Noncommutative geometry, deformation quantization and the string worldsheet

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Abstract

We describe the noncommutative worldsheet of a bosonic string using Connes' theory of noncommutative geometry. A spectral triple is constructed and the noncommutative version of the Polyakov action is written as a Dixmier trace. We show that the original Polyakov action can also be written as a Dixmier trace. This involves Fredholm modules, which are the algebraic generalizations of conformal manifolds. The noncommutative string worldsheet is defined using deformation quantization. In the special case of the Moyal star product, we use Weyl correspondence to define the noncommutative cylinder. For general Poisson manifolds, we define the covariant star product and Kontsevich's star product. It turns out that the latter can be described using path integral methods. Finally, we use the GNS-construction to obtain representations of the deformed algebra of smooth functions on the string worldsheet.

Contents

Chapter 1

Introduction

The formulation of a classical physical system in phase-space provides a geometrical description of the dynamics of this system. The transition from classical mechanics to quantum mechanics is usually done by replacing the position and momentum variables x and p by noncommuting operators $\boldsymbol{x}$ and $\boldsymbol{p}$, respectively. This leads immediately to the question what geometry is described by these operators. An answer can be found in Connes' definition of noncommutative geometry [6]. It involves spectral triples, which are the algebraic generalizations of Riemannian manifolds. Furthermore, it provides a way to formulate the standard model and general relativity in a setting of noncommutative geometry.

Recently, it turned out that noncommutative geometry also arises in string theory. In 1998 Polchinski argued that noncommutative geometry plays an important role in the dynamics of D-branes in the presence of constant magnetic fields on the branes ([17]). A more string theoretic view of this is given by Seiberg and Witten in [20]. Other examples of noncommutative geometry in string theory come from M(atrix) theory, where the noncommutative torus arises naturally in toroidial compactifications. Reviews on noncommutative geometry in M(atrix) theory are e.g. Douglas and Nekrasov [7] and Konechny and Schwarz [9].

Noncommutative geometry, however, does also appear in condensed matter physics. Let us consider a point particle in a large magnetic field. The action for this point particle reads

$$S = \int \mathrm{d}t \left(\frac{1}{2} m \dot{x}_\mu \dot{x}^\mu + e B_{\mu\nu} x^\mu \dot{x}^\nu \right).$$

The conjugate momentum of x^μ is $\pi_\mu = m\dot{x}_\mu + eB_{\mu\nu}x^\nu$. In the limit of a large magnetic field, one can neglect the kinetic term in the action, i.e., formally put $m = 0$. Then, the canonical commutation relations become

$$[x^\mu, x^\nu] = \frac{\mathrm{i}}{e}(B^{-1})^{\mu\nu}.$$

Thus, at energies much less than the cyclotron frequency $e|B|/m$ we effectively have noncommuting coordinates. This regime corresponds to the lowest Landau level, which is why noncommutative geometry appears in the quantum Hall effect.

Besides all these physical arguments, it is also quite natural to think of geometry as a special case of noncommutative geometry, from a mathematical point of view. The Gel'fand-Naimark theorem shows that there is a close relation between commutative algebras and topological (Hausdorff) spaces. One can construct the topology of a Hausdorff space from the algebra of smooth functions $\mathcal{A}$ on this space. This lifts geometry into an algebraic setting, and it is quite natural to ask what geometry is described if $\mathcal{A}$ is replaced by a noncommutative algebra. However, in order to describe a Riemannian manifold in an algebraic setting, the function algebra is not enough, as a manifold carries more structure than mere Hausdorff space. Therefore, Connes represents the algebra in a Hilbert space and combines the two with an unbounded operator in a spectral triple. With spectral triples he develops a new calculus, which replaces the usual differential and integral calculus.

In this thesis, we try to apply Connes' calculus to string theory. In particular, we try to describe the noncommutative string worldsheet by a spectral triple. Our starting point here will be deformation quantization of the worldsheet. We consider different approaches, hoping that one of them will eventually lead us to an operator algebra which enables the formulation of the noncommutative worldsheet in Connes' setting.

In chapter 2 we give a short overview of the theory on C^*-algebras and Hilbert spaces.

This is used in chapter 3 to define spectral triples. Spectral triples generalize the concept of a Riemannian manifold to a noncommutative geometry. A generalization of the integral is made in terms of the Dixmier trace (section IV.2.β of Connes, [6]).

In chapter 4 the Polyakov action is written as a Dixmier trace. This is done in the setting of Fredholm modules. The Fredholm module which arises in chapter 4 makes conformal invariance of the worldsheet explicit.

In chapter 5 we will consider noncommutative geometry as it arises in string theory. Kamani considers in [8] the worldsheet of a string as a noncommutative geometry. The noncommutative worldsheet is defined by deformation of the algebra of smooth functions $C^\infty(\Sigma)$ on the worldsheet Σ by a star product. Here, this idea will be extended in the case that the worldsheet is a cylinder C to construct the operator algebra which is homomorphic to the deformed function algebra on C, using Weyl correspondence. This is similar to the formulation of quantum mechanics in phase-space (Zachos, [29]). In order to describe other noncommutative geometries than the noncommutative cylinder, we define a deformation of the function algebra on a general Poisson manifold (Bayen et al. [2]). Unfortunately, it is not clear how to construct the corresponding operator algebra, since the Weyl correspondence seems to fail here.

We try another approach to deformation quantization in chapter 6. First, we define Kontsevich's star product on a Poisson manifold (Kontsevich [10]). Then, we claim that in the case of a symplectic manifold, the star product of two functions can be given by the semiclassical expansion of a path integral. We show that the coefficients in this expansion are finite and that the star product is associative.

In chapter 7 we try to formulate deformation quantization of the worldsheet in Connes' setting of spectral triples. Here, we use the GNS-construction to obtain a representation of the deformed function algebra.

Chapter 2

C^*-algebras and Hilbert spaces

In this chapter, we will give a very short introduction to the theory of C^*-algebras and Hilbert spaces. We refer to Landi [11], Landsman, [12] and Pedersen [16] for more information on the subject.

2.1 Basic definitions

We first recall some definitions on algebras and vector spaces, assuming some basics about vector spaces. Then, we will give definitions for C^*-algebras and Hilbert spaces.

Definition 2.1 A **norm** on a vector space V is a map $\| \, \|: V \to \mathbb{R}$ such that

1. $\|v\| \geq 0$ for all $v \in V$,
2. $\|v\| = 0 \iff v = 0$,
3. $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{C}$ and $v \in V$,
4. $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality).

A norm on V defines a metric $d: V \times V \to V$ on V by $d(v, w) := \|v - w\|$.

Definition 2.2 Let (v_n) be a sequence in V. We say that (v_n) is a **Cauchy sequence** if for every $\epsilon > 0$ there exists an N such that $\|v_n - v_m\| < \epsilon$ for every $n, m \geq N$. A vector space V is called **complete** (or a **Banach space**) if every Cauchy sequence in V converges.

If we want to be able to multiply elements of a vector space, we need the notion of an algebra.

Definition 2.3 An **algebra** over $\mathbb{C}$ is a vector space $\mathcal{A}$ with a linear, associative product $(a, b) \in \mathcal{A} \times \mathcal{A} \mapsto ab \in \mathcal{A}$ which satisfies

$$a(b + c) = ab + ac, \tag{2.1}$$

$$(a + b)c = ac + bc, \tag{2.2}$$

for all $a, b, c \in \mathcal{A}$.

A **unital algebra** is an algebra $\mathcal{A}$ which contains an element $\mathbb{I} \in \mathcal{A}$ satisfying $\mathbb{I}a = a\mathbb{I} = a$, for all $a \in \mathcal{A}$. $\mathbb{I}$ is called the **unit element** of $\mathcal{A}$.

For later use, we give the definition of an ideal.

Definition 2.4 A (two-sided) **ideal** in an algebra $\mathcal{A}$ is a linear subspace $I \subset \mathcal{A}$ such that $a \in I$ implies $ba \in I$ and $ab \in I$ for all $b \in \mathcal{A}$. A **left-ideal** of $\mathcal{A}$ is a linear subspace $I \subset \mathcal{A}$ such that $a \in I$ implies $ba \in I$ for all $b \in \mathcal{A}$. A **right-ideal** of $\mathcal{A}$ is a linear subspace $I \subset \mathcal{A}$ such that $a \in I$ implies $ab \in I$ for all $b \in \mathcal{A}$.

In particular, an ideal is itself an algebra. An ideal $I \subset \mathcal{A}$ that contains an invertible element a must coincide with $\mathcal{A}$. Since $a^{-1}a = \mathbb{I}$ must lie in I, all $b = b\mathbb{I}$ must lie in I.

Definition 2.5 An **involution** on an algebra $\mathcal{A}$ is a map $^*\colon\ a \in \mathcal{A} \mapsto a^* \in \mathcal{A}$ which satisfies

$$\begin{aligned} a^{**} &= a, \\ (ab)^* &= b^*a^*, \\ (\lambda a)^* &= \bar{\lambda}a^*. \end{aligned}$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.
A ***-algebra** or **involutive algebra** is an algebra with an involution.

Example 2.6 *An example of a *-algebra is $M_n(\mathbb{C})$, the algebra of $n \times n$-matrices with complex entries. The involution is given by hermitian conjugation, i.e., $a^* = (a^*_{ij}) := (\overline{a_{ji}})$ for $a \in \mathcal{A}$. It is also a normed algebra (see below), with the norm $\| \ \|$ defined as*

$$\|a\| = \sup_{v \in \mathbb{C}^n} \{\langle av, av\rangle : \langle v, v\rangle = 1\}$$

where sup *is the supremum, i.e., least upper bound, and $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{C}^n$.*

As we saw in the previous example, the norm on a vector space can be extended to a norm on an algebra. Only one additional condition is needed.

Definition 2.7 A **normed algebra** $\mathcal{A}$ is an algebra which is a normed vector space with norm $\| \ \|$ which satisfies in addition

$$\|ab\| \leq \|a\|\|b\|.$$

A **Banach algebra** is a normed algebra which is complete.

We see that $M_n(\mathbb{C})$ is indeed a normed algebra. We are now finally ready to give the definition of a C^*-algebra and a Hilbert space.

Definition 2.8 A **C^*-algebra** is a Banach algebra $\mathcal{A}$, which is also a *-algebra such that for all $a \in \mathcal{A}$ one has

$$\|a^*a\| = \|a\|^2.$$

When we combine this with definition 2.7, we derive $\|a\| \leq \|a^*\|$. Replacing a by a^*, we infer for all elements in a C^*-algebra $\mathcal{A}$,

$$\|a^*\| = \|a\|. \tag{2.3}$$

Example 2.9 *The algebra $C(M)$ of continuous functions on a compact Hausdorff topological space M is a C^*-algebra. The involution * is given by complex conjugation and the norm is the supremum norm $\| \ \|_\infty$, i.e.,*

$$f^*(x) = \overline{f(x)}, \quad \|f\|_\infty := \sup_{x \in M} |f(x)|, \quad (f \in C(M), x \in M)$$

One easily proves that $C(M)$ is a *-algebra and a Banach algebra.

Definition 2.10 A **Hilbert space** $\mathcal{H}$ is a vector space with a hermitian inner product, which is also a Banach space in the associated norm. If $\langle \ , \ \rangle$ denotes the inner product on $\mathcal{H}$, the associated norm is given by $\|\chi\| := \langle \chi, \chi \rangle^{1/2}$ where $\chi \in \mathcal{H}$.

Before we go into representations of C^*-algebras, we first state some standard definitions about the resolvent and the spectrum of an element in an algebra.

Definition 2.11 Let $\mathcal{A}$ be a unital Banach algebra. The **resolvent set** $\rho(a)$ of $a \in \mathcal{A}$ is the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda\mathbb{I}$ is invertible. If $\lambda \in \rho(a)$, then $(a - \lambda\mathbb{I})^{-1}$ is called the **resolvent** of a at λ.
The **spectrum** $\sigma(a)$ of $a \in \mathcal{A}$ is the complement of $\rho(a)$ in $\mathbb{C}$, in other words, $\sigma(a)$ is the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda\mathbb{I}$ is not invertible.

When $\mathcal{A}$ is the algebra $M_n(\mathbb{C})$, the spectrum of $a \in \mathcal{A}$ is just the set of eigenvalues of a.

2.2 Representations of C^*-algebras

As is usual in representation theory of algebras, one considers maps π from the algebra to the space of linear operators acting on some vector space. In analogy with this, we represent C^*-algebras in the algebra of bounded operators on a Hilbert space. Before we can do that, we need to define these.

Definition 2.12 A **bounded operator** on a Hilbert space $\mathcal{H}$ is a linear map $a\colon \mathcal{H} \to \mathcal{H}$ for which there is a constant $C > 0$ such that $\|a\chi\| \leq C\|\chi\|$. The operator norm of a bounded operator is defined by

$$\|a\| := \sup_{\chi \in \mathcal{H}} \{\|a\chi\|_{\mathcal{H}} : \|\chi\| = 1\} = \sup_{\chi \in \mathcal{H}} \left\{ \frac{\|a\chi\|}{\|\chi\|} : \chi \neq 0 \right\}.$$

The set of all bounded operators is denoted by $\mathcal{B}(\mathcal{H})$. The involution on $\mathcal{B}(\mathcal{H})$ is given by the **adjoint**. Recall that the adjoint of an operator T on $\mathcal{H}$ is the unique operator T^* on $\mathcal{H}$ which satisfies

$$\langle T\chi, \psi \rangle = \langle \chi, T^*\psi \rangle, \quad (\chi, \psi \in \mathcal{H}).$$

An operator T on $\mathcal{H}$ which satisfies $T^* = T$ is called **self-adjoint**.

From the definition of the operator norm, we have

$$\|a\chi\| \leq \|a\|\|\chi\| \tag{2.4}$$

for all $\chi \in \mathcal{H}$. To show that $\mathcal{B}(\mathcal{H})$ is a Banach algebra, we use equation (2.4) twice for $a, b \in \mathcal{A}$ yielding $\|ab\chi\| \leq \|a\|\|b\chi\| \leq \|a\|\|b\|\|\chi\|$. Hence

$$\|ab\| \leq \|a\|\|b\|. \tag{2.5}$$

This shows that $\mathcal{B}(\mathcal{H})$ is a Banach algebra.

To prove that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, we use the Cauchy-Schwarz inequality on inner product spaces $(V, \langle\ ,\ \rangle)$,

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle \tag{2.6}$$

for all $v, w \in V$. For the Hilbert space $\mathcal{H}$, we use this to estimate for $a \in \mathcal{A}, \chi \in \mathcal{H}$,

$$\|a\chi\|^2 = \langle a\chi, a\chi\rangle = \langle\chi, a^*a\chi\rangle \leq \|\chi\|\|a^*a\chi\| \leq \|a^*a\|\|\chi\|^2 \tag{2.7}$$

where we used formula (2.4). Hence $\|a\|^2 \leq \|a^*a\|$ and with equation (2.5), we infer

$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\|. \tag{2.8}$$

This leads to $\|a\| \leq \|a^*\|$. Replacing a by a^*, and using $a^{**} = a$, yields $\|a^*\| \leq \|a\|$, so that $\|a^*\| = \|a\|$. When we substitute this in the right hand side of equation (2.8), we see that $\|a^*a\| = \|a\|^2$. This proves that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Definition 2.13 Let $\mathcal{A}$ be a C^*-algebra. A **representation** of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a pair $(\mathcal{H}, \pi)$ where π is a linear map $\pi\colon\ \mathcal{A} \to \mathcal{B}(\mathcal{H})$ which satisfies

$$\begin{aligned} \pi(ab) &= \pi(a)\pi(b), \\ \pi(a^*) &= \pi(a)^* \end{aligned}$$

for all $a, b \in \mathcal{A}$.

The representation $(\mathcal{H}, \pi)$ is called **irreducible** if the only closed subspaces of $\mathcal{H}$ which are invariant under the action of $\pi(\mathcal{A})$ are the trivial subspaces $\{0\}$ and $\mathcal{H}$. There is an analogue of Schur's lemma, which states that a representation is irreducible if and only if the elements in $\pi(\mathcal{A})$ which commute with all other elements in $\mathcal{B}(\mathcal{H})$ are multiples of the unit element $\mathbb{I}$.

Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are called **equivalent** if there exists a unitary operator [1] $U\colon\ \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_1(a) = U^*\pi_2(a)U \tag{2.9}$$

for all $a \in \mathcal{A}$.

[1] Recall that an operator $U\colon\ \mathcal{H}_1 \to \mathcal{H}_2$ is called unitary if $U^*U = \mathbb{I}_{\mathcal{H}_1}$ and $UU^* = \mathbb{I}_{\mathcal{H}_2}$.

Chapter 3

Spectral triples and Fredholm modules

In noncommutative geometry, one seeks for a generalization of the usual calculus on a manifold. As we shall see, the spectral triple introduced by Connes in [6], is a powerful tool in describing noncommutative geometry.

3.1 Spectral triple

Before we give the definition of a spectral triple, we consider the canonical triple. It gives a description of a Riemannian manifold. In this section, we will look back several times to this special case of the spectral triple.

3.1.1 Canonical triple

In quantum field theory, one considers bosonic and fermionic fields in spacetime. It provides a description of particles moving in spacetime. Let us define spacetime M as a four-dimensional Riemannian spin manifold. The spin structure is required in order to describe fermionic fields and we will define it in a moment. As M is a Riemannian manifold, it carries a (positive-definite) metric g, which can be written in local coordinates (x^μ) as

$$g := g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{3.1}$$

For now, we choose M to be a flat Euclidean spacetime such that $g_{\mu\nu} = \delta_{\mu\nu}$.

The algebra of smooth functions on M plays an important role in our considerations. An important result here is the Gel'fand-Naimark theorem. It states that the topology of a manifold M can be constructed from the algebra of smooth functions on M, denoted by $C^\infty(M)$. At the intuitive level, this can be seen from the fact that the coordinate functions (x^μ) are contained in the algebra of smooth functions on M. The first building block for the canonical triple is this **algebra** $C^\infty(M)$.

Furthermore, to describe fermionic fields, we need a spin structure. A manifold M is said to have a spin structure, if it admits a spin bundle $S \to M$. Recall that in our case, a spin bundle is a vector bundle which has as a structure group the universal covering group of $SO(4)$, i.e. $SU(2) \times SU(2)$. The typical fibre of the spin bundle is spinor space $\mathbb{C}^4$. Furthermore, the set of all square integrable sections of this spin bundle form a

Hilbert space $L^2(M,S)$, consisting of spinor fields. The scalar product in $L^2(M,S)$ is given by

$$(\psi,\phi) = \int \mathrm{d}^4x\overline{\psi(x)}\phi(x), \qquad \psi,\phi \in L^2(M,S), \tag{3.2}$$

with the bar indicating complex conjugation. In equation (3.2), there is a summation implied over internal indices in spinor space $\mathbb{C}^4$. The **Hilbert space** $L^2(M,S)$ is the second building block of the canonical triple. We define an action of $C^\infty(M)$ on $\mathcal{H}$ by

$$(f\psi)(x) := f(x)\psi(x), \qquad (f \in C^\infty(M), \psi \in L^2(M,S)). \tag{3.3}$$

To construct spinor representations, we introduce Dirac matrices. These are four 4×4 matrices γ^μ which satisfy the anticommutation relations

$$\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}. \tag{3.4}$$

With this, we can construct the well known Dirac operator, written in local coordinates,

$$\not{\partial} = \gamma^\mu\partial_\mu. \tag{3.5}$$

The **Dirac operator** $\not{\partial}$ is the last ingredient for our canonical triple.

Definition 3.1 The canonical triple is a triple $(\mathcal{A},\mathcal{H},D)$, where

- $\mathcal{A}$ is the associative algebra $C^\infty(M)$
- $\mathcal{H}$ is the Hilbert space $L^2(M,S)$
- D is the Dirac operator $\not{\partial}$

Furthermore, there is the following grading on $\mathcal{H}$. First we define

$$\gamma^5 := -\gamma^1\gamma^2\gamma^3\gamma^4. \tag{3.6}$$

Since $(\gamma^5)^2 = \mathbb{I}$, the eigenvalues of γ^5, called the **chirality**, must be ± 1. Thus, the Hilbert space is decomposed into two eigenspaces (with respective eigenvalues $+1$ and -1), namely

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-. \tag{3.7}$$

In this setting γ^5 is called a grading operator, and it anticommutes with the Dirac operator $\not{\partial}$:

$$\gamma^5\not{\partial} + \not{\partial}\gamma^5 = 0 \tag{3.8}$$

In the next section, we give the generalization of the canonical triple: the spectral triple.

3.1.2 Spectral triple

Before we can give the definition of a spectral triple, we need to define compact operators. Recall the definition of the operator norm from the previous chapter.

Definition 3.2 An operator T on $\mathcal{H}$ is said to be **compact** if for every $\epsilon > 0$, there exists a finite dimensional subspace $E \subset \mathcal{H}$ such that $\|T|_{E^\perp}\| < \epsilon$.

