CHAPTER 3

Yang–Mills theory in noncommutative geometry

In the previous chapter we have seen how to associate to a spectral triple on an algebra $\mathcal{A}$ a spectral triple on a Morita equivalent algebra $\mathcal{B}$. We will now apply this to $\mathcal{A} = C^\infty(M)$ and find that a classical Yang–Mills auge theory (with gauge group $SU(n)$) is nicely captured by the notion of Morita equivalence.

1. Gauge theory as a Morita equivalence

Suppose then that $\mathcal{A} \simeq \mathcal{B}$, so that $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$. Here $\mathcal{E}$ is a finite projective $\mathcal{A}$-module, and via Serre–Swan Theorem, it is the space of sections of a vector bundle E, *i.e.* $\mathcal{E} = \Gamma(E)$. Moreover, $\mathcal{B} = \Gamma(B)$ where $B = \mathrm{End}\, E$ is the endomorphism bundle of E. Since any vector bundle has constant rank, the typical fiber of B is $M_n(\mathbb{C})$ with n the rank of E. Note that there is a determinant $\det : \mathcal{B} \to \mathcal{A}$, which is the fiberwise determinant. We define

$$\mathcal{SU}(\mathcal{B}) := \{u \in \mathcal{B} : u^* u = 1 = uu^*,\ \det u = 1\} \subset \mathcal{U}(\mathcal{B})$$

Similary, there is a fiberwise trace, and we define a Lie algebra

$$\mathfrak{u}(\mathcal{B}) := \{X \in \mathcal{B} : X^* = -X\}$$
$$\mathfrak{su}(\mathcal{B}) := \{X \in \mathcal{B} : X^* = -X, \mathrm{Tr}\, X = 0\} \subset \mathfrak{u}(\mathcal{B}).$$

PROPOSITION 19. *If $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$, then $\mathcal{SU}(\mathcal{B})$ are the sections of a fiber bundle of groups, with typical fiber $SU(n)$. Moreover, this fiber bundle is associated to a principal $PSU(n)$-bundle P as*

$$\mathrm{Ad}\, P = P \times_{PSU(n)} SU(n).$$

PROOF. Let us proof the second statement; the first follows from this. Consider the algebra bundle $B \to M$ for which $\Gamma(B) = \mathcal{B}$. Since $B = \mathrm{End}(E)$ for a hermitian vector bundle E, the transition functions are involutive automorphisms of the fibers. Since the fibers are isomorphic to $M_n(\mathbb{C})$ these transition functions take values in $PSU(n) = \mathrm{Aut}(M_n(\mathbb{C}))$. Hence, one can construct a principal $PSU(n)$-bundle by glueing together copies $U_i \times PSU(n)$ on local charts U_i using the transition functions. Obviously, $B = P \times_{PSU(n)} M_n(\mathbb{C})$, with $PSU(n)$ acting on $M_n(\mathbb{C})$ by conjugation so that the special unitary sections of B are of the desired form. □

Similarly, one can show the following

PROPOSITION 20. *If $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$, then $\mathfrak{su}(\mathcal{B})$ are the sections of a Lie algebra bundle, with typical fiber $su(n)$. Moreover, this bundle is associated to a principal $PSU(n)$-bundle P as*

$$\mathrm{ad}\, P = P \times_{PSU(n)} su(n)$$

Let us now apply the argument of the previous section (cf. Theorem 14) to obtain a spectral triple on the algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ from the canonical triple on $\mathcal{A} = C^\infty(M)$. Given a connection on $\mathcal{E}$, this is given as in Theorem 14:

$$(\mathrm{End}_{\mathcal{A}}(\mathcal{E}), \mathcal{E} \otimes_{\mathcal{A}} L^2(M,S) \otimes_{\mathcal{A}} \mathcal{E}^\circ, \nabla \otimes 1 \otimes 1 + 1 \otimes \partial\!\!\!/ \otimes 1 + 1 \otimes 1 \otimes \overline{\nabla}) \simeq (\Gamma(B), L^2(S \otimes B), \partial\!\!\!/ \otimes 1 + 1 \otimes \nabla_0).$$

