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Principal Fibrations from Noncommutative Spheres

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Abstract: We construct noncommutative principal fibrations $S^7_\theta \to S^4_\theta$ which are deformations of the classical $SU(2)$ Hopf fibration over the four sphere. We realize the noncommutative vector bundles associated to the irreducible representations of $SU(2)$ as modules of coequivariant maps and construct corresponding projections. The index of Dirac operators with coefficients in the associated bundles is computed with the Connes-Moscovici local index formula. "The algebra inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_\theta)$ is an example of a not-trivial quantum principal bundle."

1. Introduction

The ADHM construction [2, 3] of instantons in Yang-Mills theory has at its heart the theory of connections on principal and associated bundles. A central example is the basic $SU(2)$-instanton on S^4 which is described by the well-known Hopf $SU(2)$-principal bundle $S^7 \to S^4$ and connections thereon.

In this paper, we consider a noncommutative version of this Hopf fibration, in the framework of the isospectral deformations introduced in [11], while trying to understand the structure behind the noncommutative instanton bundle found there.

In Sect. 2 we will review the construction of θ-deformed spheres where θ is an anti-symmetric real-valued matrix. Apart from the noncommutative spheres S^m_θ, we also introduce differential calculi $\Omega(S^m_\theta)$ as quotients of the universal differential calculi. On the sphere S^m_θ one constructs a noncommutative Riemannian spin geometry $(C^\infty(S^m_\theta), D, \mathcal{H})$ in which the Dirac operator D is the classical one and $\mathcal{H} = L^2(S^m, \mathcal{S})$ is the usual Hilbert space of spinors. Then the deformations are isospectral, as mentioned. Furthermore, one also constructs a Hodge star operator $*_\theta$ acting on the differential calculus $\Omega(S^m_\theta)$ which is most easily defined using the so-called splitting homomorphism [10].

In Sect. 3, we focus on two noncommutative spheres S^4_θ and $S^7_{\theta'}$ starting from the algebras $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ of polynomial functions on them. The latter algebra carries an action of the (classical) group $SU(2)$ by automorphisms in such a way that its invariant elements are exactly the polynomials on S^4_θ. The anti-symmetric 2×2 matrix θ is given by a single real number also denoted by θ. On the other hand, the requirements that $SU(2)$ acts by automorphisms and that S^4_θ makes the algebra of invariant functions, give the matrix θ' in terms of θ. This yields a one-parameter family of noncommutative Hopf fibrations.

For each irreducible representation $V^{(n)} := \mathrm{Sym}^n(\mathbb{C}^2)$ of $SU(2)$ we construct the noncommutative vector bundles $E^{(n)}$ associated to the fibration $S^7_{\theta'} \to S^4_\theta$. By dualizing the classical construction, these bundles are described by the module of coequivariant maps from $\mathbb{C}^2$ to $\mathcal{A}(S^7_{\theta'})$. As expected, these modules are finitely generated projective and we construct explicitly the projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ such that these modules are isomorphic to the image of $p_{(n)}$ in $\mathcal{A}(S^4_\theta)^{4^n}$. Then, one defines connections $\nabla = p_{(n)}\mathrm{d}$ as maps from $\Gamma(S^4_\theta, E^{(n)})$ to $\Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$, where $\Omega^*(S^4_\theta)$ is the quotient of the universal differential calculus mentioned above. The corresponding connection one-form A turns out to be valued in a representation of the Lie algebra $su(2)$.

By using the projection $p_{(n)}$, the Dirac operator with coefficients in the noncommutative vector bundles $E^{(n)}$ is given by $D_{p_{(n)}} := p_{(n)} D p_{(n)}$. In order to compute its index, we first show that the local index theorem of Connes and Moscovici [12] takes a very simple form in the case of isospectral deformations. Indeed, for these deformations and with any projection e, one finds,

$$\begin{aligned}\mathrm{Ind}\, D_e = &\operatorname*{Res}_{z=0} z^{-1}\mathrm{Tr}\Big(\gamma\big(e - \frac{1}{2}\big)|D|^{-2z}\Big)\\ &+ \sum_{k\geq 1} c_k \operatorname*{Res}_{z=0} \mathrm{Tr}\Big(\gamma\big(e - \frac{1}{2}\big)[D, e]^{2k}|D|^{-2(k+z)}\Big)\end{aligned} \tag{1}$$

with some proper coefficients c_k. When applied to the projections $p_{(n)}$ on S^4_θ, we obtain exactly as in the classical case,

$$\mathrm{Ind}\, D_{p_{(n)}} = \frac{1}{6} n(n+1)(n+2). \tag{2}$$

Finally, in Sect. 5 we show that the fibration $S^7_{\theta'} \to S^4_\theta$ is a 'not-trivial principal bundle with structure group $SU(2)$'. This means that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft Hopf-Galois extension [21, 28]; in fact, it is a principal extension [6]. On this extension, we find an explicit form of the (strong) connection which induces connections on the associated bundles $E^{(n)}$ as maps from $\Gamma(S^4_\theta, E^{(n)})$ to $\Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(\mathcal{A}(S^4_\theta))$, where $\Omega^*(\mathcal{A}(S^4_\theta))$ is the universal differential calculus on $\mathcal{A}(S^4_\theta)$. We show that these connections coincide with the Grassmannian connections $\nabla = p_{(n)}\mathrm{d}$ on the quotient $\Omega(S^4_\theta)$ of the universal differential calculus alluded to before.

2. Noncommutative Spherical Manifolds

In this section, we will recall the construction of the noncommutative spheres S^n_θ as introduced in [11] and elaborated in [10]. Essentially, these θ-deformations are a natural extension of the noncommutative torus (for a review see [29]) to (compact) Riemannian manifolds carrying an action of the n-torus $\mathbb{T}^n$. In this paper we will restrict only to the cases of planes and spheres.

For $\lambda^{\mu\nu} = e^{2\pi i\theta_{\mu\nu}}$, where $\theta_{\mu\nu}$ is an anti-symmetric real-valued matrix, the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ of polynomial functions on the noncommutative $2n$-plane is defined to be the unital $*$-algebra generated by $2n$ elements $z^\mu, \bar{z}^\mu (\mu = 1, \ldots, n)$ with relations

$$z^\mu z^\nu = \lambda^{\mu\nu} z^\nu z^\mu; \quad \bar{z}^\mu z^\nu = \lambda^{\nu\mu} z^\nu \bar{z}^\mu; \quad \bar{z}^\mu \bar{z}^\nu = \lambda^{\mu\nu} \bar{z}^\nu \bar{z}^\mu. \tag{3}$$

The involution $*$ is defined by putting $z^{\mu *} = \bar{z}^\mu$. For $\theta = 0$ one recovers the commutative $*$-algebra of complex polynomial functions on $\mathbb{R}^{2n}$.

Let $\mathcal{A}(S^{2n-1}_\theta)$ be the $*$-quotient of $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by the two-sided ideal generated by the central element $\sum_\mu z^\mu \bar{z}^\mu - 1$. We will denote the images of z^μ under the quotient map again by z^μ.

A key role in what follows is played by the action of the abelian group $\mathbb{T}^n$ on $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by automorphisms. For $s = (s_\mu) \in \mathbb{T}^n$, the $*$-automorphism σ_s is defined on the generators by $\sigma_s(z^\mu) = e^{2\pi i s_\mu} z^\mu$. Clearly, $s \mapsto \sigma_s$ is a group-homomorphism from $\mathbb{T}^n \to \mathrm{Aut}(\mathcal{A}(\mathbb{R}^{2n}_\theta))$. In the special case that $\theta = 0$, we see that σ is induced by a smooth action of $\mathbb{T}^n$ on the manifold $\mathbb{R}^{2n}$. Since the ideal generating $\mathcal{A}(S^{2n-1}_\theta)$ is invariant under the action of $\mathbb{T}^n$, σ induces a group-homomorphism from $\mathbb{T}^n$ into the group of automorphisms on the quotient $\mathcal{A}(S^{2n-1}_\theta)$ as well.

We continue by defining the unital $*$-algebra $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ of polynomial functions on the noncommutative $(2n+1)$-plane which is given by adjoining a central self-adjoint generator x to the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$, i.e. $x^* = x$ and $xz^\mu = z^\mu x$ $(\mu = 1, \ldots, n)$. The action of the group $\mathbb{T}^n$ is extended trivially by $\sigma_s(x) = x$. Let $\mathcal{A}(S^{2n}_\theta)$ be the $*$-quotient of $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ by the ideal generated by the central element $\sum z^\mu \bar{z}^\mu + x^2 - 1$. As before, we will denote the canonical images of z^μ and x again by z^μ and x, respectively. Since $\mathbb{T}^n$ leaves this ideal invariant, it induces an action by $*$-automorphisms on the quotient $\mathcal{A}(S^{2n}_\theta)$.

Example 1. For $n = 2$ we obtain the noncommutative sphere S^4_θ, which was found in [11]. We adopt the notation used therein and let $\mathcal{A}(S^4_\theta)$ be generated by α, β and a central x with $x = x^*$ and relations

$$\alpha\beta = \lambda\beta\alpha, \quad \alpha\beta^* = \bar{\lambda}\beta^*\alpha, \quad \alpha\alpha^* = \alpha^*\alpha, \quad \beta\beta^* = \beta^*\beta, \tag{4}$$

together with the spherical relation $\alpha\alpha^* + \beta\beta^* + x^2 = 1$. Here $\lambda = e^{2\pi i\theta}$ with θ a real number.

We will now construct a differential calculus on $\mathbb{R}^m_\theta$. For $m = 2n$, the complex unital associative graded $*$-algebra $\Omega(\mathbb{R}^{2n}_\theta)$ is generated by $2n$ elements $z^\mu, \bar{z}^\mu$ of degree 0 and $2n$ elements $dz^\mu, d\bar{z}^\mu$ of degree 1 with relations:

$$\begin{gathered} dz^\mu dz^\nu + \lambda^{\mu\nu} dz^\nu dz^\mu = 0;\ d\bar{z}^\mu dz^\nu + \lambda^{\nu\mu} dz^\nu d\bar{z}^\mu = 0;\ d\bar{z}^\mu d\bar{z}^\nu + \lambda^{\mu\nu} d\bar{z}^\nu d\bar{z}^\mu = 0; \\ z^\mu dz^\nu = \lambda^{\mu\nu} dz^\nu z^\mu;\ \bar{z}^\mu dz^\nu = \lambda^{\nu\mu} dz^\nu \bar{z}^\mu;\ \bar{z}^\mu d\bar{z}^\nu = \lambda^{\mu\nu} d\bar{z}^\nu \bar{z}^\mu. \end{gathered} \tag{5}$$

There is a unique differential d on $\Omega(\mathbb{R}^{2n}_\theta)$ such that $\mathrm{d} : z^\mu \mapsto dz^\mu$. The involution $\omega \mapsto \omega^*$ for $\omega \in \Omega(\mathbb{R}^{2n}_\theta)$ is the graded extension of $z^\mu \mapsto \bar{z}^\mu$, i.e. it is such that $(\mathrm{d}\omega)^* = \mathrm{d}\omega^*$ and $(\omega_1\omega_2)^* = (-1)^{p_1 p_2}\omega_2^*\omega_1^*$ for $\omega_i \in \Omega^{p_i}(\mathbb{R}^{2n}_\theta)$. For $m = 2n+1$, we adjoin to $\Omega(\mathbb{R}^{2n}_\theta)$ one generator x of degree 0 and one generator dx of degree 1 such that

$$x dx = dx x; \quad x\omega = \omega x; \quad dx\omega = (-1)^{|\omega|}\omega dx. \tag{6}$$

We extend the differential d and the graded involution $\omega \mapsto \omega^*$ of $\Omega(\mathbb{R}^{2n}_\theta)$ to $\Omega(\mathbb{R}^{2n+1}_\theta)$ by setting $x^* = x$ and $(\mathrm{d}x)^* = \mathrm{d}x$, so that $(dx)^* = dx$.