The set $\mathcal{K}(\mathcal{H})$ will denote the set of all compact operators on the Hilbert space $\mathcal{H}$. One can easily verify that it is an ideal in $\mathcal{B}(\mathcal{H})$.

Definition 3.3 A **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, together with a self-adjoint (unbounded) operator $D = D^*$ on $\mathcal{H}$ with the following properties,

1. the resolvent $(D - \lambda)^{-1}$ is a compact operator on $\mathcal{H}$ for all $\lambda \notin \mathbb{R}$,

2. for any $a \in \mathcal{A}$, the commutator $[D, a] := Da - aD \in \mathcal{B}(\mathcal{H})$.

The triple is said to be **even** if there is a $\mathbb{Z}_2$ grading of $\mathcal{H}$, namely an operator Γ on $\mathcal{H}$ with $\Gamma = \Gamma^*$ and $\Gamma^2 = 1$, such that

$$\begin{aligned} \Gamma D + D\Gamma &= 0, \\ \Gamma a - a\Gamma &= 0, \qquad \text{for all } a \in \mathcal{A}. \end{aligned} \tag{3.9}$$

If such a grading does not exist, the triple is said to be **odd**.

Indeed, these definitions make the canonical triple from the previous section an even spectral triple.

3.2 Compact operators as infinitesimals

We would like to extend our calculus with some sort of 'integral'. In the usual (commutative) calculus, an important role is played by infinitesimals. It will turn out that compact operators on $\mathcal{H}$ will play the role of such infinitesimals. Therefore, we need to develop a trace, which replaces the usual integral on a manifold.

We note that every compact operator T can be decomposed in a partial isometry U and a hermitian operator $|T|$, namely

$$T = U|T| \quad \text{with } |T| := \sqrt{T^*T}. \tag{3.10}$$

For more details on the theory of compact operators, we refer to Reed and Simon [18].

If $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal set in $\mathcal{H}$ (not necessarily complete), then $\{\psi_n\}_{n\in\mathbb{N}}$ with

$$\psi_n := U\phi_n, \tag{3.11}$$

is also an orthonormal set in $\mathcal{H}$. This leads to the following proposition.

Proposition 3.4 *Let $T \in \mathcal{K}(\mathcal{H})$. Then, it has a uniformly convergent expansion*

$$T = \sum_{n \geq 0} \mu_n(T) |\psi_n\rangle\langle\phi_n|,$$

where $\mu_n(T)$ are the eigenvalues of $|T|$. They are ordered as $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_n \geq \mu_{n+1} \geq \cdots \geq 0$.

Because the sequence $\mu_n(T) \to 0$ as $n \to \infty$, compact operators are in a sense 'small'. It is easily verified that the norm of T is given by $\|T\| = \mu_0(T)$.

One way to examine the structure of the set $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$, is by looking at its ideals.

Definition 3.5 The **Schatten-von Neumann ideals** of $\mathcal{K}(\mathcal{H})$ are for $p \in [1, \infty)$

$$\mathcal{L}^p(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{\infty} \mu_n(T)^p < \infty \right\}$$

where $\mu_n(T)$ are the eigenvalues of $|T|$.

The operators in the ideal $\mathcal{L}^1$ are called trace-class, and the operators in the ideal $\mathcal{L}^2$ are called Hilbert-Schmidt.[1]

The Schatten-von Neumann ideals are (two-sided) ideals in $\mathcal{B}(\mathcal{H})$, as for $T \in \mathcal{K}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$, one has

$$\begin{aligned} \mu_n(TB) &\leq \|B\| \mu_n(T), \\ \mu_n(BT) &\leq \|B\| \mu_n(T). \end{aligned} \tag{3.12}$$

We have the following chain of ideals in $\mathcal{K}(\mathcal{H})$

$$\mathcal{L}^1 \subset \mathcal{L}^2 \subset \cdots \subset \mathcal{K}(\mathcal{H}) \tag{3.13}$$

To obtain the trace class operators $\mathcal{L}^1$ the following inequality is very useful.

Proposition 3.6 (Hölder inequality) *For $p, q \in [1, \infty)$ we have the following relation*

$$\mathcal{L}^p \mathcal{L}^q \subset \mathcal{L}^r \qquad \text{where } \frac{1}{r} := \frac{1}{p} + \frac{1}{q}.$$

We return to the canonical spectral triple to obtain the correct definition of a trace of operators, which generalizes the usual integration on a Riemannian manifold.

Let $\nabla\!\!\!/ := \gamma^\mu \nabla_\mu$ be the Dirac operator on a spin bundle $S \to M$ over a 4-dimensional Riemannian manifold (M, g). M is not necessarily flat and ∇ is the covariant derivative. In local coordinates (x^μ), the tangent bundle T_pM at $p \in M$, is spanned by $(\partial/\partial x^\mu)$.

[1] When there is no reason for confusement, we abbreviate $\mathcal{L}^p := \mathcal{L}^p(\mathcal{H})$.

However, there may be an alternative choice. If we define **vierbeins** $e_a^\mu \in GL(4,\mathbb{R})$, we can define vectors e_a as

$$e_a := e_a^\mu \frac{\partial}{\partial x^\mu}, \quad \det e_a^\mu > 0 \tag{3.14}$$

which preserves the orientation. We require that (e_a) be orthonormal,

$$g(e_a, e_b) = e_a^\mu e_b^\nu g_{\mu\nu} = \delta_{ab}. \tag{3.15}$$

We can also introduce the dual basis (θ^a) of T_p^*M,

$$\theta^a := e_\mu^a \mathrm{d}x^\mu. \tag{3.16}$$

where $(\mathrm{d}x^\mu)$ is of course the dual basis of $(\partial/\partial x^\mu)$. The bases (e_a) and (θ^a) are called the **non-coordinate bases**. More details on this can be found in Nakahara [15].

We introduce Dirac gamma matrices γ_a which satisfy

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}. \tag{3.17}$$

The curved spacetime counterparts of the Dirac matrices are defined as $\gamma^\mu := e_a^\mu \gamma^a$. These curved gamma matrices satisfy,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

as can be easily verified.

One associates a certain Fredholm[2] operator F with the Dirac operator as $F = \Delta^{-1/2}\nabla\!\!\!/$. Here we introduced the Laplacian $\Delta := (\mathrm{i}\nabla\!\!\!/)^2$.

As we want to generalize integration to an algebraic setting, we have to generalize 4-forms on M to algebraic expressions. Then, we can replace the integral of 4-forms by a trace of such generalized objects. If we define a differential as $\mathrm{d}x^\mu := [F, x^\mu]$ we write for the generalized 4-form α,

$$\alpha = f\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}x^4. \tag{3.18}$$

Note that this is an algebraic expression.

We want α to be trace class. If we consider the eigenvalues of Δ,

$$\mu_n(\Delta) \sim \sqrt{n}, \qquad \text{for } n \to \infty, \tag{3.19}$$

we find that $\mathrm{d}x^\mu = [F, x^\mu] \in \mathcal{L}^p$ for $p > 4$. By Hölder inequality, we infer that $\alpha \notin \mathcal{L}^1$, i.e., that α is not trace class. We conclude that if we want to generalize integration of 4-forms on a Riemannian spin bundle, we need a better trace. It is given in the next section.

[2] See section 3.4.

3.3 Dixmier trace

As we mentioned in the previous section, we would like to generalize the integration of 4-forms on a 4-dimensional manifold. Therefore, we need to define a trace which has a more extended domain than $\mathcal{L}^1$. Let us first define the notion of order of a compact operator.

Definition 3.7 For any $\alpha \in \mathbb{R}^+$, a compact operator $T \in \mathcal{K}(\mathcal{H})$ is said to be of **order** α if its eigenvalues $\mu_n(T)$ obey

$$\mu_n(T) = \mathcal{O}(n^{-\alpha}), \qquad \text{as } n \to \infty, \tag{3.20}$$

i.e., there exists a $C > 0$ such that $\mu_n(T) \leq Cn^{-\alpha}$ for all $n \geq 1$. The ideal of compact operators of order $1/p$ is denoted by $\mathcal{L}^{(p,\infty)}$.

Note the difference with the Schatten-von Neumann ideals of the previous section. We have the following inclusion

$$\mathcal{L}^p \subset \mathcal{L}^{(p,\infty)}. \tag{3.21}$$

Compared with equation (3.13) there is a similar chain of ideals $\mathcal{L}^{(p,\infty)}$ in $\mathcal{K}(\mathcal{H})$

$$\mathcal{L}^{(1,\infty)} \subset \mathcal{L}^{(2,\infty)} \subset \cdots \subset \mathcal{K}(\mathcal{H}), \tag{3.22}$$

and also, an analogue of the Hölder inequality exists

$$\mathcal{L}^{(p,\infty)}\mathcal{L}^{(q,\infty)} \subset \mathcal{L}^{(r,\infty)}, \qquad \text{where } \frac{1}{r} := \frac{1}{p} + \frac{1}{q}.$$

The following property

$$\mu_{n+m}(T_1T_2) \leq \mu_n(T_1)\mu_m(T_2)$$

of two operators T_1 and T_2 of order α_1 and α_2, respectively, shows that we have indeed that T_1T_2 is of order $\alpha_1 + \alpha_2$

The **Dixmier trace** $\mathrm{tr}_\omega\colon\ \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ is constructed in such a way that

1. Compact operators of order 1 are in the domain of the Dixmier trace.
2. Compact operators of order > 1 have vanishing Dixmier trace.

It is defined as

$$\mathrm{tr}_\omega(T) = \mathrm{Lim}_\omega \frac{1}{\ln N} \sum_0^{N-1} \mu_n(T) \tag{3.23}$$

where T is a compact operator and Lim_ω a generalization of the usual limit procedure, in order to obtain finite results, even for divergent (but bounded) series $\sum \mu_n(T)$. Compact operators in $\mathcal{L}^{(1,\infty)}$ are said to be Dixmier trace class.

The generalized 4-form α on the Riemannian manifold M defined in the previous section, is Dixmier trace class. We define the **integral** $⨍$ of the related function f as

$$⨍ f := \mathrm{tr}_\omega(f\Delta^{-2}) \tag{3.24}$$

The eigenvalues of Δ^{-2} in $\mathcal{H}$ are (see [14])

$$\mu_n(\Delta^{-2}) \sim \frac{1}{n}, \qquad \text{for } n \to \infty, \tag{3.25}$$

which makes $f\Delta^{-2}$ a compact operator of order 1, and thus Dixmier trace class $\mathcal{L}^{(1,\infty)}$. One can ask if this integral coincides with the Riemannian integral on M. The following proposition shows their relation for a compact Riemannian manifold of dimension 4 (the general case can be found in Landi [11]).

Proposition 3.8 *Let $S \to M$ be a spin bundle over a Riemannian manifold M and let α be the generalized 4-form from equation (3.18). The integral (3.24) is proportional to the Riemannian integral of $f \in C^\infty(M)$ on M,*

$$⨍ f \sim \int_M \mathrm{d}^4x f(x) \tag{3.26}$$

in local coordinates (x^μ).

Proof. As can be found in Landi [11], we have for a pseudodifferential operator[3] T of rank $-n$ acting on sections of a vector bundle $E \to M$, over a general Riemannian manifold M,

$$\mathrm{tr}_\omega T = \frac{1}{n(2\pi)^n} \int_{S^*M} \mathrm{tr}_E \sigma_{-n}(T) \mathrm{d}\mu. \tag{3.27}$$

Here $\sigma_{-n}(T)$ is the principal symbol of T and S^*M the co-sphere $S^*M := \{(x,\xi) \in T^*M : \|\xi\| = 1\} \subset T^*M$ with measure $\mathrm{d}\mu = \mathrm{d}x\mathrm{d}\xi$. The trace tr_E is a matrix trace over 'internal indices'.

Consider this formula for $T = f\Delta^{-2}$ acting on sections of the spin bundle $S \to M$, over a 4-dimensional manifold M. The principal symbol is $f(x)\|\xi\|^{-4}$ which reduces to the matrix $f(x)\mathbb{I}_4$ where 4 is the dimension of the fibre of the spin bundle $S \to M$. We get

$$\mathrm{tr}_\omega(f\Delta^{-2}) = \frac{1}{4(2\pi)^4} \int_{S^*M} \mathrm{tr}_S(f(x)\mathbb{I}_4)\mathrm{d}x\mathrm{d}\xi \tag{3.28}$$

$$= \frac{1}{(2\pi)^4} \left(\int_{S^3} \mathrm{d}\xi \right) \int_M f(x)\mathrm{d}x \sim \int_M f(x)\mathrm{d}x \tag{3.29}$$

□

Now that we have generalized integration on a Riemannian manifold to the setting of the canonical spectral triple, we can define integration for a general spectral triple. Before we can do that, we need to define the dimension of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Definition 3.9 A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be of **dimension** n (or n-**summable**), if $|D|^{-n}$ is an infinitesimal of order 1.

[3] More details on the theory of pseudodifferential operators can be found in section 4.1.

Finally, we can state the generalization of the integral.

Definition 3.10 Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple of dimension n. The **integral** $⨍$ of any $a \in \mathcal{A}$ is defined by

$$⨍ a := \frac{1}{V} tr_\omega a|D|^{-n},$$

Here V is a normalization constant determined by

$$\frac{1}{V}\mathrm{tr}_\omega |D|^{-n} = ⨍ \mathbb{I} \equiv 1.$$

We see that the role of the operator $|D|^{-n}$ is just to bring the bounded operator a to an infinitesimal of order 1, such that the Dixmier trace makes sense.

3.4 Fredholm modules

In section 3.2, we defined a Fredholm operator F which made it possible to replace the differential $\mathrm{d}x^\mu$ by a commutator $[F, x^\mu]$, i.e., an algebraic expression. We will now go into more detail on this. We start with a definition similar to that of a spectral triple.

Definition 3.11 Let $\mathcal{A}$ be an algebra. A **Fredholm module** $(\mathcal{H}, F)$ over $\mathcal{A}$ is given by:

1. a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$,
2. an operator $F = F^*$, $F^2 = 1$, on $\mathcal{H}$ such that $[F, a]$ is a compact operator for any $a \in \mathcal{A}$.

A Fredholm module is said to be **even** if there is a $\mathbb{Z}_2$ grading Γ on $\mathcal{H}$ with $\Gamma = \Gamma^*$ and $\Gamma^2 = 1$, such that

1. $\Gamma F + F\Gamma = 0$,
2. $\Gamma a - a\Gamma = 0, \quad$ for all $a \in \mathcal{A}$.

If such a grading does not exist, the Fredholm module is said to be **odd**.

We introduce an anologue of dimension of a spectral triple,

Definition 3.12 A Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$ is called p**-summable** if all the operators $\mathrm{d}a = [F, a]$ belong to the Schatten-von Neumann ideal $\mathcal{L}^p(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$.

When we start with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, one can canonically associate a Fredholm module with it. We define

- $\mathcal{H}' = \mathcal{H} \oplus \ker D$,
- $a(\xi, \eta) = (a\xi, 0), \qquad$ for all $a \in \mathcal{A}, \xi \in \mathcal{H}, \eta \in \ker D$,
- $F = (\mathrm{Sign} D) \oplus F_1$,

where F_1 exchanges the two copies of $\ker D$. The pair $(\mathcal{H}, F)$ is then a Fredholm module over $\mathcal{A}$.

Chapter 4

Polyakov action and Fredholm modules

The Polyakov action in string theory describes bosons on the worldsheet Σ. It is given by[1]

$$S_p = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\ g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu$$

where X is a differentiable map from Σ to M. The (σ^α) are local coordinates on Σ and the (X^μ) are local coordinates on M. Therefore, the map X may be described by functions $X^\mu(\sigma^\alpha)$ so that the Polyakov action makes sense.

The bosonic string has, besides reparametrization invariance, an invariance under rescaling of the metric. This invariance is called the conformal symmetry. It means that the string can be described by a class of metrics, and not by one special metric. As we saw that a spectral triple generalizes a Riemannian manifold, it will turn that a Fredholm module generalizes a conformal manifold. In the first section we will consider the algebraic formulation of the Polyakov action, in the second section we study the 'conformal nature' of Fredholm modules.

4.1 Polyakov action

Let Σ be a 2-dimensional compact Riemannian spin manifold, i.e. the string worldsheet. The spin bundle is denoted by $S \to \Sigma$. The worldsheet is embedded in a D-dimensional Riemannian manifold (M, g) by a map $X : \Sigma \to M$. We have in local coordinates on M, $X^\mu : \Sigma \to M$ $(\mu = 1, \ldots, D)$. Let (σ^1, σ^2) be the coordinates on Σ and set $\partial_\alpha = \frac{\partial}{\partial\sigma^\alpha}$ for $\alpha = 1, 2$. From Connes [6] we take the following theorem.

Theorem 4.1 *Let $F := \Delta^{-1/2}\not\nabla$ with $\Delta := (\mathrm{i}\not\nabla)^2$ the Laplacian and $\not\nabla = \gamma^\alpha\partial_\alpha$ the Dirac operator on Σ. Then one has for the Dixmier trace* tr_ω

$$\mathrm{tr}_\omega\big(g_{\mu\nu}[F, X^\mu][F, X^\nu]\big) = -\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\ g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu.$$

[1] In general, there is a factor $\sqrt{\det\eta}$ in the Polyakov action, where η is the metric on the worldsheet. We used conformal invariance of this action to rescale the metric into Euclidean form, for which $\det\eta = 1$.

The X^μ are in the algebra $C^\infty(\Sigma)$ of smooth functions on Σ. This algebra will be considered as an operator algebra acting on a Hilbert space in a way similar to the canonical triple, see equation (3.3). The expression $g_{\mu\nu}[F, X^\mu][F, X^\nu]$ is then an operator acting on the Hilbert space $L^2(\Sigma, S)$ of smooth sections. The Dixmier trace tr_ω is defined in the previous chapter. As the eigenvalues of the Fredholm operator F are proportional to $\frac{1}{\sqrt{j}}$ for $j \to \infty$ in two dimensions, $[F, X^\mu]$ is in the ideal $\mathcal{L}^{(2,\infty)}$. Therefore, by Hölder inequality, $g_{\mu\nu}[F, X^\mu][F, X^\nu]$ is Dixmier trace class.

To prove the theorem, we need some theory of pseudodifferential operators, which we take from Treves [22]. This theory is inspired by that of differential operators. There it is possible to consider differential operators on a manifold M as symbols, i.e. polynomial functions in coordinates (ξ_j) of covectors in the cotangent bundle T_x^*M. Here $j = 1, \ldots, \dim M$ and $x \in M$. A **differential operator of rank** m P on an n-dimensional manifold M is defined in local coordinates as

$$P = \sum_{|\alpha| \leq m} A_\alpha(x)(-\mathrm{i})^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{4.1}$$

Here α is a multi-index (see Appendix A) and $|\alpha| = \sum_{j=1}^n \alpha_j$. We assume there is a coefficient $A_\alpha \neq 0$ for some α with $|\alpha| = m$.

Consider a covector $\xi = \sum_j \xi_j \mathrm{d}x_j$ in the cotangent space T_x^*M. The **complete symbol** p^P of P is defined as

$$p^P(x, \xi) = \sum_{|\alpha| \leq m} A_\alpha(x) \xi^\alpha, \tag{4.2}$$

and its leading term is called the **principal symbol**

$$\sigma^P(x, \xi) = \sum_{|\alpha| = m} A_\alpha(x) \xi^\alpha. \tag{4.3}$$

Example 4.2 *The Dirac operator* $\nabla\!\!\!/$ *on a spin manifold* M *is locally written as*

$$\nabla\!\!\!/ = \gamma^\alpha \partial_\alpha. \tag{4.4}$$

Its principal symbol is

$$\sigma^{\nabla\!\!\!/} = \mathrm{i}\gamma^\alpha \xi_\alpha. \tag{4.5}$$

To each differential operator, there corresponds a unique symbol as we just substitute ξ_j for each partial derivative times $-\mathrm{i}$. The description of differential operators by symbols, makes the following generalization possible. We define so-called pseudodifferential operators by symbols which belong to a more general class. For example, we could consider the inverse of the Dirac operator, via the well-defined symbol $(\mathrm{i}\gamma^\alpha \xi_\alpha)^{-1}$. With this definition, $\nabla\!\!\!/^{-1}$ is a pseudodifferential operator. For more details on the theory, we refer to [22].

As we saw above, symbols make it possible to define pseudodifferential operators. In the Fredholm operator $F := \Delta^{-1/2}\not\nabla$, the factor $\Delta^{-1/2}$ is defined exactly in this way. It is the pseudodifferential operator on Σ corresponding to the symbol

$$p^{\Delta^{-1/2}}(\sigma,\xi) = \left(\eta^{\alpha\beta}\xi_\alpha\xi_\beta\right)^{-1/2} =: \frac{1}{\|\xi\|}, \tag{4.6}$$

where $\eta^{\alpha\beta}$ is the metric on the worldsheet, which we will take to be euclidean: $\eta^{\alpha\beta} = \delta^{\alpha\beta}$. Here $\xi \in T^*_\sigma\Sigma$ for $\sigma \in \Sigma$, and $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$.[2]

As is easy to see with differential operators, multiplication of pseudodifferential operators also leads to multiplication of the corresponding principal symbols (but not of the complete symbols),

$$\sigma^{P\circ Q}(x,\xi) = \sigma^P(x,\xi)\sigma^Q(x,\xi). \tag{4.7}$$

Using this, we see that the principal symbol of our Fredholm operator F equals

$$\sigma^F(\sigma,\xi) = \mathrm{i}\frac{\gamma^\alpha\xi_\alpha}{\|\xi\|}. \tag{4.8}$$

From this we see that F is a pseudodifferential operator of rank 0.

