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The noncommutative cylinder and its K-theory

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Abstract

We prove that the cylinder has rigid K-theory under deformation quantization. This means that the noncommutative cylinders, defined by strict deformation quantization of the cylinder $\mathbb{R}^d \times \mathbb{T}^d$, have the same K-groups as the ordinary cylinder. Two old examples of this phenomenon are revisited: Euclidean space $\mathbb{R}^{2n}$ and the torus $\mathbb{T}^d$. We construct an isomorphism between the noncommutative cylinder and the crossed product algebra $C(\mathbb{T}^d) \rtimes \mathbb{R}^d$, enabling us to calculate the K-groups of the noncommutative cylinder. We discuss simplicity of the noncommutative cylinders and compare with the two old examples.

1 Introduction

The quantization of the phasespace $\mathbb{R}^{2n}$, which plays an essential role in the formulation of basic quantum mechanics, has been understood since the 1930's (cf. [6] and references therein). More recently, there has been increasing interest in the quantization of other manifolds. An example is the torus, which plays a rôle in string theory and M(atrix) theory. Its quantization is called the noncommutative torus [10, 12].

String theory offers even more examples of noncommutative geometries. In 1998 Polchinski argued that these play an important role in the dynamics of D-branes in the presence of constant magnetic fields on the branes ([9]). A more string-theoretic view of this is given by Seiberg and Witten in [15]. They considered open strings in the presence of a magnetic field and showed that the effective action is described by making spacetime noncommutative.

In string theory, spacetime is a manifold of dimension higher than four. This dimension follows from certain consistency conditions (see Polchinski [9]). For example, the superstring can only be defined in a ten-dimensional background, say $\mathbb{R}^{10}$. It is usually toroidally compactified to $\mathbb{R}^4 \times \mathbb{T}^6$, in order for the theory to make sense. This means that 6 dimensions are rolled up to the 6-torus $\mathbb{T}^6$. To describe a noncommutative background for a superstring, one needs to quantize the (generalized) cylinder $\mathbb{R}^4 \times \mathbb{T}^6$.

Another motivation to quantize the cylinder comes from an idea of Kamani. In [5], he studied the worldsheet of a superstring in a D-brane background as a noncommutative geometry. In this case, one quantizes the worldsheet, which is an ordinary cylinder $\mathbb{R} \times \mathbb{T}$.

Apart from such physical arguments, the quantization of the cylinder is also interesting from a mathematical point of view. It turns out that the C^*-algebras occuring in the quantization of the plane and of the torus are rather different. As the cylinder in some sense lies in between the plane and the torus, it will be interesting to examine the C^*-algebras occuring in its quantization. Furthermore, the noncommutative cylinder provides another example in the scarce list of finite-dimensional noncommutative geometries.

In this paper we first summarize the strict deformation quantization of Euclidean space $\mathbb{R}^{2n}$ and of the torus $\mathbb{T}^d$. This requires a real parameter $\hbar$ and a skew-symmetric matrix θ, and will lead to the definition of two families of C^*-algebras, the noncommutative Euclidean spaces $\{\mathbb{R}^{2n}_\hbar\}$ and the noncommutative tori $\{\mathbb{T}^d_{\hbar\theta}\}$. These families are parametrized by $\hbar \in \mathbb{R}$. Following Rieffel ([11, 12, 13]), we relate these to the theory of crossed product algebras, and use this to discuss simplicity and K-theory of the noncommutative plane and torus. In both cases, it turns out that K-theory is rigid under quantization, that is, independent of $\hbar$, including $\hbar = 0$.

Subsequently, we give the strict deformation quantization of the generalized cylinder $\mathbb{R}^n \times \mathbb{T}^d$. Once again, this will lead to a family of C^*-algebras, the noncommutative cylinders $\{C^{(n,d)}_{\hbar\theta}\}$, parametrized by $\hbar$. For the special case of $n = d$ we prove that $C^{2d}_\hbar := C^{(d,d)}_\hbar$ is isomorphic to the crossed product algebra $C(\mathbb{T}^d) \rtimes \mathbb{R}^d$. We use this to discuss simplicity and to show that for all $\hbar$ one has

$$K_0(C^{2d}_\hbar) \cong K^0(\mathbb{R}^d \times \mathbb{T}^d) \cong \mathbb{Z}^{2^{d-1}};$$
$$K_1(C^{2d}_\hbar) \cong K^1(\mathbb{R}^d \times \mathbb{T}^d) \cong \mathbb{Z}^{2^{d-1}},$$

and conclude that the K-theory of the cylinder is rigid under quantization as well.

