CHAPTER 2

Basics of noncommutative geometry

DEFINITION 7. *A* **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ *is given by an unital $*$- algebra $\mathcal{A}$ represented as operators in a Hilbert space $\mathcal{H}$ and a self-adjoint operator D such that $(1+D^2)^{-1}$ is a compact operator and $[D,a]$ bounded for $a \in \mathcal{A}$.*

A spectral triple is **even** *if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading γ such that $[\gamma, a] = 0$ and $\{\gamma, D\} = 0$.*

A **real structure** *of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$ on a spectral triple is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that*

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon''\gamma J \quad \text{(even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n \mod 8$:

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

Moreover, with $b^0 = Jb^*J^{-1}$ we impose that

$$[a, b^0] = 0, \qquad [[D,a], b^0] = 0, \qquad \forall\, a, b \in \mathcal{A},$$

A spectral triple with a real structure is called a **real spectral triple**.

The basic example is the commutative spin geometry of a Riemannian spin manifold given by the triple

- $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions on M.
- $\mathcal{H} = L^2(M,S)$, the Hilbert space of square integrable sections of a spinor bundle $S \to M$.
- D, the Dirac operator associated with the Levi-Civita connection.

If the manifold is even dimensional, there is a grading defined by $\Gamma := -\gamma^1\gamma^2\cdots\gamma^{\dim M}$, where γ^μ are the Dirac gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The real structure is given by the charge conjugation.

A spectral triple is called **regular** (or **smooth**) if the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ lies within the smooth domain $\bigcap_{n=0}^\infty \operatorname{Dom} \delta^n$ of the operator derivation $\delta(T) := |D|T - T|D|$. This condition permits to introduce the analogue of Sobolev spaces $\mathcal{H}^s := \operatorname{Dom}(1+D^2)^{s/2}$ for $s \in \mathbb{R}$. One can develop this theory to an abstract differential calculus, cf. the notes by Higson.

DEFINITION 8. *The* **dimension spectrum** *of a regular spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the subset $\Sigma \subset \mathbb{C}$ of singularities of the meromorphic functions*

$$\zeta_b(z) = \operatorname{Tr}(b|D|^{-z})$$

where b is an element in the algebra generated by $\delta^k(\mathcal{A})$ and $\delta^k([D, \mathcal{A}])$ for all $k \geq 0$.

Corresponding to the direct product of manifolds, one can take the product of spectral triples as follows. Suppose $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ are even real spectral triples, then we define the (exterior) **product spectral triple** by

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 \\ \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \\ D &= D_1 \otimes 1 + \gamma_1 \otimes D_2 \\ \gamma &= \gamma_1 \otimes \gamma_2 \\ J &= J_1 \otimes J_2 \end{aligned}$$

Note that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$ since the cross-terms vanish due to $\gamma_1 D_1 = -D_1 \gamma_1$. The dimension spectrum Σ of the product is the sum (in $\mathbb{C}$) of $\Sigma_1 + \Sigma_2$.

1. Noncommutative differential forms

p:diff-calc

Let $\mathcal{A}$ be an algebra with unit over $\mathbb{C}$. The universal differential algebra $\Omega_{\mathrm{un}}(\mathcal{A})$ is the graded algebra generated by $a \in \mathcal{A}$ of degree 0 and symbols δa, $a \in \mathcal{A}$ of degree 1, such that

$$\delta(ab) = (\delta a)b + a\delta b \quad \delta(\alpha a + \beta b) = \alpha\delta a + \beta\delta b; \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}).$$

We can write $\Omega_{\mathrm{un}}(\mathcal{A})$ as a direct sum of subspaces $\Omega^p_{\mathrm{un}}(\mathcal{A})$ generated by linear combinations of $a_0 \delta a_1 \cdots \delta a_p$. Furthermore, there is the isomorphism of vector spaces

orms-chains

$$\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes p} \simeq \Omega^p_{\mathrm{un}}(\mathcal{A}), \tag{1.1}$$

where $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{C}\mathbb{I}$. The operator δ is defined on $\Omega_{\mathrm{un}}(\mathcal{A})$ by

$$\begin{aligned} \delta(a_0 \delta a_1 \cdots \delta a_p) &= \delta a_0 \delta a_1 \cdots \delta a_p, \\ \delta(\delta a_1 \cdots \delta a_p) &= 0. \end{aligned}$$

By construction, the algebra $\Omega_{\mathrm{un}}(\mathcal{A})$ is also a $\mathcal{A}$-bimodule. As the name suggests, the universal differential algebra satisfies the following universal property.

