-symmetric matrix. Since $\mathbb{T}^n$ acts by isometries, each $p_\mu$ commutes with $D$ so that the latter is of degree 0 and $L_\theta([D, a]) = [D, L_\theta(a)]$ for $a \in \mathcal{C}^\infty(M)$. It was shown in [11] (to which we refer for more details) that $(L_\theta(\mathcal{C}^\infty(M)), \mathcal{H}, D)$ satisfies all axioms of a noncommutative spin geometry (there is also a grading $\gamma = \gamma_5$ and a real structure $J$). In fact, the algebra $L_\theta(\mathcal{C}^\infty(M))$ is isomorphic to the algebra $\mathcal{C}^\infty(M_0)$ of the previous section.

As a next step, we write the $\phi^{2k}$ that define the local index formula in (34), by means of the twist $L_\theta$. Let $f^0, \ldots, f^{2k} \in \mathcal{C}^\infty(M)$ and suppose that the operator $f^0[D, f^1] \cdots [D, f^{2k}]$ is a homogeneous element of degree $r$. Then, as in (38)

$$L_\theta(f^0[D, f^1] \cdots [D, f^{2k}]) = f^0[D, f^1] \cdots [D, f^{2k}] U(r_\mu \theta_{\mu 1}, \ldots, r_\mu \theta_{\mu n}).$$

Each term in the local index formula for $(L_\theta(\mathcal{C}^\infty(M)), \mathcal{H}, D)$ then takes the form

$$\text{Res} \text{Tr}(\gamma f^0[D, f^1]^{(a_1)} \cdots [D, f^{2k}]^{(a_{2k})} D^{-2(|a| + k + z)} U(s))$$

for $s \mu = r_\mu \theta_{\mu 1}$, so that $s \in \mathbb{T}$. The appearance of $U(s)$ here, is a consequence of the close relation with the index formula for a $\mathbb{T}_n$-equivariant Dirac spectral triple on $M$. In [8], Chern and Hu considered an even dimensional compact spin manifold $M$ on which a (connected compact) Lie group $G$ acts by isometries. The equivariant Chern character was defined as an equivariant version of the JLO-cocycle, the latter being an element in equivariant entire cyclic cohomology. The essential point is that they obtained an explicit formula for the above residues. In the case of the previous $\mathbb{T}_n$-action on $M$, one gets

$$\text{Res} \text{Tr}(\gamma f^0[D, f^1]^{(a_1)} \cdots [D, f^{2k}]^{(a_{2k})} D^{-2(|a| + k + z)} U(s))$$

$$= \Gamma(|a| + k) \lim_{s \to 0} s^{a_0} \text{Tr}(\gamma f^0[D, f^1]^{(a_1)} \cdots [D, f^{2k}]^{(a_{2k})} / s - D^2 U(s))$$

for every $s \in \mathbb{T}$; moreover, this limit vanishes when $|a| \neq 0$ (Thm 2 in [8]). Combining these results, we arrive at the following lemma.

**Lemma 4.** Let $(L_\theta(\mathcal{C}^\infty(M)), \mathcal{H}, D)$ be the spectral triple defined above. Then all terms in $\phi^k$ with $|a| \neq 0$ vanish and the local index formula takes the form:

$$\phi^{2k}(a^0, \ldots, a^{2k}) = c_k \text{Res} \text{Tr}(\gamma a^0[D, a^1] \cdots [D, a^{2k}] D^{-2(k + z)})$$

where $c_k = (k - 1)!/(2k)!$.

In our case of interest, the index of the Dirac operator on $S_\theta^4$ with coefficients in some noncommutative vector bundle determined by $e \in K_0(C(S^4_\theta))$, we obtain

$$\text{Ind} D_e = \langle \phi^*, \text{ch}_e(e) \rangle = \text{Res} \text{Tr}(\gamma \pi_D(\text{ch}_0(e)) D^{-2z})$$

$$+ \frac{1}{2i} \text{Res} \text{Tr}(\gamma \pi_D(\text{ch}_1(e)) D^{-2z - 2})$$

$$+ \frac{1}{4i} \text{Res} \text{Tr}(\gamma \pi_D(\text{ch}_2(e)) D^{-4z - 2}).$$

Here $\pi_D$ is the representation of the universal differential calculus given by

$$\pi_D : \Omega^a_m(A(S^4_\theta)) \to \mathcal{B}(\mathcal{H}), \quad a^0 \delta a^1 \cdots \delta a^p \mapsto a^0[D, a^1] \cdots [D, a^p].$$
Let us examine at which quotients of $\Omega_{un}(A(S^4_\theta))$ this representation $\pi_D$ is well-defined. Unfortunately, $\pi_D$ is not well-defined on the quotient $\Omega(S^4_\theta)$ defined in the previous section. For example already $[D, \alpha][D, \alpha] \neq 0$ whereas $d\alpha d\alpha = 0$ in $\Omega(S^4_\theta)$. This was already noted in [10] and in fact
\[ \Omega(S^4_\theta) \simeq \pi_D(\Omega_{un}(A(S^4_\theta)))/\pi_D(\delta J_0), \]
(44)
where $J_0 := \{ \omega \in \Omega_{un}(A(S^4_\theta))|\pi_D(\omega) = 0 \}$ are the so-called ‘junk-forms’ [9]. We will avoid a discussion on junk-forms and introduce instead a different quotient of $\Omega_{un}(A(S^4_\theta))$. We define $\Omega_D(S^4_\theta)$ to be $\Omega_{un}(A(S^4_\theta))$ modulo the relations
\[ \alpha\delta\beta - \lambda(\delta\beta)\alpha = 0, \quad (\delta\alpha)\beta - \lambda\beta\delta\alpha = 0, \]
\[ a\delta x - (\delta x)a = 0, \quad \forall a \in A(S^4_\theta), \]
(45)
avoiding the second order relations that define $\Omega(S^4_\theta)$. Using the splitting homomorphism one proves that the above relations are in the kernel of $\pi_D$, for instance, $\alpha[D, \beta] - \lambda[D, \beta]\alpha = 0$ so that $\pi_D$ is well-defined on $\Omega_D(S^4_\theta)$.

