The BV Formalism for Matrix Models: a Noncommutative Geometric Approach

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus prof. dr. Th.L.M. Engelen,
volgens besluit van het college van decanen
in het openbaar te verdedigen op Mandaag 19 Oktober 2015
om 16.30 uur precies

door

Roberta Anna Iseppi

geboren op 16 November 1986
te Milaan, Italië

Promotor:

Prof. dr. N.P. Landsman

Copromotor:

Dr. W.D. van Suijlekom

Manuscriptcommissie:

Prof. dr. G. Heckman,
Voorzitter

Prof. dr. G. Cornelissen,
Universiteit Utrecht, the Netherlands

Prof. dr. G. Felder,
Eidgenössische Technische Hochschule, Zurich, Switzerland

Prof. dr. M. Marcolli,
California Institute of Technology, Pasadena, United States

Prof. dr. S. Shadrin,
Universiteit van Amsterdam, the Netherlands

The research presented in this thesis was supported by the Netherlands Organization for Scientific Research (NWO), through Vrije Competitie project number 613.000.910.

ISBN 978-90-9029238-0

The BV Formalism for Matrix Models:
a Noncommutative Geometric Approach

DOCTORAL THESIS

to obtain the degree of doctor
from Radboud University Nijmegen
on the authority of the Rector Magnificus
Prof. dr. Th.L.M. Engelen,
according to the decision of the Council of Deans
to be defended in public on Monday, October 19, 2015
at 16.30 hours

by

Roberta Anna Iseppi

born on 16 November 1986
in Milan, Italy.

Supervisor:

Prof. dr. N.P. Landsman

Co-supervisor:

Dr. W.D. van Suijlekom

Doctoral Thesis Committee:

Prof. dr. G. Heckman,
Chairman

Prof. dr. G. Cornelissen,
Universiteit Utrecht, the Netherlands

Prof. dr. G. Felder,
Eidgenössische Technische Hochschule, Zurich, Switzerland

Prof. dr. M. Marcolli,
California Institute of Technology, Pasadena, United States

Prof. dr. S. Shadrin,
Universiteit van Amsterdam, the Netherlands

Contents

Chapter 1

Introduction

The quantization of non-abelian gauge theories is an interesting subject both from a mathematical and from a physical point of view. The importance of having a precise formulation of a procedure for quantizing gauge theories comes from the fact that all known fundamental interactions appearing in Nature are governed by gauge theories.
In this thesis we focus on a particular class of gauge theories that are naturally derived from 0-dimensional noncommutative manifolds. For these models we analyze the so-called BV (Batalin-Vilkovisky) formalism and we discuss the corresponding BRST (Becchi-Rouet-Stora-Tyutin) cohomology complex. We also present a novel approach to include the BV formalism in the setting of noncommutative geometry.

The BV formalism

The context in which the BV formalism was first discovered is the quantization of non-abelian gauge theories via the path integral approach. In this short introduction to the BV formalism, we briefly present the physical motivation that originally led to the discovery of this formalism. We emphasize that this introduction is not supposed to be exhaustive, whereas it has the aim of giving an idea of the "physical flavor" behind this thesis. For a more complete and formal explanation of the concepts coming from quantum fields theory, we refer to [56] while, for the BRST quantization of gauge theories, we refer to [32], [39].

Let the pair (X_0, S_0) be a gauge theory, consisting of an *initial (field) configuration space* X_0 and an *initial action* S_0, which is invariant under the action of a gauge group $\mathcal{G}$. As already mentioned, the BV formalism has been invented with the aim of solving the problem of quantizing a (infinite-dimensional) non-abelian gauge theory using the path integral approach. In fact, the quantization of a theory via the path integral approach in the Euclidean set-up usually leads to the problem of computing integrals of the following type:

$$\langle g \rangle = \int_{X_0} g e^{-S_0} [d\mu] \tag{1.1}$$

where g is a functional on X_0, $d\mu$ denotes a measure on the configuration space X_0, while $\langle g \rangle$ is known as the *expectation value* of the functional g.
This kind of integral is known in the physics literature as a *path integral.*

Two crucial problems appear trying to quantize a non-abelian gauge theory via the computation of a path integral.
The first problem is not specifically due to the presence of a gauge invariance but is related to the notion of path integral itself: the path integral is not mathematically well defined, since the measure in (1.1) in the case of an infinite-dimensional configuration space X_0, in general is not well defined. One way to face this problem is through rigorously defined methods coming from perturbation theory. Therefore, even without the presence of a gauge invariance, the path integral of an infinite-dimensional physical theory appears to be ill-defined. In contrast, in the case of a finite-dimensional theory, the measure on a finite-dimensional configuration space X_0 can be rigorously defined. Thus already at the level of a generic quantum field theory, even without the presence of a gauge invariance, there appears to be a major difference between the finite and the infinite-dimensional cases.

Ignoring the problem of defining the measure, when the first attempt was made to quantize a gauge theory via the path integral approach – by Feynman [29] in 1963 – it immediately turned out that new difficulties appear, which Feynman to some extent addressed: gauge invariance of the action functional causes a degeneracy and the quantization via the path integral approach cannot be applied straightforwardly to the theory. More precisely, under suitable conditions on the action of the gauge group (i.e. under the condition of $\mathcal{G}$ acting freely on the space X_0 and the $\mathcal{G}$-invariance of the action S_0), the integral in (1.1) can be seen as products of two integrals, one computed on the quotient $X_0/\mathcal{G}$ and one computed along the gauge directions: this last integral is proportional

to the volume of the gauge group $\mathcal{G}$ and hence is infinite whereas $\mathcal{G}$ is infinite-dimensional (or even non-compact).
Thus the redundant gauge variables must be removed from the theory: to achieve this goal, some gauge-fixing procedure needs to be performed. However, after this procedure the gauge invariance of the theory is lost and there appears to be less control of the physical meaning of what we are computing with the path integral.

The central idea of the BRST construction [10], [57], is to replace the gauge symmetry with a new symmetry, the *BRST symmetry* to recover the lost gauge symmetry in some sense. This goal is achieved by introducing extra (non-physical) fields, which are known as *ghost fields*. The idea of adding extra fields to the configuration space was first suggested in 1967 by Faddeev and Popov [27]: therefore, these ghost fields are known also as *Faddeev-Popov ghosts*. Their main idea was to introduce extra fields in the theory in order to cancel the local symmetries and hence to be able to compute the path integral. Once again, techniques coming from perturbation theory are needed to compute (or even define) this path integral but then the introduction of these extra fields eliminates the degeneracy of the propagator that causes the failure of the perturbative approach in presence of a gauge symmetry. Nonetheless, even with the introduction of these ghost fields, the path integral remains ill-defined and is computable only as a perturbation series.

A few years later, in 1975 Becchi, Rouet, Stora [10], [11] and, independently, Tyutin [57], discovered that these extra fields led to a particular kind of transformation, now called a *BRST transformation*. Moreover, they discovered that the ghost fields are generators of a cohomology complex, known as the *BRST-cohomology complex*.
These BRST transformations were also investigated by Zinn-Justin during his study on the renormalization of Yang-Mills theories [60]. He was the first to introduce an (odd) symplectic structure in the space of fields. These ideas were further developed by Batalin and Vilkovisky, who discovered a quantization procedure known as the *antibracket formalism*, or also as the *Batalin-Vilkovisky* or *BV formalism*.

The key step in this approach to the quantization of gauge theories is to enlarge the configuration space X_0 to an extended configuration space $\widetilde{X}$ via the introduction of ghost fields. Then these extra fields are used to construct an extended action $\widetilde{S}$ by adding terms involving the ghost fields to the initial action

S_0. The condition imposed on this new extended action $\widetilde{S}$ is that it has to be BRST invariant. More explicitly, let δ_B denote the BRST symmetry, which acts on $\mathcal{O}_{\widetilde{X}}$, i.e. the space of regular functions defined on the extended configuration space; we have to require that

$$\delta_B(\widetilde{S}) = 0.$$

The BV approach [7], [8] provides a method to construct the extended pair $(\widetilde{X}, \widetilde{S})$, starting from the initial gauge theory (X_0, S_0). This method is based on the idea that for each field and each ghost field in the extended configuration space, it is necessary to introduce a corresponding antifield and antighost field, respectively. Then a so-called antibracket is defined, giving an odd non-degenerate symplectic form on the total space of fields and antifields.
Schematically, the BV construction can be summarized as follows:

$$\begin{aligned} X_0 &\rightsquigarrow \widetilde{X} = X_0 + \{\text{antifields, ghost fields and antighost fields}\}; \\ S_0 &\rightsquigarrow \widetilde{S} = S_0 + \text{ terms involving antifields, ghosts and antighosts.} \end{aligned}$$

In order to proceed with an analysis of the gauge theory using perturbation theory, it is then necessary to apply a gauge-fixing procedure, which allows one to compute correlation functions and scattering amplitudes. This process eliminates the antifields that appear in the extended action $\widetilde{S}$, replacing them by expressions depending only on the fields. When the gauge-fixing procedure is appropriately implemented, the usual Feynman graph method can be used. This makes the BV formalism a powerful method for quantizing a gauge theory, at least perturbatively.

The two fundamental properties of the BRST symmetry are the following:

- The BRST symmetry is still present also after the gauge-fixing procedure has been implemented: if Ψ is the gauge-fixing fermion used to perform the gauge-fixing procedure and $\widetilde{S}_\Psi$ is the gauge-fixed action, then

$$\delta_B(\widetilde{S}_\Psi) = 0.$$

- δ_B is a linear differential operator of degree 1 and the same applies to the gauge-fixed BRST operator $\delta_{B,\Psi}$:

$$\delta_B^2 = 0, \qquad \delta_{B,\Psi}^2 = 0.$$

These last two conditions imply that both δ_B and $\delta_{B,\Psi}$ can be seen as coboundary operators for a cohomology complex: δ_B is the coboundary operator for the *BRST cohomology complex*, while $\delta_{B,\Psi}$ plays this role for the *gauge-fixed BRST cohomology complex*. (We postpone all the formal definitions to Chapter 3.)

It is precisely via this gauge-fixed BRST cohomology complex that the gauge symmetry is in some sense recovered: the cohomology group of degree 0 of this theory describes the gauge-invariant functions of the initial gauge theory (X_0, S_0), i.e. the elements that in the physics literature are known as the *observables* of the theory:

$$H^0(\widetilde{X}, \delta_{B,\Psi}) = \{\text{Observables of the initial gauge theory } (X_0, S_0)\}.$$

The discovery of the existence of the BRST symmetry for gauge theories extended with ghost fields made it evident that the ghost fields, which were originally introduced as a tool to solve the specific problem of defining and computing path integrals, could also play a more significant role as generators of a cohomology theory with physical relevance, at least for 4-dimensional theories.

To conclude, the BV approach to the BRST construction is a procedure used to face the problem of having infinite terms in the path integral when we consider infinite-dimensional gauge theory: we loose the gauge symmetry via a gauge fixing but in exchange we introduce other non-physical fields, which allow the recovery of the gauge invariance of the theory via the cohomology groups of the cohomology theory defined by a new symmetry, namely the gauge-fixed BRST symmetry. This is the main idea behind all BRST-type constructions.

Once again we stress that the initial motivation that first led to the formulation of these techniques was the difficulty of proceeding straightforwardly with the quantization of infinite-dimensional gauge theories via the path integral approach. These problems do not appear when we want to quantize a finite-dimensional gauge theory. Therefore, strictly speaking, the BRST or BV constructions are not needed in the context of finite-dimensional gauge theories. Since in what follows we will discuss in detail how to perform this construction for a particular type of finite-dimensional gauge theories coming from a 0-dimensional noncommutative manifold, let us explain our goals: indeed our aim is not to proceed with the quantization of the theory but to investigate the BV construction in this simple setting, with the purpose of better understanding the construction itself and the relation between the initial theory (X_0, S_0)

and the extended one $(\widetilde{X}, \widetilde{S})$, in a context in which everything is mathematically well defined. Our hope is that the analysis of this particular case will give some insight on how to better understand the BV construction also for infinite-dimensional gauge theories, where a mathematically rigorous understanding of the procedure is still needed, since already the starting point, namely the path integral, is not mathematically rigorously defined.

The purpose of the first half of this thesis is to study the geometric structure of the ghost fields and describe the BRST cohomology from a novel point of view, introducing a generalized notion of Lie algebra cohomology, which gives a more explicit description of the space of ghosts and a better understanding of its structure.

For completeness, we mention that there is a large literature on BRST cohomology, which has been studied from many different points of view: for example, the BRST cohomology has been extensively analyzed in the context of constrained quantization, e.g.. and references therein [38], [40].

Gauge theories and noncommutative geometry

Since the early days of noncommutative geometry [20] it has been clear that there exists a strong connection between this mathematical theory and gauge theories in physics. Without any doubt, the greatest achievement in this direction is the description of the full Standard Model in the framework of noncommutative geometry [21].
However, the connection between noncommutative geometry and gauge theories should not be attributed only to a specific case, despite its importance in physics. In fact, gauge theories are naturally induced by *spectral triples*, which are the main technical device in contemporary noncommutative geometry. Thus it is reasonable to try to insert in the setting of noncommutative geometry also other procedures and techniques which have been developed for the analysis of gauge theories.
In the second part of this thesis we take a first step in this direction by incorporating the BV approach to the BRST quantization of non-abelian gauge theories into the framework of finite-dimensional spectral triples. The driving force of this attempt and the hope underlying it are that noncommutative geometry might give new insight in the BRST quantization procedure, helping to better determine the relationship between the initial gauge theory and its

BRST-cohomology complex.

Outline

In this thesis we focus on gauge theories described as pairs consisting of a configuration space, which is supposed to be given by a nonsingular algebraic variety, and an action functional, which is a regular function on the variety in question, invariant under the action of a gauge group. In particular, we focus on gauge theory naturally induced by 0-dimensional noncommutative manifold. In this context, the configuration space is given by matrices. These kind of gauge theories are also known as *matrix models*. Such models have also been treated in other physical contexts, such as 2-dimensional gravity theories, [31].

In this thesis, after a general introduction to the BV approach to the quantization of gauge-invariant theories defined on algebraic varieties (following [28]), we consider a $U(2)$-matrix model as an example to which we apply the BV construction. Moreover, the BRST-cohomology complex defined by this model is constructed and the corresponding cohomology groups are explicitly computed and related to a new generalized notion of Lie algebra cohomology.

The final part of this thesis is devoted to present a possible method to incorporate the BV approach to the quantization of non-abelian gauge theories in the framework of noncommutative geometry. We restrict ourselves to a $U(2)$-gauge invariant matrix model that is naturally obtained from a finite-dimensional spectral triple on the matrix algebra $M_2(\mathbb{C})$, and construct spectral triples for the antifields coming from the BV formalism.

In more detail, the structure of this thesis is as follows.

Chapter 2

In this chapter, the main notions regarding *spectral triples*, which are the main technical device in contemporary noncommutative geometry, and *gauge theories* are stated. Then we focus on finite-dimensional spectral triples, for whose analysis a graphical method is presented, in terms of *Krajewski diagrams*. Finally, the close relation existing among spectral triples and gauge theories is explained by describing how each spectral triple naturally induces a gauge theory. As an example of this construction, a finite spectral triple on the algebra $M_n(\mathbb{C})$ is considered: the gauge theory induced by this spectral triple is the $U(n)$-matrix model that is analyzed in detail in the second part of this thesis, for $n = 2$.

Chapter 3

The aim of this chapter is to review the BV approach to the quantization of non-abelian gauge theories, following [34]. First, a generalization of the notion of *BV variety*, introduced by Felder and Kazhdan [28], is presented as the mathematical object to describe the theory obtained as the extension of an initial gauge theory through the introduction of ghost fields and antighost fields. Even though the ghost fields are introduced to eliminate the gauge degrees of freedom of the theory, the extended theory has some residual symmetry, called *BRST symmetry*: this notion of symmetry is stated and it is shown how the presence of this symmetry determines a cohomology theory, known as *BRST cohomology*. Then, the *gauge-fixing procedure* is described: first, the physical motivation for its introduction is presented and then the gauge-fixing procedure itself is mathematically illustrated using the BV formalism. Also after having carried out the gauge-fixing procedure, a residual symmetry is still present, known as the *gauge-fixed BRST symmetry*, which defines a corresponding cohomology complex, namely the *gauge-fixed BRST-cohomology complex*. At the end of the chapter, the *auxiliary fields* are introduced, which are an important technical tool used to perform the gauge-fixing procedure without modifying the corresponding gauge-fixed BRST-cohomology complex.