PROPOSITION 9. *Let (Ω, d) be a graded differential algebra and let ρ be a morphism of unital algebras. Then, there exists a unique extension of ρ to a morphism of graded differential algebras $\widetilde{\rho} : \Omega_{\mathrm{un}}(\mathcal{A}) \to \Omega$ such that $\widetilde{\rho} \circ \delta = d \circ \widetilde{\rho}$.*

An example of a frequently used differential calculus in the text and more generally, in noncommutative geometry, is Connes' differential calculus [?]. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The $\mathcal{A}$-bimodule $\Omega^p_D(\mathcal{A})$ of Connes' differential p-forms is made of classes of operators of the form

$$\omega = \sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j], \quad a_i^j \in \mathcal{A},$$

modulo the sub-bimodule of operators

$$\Big\{ \sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{p-1}^j] : \ b_i^j \in \mathcal{A}, \ b_0^j [D, b_1^j] \cdots [D, b_{p-1}^j] = 0 \Big\}.$$

The exterior differential d_D is given by

$$d_D \Big[\sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j] \Big] = \sum_j [D, a_0^j][D, a_1^j] \cdots [D, a_p^j].$$

In the case of the canonical triple $(C^\infty(M), \mathcal{H}, D)$ of a Riemannian spin manifold M, this differential calculus is isomorphic to the de Rham differential calculus.

2. Modules and connections

connections

We recall some basic definitions on modules and connections thereon. We derive a general Bianchi identity for the curvature of such connections and link with gauge theory.

2.1. Modules

Let $\mathcal{A}$ be an algebra over the complex numbers $\mathbb{C}$.

Definition 10. *A* right module $\mathcal{E}$ *is a vector space over* $\mathbb{C}$ *that carries a right representation of* $\mathcal{A}$*, i.e. there is a map* $\mathcal{E} \times \mathcal{A} \ni (\eta, a) \to \eta a$ *such that*

$$\eta(ab) = (\eta a)b,$$
$$\eta(a+b) = \eta a + \eta b,$$
$$(\eta + \xi)a = \eta a + \xi a,$$

for any $\eta, \xi \in \mathcal{E}$ *and* $a, b \in \mathcal{A}$*.*

There is the natural notion of a morphism of (right) $\mathcal{A}$-modules as linear maps that respect this structure. Thus, a morphism between two (right) $\mathcal{A}$-modules $\mathcal{E}$ and F is a linear map $\rho : \mathcal{E} \to \mathcal{F}$ that is also right $\mathcal{A}$-linear:

$$\rho(\eta a) = \rho(\eta)a; \qquad \forall \eta \in \mathcal{E}, a \in \mathcal{A}.$$

Left modules and morphisms of left modules are defined similarly. A *bimodule* over an algebra $\mathcal{A}$ is both a left and a right $\mathcal{A}$-module such that the left and right action of $\mathcal{A}$ commute:

$$(a\eta)b = a(\eta b); \qquad \forall \eta \in \mathcal{E}, a, b \in \mathcal{A}.$$

Given a right $\mathcal{A}$-module, we define its *dual module* $\mathcal{E}'$ as the collection of all morphisms from $\mathcal{E}$ into $\mathcal{A}$, where $\mathcal{A}$ is seen as the trivial right $\mathcal{A}$-module; in other words:

$$\mathcal{E}' := \left\{ \phi : \mathcal{E} \to \mathcal{A} \mid \phi(\eta a) = \phi(\eta)a, \ \eta \in \mathcal{E}, a \in \mathcal{A} \right\} .$$

Definition 11. *A right* $\mathcal{A}$*-module* $\mathcal{E}$ *is said to be* finite projective *if there exists an idempotent* $p = p^2 \in M_N(\mathcal{A})$ *such that* $\mathcal{E} \simeq p\mathcal{A}^N$ *as right* $\mathcal{A}$*-modules.*

Here $M_N(\mathcal{A}) \simeq M_N(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}$ denotes the algebra of $N \times N$ matrices with entries in $\mathcal{A}$ whereas $\mathcal{A}^N := \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}$ which can be thought of as the set of N-dimensional vectors with entries in $\mathcal{A}$, and is clearly a right $\mathcal{A}$-module.