The differential calculi $\Omega(S^m_\theta)$ on the noncommutative spheres S^m_θ are defined to be the quotients of $\Omega(\mathbb{R}^{m+1}_\theta)$ by the differential ideals generated by the central elements $\sum_\mu z^\mu \bar{z}^\mu - 1$ and $\sum z^\mu \bar{z}^\mu + x^2 - 1$, for $m = 2n - 1$ and $m = 2n$ respectively.

The action of $\mathbb{T}^n$ by $*$-automorphisms on $\mathcal{A}(M_\theta)$ can be easily extended to the differential calculi $\Omega(M_\theta)$, for $M = \mathbb{R}^m_\theta$ and $M = S^m_\theta$, by imposing $\sigma_s \circ \mathrm{d} = \mathrm{d} \circ \sigma_s$.

In [10], the so-called splitting homomorphism was introduced. For the cases $M = \mathbb{R}^m$ or $M = S^m$, this map identifies $\mathcal{A}(M_\theta)$ with a subalgebra of $\mathcal{A}(M) \otimes \mathcal{A}(\mathbb{T}^n_\theta)$, and this identification allows one to use techniques from commutative differential geometry on $\mathcal{A}(M)$ and extend it to $\mathcal{A}(M_\theta)$. Let us recall the definition of the noncommutative n-torus $\mathbb{T}^n_\theta$. The unital $*$-algebra $\mathcal{A}(\mathbb{T}^n_\theta)$ of polynomial functions is generated by n unitary elements U^μ with relations

$$U^\mu U^\nu = \lambda^{\mu\nu} U^\nu U^\mu, \quad (\mu, \nu = 1, \dots, n) \tag{7}$$

with $\lambda^{\mu\nu} = e^{2\pi \mathrm{i}\theta_{\mu\nu}}$ as before. There is a natural action of $\mathbb{T}^n$ on $\mathcal{A}(\mathbb{T}^n_\theta)$ by $*$-automorphisms given by $\tau_s(U^\mu) = e^{2\pi \mathrm{i} s_\mu} U^\mu$ with $s = (s_\mu) \in \mathbb{T}^n$. This allows one to define a diagonal action $\sigma \times \tau^{-1}$ of $\mathbb{T}^n$ on $\mathcal{A}(\mathbb{R}^m \times \mathbb{T}^n_\theta) := \mathcal{A}(\mathbb{R}^m) \otimes \mathcal{A}(\mathbb{T}^n_\theta)$ by $s \mapsto \sigma_s \otimes \tau_{-s}$. That is, $s \mapsto (\sigma \times \tau^{-1})_s$ is a group-homomorphisms of $\mathbb{T}^n$ into $\mathrm{Aut}(\mathcal{A}(\mathbb{R}^m \times \mathbb{T}^n_\theta))$.

If $z^\mu_{(0)}$ denote the classical coordinates of $\mathbb{R}^n$ corresponding to z^μ for $\theta = 0$, one defines the splitting homomorphism on the generators of $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by

$$\begin{aligned} \mathrm{st} : \mathcal{A}(\mathbb{R}^{2n}_\theta) &\to \mathcal{A}(\mathbb{R}^{2n}) \otimes \mathcal{A}(\mathbb{T}^n_\theta); \\ z^\mu &\mapsto z^\mu_{(0)} \otimes U^\mu. \end{aligned} \tag{8}$$

One checks that st induces an isomorphism between the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ and the subalgebra $\mathcal{A}(\mathbb{R}^{2n} \times \mathbb{T}^n_\theta)^{\sigma \times \tau^{-1}}$ of $\mathcal{A}(\mathbb{R}^{2n} \times \mathbb{T}^n_\theta)$ consisting of fixed points of the previous diagonal action of $\mathbb{T}^n$. By setting $\mathrm{st}(x) = x_{(0)} \otimes 1$ the splitting homomorphism extends trivially to a map from $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ to $\mathcal{A}(\mathbb{R}^{2n+1}) \otimes \mathcal{A}(\mathbb{T}^n_\theta)$, giving an algebra isomorphism $\mathcal{A}(\mathbb{R}^{2n+1}_\theta) \simeq \mathcal{A}(\mathbb{R}^{2n+1} \times \mathbb{T}^n_\theta)^{\sigma \times \tau^{-1}}$.

Furthermore, the map st will pass to the quotient, for $m = 2n, 2n + 1$,

$$\mathrm{st} : \mathcal{A}(S^m_\theta) \to \mathcal{A}(S^m) \otimes \mathcal{A}(\mathbb{T}^n_\theta) =: \mathcal{A}(S^m \times \mathbb{T}^n_\theta), \tag{9}$$

giving isomorphisms $\mathcal{A}(S^m_\theta) \simeq \mathcal{A}(S^m \times \mathbb{T}^n_\theta)^{\sigma \times \tau^{-1}}$.

The splitting homomorphism allows one to introduce algebras of smooth functions $C^\infty(M_\theta)$, for $M = \mathbb{R}^m$ or $M = S^m$. They are defined to be the fixed point subalgebras of the diagonal action of $\mathbb{T}^n$ on $C^\infty(M) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$. Here $C^\infty(\mathbb{T}^n_\theta)$ is the nuclear Fréchet algebra of smooth functions on $\mathbb{T}^n_\theta$ and $\widehat{\otimes}$ denotes the completion of the tensor product in the projective tensor product topology (see [10] for more details).

The extension of the splitting homomorphism to the differential calculi yields isomorphisms $\Omega(M_\theta) \simeq \left(\Omega(M) \otimes \mathcal{A}(\mathbb{T}^n_\theta)\right)^{\sigma \otimes \tau^{-1}}$, with M as above. This allows one to introduce a Hodge star operator on $\Omega(M_\theta)$. Let $*$ be the Hodge star operator on $\Omega(M)$ defined with a σ-invariant metric on M. The operator $* \otimes \mathrm{id}$ on $\Omega(M) \otimes \mathcal{A}(\mathbb{T}^n_\theta)$ restricted to the fixed point subalgebra of the diagonal action, defines the Hodge star operator $*_\theta$ on

$\Omega(M_\theta)$. Using this operator, one defines a hermitian structure on $\Omega(M_\theta)$ in the following way. If $\omega, \eta \in \Omega^p(M_\theta)$, then

$$\langle \omega, \eta \rangle := *_\theta(\overline{\omega} *_\theta \eta) \tag{10}$$

takes values in $\mathcal{A}(M_\theta)$ and fulfills all properties of a hermitian structure on $\Omega^p(M_\theta)$.

Finally, for the Dirac operator one has the following construction. Suppose for convenience that $M = S^m$ and equip S^m with a Riemannian metric such that $\mathbb{T}^n$ acts isometrically (this is always possible, for instance by averaging). Let $\mathcal{S}$ be a spin bundle over the spin manifold S^m and D the Dirac operator on $\Gamma^\infty(S^m, \mathcal{S})$. The action of the group $\mathbb{T}^n$ on S^m does not lift directly to the spinor bundle. Rather, there is a double cover $\pi : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a group-homomorphism $\tilde{s} \to V_{\tilde{s}}$ of $\widetilde{\mathbb{T}}^n$ into $\mathrm{Aut}(\mathcal{S})$ covering the action of $\mathbb{T}^n$ on M:

$$V_{\tilde{s}}(f\psi) = \sigma_{\pi(s)}(f) V_{\tilde{s}}(\psi), \tag{11}$$

for $f \in C^\infty(S^m)$ and $\psi \in \Gamma^\infty(M, \mathcal{S})$. It turns out that the proper notion of smooth sections $\Gamma^\infty(S^m_\theta, \mathcal{S})$ of a spinor bundle on S^m_θ is given by the subalgebra of $\Gamma^\infty(S^m, \mathcal{S}) \widehat{\otimes} C^\infty(\mathbb{T}^n_{\theta/2})$ made of elements which are invariant under the diagonal action $V \times \tilde{\tau}^{-1}$ of $\widetilde{\mathbb{T}}^n$. Here $\tilde{s} \mapsto \tilde{\tau}_{\tilde{s}}$ is the canonical action of $\widetilde{\mathbb{T}}^n$ on $\mathcal{A}(\mathbb{T}^n_{\theta/2})$. Since the Dirac operator D will commute with $V_{\tilde{s}}$ one can restrict $D \otimes \mathrm{id}$ to the fixed point subalgebra $\Gamma^\infty(S^m_\theta, \mathcal{S})$.

Next, let $L^2(S^m, \mathcal{S})$ be the space of square integrable spinors on S^m and let $L^2(\mathbb{T}^n_{\theta/2})$ be the completion of $C^\infty(\mathbb{T}^n_{\theta/2})$ in the norm $f \mapsto \|f\| = \tau(f^* f)^{1/2}$, with τ the usual trace on $C^\infty(\mathbb{T}^n_{\theta/2})$. The diagonal action $V \times \tilde{\tau}^{-1}$ of $\widetilde{\mathbb{T}}^n$ extends to $L^2(S^m, \mathcal{S}) \otimes L^2(\mathbb{T}^n_{\theta/2})$ and defines $L^2(S^m_\theta, \mathcal{S})$ to be the fixed point Hilbert subspace. If D also denotes the closure of the Dirac operator on $L^2(S^m, \mathcal{S})$, we denote the operator $D \otimes \mathrm{id}$ on $L^2(S^m, \mathcal{S}) \otimes L^2(\mathbb{T}^n_{\theta/2})$ when restricted to $L^2(S^m_\theta, \mathcal{S})$ by D.

The triple $(C^\infty(S^m_\theta), L^2(S^m_\theta, \mathcal{S}), D)$ satisfies all axioms of a noncommutative spin geometry (there is also a real structure J). In fact, this construction on S^m_θ can be generalized to any compact Riemannian spin manifold, carrying an isometrical action of $\mathbb{T}^n$. For more details, we refer to [11, 10].