We have used that $\mathcal{A}$ is commutative so that $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^\circ \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E}) = \Gamma(B)$. Also, ∇_0 denotes the connection on $\mathrm{End}(E)$ induced by ∇ on E and the inner product on $\Gamma(B)$ is the Hilbert–Schmidt inner product. Let us summarize this in the following

PROPOSITION 21. *Let $B = \mathrm{End}(E)$ for a hermitian vector bundle E on a Riemannian spin manifold M, and let ∇_0 be a connection on B (with respect to $\Omega^1_{\partial\!\!\!/}(\mathcal{A})$). Then $(\Gamma(B), L^2(S \otimes B), \partial\!\!\!/ \otimes 1 + 1 \otimes \nabla_0)$ is a spectral triple. A real structure is given by $J = J_M \otimes (\cdot)^*$ with J_M the real structure on M and a grading is defined by $\gamma = \gamma \otimes 1$.*

We already identified the gauge group $\mathcal{SU}(\Gamma(B))$ as the adjoint bundle $\mathrm{Ad}\, P$ to a $PSU(n)$-principal bundle P. Let us find the gauge fields as the inner perturbations of the above spectral triple.

THEOREM 22. *The inner fluctuations of the real spectral triple $(\Gamma(B), L^2(S \otimes B), D, \gamma, J)$ with $D = \partial\!\!\!/ \otimes 1 + 1 \otimes \nabla_0$ are parametrized by a section $\mathbb{A}$ of $\Lambda^1 \times \mathrm{ad}\, P$ as*

$$D_A = D + i\gamma \circ \mathbb{A}$$

with γ indicating Clifford multiplication.

PROOF. We claim that any $A \in \Omega^1_D(\mathcal{B})$ is the image under Clifford multiplication of an $\mathcal{B}$-valued one-form on M. Indeed, with $\mathcal{B} = \Gamma(B)$ we can write $D = \gamma \circ \nabla^{S \otimes B}$ with$\nabla^{S\otimes B}$ the tensor product of the spin connection with the connection on B that satisfies $\nabla_0 = \gamma \circ \nabla^B$. Note that $\nabla^B : \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M)$. Thus, with $A = \sum_j a_j [D, b_j]$ we have

$$A = \gamma \circ \sum_j a_j \nabla^B(b_j).$$

The sum $\sum_j a_j \nabla^B(b_j)$ takes values in $\mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M) \simeq \Gamma(\Lambda^1 \otimes B)$ as claimed. Since $J\gamma^\mu J^{-1} = -\epsilon' \gamma^\mu$, we have similarly

$$\epsilon' JAJ^{-1} = -\gamma \circ (\cdot)^* \sum_j a_j \nabla^B(b_j)(\cdot)^*$$

In other words, if $A^* = A$, the combination $A + \epsilon' JAJ^{-1}$ acts in the adjoint on the Hilbert space:

$$A + \epsilon' JAJ^{-1}(\psi \otimes \eta) = \gamma^\mu \psi \otimes (A\eta - \eta A) \qquad (\psi \otimes \eta \in L^2(S \otimes B)).$$

If we define the $\mathcal{B}$-valued one-form $\mathbb{A}$ by $i\gamma \circ \mathbb{A} = (A + \epsilon' JAJ^{-1})$ we observe that $\mathbb{A}^* = -\mathbb{A}$. Moreover, the trace part of $\mathbb{A} \in \Gamma(B \otimes \Lambda^1)$ in each fiber of the bundle B acts trivially on $L^2(B \otimes S)$. Thus, effectively, D_A is parametrized by a traceless, skew-hermitian $\mathcal{B}$-valued one-form $\mathbb{A}$ on M. In other words, $\mathbb{A} \in \mathfrak{su}(\mathcal{B}) \otimes_{\mathcal{A}} \Omega^1(M) \simeq \Gamma(\Lambda^1 \otimes \mathrm{ad}\, P)$. □