2 Old examples

We start with a brief recapitulation of the definition of strict deformation quantization. Subsequently, we review the strict deformation quantization of Euclidean space and of the torus, both due to Rieffel [10, 11, 12, 13].

Definition 1 Let M be a manifold with Poisson bracket $\{\ ,\ \}$ and let $\mathcal{A}$ be a dense $*$-subalgebra of $C_0(M)$. A **strict deformation quantization** of M in the direction of $\{\ ,\ \}$, consists of an open interval $I \subseteq \mathbb{R}$ with 0 as an accumulation point, together with, for each $\hbar \in I$, an associative product $*_\hbar$, an involution $^{*_\hbar}$, and a C^*-norm $\|\ \|_\hbar$ (for $*_\hbar$ and $^{*_\hbar}$) on $\mathcal{A}$, which for $\hbar = 0$ are the original pointwise product, complex conjugation involution, and supremum norm, such that

1. The family $\{\mathcal{A}_\hbar\}_{\hbar \in I}$ forms a continuous field of C^*-algebras over I. Here $\mathcal{A}_\hbar$ denotes the C^*-completion of $\mathcal{A}$ with respect to $\|\ \|_\hbar$.

2. For every $f, g \in \mathcal{A}$,
$$\|(f *_\hbar g - g *_\hbar f)/\mathrm{i}\hbar - \{f, g\}\|_\hbar$$

converges to 0 as $\hbar$ goes to 0. (Dirac's condition)

2.1 Weyl quantization

We consider even-dimensional Euclidean space $\mathbb{R}^{2n}$. Let $\mathcal{S}(\mathbb{R}^{2n})$ denote the algebra of Schwartz functions on $\mathbb{R}^{2n}$. The pointwise product in this algebra is deformed to the Moyal star product, which reads in Fourier space for any $\hbar \in \mathbb{R}$

$$(\phi *_\hbar \psi)(p,q) = \int_{\mathbb{R}^{2n}} \mathrm{d}^n p' \ \mathrm{d}^n q' \ \phi(p',q')\psi(p-p',q-q')e^{-\mathrm{i}\hbar(q'\cdot p - p'\cdot q)}. \tag{1}$$

The involution we use on $\mathcal{S}(\mathbb{R}^{2n})$ is defined by $\phi^*(p,q) = \overline{\phi(-p,-q)}$, which is independent of $\hbar$. We let $\pi_\hbar$ denote the left regular representation of $\mathcal{S}(\mathbb{R}^{2n})$ on $L^2(\mathbb{R}^{2n})$ via $*_\hbar$, i.e. for $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\Psi \in L^2(\mathbb{R}^{2n})$,

$$\pi_\hbar(\phi)\Psi := \phi *_\hbar \Psi.$$

We define a norm $\| \ \|_\hbar$ on $\mathcal{S}(\mathbb{R}^{2n})$ as the operator norm for this representation. The completion of $\mathcal{S}(\mathbb{R}^{2n})$ with respect to this norm is a C^*-algebra, denoted by $\mathbb{R}^{2n}_\hbar$. By rewriting formula (1) in terms of partial Fourier transforms, one can show the following.

Proposition 2 *The C^*-algebra $\mathbb{R}^{2n}_\hbar$ is isomorpic to the crossed product algebra*

$$C_0(\mathbb{R}^n) \rtimes_\hbar \mathbb{R}^n,$$

where $\mathbb{R}^n$ acts on $\mathbb{R}^n$ by translation, $x \mapsto x + \hbar y$ $(x, y \in \mathbb{R}^n)$. □

Theorem 3 *The C^*-algebra $\mathbb{R}^{2n}_\hbar$ is isomorphic to $\mathcal{B}_0(L^2(\mathbb{R}^n))$, the C^*-algebra of compact operators on $L^2(\mathbb{R}^n)$.*