3. Hopf Fibration and Associated Bundles on S^4_θ

We will now construct a θ-deformation of the Hopf fibration $SU(2) \to S^7 \to S^4$. For convenience, the classical fibration is described in some detail in App. A. Firstly, we recall that while there is a θ-deformation of the manifold $S^3 \simeq SU(2)$, to a sphere S^3_θ, on the latter there is no compatible group structure so that there is no θ-deformation of the group $SU(2)$ [10]. Therefore, we must choose the matrix $\theta'_{\mu\nu}$ in such a way that the noncommutative 7-sphere $S^7_{\theta'}$ carries a classical $SU(2)$ action, which in addition is such that the subalgebra of $\mathcal{A}(S^7_{\theta'})$ consisting of $SU(2)$-invariant polynomials is exactly $\mathcal{A}(S^4_\theta)$. As expected, we will find that θ' is expressed in terms of θ. Then we construct the finitely generated projective modules $\Gamma(S^4_\theta, E^{(n)})$, associated to the irreducible representations $V^{(n)}$ of $SU(2)$ as the space of $SU(2)$-coequivariant maps from $V^{(n)}$ to $\mathcal{A}(S^7_{\theta'})$. We will construct projections $p_{(n)} \in \mathrm{Mat}_{4^n}(\mathcal{A}(S^4_\theta))$ such that $\Gamma(S^4_\theta, E^{(n)}) \simeq p_{(n)}(\mathcal{A}(S^4_\theta))^{4^n}$. In the special case of the defining representation, we recover the basic instanton projection on the sphere S^4_θ constructed in [11].

As mentioned, the interplay of the noncommutative spheres S^4_θ and $S^7_{\theta'}$ is in that $\mathcal{A}(S^7_{\theta'})$ will be required to carry an action of $SU(2)$ by automorphisms and this action is such that

$$\mathcal{A}(S^4_\theta) = \mathrm{Inv}_{SU(2)}(\mathcal{A}(S^7_{\theta'})). \tag{12}$$

These requirements will restrict the values of $\lambda'^{ij} = e^{2\pi \mathrm{i}\theta'_{ij}}$ is such a manner that there is essentially only 'one' noncommutative 7-sphere such that the invariance condition (12) is satisfied, with a compatible right $SU(2)$ action on $S^7_{\theta'}$. This action on the generators of $\mathcal{A}(S^7_{\theta'})$ is simply defined by

$$\alpha_w : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \qquad w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix}. \tag{13}$$

Here w^1 and w^2, satisfying $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$, are the coordinates on $SU(2)$. By imposing that the map $w \mapsto \alpha_w$ embeds $SU(2)$ in $\mathrm{Aut}(\mathcal{A}(S^7_{\theta'}))$ we find that $\lambda'^{12} = \lambda'^{34} = 1$ and $\lambda'^{14} = \lambda'^{23} = \lambda'^{24} = \lambda'^{13} =: \lambda'$.

In terms of the splitting homomorphism, this means that we can identify $\mathcal{A}(S^7_{\theta'})$ with a certain subalgebra of $\mathcal{A}(S^7 \times \mathbb{T}^2_{\theta'})$ instead of $\mathcal{A}(S^7 \times \mathbb{T}^4_{\theta'})$. In fact, we can write

$$\begin{aligned} z^1 &= z^1_{(0)} \otimes u, \quad & z^3 &= z^3_{(0)} \otimes v, \\ z^2 &= z^2_{(0)} \otimes u, \quad & z^4 &= z^4_{(0)} \otimes v, \end{aligned} \tag{14}$$

for two unitaries u, v satisfying $uv = \lambda' vu$, i.e. the generators of $\mathcal{A}(\mathbb{T}^2_{\theta'})$.

The subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7_{\theta'})$ can be found in the following way. By using the splitting homomorphism, a general element $a \in \mathcal{A}(S^7_{\theta'})$ can be written as a finite sum: $a = \sum a^i_{(0)} \otimes u^i$, where $a^i_{(0)} \in \mathcal{A}(S^7)$ and $u^i \in \mathcal{A}(\mathbb{T}^2_{\theta'})$. Then, from the diagonal nature of the action of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ and the above formulæ for $z^1, \ldots, z^4$ we have that $\alpha_w(a) = \sum \alpha_w(a^i_{(0)}) \otimes u^i$, encoding the fact that $SU(2)$ essentially acts classically. But this means that any invariant polynomial $a = \alpha_w(a)$ induces a classical invariant polynomial $a_{(0)}$. Hence, the subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7_{\theta'})$ is completely determined by the classical subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7)$. From App. A we can conclude that

$$\mathrm{Inv}_{SU(2)}(\mathcal{A}(S^7_\theta)) = \mathbb{C}[\, 1, z^1\overline{z}^3 + z^2\overline{z}^4, -z^1z^4 + z^2z^3, z^1\overline{z}^1 + z^2\overline{z}^2 \,] \tag{15}$$

modulo the relations in the algebra $\mathcal{A}(S^7_{\theta'})$. We identify

$$\begin{aligned} &\alpha = 2(z^1\overline{z}^3 + z^2\overline{z}^4), \qquad \beta = 2(-z^1z^4 + z^2z^3), \\ &x = z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4, \end{aligned} \tag{16}$$

and compute that $\alpha\alpha^* + \beta\beta^* + x^2 = 1$. By imposing commutation rules $\alpha\beta = \lambda\beta\alpha$ and $\alpha\beta^* = \overline{\lambda}\beta^*\alpha$, as in Example 1, we infer that $\lambda'^{14} = \lambda'^{23} = \lambda'^{24} = \lambda'^{13} = \sqrt{\lambda} =: \mu$ on $S^7_{\theta'}$. We conclude that $\mathrm{Inv}_{SU(2)}(\mathcal{A}(S^7_{\theta'})) = \mathcal{A}(S^4_\theta)$ for $\lambda'^{ij} = e^{2\pi\mathrm{i}\theta'_{ij}}$ of the following form:

$$\lambda'_{ij} = \begin{pmatrix} 1 & 1 & \mu & \mu \\ 1 & 1 & \mu & \mu \\ \overline{\mu} & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \overline{\mu} & 1 & 1 \end{pmatrix}, \quad \mu = \sqrt{\lambda}, \tag{17}$$

or equivalently

$$\theta'_{ij} = \frac{\theta}{2} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}. \tag{18}$$

There is a nice description of the instanton projection constructed in [11] in terms of ket-valued polynomials on $S^7_{\theta'}$. The latter are elements in the right $\mathcal{A}(S^7_{\theta'})$-module $\mathcal{E} := \mathbb{C}^4 \otimes \mathcal{A}(S^7_{\theta'}) =: \mathcal{A}(S^7_{\theta'})^4$ with a hermitian structure given by $\langle \xi, \eta \rangle = \sum_j \xi_j^* \eta_j$. To any $|\xi\rangle \in \mathcal{E}$ one associates its dual $\langle \xi| \in \mathcal{E}^*$ by setting $\langle \xi|(\eta) := \langle \xi, \eta \rangle, \forall \eta \in \mathcal{E}$.

Similarly to the classical case (see App. A), we define $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{A}(S^7_{\theta'})^4$ by

$$|\psi_1\rangle = (z^1, -\bar{z}^2, z^3, -\bar{z}^4)^{\mathrm{t}}, \quad |\psi_2\rangle = (z^2, \bar{z}^1, z^4, \bar{z}^3)^{\mathrm{t}}, \tag{19}$$

with t denoting transposition. They satisfy $\langle \psi_k | \psi_l \rangle = \delta_{kl}$, so that the 4×4-matrix $p = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$ is a projection, $p^2 = p = p^*$, with entries in $\mathcal{A}(S^4_\theta)$. Indeed, let us introduce the matrix

$$u = (|\psi_1\rangle, |\psi_2\rangle) = \begin{pmatrix} z^1 & z^2 \\ -\bar{z}^2 & \bar{z}^1 \\ z^3 & z^4 \\ -\bar{z}^4 & \bar{z}^3 \end{pmatrix}. \tag{20}$$

Then $u^* u = \mathbb{I}_2$ and $p = uu^*$. The action (13) becomes

$$\alpha_w(u) = uw, \tag{21}$$

from which the invariance of the entries of p follows at once. Explicitly one finds

$$p = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\mu\beta^* & \bar{\mu}\alpha^* \\ \alpha^* & -\bar{\mu}\beta & 1-x & 0 \\ \beta^* & \mu\alpha & 0 & 1-x \end{pmatrix}. \tag{22}$$

The projection p is easily seen to be equivalent to the projection describing the instanton on S^4_θ constructed in [11]. Indeed, if one defines

$$|\widetilde{\psi}_1\rangle = (z^1, -\mu\bar{z}^2, z^3, -\bar{z}^4)^{\mathrm{t}}, \quad |\widetilde{\psi}_2\rangle = (z^2, \mu\bar{z}^1, z^4, \bar{z}^3)^{\mathrm{t}}, \tag{23}$$

one obtains exactly the projection obtained therein, that is,

$$\widetilde{p} = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\lambda\beta^* & \alpha^* \\ \alpha^* & -\bar{\lambda}\beta & 1-x & 0 \\ \beta^* & \alpha & 0 & 1-x \end{pmatrix}. \tag{24}$$

We will denote the image of p in $\mathcal{A}(S^4_\theta)^4$ by $\Gamma(S^4_\theta, E) = p\mathcal{A}(S^4_\theta)^4$ which is clearly a right $\mathcal{A}(S^4_\theta)$-module. Another description of the module $\Gamma(S^4_\theta, E)$ comes from considering coequivariant maps from $\mathbb{C}^2$ to $\mathcal{A}(S^7_{\theta'})$ [16]. The defining left representation of $SU(2)$ on $\mathbb{C}^2$ is given by $SU(2) \times \mathbb{C}^2 \to \mathbb{C}^2; (w, v) \mapsto w \cdot v$. The collection $\mathrm{Hom}_{SU(2)}(\mathbb{C}^2, \mathcal{A}(S^7_{\theta'}))$ of coequivariant maps, i.e. of maps $\phi : \mathbb{C}^2 \to \mathcal{A}(S^7_{\theta'})$, such that

$$\phi(w^{-1} \cdot v) = \alpha_w(\phi(v)), \tag{25}$$

is a right $\mathcal{A}(S^4_\theta)$-module (it is in fact also a left $\mathcal{A}(S^4_\theta)$-module).