PROPOSITION 23. *Let ∇^B be locally given by $\mathrm{d} + \mathbb{A}_0$. The gauge group $\mathcal{SU}(\mathcal{B})$ acts on the $\mathbb{A}_0$ and on the $\mathbb{A} \in \Gamma(\Lambda^1 \otimes \mathrm{ad}\, P)$ that parametrizes D_A as*

$$\mathbb{A}_0 \mapsto u\mathbb{A}_0 u^* + u[D, u^*]; \qquad \mathbb{A} \mapsto u\mathbb{A}u^*.$$

We thus have all the ingredients for a $(P)SU(n)$ Yang–Mills gauge theory on M: a principal bundle P, the gauge group given as sections of $\mathrm{Ad}\, P$ and the gauge fields parametrized by $\mathbb{A} \in \Gamma(\Lambda^1 \otimes \mathrm{ad}\, P)$. We conclude this section by actually deriving the Yang–Mills action for these gauge fields as the spectral action applied to the above spectral triple.

Intermezzo: heat kernel expansion on a manifold

Let M be a compact Riemannian spin manifold of dimension m and let $S \to M$ be the spinor bundle with the spinor connection ∇^S. Let V be a hermitian vector bundle over M with a compatible connection ∇_V. First, we state a more general result (cf. Gilkey for more details).

Theorem 24 (Generalized Lichnerowicz formula). *Let $\not\partial_V$ be the twisted Dirac operator on $S \otimes V$. Then*

$$\not\partial_V^2 = \Delta^{S\otimes V} - \frac{1}{4}R + R_V$$

where $\Delta^{S\otimes V} = \nabla^{S\otimes V *}\nabla^{S\otimes V}$ is the Laplacian of $S \otimes V$ and R_V is the bundle endomorphism

$$R_V = \sum_{\mu<\nu} \gamma_\mu \gamma_\nu \otimes [\nabla^V_\mu, \nabla^V_\nu]$$

In preparation to the next section, where we will compute the spectral action, let us give some convenient expression for the residues of the zeta functions $\operatorname{Tr} |\not\partial_V|^{-z}$.

Theorem 25. *Let M be an m-dimensional compact Riemannian spin manifold. If $P = \nabla^*\nabla - E$ for a connection ∇ and an endomorphism E on some vector bundle, then*

$$\operatorname{Tr} P^{-z/2}$$

has simple poles at $z = 1, 2, \ldots, m$. The residues can be computed explicitly as

$$\operatorname{res}_{z=k} \operatorname{Tr} P^{-z/2} = \frac{2}{\Gamma(k/2)} \int_M a_{m-k}(x,P)\, d\mu(x)$$

in terms of the Seeley–DeWitt coefficients:

$$a_0(x,P) = (4\pi)^{-m/2} \operatorname{Tr}(1)$$

$$a_2(x,P) = (4\pi)^{-m/2} \operatorname{Tr}\left(-\frac{R}{6} + E\right)$$

$$a_4(x,P) = (4\pi)^{-m/2} \frac{1}{360} \operatorname{Tr}\Big(-60RE + 180E^2 + 60E;_\mu{}^\mu + 30\Omega_{\mu\nu}\Omega^{\mu\nu} - 12R;_\mu{}^\mu + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big)$$

Corollary 26. *Let $\not\partial_V$ be a twisted Dirac operator on a 4-dimensional spin manifold, and let N be the rank of V. The zeta function*

$$\operatorname{Tr} |\not\partial_V|^{-z}$$

has simple poles at $\{1, 2, \ldots, m\}$. The first three non-zero residues can be expressed in terms of the curvature F of ∇^V as

$$\operatorname{res}_{z=0} \operatorname{Tr} |\not\partial_V|^{-z} = -\frac{N^2}{160\pi^2} \int_M C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} d\mu(x) + \frac{1}{24\pi^2} \int_M \operatorname{Tr} F_{\mu\nu}\overline{F}^{\mu\nu}\, d\mu(x)$$

$$\operatorname{res}_{z=2} \operatorname{Tr} |\not\partial_V|^{-z} = \frac{N^2}{24\pi^2} \int_M R\, d\mu(x) \qquad \operatorname{res}_{z=4} \operatorname{Tr} |\not\partial_V|^{-z} = \frac{N^2}{2\pi^2} \int_M d\mu(x)$$

2. Yang–Mills action as a spectral action

The previous intermezzo makes us well-prepared for a computation of the spectral action in the present case, given by the general expansion of Corollary 18.