We state two useful theorems, as taken from Treves [22] and Connes [6].

Theorem 4.3 *Let P and Q be two pseudodifferential operators of rank m and m' respectively. Then the commutator $[P,Q]$ is a pseudodifferential operator of rank $m+m'-1$ with principal symbol*

$$p^{[P,Q]} = \{p^P, p^Q\} := \sum_{\alpha=1,2}\left(\frac{\partial p^P}{\partial\xi_\alpha}\frac{\partial p^Q}{\partial\sigma^\alpha} - \frac{\partial p^P}{\partial\sigma^\alpha}\frac{\partial p^Q}{\partial\xi_\alpha}\right),$$

Notice that $\{\ ,\ \}$ denotes the Poisson bracket.

Theorem 4.4 *Let M be an n-dimensional Riemannian manifold and let T be a pseudodifferential operator of rank $-n$ acting on sections of a vector bundle $E \to M$. Then we have for the Dixmier trace tr_ω of T,*

$$\mathrm{tr}_\omega(T) = \frac{1}{n(2\pi)^n}\int_{S^*M} \mathrm{tr}_E\sigma^T \mathrm{d}^n x \mathrm{d}^n\xi.$$

where tr_E is the trace over internal indices and S^*M the unit co-sphere[3]*, defined as*

$$S^*M := \{(x,\xi) \in T^*M : \|\xi\| = 1\} \subset T^*M.$$

[2] The γ^α are usually represented in 2 dimensions as $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$. See [26] or [27] for more general considerations.

[3] A Minkowski metric would not work in these considerations. In two dimensions $S^*\Sigma$ is with Minkowskian signature a hyperboloide and thus not a compact region. To avoid this, we take the Euclidean signature and integrate over the 1-sphere S^1. Via Wick-rotation we can get from the Euclidean to the Minkowski metric and vice versa.

In order to prove theorem 4.1, we use this last theorem for the spin bundle $S \to \Sigma$ with $T = G$ defined by

$$G := g_{\mu\nu}[F, X^\mu][F, X^\nu],$$

which is indeed a pseudodifferential operator of rank -2. To obtain its principal symbol, we first note that for functions X^μ we have $\sigma^{X^\mu} = X^\mu$. Using theorem 4.3 yields

$$\sigma^{[F,X^\mu]} = \mathrm{i}\frac{\gamma^\alpha \partial_\alpha X^\mu}{\|\xi\|} - \mathrm{i}\frac{\gamma^\beta \xi_\beta}{\|\xi\|^3}\xi^\alpha \partial_\alpha X^\mu. \tag{4.9}$$

As noted in formula (4.7), we can just multiply principal symbols in a product of operators. This leads to the following principal symbol

$$\sigma^G = -g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu \frac{1}{\|\xi\|^2} + g_{\mu\nu}\xi^\alpha \partial_\alpha X^\mu \xi^\beta \partial_\beta X^\nu \frac{1}{\|\xi\|^4}. \tag{4.10}$$

We are now ready to use theorem 4.4 in the case of the 2-dimensional worldsheet and the Fredholm operator $F = \Delta^{-1/2}\nabla\!\!\!/$. This theorem yields

$$\mathrm{tr}_\omega G = \frac{1}{2(2\pi)^2}\int_{S^1} \mathrm{d}^2\xi \int_\Sigma \mathrm{d}^2\sigma \ \mathrm{tr}_S \sigma^G. \tag{4.11}$$

Note the difference between the σ's in the measure $\mathrm{d}^2\sigma$ and in the principal symbol σ^G. Substituting equation (4.10) in (4.11) yields

$$\begin{aligned} \mathrm{tr}_\omega G &= -\frac{1}{(2\pi)^2}\int_{S^1} \mathrm{d}^2\xi \int_\Sigma \mathrm{d}^2\sigma \ g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu \qquad (4.12)\\ &\quad + \frac{1}{(2\pi)^2}\int_{S^1} \mathrm{d}^2\xi (\xi^1)^2 \int_\Sigma \mathrm{d}^2\sigma \ g_{\mu\nu}\partial_1 X^\mu \partial^1 X^\nu \\ &\quad + \frac{1}{(2\pi)^2}\int_{S^1} \mathrm{d}^2\xi (\xi^2)^2 \int_\Sigma \mathrm{d}^2\sigma \ g_{\mu\nu}\partial_2 X^\mu \partial^2 X^\nu \\ &= -\frac{\pi}{(2\pi)^2}\int_\Sigma \mathrm{d}^2\sigma \ g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu \end{aligned}$$

where the internal trace tr_S lead to a factor 2 as the gamma-matrices are 2-dimensional. The integral $\int \mathrm{d}^2\xi \ \xi^1\xi^2$ vanishes on S^1 by symmetry considerations. Finally, this leads to the Polyakov action,

$$\mathrm{tr}_\omega\big(g_{\mu\nu}[F, X^\mu][F, X^\nu]\big) = -\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma \ g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu. \tag{4.13}$$

4.2 Conformal structure and Fredholm modules

It is not so difficult to see that the Fredholm operator $F = \Delta^{-1/2}\nabla\!\!\!/$ is conformally invariant. Let us first review some definitions on conformal structures, which we take from Nakahara [15]. Let M be a Riemannian manifold with metric g.

Definition 4.5 A diffeomorphism $f : M \to M$ is called a **conformal transformation** at $p \in M$ if it preserves the metric up to a scale, i.e., if there is a $\lambda \in C^\infty(M)$ such that

$$f^*g|_{f(p)} = \mathrm{e}^{2\lambda} g|_p$$

where f^*g is the pullback of g by f.

Two metrics g and g' on M are said to be **conformally related** at $p \in M$ if there is a $\lambda \in C^\infty(M)$ such that

$$g'|_p = \mathrm{e}^{2\lambda} g|_p \tag{4.14}$$

Definition 4.6 A **conformal structure** is an equivalence class of metrics, where

$$g_1 \sim g_2 \iff g_1 = \mathrm{e}^{2\lambda} g_2.$$

The transformation $g \mapsto \mathrm{e}^{2\lambda} g$ is called a **Weyl rescaling**.

We write the metric in local coordinates (x^μ) at $p \in M$ as

$$g = g_{\mu\nu}(p) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu.$$

If f is a conformal transformation $f\colon\ x \in M \mapsto \tilde{x} \in M$, we write

$$f^*g = \tilde{g}_{\alpha\beta}(f(p)) \mathrm{d}\tilde{x}^\alpha \otimes \mathrm{d}\tilde{x}^\beta.$$

From definition 4.5 we infer that the compontents $g_{\mu\nu}$ transform as

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(f(p)) = \mathrm{e}^{2\lambda(p)} g_{\mu\nu}(p). \tag{4.15}$$

Let us now go back to our two-dimensional worldsheet Σ. We write the metric η in local coordinates (σ^α) at $\sigma \in \Sigma$ as

$$\eta = \eta_{\alpha\beta}(\sigma) \mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta.$$

If we consider the conformal transformation $\sigma \mapsto \tilde{\sigma}$ this leads to the following transformations

$$\mathrm{d}\tilde{\sigma}^\alpha = \left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\beta}\right) \mathrm{d}\sigma^\beta \tag{4.16}$$

$$\tilde{\partial}_\alpha := \frac{\partial}{\partial \tilde{\sigma}^\alpha} = \left(\frac{\partial \sigma^\beta}{\partial \tilde{\sigma}^\alpha}\right) \frac{\partial}{\partial \sigma^\beta} \tag{4.17}$$

$$\tilde{\eta}^{\alpha\beta}(\tilde{\sigma}) = \left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\delta}\right) \left(\frac{\partial \tilde{\sigma}^\beta}{\partial \sigma^\kappa}\right) \mathrm{e}^{-2\lambda(\sigma)} \eta^{\delta\kappa}(\sigma). \tag{4.18}$$

where we used formula (4.15). Because the gamma-matrices γ^α on Σ satisfy $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$, they transform as

$$\tilde{\gamma}^\alpha(\tilde{\sigma}) = \left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\delta}\right) \mathrm{e}^{-\lambda(\sigma)} \gamma^\delta(\sigma). \tag{4.19}$$

The Fredholm operator $F := \Delta^{-1/2}\nabla\!\!\!/$ reads in local coordinates

$$F = \frac{\gamma^\alpha \partial_\alpha}{\sqrt{\mathrm{i}^2 \eta^{\alpha\beta} \partial_\alpha \partial_\beta}} \tag{4.20}$$

which has meaning as a pseudodifferential operator. We obtain for the conformally transformed $\tilde{F}$,

$$\tilde{F}(\tilde{\sigma}) = \frac{\tilde{\gamma}^\alpha \tilde{\partial}_\alpha}{\sqrt{\mathrm{i}^2 \tilde{\eta}^{\alpha\beta} \tilde{\partial}_\alpha \tilde{\partial}_\beta}} = \frac{\mathrm{e}^{-\lambda} \gamma^\beta \partial_\beta}{\sqrt{\mathrm{i}^2 \mathrm{e}^{-2\lambda} \eta^{\delta\kappa} \partial_\delta \partial_\kappa}} = F(\sigma). \tag{4.21}$$

We can say that F contains the conformal structure of the underlying manifold. A similar thing happens when one considers the canonical triple. Then, the Dirac operator contains the metric structure of the manifold, as we saw in the previous chapter.

Chapter 5

Deformation quantization of the worldsheet

Deformation quantization makes it possible to formulate quantum mechanics in phase-space. The product of classical functions on phase-space Γ is deformed to a so called star product,

$$(f * g)(x,p) := f(x,p) e^{\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x})} g(x,p) \tag{5.1}$$

for functions f and g on phase-space, see Bayen et al. [2]. There exists an isomorphism from the algebra of smooth functions $C^\infty(\Gamma)$ to the algebra of operators on a Hilbert space (Zachos, [29]). We will consider such a deformation of smooth functions on a two-dimensional Riemannian manifold: the worldsheet of a bosonic string.

5.1 Star product

Before we consider the worldsheet as a noncommutative geometry, we take a closer look at the two-dimensional Euclidean plane $\mathbb{R}^2$. We describe the noncommutative plane by deformation of the algebra of smooth functions $C^\infty(\mathbb{R}^2)$. We introduce a star product $*$,

$$(f * g)(\xi^1, \xi^2) := \exp\left(\frac{\mathrm{i}}{2}\theta^{\alpha\beta}\frac{\partial}{\partial\zeta^\alpha}\frac{\partial}{\partial\eta^\beta}\right) f(\zeta^1,\zeta^2) g(\eta^1,\eta^2)\bigg|_{\zeta=\eta=\xi}. \tag{5.2}$$

where $(\xi^\alpha) = (x,y)$ are coordinates on $\mathbb{R}^2$ and $\theta^{\alpha\beta}$ is a constant anti-symmetric 2×2-matrix $(\alpha,\beta = 1,2)$. With $\theta := \theta^{12}$, it can also be written as

$$(f * g)(x,y) = f(x,y) e^{\frac{\mathrm{i}\theta}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x})} g(x,y). \tag{5.3}$$

We define the **Moyal bracket** as

$$[f,g]_*(x,y) := (f * g)(x,y) - (g * f)(x,y). \tag{5.4}$$

For the coordinates functions, this gives

$$[x,y]_* := x * y - y * x = \mathrm{i}\theta. \tag{5.5}$$

Finally we state Baker's formula, which is a useful way to write the star product via its Fourier transform (Baker [1]),

$$(f * g)(x,y) = \frac{1}{\theta^2\pi^2}\int \mathrm{d}x'\mathrm{d}x''\mathrm{d}y'\mathrm{d}y'' f(x',y')g(x'',y'') \\ \times \exp\left(\frac{-2\mathrm{i}}{\theta}\big(y(x'-x'')+y'(x''-x)+y''(x-x')\big)\right). \tag{5.6}$$

5.2 Weyl correspondence

In quantum mechanics we have an operator algebra, acting on a Hilbert space. It turns out that this algebra is in one-to-one correspondence with the deformed algebra of smooth functions on phase-space [29].

Let $f(x,y)$ be a smooth function on $\mathbb{R}^2$. It is mapped to an operator $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ by Weyl correspondence:

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{(2\pi)^2}\int \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}x\mathrm{d}y f(x,y)e^{\mathrm{i}\xi(\boldsymbol{x}-x)+i\eta(\boldsymbol{y}-y)}. \tag{5.7}$$

This operator is Weyl ordered (completely symmetric) in $\boldsymbol{x}$ and $\boldsymbol{y}$. For example,

$$6x^2y^2 \mapsto \boldsymbol{x}^2\boldsymbol{y}^2+\boldsymbol{y}^2\boldsymbol{x}^2+\boldsymbol{xyxy}+\boldsymbol{yxyx}+\boldsymbol{x}\boldsymbol{y}^2\boldsymbol{x}+\boldsymbol{y}\boldsymbol{x}^2\boldsymbol{y} \tag{5.8}$$

As a special case we have $x\mapsto\boldsymbol{x}, y\mapsto\boldsymbol{y}$ and $1\mapsto\mathbb{I}$, the idenity on the Hilbert space.

Theorem 5.1 *The algebra of smooth functions $C^\infty(\mathbb{R}^2)$ under star product, is homomorphic to the operator algebra constructed via the Weyl correspondence, under operator product.*

Proof. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be the operators corresponding to functions f and g, respectively (see equation (5.7)). Using the latter equation, we have

$$\boldsymbol{fg} = \frac{1}{(2\pi)^4}\int \mathrm{d}\zeta\mathrm{d}\zeta'\mathrm{d}\chi\mathrm{d}\chi'\mathrm{d}x'\mathrm{d}x''\mathrm{d}y'\mathrm{d}y'' f(x',y')g(x'',y'') \\ \times e^{\mathrm{i}\left(\zeta(\boldsymbol{x}-x')+\chi(\boldsymbol{y}-y')\right)}e^{\mathrm{i}\left(\zeta'(\boldsymbol{x}-x'')+\chi'(\boldsymbol{y}-y'')\right)}. \tag{5.9}$$

Using the Baker-Campbell-Hausdorff formula, we get

$$\boldsymbol{fg} = \frac{1}{(2\pi)^4}\int \mathrm{d}\zeta\mathrm{d}\zeta'\mathrm{d}\chi\mathrm{d}\chi'\mathrm{d}x'\mathrm{d}x''\mathrm{d}y'\mathrm{d}y'' f(x',y')g(x'',y'') \\ \times e^{\mathrm{i}\left((\zeta+\zeta')\boldsymbol{x}+(\chi+\chi')\boldsymbol{y}\right)}e^{\mathrm{i}\left(-\zeta x'-\chi y'-\zeta' x''-\chi' y''-\frac{\theta}{2}(\zeta\chi'-\chi\zeta')\right)}. \tag{5.10}$$

Changing the variables to

$$\zeta' := \frac{2}{\theta}(y'-y), \qquad \zeta := \xi-\frac{2}{\theta}(y'-y), \\ \chi' := \frac{2}{\theta}(x-x'), \qquad \chi := \eta-\frac{2}{\theta}(x-x'), \tag{5.11}$$

reduces the above integral to

$$\begin{aligned} \boldsymbol{fg} &= \frac{(2/\theta)^2}{(2\pi)^4}\int \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}x''\mathrm{d}y'\mathrm{d}y'' f(x',y')g(x'',y'')\ e^{\mathrm{i}\xi(\boldsymbol{x}-x)+\mathrm{i}\eta(\boldsymbol{y}-y)} \quad (5.12)\\ &\times \exp\left(\frac{-2\mathrm{i}}{\theta}\big(y(x'-x'')+y'(x''-x)+y''(x-x')\big)\right). \end{aligned}$$

With Baker's formula (5.6), this becomes

$$\boldsymbol{fg} = \frac{1}{(2\pi)^2}\int \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}x\mathrm{d}y\ e^{\mathrm{i}\xi(\boldsymbol{x}-x)+\mathrm{i}\eta(\boldsymbol{y}-y)}(f*g)(x,y), \tag{5.13}$$

□

In particular, the Moyal bracket of x and y is mapped to the commutator of the operators $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$[\boldsymbol{x},\boldsymbol{y}] = \boldsymbol{x}\circ\boldsymbol{y} - \boldsymbol{y}\circ\boldsymbol{x} = \frac{1}{(2\pi)^2}\int \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}y\mathrm{d}x e^{\mathrm{i}\xi(\boldsymbol{x}-x)+\mathrm{i}\eta(\boldsymbol{y}-y)}[x,y]_* = \mathrm{i}\theta\mathbb{I}. \tag{5.14}$$

where $\mathbb{I}$ is the operator corresponding to the function $f(x,y)=1$.

5.3 Noncommutative cylinder

In string theory, a worldsheet is a two-dimensional Riemannian manifold (Σ,η) with metric η. It is embedded in a D-dimensional target space M, spacetime, via maps $X^\mu : \Sigma \to M$ with $\mu = 1,\cdots,D$.

Seiberg and Witten [20] considered spacetime as a noncommutative geometry, which was defined by deformation of the algebra of smooth functions on M. They introduced a star product $\hat{*}$ on $C^\infty(M)$, which reads for local coordinates (X^μ) on M

$$F(X)\hat{*}G(X) := \exp\left(\frac{\mathrm{i}}{2}\Theta^{\mu\nu}\frac{\partial}{\partial Y^\mu}\frac{\partial}{\partial Z^\nu}\right)F(Y)G(Z)\bigg|_{Z=Y=X}. \tag{5.15}$$

Here $\Theta^{\mu\nu}$ is a constant, anti-symmetric $D\times D$-matrix and $F,G\in C^\infty(M)$. In particular, this yields for the commutator of the coordinate functions X^μ and X^ν,

$$X^\mu\hat{*}X^\nu - X^\nu\hat{*}X^\mu = \mathrm{i}\Theta^{\mu\nu}. \tag{5.16}$$

In analogy with this, we consider here the worldsheet Σ of a string as a noncommutative geometry. We will restrict to a closed string in the case that Σ is a cylinder C. Let C be parametrized by $\tau\in\mathbb{R}$ and $\sigma\in[0,2\pi)$.

As we did for noncommutative $\mathbb{R}^2$, we deform the algebra of smooth functions $C^\infty(C)$ on C. We introduce a star product $*$,

$$(f*g)(\tau,\sigma) = f(\tau,\sigma)e^{\frac{\mathrm{i}\theta}{2}(\overleftarrow{\partial_\tau}\overrightarrow{\partial_\sigma}-\overleftarrow{\partial_\sigma}\overrightarrow{\partial_\tau})}g(\tau,\sigma). \tag{5.17}$$

The Moyal bracket is given by

$$[f,g]_*(\tau,\sigma) := (f*g)(\tau,\sigma) - (g*f)(\tau,\sigma). \tag{5.18}$$

For the coordinates functions, this gives

$$[\tau,\sigma]_* := \tau*\sigma - \sigma*\tau = \mathrm{i}\theta. \tag{5.19}$$

Note that these definitions are the same as in section 5.1 in the case of $\mathbb{R}^2$.

As we did in section 5.2, we want to map a smooth function $f(\tau,\sigma)$ on C to an operator $\boldsymbol{f}(\boldsymbol{\tau},\boldsymbol{\sigma})$ acting on a Hilbert space. Once again, this is is done by Weyl corresponce. Because σ is restricted to $[0,2\pi)$, there is a slight modification.

$$\boldsymbol{f}(\boldsymbol{\tau},\boldsymbol{\sigma}) := \frac{1}{(2\pi)^2}\sum_{n\in\mathbb{Z}}\int \mathrm{d}\xi\mathrm{d}\tau\mathrm{d}\sigma f(\tau,\sigma)e^{\mathrm{i}\xi(\boldsymbol{\tau}-\tau)+\mathrm{i}n(\boldsymbol{\sigma}-\sigma)}. \tag{5.20}$$

The operator $\boldsymbol{f}(\boldsymbol{\tau},\boldsymbol{\sigma})$ is Weyl ordered in $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. Furthermore, we have $\tau\mapsto\boldsymbol{\tau}, \sigma\mapsto\boldsymbol{\sigma}$ and $1\mapsto\mathbb{I}$, the idenity on the Hilbert space.

Theorem 5.2 *The algebra of smooth functions $C^\infty(C)$ under star product, is homomorphic to the operator algebra constructed via the Weyl correspondence (5.20), under operator product.*

The proof differs from the proof of theorem 5.1 as there is no analogue of Baker's formula on a cylinder. The proof can be found in appendix C.