Chapter 4

This chapter is devoted to giving a mathematical description of the procedure of extending a gauge theory through the introduction of ghost fields. The construction explained in this chapter has been inspired by the one, presented in [28], of a BV variety associated to a physical theory. This construction is based on Tate's algorithm, of which a brief presentation may be found in Appendix B. Moreover, we explain how the BV algorithm gives a mathematical interpretation of physical properties of the theory such as the minimal number of ghost fields that need to be introduced, their ghost degree, and their parity.

Chapter 5

In this chapter we describe in detail the procedure of Chapter 4 in the case where the configuration space is given by self-adjoint complex 2×2 matrices and the

gauge group is the unitary group $U(2)$. This model has been already introduced in Chapter 2, where it was described as the gauge theory induced by a finite spectral triple on the algebra $M_2(\mathbb{C})$. For this model, we first determine the most general minimally-extended theory and then we explicitly describe the related gauge-fixed BRST-cohomology complex, while the detailed computations of the cohomology groups has been collected in Appendix D.
The second part of the chapter is devoted to the analysis of the BRST-cohomology complex of the $U(2)$-matrix model from a different point of view: by the introduction of a new generalized notion of Lie algebra cohomology, we prove that in our generalized Lie algebra cohomology setting, the BRST-cohomology complex found for the model coincides, at the level of cochain spaces and coboundary operators, with a shifted double complex. Subsequently, the properties of the shifted double complex are analyzed and their relations with the BRST-cohomology complex are determined also at the level of the corresponding cohomology groups. One of the interesting results achieved with this approach to the BRST complex is the determination of the role played by the different kinds of ghost fields introduced, as well as the translation of the physical properties of these ghosts, such as their ghost degree and their parity, in terms of properties of the double complex structure.

Chapter 6

The purpose of this chapter is to explain how to extend the analysis that was described in the previous chapter for a $U(2)$-matrix model to the general setting, i.e., to matrix models obtained from a finite spectral triple on the algebra $M_n(\mathbb{C})$, with a $U(n)$-gauge invariance, for $n \in \mathbb{N}$. The main result is a relation between the gauge group $U(n)$ acting on the configuration space, and the minimal number of ghost fields that need to be introduced to obtain an extended theory $(\widetilde{X}, \widetilde{S})$ amenable to the techniques already developed, such as generalized Lie algebra cohomology.

Chapter 7

The main part of this chapter is devoted to the introduction and the description of a so-called *BV-spectral triple*. We introduce this notion with the aim of incorporating the BRST formalism in the setting of noncommutative geometry: this goal will be achieved in the case of a $U(2)$-gauge invariant matrix model induced by a finite-dimensional spectral triple on the matrix algebra $M_2(\mathbb{C})$. The main result obtained with this approach is that all physical properties of

the ghost fields, such as their bosonic or fermionic character, have a natural interpretation in terms of the spectral triple itself.
In the final part of the chapter, also the device of trivial pairs is incorporated in the setting of noncommutative geometry via the introduction of a so-called *BV-auxiliary spectral triple*. It is interesting to notice that the structure that appears at the level of the BV-spectral triple emerges also for the BV-auxiliary spectral triple.

Appendix A

The main characters of this appendix are the auxiliary fields: more precisely, in this appendix we justify the method used to introduce the auxiliary fields, restricting ourselves to the context of gauge theories with level of reducibility $L = 1$.

Appendix B

This appendix is dedicated to a brief review of Tate's algorithm.

Appendix C

Here we give proofs of some technical lemmas stated in Chapter 4, where they are used for the construction of the algorithm to determine the extended action.

Appendix D

In this appendix the explicit computations of the gauge-fixed BRST-cohomology groups of the $U(2)$-matrix model, which were first introduced in Chapter 5, are presented in detail.

The main new results presented in this thesis are briefly stated in the following list:

- ► A new procedure, inspired by the construction presented in [28], is explained to determine an *extended variety* associated to an initial gauge theory. This method, which is applicable to a suitable type of initial gauge theories, may allows to select a finite number of ghost fields and antighost fields. They are used to enlarge the configuration space and to define the extra terms, which added to the initial action determine a solution of the classical master equation on the extended configuration space.

- The BV approach to the quantization of non-abelian gauge theories has been applied to a $U(2)$-matrix model, which was derived by a finite-dimensional spectral triple over the algebra $M_2(\mathbb{C})$, determining the minimally extended theory corresponding to it. Moreover, a possible approach to the construction of the minimally extended theory is described also for general $U(n)$-matrix model, for any $n \in \mathbb{N}$.

- A new notion of *generalized Lie algebra cohomology* has been introduced, which we used to describe the BRST-cohomology complex for a $U(2)$-matrix model in a new way. Through this, a richer structure of the BRST complex has emerged, namely a double complex structure. Moreover, with this approach, a geometric interpretation of the ghost fields and their properties, such as their ghost degree and their parity, has been obtained.

- The BRST-cohomology groups have been explicitly computed for the $U(2)$-matrix model and hence been related to the cohomology groups of a suitable generalized Lie algebra cohomology complex.

- A possible approach to the problem of describing the BV construction for gauge theories in the setting of noncommutative geometry is presented: this approach is based on the introduction of a so-called *BV-spectral triple*. Even though the solution of this problem has been given only in the case of a $U(2)$-gauge invariant matrix model, this approach suggests a possible way to address the problem in a more general setting.

- A so-called *BV-auxiliary spectral triple* has been introduced for a $U(2)$-matrix model, enabling one to include also the device of auxiliary fields in the setting of noncommutative geometry.

Acknowledgment: the content presented in Chapter 6 was carried out during the author's visit to the California Institute of Technology, under the supervision of Prof. Dr. Matilde Marcolli. Chapter 7 is based on a joint work with Dr. Walter D. van Suijlekom.

Part I

General theory

Chapter 2

Noncommutative geometry and matrix models

The purpose of this chapter is to recall the main notions in noncommutative geometry that will be used in the rest of this thesis. In particular we focus on finite spectral triples and on their properties.

More precisely, Section 2.1 will be devoted to define the notions of a *spectral triple*, of a *real spectral triple*, and of the *fermionic action* while in Section 2.2 we focus on *finite spectral triples*, that is, on spectral triples that as algebras are sums of matrix algebras, and as Hilbert spaces are finite-dimensional. For this particular kind of spectral triples we present a graphical method used to classify them, which is based on the notion of *Krajewski diagram*.
Finally, in Section 2.3 we explain how a spectral triple naturally gives rise to a gauge theory and we introduce the interesting example of a $U(n)$-gauge invariant matrix model, which is naturally defined by a finite spectral triple on the algebra $M_n(\mathbb{C})$.

This chapter is mainly based on [21], [46], and [53], with the exception of the notion of fermionic action, which we introduce in a slightly more general version, suitable for the constructions that will be presented in Chapter 7.

2.1 The noncommutative geometry setting

In this section we focus on the notion of a *spectral triple* and, more specifically, of a *real spectral triple* [21]. The notion of a spectral triple, stated in its full generality, involves some concepts from the theory of operators and operator algebras. For completeness we state these definitions in the way they are usually presented, namely for a possibly infinite-dimensional Hilbert space. However, in what follows we focus on finite spectral triples: under the hypothesis that the Hilbert space is finite-dimensional, some of the conditions appearing in the general definition of a spectral triple will be automatically satisfied. For this reason we prefer not to explain in detail the full theory necessary to understand the definition of spectral triple in the general context: we simply state the definition, referring to e.g.. [49] and [22] for those aspects concerning functional analysis and operator theory.

Definition 1. *A* spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *is a triple consisting of an algebra* $\mathcal{A}$*, a Hilbert space* $\mathcal{H}$ *and an operator* D *where:*

- ▶ $\mathcal{A}$ *is an involutive unital algebra;*
- ▶ $\mathcal{H}$ *is a Hilbert space such that the algebra* $\mathcal{A}$ *is faithfully represented as operators on it;*
- ▶ D *is a (possibly unbounded) self-adjoint operator on* $\mathcal{H}$ *with compact resolvent, such that all commutators* $[D, a]$ *are bounded operators, for* $a \in A$.

Definition 2. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is said to be* even *if the Hilbert space* $\mathcal{H}$ *is endowed with a* $\mathbb{Z}/2$*-grading* γ *that commutes with any element* a *in* $\mathcal{A}$ *and anticommutes with the operator* D*. More explicitly there exists a linear map:*

$$\gamma : \mathcal{H} \longrightarrow \mathcal{H}$$

such that the following conditions hold for any element a *in* $\mathcal{A}$ *and* φ *in* $\mathcal{H}$*:*

$$\gamma(a\varphi) = a\gamma(\varphi) \qquad D(\gamma(\varphi)) = -\,\gamma(D(\varphi)).$$

Definition 3. *A* real structure *of KO-dimension* $n \in \mathbb{Z}/8$ *on a spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is an antilinear isometry on* $\mathcal{H}$*,*

$$J : \mathcal{H} \longrightarrow \mathcal{H}$$

satisfying the following properties:

- ► $J^2 = \epsilon$;
- ► $JD = \epsilon' DJ$;
- ► $J\gamma = \epsilon'' \gamma J$, *(in the even case).*

Here the numbers ϵ, ϵ' *and* ϵ'' *are either* 1 *or* -1 *and their value is determined by the KO-dimension* n *(mod 8) as follows:*

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Moreover, the action of the algebra $\mathcal{A}$ *satisfies the following commutation rule:*

$$\left[a, Jb^* J^{-1}\right] = 0 \qquad \forall a, b \in \mathcal{A}, \tag{2.1}$$

and the operator D *satisfies the so-called* first-order condition*:*

$$\left[\left[D, a\right], Jb^* J^{-1}\right] = 0 \qquad \forall a, b \in \mathcal{A}. \tag{2.2}$$

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *endowed with a real structure* J *is called a* real spectral triple*, denoted by* $(\mathcal{A}, \mathcal{H}, D, J)$.

Remark 1
Given a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$, the antilinear isometry J induces a right action of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, defined by

$$a \mapsto Ja^* J^{-1}, \qquad a \in \mathcal{A}.$$

Equivalently, we say that $a^\circ := Ja^* J^{-1}$ defines a left action of the opposite algebra $\mathcal{A}^\circ$ on $\mathcal{H}$. We recall that, by definition, the opposite algebra $\mathcal{A}^\circ$ coincides with $\mathcal{A}$ as a vector space, but its product is the opposite of the one defined in $\mathcal{A}$:

$$a \circ b := b \cdot a$$

with $a, b \in \mathcal{A}^\circ = \mathcal{A}$ and $\cdot$ being the product in $\mathcal{A}$.

2.2 Finite spectral triples

Having recalled the notion of a (general) spectral triple, in this section we focus on *finite spectral triples*: these are spectral triples $(\mathcal{A}, \mathcal{H}, D)$ in which both the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ are finite-dimensional.

Definition 4. *A* finite spectral triple *is a triple* $(\mathcal{A}, \mathcal{H}, D)$ *consisting of an involutive unital algebra* $\mathcal{A}$ *represented faithfully on a finite-dimensional Hilbert space* $\mathcal{H}$*, together with a symmetric operator* $D : \mathcal{H} \to \mathcal{H}$*.*

The conditions imposed on the algebra $\mathcal{A}$ in the definition of a finite spectral triple are such that the algebra is forced to be a direct sum of matrix algebras, as precisely stated in the following classical lemma (for a proof we refer to [53, Lemma 2.20]).

Lemma 1
Let $\mathcal{A}$ be an involutive unital algebra that acts faithfully on a finite-dimensional Hilbert space. Then $\mathcal{A}$ is a matrix algebra of the following form:

$$\mathcal{A} \simeq \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}), \tag{2.3}$$

with $n_1, \ldots, n_k \in \mathbb{N}$.

Given a possibly real spectral triple $(\mathcal{A}, \mathcal{H}, D, (J))$, there are two notions of action functionals related to it: the *spectral action* and the *fermionic action*. Even though both may be defined for a general spectral triple, since throughout the whole thesis they will be used only in the context of finite spectral triples, we decided to state both of them only in this context. However, while the notion of a spectral action considerably simplifies for finite-dimensional spectral triples, allowing us to avoid the introduction of further tools coming from operator theory, this simplification does not occur for the notion of fermionic action: indeed, the fermionic action does not depend on the Hilbert space or the algebra being finite or infinite-dimensional, though it is related to the KO-dimension of the spectral triple, as explained below.
For the definition of the spectral action in its full generality we refer to [16], [17], while the definition of fermionic action stated here is a generalization of the notion given in [21].

Definition 5. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a finite spectral triple, and let* f *be a polynomial in one real variable. Then the* spectral action S_0 *is defined by*

$$S_0[D + \varphi] := Tr\big(f(D + \varphi)\big),$$

with φ a self-adjoint element of $\Omega^1_D(\mathcal{A})$, which is defined to be the space of the following finite sums:

$$\Omega^1_D(\mathcal{A}) := \Big\{ \sum_j a_j[D, b_j] : a_j, b_j \in \mathcal{A} \Big\}. \tag{2.4}$$

Note: in a finite spectral triple, the operator D is simply a hermitian matrix. In that case, the trace in the definition of the spectral action is the usual trace of matrices.

Definition 6. *Let $(\mathcal{A}, \mathcal{H}, D, (J))$ be a finite (possibly real) spectral triple and fix an Hilbert subspace $\mathcal{H}' \subseteq \mathcal{H}$. Then the* fermionic action *is defined by:*

$$S_{ferm}[\varphi] = \frac{1}{2}\langle (J)\varphi, D\varphi\rangle, \qquad \varphi \in \mathcal{H}'. \tag{2.5}$$

Note: the subspace $\mathcal{H}' \subseteq \mathcal{H}$, which appears in the definition of a fermionic action, depends on the KO-dimension of the real spectral triple, as explained in the following remark.

Remark 2
The notion of a fermionic action is usually introduced for real spectral triples of KO-dimension 2 (mod 8), i.e., for real spectral triples endowed with a grading γ and satisfying specific conditions on the signs appearing in the commutation relations among J, D and γ (see [21, Definition 1.216]). In this more usual definition the subspace $\mathcal{H}'$ is assumed to be the even part of the Hilbert space $\mathcal{H}$, which is denoted by $\mathcal{H}^+$ and is determined by the grading γ as follows:

$$\mathcal{H}^+ = \big\{\varphi \in \mathcal{H},\ \gamma(\varphi) = \varphi\big\}.$$

Moreover, the elements in $\mathcal{H}' = \mathcal{H}^+$ are supposed to be classical fermions, i.e., Grassmannian variables. This is the reason why this kind of action has been called *fermionic action*: it is defined on fermionic vectors.
The reason that forces to make this assumption on the parity of the vectors in $\mathcal{H}'$ lies in the commutation relations among J and D imposed by the condition of having KO-dimension 2.
However, the restriction to real spectral triples with KO-dimension 2 is not necessary, at least in the finite-dimensional case. Indeed, fixing a different subspace $\mathcal{H}'$ and eventually imposing the Grassmannian parity to some of the components of the operator D, the fermionic action can be defined also for finite real spectral triple with KO-dimension not necessarily equals to 2 (as it will be done in Chapter 7). Since a more general construction is possible, we introduce a more general notion of fermion action.

Given two finite spectral triples, the most natural notion of equivalence between them is *unitary equivalence*, whose definition we recall here.

Definition 7. *Two finite spectral triples* $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ *and* $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ *are said to be* unitarily equivalent *if both of the following conditions are satisfied:*

- $\mathcal{A}_1 = \mathcal{A}_2$;
- *there exists a unitary operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that:*

$$U\pi_1(a)U^* = \pi_2(a), \quad \forall a \in \mathcal{A}_1 \qquad \text{and} \qquad UD_1U^* = D_2,$$

  where π_1 *and* π_2 *are, respectively, the action of the algebra* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *on the Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*.*

Given this notion of equivalence, the natural question that rises is whether it is possible to classify the finite spectral triples up to unitary equivalence. The remaining part of this section is devoted to answer to this question.

2.2.1 Krajewski diagrams for finite real spectral triples

In this section we present a graphical method, using *Krajewski diagrams*, to classify finite real spectral triples up to unitary equivalence. This graphical approach turns out to be very helpful to check if all the properties required to have a real spectral triple are satisfied. More precisely, these diagrams allow us to immediately verify if, given $\mathcal{A}$, $\mathcal{H}$, D and J such that

- $\mathcal{A}$ is a finite-dimensional involutive unital algebra;
- $\mathcal{H}$ is a finite-dimensional Hilbert space on which $\mathcal{A}$ is faithfully represented;
- $D : \mathcal{H} \longrightarrow \mathcal{H}$ is a self-adjoint operator on $\mathcal{H}$;
- $J : \mathcal{H} \longrightarrow \mathcal{H}$ is an antilinear map on $\mathcal{H}$, such that $[a,\ Jb^*J^{-1}] = 0$, $\forall a, b \in \mathcal{A}$,

they form a finite real spectral triple. In particular, this method helps to immediately check whether the operator D satisfies the first-order condition, which otherwise could require long computations to be verified.