In App. B we compute the Chern characters as elements in $\Omega_D(S^4_\theta)$, which results in the following lemma.

**Lemma 5.** The following formulæ hold for the images under $\pi_D$ of the Chern characters of $p(n)$:
\[ \pi_D(ch_0(p(n))) = n + 1; \]
\[ \pi_D(ch_1(p(n))) = 0; \]
\[ \pi_D(ch_2(p(n))) = \frac{1}{6}n(n + 1)(n + 2)\pi_D(ch_2(p(1)))); \]
up to the coefficients $\mu_k = (-1)^k \frac{(2k)!}{k!}$. □

Combining this with the simple form of the index formula while taking the proper coefficients, we find that
\[ \text{Ind } D_{p(n)} = \frac{1}{4!} \frac{1}{27} n(n + 1)(n + 2) \text{Res } \sum_{z=0} \text{Tr}(\gamma \pi_D(ch_2(p(1))))|D|^{-4-2z}. \]
(46)
where for the vanishing of the first term, we used the fact that $\text{Ind } D = 0$, since the first Pontrjagin class on $S^4$ vanishes. Theorem I.2 in [12] allows one to express the residue as a Dixmier trace. Combining this with $\pi_D(ch_2(p(1))) = 3\gamma$ (as computed in [11]), we obtain
\[ 3 \cdot \text{Res } \sum_{z=0} \text{Tr}(|D|^{-4-2z}) = 6 \cdot \text{Tr}_m(|D|^{-4}) = 2 \]
since the Dixmier trace of $|D|^{-m}$ on the $m$-sphere equals $8/m!$ (cf. for instance [17, 23]). This combines to give:

**Proposition 6.** The index of the Dirac operator on $S^4_\theta$ with coefficients in $E^{(n)}$ is given by:
\[ \text{Ind } D_{p(n)} = \frac{1}{6}n(n + 1)(n + 2). \]
□

Note that this coincides with the classical result.
5. The Noncommutative Principal Bundle

In this section, we apply the general theory of Hopf-Galois extensions [21, 28] to the inclusion \( \mathcal{A}(S^0_\theta) \hookrightarrow \mathcal{A}(S^0_p) \). Such extensions can be understood as noncommutative principal bundles. We will first dualize the construction of the previous section, i.e. replace the action of \( SU(2) \) on \( \mathcal{A}(S^0_\theta) \) by a coaction of \( \mathcal{A}(SU(2)) \). Then, we will recall some definitions involving Hopf-Galois extensions and principality ([16]) of such extensions. We show that \( \mathcal{A}(S^0_\theta) \hookrightarrow \mathcal{A}(S^0_p) \) is a not-cleft (i.e. not-trivial) principal Hopf-Galois extension and compare the connections on the associated bundles, induced from the strong connection, with the Grassmannian connection defined in Sect. 3.

The action of \( SU(2) \) on \( \mathcal{A}(S^0_\theta) \) by automorphisms can be easily dualized to a coaction \( \Delta_R : \mathcal{A}(S^0_p) \to \mathcal{A}(S^0_p) \otimes \mathcal{A}(SU(2)) \), where now \( \mathcal{A}(SU(2)) \) is the unital complex \( \ast \)-algebra generated by \( w^1, w^2, w^3 \) with relation \( w^1 w^2 + w^2 w^1 = 1 \). Clearly, \( \mathcal{A}(SU(2)) \) is a Hopf algebra with comultiplication

\[
\Delta : \begin{pmatrix} w^1 & w^2 \\ -w^2 & w^1 \end{pmatrix} \mapsto \begin{pmatrix} w^1 & w^2 \\ -w^2 & w^1 \end{pmatrix} \otimes \begin{pmatrix} w^1 & w^2 \\ -w^2 & w^1 \end{pmatrix},
\]

and antipode \( S(w^1) = w^2 \), \( S(w^2) = -w^1 \) and counit \( \epsilon(w^1) = \epsilon(w^2) = 1, \epsilon(w^2) = 0 \). The coaction of \( \mathcal{A}(SU(2)) \) on \( \mathcal{A}(S^0_p) \) is given by

\[
\Delta_R : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \otimes \begin{pmatrix} u^1 & 0 & 0 & 0 \\ 0 & u^2 & 0 & 0 \\ 0 & 0 & w^1 & w^2 \\ 0 & 0 & -w^2 & w^1 \end{pmatrix}.
\]

The algebra of coinvariants in \( \mathcal{A}(S^0_p) \), which consists of elements \( p \in \mathcal{A}(S^0_p) \) satisfying \( \Delta_R(p) = p \otimes 1 \), can be identified with \( \mathcal{A}(S^0_\theta) \) for the particular values of \( \theta_{ij} \) found before, in the same way as in Sect. 3.