2.2. Connections

Let us suppose we have an algebra $\mathcal{A}$ with a differential calculus $(\Omega(\mathcal{A}) = \oplus_p \Omega^p(\mathcal{A}), \mathrm{d})$. We now review the notion of a (gauge) connection on a (finite projective) module $\mathcal{E}$ over $\mathcal{A}$ with respect to the given calculus; we take a right module structure.

A *connection* on the right $\mathcal{A}$-module $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) \mapsto \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}(\mathcal{A}) ,$$

defined for any $p \geq 0$, and satisfying the Leibniz rule

$$\nabla(\omega\rho) = (\nabla\omega)\rho + (-1)^p \omega \mathrm{d}\rho , \ \forall \ \omega \in \Omega^p(\mathcal{A}) , \ \rho \in \Omega(\mathcal{A}) .$$

A connection is completely determined by its restriction

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) ,$$

which satisfies

$$\nabla(\eta a) = (\nabla\eta)a + \eta \otimes_{\mathcal{A}} \mathrm{d}a , \ \forall \ \eta \in \mathcal{E} , \ a \in \mathcal{A} ,$$

and which is extended by the Leibniz rule. It is the latter property that implies that the composition,

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^2(\mathcal{A}) \ ,$$

is $\Omega(\mathcal{A})$-linear. Indeed, for any $\omega \in \Omega^p(\mathcal{A}),\ \rho \in \Omega(\mathcal{A})$,

$$\begin{aligned} \nabla^2(\omega\rho) &= \nabla\left((\nabla\omega)\rho + (-1)^p \omega \mathrm{d}\rho\right) \\ &= (\nabla^2\omega)\rho + (-1)^{p+1}(\nabla\omega)\mathrm{d}\rho + (-1)^p(\nabla\omega)\mathrm{d}\rho + \omega \mathrm{d}^2\rho \\ &= (\nabla^2\omega)\rho \ . \end{aligned}$$

The restriction of ∇^2 to $\mathcal{E}$ is the *curvature*

$$F : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}) \ , \tag{2.1}$$

of the connection. It is $\mathcal{A}$-linear, $F(\eta a) = F(\eta)a$ for any $\eta \in \mathcal{E}, a \in \mathcal{A}$, and satisfies

$$\nabla^2(\eta \otimes_{\mathcal{A}} \rho) = F(\eta)\rho \ , \ \forall\, \eta \in \mathcal{E} \ , \ \rho \in \Omega(\mathcal{A}) \ .$$

Thus, $F \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}))$, the collection of (right) $\mathcal{A}$-linear endomorphisms of $\mathcal{E}$, taking values in the two-forms $\Omega^2\mathcal{A}$.

Connections always exist on a projective module. On the free module $\mathcal{E} = \mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A} \simeq \mathcal{A}^N$, a connection is given by the operator

$$\nabla_0 = 1 \otimes \mathrm{d} : \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p(\mathcal{A}) \to \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1}(\mathcal{A}) \ .$$

With the canonical identification $\mathbb{C}^N \otimes_{\mathbb{C}} \otimes_{\mathcal{A}} = (\mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A}) \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \ \simeq \ (\Omega(\mathcal{A}))^N$, one thinks of ∇_0 as acting on $(\Omega(\mathcal{A}))^N$ as the operator $\nabla_0 = (\mathrm{d}, \mathrm{d}, \cdots, \mathrm{d})$ (N-times).
For a generic projective module $\mathcal{E}$ one has a canonical inclusion map, $\lambda : \mathcal{E} \to \mathcal{A}^N$, which identifies $\mathcal{E}$ as a direct summand of the free module $\mathcal{A}^N$ and a canonical idempotent $p : \mathcal{A}^N \to \mathcal{E}$ which allows to identify $\mathcal{E} = p\mathcal{A}^N$. Using these maps as well as their natural extension to $\mathcal{E}$-valued forms, on $\mathcal{E}$ a connection ∇_0 is given by the composition

$$\mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) \ \xrightarrow{\lambda}\ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p(\mathcal{A}) \ \xrightarrow{1\otimes \mathrm{d}}\ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1}(\mathcal{A}) \ \xrightarrow{p}\ \Omega^{p+1}(\mathcal{A})$$