Since $SU(2)$ acts classically on $\mathcal{A}(S^7_{\theta'})$, one sees that the coequivariant maps are given on the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ by $\phi(e_k) = \langle\psi_k|f\rangle$ for $|f\rangle = |f_1, f_2, f_3, f_4\rangle^{\mathrm{t}}$, with $f_i \in \mathcal{A}(S^4_\theta)$ (cf. App. A). We then have the following isomorphism

$$\begin{aligned} &\Gamma(S^4_\theta, E) \simeq \mathrm{Hom}_{SU(2)}(\mathbb{C}^2, \mathcal{A}(S^7_{\theta'})), \\ &\sigma = p|f\rangle \leftrightarrow \phi : e_k \mapsto \langle\psi_k|f\rangle. \end{aligned} \tag{26}$$

More generally, one can define the right $\mathcal{A}(S^4_\theta)$-module $\Gamma(S^4_\theta, E^{(n)})$ associated with any irreducible representation $\rho_n : SU(2) \to GL(V^{(n)})$, with $V^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$, for a positive integer n. The module of coequivariant maps $\mathrm{Hom}_{\rho_n}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$ consists of maps $\phi : V^{(n)} \to \mathcal{A}(S^7_{\theta'})$ satisfying

$$\phi(\rho_n^{-1}(w) \cdot v) = \alpha_w(\phi(v)). \tag{27}$$

It is easy to see that these maps are of the form $\phi_{(n)}(e_k) = \langle\phi_k|f\rangle$ on the basis $\{e_1, \ldots, e_{n+1}\}$ of $V^{(n)}$, where now $|f\rangle \in \mathcal{A}(S^4_\theta)^{4^n}$ and

$$|\phi_k\rangle = \frac{1}{a_k}|\psi_1\rangle^{\otimes(n-k+1)} \otimes_S |\psi_2\rangle^{\otimes(k-1)} \quad (k = 1, \ldots, n+1), \tag{28}$$

with $\otimes_S$ denoting symmetrization and a_k are suitable normalization constants. These vectors $|\phi_k\rangle \in \mathbb{C}^{4^n} \otimes \mathcal{A}(S^7_{\theta'}) =: \mathcal{A}(S^7_{\theta'})^{4^n}$ are orthogonal (with the natural hermitian structure), and with $a_k^2 = \binom{n}{k-1}$ they are also normalized. Then

$$p_{(n)} := |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{n+1}\rangle\langle\phi_{n+1}| \in \mathrm{Mat}_{4^n}(\mathcal{A}(S^4_\theta)) \tag{29}$$

defines a projection $p^2 = p = p^*$. That its entries are in $\mathcal{A}(S^4_\theta)$ and not in $\mathcal{A}(S^7_{\theta'})$ is easily seen. Indeed, much as it happens for the vector u in (21), for every $i = 1, \ldots, 4^n$, the vector $u_{(i)} = \big(|\phi_1\rangle_i, |\phi_2\rangle_i, \ldots, |\phi_{n+1}\rangle_i\big)$ transforms under the action of $SU(2)$ to the vector $\big(|\phi_1\rangle_i, \ldots, |\phi_{n+1}\rangle_i\big) \cdot \rho_{(n)}(w)$ so that each entry $\sum_k |\phi_k\rangle_i\langle\phi_k|_j$ of $p_{(n)}$ is $SU(2)$-invariant and hence an element in $\mathcal{A}(S^4_\theta)$. With this we proved the following.

Proposition 2. *The module of coequivariant maps* $\mathrm{Hom}_{\rho_n}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$ *is isomorphic to* $\Gamma(S^4_\theta, E^{(n)}) := p_{(n)}(\mathcal{A}(S^4_\theta)^{4^n})$ *(as right-*$\mathcal{A}(S^4_\theta)$ *modules) with the isomorphism given explicitly by:*

$$\begin{aligned} \Gamma(S^4_\theta, E^{(n)}) &\simeq \mathrm{Hom}_{\rho_n}(V^{(n)}, \mathcal{A}(S^7_{\theta'})), \\ \sigma_{(n)} = p_{(n)}|f\rangle &\leftrightarrow \phi_{(n)} : e_k \mapsto \langle\phi_k|f\rangle. \end{aligned}$$

Using the splitting homomorphisms of the previous section, one can lift this whole construction to the smooth level. One proves that the $C^\infty(S^4_\theta)$-module $\Gamma^\infty(S^4_\theta, E^{(n)})$ defined by $p_{(n)}(C^\infty(S^4_\theta))^{4^n}$ is isomorphic to $\mathrm{Hom}_{\rho_n}(V^{(n)}, C^\infty(S^7_{\theta'}))$.

With the projections $p_{(n)}$ one associates (Grassmannian) connections on the modules $\Gamma(S^4_\theta, E^{(n)})$ in a canonical way:

$$\nabla = p_{(n)} \circ \mathrm{d} : \Gamma(S^4_\theta, E^{(n)}) \to \Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta), \tag{30}$$

where $(\Omega^*(S^4_\theta), \mathrm{d})$ is the differential calculus defined in the previous section. An expression for these connections as acting on coequivariant maps can be obtained using the above isomorphism and results in:

$$\nabla(\phi)(e_k) = \mathrm{d}(\phi(e_k)) + A_{kl}\phi(e_l), \tag{31}$$

where $A_{kl} = \langle \phi_k | \mathrm{d}\phi_l \rangle \in \Omega^1(S^7_{\theta'})$. The corresponding matrix A is called the connection one-form; it is clearly anti-hermitian, and it is valued in the derived representation space, $\rho'_n : su(2) \to \mathrm{End}(V^{(n)})$, of the Lie algebra $su(2)$.

The case $n = 1$ describes classically (i.e. $\theta = 0$) the charge -1 instanton [2]. An instanton is defined as a connection on $\Gamma(S^4, E)$ with (anti-)selfdual curvature F, i.e $*F = \pm F$ with $*$ the Hodge star operator. In physics, instantons are of importance since they are extrema of the Yang-Mills action. The equation of motion obtained from this action by a variatonal method is called the Yang-Mills equation $[\nabla, *F] = 0$. In the case that F is (anti-)selfdual, this equation of motion follows directly from the Bianchi identity $[\nabla, F] = 0$. There is no need to stress the huge importance of instantons (and in general Yang-Mills gauge theory) both in physics and in mathematics.

On the noncommutative sphere S^4_θ, the curvature of the connection $p \circ \mathrm{d}$ constructed above satisfies the following anti-selfdual equation [10] (see also [1, 25])[1]:

$$*_\theta\, p(\mathrm{d}p)^2 = -p(\mathrm{d}p)^2. \tag{32}$$

In order to fully justify the name instanton, one should find a noncommutative analogue of the Yang-Mills action such that connections with an (anti-)selfdual curvature are its extrema. This will be discussed elsewhere [26].

4. Index of Dirac Operators

We know from Sect. 2 that there is a structure of noncommutative spin geometry on the sphere S^4_θ given by a 'triple' $(C^\infty(S^4_\theta), L^2(S^4_\theta, \mathcal{S}), D, \gamma)$ with $\gamma = \gamma_5$ the grading. In this section we shall compute explicitly the index of the Dirac operator with coefficients in the bundles $E^{(n)}$, that is the index of the operator of $D_{p_{(n)}} := p_{(n)}(D \otimes \mathbb{I}_{4^n})p_{(n)}$. In order to do that, we will use the ('even dimensional' version of the) local index formula of Connes and Moscovici [12] which we shall briefly describe.

Suppose in general that $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is an even p-summable spectral triple with discrete simple dimension spectrum. Let $C_*(\mathcal{A})$ be the complex consisting of cycles over the algebra $\mathcal{A}$, that is in degree n, $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$. On this complex there are defined the Hochschild operator $b : C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A})$ and the boundary operator $B : C_n(\mathcal{A}) \to C_{n+1}(\mathcal{A})$, satisfying $b^2 = 0$, $B^2 = 0$, $bB + Bb = 0$; thus $(b + B)^2 = 0$. From general homological theory, one defines a bicomplex $CC_*(\mathcal{A})$ by $CC_{(n,m)}(\mathcal{A}) :=$

[1] An early attempt to write self-duality equations in terms of projections was done in [15].

$CC_{n-m}(\mathcal{A})$ in bi-degree (n, m). Dually, one defines $CC^*(\mathcal{A})$ as functionals on $CC_*(\mathcal{A})$, equipped with the dual Hochschild operator b and coboundary operator B (we refer to [9] and [27] for more details on this).

Theorem 3 (Connes–Moscovici [12]).

(a) An even cocycle $\phi^* = \sum_{k\geq 0} \phi^k$ in $CC^*(\mathcal{A})$, $(b+B)\phi^* = 0$, is defined by the following formulæ. For $k = 0$,

$$\phi^0(a) := \operatorname*{Res}_{z=0} z^{-1}\mathrm{Tr}(\gamma a|D|^{-2z}); \tag{33}$$

whereas for $k \neq 0$,

$$\phi^{2k}(a^0, \dots, a^{2k}) := \sum_\alpha c_{k,\alpha} \times \operatorname*{Res}_{z=0} \mathrm{Tr}\big(\gamma a^0[D, a^1]^{(\alpha_1)} \cdots [D, a^{2k}]^{(\alpha_{2k})}|D|^{-2(|\alpha|+k+z)}\big), \tag{34}$$

where

$$c_{k,\alpha} = (-1)^{|\alpha|}\Gamma(k + |\alpha|)\big(\alpha!(\alpha_1 + 1)(\alpha_1 + \alpha_2 + 2)\cdots(\alpha_1 + \cdots + \alpha_{2k} + 2k)\big)^{-1}$$

and $T^{(j)}$ denotes the j^{th} iteration of the derivation $T \mapsto [D^2, T]$.
(b) For $e \in K_0(\mathcal{A})$, the Chern character $\mathrm{ch}_*(e) = \sum_{k\geq 0} \mathrm{ch}_k(e)$ is the even cycle in $CC_*(\mathcal{A})$, $(b+B)\mathrm{ch}_*(e) = 0$, defined by the following formulæ. For $k = 0$,

$$\mathrm{ch}_0(e) := \mathrm{Tr}(e); \tag{35}$$

whereas for $k \neq 0$

$$\mathrm{ch}_k(e) := (-1)^k \frac{(2k)!}{k!} \sum (e_{i_0 i_1} - \frac{1}{2}\delta_{i_0 i_1}) \otimes e_{i_1 i_2} \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2k} i_0}. \tag{36}$$

(c) The index is given by the natural pairing between cycles and cocycles

$$\mathrm{Ind}\, D_e = \langle \phi^*, \mathrm{ch}_*(e)\rangle. \tag{37}$$

We concentrate on a compact Riemannian spin manifold M of even dimension carrying an isometric action of an n-torus. Set $\mathcal{H} := L^2(M, \mathcal{S})$ and recall the grading on $\mathcal{B}(\mathcal{H})$ with respect to the action of $\mathbb{T}^n$ [11]. An element $T \in \mathcal{B}(\mathcal{H})$ that is smooth for the action of $\mathbb{T}^n$, (i.e. such that the map $\mathbb{T}^n \ni s \mapsto \alpha_s(T)$ with α_s defined by $\alpha_s(T) := U(s)TU(s)^{-1}$ is smooth for the norm topology) can be expanded as $T = \sum T_r$ with $r = (r_1, r_2, \dots, r_n)$ a multi-index, and with each T_r of homogeneous degree r under the action of $\mathbb{T}^n$, i.e.

$$\alpha_s(T_r) = e^{2\pi \mathrm{i}(\sum_{\mu=1}^n r_\mu s_\mu)} T_r \quad (s \in \mathbb{T}^n).$$

Note that α_s coincides on $\pi(C^\infty(M)) \subset \mathcal{B}(\mathcal{H})$ with the automorphism σ_s defined in Sect. 2. Then, let $(p_1, p_2, \dots, p_n)$ be the infinitesimal generators of the action of $\mathbb{T}^n$ so that $U(s) = \exp 2\pi \mathrm{i}(\sum_{\mu=1}^n s_\mu p_\mu)$. For $T \in \mathcal{B}(\mathcal{H})$ we define a twisted representation on $\mathcal{H}$ by

$$L_\theta(T) := \sum_r T_r U(r_\mu\theta_{\mu 1}, \dots, r_\mu\theta_{\mu n}) = \sum_r T_r \exp\big\{2\pi \mathrm{i} \sum_\mu r_\mu \theta_{\mu\nu} p_\nu\big\} \tag{38}$$

with θ an $n \times n$ anti-symmetric matrix. Since $\mathbb{T}^n$ acts by isometries, each p_μ commutes with D so that the latter is of degree 0 and $L_\theta([D, a]) = [D, L_\theta(a)]$ for $a \in C^\infty(M)$. It was shown in [11] (to which we refer for more details) that $(L_\theta(C^\infty(M)), \mathcal{H}, D)$ satisfies all axioms of a noncommutative spin geometry (there is also a grading $\gamma = \gamma_5$ and a real structure J). In fact, the algebra $L_\theta(C^\infty(M))$ is isomorphic to the algebra $C^\infty(M_\theta)$ of the previous section.