The associated modules \( \Gamma(S^0_p, E^{(n)}) \) are described in the following way. Given an irreducible corepresentation of \( \mathcal{A}(SU(2)) \), \( \rho^{(n)} : V^{(n)} \to \mathcal{A}(SU(2)) \otimes V^{(n)} \) with \( V^{(n)} = \text{Sym}^n(C^2) \), we denote \( \rho^{(n)}(v) = v(0) \otimes v(1) \). Then, the module of coequivariant maps \( \text{Hom}^{\rho^{(n)}}(V^{(n)}, \mathcal{A}(S^0_p)) \) consists of maps \( \phi : V^{(n)} \to \mathcal{A}(S^0_p) \) satisfying

\[
\phi(v(1)) \otimes Sv(0) = \Delta_R \phi(v); \quad v \in C^2.
\]

Again, such maps are \( C \)-linear maps of the form \( \phi_{(e_k)}(e_k) = \langle \phi_k | f \rangle \) on the basis \( \{e_1, \ldots, e_{n+1}\} \) of \( V^{(n)} \) in the notation of the previous section. Also, Proposition 2 above translates straightforwardly into the isomorphism \( \text{Hom}^{\rho^{(n)}}(V^{(n)}, \mathcal{A}(S^0_p)) \cong p^{(n)}(\mathcal{A}(S^0_p))^{4^n} \) for the projections defined in Eq. (29).

Before we proceed, recall that for an algebra \( P \) and a subalgebra \( B \subset P \), \( P \otimes_B P \) denotes the quotient of the tensor product \( P \otimes P \) by the ideal generated by expressions \( p \otimes bp' - pb \otimes p' \), for \( p, p' \in P, b \in B \).

**Definition 7.** Let \( H \) be a Hopf algebra and \( P \) a right \( H \)-comodule algebra, i.e. such that the coaction \( \Delta_R : P \to P \otimes H \) is an algebra map. Let the algebra of coinvariants be \( B := \text{Coinv}_{\Delta_R}(P) := \{ p \in P : \Delta_R(p) = p \otimes 1 \} \). One says that \( B \hookrightarrow P \) is a **Hopf-Galois extension** if the canonical map

\[
\chi : P \otimes_B P \to P \otimes H; \quad p \otimes_B p' \mapsto p' \Delta_R(p) = p' p(0) \otimes p(1)
\]

is bijective.
We use Sweedler-like notation for the coaction: \( \Delta_R(p) = p(0) \otimes p(1) \). The canonical map is left \( P \)-linear and right \( H \)-colinear and is a morphism (an isomorphism for Hopf-Galois extensions) of left \( P \)-modules and right \( H \)-comodules. It is also clear that \( P \) is both a left and a right \( B \)-module.

Classically, the notion of Hopf-Galois extension corresponds to freeness of the action of a Lie group \( G \) on a manifold \( P \). Indeed, freeness can be translated into bijectivity of the map

\[
\tilde \chi : P \times G \to (p, g) \mapsto (p, p \cdot g).
\] (51)

where \( P \times_G P \) denotes the fibre direct product consisting of elements \((p, p')\) with the same image under the quotient map \( P \to P/G \).

For a Hopf algebra \( H \) which is cosemisimple, surjectivity of the canonical map (50) implies its bijectivity [31]. Moreover, in order to prove surjectivity of \( \chi \), it is enough to prove that for any generator \( h \) of \( H \), the element \( 1 \otimes h \) is in the image of the canonical map. Indeed, if \( \chi(g_k \otimes_B g'_k) = 1 \otimes g \) and \( \chi(h_l \otimes_B h'_l) = 1 \otimes h \) for \( g, h \in H \), then \( \chi(g_k h_l \otimes_B g'_k h'_l) = g_k h_l \chi(1 \otimes_B h'_l g'_l) = 1 \otimes h g \), using the fact that the canonical map restricted to \( 1 \otimes_B P \) is a homomorphism. Extension to all of \( P \otimes_B P \) then follows from left \( P \)-linearity of \( \chi \). It would also be easy to write down an explicit expression for the inverse of the canonical map. Indeed, one has \( \chi^{-1}(1 \otimes h g) = g_k h_l \otimes_B h'_l g'_l \) in the above notation so that the general form of the inverse follows again from left \( P \)-linearity.

**Proposition 8.** The inclusion \( \mathcal{A}(S^4_B) \hookrightarrow \mathcal{A}(S^7_B) \) is a Hopf-Galois extension.

**Proof.** Since \( \mathcal{A}(SU(2)) \) is cosemisimple, we can rely for a proof of this statement on the previous remarks. On the other hand, it is straightforward to check that in terms of the ket-valued polynomials defined in (19) we have

\[
\chi \left( \sum_i (|\psi_1_i \rangle \otimes_{\mathcal{A}(S^4_B)} |\psi_1_i \rangle) \right) = 1 \otimes w^1; \quad \chi \left( \sum_i (|\psi_2_i \rangle \otimes_{\mathcal{A}(S^4_B)} |\psi_2_i \rangle) \right) = 1 \otimes w^2;
\]

\[
\chi \left( \sum_i (|\psi_2_i \rangle \otimes_{\mathcal{A}(S^7_B)} |\psi_1_i \rangle) \right) = -1 \otimes w^2; \quad \chi \left( \sum_i (|\psi_2_i \rangle \otimes_{\mathcal{A}(S^7_B)} |\psi_2_i \rangle) \right) = 1 \otimes w^1. \quad \square
\]

In the definition of a principal bundle in differential geometry there is much more than the requirement of bijectivity of the canonical map. It turns out that our ‘structure group’ being \( H = \mathcal{A}(SU(2)) \) which, besides being cosemisimple has also bijective antipode, all additional desired properties follow from the surjectivity of the canonical map which we have just established. We refer to [30, 6] for the full fledged theory while giving only the basic definitions that we shall need.