Proof. We define the map $Q^W_\hbar : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{B}(L^2(\mathbb{R}^n))$ by its action on $L^2(\mathbb{R}^n)$,

$$Q^W_\hbar(\phi)\chi(x) := \int_{\mathbb{R}^{2n}} \frac{\mathrm{d}^n p \ \mathrm{d}^n y}{(2\pi\hbar)^n} \ e^{\mathrm{i}p(x-y)/\hbar}\phi\big(p, \frac{1}{2}(x+y)\big)\chi(y).$$

where $\chi \in L^2(\mathbb{R}^n)$. This map is known as Weyl quantization, and satisfies

$$Q^W_\hbar(\phi *_\hbar \psi) = Q^W_\hbar(\phi)Q^W_\hbar(\psi).$$

Furthermore, $Q^W_\hbar(\phi)\chi(x)$ can be written as an integral operator with kernel in $\mathcal{S}(\mathbb{R}^{2n})$ (see [6]). Thus, $Q^W_\hbar$ maps $\mathcal{S}(\mathbb{R}^{2n})$ onto $\mathcal{B}_2(L^2(\mathbb{R}^n))$, the algebra of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. The following lemma completes the proof.

Lemma 4 *Let A and B be C^*-algebras. Let E and F be dense $*$-subalgebras of A and B, respectively. If $E \cong F$ as pre-C^*-algebras, then $A \cong B$ as C^*-algebras.* □

□

For $\hbar = 0$, formula (1) reduces to the ordinary convolution product. The closure of $\mathcal{S}(\mathbb{R}^{2n})$ with this product in the operator norm, is $\mathbb{R}^{2n}_0$. With Lemma 4, we have $\mathbb{R}^{2n}_0 \cong C_0(\mathbb{R}^{2n})$. The Weyl maps $Q^W_\hbar$ then define a strict deformation quantization of $\mathbb{R}^{2n}$ [11, 13] (also cf. [6]). With Proposition 2 above, continuity of the field $\{\mathbb{R}^{2n}_\hbar\}$ follows from Lemma 1 in [7].

The following corollaries are immediate.

Corollary 5 *The C^*-algebras $\mathbb{R}^{2n}_\hbar$ $(\hbar \neq 0)$ are simple algebras, and are all isomorphic to each other.* $\square$

Corollary 6 *Euclidean space $\mathbb{R}^{2n}$ has rigid K-theory under quantization, i.e., for all $\hbar$ one has*

$$K^0(\mathbb{R}^{2n}) \cong K_0(\mathbb{R}^{2n}_\hbar) \cong \mathbb{Z};$$
$$K^1(\mathbb{R}^{2n}) \cong K_1(\mathbb{R}^{2n}_\hbar) \cong 0.$$

$\square$

2.2 Noncommutative tori

Let $\mathbb{T}^d$ be the d-dimensional torus, and let θ be a real skew-symmetric $d \times d$ matrix. Instead of deforming the pointwise product in the space of smooth functions on $\mathbb{T}^d$, we deform the product in its Fourier space $\mathcal{S}(\mathbb{Z}^d)$. For $\hbar \in \mathbb{R}$, the star product reads

$$(\phi *_\hbar \psi)(n) = \sum_{m \in \mathbb{Z}^d} \phi(m)\psi(n-m)e^{2\pi i \hbar \theta(m,n)}. \tag{2}$$

Here θ is the skew bilinear form defined by

$$\theta(m,n) := \sum_{j,k} \theta_{jk} m_j n_k. \tag{3}$$

We set $\phi^*(n) := \overline{\phi(-n)}$, which is independent of $\hbar$. We let $\mathcal{S}(\mathbb{Z}^d)$ act on $L^2(\mathbb{Z}^d)$ by left multiplication via $*_\hbar$. The completion of $\mathcal{S}(\mathbb{Z}^d)$ with respect to the operator norm $\| \, \|_\hbar$, equipped with this star product is the **noncommutative torus**, denoted by $\mathbb{T}^d_{\hbar\theta}$. For fixed θ, the family $\{\mathbb{T}^d_{\hbar\theta}\}$ provides a strict deformation quantization of $\mathbb{T}^d$ [10]. When $d = 2$, the skew-symmetric matrix θ is just determined by a real number, denoted by θ as well. From [12] we take the following.

Proposition 7 *The noncommutative torus $\mathbb{T}^2_\theta$ is isomorphic to the crossed product algebra $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, where $\alpha(f)(t) := f(t+\theta)$.* $\square$

Proposition 8 *The C^*-algebra $\mathbb{T}^2_\theta$ is simple if and only if θ is irrational. If $\theta \neq \theta'$, both irrational with $0 < \theta, \theta' < \frac{1}{2}$, then $\mathbb{T}^2_\theta \not\cong \mathbb{T}^2_{\theta'}$.* $\square$

It came as a surprise that the K-groups of $\mathbb{T}^d_\theta$ do not depend on θ.

