CHAPTER 3

Yang–Mills theory in noncommutative geometry

In the previous chapter we have seen how to associate to a spectral triple on an algebra $A$ a spectral triple on a Morita equivalent algebra $B$. We will now apply this to $A = C^\infty(M)$ and find that a classical Yang–Mills auge theory (with gauge group $SU(n)$) is nicely captured by the notion of Morita equivalence.

1. Gauge theory as a Morita equivalence

Suppose then that $A \simeq B$, so that $B = \text{End}_A(\mathcal{E})$. Here $\mathcal{E}$ is a finite projective $A$-module, and via Serre–Swan Theorem, it is the space of sections of a vector bundle $E$, i.e. $\mathcal{E} = \Gamma(E)$. Moreover, $B = \Gamma(B)$ where $B = \text{End}E$ is the endomorphism bundle of $E$. Since any vector bundle has constant rank, the typical fiber of $B$ is $M_n(\mathbb{C})$ with $n$ the rank of $E$. Note that there is a determinant $\det : B \to A$, which is the fiberwise determinant. We define

$$SU(B) := \{u \in B : u^*u = 1 = uu^*, \text{ det } u = 1\} \subset U(B)$$

Similarly, there is a fiberwise trace, and we define a Lie algebra

$$u(B) := \{X \in B : X^* = -X\}$$
$$su(B) := \{X \in B : X^* = -X, \text{ Tr } X = 0\} \subset u(B).$$

**Proposition 19.** If $B$ is Morita equivalent to $A$, then $SU(B)$ are the sections of a fiber bundle of groups, with typical fiber $SU(n)$. Moreover, this fiber bundle is associated to a principal $PSU(n)$- bundle $P$ as

$$\text{Ad } P = P \times_{PSU(n)} SU(n).$$

**Proof.** Let us proof the second statement; the first follows from this. Consider the algebra bundle $B \to M$ for which $\Gamma(B) = B$. Since $B = \text{End}(E)$ for a hermitian vector bundle $E$, the transition functions are involutive automorphisms of the fibers. Since the fibers are isomorphic to $M_n(\mathbb{C})$ these transition functions take values in $PSU(n) = \text{Aut}(M_n(\mathbb{C}))$. Hence, one can construct a principal $PSU(n)$-bundle by glueing together copies $U_i \times PSU(n)$ on local charts $U_i$ using the transition functions. Obviously, $B = P \times_{PSU(n)} M_n(\mathbb{C})$, with $PSU(n)$ acting on $M_n(\mathbb{C})$ by conjugation so that the special unitary sections of $B$ are of the desired form. \[\square\]

Similarly, one can show the following

**Proposition 20.** If $B$ is Morita equivalent to $A$, then $su(B)$ are the sections of a Lie algebra bundle, with typical fiber $su(n)$. Moreover, this bundle is associated to a principal $PSU(n)$-bundle $P$ as

$$\text{ad } P = P \times_{PSU(n)} su(n)$$

Let us now apply the argument of the previous section (cf. Theorem 14) to obtain a spectral triple on the algebra $B = \text{End}_A(\mathcal{E})$ from the canonical triple on $A = C^\infty(M)$. Given a connection on $\mathcal{E}$, this is given as in Theorem 14:

$$(\text{End}_A(\mathcal{E}), \mathcal{E} \otimes_A L^2(M, S) \otimes_A \mathcal{E}^0, \nabla \otimes 1 \otimes 1 + 1 \otimes \varphi \otimes 1 + 1 \otimes 1 \otimes \nabla) \simeq (\Gamma(B), L^2(S \otimes B, \varphi \otimes 1 + 1 \otimes \nabla_0).$$
We have used that $\mathcal{A}$ is commutative so that $\mathcal{E} \otimes \mathcal{A} \mathcal{E}^\circ \simeq \text{End}_{\mathcal{A}}(\mathcal{E}) = \Gamma(B)$. Also, $\nabla_0$ denotes the connection on $\text{End}(E)$ induced by $\nabla$ on $E$ and the inner product on $\Gamma(B)$ is the Hilbert–Schmidt inner product. Let us summarize this in the following

**Proposition 21.** Let $B = \text{End}(E)$ for a hermitian vector bundle $E$ on a Riemannian spin manifold $M$, and let $\nabla_0$ be a connection on $B$ (with respect to $\Omega_B^0(\mathcal{A})$). Then $(\Gamma(B), L^2(S \otimes B), \theta \otimes 1 + 1 \otimes \nabla_0)$ is a spectral triple. A real structure is given by $J = J_M \otimes (\cdot)^\ast$ with $J_M$ the real structure on $M$ and a grading is defined by $\gamma = \gamma \otimes 1$.