As a next step, we write the ϕ^{2k} that define the local index formula in (34), by means of the twist L_θ. Let $f^0, \dots, f^{2k} \in C^\infty(M)$ and suppose that the operator $f^0[D, f^1]\cdots[D, f^{2k}]$ is a homogeneous element of degree r. Then, as in (38)

$$L_\theta(f^0[D, f^1]\cdots[D, f^{2k}]) = f^0[D, f^1]\cdots[D, f^{2k}]U(r_\mu\theta_{\mu 1}, \dots, r_\mu\theta_{\mu n}). \tag{39}$$

Each term in the local index formula for $(L_\theta(C^\infty(M)), \mathcal{H}, D)$ then takes the form

$$\operatorname*{Res}_{z=0} \operatorname{Tr}\big(\gamma f^0[D, f^1]^{(\alpha_1)}\cdots[D, f^{2k}]^{(\alpha_{2k})}|D|^{-2(|\alpha|+k+z)}U(s)\big)$$

for $s_\nu = r_\mu\theta_{\mu\nu}$ so that $s \in \mathbb{T}^n$. The appearance of $U(s)$ here, is a consequence of the close relation with the index formula for a $\mathbb{T}^n$-equivariant Dirac spectral triple on M. In [8], Chern and Hu considered an even dimensional compact spin manifold M on which a (connected compact) Lie group G acts by isometries. The equivariant Chern character was defined as an equivariant version of the JLO-cocycle, the latter being an element in equivariant entire cyclic cohomology. The essential point is that they obtained an explicit formula for the above residues. In the case of the previous $\mathbb{T}^n$-action on M, one gets

$$\begin{aligned}&\operatorname*{Res}_{z=0} \operatorname{Tr}\big(\gamma f^0[D, f^1]^{(\alpha_1)}\cdots[D, f^{2k}]^{(\alpha_{2k})}|D|^{-2(|\alpha|+k+z)}U(s)\big)\\ &\qquad = \Gamma(|\alpha|+k)\lim_{t\to 0} t^{|\alpha|+k}\operatorname{Tr}\big(\gamma f^0[D, f^1]^{(\alpha_1)}\cdots[D, f^{2k}]^{(\alpha_{2k})}e^{-tD^2}U(s)\big)\end{aligned} \tag{40}$$

for every $s \in \mathbb{T}^n$; moreover, this limit vanishes when $|\alpha| \neq 0$ (Thm 2 in [8]). Combining these results, we arrive at the following lemma.

Lemma 4. *Let $(L_\theta(C^\infty(M)), \mathcal{H}, D)$ be the spectral triple defined above. Then all terms in ϕ^* with $|\alpha| \neq 0$ vanish and the local index formula takes the form:*

$$\phi^{2k}(a^0, \dots, a^{2k}) = c_k \operatorname*{Res}_{z=0} \operatorname{Tr}\big(\gamma a^0[D, a^1]\cdots[D, a^{2k}]|D|^{-2(k+z)}\big), \tag{41}$$

where $c_k = (k-1)!/(2k)!$.

In our case of interest, the index of the Dirac operator on S^4_θ with coefficients in some noncommutative vector bundle determined by $e \in K_0(C(S^4_\theta))$, we obtain

$$\begin{aligned}\operatorname{Ind} D_e = \langle\phi^*, \mathrm{ch}_*(e)\rangle &= \operatorname*{Res}_{z=0} z^{-1}\operatorname{Tr}\big(\gamma\pi_D(\mathrm{ch}_0(e))|D|^{-2z}\big)\\ &\quad + \frac{1}{2!}\operatorname*{Res}_{z=0} \operatorname{Tr}\big(\gamma\pi_D(\mathrm{ch}_1(e))|D|^{-2-2z}\big)\\ &\quad + \frac{1}{4!}\operatorname*{Res}_{z=0} \operatorname{Tr}\big(\gamma\pi_D(\mathrm{ch}_2(e))|D|^{-4-2z}\big).\end{aligned} \tag{42}$$

Here π_D is the representation of the universal differential calculus given by

$$\pi_D : \Omega^p_{\mathrm{un}}(\mathcal{A}(S^4_\theta)) \to \mathcal{B}(\mathcal{H}), \quad a^0\delta a^1\cdots\delta a^p \mapsto a^0[D, a^1]\cdots[D, a^p]. \tag{43}$$

Let us examine at which quotients of $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ this representation π_D is well-defined. Unfortunately, π_D is not well-defined on the quotient $\Omega(S^4_\theta)$ defined in the previous section. For example already $[D,\alpha][D,\alpha] \neq 0$ whereas $d\alpha d\alpha = 0$ in $\Omega(S^4_\theta)$. This was already noted in [10] and in fact

$$\Omega(S^4_\theta) \simeq \pi_D\big(\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))\big)/\pi_D(\delta J_0), \tag{44}$$

where $J_0 := \{\omega \in \Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta)) | \pi_D(\omega) = 0\}$ are the so-called 'junk-forms' [9]. We will avoid a discussion on junk-forms and introduce instead a different quotient of $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$. We define $\Omega_D(S^4_\theta)$ to be $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ modulo the relations

$$\begin{aligned}
&\alpha\delta\beta - \lambda(\delta\beta)\alpha = 0, \quad (\delta\alpha)\beta - \lambda\beta\delta\alpha = 0,\\
&\alpha\delta\beta^* - \bar\lambda(\delta\beta^*)\alpha = 0, \quad (\delta\alpha^*)\beta - \bar\lambda\beta\delta\alpha^* = 0,\\
&a\delta x - (\delta x)a = 0, \qquad \forall a \in \mathcal{A}(S^4_\theta),
\end{aligned} \tag{45}$$

avoiding the second order relations that define $\Omega(S^4_\theta)$. Using the splitting homomorphism one proves that the above relations are in the kernel of π_D, for instance, $\alpha[D,\beta] - \lambda[D,\beta]\alpha = 0$ so that π_D is well-defined on $\Omega_D(S^4_\theta)$.

In App. B we compute the Chern characters as elements in $\Omega_D(S^4_\theta)$, which results in the following lemma.

Lemma 5. *The following formulæ hold for the images under π_D of the Chern characters of $p_{(n)}$:*

$$\begin{aligned}
\pi_D(\mathrm{ch}_0(p_{(n)})) &= n+1;\\
\pi_D(\mathrm{ch}_1(p_{(n)})) &= 0;\\
\pi_D(\mathrm{ch}_2(p_{(n)})) &= \frac{1}{6}n(n+1)(n+2)\pi_D(\mathrm{ch}_2(p_{(1)}));
\end{aligned}$$

up to the coefficients $\mu_k = (-1)^k \frac{(2k)!}{k!}$. $\square$

Combining this with the simple form of the index formula while taking the proper coefficients, we find that

$$\mathrm{Ind}\, D_{p_{(n)}} = \frac{1}{4!}\frac{4!}{2!}\frac{1}{6}n(n+1)(n+2)\operatorname*{Res}_{z=0} \mathrm{Tr}\big(\gamma\pi_D(\mathrm{ch}_2(p_{(1)}))|D|^{-4-2z}\big), \tag{46}$$

where for the vanishing of the first term, we used the fact that $\mathrm{Ind}\, D = 0$, since the first Pontrjagin class on S^4 vanishes. Theorem I.2 in [12] allows one to express the residue as a Dixmier trace. Combining this with $\pi_D(\mathrm{ch}_2(p_{(1)})) = 3\gamma$ (as computed in [11]), we obtain

$$3 \cdot \operatorname*{Res}_{z=0} \mathrm{Tr}(|D|^{-4-2z}) = 6 \cdot \mathrm{Tr}_\omega(|D|^{-4}) = 2$$

since the Dixmier trace of $|D|^{-m}$ on the m-sphere equals $8/m!$ (cf. for instance [17, 23]). This combines to give:

Proposition 6. *The index of the Dirac operator on S^4_θ with coefficients in $E^{(n)}$ is given by:*

$$\mathrm{Ind}\, D_{p_{(n)}} = \frac{1}{6}n(n+1)(n+2). \quad \square$$

Note that this coincides with the classical result.

5. The Noncommutative Principal Bundle

In this section, we apply the general theory of Hopf-Galois extensions [21, 28] to the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$. Such extensions can be understood as noncommutative principal bundles. We will first dualize the construction of the previous section, i.e. replace the action of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ by a coaction of $\mathcal{A}(SU(2))$. Then, we will recall some definitions involving Hopf-Galois extensions and principality ([6]) of such extensions. We show that $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft (i.e. not-trivial) principal Hopf-Galois extension and compare the connections on the associated bundles, induced from the strong connection, with the Grassmannian connection defined in Sect. 3.

The action of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ by automorphisms can be easily dualized to a coaction $\Delta_R : \mathcal{A}(S^7_{\theta'}) \to \mathcal{A}(S^7_{\theta'}) \otimes \mathcal{A}(SU(2))$, where now $\mathcal{A}(SU(2))$ is the unital complex $*$-algebra generated by $w^1, \overline{w}^1, w^2, \overline{w}^2$ with relation $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$. Clearly, $\mathcal{A}(SU(2))$ is a Hopf algebra with comultiplication

$$\Delta : \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \mapsto \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \otimes \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix}, \tag{47}$$

antipode $S(w^1) = \overline{w}^1, S(w^2) = -w^2$ and counit $\epsilon(w^1) = \epsilon(\overline{w}^1) = 1, \epsilon(w^2) = \epsilon(\overline{w}^2) = 0$. The coaction of $\mathcal{A}(SU(2))$ on $\mathcal{A}(S^7_{\theta'})$ is given by

$$\Delta_R : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \otimes \begin{pmatrix} w^1 & w^2 & 0 & 0 \\ -\overline{w}^2 & \overline{w}^1 & 0 & 0 \\ 0 & 0 & w^1 & w^2 \\ 0 & 0 & -\overline{w}^2 & \overline{w}^1 \end{pmatrix}. \tag{48}$$

The algebra of coinvariants in $\mathcal{A}(S^7_{\theta'})$, which consists of elements $p \in \mathcal{A}(S^7_{\theta'})$ satisfying $\Delta_R(p) = p \otimes 1$, can be identified with $\mathcal{A}(S^4_\theta)$ for the particular values of θ'_{ij} found before, in the same way as in Sect. 3.