Proposition 9 *The torus $\mathbb{T}^d$ has rigid K-theory under quantization, i.e., for all $\hbar$ one has*

$$K^0(\mathbb{T}^d) \cong K_0(\mathbb{T}^d_{\hbar\theta}) \cong \mathbb{Z}^{2^{d-1}};$$
$$K^1(\mathbb{T}^d) \cong K_1(\mathbb{T}^d_{\hbar\theta}) \cong \mathbb{Z}^{2^{d-1}}.$$

Proof. For the calculation of $K_0(\mathbb{T}^d_{\hbar\theta})$ and $K_1(\mathbb{T}^d_{\hbar\theta})$ we refer to a note at the end of Chapter 12 in [3]. For the K-groups of $\mathbb{T}^d$, we note that

$$K_0(C(\mathbb{T},\mathcal{A})) \cong K_1(C(\mathbb{T},\mathcal{A})) \cong K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$$

for any C^*-algebra $\mathcal{A}$ (cf. Exercise 10.1 in [14]). Since $C(\mathbb{T}^d) \cong C(\mathbb{T}, C(\mathbb{T}^{d-1}))$, this yields by induction

$$K_0(C(\mathbb{T}^d)) \cong K_1(C(\mathbb{T}^d)) \cong \mathbb{Z}^{2^{d-1}}$$

□

3 Deformation quantization of cylinders

We consider the cylinder in a generalized form. The (n,d)**-dimensional cylinder** $C^{(n,d)}$ is defined as

$$C^{(n,d)} := \mathbb{R}^n \times \mathbb{T}^d. \tag{4}$$

In the case $n = d = 1$ we obtain $C := \mathbb{R} \times S^1$, which is of course the familiar two-dimensional cylinder.

Let Λ be a Poisson structure on $\mathbb{R}^n \times \mathbb{T}^d$. For $j = 1, \cdots, n+d$, let ∂_{x_j} denote the vector field on $\mathbb{R}^n \times \mathbb{T}^d$ corresponding to differentiation in the j^{th} direction. We can write the Poisson structure as

$$\Lambda = -\pi^{-1} \sum_{i<j} \theta_{ij} \partial_{x_i} \wedge \partial_{x_j}. \tag{5}$$

The factor π^{-1} has been included for later convenience. Here θ_{ij} is a real skew-symmetric matrix. For later use, we define a skew bilinear form θ on $\mathbb{R}^n \times \mathbb{Z}^d$,

$$\theta(l,k) = \sum_{i,j} \theta_{ij} l_i k_j, \qquad (l, k \in \mathbb{R}^n \times \mathbb{Z}^d). \tag{6}$$

Let λ denote Lebesgue measure on $\mathbb{R}^n \times \mathbb{T}^d$. The Fourier transform $\hat{f}$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{T}^d)$ is given by

$$\hat{f}(k) = \int_{\mathbb{R}^n \times \mathbb{T}^d} \mathrm{d}\lambda(x) e^{-2\pi \mathrm{i} k \cdot x} f(x). \tag{7}$$

For $i = 1, \dots, n$ we have $k_i \in \mathbb{R}$, for $i = n+1, \dots, n+d$ we have $k_i \in \mathbb{Z}$. Furthermore, we have the following result.

Lemma 10 *The Fourier transform maps $\mathcal{S}(\mathbb{R}^n \times \mathbb{T}^d)$ isomorphically to $\mathcal{S}(\mathbb{R}^n \times \mathbb{Z}^d)$.* □

To integrate over $\mathbb{R}^n$ and sum over $\mathbb{Z}^d$ in the product $\mathbb{R}^n \times \mathbb{Z}^d$, we introduce the measure μ on $\mathbb{R}^n \times \mathbb{Z}^d$, defined as the product of Lebesgue measure on $\mathbb{R}^n$ and the counting measure on $\mathbb{Z}^d$.

For functions in Fourier space, the Poisson bracket is given by

$$\begin{aligned} \{\phi, \psi\}(k) &= 4\pi \int_{\mathbb{R}^n \times \mathbb{Z}^d} \mathrm{d}\mu(l) \sum_{i,j} \theta_{ij} l_i \phi(l)(k_j - l_j)\psi(k-l) \\ &= 4\pi \int_{\mathbb{R}^n \times \mathbb{Z}^d} \mathrm{d}\mu(l)\ \phi(l)\psi(k-l)\theta(l,k) \end{aligned} \tag{8}$$

where $k, l \in \mathbb{R}^n \times \mathbb{Z}^d$ and θ is the bilinear form defined in equation (6).