As the Weyl corresponce is a homomorphism, the Moyal bracket of τ and σ is mapped to the commutator of the operators $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$,

$$[\boldsymbol{\tau},\boldsymbol{\sigma}] = \frac{1}{(2\pi)^2}\sum_{n}\int \mathrm{d}\xi\mathrm{d}\tau\mathrm{d}\sigma e^{\mathrm{i}\xi(\boldsymbol{\sigma}-\sigma)+\mathrm{i}n(\boldsymbol{\tau}-\tau)}[\tau,\sigma]_* = \mathrm{i}\theta\mathbb{I}. \tag{5.21}$$

where $\mathbb{I}$ is the operator corresponding to the function $f(\tau,\sigma)=1$.

We see that on a cylinder C, it is also possible to construct an operator algebra which is homomorphic to the algebra of smooth functions on C. Can this be extended to general two-dimensional manifolds? As string theory describes more complicated surfaces than $\mathbb{R}^2$ and C, this is an important question.

On a general two-dimensional manifold Σ with coordinates (ξ^1,ξ^2), we define an anti-symmetric matrix $\theta^{\alpha\beta}$. This matrix $\theta^{\alpha\beta}$ transforms as a tensor under coordinate transformations $(\xi^1,\xi^2)\mapsto(\xi'^1,\xi'^2)$,

$$\theta'^{\alpha'\beta'} = \frac{\partial\xi'^{\alpha'}}{\partial\xi^\alpha}\frac{\partial\xi'^{\beta'}}{\partial\xi^\beta}\theta^{\alpha\beta}. \tag{5.22}$$

It follows that $\theta^{\alpha\beta}$ is not necessarily constant in each coordinate system on Σ. As we want the theory to be invariant under reparametrizations, we have to consider more general star products with $\theta^{\alpha\beta}$ not constant. In the next section we first prove a no-go theorem, then we will construct a general form of the star product.

5.4 Associativity

We return to the general case. Let M be an n-dimensional manifold. We define the star product on $C^\infty(M)$ as

$$f * g(x) := \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}(x)\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\right) f(y)g(z)\Big|_{y=z=x}. \tag{5.23}$$

for $f, g \in C^\infty(M)$ and $\theta^{\mu\nu} = \theta^{\mu\nu}(x)$ an anti-symmetric nondegenerate tensor field where $\mu, \nu = 1, 2, \dots, n$. These properties of $\theta^{\mu\nu}$ imply that n must be even. The following theorem appears in Ydri [28] in the case of four dimensions.

Theorem 5.3 *Let $\mathcal{A}$ be the algebra of smooth functions $C^\infty(M)$ on a manifold M with star product (5.23). Then the algebra $\mathcal{A}$ is associative if and only if $\theta^{\mu\nu}(x)$ is constant.*

Proof. We refer to Bayen et al. [2] for the 'if' statement and only prove the 'only if' statement. If $\mathcal{A}$ is associative, then we have by definition

$$(f * g) * h = f * (g * h) \tag{5.24}$$

for all $f, g, h \in \mathcal{A}$. We write the star product as

$$* := \mathrm{e}^{\overleftarrow{\partial_\mu}\frac{\mathrm{i}}{2}\hbar\theta^{\mu\nu}(x)\overrightarrow{\partial_\nu}} \tag{5.25}$$

where we introduced a formal parameter $\hbar \in \mathbb{R}$. Next, we consider the expression

$$I = (\mathrm{e}^{\mathrm{i}px} * \mathrm{e}^{\mathrm{i}kx}) * \mathrm{e}^{\mathrm{i}lx} - \mathrm{e}^{\mathrm{i}px} * (\mathrm{e}^{\mathrm{i}kx} * \mathrm{e}^{\mathrm{i}lx}). \tag{5.26}$$

We expand I in a formal power series in $\hbar$. To do so, we first expand the star product in powers of $\hbar$,

$$\begin{aligned} * \;=\; & 1 + \frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu} - \frac{\hbar^2}{4}\Big(\overleftarrow{\partial_\mu}\overleftarrow{\partial_\kappa}\theta^{\mu\nu}\theta^{\kappa\lambda}\overrightarrow{\partial_\nu}\overrightarrow{\partial_\lambda} + \overleftarrow{\partial_\mu}(\partial_\kappa\theta^{\mu\nu})(\partial_\nu\theta^{\kappa\lambda})\overrightarrow{\partial_\lambda} \\ & + \overleftarrow{\partial_\mu}\overleftarrow{\partial_\kappa}\theta^{\mu\nu}(\partial_\nu\theta^{\kappa\lambda})\overrightarrow{\partial_\lambda} + \overleftarrow{\partial_\mu}(\partial_\kappa\theta^{\mu\nu})\theta^{\kappa\lambda}\overrightarrow{\partial_\nu}\overrightarrow{\partial_\lambda}\Big) + \mathcal{O}(\hbar^3). \end{aligned} \tag{5.27}$$

With this we get

$$\begin{aligned} \mathrm{e}^{\mathrm{i}px} * \mathrm{e}^{\mathrm{i}kx} \;=\; & \Big[1 - \frac{\mathrm{i}\hbar}{2}p_\mu\theta^{\mu\nu}k_\nu - \frac{\hbar^2}{4}\Big(p_\mu p_\kappa\theta^{\mu\nu}\theta^{\kappa\lambda}k_\nu k_\lambda + p_\mu(\partial_\kappa\theta^{\mu\nu})(\partial_\nu\theta^{\kappa\lambda})k_\lambda \\ & + p_\mu p_\kappa\theta^{\mu\nu}(\partial_\nu\theta^{\kappa\lambda})k_\lambda + p_\mu(\partial_\kappa\theta^{\mu\nu})\theta^{\kappa\lambda}k_\nu k_\lambda\Big)\Big]\mathrm{e}^{\mathrm{i}(p+k)x} + \mathcal{O}(\hbar^3) \end{aligned} \tag{5.28}$$

Eventually, I becomes a polynomial in p, k, l and an expansion in $\hbar$,

$$I = \sum_{n_i, m \in \mathbb{N}} a_{n_1 n_2 n_3, m}\; p^{n_1} k^{n_2} l^{n_3} \hbar^m. \tag{5.29}$$

We will only consider the term linear in p, k and l and quadratic in $\hbar$. As we have $I = 0$ for arbitrary p, k, l and $\hbar$, we have in particular

$$a_{111,2} = 0. \tag{5.30}$$

The first term in the right-hand side of equation (5.26) reads

$$\begin{aligned}\left(e^{ipx} * e^{ikx}\right) * e^{ilx} &= \left[\left(1 - \frac{i\hbar}{2}p_\mu\theta^{\mu\nu}k_\nu - \frac{\hbar^2}{4}p_\mu(\partial_\kappa\theta^{\mu\nu})(\partial_\nu\theta^{\kappa\lambda})k_\lambda\right)e^{i(p+k)x}\right] \quad (5.31)\\ &\times\left(1 - \frac{\hbar}{2}\overleftarrow{\partial_\gamma}\theta^{\gamma\delta}l_\delta - \frac{i\hbar^2}{4}\overleftarrow{\partial_\gamma}(\partial_\alpha\theta^{\gamma\delta})(\partial_\delta\theta^{\alpha\beta})l_\beta\right)e^{ilx} + \left(\text{other terms } \not\propto pkl\hbar^2\right).\end{aligned}$$

where the derivatives act on the term between brackets. The only term proportional to $pkl\hbar^2$ comes from the product of the two terms linear in $\hbar$,

$$\left(e^{ipx} * e^{ikx}\right) * e^{ilx} = \frac{i\hbar^2}{4}p_\mu(\partial_\gamma\theta^{\mu\nu})k_\nu\theta^{\gamma\lambda}l_\lambda\ e^{i(p+k+l)x} + \cdots. \tag{5.32}$$

Similarly, we have for the second term in the right-hand side of equation (5.26),

$$\begin{aligned}e^{ipx} * \left(e^{ikx} * e^{ilx}\right) &= e^{ipx}\left(1 - \frac{\hbar}{2}p_\gamma\theta^{\gamma\delta}\overrightarrow{\partial_\delta} - \frac{i\hbar^2}{4}p_\gamma(\partial_\alpha\theta^{\gamma\delta})(\partial_\delta\theta^{\alpha\beta})\overrightarrow{\partial_\beta}\right) \quad (5.33)\\ &\times\left[\left(1 - \frac{i\hbar}{2}k_\mu\theta^{\mu\nu}l_\nu - \frac{\hbar^2}{4}k_\mu(\partial_\kappa\theta^{\mu\nu})(\partial_\nu\theta^{\kappa\lambda})l_\lambda\right)e^{i(k+l)x}\right] + \left(\text{other terms } \not\propto pkl\hbar^2\right)\end{aligned}$$

which reduces to

$$e^{ipx} * \left(e^{ikx} * e^{ilx}\right) = \frac{i\hbar^2}{4}p_\mu\theta^{\mu\gamma}k_\nu(\partial_\gamma\theta^{\nu\lambda})l_\lambda\ e^{i(p+k+l)x} + \cdots. \tag{5.34}$$

With this, we can write equation (5.30) as

$$\theta^{\gamma\lambda}\partial_\gamma\theta^{\mu\nu} + \theta^{\gamma\mu}\partial_\gamma\theta^{\nu\lambda} = 0 \tag{5.35}$$

Cyclic permutation of this expression gives

$$-\theta^{\gamma\mu}\partial_\gamma\theta^{\nu\lambda} - \theta^{\gamma\nu}\partial_\gamma\theta^{\lambda\mu} = 0, \tag{5.36}$$
$$\theta^{\gamma\nu}\partial_\gamma\theta^{\lambda\mu} + \theta^{\gamma\lambda}\partial_\gamma\theta^{\mu\nu} = 0. \tag{5.37}$$

When we add these three expressions, we obtain $\theta^{\gamma\lambda}\partial_\gamma\theta^{\mu\nu} = 0$. Since $\theta^{\gamma\lambda}$ is invertible, this implies that $\theta^{\mu\nu}$ is constant. □

5.5 Poisson connections and covariant star product

In the previous section, we showed that the star product (5.23) is associative if and only if the matrix $\theta^{\mu\nu}$ is constant. In this section, we define a star product which is invariant under diffeomorphisms and which is also an associative product on the algebra of smooth functions on a manifold. We will call this star product the **covariant star product**.

In order to define the latter, we need some theory on Poisson manifolds and symplectic manifolds, which is taken from Vaisman [23] and Westenholz [24]. The approach we take towards the star product, is based on [2].

5.5.1 Poisson manifolds

In classical mechanics, the dynamics of physical systems can be described by Hamilton equations. These can be written as Poisson brackets of functions on phase space. The Poisson brackets give a Lie structure to the space of functions on phase space. In general, we have the following.

Definition 5.4 Let M be a differentiable manifold. A **bivector field** Λ on M is an anti-symmetric, bilinear form on T^*M, i.e., a map $\Lambda : T^*M \times T^*M \to \mathbb{C}$, that is linear in both entries. It is written in local coordinates (x^α) on M as

$$\Lambda = \Lambda^{\alpha\beta}\partial_\alpha \wedge \partial_\beta \qquad \alpha, \beta = 0, \ldots, n-1 \tag{5.38}$$

where $n = \dim M$. A **Poisson bivector field** is a bivector field on M that satisfies

$$\Lambda^{\gamma\alpha}\partial_\gamma\Lambda^{\beta\kappa} + \Lambda^{\gamma\beta}\partial_\gamma\Lambda^{\kappa\alpha} + \Lambda^{\gamma\kappa}\partial_\gamma\Lambda^{\alpha\beta} = 0 \tag{5.39}$$

A **Poisson manifold** is a pair (M, Λ). A Poisson bivector field Λ defines a Poisson bracket,

$$\{f, g\} := \Lambda(\mathrm{d}f, \mathrm{d}g) := \Lambda^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial x^\beta}, \tag{5.40}$$

for $f, g \in C^\infty(M)$. This bracket gives a Lie structure to the algebra of smooth functions $C^\infty(M)$. The Jacobi identity,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{5.41}$$

is equivalent with equation (5.39).

We recall the definition of a **symplectic manifold**: it is a pair (M, ω) with M a manifold and ω a closed, non-degenerate 2-form; ω is called the **symplectic form**. We note that every orientable two-dimensional manifold M with a volume form ω is a symplectic manifold. Furthermore, when we concentrate on string theory, we can combine the theory of symplectic manifolds and Poisson manifolds.

Theorem 5.5 *Let (M, Λ) be a two-dimensional Poisson manifold with $\Lambda \neq 0$, then (M, ω) is a symplectic manifold with symplectic form,*

$$\omega := \Lambda_{\alpha\beta}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta; \qquad (\Lambda_{\alpha\beta}) = (\Lambda^{\alpha\beta})^{-1}. \tag{5.42}$$

Proof. ω is closed because there are no 3-forms in 2 dimensions. As $\Lambda \neq 0$, $\det(\Lambda^{\alpha\beta}) \neq 0$, and ω is non-degenerate. □

As we saw before, problems arise when we consider derivatives in expressions which should be invariant under diffeomorphisms (for example in a star product). But we're already familiar with this: it also arises in general relativity. There we've seen that the derivative of a tensor does not transform as a tensor, but that the covariant derivative of a tensor does so. In order to define a covariant derivative on a Poisson manifold, we need a connection.

Definition 5.6 A **Poisson connection** Γ on the Poisson manifold (M, Λ), is a linear connection *without torsion*, such that $\nabla\Lambda = 0$, where ∇ is the covariant derivative defined by Γ. If (x^α) are local coordinates on M, then the components of the connection are $\Gamma^\alpha_{\beta\gamma}$. The torsion tensor T is defined as

$$T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}. \tag{5.43}$$

From [2] we take the following proposition, which is similar to the Darboux theorem on symplectic manifolds.

Proposition 5.7 *Let (M, Λ) be a Poisson manifold. Then there exist atlases with charts such that the components of Λ with respect to these charts are constant. These charts are called **natural charts**.*

Let (x^i) be a natural chart on M and denote the Christoffel symbols of the Poisson connection Γ in this chart by Γ^i_{jk}. Note that Γ^i_{jk} is symmetric in the lower indices. The covariant derivative of a contravariant vector a^i is

$$\nabla_k a^i := \partial_k a^i + \Gamma^i_{mk} a^m. \tag{5.44}$$

The covariant derivative of the components Λ^{ij} of the Poisson bivector field Λ reads[1]

$$\nabla_k \Lambda^{ij} = \partial_k \Lambda^{ij} + \Gamma^i_{mk}\Lambda^{mj} + \Gamma^j_{mk}\Lambda^{im}. \tag{5.45}$$

The curvature tensor R is defined by,

$$R^i_{ljk} := \frac{\partial \Gamma^i_{kl}}{\partial x^j} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{km}\Gamma^m_{jl}. \tag{5.46}$$

For later use, we recall the following result for a torsionless connection,

$$\nabla_k \nabla_j a^i - \nabla_j \nabla_k a^i = R^i_{lkj} a^l. \tag{5.47}$$

In general relativity the Levi-Civita connection is completely determined by the metric and its derivatives. The Poisson connection, however, cannot be determined uniquely from Λ^{ij} and its derivatives. The following theorem provides a way to find a suitable Poisson connection Γ for which $\nabla\Lambda = 0$. Define quantities[2]

$$\Gamma^{ijk} := \Gamma^i_{lm}\Lambda^{jl}\Lambda^{km}. \tag{5.48}$$

Theorem 5.8 *A connection Γ is a Poisson connection if and only if Γ^{ijk} is completely symmetric.*

---

[1] Of course, this can be written down for general chart (x^α) but we don't need that here.

[2] As the Christoffel symbols are not tensors, these Γ^{ijk} do not transform like tensors. A little calculation shows that in n dimensions, there are $n^2 + \frac{1}{3!}n(n-1)(n-2)$ independent components Γ^{ijk}. For example, in two dimensions, there are 4 independent components: $\Gamma^{000}, \Gamma^{100}, \Gamma^{110}$ and Γ^{111}.

Proof. As (x^i) is a natural chart, Λ^{ij} is a constant matrix. Equation (5.45) contracted with Λ^{lk}, yields

$$\Lambda^{lk}\nabla_k\Lambda^{ij} = 0 + \Gamma^i_{mk}\Lambda^{mj}\Lambda^{lk} + \Gamma^j_{mk}\Lambda^{im}\Lambda^{lk} \tag{5.49}$$

Using formula (5.48) we obtain

$$\Lambda^{lk}\nabla_k\Lambda^{ij} = -\Gamma^{ijl} + \Gamma^{jil} \tag{5.50}$$

which proves the 'if' statement for invertible Λ^{lk}.

For the proof of the 'only if' statement, we rewrite formula (5.49) by using the symmetry of the Christoffel symbols in the lower indices to obtain

$$\begin{aligned}
\Lambda^{lk}\nabla_k\Lambda^{ij} &= -\Gamma^i_{mk}\Lambda^{jm}\Lambda^{lk} + \Gamma^j_{mk}\Lambda^{im}\Lambda^{lk} && (5.51)\\
&= -\Gamma^i_{km}\Lambda^{jm}\Lambda^{lk} + \Gamma^j_{km}\Lambda^{im}\Lambda^{lk} \\
&= -\Gamma^{ilj} + \Gamma^{jli}. && (5.52)
\end{aligned}$$

If $\nabla_k\Lambda^{ij} = 0$, we have symmetry in the first and last index of Γ^{ilj}. From equation (5.50) we have symmetry in the first two indices and the symmetry in the last two indices follows from the definition of Γ^{ijk} (see (5.48)). Hence Γ^{ilj} is completely symmetric. □

5.5.2 Covariant star product

We are finally ready to define the covariant star product, which is associative in each coordinate system on a Poisson manifold (M, Λ). We refer to [2] for more details.

Theorem 5.9 *Let (M,Λ) be a Poisson manifold, Γ a Poisson connection and (x^α) local coordinates on M. If the curvature of Γ is zero, then the corresponding covariant derivative ∇, defines an associative star product,*

$$* := e^{\left(\frac{\mathrm{i}}{2}\Lambda^{\alpha\beta}\overleftarrow{\nabla}_\alpha\overrightarrow{\nabla}_\beta\right)} \tag{5.53}$$

on the algebra of smooth functions $C^\infty(M)$. Thus, for $f, g \in C^\infty(M)$ we have

$$f * g = \sum_n \frac{(\mathrm{i}/2)^n}{n!}\Lambda^{\alpha_1\beta_1}\cdots\Lambda^{\alpha_n\beta_n}(\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}f)\ (\nabla_{\beta_1}\cdots\nabla_{\beta_n}g). \tag{5.54}$$

The proof follows by considering $(f*g)*h - f*(g*h)$ and can be found in [2]. We note that it is sufficient to concentrate on natural charts, as $\nabla_k\Lambda^{ij}$ transforms as a tensor. Hence, if it is zero in a natural chart, it is zero in each chart on M. The same argument applies to the vanishing curvature tensor.

It is always possible to choose a Poisson connection with vanishing curvature tensor. For example, let us define $\Gamma^i_{jk} := 0$ in a natural chart. As the Poisson connection transforms like

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^i_{jk}\frac{\partial x^j}{\partial\tilde{x}^\beta}\frac{\partial x^k}{\partial\tilde{x}^\gamma}\frac{\partial\tilde{x}^\alpha}{\partial x^i} + \frac{\partial^2 x^i}{\partial\tilde{x}^\beta\partial\tilde{x}^\gamma}\frac{\partial\tilde{x}^\alpha}{\partial x^i} \tag{5.55}$$

under transformations $(x^i) \mapsto (\tilde{x}^\alpha)$, this leads in general to nonzero connections on M.

With this, we have solved the problem of the previous section. As we now have a coordinate independent expression for the star product, deformation quantization is possible on any Poisson manifold.

We conclude with a theorem which links Levi-Civita connections to Poisson connection (Vaisman [23]).

Theorem 5.10 *Let (M, g) be a Riemannian manifold endowed with the Levi-Civita connection Γ_g with associated covariant derivative ∇_g. Let Λ be a Poisson bivector field on M. If there is a tensor field $A^{\alpha}_{\beta\gamma}$ such that $A^{\alpha}_{\beta\gamma} = A^{\alpha}_{\gamma\beta}$, and*

$$(\nabla_g)_\kappa \Lambda^{\beta\gamma} + A^{\beta}_{\delta\kappa}\Lambda^{\delta\gamma} + A^{\gamma}_{\delta\kappa}\Lambda^{\beta\delta} = 0, \tag{5.56}$$

then $\Gamma_g + A$ is a Poisson connection on (M, Λ).

5.6 Spectral triple

The Weyl correspondence described in section 5.2 is only possible for constant $\theta^{\alpha\beta}$. It turned out to be rather difficult and for now impossible to write the Weyl corresponce for the covariant star product (5.53). For an overview, we refer to Sternheimer [21]. In this section, we discuss the noncommutative cylinder in Connes' setting of spectral triples.

In the language of the previous section, a noncommutative worldsheet is a two-dimensional Poisson manifold (Σ, Λ), where a star product is defined on $C^\infty(\Sigma)$ as

$$f * g = \sum_n \frac{(\mathrm{i}/2)^n}{n!}\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_n\beta_n}(\nabla_{\alpha_1}\cdots\nabla_{\alpha_n} f)\ (\nabla_{\beta_1}\cdots\nabla_{\beta_n} g). \tag{5.57}$$

for $f, g \in C^\infty(\Sigma)$. The components of the Poisson bivector field Λ are thus $\theta^{\alpha\beta}$.