(we have also used canonical identifications for the free module). This connection is called the *Levi-Civita* or *Grassmann connection* and is explicitly given by

$$\nabla_0 = p \circ (1 \otimes \mathrm{d}) \circ \lambda$$

although one simply indicates it by

ugras
$$\nabla_0 = p\mathrm{d}. \tag{2.2}$$

3. Unitary and Morita equivalence of spectral triples

In the previous chapter we have seen the prominent role that is played by symmetries in physics. We now consider symmetries of $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$, an even real spectral triple. The first candidate is unitary equivalence.

DEFINITION 12. *Two spectral triples* $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ *and* $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ *are called* **unitary equivalent** *if* $\mathcal{A}_1 \simeq \mathcal{A}_2$ *and there exists a unitary intertwining operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that*

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a) \qquad (a \in \mathcal{A}_1) \\ UD_1U^* &= D_2 \end{aligned}$$

If there exist grading operators γ_1, γ_2 *then we also demand that* $U\gamma_1U^* = \gamma_2$. *If there exist real structures* J_1, J_2 *then we also demand that* $UJ_1U^* = J_2$.

As a special case, we consider the **gauge group** $\mathcal{U}(\mathcal{A})$, defined as the unitary elements in the algebra $\mathcal{A}$ of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. It implements unitary transformations from the spectral triple to itself, transforming

$$D \mapsto D + u[D, u^*].$$

If the spectral triple is real, the unitary intertwiner is given by $U = uJu^*J^{-1}$ for $u \in \mathcal{U}(\mathcal{A})$, thus transforming

$$D \mapsto D + u[D, u^*] + \epsilon' Ju[D, u^*]J^{-1}.$$

Effectively, the unitary group acts as automorphisms on $\mathcal{A}$ by conjugation, $a \mapsto uau^*$. Such automorphisms are called **inner**, in contrast to the outer automorphisms which are defined as the quotient $\mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$. This is nicely summarized by

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1.$$

Note that if $\mathcal{A} = C^\infty(M)$ is commutative, there are no non-trivial inner automorphisms and $\mathrm{Out}(\mathcal{A}) = \mathrm{Diff}(M)$.

We have seen that a non-abelian gauge group appears naturally when $\mathcal{A}$ is noncommutative. Even more, noncommutative algebras allow for a more general – and more natural – notion of equivalence than automorphisms, namely, Morita equivalence. Let us see if we can lift Morita equivalence to the level of spectral triples.

Recall that given an algebra $\mathcal{A}$, a Morita equivalent algebra $\mathcal{B}$ is the algebra of endomorphisms of a finite projective (right) module $\mathcal{E}$ over $\mathcal{A}$,

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}).$$

Let $(\mathcal{A}, \mathcal{H}, D)$ be a given spectral triple and try to construct a spectral triple $(\mathcal{B}, \mathcal{H}', D')$. Naturally, $\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ carries an action of $\phi \in \mathcal{B}$:

$$\phi(\eta \otimes \psi) = \phi(\eta) \otimes \psi \qquad (\eta \in \mathcal{E}, \psi \in \mathcal{H}).$$

The naive choice of an operator D' by $D'(\eta \otimes \psi) = \eta \otimes D\psi$ will not do, because it does not respect the ideal defining the tensor product over $\mathcal{A}$, being generated by elements of the form

$$\eta a \otimes \psi - \eta \otimes a\psi; \qquad (\eta \in \mathcal{E}, a \in \mathcal{A}, \psi \in \mathcal{H}).$$