Theorem 27. *The spectral action $S_b[\mathbb{A}]$ for the spectral triple $(\Gamma(B), L^2(S \otimes B), D_A)$ is given by the expansion*

$$S_b[\mathbb{A}] = \frac{1}{4\pi^2} \int_M \left(2N^2\Lambda^4 f_4 + \frac{N^2}{6}\Lambda^2 f_2 R - \frac{f_0}{6} \operatorname{Tr} \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} - \frac{N^2 f_0}{80} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \right) d\mu(x)$$

where $\mathbb{F}$ is the curvature of the connection $\nabla^B + \mathbb{A}$ on the bundle B.

Besides the gravitational terms – among which the Einstein–Hilbert action of general relativity! – we find the Yang–Mills action for a gauge field on a non-trivial bundle B.

Bis: The Standard Model as a noncommutative manifold

The previous model generalizes to actually geometrically describe the full Standard Model, including Higgs boson. We give an extremely brief account on this derivation, referring to Connes-Marcolli[1]for more details.

The spectral triple is now given by

$$(C^\infty(M)\otimes(\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}),\ L^2(M,S)\otimes\mathbb{C}^{96},\ \partial\!\!\!/\otimes 1+\gamma_5\otimes D_F).$$

Here 96 is 2 (particles and anti-particles) times 3 (families) times 4 leptons times 4 quarks with 3 colors each. We write the representation of $\mathcal{A}$ in terms of the suggestive basis of $\mathbb{C}^{96}$:

$$\left(\nu_L\ e_L\ \nu_R\ e_R\ u_L\ d_L\ u_R\ d_R\ \overline{\nu}_L\ \overline{e}_L\ \overline{\nu}_R\ \overline{e}_R\ \overline{u}_L\ \overline{d}_L\ \overline{u}_R\ \overline{d}_R\right)^t.$$

Then, for an element $(\lambda,q,m)\in\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$

$$\pi(\lambda,q,m)=\begin{pmatrix} q &&&&& \\ & \begin{bmatrix}\lambda & \\ & \overline{\lambda}\end{bmatrix} &&&& \\ && q\otimes 1_3 &&& \\ &&& \begin{bmatrix}\lambda & \\ & \overline{\lambda}\end{bmatrix}\otimes 1_3 && \\ &&&& \lambda 1_4 & \\ &&&&& 1_4\otimes\overline{m}\end{pmatrix}$$

Here, the quaternion q is considered as a 2×2-matrix. The 96×96-matrix D_F is of the following form: $D_F=\begin{pmatrix} S & T^* \\ T & \overline{S}\end{pmatrix}$ where

$$S=\begin{pmatrix} & \begin{bmatrix}\Upsilon_v & \\ & \Upsilon_e\end{bmatrix} && \\ \begin{bmatrix}\Upsilon_v^* & \\ & \Upsilon_e^*\end{bmatrix} &&& \\ &&& \begin{bmatrix}\Upsilon_u\otimes 1_3 & \\ & \Upsilon_d\otimes 1_3\end{bmatrix} \\ && \begin{bmatrix}\Upsilon_u^*\otimes 1_3 & \\ & \Upsilon_d^*\otimes 1_3\end{bmatrix} & \end{pmatrix};\quad T=\begin{pmatrix}\begin{bmatrix}0 &&& \\ & 0 && \\ && \Upsilon_R & \\ &&& 0\end{bmatrix} & \\ & 0_4\end{pmatrix}$$

in terms of the 3×3 *Yukawa-mixing-matrices* $\Upsilon_\nu,\Upsilon_e,\Upsilon_u,\Upsilon_d$ and a real constant Υ_R responsible for neutrino mass terms.

One can further enrich this spectral triple by a grading γ_F which is $+1$ on all L-particles, and -1 on all R-particles; the total grading is then $\gamma_5\otimes\gamma_F$. The anti-linear operator J is a combination of charge conjugation on S and the (anti-linear) matrix $J_F=\begin{pmatrix} & 1_{48} \\ 1_{48} & \end{pmatrix}$.