By conformal invariance of the Polyakov action (see the previous chapter), one can always rescale the metric η on Σ into Euclidean form $\eta_{\alpha\beta} = \delta_{\alpha\beta}$. Unfortunately, it is not necesarily true that the coordinates τ and σ on Σ form a natural chart for which $\theta_{\alpha\beta}$ is a constant matrix. This is due to the fact that the two symmetry groups Diff $\ltimes$ Weyl and $Sp(1) \cong SL(2, \mathbb{R})$, are different. Therefore, we will only discuss the noncommutative cylinder as a special case of a noncommutative worldsheet of a closed (bosonic) string.

The cylinder is parametrized by coordinates $\tau \in \mathbb{R}$ and $\sigma \in [0, 2\pi)$. The covariant star product reduces to

$$f * g = \sum_n \frac{(\mathrm{i}/2)^n}{n!}\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_n\beta_n}(\partial_{\alpha_1}\cdots\partial_{\alpha_n} f)\ (\partial_{\beta_1}\cdots\partial_{\beta_n} g). \tag{5.58}$$

which is the Moyal star product as defined in section 5.3.

We saw before that the algebra of smooth functions under Moyal star product is homomorphic to an operator algebra. The latter is defined by its action on a Hilbert space, i.e., by a representation on $\mathcal{H}$. For the commutator of the operators corresponding to the coordinates τ and σ, we have the relation

$$[\boldsymbol{\tau}, \boldsymbol{\sigma}] = \mathrm{i}\theta\mathbb{I}. \tag{5.59}$$

We define operators $\boldsymbol{c}$ and $\boldsymbol{c}^\dagger$,

$$\boldsymbol{c} = \frac{1}{\sqrt{2\theta}}(\boldsymbol{\tau} + \mathrm{i}\boldsymbol{\sigma}), \qquad \boldsymbol{c}^\dagger = \frac{1}{\sqrt{2\theta}}(\boldsymbol{\tau} - \mathrm{i}\boldsymbol{\sigma}), \tag{5.60}$$

which satisfy

$$[\boldsymbol{c}, \boldsymbol{c}^\dagger] = \mathbb{I}. \tag{5.61}$$

These are the defining relations for creation and annihilation operators on a Hilbert space $\mathcal{H}$. We define a number operator $\boldsymbol{N} = \boldsymbol{c}^\dagger\boldsymbol{c} + 1/2$, and denote its eigenvectors in $\mathcal{H}$ by $|n\rangle$ with eigenvalue $n + 1/2$ ($n \in \mathbb{N}$). The eigenvector with eigenvalue $1/2$ is annihilated by $\boldsymbol{c}$, $\boldsymbol{c}|0\rangle = 0$. Furthermore, we have

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}|n\rangle, \tag{5.62}$$

with the action of $\boldsymbol{c}$ and $\boldsymbol{c}^\dagger$ defined as

$$\boldsymbol{c}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \boldsymbol{c}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{5.63}$$

The inproduct on $\mathcal{H}$ is given by $\langle\ |\ \rangle$. With respect to this inproduct, the vectors $|n\rangle$ are orthonormal,

$$\langle n|m\rangle = \delta_{nm}. \tag{5.64}$$

From equation (5.63) it follows that $\boldsymbol{c}$ and $\boldsymbol{c}^\dagger$ are unbounded operators on $\mathcal{H}$. This means that no connection can be made with the theory on spectral triples, as they require an algebra of bounded operators acting on a Hilbert space. What to do? The answer can be found in quantum mechanics. There the operators $\boldsymbol{x}$ and $\boldsymbol{p}$ corresponding to position and momentum, are interpreted as generators of a Lie group. This group consist of the exponentiation of these generators. The exponentiations are well-defined unitary operators and therefore bounded.

In the case of the cylinder C, we have something similar. The operators $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ form a Lie algebra defined by equation (5.59). We define unitary operators U_α and V as

$$U_\alpha = e^{\mathrm{i}\alpha\boldsymbol{\tau}}, \qquad V = e^{\mathrm{i}\boldsymbol{\sigma}} \tag{5.65}$$

where α is a real parameter. They obey the following relation,

$$U_\alpha V = e^{-\mathrm{i}\alpha\theta} V U_\alpha. \tag{5.66}$$

and are represented in Schwartz space $\mathcal{S}(\mathbb{R})$. This space consists of smooth funtions $f(\tau)$ on $\mathbb{R}$ that are required to go to zero, as $\tau \to \infty$, faster than any inverse power of τ, and so are all their derivatives (see for example Richtmyer [19]). In other words,

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \sup \left| \tau^p \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^k f(\tau) \right| < \infty, \text{ for all } p, k \in \mathbb{N} \right\} \tag{5.67}$$

The action of U_α and V on functions $f \in \mathcal{S}(\mathbb{R})$ is given by

$$U_\alpha \colon \; f(\tau) \mapsto f(\tau) e^{\mathrm{i}\alpha\tau}, \qquad V \colon \; f(\tau) \mapsto f(\tau + \theta). \tag{5.68}$$

Finally, the **noncommutative cylinder** C_θ is defined as the algebra

$$C_\theta := \Big\{ \sum_{n,m} a_{nm} (U_\alpha)^n V^m \, : a \in \mathcal{S}(\mathbb{Z}^2) \Big\} \tag{5.69}$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the vector space of sequences $(a_{nm})_{n,m \in \mathbb{Z}}$ that decay faster than the inverse of any polynomial in (n, m).

In order to complete the definition of the spectral triple, we need to define an unbounded self-adjoint operator. In the Fock space representation we constructed the number operator which reads in the Schwartz space representation

$$\boldsymbol{N} = \frac{1}{2\theta} \left[(-\mathrm{i}\theta)^2 \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \right)^2 + \tau^2 \right]. \tag{5.70}$$

The completion $L^2(\mathbb{R})$ of Schwartz space $\mathcal{S}(\mathbb{R})$ is a Hilbert space. $\boldsymbol{N}$ is an unbounded self-adjoint operator on this Hilbert space and the triple $(C_\theta, L^2(\mathbb{R}), \boldsymbol{N})$ is a spectral triple. As the eigenvalues of $\boldsymbol{N}^{-1}$ are $\frac{1}{n+1/2}$, its dimension is 1. The integral of elements in C_θ is given by

$$⨍ \Big(\sum_{n,m} a_{nm} (U_\alpha)^n V^m \Big) := \mathrm{tr}_\omega \Big(\sum_{n,m} a_{nm} (U_\alpha)^n V^m |\boldsymbol{N}|^{-1} \Big). \tag{5.71}$$

A spectral triple of dimension 2 can be obtained by using the self-adjoint operator $\sqrt{\boldsymbol{N}}$ instead of $\boldsymbol{N}$. Actually, one can define a whole class of spectral triples of dimension 2 by operators similar to $\sqrt{\boldsymbol{N}}$. All these spectral triples can be examined on their own, as each triple provides a description of the noncommutative cylinder in Connes' setting.

Chapter 6

Path integral approach to deformation quantization

It is rather difficult to write down the Weyl correspondence for the covariant star product (5.53). In order to quantize a manifold, we try another approach using a path integral. In quantum field theory it is quite natural to use path integral methods to quantize the field theory. Therefore, we will consider in this chapter path integral quantization of a manifold. Before we introduce this path integral, we will define an associative star product by means of Kontsevich's formula. This star product is different from the covariant star product we considered in the previous chapter (see Kontsevich, [10]). The semiclassical expansion of the above mentioned path integral equals Kontsevich's star product.

6.1 Deformation quantization of Kontsevich

In the previous chapter we constructed a generalization of the star product on a Poisson manifold (M, Λ), where we used a Poisson connection Γ and the corresponding covariant derivative. This led to an associative, covariant star product but it turned out to be difficult to construct an operator algebra, via Weyl's correspondence. For now, we will leave this star product for what it is, and consider a different approach to deformation quantization by defining another star product.

Kontsevich writes in [10] a star product on the space of smooth functions $C^\infty(M)$ on a Poisson manifold (M, Λ) for constant coefficients $\Lambda^{\alpha\beta}$ as

$$f * g = fg + \left(\frac{\mathrm{i}\hbar}{2}\right)\Lambda^{\alpha\beta}(\partial_\alpha f)(\partial_\beta g) + \frac{1}{2}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \Lambda^{\alpha_1\beta_1}\Lambda^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2} f)(\partial_{\beta_1}\partial_{\beta_2} g) + \cdots \quad (6.1)$$

where $f, g \in C^\infty(M)$. Of course, this is just the Moyal star product (5.58), with $\Lambda^{\alpha\beta} = \theta^{\alpha\beta}$. For local coordinates (x^α), we have $\Lambda = \Lambda^{\alpha\beta}\partial_\alpha \wedge \partial_\beta$. Kontsevich's generalization of equation (6.1) to variable coefficients $\Lambda^{\alpha\beta}(x)$ reads

$$\begin{aligned} f \star g &= fg + \left(\frac{\mathrm{i}\hbar}{2}\right)\Lambda^{\alpha\beta}(\partial_\alpha f)(\partial_\beta g) + \frac{1}{2}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \Lambda^{\alpha_1\beta_1}\Lambda^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2} f)(\partial_{\beta_1}\partial_{\beta_2} g) \\ &+\frac{1}{3}\left(\frac{\mathrm{i}\hbar}{2}\right)^2\Big(\Lambda^{\alpha_1\beta_1}(\partial_{\beta_1}\Lambda^{\alpha_2\beta_2})\big[(\partial_{\alpha_1}\partial_{\alpha_2} f)(\partial_{\beta_2} g) - (\partial_{\alpha_2} f)(\partial_{\alpha_1}\partial_{\beta_2} g)\big]\Big) + \mathcal{O}(\hbar^3). \end{aligned} \tag{6.2}$$

Up to second order this product is associative, i.e.,

$$(f \star g) \star h = f \star (g \star h) + \mathcal{O}(\hbar^3). \tag{6.3}$$

'Bad terms' like $(\partial_{\alpha_2}\Lambda^{\alpha_1\beta_1})(\partial_{\beta_1}\Lambda^{\alpha_2\beta_2})$, which would have come from expanding the exponential in equation (5.23), do not occur in equation (6.2).

In order to describe higher order terms, proportional to $\hbar^n$, Kontsevich introduces a special class G_n of oriented labeled graphs. Before we define G_n, let us review some definitions from graph theory (see for example Bollobás /[3] for more details).

Definition 6.1 An (oriented) **graph** Γ is a pair (V_Γ, E_Γ) of two finite sets such that E_Γ is a subset of $V_\Gamma \times V_\Gamma$.
Elements of V_Γ are **vertices** of Γ and elements of E_Γ are **edges** of Γ. For an edge $e = (v_1, v_2) \in E_\Gamma$ we say that e starts at the vertex v_1 and ends at the vertex v_2.

The class G_n is defined as follows.

Definition 6.2 A labeled graph Γ belongs to G_n $(n = 0, 1, 2, \ldots)$ if

1. Γ has $n + 2$ vertices and $2n$ edges,
2. the set vertices V_Γ is $\{1, \ldots, n\} \sqcup \{L, R\}$, where L, R are just two distinct symbols (think e.g. of left and right),
3. the edges of Γ are labeled by symbols e_k^l, where $k \in \{1, \ldots, n\}$ and $l \in \mathcal{B} \subset \{1, \ldots, n\} \sqcup \{L, R\}$ with cardinality $\#\mathcal{B} = 2$,
4. for every $k \in \{1, \ldots, n\}$ and $l \in \mathcal{B}$, edges labeled by e_k^l start at the vertex k and end at the vertex l,
5. for any $v \in V_\Gamma$, the ordered pair (v, v) is not an edge of Γ.

The set G_n is finite, it has $(n(n+1))^n$ elements for $n \geq 1$ and 1 element for $n = 0$. To each graph we associate a bidifferential operator $B_{\Gamma,\Lambda}$, acting on $f, g \in C^\infty(M)$. We illustrate this with one example.

Let $n = 3$ and let the list of edges be

$$(e_1^L, e_1^R, e_2^R, e_2^3, e_3^L, e_3^R) = \big((1, L), (1, R), (2, R), (2, 3), (3, L), (3, R)\big). \tag{6.4}$$

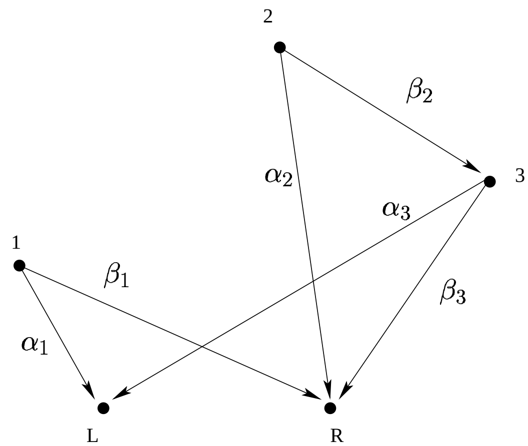

Figure 6.1.1

In the picture of Γ (figure 6.1.1) we put independent indices $1 \leq \alpha_i, \beta_i \leq \dim M$ on edges, instead of labels e_k^l, and associate a partial derivative to each edge. The operator $B_{\Gamma,\Lambda}$ corresponding to the graph in figure 6.1.1 is

$$(f,g) \mapsto \Lambda^{\alpha_1\beta_2}\Lambda^{\alpha_2\beta_2}(\partial_{\beta_2}\Lambda^{\alpha_3\beta_3})(\partial_{\alpha_1}\partial_{\alpha_3}f)(\partial_{\beta_1}\partial_{\alpha_2}\partial_{\beta_3}g). \tag{6.5}$$

Kontsevich's formula (6.2) is represented in figure 6.1.2. If we finally associate a weight ω_Γ with every graph Γ,[1] we can write down Kontsevich's star product for all orders in $\hbar$ as follows.

Theorem 6.3 (Kontsevich) *Let (M, Λ) be a Poisson manifold. Then the star product*

$$f \star g := \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^2 \sum_{\Gamma \in G_n} \omega_\Gamma B_{\Gamma,\Lambda}(f,g) \tag{6.6}$$

is associative on $C^\infty(M)$.

6.2 Path integral

Cattaneo and Felder [5] claim that Kontsevich's expression (6.6) on a Poisson manifold (M, Λ) can be given by the semiclassical expansion of a path integral. Inspired by this, we will consider a path integral whose semiclassical expansion yields Kontsevich's star product on a symplectic manifold.

Let (M, ω) be an n-dimensional symplectic manifold. We define paths $\gamma : t \in \mathbb{R} \mapsto \gamma(t) \in M$ on M with boundary conditions $\gamma(\pm\infty) = x$. We write γ^α in local coordinates (y^α). Before we can introduce the path integral, we need the following lemma.

[1] We won't discuss this weight here. See Kontsevich [10] for the definition of ω_Γ.

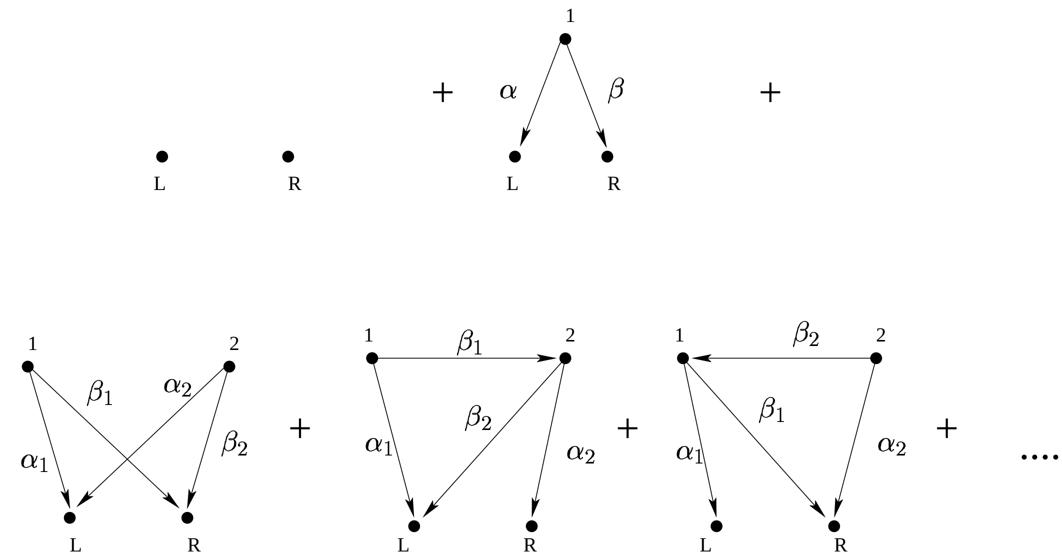

Figure 6.1.2

Proposition 6.4 (Poincaré) *Let ω be a k-form on M with $\mathrm{d}\omega = 0$. Then for all $p \in M$, there exists a neighbourhood U of p and a $(k-1)$-form α on U such that $\mathrm{d}\alpha = \omega$ on U.*

The one-form α is called the **symplectic one-form**. It satisfies $\mathrm{d}\alpha = \omega$ with ω the symplectic form. Poincaré lemma states that such a one-form does always exist locally. We can write down α explicitly,

$$\alpha(y) := \int_0^1 \mathrm{d}\tau\tau\ \omega_{\alpha\beta}(\tau(y-x)+x)\ y^\alpha \mathrm{d}y^\beta \tag{6.7}$$

with (y^α) local coordinates on M. In two dimensions this can be easily seen:

$$\begin{aligned}
\mathrm{d}\alpha(y) &= \mathrm{d}\Big[\int_0^1 \mathrm{d}\tau\tau\ \omega_{12}(\tau(y-x)+x)(y^1\mathrm{d}y^2 - y^2\mathrm{d}y^1)\Big] \\
&= \int_0^1 \mathrm{d}\tau\big\{\tau^2 \frac{\partial\omega_{12}(\tau(y-x)+x)}{\partial(\tau y^\kappa)}\mathrm{d}y^\kappa \wedge (y^1\mathrm{d}y^2 - y^2\mathrm{d}y^1) \\
&\qquad +\tau\omega_{12}(\tau(y-x)+x)\mathrm{d}(y^1\mathrm{d}y^2 - y^2\mathrm{d}y^1)\big\} \\
&= \int_0^1 \mathrm{d}\tau\big\{\tau^2 \frac{\partial\omega_{12}(\tau(y-x)+x)}{\partial(\tau y^\kappa)} y^k \mathrm{d}y^1 \wedge \mathrm{d}y^2 \\
&\qquad +2\tau\omega_{12}(\tau(y-x)+x)\mathrm{d}y^1 \wedge \mathrm{d}y^2\big\} \\
&= \int_0^1 \mathrm{d}\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\big[\tau^2\omega_{12}(\tau(y-x)+x)\big]\mathrm{d}y^1\wedge\mathrm{d}y^2 = \omega_{12}(y)\mathrm{d}y^1\wedge\mathrm{d}y^2 = \omega.
\end{aligned} \tag{6.8}$$

We consider the path integral

$$Z := \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma e^{\frac{\mathrm{i}}{\hbar}S[\gamma]}, \tag{6.9}$$

with

$$S[\gamma] := \frac{1}{2}\int_{\mathbb{R}} \gamma^*\alpha \tag{6.10}$$

where $\gamma^*\alpha$ is the pullback of α by γ. It is given by

$$\gamma^*\alpha(t) = \int_0^1 \mathrm{d}\tau\tau\ \omega_{\alpha\beta}(\tau(\gamma(t)-x)+x)\ \gamma^\alpha(t)\frac{\mathrm{d}\gamma^\beta(t)}{\mathrm{d}t}\mathrm{d}t, \tag{6.11}$$

which is a one-form on $\mathbb{R}$. We note that the integral in equation (6.10) is well-defined even though α is only defined locally. If we define intervals $I_i \subset \mathbb{R}$ for which a symplectic one-form α_i on $\gamma(I_i)$ is given by Poincaré lemma, the integral can be written as a sum of integrals $\int_{I_i}\gamma^*\alpha_i$.

Before we consider the general case, we first state a theorem concerning the case where $\omega_{\alpha\beta}$ is constant. Then, equation (6.7) reads

$$\alpha(y) = \omega_{\alpha\beta}y^\alpha \mathrm{d}y^\beta. \tag{6.12}$$

Theorem 6.5 *Let* $\dim M = 2$ *and let* $\omega_{\alpha\beta}$ *be constant. Then the Moyal star product is given by*

$$f * g(x) = \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0))e^{\frac{\mathrm{i}}{\hbar}S[\gamma]},$$

where 1 and 0 are two points on $\mathbb{R}$. *The action is*

$$S[\gamma] := \frac{1}{2}\int \mathrm{d}t\ \omega_{\alpha\beta}\gamma^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}\gamma^\beta(t).$$

This path integral can be computed exactly by discretization (see appendix B). We will concentrate on the semiclassical expansion of this path integral, as we want to extend the star product to the case when $\omega_{\alpha\beta}$ is not constant.