A better definition is

$$D'(\eta \otimes \psi) = \eta \otimes D\psi + \nabla(\eta)\psi.$$

where $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_D(\mathcal{A})$ is a connection associated to the differential $d : a \mapsto [D, a]$ $(a \in \mathcal{A})$.

m:morita-st THEOREM 13. *If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and ∇ is a connection on a finite projective right $\mathcal{A}$-module $\mathcal{E}$, then $(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, \nabla \otimes 1 + 1 \otimes D)$ is a spectral triple.*

Analogously, we define for a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ a real spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$ by setting $\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ$. Here $\mathcal{E}^\circ$ is the **conjugate module** to $\mathcal{E}$:

$$\mathcal{E}^\circ = \{\overline{\xi} : \xi \in \mathcal{E}\}$$

with a left $\mathcal{A}$ action defined by $a\overline{\xi} = \overline{\xi a^*}$ for any $a \in \mathcal{A}$. Still $\phi \in \mathcal{B}$ acts on $\mathcal{H}'$ by

$$\phi(\eta \otimes \psi \otimes \overline{\xi}) = \phi(\eta) \otimes \psi \otimes \overline{\xi}$$

and

$$D'(\eta \otimes \psi \otimes \overline{\xi}) = (\nabla\eta)\psi \otimes \overline{\xi} + \eta \otimes D\psi \otimes \overline{\xi} + \eta \otimes \psi(\overline{\nabla\xi})$$
$$J'(\eta \otimes \psi \otimes \overline{\xi}) = \xi \otimes J\psi \otimes \overline{\eta}$$

ita-st-real THEOREM 14. *If $(\mathcal{A}, \mathcal{H}, D, J)$ is a real spectral triple, then $(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ, \nabla \otimes 1 \otimes 1 + 1 \otimes D \otimes 1 + 1 \otimes 1 \otimes \overline{\nabla})$ is a real spectral triple.*

Finally, for even spectral triples one defines a grading $\gamma' = 1 \otimes \gamma \otimes 1$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ$.

We now focus on **Morita self-equivalences**, for which $\mathcal{B} = \mathcal{A}$ and consequently $\mathcal{E} = \mathcal{A}$. Let us look at connections

$$\nabla : \mathcal{A} \to \Omega^1_D(\mathcal{A}).$$

Clearly, by the Leibniz rule $\nabla = d + A$ where $A \equiv \nabla(1) = \sum_j a_j[D, b_j]$ is a generic element in $\Omega^1_D(\mathcal{A})$. Similarly, $\psi\overline{\nabla}\overline{a} = (\epsilon' J da J^{-1} + \epsilon' J A a J^{-1})\psi$. Since $\mathcal{H}' \simeq \mathcal{H}$ we have

$$D'(\psi) \equiv D'(1 \otimes \psi \otimes \overline{1}) = \nabla(1)\psi + \psi\overline{\nabla}(\overline{1}) + D\psi = D\psi + A\psi + \epsilon' J A J^{-1}\psi.$$

In other words, D is 'innerly perturbed' to $D_A := D + A + \epsilon' J A J^{-1}$ where $A^* = A \in \Omega^1_D(\mathcal{A})$ is called the **gauge field**. Another name used for these fields is **inner fluctuations** of the Dirac operator, since it is the algebra $\mathcal{A}$ that generates – through Morita self-equivalences – the fields A.

On the new spectral triple $(\mathcal{A}, \mathcal{H}, D_A)$ there is an action of the unitary group $\mathcal{U}(\mathcal{A})$ by unitary equivalences. Recall that $U = uJu^*J^{-1}$ so that

$$D_A \mapsto U D_A U^*$$

or, equivalently,

$$A \mapsto uAu^* + u[D, u^*]$$

which is the usual rule for a gauge transformation on a gauge field.

3.1. Spectral action functional

Having identified the gauge group canonically assoaciated to a spectral triple, and derived the gauge fields, we are ready to find action functionals of these fields that are invariant under the gauge group.