For our purposes, a better algebraic translation of the notion of a principal bundle is encoded in the requirement that the extension \( B \subset P \), besides being Hopf-Galois, is also faithfully flat. We recall [22] that a right module \( P \) over a ring \( R \) is said to be **faithfully flat** if the functor \( P \otimes_R - \) is exact and faithful on the category \( \mathcal{M} \) of left \( R \)-modules. Flatness means that the functor associates exact sequences of abelian groups to exact sequences of \( R \)-modules and the functor is faithful if it is injective on morphisms. Equivalently one could state that a right module \( P \) over a ring \( R \) is faithfully flat if a sequence \( M' \to M \to M'' \) in \( \mathcal{M} \) is exact if and only if \( P \otimes_R M' \to P \otimes_R M \to P \otimes_R M'' \) is exact.
As mentioned, from the fact that $H = A(SU(2))$ is both cosemisimple and has also bijective antipode, the faithful flatness of $A(S^7_\theta)$ as a right (as well as left) $A(S^4_\theta)$-module follows from the surjectivity of the canonical map ([31], Th. I).

One says that a principal Hopf-Galois extension is **cleft** if there exists a (unital) convolution-invertible colinear map $\phi : H \to P$, called a **cleaving map** [13, 30]. Classically, this notion is close (although not equivalent) to triviality of a principal bundle [14]. In [7] (cf. [19]) it is shown that if a principal Hopf-Galois extension is cleft, its associated modules are trivial, i.e. isomorphic to the free module $B^N$ for some $N$. In our case, we can conclude the following.

**Proposition 9.** The Hopf-Galois extension $A(S^4_\theta) \hookrightarrow A(S^7_\theta)$ is not cleft.

**Proof.** This is a simple consequence of the nontriviality of the Chern characters of the projection $p(n)$ as seen in Sect. 4. Indeed, this implies that the associated modules are nontrivial. $\square$

Summing up what we have shown up to now, we have the following

**Theorem 10.** The inclusion $A(S^4_\theta) \hookrightarrow A(S^7_\theta)$ is a not-cleft faithfully flat $A(SU(2))$-Hopf-Galois extension.

An important consequence is the existence of a so-called **strong connection** [18, 13]. In fact, the existence of such a connection could be used to give a more intuitive definition of ‘principality of an extension’[6]. Let us first recall that if $H$ is cosemisimple and has a bijective antipode, then a $H$-Hopf-Galois extension $B \hookrightarrow P$ is **equivariantly projective**, that is, there exists a left $B$-linear right $H$-colinear splitting $s : P \to B \otimes P$ of the multiplication map $m : B \otimes P \to P$, $m \circ s = id_P$ [30]. Such a map characterizes a strong connection.

**Definition 11.** Let $B \hookrightarrow P$ be a $H$-Hopf-Galois extension. A **strong connection one-form** is a map $\omega : P \to \Omega^1_{um}P$ satisfying

1. $\bar{\gamma} \circ \omega = 1 \otimes (id - \epsilon)$, (fundamental vector field condition),
2. $\Delta_{\Omega^1_{um}}(p) \circ \omega = (\omega \otimes id) \circ \text{Ad}_R$, (right adjoint colinearity),
3. $\delta p - p(0)\omega(p(1)) \in (\Omega^1_{um} B)P$, $\forall p \in P$, (strongness condition).

Here $\Delta_R : P \to P \otimes H$, $\Delta_R(p) = p(0) \otimes p(1)$, is extended to $\Delta_{\Omega^1_{um}}(p)$ on $\Omega^1_{um}P \subset P \otimes P$ in a natural way by

$$\Delta_{\Omega^1_{um}}(p)(p' \otimes p) \mapsto p'(0) \otimes p(0) \otimes p'(1)p(1).$$

(52)

and $\text{Ad}_R(h) = h(1) \otimes S(h(2))h(3)$ is the right adjoint coaction of $H$. Finally, the map $\bar{\gamma} : P \otimes P \to P \otimes H$ is defined like the canonical map as $\bar{\gamma}(p' \otimes p) = p'p(0) \otimes p(1)$.

As shown in [6] (cf. [5, 20]), a strong connection can always be given by a map $\ell : H \to P \otimes P$ satisfying

$$\ell(1) = 1 \otimes 1,$$
$$\bar{\gamma}(\ell(h)) = 1 \otimes h,$$
$$((\ell \otimes id) \circ \Delta = (id \otimes \Delta_R) \circ \ell,$$
$$((id \otimes \ell) \circ \Delta = (\Delta_L \otimes id) \circ \ell,$$

(53)
where $\Delta_L : P \rightarrow H \otimes P$, $p \mapsto S^{-1} p_{(1)} \otimes p_{(0)}$. Then, one defines the connection one-form by

$$\omega : h \mapsto \ell(h) - \epsilon(h) P \otimes 1.$$  \hspace{1cm} (54)

Indeed, if one writes $\ell(h) = h^{(1)} \otimes h^{(2)}$ (summation understood) and applies $\text{id} \otimes \epsilon$ to the second formula in (53), one has $h^{(1)} h^{(2)} = \epsilon(h)$. Therefore,

$$\omega(h) = h^{(1)} \delta h^{(2)},$$  \hspace{1cm} (55)

where $\delta : P \rightarrow \Omega^1_{\text{un}} P$, $p \mapsto 1 \otimes p - p \otimes 1$. Equivariant projectivity of $B \hookrightarrow P$ follows by taking as splitting of the multiplication the map $s : P \rightarrow B \otimes P$, $p \mapsto p_{(0)} \ell(p_{(1)})$.

For later use, we prove the following lemma, analogous to the strongness Condition 3 above.

**Lemma 12.** Let $\omega$ be a strong connection one-form on a $H$-Hopf-Galois extension $B \hookrightarrow P$ with the antipode of $H$ invertible. Then $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)} \in P \Omega^1_{\text{un}} B$, $\forall p \in P$.