The first results in this direction were obtained by Krajewski in [46], where the case of KO-dimension 0 is analyzed; the generalization of this approach to finite real spectral triple of any KO-dimension is explained in detail in [53]. This is the reference that we follow.

Definition 8. *Let $\Gamma^{(0)}$ be a collection of points and let $\Gamma^{(1)}$ be a subset of $\Gamma^{(0)} \times \Gamma^{(0)}$. The elements of $\Gamma^{(0)}$ are called* vertices*, while an element of $\Gamma^{(1)}$ is called an* edge*. The ordered pair $(\Gamma^{(0)}, \Gamma^{(1)})$ is a* graph*.*

In the notation just introduced, if an edge e is of the form $e = (v, v)$ for v a vertex, then e is called a *loop*.

Note: let $\mathcal{A}$, $\mathcal{H}$, D, J be as above. Then, since the algebra $\mathcal{A}$ is finite-dimensional and is faithfully represented on a finite-dimensional Hilbert space, it can always be decomposed as direct sum of a finite number of matrix algebras, as already recalled in Lemma 1. Moreover, we are assuming to have a map J, which defines a left action of $\mathcal{A}^\circ$ on $\mathcal{H}$ and which satisfies the commutation relation (2.1). Since this condition imposes that the representation of $\mathcal{A}$ on $\mathcal{H}$ commutes with the representation of $\mathcal{A}^\circ$ on $\mathcal{H}$, not only $\mathcal{A}$ but also $\mathcal{A}^\circ$ is faithfully represented on $\mathcal{H}$: thus we have to consider the irreducible representations of $\mathcal{A} \otimes \mathcal{A}^\circ$. Hence the Krajewski diagram corresponding to $\mathcal{A}$, $\mathcal{H}$, D, J is two-dimensional.

Given $\mathcal{A}$, $\mathcal{H}$, D, J as above, the corresponding Krajewski diagram is defined by the following steps.

Step 1: The labels

The first step in the construction of a Krajewski diagram is to determine the labels of the vertices. These labels are determined by the algebra $\mathcal{A}$: the coordinates are labeled by a pair of integers (n_i, n_j°), where n_i denotes the irreducible representation of $\mathcal{A}$ on $\mathbb{C}^{n_i}$, while n_j° denotes the irreducible representation of the opposite algebra $\mathcal{A}^\circ$ on $\mathbb{C}^{n_j}$.

Note that a matrix algebra $M_{n_i}(\mathbb{C})$ of dimension n_i could appear in the decomposition (2.3) with multiplicity higher than 1. Even though the integer n_i is the same, each of these copies of the algebra $M_{n_i}(\mathbb{C})$ will define a label for the vertices in the diagram.

So, up to this point, the Krajewski diagram has the structure described in Figure 2.1.

Step 2: The nodes

The second step in the construction of a Krajewski diagram is to determine the nodes. To do this we have to consider the Hilbert space $\mathcal{H}$. In view of the

$$
\begin{array}{c|ccccccc}
 & n_1 & \dots & n_i & \dots & n_j & \dots & n_k \\
\hline
n_1^\circ & & & & & & & \\
\vdots & & & & & & & \\
n_i^\circ & & & & & & & \\
\vdots & & & & & & & \\
n_j^\circ & & & & & & & \\
\vdots & & & & & & & \\
n_k^\circ & & & & & & &
\end{array}
$$

Figure 2.1: First step in the construction of a Krajewski diagram: the labels. These labels are determined by the dimension of the matrix algebras appearing as irreducible representations of $\mathcal{A} \otimes \mathcal{A}^\circ$.

irreducible representations of $\mathcal{A} \otimes \mathcal{A}^\circ$, the Hilbert space $\mathcal{H}$ can be decomposed as follows into irreducible representations:

$$\mathcal{H} \simeq \bigoplus_{i,j=1}^{k} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \tag{2.6}$$

where $\mathbb{C}^{n_j \circ}$ denotes the unique irreducible representation of $M_{n_j}(\mathbb{C})^\circ$ on $\mathbb{C}^{n_j \circ}$, while V^{ij} is a vector space whose dimension is the multiplicity of the representation $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$.

To determine the nodes in the diagram we look at the decomposition (2.6) of the Hilbert space $\mathcal{H}$ and for each summand $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$ in the decomposition we draw a node in the diagram at the coordinate (n_i, n_j°). This implies that, if a representation has multiplicity higher than 1, we have to draw as many nodes as indicated by the multiplicity.
Up to this point, the diagram which we are constructing would appear similar to the one drawn in Figure 2.2.

Hence, a node in a position (n_i, n_j°) indicates that the summand $M_{n_i}(\mathbb{C})$ acts on $\mathcal{H}$ by the product on the left by a matrix of size n_i but, if we consider

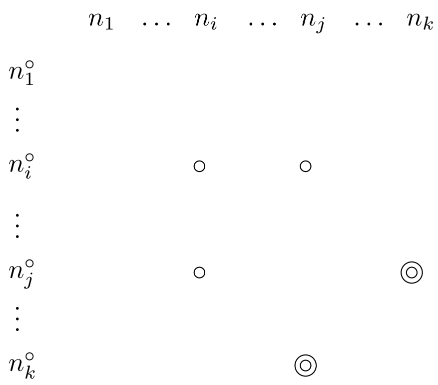


Figure 2.2: The presence of a double node in position (n_j, n_k°) and (n_k, n_j°) indicates that the representations $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_k\circ}$ and $\mathbb{C}^{n_k} \otimes \mathbb{C}^{n_j\circ}$ appear in the decomposition of the Hilbert space $\mathcal{H}$ with multiplicity 2 or, equivalently, the vector spaces V_{jk} and V_{kj} have dimension 2.

the corresponding right action, this is the action of the summand $M_{n_j}(\mathbb{C})$ by the product on the right by a matrix of size n_j.

Step 3: The edges

The last step in the construction of the Krajewski diagram corresponding to a collection $(\mathcal{A}, \mathcal{H}, D, J)$ is to determine the edges connecting the nodes. The edges are established by the behavior of the operator D. Since D is an operator on $\mathcal{H}$, to the decomposition (2.6) of the Hilbert space $\mathcal{H}$ there corresponds a decomposition of D as a matrix composed of blocks. Thus D is the sum of summands of the following type:

$$D_{ij,pq} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j} \otimes V_{ij} \longrightarrow \mathbb{C}^{n_p} \otimes \mathbb{C}^{n_q} \otimes V_{pq}.$$

So, for each non-zero matrix $D_{ij,pq}$ we draw a line connecting the node with coordinate (n_i, n_j°) and the node with coordinate (n_p, n_q°).
Note that the condition of being self-adjoint for the operator D ensures that this construction is well defined. In fact, if the matrix $D_{ij,pq}$ is non-zero the same holds for the matrix $D_{pq,ij}$ so that we can simply consider edges and not oriented edges.

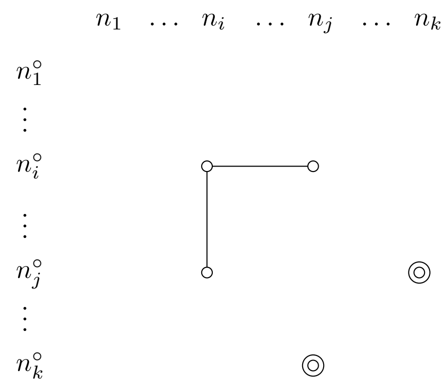


Figure 2.3: Example of a Krajewski diagram for a finite triple $(\mathcal{A}, \mathcal{H}, D, J)$. Since the diagram satisfies all conditions listed in Theorem 1, this triple is a real spectral triple.

Moreover, multiple edges represent a component $D_{ij,pq}$ of the operator D which acts among representations in $\mathcal{H}$ with multiplicity higher than 1. In other words, there would be a multiple edge connecting the nodes in positions (n_i, n_j°) and (n_p, n_q°) if the matrix $D_{ij,pq}$ is not zero and if either the representation $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j} \otimes V_{ij}$ on which $D_{ij,pq}$ is defined or the representation $\mathbb{C}^{n_p} \otimes \mathbb{C}^{n_q} \otimes V_{pq}$ in which it takes values has multiplicity higher than 1.

In case that a non-zero term $D_{ij,ij}$ appears in the decomposition of D, we draw a loop with the node in the position labeled by (n_i, n_j°) as a base.
Thus at this point the Krajewski diagram for $\mathcal{A}$, $\mathcal{H}$, D and J could be similar to the one drawn in Figure 2.3.

Up to now, we have explained how to determine the edges, the vertices and the labels of a Krajewski diagram. However, a Krajewski diagram is not completely determined by these data: to complete the construction of a Krajewski diagram also the operator D has to be inserted in the diagram itself, as stated in the following formal definition of a *Krajewski diagram.*

Definition 9. *A* Krajewski diagram *is given by a pair* (Γ, Λ) *where* Γ *is a finite graph, while* Λ *is a finite set of pairs of positive integers such that:*

- *to each vertex* $v \in \Gamma^{(0)}$, *a pair of positive integers* $(n,m)(v)$ *in* Λ *is associated;*
- *to each edge* $e = (v_i, v_j) \in \Gamma^{(1)}$, *two operators are associated, namely an operator* D_e *with*

$$D_e : \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i} \longrightarrow \mathbb{C}^{n_j} \otimes \mathbb{C}^{m_j},$$

as well as its conjugate-transpose D_e^* *with:*

$$D_e^* : \mathbb{C}^{n_j} \otimes \mathbb{C}^{m_j} \longrightarrow \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}.$$

We want to stress that, up to now, we have only considered the consequences to the diagram of the condition in (2.1) that is satisfied by the antilinear isometry J. However, in order for J to be a real structure for the finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$, it has to satisfy also other conditions, as stated in Definition 3.

In the following theorem we explain how these properties of the real structure impose further conditions on the Krajewski diagram (for the proof we refer to [53, Lemmas 3.8, 3.10]).

Theorem 1. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a finite spectral triple and let* J *be an antilinear isometry on* $\mathcal{H}$ *such that:*

$$\left[a, Jb^*J^{-1}\right] = 0, \qquad \forall a, b \in \mathcal{A}.$$

Let (Γ, Λ) *be the diagram corresponding to* $(\mathcal{A}, \mathcal{H}, D)$ *with:*

- $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)})$ *where* $\Gamma^{(0)}$ *is the set of nodes defined by the decomposition of* $\mathcal{H}$ *in irreducible representations and* $\Gamma^{(1)}$ *is the set of edges, defined by the decomposition of* D *in matrices;*
- Λ *is the set of the coordinates* (n_i, n_j°), *where the labels are determined by the decomposition of the algebra as direct sum of matrix algebras.*

Then the following hold:

(1) $J^2 = \pm Id$ *if and only if the diagram* (Γ, Λ) *is symmetric with respect to the diagonal, that is, if the following conditions are satisfied:*

  - *given a vertex* v *in* $\Gamma^{(0)}$ *with coordinates* (n_i, n_j°) *and with multiplicity* m, *there exists another vertex* v' *in* $\Gamma^{(0)}$ *with coordinates* (n_j, n_i°) *and with multiplicity* m;

- *for each edge e of multiplicity r connecting the vertex with coordinates (n_i, n_j°) to the one with coordinates (n_p, n_q°), there exists another edge e' with the same multiplicity r, connecting the vertices (n_j, n_i°) and (n_q, n_p°).*

(2) $JD = \pm DJ$ and $\big[[D,a], Jb^*J^{-1}\big] = 0$, for all a, b in $\mathcal{A}$, if and only if the edges in the diagram connecting two different vertices are either vertical or horizontal, maintaining the symmetry with respect to the diagonal.

Remark 3
In Theorem 1 we listed necessary and sufficient conditions on the diagram (Γ, Λ) to conclude that an antilinear isometry J, defined on a Hilbert space $\mathcal{H}$, with $(\mathcal{A}, \mathcal{H}, D)$ a finite spectral triple, is a real structure. It is possible to prove an even stronger theorem, which states the existence of a one-to-one correspondence between real finite spectral triples, up to unitary equivalence, and Krajewski diagrams satisfying conditions (1) and (2).
We are not going any further into the discussion of this correspondence, referring to [53] for full details. For us, the statement in Theorem 1 is sufficient: in fact, this graphical method allows to immediately check if the conditions to have a finite real spectral triple are satisfied. Moreover, in the case in which the Hilbert space $\mathcal{H}$, the operator D and the real structure J have already been fixed, the method given by the Krajewski diagrams can be used to determine a suitable algebra $\mathcal{A}$ to complete $(\mathcal{H}, D, J)$ to a finite real spectral triple. This is, indeed, the context in which we will apply this method (see Section 7.1, 7.2) and the reason for which we decided to describe it.

2.3 Gauge theories from spectral triples

In this section we state the fundamental notion of *gauge theory*. Even though gauge theories are usually defined over a manifold, we restrict on the 0-dimensional case, i.e., as base manifold we consider a point. Hence we work with the following notion of gauge theory.

Definition 10. *Let $(X_0, S_0, \mathcal{G})$ be a triple where:*

- *X_0 is a vector space over $\mathbb{R}$;*
- *S_0 is a functional on X_0 with values in $\mathbb{R}$, $S_0 : X_0 \longrightarrow \mathbb{R}$;*
- *$\mathcal{G}$ is a group acting on X_0 through the action:*

$$F : \mathcal{G} \times X_0 \longrightarrow X_0.$$

The triple $(X_0, S_0, \mathcal{G})$ *is a* gauge theory with gauge group $\mathcal{G}$ *if the functional* S_0 *is invariant under the action of the group* $\mathcal{G}$*, that is, if for any element* $g \in \mathcal{G}$*,*

$$S_0(F(g,\varphi)) = S_0(\varphi), \qquad \forall \varphi \in X_0.$$

Note: in the physics literature, X_0 is called the *configuration space*, while an element φ in X_0 is a *gauge field*. The functional S_0 is called the *action* and $\mathcal{G}$ is known as the *gauge group*.
In what follows, next to the notation $(X_0, S_0, \mathcal{G})$, a gauge theory will also be denoted by the shorter (X_0, S_0), where we keep track only of the configuration space X_0 and of the action S_0, while the gauge group $\mathcal{G}$ does not appear explicitly in the notation.

The aim of this section is to explain how a spectral triple naturally gives rise to a gauge theory. We restrict our discussion to the case of finite spectral triples. However, a similar construction can be done also in the general setting of spectral triples defined for infinite-dimensional algebras and Hilbert spaces [53].

We emphasize that in general a configuration space is not required to have a vector space structure. However, for the particular case of gauge theory induced by a finite spectral triple, the configuration space X_0 is always equipped with a real vector space structure, as will be made clear by the construction described in the following proposition.

Proposition 1
Given a finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$, it induces a gauge theory $(X_0, S_0, \mathcal{G})$, defined as follows:

- $X_0 := \left\{\varphi = \sum_j a_j \left[D, b_j\right] : \varphi^* = \varphi,\ a_j, b_j \in \mathcal{A}\right\}$
  where $*$ denotes the involution defined on the involutive algebra $\mathcal{A}$;
- $\mathcal{G} := \mathcal{U}(\mathcal{A}) = \left\{u \in \mathcal{A} : uu^* = u^*u = 1\right\}$,
  where $\mathcal{G}$ acts on X_0 as follows:

$$\begin{array}{ccc} \mathcal{G} \times X_0 & \longrightarrow & X_0 \\ (u,\varphi) & \mapsto & u\varphi u^* + u[D, u^*]. \end{array} \tag{2.7}$$

- $S_0[D + \varphi] := Tr\big(f(D+\varphi)\big)$,
  with f is a polynomial in one real variable, while Tr denotes the usual trace of matrices.

Proof. The proof of the proposition consists of elementary computations. □

Note: in the construction described in Proposition 1, we have that:

- ▶ The configuration space X_0 is defined to be the space of self-adjoint elements in the space of 1-forms $\Omega^1(\mathcal{A})$, whose definition was already given in (2.4). The set of self-adjoint elements in $\Omega^1(\mathcal{A})$ is also known as the space of the *inner fluctuations* [19].

- ▶ As an action S_0 we consider the *spectral action* of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. We recall that the notion of a spectral action has already been stated in Definition 5.