Definition 15. *Let f be a positive and even function from $\mathbb{R}$ to $\mathbb{R}$. The* **spectral action** *is defined by*

$$S_b[A] := \operatorname{Tr} f(D_A/\Lambda)$$

where Λ is a real cutoff parameter. The fermionic action is defined as the inner product

$$S_f[A, \psi] := (\psi, D_A\psi).$$

We will assume that f is given by a Laplace–Stieltjes transform:

$$f(x) = \int_{t>0} e^{-tx^2} d\mu(t).$$

with μ a measure on $\mathbb{R}$, and further that there exists the following **heat kernel expansion**:

$$\operatorname{Tr} e^{-tD^2} = \sum_\alpha t^\alpha c_\alpha$$

as $t \to 0$. Note that this is defined for the unperturbed operator D, but similar expression hold for any bounded perturbation such as D_A of D.

Lemma 16. *For $\alpha < 0$ we have*

$$\operatorname{res}_{z=-2\alpha}\zeta_1(z) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

with $\zeta_b(z) = \operatorname{Tr} b|D|^{-z}$ as before.

PROOF. This follows from the Mellin transform:

$$|D|^{-z} = \frac{1}{\Gamma(z/2)} \int_0^\infty e^{-tD^2} t^{z/2-1}\, dt$$

or, after inserting the heat expansion:

$$\begin{aligned}
\mathrm{Tr}\,|D|^{-z} &= \frac{1}{\Gamma(z/2)} \sum_\alpha \int_0^\infty c_\alpha t^{\alpha+z/2-1}\, dt \\
&= \frac{1}{\Gamma(z/2)} \sum_\alpha \int_0^1 c_\alpha t^{\alpha+z/2-1}\, dt + \text{ holomorphic} \\
&= \sum_\alpha \frac{c_\alpha}{\Gamma(z/2)(\alpha+z/2)}.
\end{aligned}$$

Taking resisues at $z = -2\alpha$ on both sides gives the desired result. □

Using the Laplace–Stieltjes transform, we now derive an asymptotic expansion of the spectral in terms of the heat coefficients c_α.

PROPOSITION 17. *Let Σ be the dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$. Then*

$$\mathrm{Tr}\, f(D/\Lambda) = \sum_{\beta \in \Sigma \cup 0} f_\beta \Lambda^\beta \frac{\Gamma(\beta/2)}{2} c_{-\frac{1}{2}\beta} + \mathcal{O}(\Lambda^{-1})$$

where $f_\beta := \int_0^\infty f(v) v^{\beta-1} dv$ and $f_0 := f(0)$.

PROOF. Inserting the heat expansion in the Laplace–Stieltjes transform gives

$$\mathrm{Tr}\, f(D/\Lambda) = \int_{t>0} t^\alpha \Lambda^{-2\alpha} c_\alpha\, d\mu(t).$$

The terms with $\alpha > 0$ are of order Λ^{-1}; if $\alpha \leq 0$, then

$$t^\alpha = \frac{1}{\Gamma(\alpha)} \int_{v>0} e^{-tv} v^{-\alpha-1}\, dv.$$

Applying this to the above intergral gives

$$\begin{aligned}
\Lambda^{-2\alpha} c_\alpha \int_{t>0} t^\alpha\, d\mu(t) &= \Lambda^{-2\alpha} c_\alpha \int_{t>0} \int_{v>0} e^{-tv} v^{-\alpha-1}\, dv d\mu(t) \\
&= \Lambda^{-2\alpha} c_\alpha \int_{t>0} \int_{v>0} e^{-tv^2} v^{-2\alpha-1}\, dv d\mu(t) \\
&= \Lambda^{-2\alpha} c_\alpha \int_{v>0} f(v) v^{-2\alpha-1}\, dv \equiv \Lambda^{-2\alpha} c_\alpha f_{-2\alpha}
\end{aligned}$$

substituting $v \mapsto v^2$ in the going to the second line. Since $c_\alpha = 0$ unless $-2\alpha \in \Sigma$ we substitute $\beta = -2\alpha$ to obtain the claimed formula. □

COROLLARY 18. *For the perturbed operator D_A we have*

$$S_b[A] = \sum_{\beta \in \Sigma \cup 0} f_\beta \Lambda^\beta \frac{\Gamma(\beta/2)}{2} \mathrm{res}_{z=\beta}\, \mathrm{Tr}\, |D_A|^{-z} + \mathcal{O}(\Lambda^{-1})$$