The rest then follows from a long calculation; the inner fluctuations are $D_A=\partial\!\!\!/+i\gamma_\mu A_\mu+\gamma_5(D_F+\mathcal{M}(\Phi))$ with

$$A_\mu=\begin{pmatrix}\frac{g_1}{2}B_\mu-\frac{g_2}{2}W_\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g_1B_\mu\end{pmatrix}\oplus\begin{pmatrix}-\frac{g_2}{2}W_\mu\otimes 1_3-\frac{g_1}{6}B_\mu\otimes_3 & 0 & 0 \\ 0 & -\frac{2g_1}{3}B_\mu\otimes 1_3 & 0 \\ 0 & 0 & \frac{g_1}{3}B_\mu\otimes 1_3\end{pmatrix}-1_4\otimes\frac{g_3}{2}V_\mu$$

$$\mathcal{M}(\Phi)=\begin{pmatrix} & \begin{matrix}\Upsilon_\nu\phi_1 & \Upsilon_\nu\phi_2 \\ -\Upsilon_e\overline{\phi}_2 & \Upsilon_e\overline{\phi}_1\end{matrix} \\ \text{(h.c.)} & \end{pmatrix}\oplus\begin{pmatrix} & \begin{matrix}\Upsilon_u\phi_1 & \Upsilon_u\phi_2 \\ -\Upsilon_d\overline{\phi}_2 & \Upsilon_d\overline{\phi}_1\end{matrix} \\ \text{(h.c.)} & \end{pmatrix}$$

[1] A. Connes and M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives.* AMS, Providence, 2008

Here B_μ, W_μ, V_μ are $U(1), SU(2)$ and $SU(3)$-gauge fields, resp. and $\Phi = (\phi_1\ \phi_2)^t$ two scalar (Higgs) fields. The spectral action is modulo gravitational terms:

$$S_\Lambda = \frac{-2af_2\Lambda^2 + ef_0}{\pi^2} \int |\phi|^2 + \frac{f_0}{2\pi^2} \int a|D_\mu\phi|^2 - \frac{f_0}{12\pi^2} \int aR|\phi|^2$$
$$- \frac{f_0}{2\pi^2} \int \left(g_3^2 G_\mu^i G^{\mu i} + g_2^2 F_\mu^a F^{\mu\nu a} + \frac{5}{3} g_1^2 B_\mu B^\mu \right) + \frac{f_0}{2\pi^2} \int b|\phi|^4 + \text{Ø}(\Lambda^{-2})$$

with a, b, c, d, e constants depending on the Yukawa parameters. For example,

$$a = \mathrm{Tr}\left(\Upsilon_\nu^*\Upsilon_\nu + \Upsilon_e^*\Upsilon_e + 3\left(\Upsilon_u^*\Upsilon_u + \Upsilon_d^*\Upsilon_d\right)\right)$$
$$b = \mathrm{Tr}\left((\Upsilon_\nu^*\Upsilon_\nu)^2 + (\Upsilon_e^*\Upsilon_e)^2 + 3\left((\Upsilon_u^*\Upsilon_u)^2 + (\Upsilon_d^*\Upsilon_d)^2\right)\right)$$

When we add the fermionic term $\langle J\psi, D_A\psi\rangle$ to S_Λ, we obtain the Standard Model Lagrangian, including the Higgs boson, provided we have

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \qquad g_3^2 = g_2^2 = \frac{5}{3} g_1^2.$$

These GUT-type relations between the coupling constants allows for predictions. For example, one identifies the mass of the W as $2M_W = \sqrt{a/2}$ so that the Higgs vacuum reads $2M/g_2$. The above relation for a then gives a postdiction for the mass of the top quark as $m_t \leq 180$ GeV. Moreover, the mass of the Higgs is $m_H = 8\lambda M^2/g_2^2$ with $\lambda = g_3^2 b/a^2$ resulting in a prediction of $m_H \sim 168$ GeV.