We define a bicharacter $\sigma_\hbar$ on $\mathbb{R}^n \times \mathbb{Z}^d$ by

$$\sigma_\hbar(l,k) = e^{2\pi \mathrm{i}\hbar\theta(l,k)}, \tag{9}$$

where $\hbar \in \mathbb{R}$, and introduce a star product $*_\hbar$ on $\mathcal{S}(\mathbb{R}^n \times \mathbb{Z}^d)$ by,

$$(\phi *_\hbar \psi)(k) = \int_{\mathbb{R}^n \times \mathbb{Z}^d} \mathrm{d}\mu(l)\ \phi(l)\psi(k-l)\sigma_\hbar(l,k). \tag{10}$$

We define an involution on $\mathcal{S}(\mathbb{R}^n \times \mathbb{Z}^d)$ by $\phi^*(k) := \overline{\phi(-k)}$, independent of $\hbar$. We represent $\mathcal{S}(\mathbb{R}^n \times \mathbb{Z}^d)$ on $L^2(\mathbb{R}^n \times \mathbb{Z}^d)$ by star product multiplication, and define the **noncommutative cylinder** as the completion of $\mathcal{S}(\mathbb{R}^n \times \mathbb{Z}^d)$ in the operator norm $\| \|_\hbar$, equipped with product $*_\hbar$. This C^*-algebra is denoted by $C_{\hbar\theta}^{(n,d)}$.

We could equally well define the noncommutative cylinder as the (completion of) the algebra $\mathcal{S}(\mathbb{R}^n \times \mathbb{T}^d)$ with product, involution and norm obtained by pulling back the product $*_\hbar$, involution * and norm $\| \|_\hbar$ through the inverse Fourier transform. Even though this makes the differences with the ordinary cylinder more clear, we will continue in Fourier space to avoid expressions involving many derivatives.

Theorem 11 *For fixed θ, the family $\{C_{\hbar\theta}^{(n,d)}\}$ provides a strict deformation quantization of $\mathbb{R}^n \times \mathbb{T}^d$ in the direction of $\{\ ,\ \}$.*

Proof. We verify Dirac's condition

$$\|(\phi *_\hbar \psi - \psi *_\hbar \phi)/\mathrm{i}\hbar - \{\phi, \psi\}\|_\hbar \to 0 \quad \text{as } \hbar \to 0, \tag{11}$$

where $\phi, \psi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{Z}^d)$. We define

$$\Delta_\hbar := (\phi *_\hbar \psi - \psi *_\hbar \phi)/\mathrm{i}\hbar - \{\phi, \psi\}.$$

With formulae (8) and (10), this reads

$$\Delta_\hbar(k) = \int_{\mathbb{R}^n \times \mathbb{Z}^d} \mathrm{d}\mu(l)\ \phi(l)\psi(k-l)\big((\sigma_\hbar(l,k) - \sigma_\hbar(k,l))/\mathrm{i}\hbar - 4\pi\theta(l,k)\big).$$

Similar to Rieffel in [10], we can estimate the expression inside ( ) so that,

$$|\Delta_\hbar(k)| \le \hbar M \int_{\mathbb{R}^n\times\mathbb{Z}^d} \mathrm{d}\mu(l)\ |\phi(l)|\ |\psi(k-l)||l|^2|k-l|^2,$$

for some constant M. This last expression is just (proportional to) the convolution product of two functions $\tilde{\phi}$ and $\tilde{\psi}$ where

$$\tilde{\phi}(k) := |k|^2|\phi(k)|, \quad \tilde{\psi}(k) := |k|^2|\psi(k)|.$$

As the L^1-norm dominates the norm $\|\ \|_\hbar$, we have

$$\|\Delta_\hbar\|_\hbar \le \hbar M \|\tilde{\phi} * \tilde{\psi}\|_1.$$

It follows that $\|\Delta_\hbar\|_\hbar \to 0$ as $\hbar \to 0$.
Continuity of the field $\{C^{(n,d)}_{\hbar\theta}\}$ follows from Lemma 1 in [11], in combination with Proposition 12 below. □

4 Properties of noncommutative cylinders

In this section, we want to discuss the algebraic properties of the noncommutative cylinder. First, we connect with the theory of crossed product algebras. Then, we discuss the K-theory of noncommutative cylinders.