We already identified the gauge group $\text{SU}(\Gamma(B))$ as the adjoint bundle $\text{Ad} P$ to a $\text{PSU}(n)$-principal bundle $P$. Let us find the gauge fields as the inner perturbations of the above spectral triple.

**Theorem 22.** The inner fluctuations of the real spectral triple $(\Gamma(B), L^2(S \otimes B), D, \gamma, J)$ with $D = \theta \otimes 1 + 1 \otimes \nabla_0$ are parametrized by a section $\mathcal{A}$ of $\Lambda^1 \times \text{ad} P$ as

$$D_A = D + i J \gamma \circ \mathcal{A}$$

with $\gamma$ indicating Clifford multiplication.

**Proof.** We claim that any $A \in \Omega_B^1(\mathcal{A})$ is the image under Clifford multiplication of a $B$-valued one-form on $M$. Indeed, with $B = \Gamma(B)$ we can write $D = \gamma \circ \nabla S \otimes B$ with $\nabla S \otimes B$ the tensor product of the spin connection with the connection on $B$ that satisfies $\nabla_0 = \gamma \circ \nabla B$. Note that $\nabla B : B \to B \otimes \mathcal{A} \Omega^1(M)$. Thus, with $A = \sum_j a_j[D,b_j]$ we have

$$A = \gamma \circ \sum_j a_j \nabla B(b_j).$$

The sum $\sum_j a_j \nabla B(b_j)$ takes values in $B \otimes \mathcal{A} \Omega^1(M) \simeq \Gamma(\Lambda^1 \otimes B)$ as claimed. Since $J \gamma^\mu J^{-1} = -\epsilon^\mu \gamma^\mu$, we have similarly

$$\epsilon^\mu J A J^{-1} = -\gamma \circ (\cdot)^\ast \sum_j a_j \nabla B(b_j)(\cdot)^\ast$$

In other words, if $A^\ast = A$, the combination $A + \epsilon^\mu J A J^{-1}$ acts in the adjoint on the Hilbert space:

$$A + \epsilon^\mu J A J^{-1}(\psi \otimes \eta) = \gamma^\mu \psi \otimes (A \eta - \eta A) \quad (\psi \otimes \eta \in L^2(S \otimes B)).$$

If we define the $B$-valued one-form $\mathcal{A}$ by $j \gamma \circ \mathcal{A} = (A + \epsilon^\mu J A J^{-1})$ we observe that $\mathcal{A}^\ast = -\mathcal{A}$. Moreover, the trace part of $\mathcal{A} \in \Gamma(B \otimes \Lambda^1)$ in each fiber of the bundle $B$ acts trivially on $L^2(B \otimes S)$. Thus, effectively, $D_A$ is parametrized by a traceless, skew-hermitian $B$-valued one-form $\mathcal{A}$ on $M$. In other words, $\mathcal{A} \in \mathfrak{su}(B) \otimes \mathcal{A} \Omega^1(M) \simeq \Gamma(\Lambda^1 \otimes \text{ad} P)$. $\square$

**Proposition 23.** Let $\nabla B$ be locally given by $d + \mathcal{A}_0$. The gauge group $\text{SU}(B)$ acts on the $\mathcal{A}_0$ and on the $\mathcal{A} \in \Gamma(\Lambda^1 \otimes \text{ad} P)$ that parametrizes $D_A$ as

$$\mathcal{A}_0 \mapsto u \mathcal{A}_0 u^\ast + u[D,u^\ast]; \quad \mathcal{A} \mapsto u \mathcal{A} u^\ast.$$ 

We thus have all the ingredients for a $(P)\text{SU}(n)$ Yang–Mills gauge theory on $M$: a principal bundle $P$, the gauge group given as sections of $\text{Ad} P$ and the gauge fields parametrized by $\mathcal{A} \in \Gamma(\Lambda^1 \otimes \text{ad} P)$. We conclude this section by actually deriving the Yang–Mills action for these gauge fields as the spectral action applied to the above spectral triple.

**Intermezzo: heat kernel expansion on a manifold**

Let $M$ be a compact Riemannian spin manifold of dimension $m$ and let $S \to M$ be the spinor bundle with the spinor connection $\nabla S$. Let $V$ be a hermitian vector bundle over $M$ with a compatible connection $\nabla_V$. First, we state a more general result (cf. Gilkey for more details).