- ▶ In the physics literature, it is usually said that the gauge group $\mathcal{G}$ acts on X_0 by so-called *gauge transformations* (2.7).

Remark 4
The construction of a gauge theory presented in Proposition 1 is typical of the noncommutative geometrical setting. Indeed, in the commutative case a gauge theory is usually defined starting with an initial pair (M, G), where M is a smooth manifold and G is a Lie group. In this setting we have that:

- ▶ the configuration space X_0 is a principal fibre bundle P over M with structure group G;

- ▶ the gauge group $\mathcal{G}$ is defined to be the set of all the principal bundle automorphisms of $P \xrightarrow{\pi} M$ over the identity map on M, $id : M \to M$.
  In other words, the gauge group is the set of all smooth and invertible maps $\varphi : P \to P$ such that

$$\pi(\varphi(p)) = \pi(p), \qquad \varphi(pg) = \varphi(p)g, \qquad \forall p \in P,\ \forall g \in G.$$

For more details on this construction and on the necessary notions of differential geometry we refer to [13], [45].

To conclude, we apply the construction presented above to an example: we show how a finite spectral triple on the algebra $M_n(\mathbb{C})$ naturally defines a $U(n)$-gauge invariant matrix model. This example, in the case of $n = 2$, will play a fundamental role in the second part of the thesis, where the induced $U(2)$-gauge theory will be analyzed using the BV formalism.

Example 1
Let us consider the following finite spectral triple on the algebra $M_n(\mathbb{C})$:

$$(M_n(\mathbb{C}), \mathbb{C}^n, D), \tag{2.8}$$

with D a self-adjoint $n \times n$-matrix. Then the induced gauge theory is given by:

- $X_0 = \big\{A \in M_n(\mathbb{C}) : A^* = A\big\}$;
- $\mathcal{G} \simeq U(n)$;
- $S_0[D+\varphi] = Tr(f(D+\varphi))$,
  with $\varphi \in X_0$ and f a polynomial in one real variable.

The construction of the configuration space is based on the fact that

$$\Omega^1(M_n(\mathbb{C})) \simeq M_n(\mathbb{C}),$$

which can be verified with a direct computation (see [53, Lemma 2.23]).
Thus the spectral triple (2.8) naturally gives rise to a $U(n)$-gauge invariant matrix model.

Chapter 3

The BV approach to gauge theories

The central topic of this chapter is the Batalin-Vilkovisky (BV) approach to gauge theories. As we already explained in Chapter 1, the fundamental idea of BV approach is the elimination of local degrees of freedom of a gauge theory via the introduction of extra fields, known as *ghost fields.* Although the motivation that led to the discovery of the notion of ghost fields came from the context of the quantization of gauge theories via the path integral approach, the aim of this chapter is to explain that the ghost fields are not simply a tool for solving a specific problem, but that they have a more fundamental role, namely as generators of a cohomology complex, known as the *BRST-cohomology complex.* To achieve this goal we will proceed as follows:

- In Section 3.1, the notion of *extended theory* is introduced as the mathematical object to describe the theory obtained as an extension of the initial gauge theory through the introduction of ghost fields and antighost fields.
- The central point of Section 3.2 will be the introduction of the notion of *classical BRST-cohomology complex.* The ghost fields play a fundamental role in this construction, since they are generators of this cohomology theory.
- Section 3.3 will be devoted to explaining the *gauge-fixing procedure*: first we will motivate the necessity for carrying out this procedure from a physical point of view, and then this procedure itself will be described in the context given by the BV formalism.

- The aim of Section 3.4 is to give an intuitive idea why the BRST cohomology may be interesting also from a physical point of view, at least for 4-dimensional gauge theories. More precisely, we explain the relation between physical aspects of the theory, such as the space of physical observables, and the corresponding BRST cohomology groups.
- The introduction of the fundamental notion of *gauge-fixed BRST cohomology* will be the main task of Section 3.5: this cohomology theory will play a very important role in the rest of the thesis.
- To conclude, in Section 3.6, we analyze the devices of the auxiliary fields and the trivial pairs, first discussing the physical reason that enforce the introduction of these auxiliary fields and then justifying why this further enlargement of the extended configuration space does not induce any changes at the level of the corresponding cohomology groups.

3.1 The extended variety

The BV construction is basically a procedure to construct an extended pair $(\widetilde{X}, \widetilde{S})$, by starting with an initial gauge theory (X_0, S_0). In this section, to describe the pair $(\widetilde{X}, \widetilde{S})$ we present the definition of an *extended variety*. This concept is crucial, since it represents the right mathematical notion to describe a suitable extension $(\widetilde{X}, \widetilde{S})$ for a given gauge-invariant theory (X_0, S_0). This notion is a generalization of the notion of *BV variety*, first introduced by Felder and Kazhdan [28].
For more details concerning the algebraic notions we refer to [47] while, for aspects related to algebraic geometry, the standard reference is [37].

Up to now, we have described a gauge theory as a pair (X_0, S_0) consisting of a configuration space X_0, endowed with a real vector space structure, together with an action functional S_0, defined on X_0 with values in $\mathbb{R}$. However, since the construction we are going to present can be applied to a more general context, we are going to use a different notation: for the whole section, by (M_0, S_0) we denote a pair in which:

- M_0 is a nonsingular algebraic variety over the field $\mathbb{K}$, which can be either $\mathbb{R}$ or $\mathbb{C}$;
- S_0 is a regular function on M_0, i.e. it is an element of the structure sheaf $\mathcal{O}_{M_0}$, such that S_0 is invariant under the action of a Lie group $\mathcal{G}$.

Definition 11. *Let B be a commutative unital ring and let $V = \oplus_{i \in \mathbb{Z}_{\geqslant 0}} V^i$ be a graded module over B with free homogeneous components V^i of finite rank and such that $V^0 = 0$. An element $a \in V^i$ is said to be* homogeneous of degree i. *The* symmetric algebra $\mathrm{Sym}(V)$ *is defined as the following quotient:*

$$\mathrm{Sym}(V) = \frac{T(V)}{K}, \tag{3.1}$$

where:

- *$T(V)$ is the tensor algebra of V;*
- *K is the B-module generated by the following relation:*

$$ab = (-1)^{deg(a)deg(b)}ba, \qquad \forall a, b \in V, \quad a, b \text{ homogeneous.}$$

Remark 5
From Definition 11, it follows that $\mathrm{Sym}(V)$ has the structure of a graded commutative algebra, where the grading is the grading in V.

In the following, with $F^q(\mathrm{Sym}(V))$ we denote the subspace of $\mathrm{Sym}(V)$ generated by elements of degree at least q. More explicitly:

$$F^q(\mathrm{Sym}(V)) = \{a \in \mathrm{Sym}(V) : deg(a) \geqslant q\} \cup \{0\}. \tag{3.2}$$

For now, we are defining this set $F^q(\mathrm{Sym}(V))$ for a $\mathbb{Z}_{\geqslant 0}$-graded module V. In what follows, this definition will be extended to the case of $\mathbb{Z}$-graded modules.

Note:

- $F^q(\mathrm{Sym}(V))$ has the structure of an ideal: from the way in which the grading is defined over $\mathrm{Sym}(V)$, the product of an element in $F^q(\mathrm{Sym}(V))$ with an element in $\mathrm{Sym}(V)$ always gives an element of degree at least q, i.e., the product is again an element in $F^q(\mathrm{Sym}(V))$.
- The collection of the ideals $\{F^q(\mathrm{Sym}(V))\}$, for $q \in \mathbb{Z}_{\geqslant 0}$, forms a descending filtration of $\mathrm{Sym}(V)$:

$$\mathrm{Sym}(V) \supseteq F^1(\mathrm{Sym}(V)) \supseteq F^2(\mathrm{Sym}(V)) \supseteq \cdots$$

Definition 12. *The* completion $\widehat{\mathrm{Sym}}(V)$ *of the graded algebra* $\mathrm{Sym}(V)$ *is the inverse limit of* $\mathrm{Sym}(V)/F^q(\mathrm{Sym}(V))$ *in the category of graded modules. More explicitly,*

$$\widehat{\mathrm{Sym}}(V) = \oplus_{i\in\mathbb{Z}}[\widehat{\mathrm{Sym}}(V)]^i,$$

with

$$[\widehat{\mathrm{Sym}}(V)]^i = \varprojlim_p [\mathrm{Sym}(V)]^i/(F^q\,\mathrm{Sym}(V) \cap [\mathrm{Sym}(V)]^i).$$

Note:

- ▶ $\widehat{\mathrm{Sym}}(V)$ again has the structure of a graded commutative algebra.
- ▶ In the case in which we are considering a module V that is graded only on $\mathbb{Z}_{\leq 0}$ or $\mathbb{Z}_{\geq 0}$, the completion $\widehat{\mathrm{Sym}}(V)$ coincides with the symmetric algebra $\mathrm{Sym}(V)$.

In the following definition we state the notion of *graded space*, which will play a fundamental role in the definition of an extended variety. This definition was introduced by Manin [48], who stated a general notion of $\mathbb{Z}/2\mathbb{Z}$-graded spaces. The generalization to $\mathbb{Z}$-graded spaces is discussed in [28].

Definition 13. *Let* M_0 *be a topological space. Let* $\mathcal{O}_{M_0}$ *be a sheaf of* $\mathbb{Z}$*-graded commutative rings on* M_0 *such that the stalk* $\mathcal{O}_{M_0,x}$*, for all* x *in* M_0*, is a local graded ring, that is to say, a ring with only one maximal proper graded ideal. The sheaf* $\mathcal{O}_{M_0}$ *is called the* structure sheaf *of* M_0*. The pair* $M = (M_0, \mathcal{O}_{M_0})$*, given by a topological space* M_0 *and its graded structure sheaf, is called a* graded space*.*

Remark 6

In the above definition, a general notion of *structure sheaf* has been introduced in the case in which the underlying space M_0 is supposed to simply be a topological space. This definition is a generalization of the usual notion of structure sheaf for an irreducible algebraic variety M_0, which we briefly recall for completeness.

Let M_0 be an irreducible algebraic variety over an algebraically closed field $\mathbb{K}$, with $M_0 \subseteq \mathbb{A}^n_{\mathbb{K}}$, $n \in \mathbb{N}$. Then, applying the Nullstellensatz, an equivalent way to describe M_0 is as the zero locus of a collection of polynomials forming a prime ideal. More precisely:

$$M_0 = V(I) = \{p \in \mathbb{A}^n_{\mathbb{K}} : f(p) = 0,\ \forall f \in I\}\ ,$$

where I is a prime ideal in the ring of polynomials $\mathrm{Pol}_{\mathbb{K}}(x_1, \dots, x_n)$, which is also often denoted by $\mathbb{K}[x_1, \dots, x_n]$.

Then the *coordinate ring* $\Gamma(M_0)$ of M_0 is defined to be the quotient of the polynomial ring $\mathrm{Pol}_{\mathbb{K}}(x_1, \ldots, x_n)$ by the ideal I:

$$\Gamma(M_0) = \frac{\mathrm{Pol}_{\mathbb{K}}(x_1, \ldots, x_n)}{I}.$$

Since I is a prime ideal, the coordinate ring $\Gamma(M_0)$ is an integral domain and so its field of fractions $Frac(\Gamma(M_0))$ can be constructed. Formally:

$$Frac(\Gamma(M_0)) = \frac{\{f/g : f, g \in \Gamma(M_0),\ g \neq 0\}}{\sim}$$

where $\sim$ is the following equivalence relation:

$$\frac{f_1}{g_1} \sim \frac{f_2}{g_2} \quad \Leftrightarrow \quad f_1 g_2 - f_2 g_1 = 0 \qquad \text{in } \Gamma(M_0).$$

Definition 14. *An element $h \in Frac(\Gamma(M_0))$ is said to be* regular at the point *x, with $x \in M_0$ if, in a neighborhood of x, there exists a representation of h as a quotient $h = f/g$, with $f, g \in \Gamma(M_0)$ such that $g(x) \neq 0$.*

Then:

- the *stalk at the point x of the structure sheaf* $\mathcal{O}_{M_0,x}$ is defined as

  $$\mathcal{O}_{M_0,x} = \{h \in Frac(\Gamma(M_0)) : h \text{ is regular at } x\}.$$

  Note: $\mathcal{O}_{M_0,x}$ is a subring of $Frac(\Gamma(M_0))$ and it is a local ring with unique maximal ideal $\mathfrak{m}_x$ defined as

  $$\mathfrak{m}_x = \{f/g \in \mathcal{O}_{M_0,x} : f(x) = 0,\ g(x) \neq 0\}.$$

- For any open set $U \subset M_0$ we define the *section of the structure sheaf* $\mathcal{O}_{M_0}$ *over U* by

  $$\mathcal{O}_{M_0}(U) = \{f \in Frac(\Gamma(M_0)) : f \text{ is regular in each point of } U\}.$$

  Note: $\mathcal{O}_{M_0}(M_0) = \Gamma(M_0)$.

  In other words, the coordinate ring, known also as the *ring of regular functions*, coincides with the ring of global sections of the structure sheaf.

This concludes the definition of the structure sheaf $\mathcal{O}_{M_0}$ of an irreducible algebraic variety M_0. Thus a pair $(M_0, \mathcal{O}_{M_0})$, with M_0 an irreducible algebraic variety and $\mathcal{O}_{M_0}$ the usual structure sheaf as described above, is a particular example of a graded space, as defined in Definition 13, considered with a trivial grading.

To conclude the description of these objects, we still have to state the natural notion of *morphism* in this context.

Definition 15. *Given two graded spaces $M = (M_0, \mathcal{O}_{M_0})$ and $N = (N_0, \mathcal{O}_{N_0})$, a* morphism *$M \to N$ is given by a pair $f = (f_0, \varphi)$ where:*

- *$f_0 : M_0 \to N_0$ is a homeomorphism;*
- *$\varphi : \mathcal{O}_{N_0} \to \mathcal{O}_{M_0}$ is a grading-preserving morphism of sheaves of rings such that, for all $x \in M_0$, φ sends the maximal ideal of $\mathcal{O}_{N_0, f_0(x)}$ to the maximal ideal of $\mathcal{O}_{M_0,x}$.*

As already said at the beginning of this section, we are interested in analyzing the case in which the underlying space M_0 is an algebraic variety. For this reason, we adapt the notion of graded space to this setting, introducing the concept of *graded algebraic variety.*

Definition 16. *A* graded algebraic variety *is a graded space $M = (M_0, \mathcal{O}_{M_0})$ such that:*

- *M_0 is an algebraic variety;*
- *for each point x in M_0 there exists an open neighborhood $U \subseteq M_0$, with $x \in U$, such that*
$$(U, \mathcal{O}_{M_0}|_U) \simeq (U, \widehat{\mathrm{Sym}}_{\mathcal{O}_{M_0}}(\mathcal{E}))$$
as graded spaces, for some free graded $\mathcal{O}_{M_0}$-module $\mathcal{E}$ with homogeneous components of finite rank, and $\mathcal{E}^0 = 0$.

The next step that needs to be taken before being able to state the notion of extended variety is to introduce a notion of *shifted cotangent bundle* associated to a nonsingular graded algebraic variety. In fact, as already for the "classical" algebraic variety, to have a well-defined notion of "tangent" we have to restrict ourselves to the case of a *nonsingular graded algebraic variety.* Moreover, we emphasize that the construction of this shifted cotangent bundle is presented for $\mathbb{Z}_{\geqslant 0}$-graded variety. To keep track of these assumptions made on the regularity and the grading of the variety we use a different notation: by $V = (V_0, \mathcal{O}_{V_0})$ we denote a graded algebraic variety such that:

- ► V_0 is a nonsingular algebraic variety;
- ► the structure sheaf $\mathcal{O}_{V_0}$ is an $\mathbb{Z}_{\geqslant 0}$-graded commutative ring, i.e.,

$$\mathcal{O}_{V_0} = \oplus_{i \in \mathbb{Z}_{\geqslant 0}} (\mathcal{O}^i_{V_0}).$$

In what follows, we refer to $V = (V_0, \mathcal{O}_{V_0})$ as a *nonsingular* $\mathbb{Z}_{\geqslant 0}$*-graded algebraic variety.*

Definition 17. *Let* $V = (V_0, \mathcal{O}_{V_0})$ *be a nonsingular* $\mathbb{Z}_{\geqslant 0}$*-graded algebraic variety. Then an* endomorphism of degree n over $\mathcal{O}_{V_0}$ *is given by a collection* $\varphi = \{\varphi_U\}$*, with* $U \subseteq V_0$ *and* U *open, where each* φ_U *is an endomorphism of graded commutative rings with*

$$\varphi_U : \mathcal{O}^i_{V_0}(U) \to \mathcal{O}^{i+n}_{V_0}(U), \qquad \forall i \in \mathbb{Z}_{\geqslant 0}.$$

Note: the collection of all endomorphisms of degree n forms a sheaf over V_0, denoted by $\prod_{i=0}^{\infty} \mathcal{H}om(\mathcal{O}^i_{V_0}, \mathcal{O}^{i+n}_{V_0})$.