The associated modules $\Gamma(S^4_\theta, E^{(n)})$ are described in the following way. Given an irreducible corepresentation of $\mathcal{A}(SU(2))$, $\rho_{(n)} : V^{(n)} \to \mathcal{A}(SU(2)) \otimes V^{(n)}$ with $V^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$, we denote $\rho_{(n)}(v) = v_{(0)} \otimes v_{(1)}$. Then, the module of coequivariant maps $\mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$ consists of maps $\phi : V^{(n)} \to \mathcal{A}(S^7_{\theta'})$ satisfying

$$\phi(v_{(1)}) \otimes S v_{(0)} = \Delta_R \phi(v); \quad v \in \mathbb{C}^2. \tag{49}$$

Again, such maps are $\mathbb{C}$-linear maps of the form $\phi_{(n)}(e_k) = \langle \phi_k | f \rangle$ on the basis $\{e_1, \ldots, e_{n+1}\}$ of $V^{(n)}$ in the notation of the previous section. Also, Proposition 2 above translates straightforwardly into the isomorphism $\mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \simeq p_{(n)}(\mathcal{A}(S^4_\theta))^{4^n}$ for the projections defined in Eq. (29).

Before we proceed, recall that for an algebra P and a subalgebra $B \subset P$, $P \otimes_B P$ denotes the quotient of the tensor product $P \otimes P$ by the ideal generated by expressions $p \otimes bp' - pb \otimes p'$, for $p, p' \in P, b \in B$.

Definition 7. *Let H be a Hopf algebra and P a right H-comodule algebra, i.e. such that the coaction $\Delta_R : P \to P \otimes H$ is an algebra map. Let the algebra of coinvariants be $B := \mathrm{Coinv}_{\Delta_R}(P) := \{p \in P : \Delta_R(p) = p \otimes 1\}$. One says that $B \hookrightarrow P$ is a* **Hopf-Galois extension** *if the canonical map*

$$\chi : P \otimes_B P \to P \otimes H; \quad p' \otimes_B p \mapsto p' \Delta_R(p) = p' p_{(0)} \otimes p_{(1)} \tag{50}$$

is bijective.

We use Sweedler-like notation for the coaction: $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$. The canonical map is left P-linear and right H-colinear and is a morphism (an isomorphism for Hopf-Galois extensions) of left P-modules and right H-comodules. It is also clear that P is both a left and a right B-module.

Classically, the notion of Hopf-Galois extension corresponds to freeness of the action of a Lie group G on a manifold P. Indeed, freeness can be translated into bijectivity of the map

$$\tilde{\chi} : P \times G \to P \times_G P, \quad (p, g) \mapsto (p, p \cdot g), \tag{51}$$

where $P \times_G P$ denotes the fibred direct product consisting of elements (p, p') with the same image under the quotient map $P \to P/G$.

For a Hopf algebra H which is cosemisimple, surjectivity of the canonical map (50) implies its bijectivity [31]. Moreover, in order to prove surjectivity of χ, it is enough to prove that for any generator h of H, the element $1 \otimes h$ is in the image of the canonical map. Indeed, if $\chi(g_k \otimes_B g_k') = 1 \otimes g$ and $\chi(h_l \otimes_B h_l') = 1 \otimes h$ for $g, h \in H$, then $\chi(g_k h_l \otimes_B h_l' g_k') = g_k h_l \chi(1 \otimes_B h_l' g_k') = 1 \otimes hg$, using the fact that the canonical map restricted to $1 \otimes_B P$ is a homomorphism. Extension to all of $P \otimes_B P$ then follows from left P-linearity of χ. It would also be easy to write down an explicit expression for the inverse of the canonical map. Indeed, one has $\chi^{-1}(1 \otimes hg) = g_k h_l \otimes_B h_l' g_k'$ in the above notation so that the general form of the inverse follows again from left P-linearity.

Proposition 8. *The inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a Hopf-Galois extension.*

Proof. Since $\mathcal{A}(SU(2))$ is cosemisimple, we can rely for a proof of this statement on the previous remarks. On the other hand, it is straightforward to check that in terms of the ket-valued polynomials defined in (19) we have

$$\chi\left(\textstyle\sum_i \langle\psi_1|_i \otimes_{\mathcal{A}(S^4_\theta)} |\psi_1\rangle_i\right) = 1 \otimes w^1; \qquad \chi\left(\sum_i \langle\psi_1|_i \otimes_{\mathcal{A}(S^4_\theta)} |\psi_2\rangle_i\right) = 1 \otimes w^2;$$

$$\chi\left(\textstyle\sum_i \langle\psi_2|_i \otimes_{\mathcal{A}(S^4_\theta)} |\psi_1\rangle_i\right) = -1 \otimes \overline{w}^2; \qquad \chi\left(\sum_i \langle\psi_2|_i \otimes_{\mathcal{A}(S^4_\theta)} |\psi_2\rangle_i\right) = 1 \otimes \overline{w}^1. \quad \square$$

In the definition of a principal bundle in differential geometry there is much more than the requirement of bijectivity of the canonical map. It turns out that our 'structure group' being $H = \mathcal{A}(SU(2))$ which, besides being cosemisemple has also bijective antipode, all additional desired properties follow from the surjectivity of the canonical map which we have just established. We refer to [30, 6] for the full fledged theory while giving only the basic definitions that we shall need.

For our purposes, a better algebraic translation of the notion of a principal bundle is encoded in the requirement that the extension $B \subset P$, besides being Hopf-Galois, is also faithfully flat. We recall [22] that a right module P over a ring R is said to be **faithfully flat** if the functor $P \otimes_R \cdot$ is exact and faithful on the category ${}_R\mathcal{M}$ of left R-modules. Flatness means that the functor associates exact sequences of abelian groups to exact sequences of R-modules and the functor is faithful if it is injective on morphisms. Equivalently one could state that a right module P over a ring R is faithfully flat if a sequence $M' \to M \to M''$ in ${}_R\mathcal{M}$ is exact if and only if $P \otimes_R M' \to P \otimes_R M \to P \otimes_R M''$ is exact.

As mentioned, from the fact that $H = \mathcal{A}(SU(2))$ is both cosemisimple and has also bijective antipode, the faithful flatness of $\mathcal{A}(S^7_{\theta'})$ as a right (as well as left) $\mathcal{A}(S^4_\theta)$-module follows from the surjectivity of the canonical map ([31], Th. I).

One says that a principal Hopf-Galois extension is **cleft** if there exists a (unital) convolution-invertible colinear map $\phi : H \to P$, called a **cleaving map** [13, 30]. Classically, this notion is close (although not equivalent) to triviality of a principal bundle [14]. In [7] (cf. [19]) it is shown that if a principal Hopf-Galois extension is cleft, its associated modules are trivial, i.e. isomorphic to the free module B^N for some N. In our case, we can conclude the following.

Proposition 9. *The Hopf-Galois extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is not cleft.*

Proof. This is a simple consequence of the nontriviality of the Chern characters of the projection $p_{(n)}$ as seen in Sect. 4. Indeed, this implies that the associated modules are nontrivial. □

Summing up what we have shown up to now, we have the following

Theorem 10. *The inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft faithfully flat $\mathcal{A}(SU(2))$-Hopf-Galois extension.*

An important consequence is the existence of a so-called **strong connection** [18, 13]. In fact, the existence of such a connection could be used to give a more intuitive definition of 'principality of an extension' [6]. Let us first recall that if H is cosemisimple and has a bijective antipode, then a H-Hopf-Galois extension $B \hookrightarrow P$ is **equivariantly projective**, that is, there exists a left B-linear right H-colinear splitting $s : P \to B \otimes P$ of the multiplication map $m : B \otimes P \to P$, $m \circ s = \mathrm{id}_P$ [30]. Such a map characterizes a strong connection.

Definition 11. *Let $B \hookrightarrow P$ be a H-Hopf-Galois extension. A* **strong connection one-form** *is a map $\omega : H \to \Omega^1_{\mathrm{un}} P$ satisfying*

1. $\overline{\chi} \circ \omega = 1 \otimes (\mathrm{id} - \epsilon)$, *(fundamental vector field condition),*
2. $\Delta_{\Omega^1_{\mathrm{un}}(P)} \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{Ad}_R$, *(right adjoint colinearity),*
3. $\delta p - p_{(0)} \omega(p_{(1)}) \in (\Omega^1_{\mathrm{un}} B) P, \ \forall p \in P$, *(strongness condition).*

Here $\Delta_R : P \to P \otimes H$, $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$, is extended to $\Delta_{\Omega^1_{\mathrm{un}}(P)}$ on $\Omega^1_{\mathrm{un}} P \subset P \otimes P$ in a natural way by

$$\Delta_{\Omega^1_{\mathrm{un}}(P)}(p' \otimes p) \mapsto p'_{(0)} \otimes p_{(0)} \otimes p'_{(1)} p_{(1)}, \tag{52}$$

and $\mathrm{Ad}_R(h) = h_{(2)} \otimes S(h_{(1)}) h_{(3)}$ is the right adjoint coaction of H. Finally, the map $\overline{\chi} : P \otimes P \to P \otimes H$ is defined like the canonical map as $\overline{\chi}(p' \otimes p) = p' p_{(0)} \otimes p_{(1)}$.

As shown in [6] (cf. [5, 20]), a strong connection can always be given by a map $\ell : H \to P \otimes P$ satisfying

$$\begin{aligned} \ell(1) &= 1 \otimes 1, \\ \overline{\chi}(\ell(h)) &= 1 \otimes h, \\ (\ell \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta_R) \circ \ell, \\ (\mathrm{id} \otimes \ell) \circ \Delta &= (\Delta_L \otimes \mathrm{id}) \circ \ell, \end{aligned} \tag{53}$$

where $\Delta_L : P \to H \otimes P$, $p \mapsto S^{-1}p_{(1)} \otimes p_{(0)}$. Then, one defines the connection one-form by

$$\omega : h \mapsto \ell(h) - \epsilon(h)1 \otimes 1. \tag{54}$$

Indeed, if one writes $\ell(h) = h^{\langle 1\rangle} \otimes h^{\langle 2\rangle}$ (summation understood) and applies $\mathrm{id} \otimes \epsilon$ to the second formula in (53), one has $h^{\langle 1\rangle}h^{\langle 2\rangle} = \epsilon(h)$. Therefore,

$$\omega(h) = h^{\langle 1\rangle}\delta h^{\langle 2\rangle}, \tag{55}$$

where $\delta : P \to \Omega^1_{\mathrm{un}}P,\ p \mapsto 1 \otimes p - p \otimes 1$. Equivariant projectivity of $B \hookrightarrow P$ follows by taking as splitting of the multiplication the map $s : P \to B \otimes P,\ p \to p_{(0)}\ell(p_{(1)})$.

For later use, we prove the following lemma, analogous to the strongness Condition 3 above.