Theorem 6.6 (General case) *Kontsevich's expression (6.6) for the star product* $f\star g$ *is given by the semiclassical expansion of the path integral*

$$f \star g(x) = \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0))e^{\frac{\mathrm{i}}{\hbar}S[\gamma]},$$

where the action reads

$$S[\gamma] = \frac{1}{2}\int \gamma^*\alpha(t) = \frac{1}{2}\int\left[\int_0^1 \mathrm{d}\tau\tau\ \omega_{\alpha\beta}(\tau(\gamma(t)-x)+x)\right]\gamma^\alpha(t)\frac{\mathrm{d}\gamma^\beta(t)}{\mathrm{d}t}\mathrm{d}t.$$

6.3 Semiclassical expansion

We first prove theorem 6.6 in the case that $\omega_{\alpha\beta}$ is a constant, i.e., the Moyal case. Let us denote the path integral by $\mathfrak{I}$,

$$\mathfrak{I} := \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0))e^{\frac{i}{\hbar}S[\gamma]}, \tag{6.13}$$

with

$$S[\gamma] := \frac{1}{2}\int \mathrm{d}t\ \omega_{\alpha\beta}\gamma^{\alpha}(t)\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{\beta}(t) \tag{6.14}$$

for the Moyal case. The equation of motion of this action is $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{\alpha}(t) = 0$. Taking the boundary conditions into account, the classical solution of the action is $\gamma(t) = x$. The semiclassical expansion is an expansion around this solution. Define $\xi(t) := \gamma(t) - x$ which has boundary conditions $\xi(\pm\infty) = 0$. We expand f and g in a Taylor series,

$$f(x+\xi(1)) = f(x) + \sum_{|\mu|=1}^{\infty} \frac{1}{\mu!}\frac{\partial^{|\mu|}f}{\partial x^{\mu}}(x)\xi^{\mu}(1), \tag{6.15}$$

$$g(x+\xi(0)) = g(x) + \sum_{|\nu|=1}^{\infty} \frac{1}{\nu!}\frac{\partial^{|\nu|}g}{\partial x^{\nu}}(x)\xi^{\nu}(0). \tag{6.16}$$

where we used multi-index notation (see appenix A). The action can be written as

$$\begin{aligned} S[\gamma] &= \frac{1}{2}\int \mathrm{d}t\omega_{\alpha\beta}\gamma^{\alpha}(t)\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{\beta}(t) \\ &= \frac{1}{2}\int \mathrm{d}t\omega_{\alpha\beta}\xi^{\alpha}(t)\frac{\mathrm{d}}{\mathrm{d}t}\xi^{\beta}(t) + \frac{1}{2}\omega_{\alpha\beta}x^{\alpha}\int \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\xi^{\beta}(t) \\ &= \frac{1}{2}\int \mathrm{d}t\omega_{\alpha\beta}\xi^{\alpha}(t)\frac{\mathrm{d}}{\mathrm{d}t}\xi^{\beta}(t) \end{aligned} \tag{6.17}$$

With this, we write

$$\mathfrak{I} = \int_{\xi(\pm\infty)=0} \mathcal{D}\xi\ f(x+\xi(1))g(x+\xi(0))e^{\frac{i}{\hbar}S[\xi]}, \tag{6.18}$$

where now

$$S[\xi] = \frac{1}{2}\int \mathrm{d}t\ \xi^{\alpha}(t)\left[\omega_{\alpha\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right]\xi^{\beta}(t). \tag{6.19}$$

We write this in suggestive matrix form,

$$S[\xi] = \frac{1}{2}\xi^{T}G^{-1}\xi \equiv \frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\xi^{\alpha}(t)G^{-1}_{\alpha\beta}(t,t')\xi^{\beta}(t') \tag{6.20}$$

where we defined

$$G^{-1}_{\alpha\beta}(t,t') := \omega_{\alpha\beta}\frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t'). \tag{6.21}$$

If we define transposition ^{T} to act on both continuous and discrete indices, G^{-1} is symmetric. The corresponding Green's function G can be found quite easily, as $\omega_{\alpha\beta}$ is invertible,

$$G^{\alpha\beta}(t,t') = \frac{1}{2}\omega^{\alpha\beta}\big[\theta(t-t') - \theta(t'-t)\big]. \tag{6.22}$$

Here θ is the Heavyside step function, which is defined as

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} \tag{6.23}$$

Note that the Green's function G is also symmetric,

$$G^{\alpha\beta}(t,t') = G^{\beta\alpha}(t',t).$$

If we add an external source term $-J_\alpha(t)\xi^\alpha(t)$ to the action, the latter reads in condensed notation

$$S[\xi] = \frac{1}{2}\xi^T G^{-1}\xi - J^T\xi. \tag{6.24}$$

If we functionally differentiate the path integral Z twice with respect to J, setting $J=0$ afterwards, we obtain two-point functions like,

$$\begin{aligned} \langle \xi^\alpha(1)\xi^\beta(0)\rangle &:= \int_{\xi(\pm\infty)=0} \mathcal{D}\xi\ \xi^\alpha(1)\xi^\beta(0)e^{\frac{\mathrm{i}}{\hbar}S[\xi]} \\ &= \left(\frac{-\hbar}{\mathrm{i}}\right)^2 \frac{\delta}{\delta J_\alpha(1)}\frac{\delta}{\delta J_\beta(0)} Z\Big|_{J=0} \end{aligned} \tag{6.25}$$

If we rewrite the action as

$$S[\xi] = \frac{1}{2}\xi^T G^{-1}\xi - \frac{1}{2}J^T G J,$$

the two point can be written as a Green's function,

$$\langle \xi^\alpha(1)\xi^\beta(0)\rangle = \mathrm{i}\hbar G^{\alpha\beta}(1,0) = \frac{\mathrm{i}\hbar}{2}\omega^{\alpha\beta} \tag{6.26}$$

In general we have for multi-indices μ and ν,

$$\int_{\xi(\pm\infty)=0} \mathcal{D}\xi\ \xi^\mu(1)\xi^\nu(0)e^{\frac{\mathrm{i}}{\hbar}S[\xi]} = \left(\frac{-\hbar}{\mathrm{i}}\right)^{|\mu|+|\nu|} \frac{\delta^{|\mu|}}{\delta J_\mu(1)}\frac{\delta^{|\nu|}}{\delta J_\nu(0)} Z\Big|_{J=0} \tag{6.27}$$

From the Taylor expansions of f and g we see that the prefactor of $\xi^\mu(1)\xi^\nu(0)$ in the expansion of $f(x+\xi(1))g(x+\xi(0))$ is

$$\frac{1}{\mu!\nu!}\frac{\partial^{|\mu|}f}{\partial x^\mu}\cdot\frac{\partial^{|\nu|}g}{\partial x^\nu}(x), \tag{6.28}$$

which is symmetric under $\mu_i \leftrightarrow \mu_j$ and $\nu_i \leftrightarrow \nu_j$ if $\mu = (\mu_1,\ldots,\mu_n)$ and $\nu = (\nu_1,\ldots,\nu_n)$. As the Green's function is antisymmetric in the discrete indices, i.e., $G^{\alpha\beta}(t,t') =$

$-G^{\beta\alpha}(t,t')$, the only contribution from the path integral comes from functional derivatives with $|\mu| = |\nu|$. This means that $\frac{\delta}{\delta J(1)}$ and $\frac{\delta}{\delta J(0)}$ have to come in pairs, every pair leading to $G^{\mu_i\nu_j}$ by equation (6.26).

When we substitute the expansions of f and g in $\mathfrak{I}$ we can finally write using equation (6.26),

$$\begin{aligned}
\mathfrak{I} &= \int_{\xi(\pm\infty)=0} \mathcal{D}\xi\ f(x+\xi(1))g(x+\xi(0))e^{\frac{i}{\hbar}S[\xi]} \qquad (6.29)\\
&= \sum_{|\mu|,|\nu|\geq 0} \left(\frac{-\hbar}{i}\right)^{|\mu|+|\nu|} \frac{1}{\mu!\nu!}\frac{\partial^{|\mu|}f}{\partial x^\mu}\cdot\frac{\partial^{|\nu|}g}{\partial x^\nu}(x)\frac{\delta^{|\mu|}}{\delta J_\mu(1)}\frac{\delta^{|\nu|}}{\delta J_\nu(0)}Z\Big|_{J=0}\\
&= \Bigg[fg(x) + i\hbar\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu}(x)G^{\mu\nu}(1,0)\\
&\qquad +\frac{2}{4}(i\hbar)^2\frac{\partial^2 f}{\partial x^\mu\partial x^\kappa}\frac{\partial^2 g}{\partial x^\nu\partial x^\lambda}(x)G^{\mu\nu}(1,0)G^{\kappa\lambda}(1,0)+\mathcal{O}(\hbar^3)\Bigg]\times Z\Big|_{J=0}\\
&= \left[fg(x) + \left(\frac{i\hbar}{2}\right)\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu}(x)\omega^{\mu\nu} + \frac{2}{4}\left(\frac{i\hbar}{2}\right)^2\frac{\partial^2 f}{\partial x^\mu\partial x^\kappa}\frac{\partial^2 g}{\partial x^\nu\partial x^\lambda}(x)\omega^{\mu\nu}\omega^{\kappa\lambda}+\mathcal{O}(\hbar^3)\right]
\end{aligned}$$

Up to all orders, we have

$$\mathfrak{I} = \sum_{k=1}^{\infty}\left(\frac{i\hbar}{2}\right)^k\frac{1}{k!}\omega^{\alpha_1\beta_1}\cdots\omega^{\alpha_k\beta_k}(\partial_{\alpha_1}\cdots\partial_{\alpha_k}f)(\partial_{\beta_1}\cdots\partial_{\beta_k}g) \qquad (6.30)$$

If we compare this with equation (6.1), we have $\mathfrak{I} = f * g$ which proves the Moyal case of theorem 6.6.

We now sketch the proof of the general case in which $\omega_{\alpha\beta}$ is not constant. We expand around $\gamma(t) = x$,

$$\omega_{\alpha\beta}(\tau\xi(t)+x) = \sum_{|\lambda|=0}^{\infty}\frac{1}{\lambda!}\frac{\partial^{|\lambda|}\omega_{\alpha\beta}}{\partial x^\lambda}(x)\tau^{|\lambda|}\xi^\lambda \qquad (6.31)$$

where we used multi-index notation only for λ. With this, the symplectic one-form becomes

$$\begin{aligned}
\alpha(\xi) &= \int_0^1 d\tau\tau\left[\sum_{|\lambda|=0}^{\infty}\frac{1}{\lambda!}\frac{\partial^{|\lambda|}\omega_{\alpha\beta}}{\partial x^\lambda}(x)\tau^{|\lambda|}\xi^\lambda\right]\xi^\alpha d\xi^\beta \qquad (6.32)\\
&= \sum_{|\lambda|=0}^{\infty}\frac{1}{\lambda!}\frac{1}{|\lambda|+2}\frac{\partial^{|\lambda|}\omega_{\alpha\beta}}{\partial x^\lambda}(x)\xi^\lambda\ \xi^\alpha d\xi^\beta
\end{aligned}$$

We will only consider the first two terms of this expression,

$$\alpha(\xi) = \omega_{\alpha\beta}\xi^\alpha d\xi^\beta + \frac{1}{3}\frac{\partial\omega_{\alpha\beta}}{\partial x^\kappa}(x)\xi^\kappa\ \xi^\alpha d\xi^\beta + \cdots \qquad (6.33)$$

with no multi-index notation. The first term leads to the Moyal star product, as we saw before. The second term will be considered as an 'interaction part'. We separate the action,

$$S = S_0 + S_1, \tag{6.34}$$

where we define

$$S_0 := \frac{1}{2}\xi^\alpha G^{-1}_{\alpha\beta}\xi^\beta \equiv \int \mathrm{d}t\mathrm{d}t'\xi^\alpha(t)G^{-1}_{\alpha\beta}(t,t')\xi^\beta(t'), \tag{6.35}$$

$$S_1 := \frac{1}{6}\int \mathrm{d}t\xi^\alpha(t)\xi^\kappa(t)(\partial_\kappa\omega_{\alpha\beta})\frac{\mathrm{d}}{\mathrm{d}t}\xi^\beta(t). \tag{6.36}$$

The perturbation expansion of the path integral is obtained by expanding $e^{\frac{\mathrm{i}}{\hbar}S_1}$. For k-point functions,

$$\begin{aligned}\langle\xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_k}\rangle &= \langle\xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_k}e^{\frac{\mathrm{i}}{\hbar}S_1}\rangle_0 \\ &= \sum_{l=0}^{\infty}\frac{\mathrm{i}^l}{\hbar^l l!}\langle\xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_k}(S_1)^l\rangle_0\end{aligned} \tag{6.37}$$

where the subscript 0 denotes the 'free field average',

$$\langle F[\xi]\rangle_0 := \int \mathcal{D}\xi e^{\frac{\mathrm{i}}{\hbar}S_0}F[\xi] \tag{6.38}$$

Wick's theorem is formulated as[2]

$$\begin{aligned}&\langle\xi^{\mu_1}(t_1)\xi^{\mu_2}(t_2)\cdots\xi^{\mu_{k-1}}(t_{k-1})\xi^{\mu_k}(t_k)\rangle_0 \\ &= \langle\xi^{\mu_1}(t_1)\xi^{\mu_2}(t_2)\rangle_0\cdots\langle\xi^{\mu_{k-1}}(t_{k-1})\xi^{\mu_k}(t_k)\rangle_0 + \text{ permutations.}\end{aligned} \tag{6.39}$$

If we use this expression and the explicit form of S_1, we can consider for example the three point function,

$$\begin{aligned}&\langle\xi^\mu(1)\xi^\nu(1)\xi^\lambda(0)e^{\frac{\mathrm{i}}{\hbar}S_1}\rangle_0 = \langle\xi^\mu(1)\xi^\nu(1)\xi^\lambda(0)\rangle_0 \\ &+\frac{\mathrm{i}}{6\hbar}\int \mathrm{d}t\langle\xi^\mu(1)\xi^\nu(1)\xi^\lambda(0)\xi^\alpha(t)\xi^\kappa(t)(\partial_\kappa\omega_{\alpha\beta})\frac{\mathrm{d}}{\mathrm{d}t}\xi^\beta(t)\rangle_0 + \cdots .\end{aligned} \tag{6.40}$$

The first term equals zero, the second term becomes with Wick's Theorem and equation

[2] The derivative in S_1 leads to a derivative of a Green's function. We neglect the derivatives in Wick's theorem and come back to them in the next section.

(6.26)

$$\begin{aligned}
&\frac{\mathrm{i}}{6\hbar}\int \mathrm{d}t \langle \xi^\mu(1)\xi^\nu(1)\xi^\lambda(0)\xi^\alpha(t)\xi^\kappa(t)(\partial_\kappa\omega_{\alpha\beta})\frac{\mathrm{d}}{\mathrm{d}t}\xi^\beta(t)\rangle_0 \\
&= \frac{\mathrm{i}}{6\hbar}\int \mathrm{d}t \langle \xi^\mu(1)\xi^\alpha(t)\rangle_0 \ \langle \xi^\nu(1)\xi^\kappa(t)\rangle_0 \ \langle \xi^\lambda(0)\frac{\mathrm{d}}{\mathrm{d}t}\xi^\beta(t)\rangle_0 \ (\partial_\kappa\omega_{\alpha\beta}) \\
&\qquad + (\mu \leftrightarrow \nu) + \text{other permutations} \\
&= \frac{\mathrm{i}}{6\hbar}\left(\frac{\mathrm{i}\hbar}{2}\right)^3 G^{\mu\alpha}(1,t)G^{\nu\kappa}(1,t)\frac{\mathrm{d}}{\mathrm{d}t}G^{\lambda\beta}(0,t)(\partial_\kappa\omega_{\alpha\beta}) + (\mu \leftrightarrow \nu) + \ldots \\
&= \frac{\mathrm{i}}{6\hbar}\left(\frac{\mathrm{i}\hbar}{2}\right)^3 \frac{\mathrm{i}}{\hbar}\omega^{\mu\alpha}\omega^{\nu\kappa}\omega^{\lambda\beta}\partial_\kappa\omega_{\alpha\beta}\theta(1-0)\theta(1-0) + (\mu \leftrightarrow \nu) + \ldots \\
&= \frac{1}{3}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \omega^{\nu\kappa}\partial_\kappa\omega^{\mu\lambda} + (\mu \leftrightarrow \nu) + \ldots
\end{aligned} \tag{6.41}$$

In the last line, we used $\omega^{\lambda\beta}\partial_\kappa\omega_{\alpha\beta} = \omega_{\alpha\beta}\partial_\kappa\omega^{\beta\lambda}$. When this three point function is multiplied by $\frac{1}{2}(\partial_\mu\partial_\nu f)(\partial_\lambda g)$ from the Taylor expansions of f and g, we obtain exactly the fourth term in equation (6.2). The factors: there is a 1/3 from the expansion of ω, 1/2 from the expansion of f and a factor 2 from the permutation $\mu \leftrightarrow \nu$. The other terms in equation (6.2) come from (6.30).

In order to fully prove theorem 6.6, one should construct an isomorphism between all Feynman diagrams and Kontsevich's graphs G_n ($n \in \mathbb{N}$). But this requires to work out the full expansion of $\omega_{\alpha\beta}$ (equation (6.31) leading to 4-vertices, 5-vertices and so on. This doesn't seem to be an elegant way to prove theorem 6.6. A more precise statement on this (concerning Poisson manifolds instead of symplectic manifolds) can be found in Cattaneo and Felder [5].

6.4 Regularization and tadpoles

So far we neglected possible divergences in the Feynman amplitudes. In this section, we will consider the finiteness of the integrals occurring in the expansion (6.37) of the k-point function.

In this expansion, every factor S_1 corresponds to a 3-vertex. It carries an integration $\int \mathrm{d}t$ and contains a derivative of a field $\frac{\mathrm{d}}{\mathrm{d}t}\xi^\beta(t)$. It is possible to write for the derivative interaction,

$$\langle \xi^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t'}\xi^\beta(t')\rangle_0 = \frac{\mathrm{d}}{\mathrm{d}t'}\langle \xi^\alpha(t)\xi^\beta(t')\rangle_0 \tag{6.42}$$

where $\langle \xi^\alpha(t)\xi^\beta(t')\rangle_0$ equals a Green's function. In our case, the Green's functions are proportional to Heavyside step functions and a time derivative leads to a delta function which cancels the integration over t,

$$\frac{\mathrm{d}}{\mathrm{d}t'}G^{\alpha\beta}(t,t') = \omega^{\alpha\beta}\delta(t-t'). \tag{6.43}$$

We are left without integration and obtain a constant, finite expression in the expansion of the k-point function.

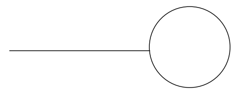

Figure 6.4.1

Problems do arise, when one considers expressions in the expansion, corresponding to Feynman diagrams which contain tadpoles (figure 6.4.1). A tadpole in a diagram corresponds to an amplitude which contains an ill-defined factor $G^{\alpha\kappa}(t,t)$. It comes from contracting the two fields ξ^α and ξ^κ in S_1. We use the point-splitting method to regularize this factor and define

$$G^{\alpha\kappa}(t,t) := \lim_{\epsilon\to 0} G^{\alpha\kappa}(t,t+\epsilon). \tag{6.44}$$

We will use the definition of the Heaviside step function, in terms of its Fourier transform,

$$\theta(t) = \int \mathrm{d}k \frac{e^{ikt}}{k+i\eta} \tag{6.45}$$

where $\eta \downarrow 0$. When we combine this with the definition of the Green's function, we have

$$\begin{aligned}
G^{\alpha\kappa}(t,t) &= \lim_{\epsilon\to 0} \omega^{\alpha\kappa}[\theta(-\epsilon) - \theta(\epsilon)] \qquad (6.46)\\
&= \lim_{\epsilon\to 0} \omega^{\alpha\kappa} \int \mathrm{d}k \frac{e^{-\mathrm{i}k\epsilon} - e^{\mathrm{i}k\epsilon}}{k+\mathrm{i}\eta}\\
&= \lim_{\epsilon\to 0} -2\omega^{\alpha\kappa} \int \mathrm{d}k \frac{\sin k\epsilon}{k+\mathrm{i}\eta}\\
&= \lim_{\epsilon\to 0} -2\omega^{\alpha\kappa} \int \mathrm{d}k\epsilon \frac{\sin k\epsilon}{k\epsilon+\mathrm{i}\epsilon\eta}\\
&= -2\omega^{\alpha\kappa} \int \mathrm{d}x \frac{\sin x}{x}.
\end{aligned}$$

The last integral is convergent and equal to a constant. The only t-dependent factor which remains in the tadpole amplitude, equals

$$\int \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \xi^\beta(t). \tag{6.47}$$

This integral vanishes because ξ^β is zero at $\pm\infty$ by definition. We see that the tadpole amplitudes do not contribute in the semiclassical expansion when they are regularized by the point-splitting method.