Definition 18. *Let* $V = (V_0, \mathcal{O}_{V_0})$ *be a nonsingular* $\mathbb{Z}_{\geqslant 0}$*-graded variety. Then:*

- ► *a section* ξ *of the sheaf* $\prod_{i=0}^{\infty} \mathcal{H}om(\mathcal{O}^i_{V_0}, \mathcal{O}^{i+n}_{V_0})$ *is said to be a* (left) derivation of $\mathcal{O}_{V_0}$ of degree n*, with* $n \in \mathbb{N}$ *if*

$$\xi(ab) = \xi(a)b + (-1)^{n \cdot deg(a)} a\xi(b), \qquad \forall a, b$$

with $a \in \mathcal{O}^{deg(a)}_{V_0,x}$*,* $b \in \mathcal{O}^{deg(b)}_{V_0,x}$*, and* $x \in V_0$*;*

- ► *the* tangent sheaf T_V *of* V *is*

$$T_V = \oplus_{n \in \mathbb{Z}_{\geqslant 0}} T^n_V := \oplus_{n \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}om^{der}(\mathcal{O}^i_{V_0}, \mathcal{O}^{i+n}_{V_0}),$$

with $\mathcal{H}om^{der}(\mathcal{O}^i_{V_0}, \mathcal{O}^{i+n}_{V_0})$ *the set of all derivations of* $\mathcal{O}_{V_0}$ *of degree* n*.*

Note: the set T^n_V can be equipped with a sheaf structure over V_0 while the tangent sheaf T_V can be seen as a sheaf of graded Lie algebras whose sections act on $\mathcal{O}_{V_0}$ by derivations.

Notation: given a graded object $\mathcal{E} = \oplus_{i \in \mathbb{Z}} \mathcal{E}^i$, by $\mathcal{E}[j]$ we indicate a new object obtained from the previous by shifting it by j: explicitly,

$$\mathcal{E}[j] = \oplus_{i \in \mathbb{Z}} [\mathcal{E}[j]]^i = \oplus_{i \in \mathbb{Z}} [\mathcal{E}]^{i+j}.$$

In the construction we are going to present, a very important role will be played by $\mathrm{Sym}_{\mathcal{O}_{V_0}} T_V[1]$, i.e. by the symmetric algebra over the ring $\mathcal{O}_{V_0}$ of the $\mathbb{Z}_{\geqslant 0}$-graded module over $\mathcal{O}_{V_0}$ given by the shifted tangent sheaf $T_V[1]$.

Note: this symmetric algebra $\mathrm{Sym}_{\mathcal{O}_{V_0}} T_V[1]$ turns out to have a very rich structure: the Lie bracket structure defined over the tangent sheaf can be extended to a Poisson bracket structure of degree 1 on the whole symmetric algebra $\mathrm{Sym}_{\mathcal{O}_{V_0}} T_V[1]$.
Thus $\mathrm{Sym}_{\mathcal{O}_{V_0}} T_V[1]$ becomes a sheaf of P_0-algebras, whose definition we briefly recall, taking in this way the opportunity to also fix the conventions on the grading.

Definition 19. *A* P_0-algebra A *over a field* $\mathbb{K}$ *is a* $\mathbb{Z}$*-graded commutative algebra* $A = \bigoplus_{d\in\mathbb{Z}} A^d$ *endowed with a degree* 1 *Poisson bracket structure:*

$$\{-,-\} : A \otimes A \longrightarrow A,$$

with $deg(\{\varphi,\psi\}) = deg(\varphi) + deg(\psi) + 1$*, for all homogeneous elements* $\varphi, \psi \in A$*.*

Definition 20. *A* differential P_0-algebra *is given by a triple* $(A, d, \{-,-\})$ *where:*

- ▶ A *is a* $\mathbb{Z}$*-graded commutative algebra over a field* $\mathbb{K}$*,* $A = \oplus_{i\in\mathbb{Z}} A^i$*;*

- ▶ d *is a differential operator of degree* 1*:*

$$d : A^i \to A^{i+1};$$

- ▶ $\{-,-\}$ *is a graded Poisson bracket defined on* A *and of degree* 1*:*

$$\{-,-\} : A^i \otimes A^j \to A^{i+j+1};$$

Moreover, A*,* d *and* $\{-,-\}$ *are required to satisfy the following axioms, for all homogeneous elements* $a, b, c \in A$*:*

1. *the bracket is a bilinear map;*

2. *the bracket is graded symmetric:*

$$\{a, b\} = -(-1)^{(deg(a)-1)(deg(b)-1)} \{b, a\};$$

3. *the bracket is a graded Poisson bracket:*

$$\{ab, c\} = a\{b, c\} + (-1)^{deg(a)deg(b)} b\{a, c\};$$

4. *the bracket satisfies the graded Jacobi identity:*

$$(-1)^{(deg(a)-1)(deg(c)-1)}\{a, \{b, c\}\} + (-1)^{(deg(b)-1)(deg(a)-1)}\{b, \{c, a\}\}$$
$$+(-1)^{(deg(c)-1)(deg(b)-1)}\{c, \{a, b\}\} = 0;$$

5. *d is a graded derivative map:*

$$d(ab) = (d(a))b + (-1)^{deg(a)} ad(b);$$

6. *the operator d is a graded distributive operator when composed together with the bracket:*

$$d(\{a, b\}) = \{d(a), b\} + (-1)^{deg(a)-1}\{a, d(b)\}.$$

Remark 7
From axioms 2. and 3. one can deduce that the Poisson bracket on a P_0-algebra A satisfies also the following property:

$$\{a, bc\} = \{a, b\} c + (-1)^{deg(b)deg(c)}\{a, c\} b,$$

$\forall a, b, c \in A$, homogeneous elements.

In the literature, one also encounters the name *BV algebra* for a differential P_0-algebra. Nevertheless, the terminology P_0-algebra is the one used in [24] and [28], to which we are referring in this section. In general, with the terminology P_j-algebra, [24] denotes a graded commutative algebra with a Poisson bracket of degree $1 - j$.

Using the notation already introduced in (3.2), for $q \in \mathbb{Z}_{\geqslant 0}$, we define:

$$F^q(Sym_{\mathcal{O}_{V_0}}(T_V[1])) = \big\{a \in Sym_{\mathcal{O}_{V_0}}(T_V[1]) : deg(a) \geqslant q\big\}. \qquad (3.3)$$

More explicitly, we denote by $F^q(Sym_{\mathcal{O}_{V_0}}(T_V[1]))$ the ideal in $Sym_{\mathcal{O}_{V_0}}(T_V[1])$ given by elements of degree at least q.

Note:

- $F^0(Sym_{\mathcal{O}_{V_0}}(T_V[1])) = Sym_{\mathcal{O}_{V_0}}(T_V[1])$;
- the sheaves of ideals given by $F^q(Sym_{\mathcal{O}_{V_0}}(T_V[1]))$, with $q \in \mathbb{Z}_{\geqslant 0}$ form a descending filtration of $Sym_{\mathcal{O}_{V_0}}(T_V[1])$.

Thus we can use this filtration to construct the completion of $Sym_{\mathcal{O}_{V_0}}(T_V[1])$, to which we refer as the *(-1)-shifted cotangent bundle* of V and which is denoted by $T^*[-1]V$.

Definition 21. *Given* $V = (V_0, \mathcal{O}_{V_0})$, *the* (-1)-shifted cotangent bundle *of* V *is the graded variety* $T^*[-1]V = (V_0, \mathcal{O}_{T^*[-1]V})$ *where the structure sheaf* $\mathcal{O}_{T^*[-1]V}$ *is defined as*

$$\mathcal{O}_{T^*[-1]V} = \lim_{\leftarrow q} Sym_{\mathcal{O}_{V_0}}(T_V[1])/F^q(Sym_{\mathcal{O}_{V_0}}(T_V[1])),$$

where the inverse limit is taken in the category of $\mathbb{Z}$*-graded sheaves.*

Note: the Poisson structure defined on $Sym_{\mathcal{O}_{V_0}}(T_V[1])$ extends also to its completion $T^*[-1]V$, making the (-1)-shifted cotangent bundle a sheaf of P_0-algebras over V_0. (see [28, Proposition 2.9]).

Definition 22. *Let* V *and* W *be two* nonsingular $\mathbb{Z}_{\geqslant 0}$-graded algebraic variety*.* *A* Poisson morphism $\varphi : T^*[-1]V \to T^*[-1]W$ *is a map of graded varieties that respects the Poisson bracket.*

Definition 23. *A* (-1)-symplectic variety *is a graded variety* $M = (M_0, \mathcal{O}_M)$ *that is locally Poisson isomorphic to a* (-1)*-shifted cotangent bundle: each point* $x \in M_0$ *has an open neighborhood* U *such that* $\mathcal{O}_M|_U$ *is Poisson isomorphic to* $\mathcal{O}_{T^*[-1]V}|_U$*, for some non-negatively graded variety* $V = (U, \mathcal{O}_V)$*.*

An important point is that it is possible to give an explicit description of a graded variety that is globally Poisson isomorphic to the (-1)-shifted cotangent bundle of a non-negatively graded variety, as precisely stated in the following proposition (for the proof, see [28]).

Proposition 2
Let $M = (M_0, \mathcal{O}_M)$ be a graded variety, with M_0 a nonsingular algebraic variety. If M is globally Poisson isomorphic to $T^*[-1]V$ for some $\mathbb{Z}_{\geqslant 0}$-graded variety V, then:

- V is isomorphic to a graded variety $(M_0, Sym_{\mathcal{O}_{M_0}}\mathcal{E})$, where $\mathcal{E}$ is a graded $\mathcal{O}_{M_0}$-module with homogeneous components $\mathcal{E}^q$ defined as follows:

$$\mathcal{E}^q = {\mathcal{O}_M}^q / F^{q+1}\mathcal{O}_M^q + I_M \cdot (I_M \cap \mathcal{O}_M^q), \qquad q \geqslant 1$$

  where $I_M = F^1\mathcal{O}_M$ and $F^q\mathcal{O}_M$ is the ideal of $\mathcal{O}_M$ generated by elements of degree at least q.

- Moreover, if $\mathcal{E}$ is also a locally free $\mathcal{O}_{M_0}$-module with homogeneous components of finite rank, then the following isomorphism holds:

$$\mathcal{O}_M \cong \widehat{Sym}_{\mathcal{O}_{M_0}}(T_{M_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]).$$

Finally we introduce the fundamental notion of a *solution of the classical master equation.*

Definition 24. *Let $M = (M_0, \mathcal{O}_M)$ be a (-1)-symplectic variety with M_0 a nonsingular algebraic variety. Given a function $S \in \Gamma(M_0, \mathcal{O}_M^0)$, i.e., a regular function in $\mathcal{O}_M$ of total degree* 0*, then the* classical master equation *for S is*

$$\{S, S\} = 0.$$

Proposition 3
Let $M = (M_0, \mathcal{O}_M)$ be a graded variety as in the above definition and let $S \in \Gamma(M_0, \mathcal{O}_M^0)$ be a solution of the classical master equation. Then the operator

$$d_S := \{S, -\}\,,$$

with

$$\begin{aligned} d_S : \quad \mathcal{O}_M^n \quad &\longrightarrow \quad \mathcal{O}_M^{n+1} \\ \varphi \quad &\mapsto \quad \{S, \varphi\}\,, \end{aligned}$$

is a differential over the sheaf of P_0-algebras $\mathcal{O}_M$.
Moreover, d_S defines a differential on the sheaf of $\mathbb{Z}_{\leq 0}$-graded algebras $\mathcal{O}_M/I_M$.

Proof. The fact that d_S is a linear operator follows immediately from the definition, since the Poisson bracket is supposed to be linear in both entries. Similarly, also the fact that d_S is a derivation of degree 1 is a consequence of the properties of the Poisson bracket: the condition of being a derivation follows from Remark 7, while being of degree 1 is a consequence of the Poisson bracket being of degree 1 and the action S being an element of degree 0.

Finally, the property $d_S^2 = 0$ follows from the graded Jacobi identity and from the fact that S is supposed to be a solution of the classical master equation. More explicitly:

$$d_S^2(\varphi) = \{S, \{S, \varphi\}\} = (-1)^{n-1} \{S, \{\varphi, S\}\} = - \{S, \{S, \varphi\}\} \tag{3.4}$$

for all $n \in \mathbb{N}_0$ and $\varphi \in \mathcal{O}_M^n$.

To conclude, since d_S is a derivation of degree 1, it preserves the ideal $I_M = F^1\mathcal{O}_M$, where $F^q\mathcal{O}_M$ denotes the ideal in $\mathcal{O}_M$ generated by elements of degree at least q. Therefore, d_S defines a differential on the sheaf of $\mathbb{Z}_{\leq 0}$-graded algebras $\mathcal{O}_M/I_M$. □

Now we state the definition of an *extended variety* in the BV formalism. As claimed at the beginning of this section, the notion of an extended variety is the translation in a mathematical structure of the idea of extending a gauge theory via the introduction of ghost field: the explanation of this statement is postponed to Section 3.1.2.

Definition 25. *Let M_0 be a nonsingular algebraic variety and let S_0 be a regular function on M_0. An* extended variety *with support (M_0, S_0) is a pair $(N, \widetilde{S})$ where:*

- *N is a (-1)-symplectic variety with support M_0, globally Poisson isomorphic to the (-1)-shifted cotangent bundle of a non-negatively graded variety, with $N = (M_0, \mathcal{O}_N)$ and*

$$\mathcal{O}_N \cong \widehat{Sym}_{\mathcal{O}_{M_0}}(T_{M_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]),$$

  for $\mathcal{E}$ a graded locally free $\mathcal{O}_{M_0}$-module with homogeneous components of finite rank.

- *$\widetilde{S}$ is an element of the structure sheaf of N of degree 0, $\widetilde{S} \in \Gamma(M_0, \mathcal{O}_N^0)$ such that:*

  1. *$\widetilde{S}|_{M_0} = S_0$, with $\widetilde{S} \neq S_0$;*
  2. *$\widetilde{S}$ is a solution of the classical master equation, i.e., $\{\widetilde{S}, \widetilde{S}\} = 0$.*

Notation: in what follows, we refer to the difference $\widetilde{S} - S_0$ as the *BV action* (denoted by S_{BV}) of the extended variety $(N, \widetilde{S})$ for a gauge theory (X_0, S_0).

Remark 8
It follows immediately from the definition that, given a generic extended variety $(N, \widetilde{S})$, the action $\widetilde{S}$ has symmetry invariance. More precisely, the action $\widetilde{S}$ is invariant under the symmetry transformation defined by the operator $d_{\widetilde{S}}$ introduced in Proposition 3. In fact, since by definition $d_{\widetilde{S}}(\widetilde{S}) = \{\widetilde{S}, \widetilde{S}\}$, it follows that $\widetilde{S}$ is invariant under $d_{\widetilde{S}}$ in the sense that

$$\widetilde{S} \longmapsto \widetilde{S} + d_{\widetilde{S}}(\widetilde{S}) = \widetilde{S},$$

due to the fact that $\widetilde{S}$ is supposed to be a solution of the classical master equation.

The notion of extended variety stated in Definition 25 has to be compared with the notion of *BV variety*, first introduced by Felder and Kazhdan in [28], which we briefly recall for completeness.

Definition 26. *Let M_0 be a nonsingular algebraic variety and let S_0 be a regular function on M_0. A* BV variety with support (M_0, S_0) *is a pair* $(N, \widetilde{S})$ *where:*

- ► *N is a (-1)-symplectic variety with support M_0;*
- ► *$\widetilde{S}$ is an element of the structure sheaf of N of degree* 0, *$\widetilde{S} \in \Gamma(M_0, \mathcal{O}_N^0)$ such that:*
    1. *$\widetilde{S}|_{M_0} = S_0$;*
    2. *$\widetilde{S}$ is a solution of the classical master equation, i.e., $\{\widetilde{S}, \widetilde{S}\} = 0$;*
    3. *the sheaf cohomology of the complex $(\mathcal{O}_N / I_N, d_{\widetilde{S}})$ vanishes in non-zero degree.*

Remark 9
Making a comparison between the notions of an *extended variety*, on one hand, and of a *BV variety*, on the other, we notice that, even though these two notions appear to be similar, there are two main differences.
First of all, in order to have that a pair $(N, \widetilde{S})$ defines an extended variety, N is required to be *globally* Poisson isomorphic to the (-1)-shifted cotangent bundle of a non-negatively graded variety, while this property is required to be satisfied only *locally* by an N in a BV variety. The reason why we introduce this condition is that, given an extended variety, the presence of a global Poisson isomorphism allows to construct the corresponding extended configuration space, as will be explained in more detail in Section 3.1.1.