**Proof.** By writing $\omega$ in terms of $\ell$ it follows that $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}$ reduces to the expression $-p \otimes 1 + \ell(S^{-1} p_{(1)}) p_{(0)}$. From the second property of $\ell$ in (53), it follows that this expression is in the kernel of $\chi$. Since $\chi$ is an isomorphism, $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}$ is in the ideal generated by expressions of the form $p \otimes bp' - pb \otimes p'$. In other words, it is an element in $P \Omega^1_{\text{un}} (B) P$. Finally, it is not difficult to show that

$$(\text{id} \otimes \Delta_R)(\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}) \equiv (\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}) \otimes 1,$$

from which we conclude that $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}$ is in fact in $P \Omega^1_{\text{un}} (B)$. \hspace{1cm} $\Box$

In our case, the existence of a strong connection follows from [30]. However, we will write an explicit expression in terms of the inverse of the canonical map. If we denote the latter when lifted to $P \otimes P$ by $\tau$ it follows that $\ell(h) = \tau(1 \otimes h)$ satisfies the same recursive relation found before for $\chi^{-1}$ (proof of Proposition 8 above): if $\ell(h) = h_l \otimes h'_l$ and $\ell(g) = g_k \otimes g'_k$, then

$$\ell(hg) = g_k h_l \otimes h'_k g'_l.$$  \hspace{1cm} (56)

It turns out that in our case the map $\ell : H \rightarrow P \otimes P$ defined in this way defines a strong connection.

**Proposition 13.** On the Hopf-Galois extension $A(S^3_\theta) \hookrightarrow A(S^7_\theta)$, the following formulae on the generators of $A(\text{SU}(2))$,

$$\ell(w^1) = \sum_i \langle \psi_1_i \otimes |\psi_1_i \rangle_i, \hspace{1cm} \ell(w^2) = \sum_i \langle \psi_1_i \otimes |\psi_2_i \rangle_i,$$

$$\ell(\overline{w}^1) = -\sum_i \langle \psi_2_i \otimes |\psi_1_i \rangle_i, \hspace{1cm} \ell(\overline{w}^1) = \sum_i \langle \psi_2_i \otimes |\psi_2_i \rangle_i$$  \hspace{1cm} (57)

define a strong connection.
Proof. We extend the expressions (57) to all of $\mathcal{A}(SU(2))$ by giving recursive relations, using formula (56). Recall the usual vector basis $\{r_{km} : k \in \mathbb{Z}, m, n \geq 0\}$ in $\mathcal{A}(SU(2))$ given by

$$r_{km} := \begin{cases} (-1)^n (w_1)^k (w_2)^m (w_3)^n & k \geq 0, \\ (-1)^n (w_2)^m (w_3)^n (w_4)^{-k} & k < 0. \end{cases}$$ (58)

The recursive expressions on this basis are explicitly given by

$$\ell(r_{km+1, mn}) = z_1 \ell(r_{km, mn}) z_1 + z_2 \ell(r_{km, mn}) z_2 + z_3 \ell(r_{km, mn}) z_3 + z_4 \ell(r_{km, mn}) z_4, \quad k \geq 0,$$

$$\ell(w_{km+1, mn}) = z_2 \ell(r_{km, mn}) z_2 + z_1 \ell(r_{km, mn}) z_1 + z_4 \ell(r_{km, mn}) z_4 - z_3 \ell(r_{km, mn}) z_3, \quad k < 0,$$

$$\ell(w_{km, mn+1}) = z_3 \ell(r_{km, mn}) z_3 - z_2 \ell(r_{km, mn}) z_2 + z_4 \ell(r_{km, mn}) z_4 + z_3 \ell(r_{km, mn}) z_3,$$

$$\ell(w_{km, mn+1}) = z_4 \ell(r_{km, mn}) z_4 - z_1 \ell(r_{km, mn}) z_1 - z_2 \ell(r_{km, mn}) z_2 + z_3 \ell(r_{km, mn}) z_3 - z_4 \ell(r_{km, mn}) z_4,$$ (59)

while setting $\ell(1) = 1 \otimes 1$. In essentially the same manner as was done in [4] (although much simpler in our case) we prove that $\ell$ defined by the above recursive relations indeed satisfies all conditions of a strong connection. \hfill \Box

The strong connection on the extension $\mathcal{A}(S_4^2) \hookrightarrow \mathcal{A}(S_7^6)$ induces connections on the associated modules in the following way [19]. For $\phi \in \text{Hom}_{\rho(n)}(V(n), \mathcal{A}(S_7^6))$, we set

$$\nabla_\omega(\phi)(v) \mapsto \delta \phi(v) + \omega(v(0)) \phi(v(1)).$$ (60)

Using the right adjoint colinearity of $\omega$ and a little algebra one shows that $\nabla_\omega(\phi)$ satisfies the following coequivariance condition

$$\nabla_\omega(\phi)(v(1)) \otimes S v(0) = \Delta_{\Omega_{\mathbb{C}^1}(P)}(\nabla_\omega(\phi)(v))$$

so that

$$\nabla_\omega : \text{Hom}^{\rho(n)}(V(n), \mathcal{A}(S_7^6)) \rightarrow \text{Hom}^{\rho(n)}(V(n), \Omega_{\mathbb{C}^1}^{1}(\mathcal{A}(S_7^6))).$$

In fact, from Lemma 12 it follows that $\nabla_\omega$ is a map from $\text{Hom}^{\rho(n)}(V(n), \mathcal{A}(S_7^6))$ to $\text{Hom}^{\rho(n)}(V(n), \mathcal{A}(S_7^6)) \otimes \Omega_{\mathbb{C}^1}^{1}(\mathcal{A}(S_7^6))$. This allows one to compare it to the Grassmannian connection of Eq. (30). It turns out that the connection one-form $\omega$ coincides with the connection one-form $A$ of Eq. (31), on the quotient $\Omega^1(S_7^6) \otimes \Omega_{\mathbb{C}^1}^{1}(\mathcal{A}(S_7^6))$. More precisely, let $\{e_{kl}^{(n)}\}$ be a basis of $V(n)$, and $e_{kl}^{(n)}$ the corresponding matrix coefficients of $\mathcal{A}(SU(2))$ in the representation $\rho(n)$. An explicit expression for $\omega(e_{kl}^{(n)})$ can be obtained from Eqs. (59); for example $\omega(e_1^{(1)}) = \langle \psi_k | d \psi_l \rangle$, $k, l = 1, 2$.