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Theorem 24 (Generalized Lichnerowicz formula). Let \( \phi_V \) be the twisted Dirac operator on \( S \otimes V \). Then

\[
\phi_V^2 = \Delta^{S \otimes V} - \frac{1}{4} R + R_V
\]

where \( \Delta^{S \otimes V} = \nabla^{S \otimes V} \star \nabla^{S \otimes V} \) is the Laplacian of \( S \otimes V \) and \( R_V \) is the bundle endomorphism

\[
R_V = \sum_{\mu < \nu} \gamma_{\mu \nu} \otimes [\nabla^V_\mu, \nabla^V_\nu]
\]

In preparation to the next section, where we will compute the spectral action, let us give some convenient expression for the residues of the zeta function

\[
\text{Tr} P^{-z/2}
\]

has simple poles at \( z = 1, 2, \ldots, m \). The residues can be computed explicitly as

\[
\text{res}_{z=k} \text{Tr} P^{-z/2} = \frac{2}{\Gamma(k/2)} \int_M a_{m-k}(x, P) \, d\mu(x)
\]

in terms of the Seeley–DeWitt coefficients:

\[
a_0(x, P) = (4\pi)^{-m/2} \text{Tr}(1)
\]

\[
a_2(x, P) = (4\pi)^{-m/2} \text{Tr} \left( -\frac{R}{6} + E \right)
\]

\[
a_4(x, P) = (4\pi)^{-m/2} \frac{1}{360} \text{Tr} \left( -60R \cdot E + 180E^2 + 60E^\mu_\mu + 30\Omega_\mu \Omega^\mu \right.
\]

\[
- 12R;_\mu^\nu + 5R^2 - 2R^\mu_\rho R^\nu_\rho + 2R^\mu_\rho^\nu_\sigma R^{\rho\sigma\rho\sigma} \left) \text{Tr} \nabla^V_\mu \right)
\]

Corollary 26. Let \( \phi_V \) be a twisted Dirac operator on a 4-dimensional spin manifold, and let \( N \) be the rank of \( V \). The zeta function

\[
\text{Tr} |\phi_V|^{-z}
\]

has simple poles at \( \{1, 2, \ldots, m\} \). The first three non-zero residues can be expressed in terms of the curvature \( F \) of \( \nabla^V \) as

\[
\text{res}_{z=0} \text{Tr} |\phi_V|^{-z} = -\frac{N^2}{160\pi^2} \int_M C_{\mu\rho\sigma} C^{\mu\rho\sigma} \mu(x) + \frac{1}{24\pi^2} \int_M \text{Tr} F_{\mu\nu} F^{\mu\nu} \mu(x)
\]

\[
\text{res}_{z=2} \text{Tr} |\phi_V|^{-z} = \frac{N^2}{24\pi^2} \int_M R \mu(x)
\]

\[
\text{res}_{z=4} \text{Tr} |\phi_V|^{-z} = \frac{N^2}{2\pi^2} \int_M \mu(x)
\]

2. Yang–Mills action as a spectral action

The previous intermezzo makes us well-prepared for a computation of the spectral action in the present case, given by the general expansion of Corollary 18.

Theorem 27. The spectral action \( S_0[\Lambda] \) for the spectral triple \( (\Gamma(B), L^2(S \otimes B), D_A) \) is given by the expansion

\[
S_0[\Lambda] = \frac{1}{4\pi^2} \int_M \left( 2N^2 \Lambda^4 f_4 + \frac{N^2}{6} \Lambda^2 f_2 R - \frac{f_0}{6} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{N^2 f_0}{80} C_{\mu\rho\sigma} C^{\mu\rho\sigma} \right) \mu(x)
\]

where \( F \) is the curvature of the connection \( \nabla^B + \Lambda \) on the bundle \( B \).

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Besides the gravitational terms – among which the Einstein–Hilbert action of general relativity! – we find the Yang–Mills action for a gauge field on a non-trivial bundle \( B \).

### Bis: The Standard Model as a noncommutative manifold

The previous model generalizes to actually geometrically describe the full Standard Model, including Higgs boson. We give an extremely brief account on this derivation, referring to Connes-Marcolli\(^1\) for more details.

The spectral triple is now given by

\[
(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})), L^2(M, S) \otimes \mathbb{C}^{96}, \partial \otimes 1 + \gamma_5 \otimes D_F).
\]

Here \( 96 \) is \( 2 \) (particles and anti-particles) times \( 3 \) (families) times \( 4 \) leptons times \( 4 \) quarks with \( 3 \) colors each. We write the representation of \( \mathcal{A} \) in terms of the suggestive basis of \( \mathbb{C}^{96} \):

\[
\begin{pmatrix}
\nu_L & e_L & \nu_R & e_R & u_L & d_L & u_R & d_R & \Psi_L & \overline{\Psi}_L & \Psi_R & \overline{\Psi}_R & \Psi_R & \overline{\Psi}_R \end{pmatrix}^t.
\]