6.5 Associativity

To examine the associativity of the star product defined by the path integral in section 6.2, we will consider two expressions, $f \star (g \star h)$ and $(f \star g) \star h$. In terms of the path integral these two read

$$f \star (g \star h) = \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma \int_{\tilde{\gamma}(\infty)=\gamma(0)} \mathcal{D}\tilde{\gamma}\; f(\gamma(1))g(\tilde{\gamma}(1))h(\tilde{\gamma}(0))e^{\frac{i}{\hbar}S[\gamma]+S[\tilde{\gamma}]} \tag{6.48}$$

and

$$(f \star g) \star h = \int_{\tilde{\gamma}(\pm\infty)=x} \mathcal{D}\tilde{\gamma} \int_{\gamma(\infty)=\tilde{\gamma}(1)} \mathcal{D}\gamma\; f(\gamma(1))g(\gamma(0))h(\tilde{\gamma}(0))e^{\frac{i}{\hbar}S[\gamma]+S[\tilde{\gamma}]} \tag{6.49}$$

with the action from theorem 6.6,

$$S[\gamma] = \frac{1}{2}\int \mathrm{d}t\; C_{\alpha\beta}(\gamma(t))\gamma^{\alpha}(t)\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{\beta}(t), \tag{6.50}$$

where we have defined

$$C_{\alpha\beta}(\gamma(t)) = \int_0^1 \mathrm{d}\tau\tau\; \omega_{\alpha\beta}(\tau(\gamma(t)-x)+x). \tag{6.51}$$

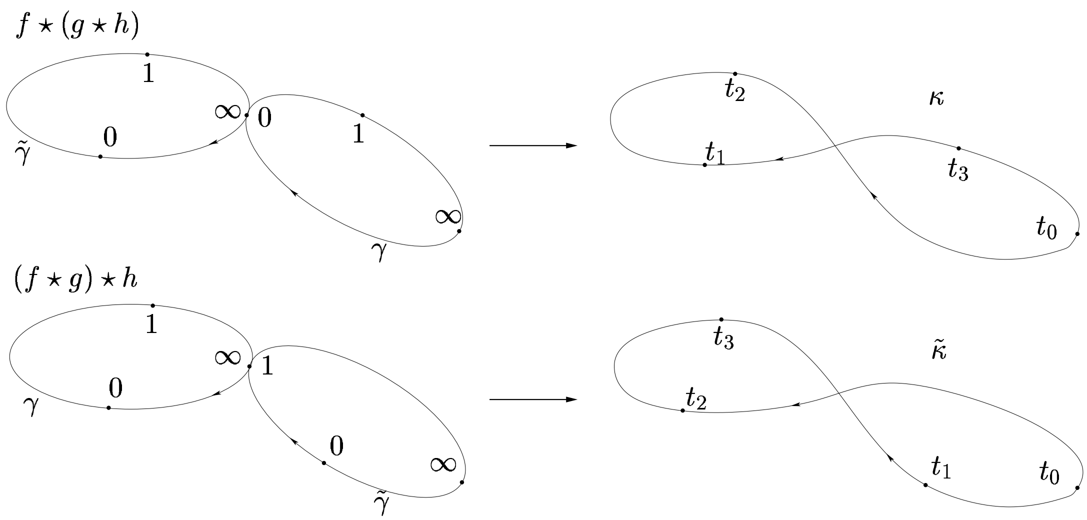

Figure 6.5.1

At the left-hand side of figure 6.5.1, the loops in the path integrals are represented. The two actions in the exponent, together with the boundary conditions between γ and $\tilde{\gamma}$, makes it possible to write the path integrals as integrals over one loop, say κ for $f \star (g \star h)$ and $\tilde{\kappa}$ for $(f \star g) \star h$. We choose new points t_0, t_1, t_2 and t_3 on $\mathbb{R}$. For the corresponce with the old labels, see figure 6.5.1. The integrals become

$$f \star (g \star h) = \int_{\kappa(t_0)=x} \mathcal{D}\kappa f(\kappa(t_3)) g(\kappa(t_2)) h(\kappa(t_1)) \times \exp\left(\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}t\ C_{\alpha\beta}(\kappa(t))\kappa^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}\kappa^\beta(t)\right) \quad (6.52)$$

$$(f \star g) \star h = \int_{\tilde\kappa(t_0)=x} \mathcal{D}\tilde\kappa f(\tilde\kappa(t_3)) g(\tilde\kappa(t_2)) h(\tilde\kappa(t_1)) \times \exp\left(\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}t\ C_{\alpha\beta}(\tilde\kappa(t))\tilde\kappa^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde\kappa^\beta(t)\right). \quad (6.53)$$

We reparametrize the first integral and let $t \mapsto \tilde{t} := t + \epsilon(t)$. Here ϵ is a smooth function on $\mathbb{R}$ with $\epsilon(t) = 0$ in a neighbourhood of t_0. We define ϵ such that $\tilde\kappa(t) = \kappa(\tilde{t})$, which makes it possible to write the first integral as

$$f \star (g \star h) = \int_{\tilde\kappa(t_0)=x} \mathcal{D}\tilde\kappa f(\tilde\kappa(t_3)) g(\tilde\kappa(t_2)) h(\tilde\kappa(t_1)) \times \exp\left(\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}\tilde{t}\ C_{\alpha\beta}(\tilde\kappa(t))\tilde\kappa^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde\kappa^\beta(t)\right). \quad (6.54)$$

As $\mathrm{d}t$ transforms opposite to $\frac{\mathrm{d}}{\mathrm{d}t}$ under $t \mapsto \tilde{t}$ in the action, the path integral equals exactly $(f \star g) \star h$. This proves associativity of Kontsevich's star product.

Chapter 7

Spectral triples and deformation quantization

In the last two chapters we concentrated on deformation quantization. This turned out to be a succesful approach to noncommutative geometry. It remained, however, too difficult to construct operator algebras for noncommutative geometries other than the noncommutative plane or the noncommutative cylinder. In this chapter we construct a representation of the deformed function algebra in a Hilbert space.

7.1 GNS-construction

Before we give the GNS-construction (Gel'fand-Naimark-Segal) to obtain representations of involutive algebras, we will first review some definitions concerning star products on function algebras.

7.1.1 Star product

We start with the algebra of smooth functions $C^\infty(M)$ on an n-dimensional Poisson manifold (M, Λ). The usual (pointwise) product in this algebra is deformed to a star product $*$,

$$f * g := \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}\hbar}{2}\right)^n B_n(f,g) \tag{7.1}$$

for $f, g \in C^\infty(M)$. Here B_n are bidifferential operators which satisfy

$$B_n(f,g) = (-1)^n B_n(g,f), \tag{7.2}$$

and furthermore $B_0(f,g) = fg$ and $B_1(f,g) - B_1(g,f) = \mathrm{i}\{f,g\} \equiv \mathrm{i}\Lambda(\mathrm{d}f, \mathrm{d}g)$. It is easily seen that these conditions are satisfied in the case of Kontsevich's star product (equation (6.2)). Thus in particular the conditions are satisfied in the case of the Moyal star product (equation (6.1)).

We consider the algebra $C^\infty(M)[[\hbar]]$ (compare Kontsevich [10]) instead of $C^\infty(M)$. It consists of formal power series in $\hbar$, i.e., we have $f \in C^\infty(M)[[\hbar]]$ if

$$f = \sum_{r \in \mathbb{N}} \hbar^r f_r \tag{7.3}$$

where $f_r \in C^\infty(M)$. In this section we will denote $C^\infty(M)[[\hbar]]$ equipped with the star product by $\mathcal{A}[[\hbar]]$.

Let us denote the complex conjugate of a function $f \in \mathcal{A}[[\hbar]]$ by $\overline{f}$.

Definition 7.1 A star product $*$ on $\mathcal{A}[[\hbar]]$ is called **hermitian** if $\overline{f * g} = \overline{g} * \overline{f}$.

For example, Kontsevich's star product is a hermitian star product. There one has $\overline{B_n(f,g)} = B_n(\overline{f},\overline{g})$ which yields

$$\overline{f \star g} = \sum_{n=0}^{\infty} \left(\frac{-i\hbar}{2}\right)^n B_n(\overline{f},\overline{g}) = \overline{g} \star \overline{f} \tag{7.4}$$

for all $f, g \in \mathcal{A}[[\hbar]]$. As we have furthermore that $\overline{\overline{f}} = f$ and $\overline{\lambda f} = \overline{\lambda}\,\overline{f}$ for all $\lambda \in \mathbb{C}$, we conclude that $\mathcal{A}[[\hbar]]$ is an involutive algebra with the involution given by the complex conjugation. The unit in $\mathcal{A}[[\hbar]]$ is given by 1, since $f * 1 = 1 * f = f$ for all $f \in \mathcal{A}[[\hbar]]$.

7.1.2 Construction of representations

The GNS-construction makes it possible to obtain representations of C^*-algebras in a Hilbert space $\mathcal{H}$. It can be found for example in Bratteli and Robinson [4] and in Landsman [13]. Unfortunately, the algebra $\mathcal{A}[[\hbar]]$ is not a C^*-algebra. For example, if we extend the supremum norm on $C^\infty(M)$ to a norm on $\mathcal{A}[[\hbar]]$ this *does* define a vector space norm, but it does not make $\mathcal{A}[[\hbar]]$ a normed algebra. It seems, however, to be possible to apply the GNS-construction to involutive algebras (Waldmann, [25]). In this section, we will construct representations of $\mathcal{A}[[\hbar]]$.

Apart from the involutive algebra, the GNS-construction requires a positive linear functional on this algebra. As functionals from $\mathcal{A}[[\hbar]]$ to $\mathbb{C}$ are not so interesting when one considers formal power series in $\hbar$, we will consider functionals

$$\omega\colon\ \mathcal{A}[[\hbar]] \to \mathbb{C}[[\hbar]]. \tag{7.5}$$

There is an order structure on $\mathbb{C}[[\hbar]]$ inherited from $\mathbb{C}$, by the definition $\sum_{r=r_0}^{\infty} \hbar^r a_r > 0$ if $a_{r_0} > 0$ ($a_r \in \mathbb{C}$).

Definition 7.2 A linear functional ω over the $*$-algebra $\mathcal{A}[[\hbar]]$ is called **positive** if

$$\omega(\overline{f} * f) \geq 0$$

for all $f \in \mathcal{A}[[\hbar]]$.

We derive, using the linearity of ω, the Cauchy-Schwarz inequality for positive linear functionals ω over $\mathcal{A}[[\hbar]]$,

$$|\omega(\overline{f} * g)|^2 \leq \omega(\overline{f} * f)\omega(\overline{g} * g). \tag{7.6}$$

In this section we assume that there exists a positive linear functional $\omega\colon\ \mathcal{A}[[\hbar]] \to \mathbb{C}[[\hbar]]$. Its existence is discussed in the next section.

First, introduce a scalar product on $\mathcal{A}[[\hbar]]$ with the aid of ω, namely

$$\langle f, g\rangle = \omega(\overline{f} * g) \tag{7.7}$$

for all $f, g \in \mathcal{A}[[\hbar]]$. Therewith we define $I_\omega \subset \mathcal{A}[[\hbar]]$ by

$$I_\omega := \{f \in \mathcal{A}[[\hbar]] : \omega(\overline{f} * f) = 0\} \tag{7.8}$$

One can show by using Cauchy-Schwarz inequality that I_ω is a left ideal of $\mathcal{A}[[\hbar]]$. Define equivalence classes ψ_f by

$$\psi_f := \{f + h : h \in I_\omega\}, \qquad (f \in \mathcal{A}[[\hbar]]). \tag{7.9}$$

The set of these classes $\mathcal{A}[[\hbar]]/I_\omega$ form a complex vector space when equipped with the operations inherited from $\mathcal{A}[[\hbar]]$, i.e.,

$$\psi_f + \psi_g := \psi_{f+g}, \qquad \alpha\psi_f := \psi_{\alpha\psi}, \qquad (\alpha \in \mathbb{C}[[\hbar]]). \tag{7.10}$$

Furthermore, we can define an inner product on $\mathcal{A}[[\hbar]]/I_\omega$ by

$$\langle \psi_f, \psi_g\rangle := \langle f, g\rangle = \omega(\overline{f} * g). \tag{7.11}$$

Once again using Cauchy-Schwarz inequality one can show that this definition is independent of the representative ψ_f. The inner product space $(\mathcal{A}[[\hbar]]/I_\omega, \langle\ ,\ \rangle)$ may be completed, i.e., linearly embedded as a dense subspace of a Hilbert space. This conserves the inner product and the completion, being a Hilbert space, is denoted by $\mathcal{H}_\omega$. The norm on $\mathcal{H}_\omega$ is

$$\|\psi_g\| = \langle \psi_g, \psi_g\rangle^{1/2}, \qquad (\psi_g \in \mathcal{H}_\omega). \tag{7.12}$$

Next, we will see that the Hilbert space $\mathcal{H}_\omega$ is a representation space of $\mathcal{A}[[\hbar]]$. The representation

$$\pi_\omega :\ f \in \mathcal{A}[[\hbar]] \to \pi_\omega(f) \in \mathcal{B}(\mathcal{H}_\omega) \tag{7.13}$$

is defined by

$$\pi_\omega(f)\psi_g = \psi_{f*g}. \tag{7.14}$$

This definition is independent of the representative ψ_g because

$$\pi_\omega(f)\psi_{g+h} = \psi_{f*g+f*h} = \psi_{f*g} = \pi_\omega(f)\psi_g.$$

for $h \in I_\omega$. The algebraic properties of π_ω follow easily, since $\pi_\omega(f_1)\pi_\omega(f_2)\psi_g = \psi_{f_1 * f_2 * g} = \pi_\omega(f_1 * f_2)\psi_g$. Hence,

$$\pi_\omega(f_1 * f_2) = \pi_\omega(f_1)\pi_\omega(f_2). \tag{7.15}$$

This completes the construction of the representation $(\mathcal{H}_\omega, \pi_\omega)$.

7.2 Noncommutative worldsheet

We have now reached the delicate question wether there exists a positive linear functional over the algebra $\mathcal{A}[[\hbar]]$. Waldmann shows in [25] that there are sufficiently many positive linear functionals under certain conditions. We will not go into details, but examine a concrete example of a positive linear functional in the special case that M is the worldsheet of a string.

Let (Σ, Λ) be a two-dimensional compact Poisson manifold, i.e. the worldsheet of a bosonic string. It is also a Riemannian manifold with metric η. The function algebra on Σ will be deformed to $\mathcal{A}[[\hbar]] = (C^\infty(\Sigma)[[\hbar]], \star)$ by Kontsevich's star product (see equation (6.6)). Up to second order it reads in local coordinates $(\xi^\alpha) = (\tau, \sigma)$:

$$\begin{aligned} f \star g \;\;=\;\; & fg + \left(\frac{\mathrm{i}\hbar}{2}\right)\Lambda^{\alpha\beta}(\partial_\alpha f)(\partial_\beta g) + \frac{1}{2}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \Lambda^{\alpha_1\beta_1}\Lambda^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2} f)(\partial_{\beta_1}\partial_{\beta_2} g) \qquad (7.16) \\ & + \frac{1}{3}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \left(\Lambda^{\alpha_1\beta_1}(\partial_{\beta_1}\Lambda^{\alpha_2\beta_2})\big[(\partial_{\alpha_1}\partial_{\alpha_2} f)(\partial_{\beta_2} g) - (\partial_{\alpha_2} f)(\partial_{\alpha_1}\partial_{\beta_2} g)\big]\right) + \mathcal{O}(\hbar^3). \end{aligned}$$

where $\Lambda^{\alpha\beta} = \Lambda^{\alpha\beta}(\tau, \sigma)$.

The Polyakov action for the noncommutative worldsheet becomes

$$S_p^\star = \frac{1}{4\pi}\int_\Sigma \mathrm{d}\mu \; g_{\mu\nu} \star \partial_\alpha X^\mu \star \partial^\alpha X^\nu. \qquad (7.17)$$

where the measure is $\mathrm{d}\mu := \mathrm{d}\tau \mathrm{d}\sigma \sqrt{\det \eta}$. This action was considerd by Kamani in [8] in the case of a flat background. He expanded the action in powers of $\hbar$, but only for constant coefficients $\Lambda^{\alpha\beta}$, which is the Moyal star product. The coordinates (τ, σ), however, do not necessarily form a natural chart on Σ in which the coefficients $\Lambda^{\alpha\beta}$ are constant. Therefore, we consider here Kontsevich's star product instead of the Moyal star product.

7.2.1 Representation and Dirac operator

We make an attempt to describe the noncommutative worldsheet using a spectral triple which is a deformation of the canonical triple $(C^\infty(\Sigma), L^2(M, S), \not{\partial})$. We do this by using the GNS-construction to obtain a representation of $\mathcal{A}[[\hbar]]$ in a Hilbert space $\mathcal{H}$. Then, we define an unbounded self-adjoint operator.

We define a linear functional ω on the algebra $\mathcal{A}[[\hbar]]$ by

$$\omega(f) = \int \mathrm{d}\mu \; f(\tau, \sigma) \qquad (7.18)$$

Note that this is indeed a positive functional because (obviously) $\omega(\overline{f} \star f) \geq 0$. Furthermore, we prove that $\omega(\overline{f} \star f) = 0$ if and only if $f = 0$. Consider for this the expansion of $\overline{f} \star f$ in powers of $\hbar$:

$$\begin{aligned}
\overline{f} \star f &= \sum_n \left(\frac{\mathrm{i}\hbar}{2}\right)^n B_n(\overline{f}, f) \qquad (7.19)\\
&= \sum_{n,r,s} \left(\frac{\mathrm{i}}{2}\right)^n \hbar^{r+s+n} B_n(\overline{f}_r, f_s)\\
&= \overline{f}_0 f_0 + \hbar\left[\overline{f}_1 f_0 + \overline{f}_0 f_1 + \frac{\mathrm{i}}{2} B_1(\overline{f}_0, f_0)\right] + \hbar^2 \left[\overline{f}_1 f_1 + \overline{f}_0 f_2 + \overline{f}_2 f_0\right.\\
&\quad \left. + \left(\frac{\mathrm{i}}{2}\right)\left(B_1(\overline{f}_1, f_0) + B_1(\overline{f}_0, f_1)\right) + \left(\frac{\mathrm{i}}{2}\right)^2 B_2(\overline{f}_0, f_0)\right] + \mathcal{O}(\hbar^3)
\end{aligned}$$

We now use the fact that the integral of this expression is also a formal power series in $\hbar$. For the first term, we have $\overline{f}_0 f_0(\tau, \sigma) \geq 0$ for all τ, σ. Thus, $\int \mathrm{d}\mu\, \overline{f}_0 f_0 = 0$ implies $f_0 = 0$. We substitute this in the right hand side of (7.19), and the $\hbar^2$-term gives $f_1 = 0$, using the same argument as above. We apply this procedure to higher order terms in (7.19) to obtain $f_r = 0$ ($r \in \mathbb{N}$). Hence $f = 0$.

We conclude that the ideal I_ω in the GNS-construction is quite trivial, $I_\omega = \{0\}$. It gives us a representation π_ω of $\mathcal{A}[[\hbar]]$ in the Hilbert space $\mathcal{H}_\omega = \mathcal{A}[[\hbar]]/I_\omega = \mathcal{A}[[\hbar]]$.

In analogy with the canonical triple, we extend the Hilbert space by introducing spinors. We define a new Hilbert space $\mathcal{H} := \mathcal{H}_\omega \oplus \mathcal{H}_\omega$ whose elements are vectors

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \qquad (7.20)$$

The components χ_a are elements in $\mathcal{A}[[\hbar]]$. The positive linear functional ω defines an inner product on $\mathcal{H}$ by

$$\langle \chi, \phi \rangle = \int \mathrm{d}\mu\, \overline{\chi}_a \star \phi_a, \qquad (\chi, \phi \in \mathcal{H}), \qquad (7.21)$$

where a summation over the spinor index a is understood. The action of $\pi_\omega(f)$ on $\mathcal{H}$ is componentswise, i.e.,

$$\pi_\omega(f) \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} f \star \chi_1 \\ f \star \chi_2 \end{pmatrix} \qquad (7.22)$$

The Dirac operator in the canonical triple was defined as $\not{\partial} = \gamma^\alpha \partial_\alpha$. We assume $\eta_{\alpha\beta} = \delta_{\alpha\beta}$ and represent the gamma matrices as

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}. \qquad (7.23)$$

The Dirac operator then reads

$$\not{\partial} = \begin{pmatrix} 0 & \partial_\tau + \mathrm{i}\,\partial_\sigma \\ \partial_\tau - \mathrm{i}\,\partial_\sigma & 0 \end{pmatrix}. \qquad (7.24)$$

It turns out that we can write derivatives on $\mathcal{A}[[\hbar]]$ using brackets $[\ ,\ ]_\star$, defined by

$$[f,g]_\star := f\star g - g\star f, \qquad (f,g\in\mathcal{A}[[\hbar]]) \tag{7.25}$$

With this and formula (7.16) we derive

$$\mathrm{ad}_\tau f(\tau,\sigma) := [\tau, f(\tau,\sigma)]_\star = \mathrm{i}\hbar\Lambda^{12}\partial_\sigma f(\tau,\sigma), \tag{7.26}$$

and

$$\mathrm{ad}_\sigma f(\tau,\sigma) := [\sigma, f(\tau,\sigma)]_\star = -\mathrm{i}\hbar\Lambda^{12}\partial_\tau f(\tau,\sigma). \tag{7.27}$$

for $f\in\mathcal{A}[[\hbar]]$. This motivates the following definition of the Dirac operator D,

$$D := \frac{1}{\hbar\Lambda^{12}}\begin{pmatrix} 0 & \mathrm{i}\ \mathrm{ad}_\sigma + \mathrm{ad}_\tau \\ \mathrm{i}\ \mathrm{ad}_\sigma - \mathrm{ad}_\tau & 0\end{pmatrix}. \tag{7.28}$$

Its action on the components of vectors in $\mathcal{H}$ is defined by equation (7.26) and (7.27). Therefore, if v is an eigenfunction of $\partial\!\!\!/$ with eigenvalue μ in the canonical triple, it is also an eigenfunction of the operator D. As we are considering formal power series, all functions in $\mathcal{H}$ of the form $\sum_{r=r_i}^{r_f}\hbar^r v$ are eigenfunctions of D with eigenvalue μ, for arbitrary integers r_i and r_f. However, as these vectors are all dependent, the eigenspace consisting of eigenvectors with eigenvalue μ remains finite dimensional.