When one observes the major differences between $\mathbb{R}^{2n}_\hbar$ and $\mathbb{T}^d_{\hbar\theta}$, one is led to the questions if the noncommutative cylinders are simple and if they are all isomorphic. We take the noncommutative cylinder for $n = d$, and denote it by $C^{2d}_\hbar$. We let $l = (x, n)$ and $k = (y, m)$, where $x, y \in \mathbb{R}^d$ and $n, m \in \mathbb{Z}^d$, and choose the following special form of the skew bilinear form on $\mathbb{R}^d \times \mathbb{Z}^d$,

$$\theta(l,k) = \frac{1}{2\pi}\sum_{i=1}^{d} y_i n_i - m_i x_i. \tag{12}$$

We want to rewrite the star product (10) in terms of partial Fourier transforms, defined by

$$\acute{\phi}(x,t) := \sum_{n\in\mathbb{Z}^d} \phi(x,n)e^{\mathrm{i}n\cdot t} \qquad (t \in \mathbb{T}^d), \tag{13}$$

which is a function on $\mathbb{R}^d \times \mathbb{T}^d$. The star product on $\mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$ then reads

$$(\acute{\phi} * \acute{\psi})(x,t) = \int_{\mathbb{R}^d} \mathrm{d}y\ \acute{\phi}(y, t + \hbar(y-x))\ \acute{\psi}(x-y, t+\hbar y), \tag{14}$$

as can be easily verified. We introduce an action β of $\mathbb{R}^d$ on $\mathbb{T}^d$ defined by $\beta_x(t) = t + \hbar x$, and write

$$(\acute{\phi} * \acute{\psi})(x,t) = \int_{\mathbb{R}^d} \mathrm{d}y\ \acute{\phi}(y, \beta_{y-x}(t))\ \acute{\psi}(x-y, \beta_y(t)). \tag{15}$$

This formulation of the star product in terms of an action β of $\mathbb{R}^d$ on $\mathbb{T}^d$ goes back to Rieffel. As is done in the examples in his paper [13], we relate this to crossed product algebras. For more details on the theory of these algebras, we refer to Pedersen [8]. Let $C(\mathbb{T}^d) \rtimes_\hbar \mathbb{R}^d$ denote the crossed product algebra for the $\hbar$-dependent action β_{2x}. Then $\mathcal{S}(\mathbb{R}^d, C^\infty(\mathbb{T}^d))$ is a dense $*$-subalgebra of this crossed product algebra. Define a map $Q : \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d) \to \mathcal{S}(\mathbb{R}^d, C^\infty(\mathbb{T}^d))$ by

$$Q(\acute{\phi})(x,t) := \acute{\phi}(x, \beta_x(t)). \tag{16}$$

Note that $\mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$ is a dense $*$-subalgebra of $C^{2d}_\hbar$. Clearly, Q is an isomorphism, in that

$$\begin{aligned} Q(\acute{\phi} *_\hbar \acute{\psi})(x,t) &= (\acute{\phi} *_\hbar \acute{\psi})(x, \beta_x(t)) \qquad (17)\\ &= \int_{\mathbb{R}^d} \mathrm{d}y\ \acute{\phi}(y, \beta_{y-x} \circ \beta_x(t))\ \acute{\psi}(x-y, \beta_y \circ \beta_x(t)) \\ &= \int_{\mathbb{R}^d} \mathrm{d}y\ Q(\acute{\phi})(y,t)\ Q(\acute{\psi})(x-y, \beta_{2x}(t)) \\ &= Q(\acute{\phi}) * Q(\acute{\psi}). \end{aligned}$$

In the case of the noncommutative cylinder, Lemma 4 now yields the following.

Proposition 12 *The noncommutative cylinder $C^{2d}_\hbar$ ($\hbar \neq 0$) is isomorphic to the crossed product $C(\mathbb{T}^d) \rtimes_\hbar \mathbb{R}^d$.* □

This allows us to use known results on crossed product algebras.