By using these and formulæ (77)–(79), one shows that

$$\pi(\omega(e_{kl}^{(n)})) = A_{kl}^{(n)} = \langle \psi_k | d \psi_l \rangle,$$

where $\pi : \Omega_{\mathbb{C}^1}(\mathcal{A}(S_7^6)) \rightarrow \Omega(S_7^6)$ is the quotient map.
A. Associated Modules

We will now construct projections $p(n)$ to $S(n)$ associated to all finite-dimensional irreducible representations of $SU(2)$. We start by recalling the Hopf fibration $\pi : S^7 \to S^3$. Let

$$S^7 := \{ z = (z^1, z^2, z^3, z^4) : |z^1|^2 + |z^2|^2 + |z^3|^2 + |z^4|^2 = 1 \},$$

$$SU(2) := \{ w \in GL(2, \mathbb{C}) : w^*w = w w^* = 1, \det w = 1 \} = \left\{ w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w^2} & \overline{w^1} \end{pmatrix} : w^1 w^1 + w^2 w^2 = 1 \right\}. \quad (61)$$

The space $S^7$ carries a right $SU(2)$-action:

$$S^7 \times SU(2) \to S^7, \quad (z, w) \mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w^1 & 0 \\ 0 & w \end{pmatrix}. \quad (62)$$

The Hopf map is defined as a map $\pi(z) : (\alpha, \beta, x) \mapsto (\alpha, \beta, x)$, where

$$\alpha = 2(z^1 \bar{z}^3 + z^2 \bar{z}^4), \quad \beta = 2(-z^1 \bar{z}^4 + z^2 \bar{z}^3),$$

$$x = z^1 \bar{z}^1 + z^2 \bar{z}^2 - z^3 \bar{z}^3 - z^4 \bar{z}^4, \quad (63)$$

and one computes $\alpha \bar{\alpha} + \beta \bar{\beta} + x^2 = (\sum_j |z^j|^2)^2 = 1$.

The finite-dimensional irreducible representations of $SU(2)$ are labeled by a positive integer $n$ with $n+1$-dimensional representation space $V(n) \cong \text{Sym}^n(\mathbb{C}^2)$. The space of smooth $SU(2)$-equivariant maps from $S^7$ to $V(n)$ is defined by

$$C_{SU(2)}^\infty(S^7, V(n)) := \{ \phi : S^7 \to V(n) : \phi(z, w) = w^{-1} \cdot \phi(z) \}. \quad (64)$$

We will now construct projections $p(n)$ as $N \times N$ matrices taking values in $C^\infty(S^4)$, such that $\Gamma^\infty(S^4, E(n)) := p(n) C_{SU(2)}^\infty(S^7, V(n))$ is isomorphic to $C_{SU(2)}^\infty(S^7, V(n))$ as right $C^\infty(S^4)$-modules. As the notation suggests, $E(n)$ is the vector bundle over $S^4$ associated with the corresponding representation. Let us first recall the case $n = 1$ from [24] and then use this to generate the vector bundles for any $n$. The $SU(2)$-equivariant maps from $S^7$ to $V(1) \cong \mathbb{C}^2$ are of the form

$$\phi(z) = \frac{z^1}{z^2} f_1 + \frac{z^2}{z^1} f_2 + \frac{z^3}{z^4} f_3 + \frac{z^4}{z^3} f_4, \quad (65)$$

where $f_1, \ldots, f_4$ are smooth functions that are invariant under the action of $SU(2)$, i.e., they are functions on the base space $S^4$.

A nice description of the equivariant maps is given in terms of ket-valued functions $|\xi\rangle$ on $S^7$, which are then elements in the free module $\mathcal{E} := C^\infty \otimes C^\infty(S^7)$. The $C^\infty(S^7)$-valued hermitian structure on $\mathcal{E}$ given by $\langle \xi, \eta \rangle = \sum_j \xi_j^* \eta_j$ allows one to associate dual elements $\langle \xi \rangle \in \mathcal{E}^*$ to each $|\xi\rangle \in \mathcal{E}$ by $\langle \xi \rangle(\eta) := \langle \xi, \eta \rangle, \ \forall \eta \in \mathcal{E}$.

If we define $|\psi_1\rangle, |\psi_2\rangle \in A(S^7)^4$ by

$$|\psi_1\rangle = (z^1, -\bar{z}^2, z^3, -\bar{z}^4), \quad |\psi_2\rangle = (z^2, z^1, z^4, \bar{z}^3), \quad (66)$$