This construction of a representation of $\mathcal{A}[[\hbar]]$ into the Hilbert space $\mathcal{H}$, and the construction of the Dirac operator, makes a connection possible between deformation quantization and Connes' theory of spectral triples. However, it is not direct that all axioms of a spectral triple are satisfied in this case. If we can impose a C^*-algebraic structure on $\mathcal{A}[[\hbar]]$, a connection is possible with Connes' theory of spectral triples.

If the triple $(\mathcal{A}[[\hbar]],\mathcal{H},D)$ can be made a spectral triple, it has dimension 2. Then, we can write the noncommutative version of the Polyakov action as a Dixmier trace,

$$S_p^\star = ⨍ g_{\mu\nu}\star[D,X^\mu]_\star\star[D,X^\nu]_\star = \mathrm{tr}_\omega\big(g_{\mu\nu}\star[D,X^\mu]_\star\star[D,X^\nu]_\star|D|^{-2}\big). \tag{7.29}$$

The proof of this then relies on theorem 4.4 and the close connection between D and $\partial\!\!\!/$. We note that when $\hbar=0$, one recovers the original (commutative) Polyakov action.

Chapter 8

Conclusions and Outlook

In this thesis, we considered several approaches to deformation quantization. In some special cases we used Weyl correspondence to construct an operator algebra acting on a Hilbert space. This led to the definition of the noncommutative plane and the noncommutative cylinder.

For a general Poisson manifold (M, Λ), however, this correspondence was not applicable. A generalization of the star product was required and we defined the covariant star product and Kontsevich's star product. We described the latter using path integral methods. The advantage of this is that the path integral gives a physical approach to deformation quantization. Furthermore, it provides an easy way to show the associativity of Kontsevich's star product. The disadvantage of the path integral approach is that it is difficult to prove exactly that the semiclassical expansion of the path integral equals Kontsevich's star product. Furthermore, the regularization of the tadpoles in the expansion was done by the point-splitting method which is not really an elegant regularization method. Maybe other methods can be applied to regularize the tadpoles.

We gave the GNS-construction for the deformed function algebra $C^\infty(M)$ (i.e., equipped with star product) to obtain a representation of this algebra. We note that although the GNS-construction was applied in two dimensions, extension to general dimension is easily made. We prepared the construction of a spectral triple that describes the noncommutative worldsheet as it was introduced by Kamani. If the triple $(\mathcal{A}[[\hbar]], \mathcal{H}, D)$ is a spectral triple, then the noncommutative version of the Polyakov action can be written as a Dixmier trace.

What are the consequences of the noncommutative worldsheet to space-time geometry? Apparently, this is a noncommutative geometry but can it be described using the same techniques?

Furthermore, the correspondence between the covariant and Kontsevich's star product should be examined. Although they both provide an associative star product on a Poisson manifold, the two definitions are quitte different.

In chapter 5, we constructed a spectal triple describing a special case of a noncommutative worldsheet: the noncommutative cylinder. We prepared the construction of a spectral triple describing a noncommutative worldsheet in general in chapter 7. If the triple constructed here can be made a spectral triple, several extensions are possible. It is then interesting to consider the superstring involving also fermions on the worldsheet. This extension can be made in analogy with the original superstring action where one adds a fermionic term $\langle \psi^\mu, \partial\!\!\!/ \psi_\mu \rangle$ to the action. With the help of the inner product on $\mathcal{H}$ one can define the fermionic action for the noncommutative superstring worldsheet

$$S_F^* := \langle \psi^\mu, D\psi_\mu \rangle.$$

Note that this inner product involves a star product and that it coincides with the fermionic term in Kamani's action for the noncommutative worldsheet.

The question arises if it is possible to define a Fredholm module instead of a spectral triple which describes the noncommutative worldsheet. Such a Fredholm module would then contain the conformal structure of the noncommutative worldsheet.

The algebraic generalization of the distance between two points in Riemannian geometry is provided by Connes' distance function. It would be interesting to consider this function for the spectral triple defined in section 5.6.

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Appendix A

Multi-index notation

To shorten the notation for expressions which contain many indices, we introduce multi-index notation in our formulas. In particular for (pseudo-)differential operators on a manifold this notation is very convenient.

Let M be a manifold with local coordinates $(x^1, \cdots, x^n)$, $n = \dim M$. A **multi-index** α is an n-tuple of integers α_j with $j = 1, \cdots, n$. $|\alpha|$ denotes its **length** $\alpha_1 + \cdots + \alpha_n$. We define $\alpha! := \alpha_1! \cdots \alpha_n!$.

For monomial expressions in the coordinates we write $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. This can be extended to partial differentiation as

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

The Taylor expansion of a smooth function f on M around $x = x_0$, is given by

$$f(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x_0)\ (x - x_0)^\alpha.$$

Appendix B

Computation of a path integral by discretization

We will show that the path integral introduced in theorem 6.5 can be computed exactly by discretization and that it equals the Moyal star product.

Let M be a two-dimensional manifold and $f, g \in C^\infty(M)$. We show that the Moyal star product $f * g(x)$ for $f, g \in C^\infty(M)$ is given by the path integral

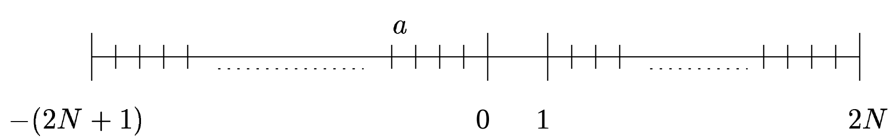
$$Z = \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f((\gamma(1))g(\gamma(0))e^{\frac{\mathrm{i}}{\hbar}S[\gamma]},$$

where the action is defined as

$$S[\gamma] = \frac{1}{2}\int \mathrm{d}t\omega_{\alpha\beta}\gamma^\alpha \frac{\mathrm{d}}{\mathrm{d}t}\gamma^\beta.$$

Here $\omega_{\alpha\beta}$ is a constant, anti-symmetric 2×2 matrix ($\alpha, \beta = 1, 2$).

The time interval is discretized by $4N + 2$ points[1]. We have $t_k = ka$ for ($k = -(2N+1), \ldots, 2N$) where the interval of length a is supposed to be small (see figure below). We write the fields γ in discretized form, $\gamma_k = \gamma(t_k)$. For a smooth function



$\gamma(t)$ the time derivative can be approximated by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t_k) = \frac{\gamma_k - \gamma_{k-1}}{a}.$$

With this, the Lagrangian can be discretized as

$$\mathcal{L}_{\text{discr}}(\gamma_k, \gamma_{k-1}) = \frac{1}{2}\omega_{\alpha\beta}\gamma_k^\alpha \left(\frac{\gamma_k^\beta - \gamma_{k-1}^\beta}{a}\right)$$

[1]The reason for this particular number of steps will become clear in a moment. For now, just note that when $N \to \infty$ time stretches from $-\infty$ to $+\infty$.

and summed up to obtain the action

$$S_{\text{discr}}[\gamma] = \sum_{-2N \le k \le 2N} a\mathcal{L}_{\text{discr}}(\gamma_k, \gamma_{k-1}).$$

Due to the anti-symmetry of $\omega_{\alpha\beta}$, the Lagrangian can be written as

$$\mathcal{L}_{\text{discr}}(\gamma_k, \gamma_{k-1}) = -\frac{1}{2a}\omega_{\alpha\beta}\gamma_k^\alpha \gamma_{k-1}^\beta$$

and thus the action reads

$$S_{\text{discr}}[\gamma] = -\frac{1}{2}\sum_{-2N \le k \le 2N} \omega_{\alpha\beta}\gamma_k^\alpha \gamma_{k-1}^\beta.$$

Note that the action is independent of the small unit a as there is only a first order time derivative in the Lagrangian. Moreover, we can write $f(\gamma(1))g(\gamma(0))$ as $f(\gamma_1)g(\gamma_0)$ as 1 and 0 were just two chosen points on $\mathbb{R}$. The path integral becomes in discretized form,

$$\begin{aligned} Z &= \lim_{N\to\infty} C^{4N} \int \mathrm{d}^2\gamma_{-2N}\mathrm{d}^2\gamma_{-2N+1}\cdots \mathrm{d}^2\gamma_{2N-2}\mathrm{d}^2\gamma_{2N-1} \\ &\qquad \times f(\gamma_1)g(\gamma_0)e^{-\frac{\mathrm{i}}{2\hbar}\omega_{\alpha\beta}\sum_{-2N\le k\le 2N}\gamma_k^\alpha\gamma_{k-1}^\beta} \end{aligned}$$

where the constants C are renormalization factors, $C = \omega_{01}/(4\pi\hbar)$. The boundary conditions are $\gamma_{-2N-1} = \gamma_{2N} = x$. We can include them in the path integral by making the sum explicit,

$$\begin{aligned} Z &= \lim_{N\to\infty} C^{4N} \int \mathrm{d}^2\gamma_{-2N}\mathrm{d}^2\gamma_{-2N+1}\cdots \mathrm{d}^2\gamma_{2N-2}\mathrm{d}^2\gamma_{2N-1} f(\gamma_1)g(\gamma_0) \\ &\qquad \times \exp\left(-\frac{\mathrm{i}}{2\hbar}\omega_{\alpha\beta}\left(x^\alpha\gamma_{2N-1}^\beta + \gamma_{2N-1}^\alpha\gamma_{2N-2}^\beta + \cdots + \gamma_{-2N+1}^\alpha\gamma_{-2N}^\beta + \gamma_{-2N}^\alpha x^\beta\right)\right). \end{aligned}$$

The integration over γ_{-2N} leads to a delta function, as it can be separated from the path integral,

$$\int \mathrm{d}^2\gamma_{-2N}\exp\left(-\frac{\mathrm{i}}{2\hbar}\omega_{\alpha\beta}\left(\gamma_{-2N}^\alpha(x^\beta - \gamma_{-2N+1}^\beta)\right)\right) = \left(\frac{4\pi\hbar}{\omega_{01}}\right)^2 \delta^2(\gamma_{-2N+1} - x).$$

This means that two integrations in the path integral are cancelled. The first one is the integration over γ_{-2N} and the second one is over γ_{-2N+1}. It cancels by the delta function, substituting x for γ_{-2N+1}.

This procedure can be applied once again to γ_{-2N+2} and γ_{-2N+3} etc. until one arrives at the integration over γ_0. The function $g(\gamma_0)$ then stops the cancellation of integrations. As every step in the described procedure requires two integrations, we must have an even number of terms on the right of the term $\gamma_1^\alpha\gamma_0^\beta$ in the path integral. By our special choice of time steps, we have $2N$ terms, which is indeed even. On the positive time side, i.e., on the left of the term $\gamma_1^\alpha\gamma_0^\beta$, we can apply the same procedure to cancel integrations until we arrive at γ_1. Here, the number of steps is $(2N-1)-1$.

When we combine all this, we get

$$Z = \lim_{N\to\infty} C^2 \int \mathrm{d}^2\gamma_0 \mathrm{d}^2\gamma_1 f(\gamma_1) g(\gamma_0) e^{-\frac{\mathrm{i}}{2\hbar}\omega_{\alpha\beta}(x^\alpha\gamma_1^\beta+\gamma_1^\alpha\gamma_0^\beta+\gamma_0^\alpha x^\beta)}.$$

Notice that the integral in this expression is independent of N. If we substitute $\omega_{01} = -4/\theta$, and denote coordinates on M by

$$(x^\alpha) = (\tau,\sigma), \quad (\gamma_1^\alpha) = (\tau',\sigma'), \quad (\gamma_0^\alpha) = (\tau'',\sigma''),$$

we obtain

$$\begin{aligned} Z &= \frac{1}{\pi^2\hbar^2\theta^2}\int \mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'') \\ &\quad \times \exp\left(\frac{2\mathrm{i}}{\hbar\theta}\big(\tau\sigma' - \sigma\tau' + \tau'\sigma'' - \sigma'\tau'' + \tau''\sigma - \sigma''\tau\big)\right) \\ &= \frac{1}{\pi^2\hbar^2\theta^2}\int \mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'') \\ &\quad \times \exp\left(\frac{-2\mathrm{i}}{\hbar\theta}\big(\sigma(\tau'-\tau'') + \sigma'(\tau''-\tau) + \sigma''(\tau-\tau')\big)\right). \end{aligned}$$

The last line is just Baker's formula (5.6) for the Moyal star product $f * g(\tau,\sigma)$. With this, we have proved theorem 6.5.

Appendix C

Weyl correspondence on a cylinder

We prove that the algebra of smooth functions $C^\infty(C)$ on a cylinder C is homomorphic to the operator algebra constructed via Weyl corresponce. Let (τ, σ) be coordinates on C.

In this appendix, we denote the operator corresponding to the function f by $\boldsymbol{W}_f$. With this, Weyl correspondence reads

$$\boldsymbol{W}_f(\boldsymbol{\tau}, \boldsymbol{\sigma}) = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \int \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}\sigma f(\tau, \sigma) e^{\mathrm{i}\xi(\boldsymbol{\tau}-\tau)+\mathrm{i}n(\boldsymbol{\sigma}-\sigma)}.$$

We have to prove that $\boldsymbol{W}$ is a homomorphism, i.e., that $\boldsymbol{W}_{f*g} = \boldsymbol{W}_f \boldsymbol{W}_g$.

As $\tau \in \mathbb{R}$ and $\sigma \in [0, 2\pi)$, the Fourier expansion of $f(\tau, \sigma)$ reads

$$f(\tau, \sigma) = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}\xi \tilde{f}_n(\xi) e^{-\mathrm{i}(\xi\tau+n\sigma)}$$

with

$$\tilde{f}_n(\xi) = \int_{\mathbb{R}} \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma f(\tau, \sigma) e^{\mathrm{i}(\xi\tau+n\sigma)}.$$

We use the following formula for the Fourier transform of $f * g$,

$$(\widetilde{f * g})_n(\xi) = \frac{1}{(2\pi)^2} \sum_{n'} \int \mathrm{d}\xi' \tilde{f}_{n'}(\xi') \tilde{g}_{n-n'}(\xi - \xi') e^{-\frac{\mathrm{i}\theta}{2}(\xi' n - n'\xi)}.$$

Using this expression, we can write $f * g$ as

$$\begin{aligned}(f * g)(\tau, \sigma) \quad &= \quad \frac{1}{(2\pi)^4} \sum_{n,n'} \int \mathrm{d}\xi \mathrm{d}\xi' \mathrm{d}\tau' \mathrm{d}\sigma' \mathrm{d}\tau'' \mathrm{d}\sigma'' f(\tau', \sigma') g(\tau'', \sigma'') \\ &\quad \times e^{-\mathrm{i}(\xi'\tau'+n'\sigma')} e^{-\mathrm{i}((\xi-\xi')\tau''+(n-n')\sigma'')} e^{-\frac{\mathrm{i}\theta}{2}(\xi' n - n'\xi)} e^{\mathrm{i}(\xi\tau+n\sigma)}.\end{aligned}$$

Ordering of the terms in the exponentials gives

$$\begin{aligned}(f * g)(\tau, \sigma) \quad &= \quad \frac{1}{(2\pi)^4} \sum_{m} \int \mathrm{d}\xi' \mathrm{d}\tau' \mathrm{d}\sigma' \mathrm{d}\tau'' \mathrm{d}\sigma'' f(\tau', \sigma') g(\tau'', \sigma'') \\ &\quad \times \delta(\tau - \tau'' + \frac{\theta}{2} m) \delta(\sigma - \sigma'' - \frac{\theta}{2}\xi') e^{\mathrm{i}(\xi'(\tau'-\tau'')-m(\sigma'-\sigma''))}\end{aligned}$$

where we replaced n' by m. We substitute the second delta-function by the integration over $\mathrm{d}\xi'$ yielding

$$(f*g)(\tau,\sigma) = \frac{2/\theta}{(2\pi)^4}\sum_m \int \mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'') \times\delta(\tau-\tau''+\frac{\theta}{2}m)e^{\mathrm{i}(-\frac{2}{\theta}(\sigma-\sigma'')(\tau'-\tau'')-m(\sigma'-\sigma''))}.$$

Using this expression, we will consider $\boldsymbol{W}_{f*g}$. It is given by

$$\boldsymbol{W}_{f*g}(\boldsymbol{\tau},\boldsymbol{\sigma}) = \frac{2/\theta}{(2\pi)^4}\sum_{m,n}\int \mathrm{d}\xi\mathrm{d}\tau\mathrm{d}\sigma\int \mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'') \times e^{\mathrm{i}\xi(\boldsymbol{\tau}-\tau)+\mathrm{i}n(\boldsymbol{\sigma}-\sigma)}\delta(\tau-\tau''+\frac{\theta}{2}m)e^{\mathrm{i}(-\frac{2}{\theta}(\sigma-\sigma'')(\tau'-\tau'')-m(\sigma'-\sigma''))}.$$

When we substitute the delta-function by the integration over $\mathrm{d}\tau$, we finally have,

$$\boldsymbol{W}_{f*g}(\boldsymbol{\tau},\boldsymbol{\sigma}) = \frac{2/\theta}{(2\pi)^4}\sum_{m,n}\int \mathrm{d}\xi\mathrm{d}\sigma\int \mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'')e^{\mathrm{i}\xi\boldsymbol{\tau}+\mathrm{i}n\boldsymbol{\sigma}} \times\exp\big\{\mathrm{i}\big(-\xi(\tau''-\frac{\theta}{2}m)-n\sigma-\frac{2}{\theta}(\sigma-\sigma'')(\tau'-\tau'')-m(\sigma'-\sigma'')\big)\big\}$$

We will now consider $\boldsymbol{W}_f\,\boldsymbol{W}_g$,

$$\boldsymbol{W}_f\,\boldsymbol{W}_g(\boldsymbol{\tau},\boldsymbol{\sigma}) = \frac{1}{(2\pi)^4}\sum_{n,n'}\int \mathrm{d}\zeta\mathrm{d}\zeta'\mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'')e^{\mathrm{i}(\zeta+\zeta')\boldsymbol{\tau}+\mathrm{i}(n+n')\boldsymbol{\sigma}} \times\exp\big\{\mathrm{i}\big(-\zeta\tau'-n\sigma'-\zeta'\tau''-n'\sigma''-\frac{\theta}{2}(\zeta n'-n\zeta')\big)\big\}.$$

When we substitute

$$\begin{array}{ll} \zeta' := -\frac{2}{\theta}(\sigma-\sigma'')+\xi, & \zeta := \frac{2}{\theta}(\sigma-\sigma''), \\ n' := k, & n := l-k, \end{array}$$

and rearrange the terms in the exponential, we obtain

$$\boldsymbol{W}_f\,\boldsymbol{W}_g(\boldsymbol{\tau},\boldsymbol{\sigma}) = \frac{2/\theta}{(2\pi)^4}\sum_{k,l}\int \mathrm{d}\xi\mathrm{d}\sigma\int \mathrm{d}\tau'\mathrm{d}\sigma'\mathrm{d}\tau''\mathrm{d}\sigma'' f(\tau',\sigma')g(\tau'',\sigma'')e^{\mathrm{i}\xi\boldsymbol{\tau}+\mathrm{i}l\boldsymbol{\sigma}} \times\exp\big\{\mathrm{i}\big(-\xi(\tau''-\frac{\theta}{2}(l-k))-l\sigma-\frac{2}{\theta}(\sigma-\sigma'')(\tau'-\tau'')-(l-k)(\sigma'-\sigma'')\big)\big\}.$$

As $k,l\in\mathbb{Z}$, $(l-k)$ can be replaced by k. When we compare this expression with the expression for $\boldsymbol{W}_{f*g}$, we conclude that $\boldsymbol{W}_{f*g}=\boldsymbol{W}_f\,\boldsymbol{W}_g$. □

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