Lemma 12. *Let ω be a strong connection one-form on a H-Hopf-Galois extension $B \hookrightarrow P$ with the antipode of H invertible. Then*

$$\delta p + \omega(S^{-1}p_{(1)})p_{(0)} \in P\Omega^1_{\mathrm{un}}B, \quad \forall p \in P.$$

Proof. By writing ω in terms of ℓ it follows that $\delta p + \omega(S^{-1}p_{(1)})p_{(0)}$ reduces to the expression $-p \otimes 1 + l(S^{-1}p_{(1)})p_{(0)}$. From the second property of ℓ in (53), it follows that this expression is in the kernel of $\overline{\chi}$. Since χ is an isomorphism, $\delta p + \omega(S^{-1}p_{(1)})p_{(0)}$ is in the ideal generated by expressions of the form $p \otimes bp' - pb \otimes p'$. In other words, it is an element in $P\Omega^1_{\mathrm{un}}(B)P$. Finally, it is not difficult to show that

$$(\mathrm{id} \otimes \Delta_R)\big(\delta p + \omega(S^{-1}p_{(1)})p_{(0)}\big) = \big(\delta p + \omega(S^{-1}p_{(1)})p_{(0)}\big) \otimes 1,$$

from which we conclude that $\delta p + \omega(S^{-1}p_{(1)})p_{(0)}$ is in fact in $P\Omega^1_{\mathrm{un}}(B)$. $\square$

In our case, the existence of a strong connection follows from [30]. However, we will write an explicit expression in terms of the inverse of the canonical map. If we denote the latter when lifted to $P \otimes P$ by τ it follows that $\ell(h) = \tau(1 \otimes h)$ satisfies the same recursive relation found before for χ^{-1} (proof of Proposition 8 above): if $\ell(h) = h_l \otimes h'_l$ and $\ell(g) = g_k \otimes g'_k$, then

$$\ell(hg) = g_k h_l \otimes h'_l g'_k. \tag{56}$$

It turns out that in our case the map $\ell : H \to P \otimes P$ defined in this way defines a strong connection.

Proposition 13. *On the Hopf-Galois extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$, the following formulæ on the generators of $\mathcal{A}(SU(2))$,*

$$\begin{aligned} \ell(w^1) &= \textstyle\sum_i \langle\psi_1|_i \otimes |\psi_1\rangle_i, \quad & \ell(w^2) &= \textstyle\sum_i \langle\psi_1|_i \otimes |\psi_2\rangle_i, \\ \ell(\overline{w}^2) &= -\textstyle\sum_i \langle\psi_2|_i \otimes |\psi_1\rangle_i, \quad & \ell(\overline{w}^1) &= \textstyle\sum_i \langle\psi_2|_i \otimes |\psi_2\rangle_i \end{aligned} \tag{57}$$

define a strong connection.

Proof. We extend the expressions (57) to all of $\mathcal{A}(SU(2))$ by giving recursive relations, using formula (56). Recall the usual vector basis $\{r^{klm} : k \in \mathbb{Z}, m, n \geq 0\}$ in $\mathcal{A}(SU(2))$ given by

$$r^{klm} := \begin{cases} (-1)^n (w^1)^k (w^2)^m (\overline{w}^2)^n & k \geq 0, \\ (-1)^n (w^2)^m (\overline{w}^2)^n (\overline{w}^1)^{-k} & k < 0. \end{cases} \tag{58}$$

The recursive expressions on this basis are explicitly given by

$$\begin{aligned}
\ell(r^{k+1,mn}) &= \overline{z}^1 \ell(r^{kmn}) z^1 + z^2 \ell(r^{kmn}) \overline{z}^2 + \overline{z}^3 \ell(r^{kmn}) z^3 + z^4 \ell(r^{kmn}) \overline{z}^4, \quad k \geq 0, \\
\ell(w^{k-1,mn}) &= \overline{z}^2 \ell(r^{kmn}) z^2 + z^1 \ell(r^{kmn}) \overline{z}^1 + \overline{z}^4 \ell(r^{kmn}) z^4 + z^3 \ell(r^{kmn}) \overline{z}^3, \quad k < 0, \\
\ell(w^{k,m+1,n}) &= \overline{z}^1 \ell(r^{kmn}) z^2 - z^2 \ell(r^{kmn}) \overline{z}^1 + \overline{z}^3 \ell(r^{kmn}) z^4 - z^4 \ell(r^{kmn}) \overline{z}^3, \\
\ell(w^{km,n+1}) &= \overline{z}^2 \ell(r^{kmn}) z^1 - z^1 \ell(r^{kmn}) \overline{z}^2 + \overline{z}^4 \ell(r^{kmn}) z^3 - z^3 \ell(r^{kmn}) \overline{z}^4,
\end{aligned} \tag{59}$$

while setting $\ell(1) = 1 \otimes 1$. In essentially the same manner as was done in [4] (although much simpler in our case) we prove that ℓ defined by the above recursive relations indeed satisfies all conditions of a strong connection. □

The strong connection on the extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ induces connections on the associated modules in the following way [19]. For $\phi \in \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$, we set

$$\nabla_\omega(\phi)(v) \mapsto \delta\phi(v) + \omega(v_{(0)})\phi(v_{(1)}). \tag{60}$$

Using the right adjoint colinearity of ω and a little algebra one shows that $\nabla_\omega(\phi)$ satisfies the following coequivariance condition

$$\nabla_\omega(\phi)(v_{(1)}) \otimes S v_{(0)} = \Delta_{\Omega^1_{\mathrm{un}}(P)}\big(\nabla_\omega(\phi)(v)\big)$$

so that

$$\nabla_\omega : \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \to \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \Omega^1_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))).$$

In fact, from Lemma 12 it follows that ∇_ω is a map from $\mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$ to $\mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \otimes \Omega^1_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$. This allows one to compare it to the Grassmannian connection of Eq. (30). It turns out that the connection one-form ω coincides with the connection one-form A of Eq. (31), on the quotient $\Omega^1(S^7_{\theta'})$ of $\Omega^1_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))$. More precisely, let $\{e^{(n)}_k\}$ be a basis of $V^{(n)}$, and $e^{(n)}_{kl}$ the corresponding matrix coefficients of $\mathcal{A}(SU(2))$ in the representation $\rho_{(n)}$. An explicit expression for $\omega(e^{(n)}_{kl})$ can be obtained from Eqs. (59); for example $\omega(e^{(1)}_{kl}) = \langle\psi_k|\delta\psi_l\rangle,\ k, l = 1, 2$.

By using these and formulæ (77)–(79), one shows that

$$\pi(\omega(e^{(n)}_{kl})) = A^{(n)}_{kl} = \langle\phi_k|\mathrm{d}\phi_l\rangle,$$

where $\pi : \Omega_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'})) \to \Omega(S^7_{\theta'})$ is the quotient map.

A. Associated Modules

We will review the classical construction of the instanton bundle on S^4 [2] taking the approach of [24]. We generalize slightly and construct complex vector bundles on S^4 associated to all finite-dimensional irreducible representations of $SU(2)$.

We start by recalling the Hopf fibration $\pi : S^7 \to S^4$. Let

$$\begin{aligned} S^7 &:= \{z = (z^1, z^2, z^3, z^4) : |z^1|^2 + |z^2|^2 + |z^3|^2 + |z^4|^2 = 1\}, \\ S^4 &:= \{(\alpha, \beta, x) : \alpha\overline{\alpha} + \beta\overline{\beta} + x^2 = 1\}, \\ SU(2) &:= \{w \in GL(2, \mathbb{C}) : w^* w = w w^* = 1, \det w = 1\} \\ &= \left\{ w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} : w^1\overline{w}^1 + w^2\overline{w}^2 = 1 \right\}. \end{aligned} \tag{61}$$

The space S^7 carries a right $SU(2)$-action:

$$S^7 \times SU(2) \to S^7, \quad (z, w) \mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}. \tag{62}$$

The Hopf map is defined as a map $\pi(z^1, z^2, z^3, z^4) \mapsto (\alpha, \beta, x)$, where

$$\begin{aligned} &\alpha = 2(z^1\overline{z}^3 + z^2\overline{z}^4), \quad \beta = 2(-z^1 z^4 + z^2 z^3), \\ &x = z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4, \end{aligned} \tag{63}$$

and one computes $\alpha\overline{\alpha} + \beta\overline{\beta} + x^2 = (\sum_i |z^i|^2)^2 = 1$.

The finite-dimensional irreducible representations of $SU(2)$ are labeled by a positive integer n with $n+1$-dimensional representation space $V^{(n)} \simeq \mathrm{Sym}^n(\mathbb{C}^2)$. The space of smooth $SU(2)$-equivariant maps from S^7 to $V^{(n)}$ is defined by

$$C^\infty_{SU(2)}(S^7, V^{(n)}) := \{\phi : S^7 \to V^{(n)} : \phi(z \cdot w) = w^{-1} \cdot \phi(z)\}. \tag{64}$$

We will now construct projections $p_{(n)}$ as $N \times N$ matrices taking values in $C^\infty(S^4)$), such that $\Gamma^\infty(S^4, E^{(n)}) := p_{(n)} C^\infty(S^4)^N$ is isomorphic to $C^\infty_{SU(2)}(S^7, V^{(n)})$ as right $C^\infty(S^4)$-modules. As the notation suggests, $E^{(n)}$ is the vector bundle over S^4 associated with the corresponding representation. Let us first recall the case $n = 1$ from [24] and then use this to generate the vector bundles for any n. The $SU(2)$-equivariant maps from S^7 to $V^{(1)} \simeq \mathbb{C}^2$ are of the form

$$\phi_{(1)}(z) = \begin{pmatrix} \overline{z}^1 \\ \overline{z}^2 \end{pmatrix} f_1 + \begin{pmatrix} -z^2 \\ z^1 \end{pmatrix} f_2 + \begin{pmatrix} \overline{z}^3 \\ \overline{z}^4 \end{pmatrix} f_3 + \begin{pmatrix} -z^4 \\ z^3 \end{pmatrix} f_4, \tag{65}$$

where $f_1, \ldots, f_4$ are smooth functions that are invariant under the action of $SU(2)$, i.e. they are functions on the base space S^4.

A nice description of the equivariant maps is given in terms of ket-valued functions $|\xi\rangle$ on S^7, which are then elements in the free module $\mathcal{E} := \mathbb{C}^N \otimes C^\infty(S^7)$. The $C^\infty(S^7)$-valued hermitian structure on $\mathcal{E}$ given by $\langle \xi, \eta \rangle = \sum_j \xi_j^* \eta_j$ allows one to associate dual elements $\langle \xi | \in \mathcal{E}^*$ to each $|\xi\rangle \in \mathcal{E}$ by $\langle \xi | (\eta) := \langle \xi, \eta \rangle, \ \forall \eta \in \mathcal{E}$.