Then, for an element \((\lambda, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\)

\[
\pi(\lambda, q, m) = \begin{pmatrix}
\lambda & q \otimes \mathbb{1}_3 & \lambda \otimes \mathbb{1}_3 \\
\lambda & \mathbb{1}_3 & \lambda \otimes \mathbb{1}_3 \\
\lambda & \mathbb{1}_3 & \lambda \otimes \mathbb{1}_3 \\
\end{pmatrix}
\]

Here, the quaternion \( q \) is considered as a \( 2 \times 2 \)-matrix. The \( 96 \times 96 \)-matrix \( D_F \) is of the following form: \( D_F = (S \ T) \) where

\[
S = \begin{pmatrix}
\gamma^y \gamma^x & 0 \\
0 & \gamma^z \gamma^x \\
\gamma^z \gamma^x & \gamma^y \\
\end{pmatrix}, \quad T = \begin{pmatrix}
0 & \gamma_R \\
\gamma_R & 0 \\
\end{pmatrix}
\]

in terms of the \( 3 \times 3 \) \textit{Yukawa-mixing-matrices} \( \gamma^y, \gamma^z, \gamma^{\mu}, \gamma^d \) and a real constant \( \gamma_R \) responsible for neutrino mass terms.

One can further enrich this spectral triple by a grading \( \gamma_F \) which is \(+1\) on all \( L \)-particles, and \(-1\) on all \( R \)-particles; the total grading is then \( \gamma_5 \otimes \gamma_F \). The anti-linear operator \( J \) is a combination of charge conjugation on \( S \) and the (anti-linear) matrix \( J_F = (148 \ 148) \).

The rest then follows from a long calculation; the inner fluctuations are \( D_A = \partial + i \gamma_\mu A_\mu + \gamma_5(D_F + \mathcal{M}(\Phi)) \) with

\[
A_\mu = \begin{pmatrix}
\frac{21}{4} B_\mu - \frac{23}{4} W_\mu & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\mathcal{M}(\Phi) = \begin{pmatrix}
\gamma_{\nu \phi_1} & \gamma_{\nu \phi_2} \\
-\gamma_{\nu \phi_1} & \gamma_{\nu \phi_2} \\
\end{pmatrix}
\]

\(^1\)A. Connes and M. Marcolli. \textit{Noncommutative Geometry, Quantum Fields and Motives}. AMS, Providence, 2008

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2. BIS: THE STANDARD MODEL AS A NONCOMMUTATIVE MANIFOLD

Here $B_\mu, W_\mu, V_\mu$ are $U(1), SU(2)$ and $SU(3)$-gauge fields, resp. and $\Phi = (\phi_1 \phi_2)^t$ two scalar (Higgs) fields. The spectral action is modulo gravitational terms:

$$S_\Lambda = \frac{-2af_2\Lambda^2 + ef_0}{\pi^2} \int |\phi|^2 + \frac{f_0}{2\pi^2} \int a|D_\mu\phi|^2 - \frac{f_0}{12\pi^2} \int aR|\phi|^2$$

$$- \frac{f_0}{2\pi^2} \int \left( g_3^2 G^a_{\mu\nu} G^{a\nu} + g_2^2 F^{a}_{\mu
u} F^{a\mu\nu} + \frac{5}{3} g_1^2 B_\mu B^\mu \right) + \frac{f_0}{2\pi^2} \int b|\phi|^4 + O(\Lambda^{-2})$$

with $a, b, c, d, e$ constants depending on the Yukawa parameters. For example,

$$a = \text{Tr} \left( \Upsilon^* \nu \Upsilon_\nu + \Upsilon^* e \Upsilon_e + 3 (\Upsilon^* u \Upsilon_u + \Upsilon^* d \Upsilon_d) \right)$$

$$b = \text{Tr} \left( (\Upsilon^* \nu \Upsilon_\nu)^2 + (\Upsilon^* e \Upsilon_e)^2 + 3 ((\Upsilon^* u \Upsilon_u)^2 + (\Upsilon^* d \Upsilon_d)^2) \right)$$

When we add the fermionic term $\langle J_\psi, D_A \psi \rangle$ to $S_\Lambda$, we obtain the Standard Model Lagrangian, including the Higgs boson, provided we have

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2.$$ 

These GUT-type relations between the coupling constants allows for predictions. For example, one identifies the mass of the $W$ as $2M_W = \sqrt{a/2}$ so that the Higgs vacuum reads $2M/g_2$. The above relation for $a$ then gives a postdiction for the mass of the top quark as $m_t \leq 180$ GeV. Moreover, the mass of the Higgs is $m_H = 8\lambda M^2/g_2^2$ with $\lambda = g_3^2 b/a^2$ resulting in a prediction of $m_H \sim 168$ GeV.