Theorem 13 *The C^*-algebra $C^{2d}_\hbar$ is isomorphic to $\mathcal{B}_0(L^2(\mathbb{T}^d)) \otimes C^*(\mathbb{Z}^d)$.*

Proof. We note that $C(\mathbb{T}^d) \rtimes_\hbar \mathbb{R}^d \cong C(\mathbb{T}^d) \rtimes_{\hbar'} \mathbb{R}^d$ for $\hbar, \hbar' \neq 0$. In particular,

$$C(\mathbb{T}^d) \rtimes_\hbar \mathbb{R}^d \cong C(\mathbb{T}^d) \rtimes \mathbb{R}^d$$

for $\hbar \neq 0$. Proposition 12 above and Corollary 2.8 of Green [4] complete the proof. □

With the isomorphism $C^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$, we have the following.

Corollary 14 *The noncommutative cylinders $C^{2d}_\hbar$ ($\hbar \neq 0$) are nonsimple C^*-algebras.*□

It is well known that any C^*-algebra A is Morita equivalent to its stabilization

$$A_S := \mathcal{B}_0(\mathcal{H}) \otimes A$$

for some Hilbert space $\mathcal{H}$. In particular, $\mathcal{B}_0(L^2(\mathbb{T}^d)) \otimes C(\mathbb{T}^d)$ is Morita equivalent to $C(\mathbb{T}^d)$. Since Morita-equivalent C^*-algebras have isomorphic K-groups, we have the following.

Corollary 15 *For the noncommutative cylinder $C^{2d}_\hbar$ one has for all $\hbar \neq 0$,*

$$K_0(C^{2d}_\hbar) \cong K_1(C^{2d}_\hbar) \cong \mathbb{Z}^{2^{d-1}}.$$

Proof. We note that these K-groups are isomorphic to $K^0(\mathbb{T}^d) \cong K^1(\mathbb{T}^d) \cong \mathbb{Z}^{2^{d-1}}$, by the above comments and the proof of Proposition 9. □

In order to compare this with the K-groups of the original cylinder $\mathbb{R}^d \times \mathbb{T}^d$ we need the following Lemma.

Lemma 16 *For the cylinder $\mathbb{R}^d \times \mathbb{T}^d$ the K-groups are*

$$K^0(\mathbb{R}^d \times \mathbb{T}^d) \cong K^1(\mathbb{R}^d \times \mathbb{T}^d) \cong \mathbb{Z}^{2^{d-1}}.$$

Proof. The proof of Proposition 9 can be adopted to show by induction that

$$K_0(C_0(\mathbb{R}^d \times \mathbb{T}^d)) \cong K_1(C_0(\mathbb{R}^d \times \mathbb{T}^d)) \cong \mathbb{Z}^{2^{d-1}}.$$

Here, one uses $K_0(C_0(\mathbb{R}^d)) \oplus K_1(C_0(\mathbb{R}^d)) \cong \mathbb{Z}$.

An alternative and more elegant proof can be constructed using Bott periodicity directly:

$$\begin{aligned} K^0(\mathbb{R}^d \times \mathbb{T}^d) &\cong K^d(\mathbb{T}^d) \cong \left\{ \begin{array}{c} K^0(\mathbb{T}^d) \text{ (d even)} \\ K^1(\mathbb{T}^d) \text{ (d odd)} \end{array} \right\} \cong \mathbb{Z}^{2^{d-1}} \\ K^1(\mathbb{R}^d \times \mathbb{T}^d) &\cong K^{d+1}(\mathbb{T}^d) \cong \left\{ \begin{array}{c} K^1(\mathbb{T}^d) \text{ (d even)} \\ K^0(\mathbb{T}^d) \text{ (d odd)} \end{array} \right\} \cong \mathbb{Z}^{2^{d-1}} \end{aligned}$$

□

Proposition 17 *The cylinder $\mathbb{R}^d \times \mathbb{T}^d$ has rigid K-theory under quantization, i.e., for all $\hbar$ one has*

$$\begin{aligned} K^0(\mathbb{R}^d \times \mathbb{T}^d) \cong K_0(C^{2d}_\hbar) \cong \mathbb{Z}^{2^{d-1}}; \\ K^1(\mathbb{R}^d \times \mathbb{T}^d) \cong K_1(C^{2d}_\hbar) \cong \mathbb{Z}^{2^{d-1}}. \end{aligned}$$

□

Note that these groups are the same as the K-groups of the torus $\mathbb{T}^d_{\hbar\theta}$. This is a consequence of Bott periodicity, cf. the above proof of Lemma 16.