with \( t \) denoting transposition, the equivariant maps in (65) are given by
\[
\phi(\mathbf{1})(z) = \begin{pmatrix} \langle \psi_1| f \rangle \\ \langle \psi_2| f \rangle \end{pmatrix},
\]
where \( |f\rangle \in (C^\infty(S^4))^4 := \mathbb{C}^4 \otimes C^\infty(S^4) \). Since \( \langle \psi_k| \psi_l \rangle = \delta_{kl} \) as is easily seen, we can define a projection in \( M_4(C^\infty(S^4)) \) by
\[
p(\mathbf{1}) = |\psi_1\rangle\langle \psi_1| + |\psi_2\rangle\langle \psi_2|.
\]
Indeed, by explicit computation we find a matrix with entries in \( C^\infty(S^4) \) which is the limit of the projection (22) for \( \theta = 0 \). Denoting the right \( C^\infty(S^4) \)-module \( p(\mathbf{1})(C^\infty(S^4)) \) by \( \Gamma_1(S^4, E^{(1)}) \), we have
\[
\Gamma(S^4, E^{(1)}) \simeq C^\infty_{SU(2)}(S^7, \mathbb{C}^2),
\]
\[
\sigma(\mathbf{1}) = p(\mathbf{1})|f\rangle \leftrightarrow \phi(\mathbf{1}) = \begin{pmatrix} \langle \psi_1| f \rangle \\ \langle \psi_2| f \rangle \end{pmatrix}.
\]
For the general case, we note that the \( SU(2) \)-equivariant maps from \( S^7 \) to \( V^{(n)} \) are of the form
\[
\phi(n)(z) = \begin{pmatrix} \langle \phi_1| f \rangle \\ \vdots \\ \langle \phi_{n+1}| f \rangle \end{pmatrix},
\]
where \( |f\rangle \in C^\infty(S^4)^{d^n} \) and \( |\phi_k\rangle \) is the completely symmetrized form of the tensor product \( |\psi_1\rangle^{\otimes-n-k+1} \otimes |\psi_2\rangle^{\otimes k-1} \) for \( k = 1, \ldots, n+1 \), normalized to have norm 1 as in formula (28). For example, for the adjoint representation \( n = 2 \), we have
\[
|\phi_1\rangle := |\psi_1\rangle \otimes |\psi_1\rangle,
|\phi_2\rangle := \frac{1}{\sqrt{2}}(|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle),
|\phi_3\rangle := |\psi_2\rangle \otimes |\psi_2\rangle.
\]
Since in general, \( \langle \phi_k| \phi_l \rangle = \delta_{kl} \), the matrix-valued function
\[
p(n) = |\phi_1\rangle\langle \phi_1| + |\phi_2\rangle\langle \phi_2| + \cdots + |\phi_{n+1}\rangle\langle \phi_{n+1}| \in M_{d^n}(C^\infty(S^4))
\]
defines a projection whose entries are in \( C^\infty(S^4) \), since each entry \( \sum_k |\phi_k\rangle_i \langle \phi_k|_j \) is \( SU(2) \)-invariant (cf. below formula (29)). We conclude that
\[
p(n)(C^\infty(S^4)^{d^n}) \simeq C^\infty_{SU(2)}(S^7, V^{(n)}),
\]
\[
\sigma(n) = p(n)|f\rangle \leftrightarrow \phi(n) = \begin{pmatrix} \langle \phi_1| f \rangle \\ \vdots \\ \langle \phi_{n+1}| f \rangle \end{pmatrix}.
\]
B. Chern Characters

We will compute the Chern characters of the projections \( p(n) \) defined in Sect. 3. It turns out to be sufficient for our purposes to obtain expressions in the differential calculus \( \Omega_D(S^q_\theta) \subset \Omega_D(S^p_\theta) \) defined in Sect. 3, which is a quotient of the universal differential calculus.

In the definition of the Chern character (36) we can replace the tensor product by the universal differential \( \delta \) by the isomorphism

\[
A \otimes \overline{\mathcal{A}}^p \simeq \Omega^p_{\text{un}}(A),
\]

where \( \overline{\mathcal{A}} := A/\mathbb{C} \) and \( \Omega^p_{\text{un}}(A) = \bigoplus p \Omega^p_{\text{un}}(A) \) is the universal differential algebra generated by \( a \in A \) and symbols \( \delta a, \alpha \in A \) of order 1 satisfying

\[
\delta(ab) = (\delta a)b + a\delta b; \quad \delta(a + b) = \delta a + \delta b; \quad (a, b \in A, \alpha, \beta \in \mathbb{C}).
\]

Then

\[
\text{ch}_k(e) := (-1)^k \frac{(2k)!}{k!} (\delta e)^{2k} \in \Omega^2_{\text{un}}(A).
\]

We recall the differential calculi \( \Omega_D(S^q_\theta) \) and \( \Omega_D(S^p_\theta) \) from Sect. 3. We defined \( \Omega_D(S^q_\theta) \) as the quotient of \( \Omega_{\text{un}}(A(S^q_\theta)) \) by the relations

\[
\begin{align*}
\alpha \delta \beta &= \lambda(\delta \beta)\alpha, \quad (\delta \alpha)\beta = \lambda(\delta \beta)\alpha, \\
\alpha \delta \beta^* &= \bar{\lambda}(\delta \beta^*)\alpha, \quad (\delta \alpha^*)\beta = \bar{\lambda}(\delta \beta^*)\alpha, \\
\alpha \delta x &= (\delta x)\alpha, \quad (a \in A(S^q_\theta)).
\end{align*}
\]

The inclusion \( A(S^q_\theta) \hookrightarrow A(S^p_\theta) \) extends to an injective map \( \Omega_D(S^q_\theta) \to \Omega_D(S^p_\theta) \), where \( \Omega_D(S^p_\theta) \) is the quotient of \( \Omega_{\text{un}}(A(S^p_\theta)) \) by only the relations in (5) of order one, that is by the relations:

\[
z^l \delta z^l = \lambda^{ij}(\delta z^l)z^i; \quad z^l \delta \bar{z}^l = \lambda^{ij}(\delta \bar{z}^l)z^i.
\]