The second main difference between these two notions lies in the presence of a third condition on the extended action $\widetilde{S}$ to be the action in a BV variety. This extra condition concerns the vanishing of the non-zero degree cohomology of a certain complex and it is not explicitly required by the BV formalism, contrary to the first two conditions, which are essential for the BV construction. Therefore, given an initial gauge theory, this third requirement on the action selects a family among the all extended theories associated to the initial gauge theory and suitable for starting an analysis of the corresponding BRST cohomology complex. This selected family satisfies additional properties concerning the BRST cohomology complex and its independence of the BV variety used to define it.
The consequences determined by the presence of this third condition on the extended action of a BV variety will be analyzed in Section 3.2, where the notion of BRST cohomology is also stated.

3.1.1 The extended configuration space of an extended variety

Let (X_0, S_0) be a gauge invariant theory, with X_0 the initial configuration space. Due to our notion of gauge theory, the configuration space X_0 is supposed to be endowed with a real vector space structure. Thus X_0 can be seen as a real affine space and hence, in particular, as a nonsingular affine variety. This implies that the theory presented in the previous section can be applied to our case of interest. Therefore, given an initial gauge theory (X_0, S_0), we are interested in constructing an extended variety $(N, \widetilde{S})$ with support (X_0, S_0). Because of the properties of X_0, the corresponding graded variety that has to be considered to determine the pair $(N, \widetilde{S})$ is

$$M = (X_0, \mathcal{O}_{X_0}) \cong (\mathbb{A}^n_{\mathbb{R}}, \mathrm{Pol}_{\mathbb{R}}(x_1, \ldots, x_n)), \qquad n = \dim_{\mathbb{R}} X_0.$$

The description of a method to construct, given an initial gauge theory, a corresponding extended variety $(N, \widetilde{S})$ is postponed to Chapter 4. The aim of this section is to describe how to determine the corresponding extended configuration space $\widetilde{X}$, once the extended variety $(N, \widetilde{S})$ has been constructed. Indeed, in the second part of this thesis, the BV construction will be seen as a method to associate to an initial gauge theory (X_0, S_0) an extended theory $(\widetilde{X}, \widetilde{S})$, where $\widetilde{X}$ is the extended configuration space while $\widetilde{S}$ is the extended action. This point of view appears to be the most appropriate to approach the analysis of the BRST cohomology complex and of its relation with the initial gauge theory.

Let $(N, \widetilde{S})$ be an extended variety associated to a fixed initial gauge theory (X_0, S_0), which satisfies the properties described above. By definition of an extended variety, there exists a global Poisson isomorphism that allows to describe the graded algebra $\mathcal{O}_N$ as the completion of a symmetric algebra:

$$\mathcal{O}_N \cong \widehat{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]),$$

with $\mathcal{E}$ a $\mathbb{Z}_{>0}$-graded and locally-free module over the ring $\mathrm{Pol}_{\mathbb{R}}(x_i)$, with homogeneous components $\mathcal{E}^p$, $p > 0$, of finite rank.
Hence, for each degree $p > 0$ it is possible to find a finite set of generators for the homogeneous component $\mathcal{E}^p$, which we denote by $\{\varphi_1^{(p)}, \dots, \varphi_{q(p)}^{(p)}\}$, that is,

$$\mathcal{E}^p = \langle \varphi_1^{(p)}, \dots, \varphi_{q(p)}^{(p)} \rangle.$$

Since the graded module $\mathcal{E}^*[1]$ is defined to be the dual of the module $\mathcal{E}$, shifted by 1 in the indices, once a set of generators for $\mathcal{E}^p$ has been determined, we have automatically also determined a set of generators for $[\mathcal{E}^*[1]]^{-p-1}$:

$$[\mathcal{E}^*[1]]^{-p-1} = \langle \varphi_1^{(p)*}, \dots, \varphi_{q(p)}^{(p)*} \rangle.$$

Then, in what follows, with the terminology *extended configuration space corresponding to the extended variety* $(N, \widetilde{S})$ we indicate the $\mathbb{Z}$-graded real vector space $\widetilde{X}$ generated by the generators $\{\varphi_i^{(p)}, \varphi_i^{(p)*}\}_{i=1,\dots q(p)}$.
More explicitly:

$$\widetilde{X} = \bigoplus_{p \in \mathbb{Z}} \widetilde{X}^p,$$

with:

- for $p > 0$, $\widetilde{X}^p = \langle \varphi_1^{(p)}, \dots, \varphi_{q(p)}^{(p)} \rangle$, where $\{\varphi_i^{(p)}\}$ are the generators of the homogeneous component $\mathcal{E}^p$;
- for $p = 0$, $\widetilde{X}^0 = X_0 = \langle \varphi_1, \dots, \varphi_q \rangle$, with X_0 the initial configuration space;
- for $p = -1$, $\widetilde{X}^{-1} = \langle \varphi_1^*, \dots, \varphi_q^* \rangle$, with $\varphi_1^*, \dots, \varphi_q^*$ the generators of the shifted tangent space $T_{X_0}[1]$;
- for $p < -1$, $\widetilde{X}^p = \langle \varphi_1^{(-p-1)*}, \dots, \varphi_{q(-p-1)}^{(-p-1)*} \rangle$, with $\{\varphi_i^{(-p-1)*}\}$ the generators of $[\mathcal{E}^*[1]]^p$ obtained as dual of the generators of the homogeneous component $\mathcal{E}^{-p-1}$.

In physics, the generators $\{\varphi_i^{(p)}, \varphi_i^{(p)*}\}$ have different names, used to distinguish the different roles played by these elements and the step of the construction in which they have been introduced:

- for $p = 0$ the generators $\varphi_1, \dots, \varphi_q$ are the generators of the initial configuration space X_0 and are usually called the *initial fields*;
- for $p = -1$, the generators $\varphi_1^*, \dots, \varphi_q^*$ are the *antifields*, corresponding to the initial fields $\varphi_1, \dots, \varphi_q$;
- usually the generators of the homogeneous components $\mathcal{E}^p$ are known as *ghost fields*, and the degree p is called the *ghost degree*;
- finally, the generators of $\mathcal{E}^*$ are collectively called *antighost fields*. Also for the antighost fields we speak about their ghost degree, which is again the degree of the corresponding homogeneous component in $\mathcal{E}^*$.

Moreover, in the physical context, next to the ghost degree, another integer number, called *parity*, is associated to a ghost or an antighost field. The parity determines if a fields behaves as a real or as a Grassmannian variable, that is to say, an anticommuting and nihilpotent variable.
Hence, in the context of 0-dimensional theories, a *ghost field* φ is a graded variable that is characterized by two integers:

$$deg(\varphi) \in \mathbb{Z} \quad \text{and} \quad \epsilon(\varphi) \in \{0, 1\}, \quad \text{with} \quad deg(\varphi) = \epsilon(\varphi) \pmod{\mathbb{Z}/2}$$

where $deg(\varphi)$ is the *ghost degree*, while $\epsilon(\varphi)$ is the *parity* of φ. In particular, if $\epsilon(\varphi) = 0$, then φ is a real variable while, in the case in which $\epsilon(\varphi) = 1$, φ is a Grassmannian variable.

Furthermore, given a field/ghost field φ, the corresponding *antifield/antighost field* is denoted by φ^* and it is completely determined by φ, by imposing the following conditions:

$$deg(\varphi^*) = -deg(\varphi) - 1, \qquad \text{and} \qquad \epsilon(\varphi^*) = \epsilon(\varphi) + 1, \pmod{\mathbb{Z}/2}.$$

3.1.2 Extended varieties and the BV formalism

We already know that, in order to solve issues coming from the quantization of a gauge theory (X_0, S_0), extra fields needed to be introduced, obtaining an extended theory $(\widetilde{X}, \widetilde{S})$. The aim that drove us to develop the techniques presented in the first part of this section was to be able to describe the extended theory $(\widetilde{X}, \widetilde{S})$ from a mathematical point of view. Having arrived at the end of the section, we still have to explain why the mathematical notion used to describe this new pair $(\widetilde{X}, \widetilde{S})$ is the notion of *extended variety.*

As briefly explained in the introduction to the BV formalism given in Chapter 1, the purpose of the BV construction is to associate to a gauge theory (X_0, S_0) a new pair $(\widetilde{X}, \widetilde{S})$, where $\widetilde{X}$ is the so-called *extended configuration space*, while $\widetilde{S}$ is the *extended action.* Both $\widetilde{X}$ and $\widetilde{S}$ are required to satisfy specific properties.

Concerning the extended configuration space $\widetilde{X}$, it has to be a $\mathbb{Z}$-graded vector space, $\widetilde{X} = \oplus_{p\in\mathbb{Z}}\widetilde{X}^p$, such that:

1. In degree 0, it coincides with the initial configuration space X_0.
2. For each field/ghost field there should be a corresponding antifield/antighost field.

Therefore, by requiring that an extended variety $(N, \widetilde{S})$ is such that $N = (X_0, \mathcal{O}_N)$, with

$$\mathcal{O}_N \cong \widehat{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]),$$

for $\mathcal{E} = \oplus_{i\geqslant 1}\mathcal{E}^i$ a graded locally free $\mathcal{O}_{X_0}$-module with homogeneous components of finite rank, we are precisely imposing the conditions required by the BV formalism for the extended configuration space.
Indeed:

- Because $\mathcal{E}$ is a positively-graded module, $\mathcal{E}^*[1]$ is a negatively graded module, and $T_{X_0}[1]$ determines the generators of degree -1, no independent generators are introduced in degree 0 so that the extended configuration space $\widetilde{X}$ coincides with X_0 in degree 0.
- To comply with the condition of introducing the antifields corresponding to the initial fields in X_0, the content of degree -1 in $\widetilde{X}$ is determined by $T_{X_0}[1]$: indeed, since X_0 has the structure of an affine space, the number of generators of the tangent space T_{X_0} coincides with the dimension of X_0. In other words, the number of the antifields that we are introducing in degree -1 coincides with the number of the initial fields.
- The presence of the two terms $\mathcal{E}$ and $\mathcal{E}^*[1]$ imposes that for any ghost field in $\mathcal{E}$, we are introducing the corresponding antighost, which is an element in $\mathcal{E}^*[1]$. The module $\mathcal{E}^*$ has been shifted by 1 as a consequence of the relation between

the ghost degree of a ghost field and the ghost degree of the corresponding antighost field.

Finally, the conditions imposed on N in the definition of an extended variety imply that $\mathcal{O}_N$ is also endowed with a graded Poisson structure, which is needed in order to be able to consider the classical master equation on $\mathcal{O}_N$.

About the extended action $\widetilde{S}$, for the BV construction it is required to satisfy the following properties:

- $\widetilde{S}$ is a proper and regular extension of the initial action S_0 to the extended configuration space $\widetilde{X}$;
- $\widetilde{S}$ coincides with S_0 when restricted to X_0;
- An operator $d_{\widetilde{S}}$ is associated to $\widetilde{S}$, such that $d_{\widetilde{S}}(\widetilde{S}) = 0$.

In other words, the extended action is constructed by adding extra terms to the initial action S_0. These extra terms are required to explicitly depend on the ghost and the antighost fields, in order to comply with the condition that $\widetilde{S}$ coincides with S_0 when it is restricted to the initial configuration space X_0. Finally, since $\widetilde{S}$ should be a proper extension of S_0, the possibility of taking $\widetilde{S} = S_0$ is not allowed. Concerning the last condition, in view of Proposition 3, this can be restated by asking that $\widetilde{S}$ is a solution of the classical master equation on $\mathcal{O}_N$.
Therefore, the conditions imposed on the extended action in the definition of an extended variety (cf. Definition 25) coincide with the conditions imposed by the BV formalism.

Hence, the notion of extended theory precisely describes the object that we are looking for in the context of the BV construction. Therefore, for the type of gauge theories we are interested in analyzing, we can restate the idea at the core of the BV formalism saying that we want to provide a method to associate an extended variety $(N, \widetilde{S})$ (and thus an extended theory $(\widetilde{X}, \widetilde{S})$) to an initial gauge theory (X_0, S_0).

Up to this point, two questions naturally arise:

- Given an initial gauge theory (X_0, S_0), is it always possible to associate a corresponding extended variety $(N, \widetilde{S})$ to it?

► In case that such an extended variety $(N, \widetilde{S})$ exists, is it well motivated from a "physical point of view"? Equivalently, is the construction of the extended variety gauge independent?

In other words, at this point questions about the existence and the uniqueness of the extended variety associated to a gauge theory need to be faced. Moreover, next to these two theoretic questions, a more practical one appears:

► Given a gauge theory (X_0, S_0), is there a method to construct a corresponding extended variety $(N, \widetilde{S})$?

Similar questions were first addressed in [28], where the authors concentrate on a particular class of extended varieties, called BV varieties (cf. Definition 26). In a few words, we can say that, for the type of gauge theories we are considering, it is possible to prove existence and uniqueness of the associated BV variety, up to the introduction of a suitable concept of "gauge equivalent" BV varieties. Moreover, the proof of the existence of this BV variety is constructive, which also gives the method to explicitly construct a BV variety $(N, \widetilde{S})$ with support the initial gauge theory (X_0, S_0). However, this approach may require the introduction of an infinite number of ghost fields also for simple models. The presence of an infinite number of ghost fields may cause the impossibility of completely determining the extended action, which would be given only as an approximation up to a certain ghost degree. Finally, this would cause difficulties in analyzing the BRST cohomology associated to the extended theory.

For this reason, in Chapter 4 we present a variation on their method to associate to an initial gauge theory a corresponding extended theory. With this method it seems to be possible to introduce only a finite number of ghost fields, at least for the class of models in which we are intersted. However, with this approach the existence of an extended variety as well as the uniqueness, up to gauge equivalence, of extended actions defined on the same extended configuration space can be proved only imposing further conditions on the models.
This second construction is used in Chapter 5 to explicitly determine an extended variety corresponding to our model of interest, that is, the $U(2)$-matrix model induced by a finite spectral triple on the algebra $M_2(\mathbb{C})$.

3.2 Classical BRST cohomology

In this section we describe how an extended variety $(N, \widetilde{S})$ automatically gives rise to a differential graded complex and a notion of cohomology, known as *classical BRST cohomology.* The choice of presenting the BRST cohomology complex in the context of extended varieties is motivated by the fact that this is the setting in which we will study this cohomology complex for our model of interest. We emphasize that the BRST cohomology complex was already known before the introduction of the notion of extended variety ([10], [57]) and it also appeared in the context of the quantization of constrained systems [38], [40].

Throughout this section, by $(N, \widetilde{S})$ we denote a extended variety associated to a gauge theory (X_0, S_0), with X_0 a nonsingular affine variety.

Definition 27. *Let $(N, \widetilde{S})$ be an extended variety. Then the* classical BRST cohomology complex *associated to $(N, \widetilde{S})$ is the cohomology complex $(\mathcal{C}^\bullet(N, d_{\widetilde{S}}), d_{\widetilde{S}})$ where:*

- $\mathcal{C}^i(N, d_{\widetilde{S}})$ *is the space of cochains of degree i, which is given by the homogeneous component of degree i of the structure sheaf of N, viz.*

$$\mathcal{C}^i(N, d_{\widetilde{S}}) = [\mathcal{O}_N]^i;$$

- $d_{\widetilde{S}}$ *is the coboundary operator defined by $\widetilde{S}$, namely:*

$$d_{\widetilde{S}} = \{\widetilde{S}, -\}.$$

The cohomology $\mathcal{H}^\bullet(N, d_{\widetilde{S}})$ defined by this complex, is called the classical BRST cohomology *associated to the extended variety $(N, \widetilde{S})$.*

Remark 10
Notice that, given an extended variety $(N, \widetilde{S})$, the corresponding classical BRST complex $(\mathcal{C}^\bullet(N, d_{\widetilde{S}}), d_{\widetilde{S}})$ is a sheaf of differential P_0-algebras. Therefore, the natural notion of cohomology in this case is the notion of *hypercohomology* for a complex with the structure of a sheaf. We decided not to state the formal notion of hypercohomology, which may be found in [15], because in the case in which we are interested this notion coincides with the more usual notion of cohomology of global sections, as stated in the following lemma, which has to be compared with Corollary 5.8 in [28].