5 Conclusions and Outlook

In this paper, we have discussed the strict deformation quantization of Euclidean space $\mathbb{R}^{2n}$ and the torus $\mathbb{T}^d$. This led to the definition of two families of C^*-algebras $\{\mathbb{R}^{2n}_\hbar\}$ and $\{\mathbb{T}^d_{\hbar\theta}\}$, parametrized by $\hbar$. We related both $\mathbb{R}^{2n}_\hbar$ and $\mathbb{T}^d_{\hbar\theta}$ to crossed product algebras and used this to discuss simplicity and K-theory of these C^*-algebras. It turned out in both cases that K-theory is rigid under quantization, that is, independent of $\hbar$, including $\hbar = 0$.

We considered strict deformation quantization of the cylinder $\mathbb{R}^n \times \mathbb{T}^d$. This led to a family of C^*-algebras $\{C_{\hbar\theta}^{(n,d)}\}$, the noncommutative cylinders. We related the noncommutative cylinder to a crossed product algebra by showing that

$$C_\hbar^{2d} \cong C(\mathbb{T}^d) \rtimes \mathbb{R}^d.$$

We used some theory on crossed product algebras to discuss simplicity and K-theory and showed that K-theory of the cylinder is rigid under quantization.

It would be interesting to describe the noncommutative cylinder as a noncommutative geometry, as defined by Connes [1]. This involves the notion of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. In the case of the noncommutative cylinder, one needs to construct an unbounded operator D, which is the analogue of the Dirac operator on the ordinary cylinder. The spectral triple $(C_\hbar^{2d}, L^2(\mathbb{R}^d \times \mathbb{T}^d), D)$ then describes the noncommutative geometry of the noncommutative cylinder. This spectral triple can be compared to the spectral triple that describes the noncommutative torus [1] and to the spectral triple that describes the noncommutative 4-sphere S_θ^4 [2].

Furthermore, the C^*-algebra $C_\hbar^{(n,d)}$ should be studied in the case $n \neq d$. One can show that if n and d are both even-dimensional, we have,

$$C_\hbar^{(n,d)} \cong C_0(\mathbb{R}^{n/2} \times \mathbb{T}^{d/2}) \rtimes (\mathbb{R}^{n/2} \times \mathbb{Z}^{d/2}).$$

Here the matrix θ has a skew diagonal form,

$$(\theta_{ij}) = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

An analogue of Theorem 13, however, does not hold in this case, because Corollary 2.8 of Green is not applicable. In order to discuss for example K-theory of this noncommutative cylinder, we need another approach.

References

[1] A. Connes. *Noncommutative Geometry.* Academic Press, 1994.

[2] A. Connes and G. Landi. Noncommutative manifolds, the instanton algebra and isospectral deformations. `arXiv: math.QA/0011194`.

[3] J.M. Gracia-Bondía, J.C. Várilly, and H. Figueroa. *Elements of Noncommutative Geometry.* Birkhäuser, 2001.

[4] P. Green. The structure of imprimitivity algebras. *J. Funct. Anal.*, 36:88–104, 1980.

[5] D. Kamani. Noncommutative superstring worldsheet. `arXiv: hep-th/0008020`.

[6] N.P. Landsman. *Mathematical Topics between Classical and Quantum Mechanics.* Springer, 1991.

[7] N.P. Landsman. Deformation quantization and the Baum-Connes conjecture. *Comm. Math. Phys.*, (to appear).

[8] G.K. Pedersen. *C^*-algebras and their Automorphism Groups.* Academic Press, 1979.

[9] J. Polchinski. *String Theory Volume I - An Introduction to the Bosonic String.* CUP, 1998.

[10] M. A. Rieffel. Deformation quantization of Heisenberg manifolds. *Comm. Math. Phys.*, 122:531–562, 1989.

[11] M.A. Rieffel. Deformation quantization and operator algebras. *Proc. Symp. Pure Math.*, 51:411–423, 1990.

[12] M.A. Rieffel. Non-commutative tori - A case study of non-commutative differentiable manifolds. *Contemp. Math.*, 105:191–212, 1990.

[13] M.A. Rieffel. Deformation quantization for actions of $\mathbb{R}^d$. *Mem. Amer. Math. Soc.*, 506, 1993.

[14] M. Rørdam, F. Larsen, and N.J.Laustsen. *An Introduction to K-theory for C^*-algebras.* CUP, 2000.

[15] N. Seiberg and E. Witten. String theory and noncommutative geometry. `arXiv: hep-th/9908142`.