Recall that the projections \( p(n) \) were defined by \( p(n) = \sum_k |\phi_k\rangle \langle \phi_k| \), where \( |\phi_k\rangle \) with \( k = 1, \ldots, n + 1 \), is given by

\[
|\phi_k\rangle = \frac{1}{ak} |\psi_1\rangle^{(n-k+1)} \otimes |\psi_2\rangle^{(k-1)}, \quad a_k^2 = \binom{n}{k-1}.
\]

Before we start the computation of the Chern characters, we state the computation rules in \( \Omega_D(S^p_\theta) \). Firstly, from the very definition of the vectors \( |\phi_k\rangle \) and the inner product in \( \mathcal{E} \otimes \mathcal{E} \otimes \cdots \otimes \mathcal{E} \), we can express, for any \( k = 1, \ldots, n + 1 \),

\[
\langle \phi_k | \delta \phi_{k-1} \rangle = \sqrt{(n-k)(k+1)} |\psi_1\rangle |\delta \psi_1\rangle, \quad \langle \phi_k | \delta \phi_{k+1} \rangle = \sqrt{(n-k-1)(k+2)} |\psi_1\rangle |\delta \psi_2\rangle, \quad \langle \phi_k | \delta \phi_k \rangle = \langle n-k-1 \rangle |\psi_1\rangle |\delta \psi_1\rangle + (k+1) |\psi_2\rangle |\delta \psi_2\rangle
\]

by using the relation \( |\psi_2\rangle |\delta \psi_2\rangle = -|\psi_1\rangle |\delta \psi_1\rangle \). The previous are in fact the only nonzero expressions for \( \langle \phi_k | \delta \phi_k \rangle \) if we apply \( \delta \) to these equations, we obtain expressions for \( \langle \delta \phi_k | \delta \phi_k \rangle \) in terms of \( |\psi_1\rangle \) and \( |\psi_2\rangle \). From this, we deduce the following result that will be central in the computation of the Chern characters.
Lemma 14. The following relations hold in $\Omega_D(S^7_\theta)$:

$$
\sum_{k,l=1}^{n+1} \langle \phi_k | \delta \phi_l \rangle \langle \delta \phi_k | \phi_l \rangle = \frac{1}{6} n(n + 1)(n + 2) \sum_{r,s=1}^2 \langle \psi_r | \delta \psi_s \rangle \langle \delta \psi_r | \psi_s \rangle.
$$

$$
\sum_{k,l,m=1}^{n+1} \langle \phi_k | \delta \phi_l \rangle \langle \delta \phi_m | \phi_l \rangle = \frac{1}{6} n(n + 1)(n + 2)
\times \sum_{r,s=1}^2 \langle \psi_r | \delta \psi_s \rangle \langle \delta \psi_r | \psi_s \rangle.
$$

Of course, there will be similar formulæ for $\langle \delta \phi_k | \delta \phi_l \rangle \langle \phi_l | \delta \phi_k \rangle$, etc.

The zeroeth Chern character is easy to compute:

$$
\text{ch}_0(p(n)) = \text{Tr}(p(n)) = \sum_k \langle \phi_k | \phi_k \rangle = n + 1.
$$

(80)

In the computation of $\text{ch}_1(p(n))$ we use the relation $\langle \delta \phi_k | \delta \phi_l \rangle = - \langle \phi_k | \delta \phi_l \rangle$, which follows from applying the derivation $\delta$ to $\langle \phi_k | \phi_l \rangle = \delta_{kl}$ and the fact that in $\Omega_D(S^7_\theta)$, $\langle \phi_k | \delta \phi_l \rangle$ commutes with any element in $A(S^7_\theta)$, in particular with $\langle \phi_m | i \rangle$. Thus,

$$
\text{ch}_1(p(n)) = \sum_k \langle \delta \phi_k | \delta \phi_k \rangle = \left( \sum_{m=1}^{n+1} \langle \delta \phi_m | \delta \phi_m \rangle - \sum_{m=1}^{n+1} \langle \delta \phi_m | \delta \phi_m \rangle \right).
$$

(82)

By using Eq. (79) and its analogue for $\langle \delta \phi_m | \delta \phi_m \rangle$, $m = 1, \ldots, n + 1$,

$$
\langle \delta \phi_m | \delta \phi_m \rangle = (k + 1) \langle \psi_1 | \delta \psi_1 \rangle + (n - k - 1) \langle \psi_2 | \delta \psi_2 \rangle,
$$

we find that

$$
\text{ch}_1(p(n)) = \frac{1}{2} n(n + 1) \left( \langle \psi_1 | \delta \psi_1 \rangle + \langle \psi_2 | \delta \psi_2 \rangle \right) = \frac{1}{2} n(n + 1) \text{ch}_1(p(1)).
$$

(81)

Note that this equation holds in the differential subalgebra $\Omega_D(S^7_\theta)$. Since $\text{ch}_1(p(1))$ was shown to vanish in [11], we proved the vanishing of the first Chern character in $\Omega_D(S^7_\theta)$.

The vanishing of $\text{ch}_1(p(1))$ can also be seen from the explicit form of $|\psi_1\rangle$ and $|\psi_2\rangle$.

A slightly more involved computation in $\Omega_D(S^7_\theta)$ shows that

$$
\text{ch}_2(p(n)) = \frac{1}{2} \sum \left\{ \delta \left( \langle \phi_k | \delta \phi_l \rangle \langle \delta \phi_l | \phi_k \rangle \langle \delta \phi_m | \phi_m \rangle \right) + \langle \delta \phi_k | \delta \phi_l \rangle \langle \delta \phi_l | \phi_k \rangle \langle \delta \phi_m | \phi_m \rangle \right\},
$$

(82)
And by using Lemma 14 we finally get

\[ \text{ch}_2(p(n)) = \frac{1}{6} n(n + 1)(n + 2) \text{ch}_2(p(1)), \] (83)

as an element in \( \Omega^4_D(S^4) \).

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