Lemma 2
Let $(N, \widetilde{S})$ be an extended variety with support (M_0, S_0), for M_0 an affine variety and let $\mathcal{H}^\bullet(N, \widetilde{S})$ be the hypercohomology of the classical BRST complex of sheaves. Then this hypercohomology coincides with the cohomology of global sections:

$$\mathcal{H}^\bullet(N, \widetilde{S}) = H^\bullet_{BRST}(\Gamma(X_0, \mathcal{O}_N), d_{\widetilde{S}}). \tag{3.5}$$

Remark 11
In general, the classical BRST cohomology complex is not determined by the initial gauge theory (X_0, S_0) but it depends on the corresponding extended variety $(N, \widetilde{S})$ considered. In other words, given an initial gauge theory, there may be different classical BRST cohomology complexes associated to it, determined by different extended varieties.
However, if we restrict ourselves to consider only a particular kind of extended varieties $(N, \widetilde{S})$, namely the BV varieties, the classical BRST cohomology sheaf $\mathcal{H}^\bullet(N, \widetilde{S})$ is uniquely determined by the initial gauge theory, as precisely stated in the following theorem, which has to be compared with [28, Corollary 4.15].

Theorem 2. *Let $(N, \widetilde{S})$ be a BV variety with support (M_0, S_0), for M_0 a non-singular affine variety. Then the classical BRST cohomology sheaf $\mathcal{H}^\bullet(N, \widetilde{S})$ is determined by the pair (X_0, S_0) up to a unique isomorphism, where $\mathcal{H}^\bullet(N, \widetilde{S})$ is the hypercohomology of the classical BRST complex of sheaves.*

Unfortunately, a similar theorem cannot be stated in the general context of extended varieties, discussed in more detail in Section 4.3.

The classical BRST cohomology is not yet the cohomology theory in which we are interested. In fact, for reasons to be explained in the next section, it is more relevant to study the residual BRST cohomology, emerging after a gauge-fixing procedure.

3.3 The gauge-fixing procedure

The main purpose of this section is to present the process of gauge-fixing a theory, extended with ghost fields, using the field-antifield formalism. Before starting with the explanation of this procedure, we briefly describe the motivation which, in the context of infinite-dimensional gauge theory, enforces a gauge-fixing process. For aspects related to quantum field theory for infinite-dimensional gauge theories, a presentation can be found in [56].

Motivation and general theory

As already mentioned in the introduction given in Chapter 1, for an infinite-dimensional gauge theory (X_0, S_0), beside the path integral being ill-defined to begin with, the presence of local symmetries makes it impossible to straightforwardly quantize the theory via the path integral approach: this was the reason for the introduction of the ghost fields in the first place. Thus we construct an extended pair $(\widetilde{X}, \widetilde{S})$ starting with the initial gauge theory (X_0, S_0), using the BV approach, one of whose fundamental aspects is that it requires the introduction of a corresponding antifield or antighost field for each field or ghost field added to the theory.

The reason why a gauge-fixing procedure needs to be carried out is that, after having performed this BV construction, the action $\widetilde{S}$ still turns out to be written in a form that is not appropriate for an analysis of the theory through methods coming from perturbation theory. From a physical point of view, the problem is that the action $\widetilde{S}$ contains antifields and antighost fields: they need to be eliminated before computing amplitudes and S-matrix elements. Therefore, the action $\widetilde{S}$ cannot be a starting point for a process of quantization via the path integral approach.

Thus the purpose of a gauge-fixing procedure is to eliminate the antifields and the antighost fields both from the extended configuration space $\widetilde{X}$ and the extended action $\widetilde{S}$. However, we immediately note that to put the antifields and the antighost fields to zero would not solve the problem, since by doing it the extended theory $(\widetilde{X}, \widetilde{S})$ would reduce to the initial theory (X_0, S_0). A way to solve the problem is by performing a *gauge-fixing procedure*, which is based on the introduction of a *gauge-fixing fermion*. The aim of this section is to present this procedure.

We emphasize once more that the reasons described above for motivating the necessity of carrying out a gauge-fixing procedure refer to infinite-dimensional gauge theories. For finite-dimensional gauge theories neither the BV construction nor the gauge-fixing procedure is required to quantize the theory via the path integral approach. However, since we will use the gauge-fixing procedure in the context of finite-dimensional gauge theory, we restrict ourselves to this particular context also for presenting this process: this allows us to describe the procedure of gauge fixing in a mathematically rigorous way, using the BV formalism. ([1], [30], [51]) For completeness we mention that the gauge-fixing procedure has been developed in a more general setting and that there are different methods to carry out a gauge-fixing procedure. For a more physical approach to the gauge-fixing procedure we refer to [34].

Summarizing, given an extended theory $(\widetilde{X}, \widetilde{S})$, we want to find a method to construct another pair $(\widetilde{X}_\Psi, \widetilde{S}_\Psi)$ such that:

- $(\widetilde{X}_\Psi, \widetilde{S}_\Psi)$ does not depend on antifields or antighost fields;
- the physically relevant quantities computed from $(\widetilde{X}_\Psi, \widetilde{S}_\Psi)$ do not depend on the choice of gauge fixing.

Throughout this section, $(\widetilde{X}, \widetilde{S})$ denotes an extended theory corresponding to a gauge invariant theory (X_0, S_0), whose initial configuration space X_0 is a real vector space. As already seen in Section 3.1.1, $\widetilde{X}$ is a $\mathbb{Z}$-graded vector space:

$$\widetilde{X} = \bigoplus_{i \in \mathbb{Z}} \widetilde{X}^i ,$$

with $\widetilde{X}^0 = X_0$ and such that each homogeneous component $\widetilde{X}^i$ is a finitely generated vector space.
Moreover:

- Since we are interested in analyzing theories for which the number of ghost fields that need to be added to the theory is finite, (see Section 4.1), there exists a positive number m in $\mathbb{N}$ such that, starting from that degree, all the homogeneous components of $\widetilde{X}$ are trivial. More explicitly:

  $$\widetilde{X}^j = \{0\} \qquad \forall j \geqslant m \quad \text{and} \quad \forall j < -m.$$

- Since, in the BV formalism, corresponding to each field and each ghost field there is a specific antifield or antighost field, the extended configuration space $\widetilde{X}$ is a $\mathbb{Z}$-graded vector space of the following form:

  $$\widetilde{X} = W \oplus W^*[1]. \tag{3.6}$$

  Here $W^*[1]$ is the graded vector space generated by the antifields and the antighost fields corresponding to the fields and ghost fields present in W. As in the previous section, the degree of an homogeneous element φ in $\widetilde{X}$, known as the *ghost degree*, is denoted by the expression $\deg(\varphi)$: therefore, given an element $\varphi \in \widetilde{X}^n$, we set $\deg(\varphi) = n$.

Remark 12
Given a $\mathbb{Z}$-graded vector space $\widetilde{X}$, it is possible to define on it a $\mathbb{Z}/2\mathbb{Z}$-grading

$$\widetilde{X} = \widetilde{W}_0 \oplus \widetilde{W}_1,$$

in a natural way as follows:

$$\widetilde{W}_0 = \bigoplus_{n\in\mathbb{Z}} \widetilde{X}^{2n}, \qquad \widetilde{W}_1 = \bigoplus_{n\in\mathbb{Z}} \widetilde{X}^{2n+1}.$$

This $\mathbb{Z}/2\mathbb{Z}$-grading is referred to as *parity*.

Thus the vector space $\widetilde{X}$ is provided with two gradings: the first is a $\mathbb{Z}$-grading, while the second is a $\mathbb{Z}/2\mathbb{Z}$-grading. To avoid misunderstanding, in what follows, given an element $\varphi \in \widetilde{X}$, with the term *degree* we will mean the degree of φ with respect to the $\mathbb{Z}$-grading. On the other hand, we will use the term *parity* for the degree of an element with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading; the parity of an element will be denoted by the symbol $\epsilon(\varphi)$, with $\epsilon(\varphi) = 0$ if $\varphi \in \widetilde{W}_0$ and $\epsilon(\varphi) = 1$ if $\varphi \in \widetilde{W}_1$.

Note: in the literature, the term *super graded vector space* is used to indicate a $\mathbb{Z}$-graded vector space $\widetilde{X}$ endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading, $\widetilde{X} = \widetilde{W}_0 \oplus \widetilde{W}_1$ such that, while the generators of the graded vector space $\widetilde{W}_0$ are considered to be real variables, the generators of $\widetilde{W}_1$ are treated as *Grassmannian variables*, that is to say, anticommuting variables: given θ, $\eta \in \widetilde{W}_1$,

$$\theta\eta = -\eta\theta.$$

In particular, for any $\theta \in \widetilde{W}_1$, $\theta^2 = 0$.

Remark 13
Let $\widetilde{X}$ be a super graded vector space and let $\mathcal{F}$ be the algebra of regular functions on $\widetilde{X}$, $\mathcal{F} = \mathcal{O}_{\widetilde{X}}$. To describe more explicitly the algebra $\mathcal{F}$, let $\{\varphi_1, \ldots, \varphi_{m_0}\}$ be the real fields that generate $\widetilde{W}_0$ and let $\{\psi_1, \ldots, \psi_{m_1}\}$ be the Grassmannian fields that generate $\widetilde{W}_1$. Notice that both the real and the Grassmannian generators, are finite in number since, by definition, $\widetilde{X}$ is a finitely generated vector space which has trivial homogeneous components starting from some finite degree m in $\mathbb{N}$. Then the graded algebra $\mathcal{F}$ can be described as the space of formal power series in $\{\varphi_i\}$ and $\{\psi_j\}$:

$$\mathcal{F} = \mathbb{K}[[\varphi_1, \ldots, \varphi_{m_0}; \psi_1, \ldots, \psi_{m_1}]].$$

Definition 28. *Let $\widetilde{X}$ be a super graded vector space with, as $\mathbb{Z}/2\mathbb{Z}$ grading, the one induced by the $\mathbb{Z}$-grading on $\widetilde{X}$, as in Remark 12. Moreover, let $\{\varphi_i\}$ denote the coordinates on W, while $\{\varphi_i^*\}$ are the corresponding coordinates on $W^*[1]$. Then the operator*

$$\Delta = \sum_i \frac{\partial}{\partial \varphi_i^*} \frac{\partial}{\partial \varphi_i}$$

is called the Batalin-Vilkovisky Laplacian.

Lemma 3
Δ is a degree 1 operator on $\mathcal{O}_{W \oplus W^*[1]}$. Moreover $\Delta^2 = 0$.

Proof. The fact that Δ is an operator of degree 1 follows immediately from the way in which the graded structure on $W^*[1]$ is defined, starting from the one on W. To show that Δ is also a coboundary operator, we start noticing that the following equality holds:

$$\frac{\partial}{\partial \varphi} \frac{\partial}{\partial \eta} = (-1)^{\epsilon(\varphi) \cdot \epsilon(\eta)} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \varphi} \ ,$$

where φ, η are generic variables in $W \oplus W^*[1]$.

Then we compute:

$$\begin{aligned} \Delta^2 &= \sum_{i,j} \tfrac{\partial}{\partial \varphi_i^*} \tfrac{\partial}{\partial \varphi_i} \tfrac{\partial}{\partial \varphi_j^*} \tfrac{\partial}{\partial \varphi_j} \\ &= \sum_{i,j} (-1)^{[\epsilon(\varphi_i) + \epsilon(\varphi_i^*)][\epsilon(\varphi_j) + \epsilon(\varphi_j^*)]} \tfrac{\partial}{\partial \varphi_j^*} \tfrac{\partial}{\partial \varphi_j} \tfrac{\partial}{\partial \varphi_i^*} \tfrac{\partial}{\partial \varphi_i} \\ &= -\sum_{i,j} \tfrac{\partial}{\partial \varphi_j^*} \tfrac{\partial}{\partial \varphi_j} \tfrac{\partial}{\partial \varphi_i^*} \tfrac{\partial}{\partial \varphi_i} \\ &= -\Delta^2. \end{aligned}$$

Therefore, $\Delta^2 = 0$.
In the above computation we used the fact that $\epsilon(\varphi_i) + \epsilon(\varphi_i^*) = 1$, for all coordinates φ_i on W. □

Definition 29. *The cohomology theory with $\mathcal{O}_{W \oplus W^*[1]}$ as a space of cochains and Δ as a coboundary operator is called the* BV cohomology *of the graded vector space $W \oplus W^*[1]$.*

From classical differential geometry it is known that, given a vector space V, the vector space $V \oplus V^*$ has a canonical symplectic structure. Analogously to what happens in the classical case, given a generic super graded vector space W, the space $W \oplus W^*[1]$ admits a canonical symplectic structure.

Definition 30. *Let $\mathcal{L}$ be a submanifold of $W \oplus W^*[1]$. Then $\mathcal{L}$ is a* Lagrangian submanifold *if it is an isotropic submanifold of maximal dimension. Equivalently, a submanifold $\mathcal{L} \subseteq W \oplus W^*[1]$ is Lagrangian if $\mathcal{L} = \mathcal{L}^\perp$ where $\mathcal{L}^\perp$ is the symplectic complement of $\mathcal{L}$, defined by*

$$\mathcal{L}^\perp = \{v \in W \oplus W^*[1] : \ \omega(v, y) = 0, \ \forall y \in \mathcal{L}\},$$

with ω the canonical symplectic form on $W \oplus W^*[1]$.

Remark 14
Another equivalent definition of a *Lagrangian submanifold* is the following: let $\mathcal{L}$ be a submanifold of $\widetilde{X}$ with $i : \mathcal{L} \hookrightarrow \widetilde{X}$ an inclusion of $\mathcal{L}$ in $\widetilde{X}$. Then $\mathcal{L}$ is a Lagrangian submanifold of $\widetilde{X}$ if and only if $i^*\omega \equiv 0$, with $2 \cdot dim(\mathcal{L}) = dim(\widetilde{X})$.

Given a Lagrangian submanifold $\mathcal{L}$ of $W \oplus W^*[1]$, we state the following proposition. A proof of an even more general statement can be found in [51] while, for a reference on symplectic geometry in general, see [2].

Proposition 4
Let $\{\varphi_i\}_{i=1,\dots,n}$ be the usual coordinates on W and $\{\varphi_i^*\}_{i=1,\dots,n}$ the corresponding coordinates on $W^*[1]$. Then the volume form $d\varphi_1 \cdots d\varphi_n d\varphi_1^* \cdots d\varphi_n^*$ on $W \oplus W^*[1]$ induces a well defined volume form $d(Vol_{\mathcal{L}})$ on Lagrangian submanifolds $\mathcal{L}$ of $W \oplus W^*[1]$. Thus the integrals

$$\int_{\mathcal{L}} g \ d(Vol_{\mathcal{L}})$$

are well defined for all Lagrangian submanifolds $\mathcal{L} \subseteq W \oplus W^*[1]$ and for all regular functions $g \in \mathcal{O}_{W \oplus W^*[1]}$.

Definition 31. *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two Lagrangian submanifolds of $\widetilde{X}$. Then $\mathcal{L}_0$ and $\mathcal{L}_1$ are said to be* homotopically equivalent *if there exists a smooth function $\Phi : [0,1] \to \widetilde{X}$ such that:*

- *$\Phi(t)$ is a Lagrangian submanifold of $\widetilde{X}$, for all $t \in [0,1]$;*
- *$\Phi(t)|_{t=0} = \mathcal{L}_0$;*
- *$\Phi(t)|_{t=1} = \mathcal{L}_1$.*

Equivalently, two Lagrangian submanifolds $\mathcal{L}_0$ and $\mathcal{L}_1$ are called homotopically equivalent if there exists a smooth deformation of Lagrangian submanifolds that sends $\mathcal{L}_0$ into $\mathcal{L}_1$.

Now we have introduced everything we need in order to state the main theorem of the BV formalism. A full proof can be found in [51].