If we define $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{A}(S^7)^4$ by

$$|\psi_1\rangle = (z^1, -\overline{z}^2, z^3, -\overline{z}^4)^t; \quad |\psi_2\rangle = (z^2, \overline{z}^1, z^4, \overline{z}^3)^t, \tag{66}$$

with t denoting transposition, the equivariant maps in (65) are given by

$$\phi_{(1)}(z) = \begin{pmatrix} \langle\psi_1|f\rangle \\ \langle\psi_2|f\rangle \end{pmatrix}, \tag{67}$$

where $|f\rangle \in (C^\infty(S^4))^4 := \mathbb{C}^4 \otimes C^\infty(S^4)$. Since $\langle\psi_k|\psi_l\rangle = \delta_{kl}$ as is easily seen, we can define a projection in $M_4(C^\infty(S^4))$ by

$$p_{(1)} = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|. \tag{68}$$

Indeed, by explicit computation we find a matrix with entries in $C^\infty(S^4)$ which is the limit of the projection (22) for $\theta = 0$. Denoting the right $C^\infty(S^4)$-module $p_{(1)}(C^\infty(S^4))^4$ by $\Gamma(S^4, E^{(1)})$, we have

$$\begin{gathered} \Gamma(S^4, E^{(1)}) \simeq C^\infty_{SU(2)}(S^7, \mathbb{C}^2), \\ \sigma_{(1)} = p_{(1)}|f\rangle \leftrightarrow \phi_{(1)} = \begin{pmatrix} \langle\psi_1|f\rangle \\ \langle\psi_2|f\rangle \end{pmatrix}. \end{gathered} \tag{69}$$

For the general case, we note that the $SU(2)$-equivariant maps from S^7 to $V^{(n)}$ are of the form

$$\phi_{(n)}(z) = \begin{pmatrix} \langle\phi_1|f\rangle \\ \vdots \\ \langle\phi_{l+1}|f\rangle \end{pmatrix}, \tag{70}$$

where $|f\rangle \in C^\infty(S^4)^{4^n}$ and $|\phi_k\rangle$ is the completely symmetrized form of the tensor product $|\psi_1\rangle^{\otimes n-k+1} \otimes |\psi_2\rangle^{\otimes k-1}$ for $k = 1, \ldots, n+1$, normalized to have norm 1 as in formula (28). For example, for the adjoint representation $n = 2$, we have

$$\begin{aligned} |\phi_1\rangle &:= |\psi_1\rangle \otimes |\psi_1\rangle, \\ |\phi_2\rangle &:= \tfrac{1}{\sqrt{2}}\big(|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle\big), \\ |\phi_3\rangle &:= |\psi_2\rangle \otimes |\psi_2\rangle. \end{aligned} \tag{71}$$

Since in general, $\langle\phi_k|\phi_l\rangle = \delta_{kl}$, the matrix-valued function

$$p_{(n)} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{n+1}\rangle\langle\phi_{n+1}| \in M_{4^n}(C^\infty(S^4))$$

defines a projection whose entries are in $C^\infty(S^4)$, since each entry $\sum_k |\phi_k\rangle_i\langle\phi_k|_j$ is $SU(2)$-invariant (cf. below formula (29)). We conclude that

$$\begin{gathered} p_{(n)}(C^\infty(S^4)^{4^n}) \simeq C^\infty_{SU(2)}(S^7, V^{(n)}), \\ \sigma_{(n)} = p_{(n)}|f\rangle \leftrightarrow \phi_{(n)} = \begin{pmatrix} \langle\phi_1|f\rangle \\ \vdots \\ \langle\phi_{n+1}|f\rangle \end{pmatrix}. \end{gathered} \tag{72}$$

B. Chern Characters

We will compute the Chern characters of the projections $p_{(n)}$ defined in Sect. 3. It turns out to be sufficient for our purposes to obtain expressions in the differential calculus $\Omega_D(S^4_\theta) \subset \Omega_D(S^7_{\theta'})$ defined in Sect. 3, which is a quotient of the universal differential calculus.

In the definition of the Chern character (36) we can replace the tensor product by the universal differential δ by the isomorphism

$$\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes p} \simeq \Omega^p_{\mathrm{un}}(\mathcal{A}),$$

where $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{C}\mathbb{I}$ and $\Omega_{\mathrm{un}}(\mathcal{A}) = \oplus_p \Omega^p_{\mathrm{un}}(\mathcal{A})$ is the universal differential algebra generated by $a \in \mathcal{A}$ and symbols δa, $a \in \mathcal{A}$ of order 1 satisfying

$$\delta(ab) = (\delta a)b + a\delta b \quad \delta(\alpha a + \beta b) = \alpha\delta a + \beta\delta b; \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}).$$

Then

$$\mathrm{ch}_k(e) := (-1)^k \frac{(2k)!}{k!}\Big\langle (e - \frac{1}{2})(\delta e)^{2k}\Big\rangle \in \Omega^{2k}_{\mathrm{un}}(\mathcal{A}). \tag{73}$$

We recall the differential calculi $\Omega_D(S^4_\theta)$ and $\Omega_D(S^7_{\theta'})$ from Sect. 3. We defined $\Omega_D(S^4_\theta)$ as the quotient of $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ by the relations

$$\begin{aligned}
\alpha\delta\beta &= \lambda(\delta\beta)\alpha, \quad (\delta\alpha)\beta = \lambda\beta\delta\alpha,\\
\alpha\delta\beta^* &= \overline{\lambda}(\delta\beta^*)\alpha, \quad (\delta\alpha^*)\beta = \overline{\lambda}\beta\delta\alpha^*,\\
a\delta x &= (\delta x)a, \qquad (a \in \mathcal{A}(S^4_\theta)).
\end{aligned} \tag{74}$$

The inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ extends to an injective map $\Omega_D(S^4_\theta) \to \Omega_D(S^7_{\theta'})$, where $\Omega_D(S^7_{\theta'})$ is the quotient of $\Omega_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))$ by only the relations in (5) of order one, that is by the relations:

$$z^i\delta z^j = \lambda^{ij}(\delta z^j)z^i; \quad z^i\delta\overline{z}^j = \lambda^{ji}(\delta\overline{z}^j)z^i. \tag{75}$$

Recall that the projections $p_{(n)}$ were defined by $p_{(n)} = \sum_k |\phi_k\rangle\langle\phi_k|$, where $|\phi_k\rangle$ with $k = 1, \dots, n+1$, is given by

$$|\phi_k\rangle = \frac{1}{a_k}|\psi_1\rangle^{\otimes(n-k+1)} \otimes_S |\psi_2\rangle^{\otimes(k-1)}, \quad a_k^2 = \begin{pmatrix} n \\ k-1 \end{pmatrix}. \tag{76}$$

Before we start the computation of the Chern characters, we state the computation rules in $\Omega_D(S^7_{\theta'})$. Firstly, from the very definition of the vectors $|\phi_k\rangle$ and the inner product in $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{E}$, we can express, for any $k = 1, \dots, n+1$,

$$\langle\phi_k|\delta\phi_{k-1}\rangle = \sqrt{(n-k)(k+1)}\langle\psi_2|\delta\psi_1\rangle, \tag{77}$$

$$\langle\phi_k|\delta\phi_{k+1}\rangle = \sqrt{(n-k-1)(k+2)}\langle\psi_1|\delta\psi_2\rangle, \tag{78}$$

$$\begin{aligned}
\langle\phi_k|\delta\phi_k\rangle &= (n-k-1)\langle\psi_1|\delta\psi_1\rangle + (k+1)\langle\psi_2|\delta\psi_2\rangle\\
&= (n-2k-2)\langle\psi_1|\delta\psi_1\rangle,
\end{aligned} \tag{79}$$

by using the relation $\langle\psi_2|\delta\psi_2\rangle = -\langle\psi_1|\delta\psi_1\rangle$. The previous are in fact the only nonzero expressions for $\langle\phi_k|\delta\phi_l\rangle$. If we apply δ to these equations, we obtain expressions for $\langle\delta\phi_k|\delta\phi_l\rangle$ in terms of ψ_1 and ψ_2. From this, we deduce the following result that will be central in the computation of the Chern characters.

Lemma 14. *The following relations hold in* $\Omega_D(S^7_{\theta'})$*:*

$$\sum_{k,l=1}^{n+1} \langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_k\rangle = \frac{1}{6}n(n+1)(n+2)\sum_{r,s=1}^{2}\langle\psi_r|\delta\psi_s\rangle\langle\psi_s|\delta\psi_r\rangle,$$

$$\sum_{k,l,m=1}^{n+1} \langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m|\delta\phi_k\rangle = \frac{1}{6}n(n+1)(n+2) \times \sum_{r,s,t=1}^{2}\langle\psi_r|\delta\psi_s\rangle\langle\psi_s|\delta\psi_t\rangle\langle\psi_t|\delta\psi_r\rangle.$$

Of course, there will be similar formulæ for $\langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_k\rangle$, etc.

The zeroeth Chern character is easy to compute:

$$\mathrm{ch}_0(p_{(n)}) = \mathrm{Tr}(p_{(n)}) = \sum_k \langle\phi_k|\phi_k\rangle = n+1. \tag{80}$$

In the computation of $\mathrm{ch}_1(p_{(n)})$ we use the relation $\langle\delta\phi_k|\phi_l\rangle = -\langle\phi_k|\delta\phi_l\rangle$, which follows from applying the derivation δ to $\langle\phi_k|\phi_l\rangle = \delta_{kl}$ and the fact that in $\Omega_D(S^7_{\theta'})$, $\langle\phi_k|\delta\phi_l\rangle$ commutes with any element in $\mathcal{A}(S^7_{\theta'})$, in particular with $\langle\phi_m|_i$. Thus,

$$\begin{aligned}\mathrm{ch}_1(p) &= \sum|\phi_k\rangle\langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m| + \sum|\phi_k\rangle\langle\delta\phi_k|\delta\phi_m\rangle\langle\phi_m| \\ &\quad -\frac{1}{2}\sum\Big(|\delta\phi_k\rangle\langle\phi_k|\delta\phi_l\rangle\langle\phi_l| + |\delta\phi_k\rangle\langle\delta\phi_k| + |\phi_k\rangle\langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l| \\ &\quad +|\phi_k\rangle\langle\delta\phi_k|\phi_l\rangle\langle\delta\phi_l|\Big) \\ &= \frac{1}{2}\sum_{m=1}^{n+1}\big(\langle\delta\phi_m|\delta\phi_m\rangle - |\delta\phi_m\rangle\langle\delta\phi_m|\big).\end{aligned}$$

By using Eq. (79) and its analogue for $|\delta\phi_m\rangle\langle\delta\phi_m|$, $m = 1, \ldots, n+1$,

$$|\delta\phi_m\rangle\langle\delta\phi_m| = (k+1)\langle\psi_1|\delta\psi_1\rangle + (n-k-1)\langle\psi_2|\delta\psi_2\rangle,$$

we find that

$$\mathrm{ch}_1(p_{(n)}) = \frac{1}{2}n(n+1)\big(\langle\psi_1|\delta\psi_1\rangle + \langle\psi_2|\delta\psi_2\rangle\big) = \frac{1}{2}n(n+1)\mathrm{ch}_1(p_{(1)}). \tag{81}$$

Note that this equation holds in the differential subalgebra $\Omega_D(S^4_\theta)$. Since $\mathrm{ch}_1(p_{(1)})$ was shown to vanish in [11], we proved the vanishing of the first Chern character in $\Omega_D(S^4_\theta)$. The vanishing of $\mathrm{ch}_1(p_{(1)})$ can also be seen from the explicit form of $|\psi_1\rangle$ and $|\psi_2\rangle$.

A slightly more involved computation in $\Omega_D(S^7_{\theta'})$ shows that

$$\begin{aligned}\mathrm{ch}_2(p_{(n)}) = \frac{1}{2}\sum\Big\{&\delta\big(\langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m|\delta\phi_k\rangle\big) + \langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m|\delta\phi_k\rangle \\ &+\langle\delta\phi_k|\delta\phi_l\rangle\langle\delta\phi_l|\delta\phi_k\rangle + \delta\big(\langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_k\rangle\big)\Big\}.\end{aligned} \tag{82}$$

And by using Lemma 14 we finally get

$$\mathrm{ch}_2(p_{(n)}) = \frac{1}{6} n(n+1)(n+2) \mathrm{ch}_2(p_{(1)}), \tag{83}$$

as an element in $\Omega^4_D(S^4_\theta)$.

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