Theorem 3. *Let $\widetilde{X}$ be a super graded vector space, that is $\widetilde{X} = W \oplus W^*[1]$, let g be a regular function on $W \oplus W^*[1]$, i.e., $g \in \mathcal{O}_{W\oplus W^*[1]}$, and let $\mathcal{L}$ be any Lagrangian submanifold of $W \oplus W^*[1]$.*
Then:

1. *if $\Delta g = 0$, then the integral $\int_{\mathcal{L}} g$ depends only on the homology class of $\mathcal{L}$;*
2. *if there exists a regular function $f \in \mathcal{O}_{W\oplus W^*[1]}$ such that $g = \Delta f$, then, for any Lagrangian submanifold $\mathcal{L}$,*

$$\int_{\mathcal{L}} g \ d(Vol_{\mathcal{L}}) = 0.$$

Remark 15
Using the theorem stated above, it is possible to conclude that the integrals $\int_{\mathcal{L}} g$ depend only on the BV-cohomology class of the function g and on the homotopy class of the Lagrange submanifold $\mathcal{L}$. Therefore, there is a pairing between homotopy classes of Lagrangian submanifolds and BV-cohomology classes of functions on the superspace $W \oplus W^*[1]$.

Remark 16
In the setting of BV formalism, a differential P_0-algebra A, as introduced in Definition 20, often appears under the name *BV algebra*, while the differential operator Δ and the bracket $\{\ ,\ \}$ are called, respectively, the *BV-Laplacian* and *BV-bracket*.

By considering the compatibility between the three operations defined on a BV algebra, namely the multiplication, the BV-Laplacian and the BV-bracket, it is possible to deduce the relation stated in the next proposition, which is also known as the *seven terms relation.*

Proposition 5
Given a generic BV algebra $(\mathcal{A}, d, \{\ ,\ \})$ the following property holds, for all $x, y, z \in \mathcal{A}$:

$$\begin{aligned}&\Delta(x \cdot y \cdot z) + (\Delta x) \cdot y \cdot z + (-1)^{\epsilon(x)} x \cdot (\Delta y) \cdot z + (-1)^{\epsilon(x)+\epsilon(y)} x \cdot y \cdot (\Delta z)\\ &= \Delta(x \cdot y) \cdot z + (-1)^{\epsilon(x)} x \cdot \Delta(y \cdot z) + (-1)^{(\epsilon(x)+1)\epsilon(y)} y \cdot \Delta(x \cdot z).\end{aligned}$$

Remark 17
Using the notation introduced in the BV formalism, the Poisson bracket defined on the space of regular functions $\mathcal{O}_{W\oplus W^*[1]}$ can be expressed as follows:

$$\{g,l\} = \sum_i (-1)^{\epsilon(\varphi_i^*)(\epsilon(l)+1)} \frac{\partial g}{\partial \varphi_i} \frac{\partial l}{\partial \varphi_i^*} - (-1)^{\epsilon(\varphi_i)(\epsilon(l)+1)} \frac{\partial g}{\partial \varphi_i^*} \frac{\partial l}{\partial \varphi_i} \ , \tag{3.7}$$

for $g, l \in \mathcal{O}_{W\oplus W^*[1]}$.

Now we are going to explain how Theorem 3 can be used to give a mathematically rigorous explanation of the gauge-fixing process for a theory defined on a 0-dimensional spacetime.

The gauge-fixed extended theory

Let the pair $(\widetilde{X}, \widetilde{S})$ be an extended physical theory, where the extended configuration space $\widetilde{X}$ has the structure of a super graded vector space, $\widetilde{X} \simeq W \oplus W^*[1]$. To describe the gauge-fixing process, let us start by introducing the notion of a *gauge-fixing fermion.*

Definition 32. *Let $\widetilde{X}$ be a super graded vector space, with $\widetilde{X} = W \oplus W^*[1]$. Then a* gauge-fixing fermion Ψ *is a Grassmannian function of total degree* -1 *depending only on fields, that is,* $\Psi \in \mathcal{O}_W^{-1}$, $\epsilon(\Psi) = 1$.

Then, given an extended theory $(\widetilde{X}, \widetilde{S})$ together with a gauge-fixing fermion Ψ, the corresponding *gauge-fixed theory* is given by the pair $(\widetilde{X}_\Psi, \widetilde{S}_\Psi)$ with:

- $\widetilde{X}_\Psi = W$.

  That is to say, the extended configuration space is restricted to the subspace generated by fields and ghost fields, while all the antifields and antighost fields have been removed;

- $\widetilde{S}_\Psi = \widetilde{S}(\varphi_i, \varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i})$.

  In other words, the gauge-fixed action $\widetilde{S}_\Psi$ is obtained by imposing the so-called *gauge-fixing condition* on all the antifields φ_i^* in $W^*[1]$, i.e.,

$$\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}. \tag{3.8}$$

By applying the gauge-fixing condition, each antifield $\varphi_i^* \in W^*[1]$ is replaced by the partial derivative of the gauge-fixing fermion Ψ computed with respect to the field φ_i corresponding to φ_i^*. It immediately follows from the definition of a gauge-fixing fermion that the gauge-fixed action $\widetilde{S}_\Psi$, obtained by imposing the condition in (3.8), is a function of the fields only: given an action $\widetilde{S} \in \mathcal{O}_{W\oplus W^*[1]}$, after the gauge-fixing process we obtain a function $\widetilde{S}_\Psi \in \mathcal{O}_W$.

To be able to ensure that the gauge-fixing procedure is well defined we have to prove that it does not depend on the choice of the gauge-fixing fermion. To be more precise, what we need to ensure is that the physically relevant quantities, namely the integrals

$$\int_{\widetilde{X}_\psi} g \; d(Vol_{\widetilde{X}_\Psi})$$

do not depend on the gauge-fixing fermion ψ used to carry out the gauge-fixing procedure. This is proved in the following proposition.

Proposition 6
Let $(\widetilde{X}, \widetilde{S})$ be an extended physical theory as above, with $\widetilde{X} \simeq W \oplus W^*[1]$, and let Ψ be a gauge-fixing fermion on W. Then, for any Δ-closed function $g \in \mathcal{O}_{W\oplus W^*[1]}$ the following equality holds:

$$\int_W g \; d(Vol_W) = \int_{\widetilde{X}_\Psi} g \; d(Vol_{\widetilde{X}_\Psi}). \tag{3.9}$$

Proof. Let us start by recalling that, given a smooth function Ψ on W, the submanifold $\widetilde{X}_\Psi$ defined by the equations (3.8) is a Lagrangian submanifold of $W \oplus W^*[1]$.
This statement can be verified by using the definition of a Lagrangian submanifold as an n-dimensional submanifold on which the pullback of the symplectic 2-form ω is identically equal to 0. In the particular case we are considering, the canonical symplectic form on $W \oplus W^*[1]$ can be rewritten as follows in terms of the coordinates:

$$\omega = d\varphi_1 \wedge d\varphi_1^* + \cdots + d\varphi_n \wedge d\varphi_n^*.$$

If we evaluate the pullback of ω on the submanifold $\widetilde{X}_\Psi$, we find that it is always zero since the points in $\widetilde{X}_\Psi$ by definition do not depend on the coordinates φ_i^*, for each $i = 1, \ldots, n$. Thus $\widetilde{X}_\Psi$ is a Lagrangian submanifold for each smooth function Ψ on W.

Moreover, any two functions Ψ_0 and Ψ_1 on W are homotopically equivalent through the homotopy

$$\Psi_t = t\Psi_1 + (1-t)\Psi_0, \qquad t \in [0,1].$$

In other words, there exists a continuous deformation

$$\begin{array}{rccc} \Phi: & [0,1] & \longrightarrow & \widetilde{X} \\ & t & \mapsto & \Phi(t) := \widetilde{X}_{\Psi_t} \end{array}$$

such that:

- $\Phi(t)$ is a Lagrangian submanifold for every value of $t \in [0,1]$;
- $\Phi(0) = \widetilde{X}_{\Psi_1}$;
- $\Phi(1) = \widetilde{X}_{\Psi_2}$.

Therefore, the two corresponding submanifolds $\widetilde{X}_{\Psi_0}$ and $\widetilde{X}_{\Psi_1}$ are in the same homology class. So, applying Theorem 3, we deduce that for each Δ-closed function g in $\mathcal{O}_{W\oplus W^*[1]}$, we have that:

$$\int_{\widetilde{X}_{\Psi_0}} g \; d(Vol_{\widetilde{X}_{\Psi_0}}) = \int_{\widetilde{X}_{\Psi_1}} g \; d(Vol_{\widetilde{X}_{\Psi_1}}) \; .$$

Then (3.9) follows immediately from observing that the space W itself can be seen as a Lagrangian submanifold defined by a gauge-fixing fermion Ψ: more precisely, W corresponds to the Lagrangian submanifold defined by a function Ψ homotopically equivalent to the zero function, $\Psi \equiv 0$. □

Note: in the statement of Proposition 6 no further hypotheses are imposed on the gauge-fixing fermion Ψ in order to conclude that the physically relevant quantities computed starting from the pair $(\widetilde{X}_\Psi, \widetilde{S}_\Psi)$ do not depend on Ψ. However, further conditions need to be imposed on Ψ in order to obtain a gauge-fixed action $\widetilde{S}_\Psi$ that describes a non-degenerate theory. This aspect of the gauge-fixing procedure is discussed in detail in Section 3.6.

3.4 The quantum master equation and the quantum BRST operator

This section is devoted to explaining why the constructions presented in the previous sections have a physical relevance (once again, considered in the context of infinite-dimensional gauge theories). More precisely, the purpose of this section is to explain the relation between physically relevant quantities of the quantized gauge theory and a cohomology theory defined on the extended theory and involving the ghost fields. In other words, even though the ghost fields were first introduced as a tool to address the problem of quantizing a gauge invariant theory via the path integral approach, they appear to have a surprising mathematical relevance from a physical point of view, as generators of a particular type of cohomology theory, known as the *quantum BRST cohomology.* (As reference for this part see for example [30]).

Let us briefly recall the notation already introduced:

- (X_0, S_0) is a gauge invariant theory;
- $(\widetilde{X}, \widetilde{S})$ is the corresponding extended theory, where $\widetilde{X}$ is a super graded vector space, $\widetilde{X} = W \oplus W^*[1]$;
- Ψ is a gauge-fixing fermion.

In Proposition 6 we proved that, in order to be independent of the gauge-fixing fermion, an integral of the form

$$\int_W g \; d(Vol_W)$$

has to be defined by a Δ-closed function g in $\mathcal{O}_{W\oplus W^*[1]}$. Equivalently, the only quantities that are meaningful from a physical point of view are the ones determined by Δ-closed functions.
In the context of quantization via a path integral approach, the integrands considered are usually of the form:

$$g = le^{\frac{i}{\hbar}\widetilde{S}} \tag{3.10}$$

where g, l are regular functions belonging to $\mathcal{O}_{W\oplus W^*[1]}$ and $\widetilde{S}$ is the extended action of the theory.

The case $l = 1$

If we consider the particular case in which $l = 1$, then the condition of being a Δ-closed function for g can we rewritten as follows:

$$0 = \Delta e^{\frac{i}{\hbar}\widetilde{S}} = \Delta\left(\sum_{n=0}^{\infty}\frac{(i\widetilde{S})^n}{\hbar^n n!}\right) = \left(\frac{i}{\hbar}\Delta\widetilde{S} - \frac{1}{2\hbar^2}\{\widetilde{S}, \widetilde{S}\}\right)e^{\frac{i}{\hbar}\widetilde{S}}.$$

Therefore, imposing the condition of being a Δ-closed function to the function $g = e^{\frac{i}{\hbar}\widetilde{S}}$ is equivalent to imposing that the action $\widetilde{S}$ satisfies the so-called *quantum master equation*

$$\{\widetilde{S}, \widetilde{S}\} - 2i\hbar\Delta\widetilde{S} = 0. \tag{3.11}$$

The general case

In order to apply the BV formalism to the general case of an integrand of the type given in Equation (3.10), we need to impose the following condition:

$$\Delta\left(le^{\frac{i}{\hbar}\widetilde{S}}\right) = 0,$$

i.e., we require the integrand being a Δ-closed function. However, assuming that $\widetilde{S}$ is a solution of the quantum master equation, then the previous equation is equivalent to

$$\Delta l + \frac{i}{\hbar}\{\widetilde{S}, l\} = 0.$$

Introducing a new operator, what we found above can be rewritten more precisely as follows.

Definition 33. *Let*

$$(\mathcal{O}_{W\oplus W^*[1]}, \cdot, d, \{\ ,\ \})$$

be a BV algebra defined as before and let $\widetilde{S}$ be an action on the extended configuration space $W \oplus W^*[1]$. Then the quantum BRST operator Ω *is an operator on $\mathcal{O}_{W\oplus W^*[1]}$ defined as follows:*

$$\Omega = -i\hbar\Delta + \{\widetilde{S}, -\}.$$

Note: it follows straightforwardly from the definition of the quantum BRST operator Ω that the BV formalism applies to all regular functions g in $\mathcal{O}_{W\oplus W^*[1]}$ that belong to $\mathrm{Ker}(\Omega)$.

Under the hypothesis that the action $\widetilde{S}$ is a solution of the quantum master equation, it is also possible to prove other properties of the quantum BRST operator, as stated in the following proposition.

Proposition 7
Let us assume that $\widetilde{S}$ is a solution of the quantum master equation. Then the

quantum BRST operator Ω is a differential. Moreover, the expression

$$\begin{array}{rccl} \langle\ \rangle: & H(\mathcal{O}_{W\oplus W},\Omega) & \longrightarrow & \mathbb{R} \\ & g & \mapsto & \langle g\rangle := \int_W g e^{\frac{i}{\hbar}\widetilde{S}}\, d(Vol_W)\ , \end{array}$$

is a linear functional on the Ω-cohomology classes of $\mathcal{O}_{W\oplus W^*[1]}$.

Proof. To prove that Ω is a differential, let us consider a generic element g in $\mathcal{O}_{W\oplus W^*[1]}$. Then we have the following equalities:

$$\begin{aligned} \Omega^2 g &= \Omega\big(-i\hbar\Delta g + \{\widetilde{S}, g\}\big) \\ &= -\hbar^2\Delta^2 g - i\hbar\Delta\{\widetilde{S}, g\} - i\hbar\{\widetilde{S}, \Delta g\} + \{\widetilde{S}, \{\widetilde{S}, g\}\} \\ &= -i\hbar\{\Delta\widetilde{S}, g\} + \tfrac{1}{2}\{\{\widetilde{S}, \widetilde{S}\}, g\} \\ &= \tfrac{1}{2}\{\{\widetilde{S}, \widetilde{S}\} - 2i\hbar\Delta\widetilde{S}, g\} = 0. \end{aligned}$$

In the previous computation we used the properties stated in Definition 20, together with the fact that Δ is a coboundary operator: this makes $\Delta^2 g$ zero for any regular function g. From this, we conclude that Ω is a coboundary operator on $\mathcal{O}_{W\oplus W^*[1]}$.
Concerning the second part of the proposition, we have already noticed that the expression $\langle - \rangle$, known also as an *expectation value*, is well defined on elements in Ker(Ω). Therefore, to conclude that $\langle - \rangle$ is a functional on the Ω-cohomology classes we only need to prove that the expectation value of an Ω-coboundary element vanishes. So, let g be an element in $\mathcal{O}_{W\oplus W^*[1]}$ that is a coboundary element with respect to the quantum BRST operator Ω: this implies that there is an element l in $\mathcal{O}_{W\oplus W^*[1]}$ such that $g = \Omega(l)$.
Then:

$$\langle g\rangle = \int_W g e^{\frac{i}{\hbar}\widetilde{S}} = \int_W (\Omega l) e^{\frac{i}{\hbar}\widetilde{S}} = -i\hbar \int_W \Delta(l e^{\frac{i}{\hbar}\widetilde{S}}) = 0,$$

where in the last equation we applied Theorem 3.
Thus we conclude that the expectation value is a well defined functional on the Ω-cohomology classes. □

Remark 18
In a quantized physical theory, the cohomology classes defined by the coboundary operator Ω are called the *observables* of the theory. The reason for this name is that the Ω-cocycles are exactly the elements for which the expectation value is well defined and independent of the choice of the gauge-fixing fermion.

Moreover, since the Ω-coboundaries have zero expectation, one concludes that the expectation value is well defined on Ω-cohomology classes.

3.5 The gauge-fixed BRST cohomology

The aim of this section is the introduction of the fundamental notion of *(gauge-fixed) BRST cohomology*. Indeed, a natural question that arises after the gauge-fixing procedure of the extended theory $(\widetilde{X}, \widetilde{S})$ is whether it still has a residual BRST-symmetry on $(\widetilde{X}_\Psi, \widetilde{S}_\Psi)$ that induces a cohomology complex. This section is devoted to answer to this question.

Throughout this section, $(\widetilde{X}, \widetilde{S})$ is the extended theory corresponding to a BV variety $(N, \widetilde{S})$. We assume that N satisfies all required conditions to induce a globally well-defined extended configuration space (see Section 3.1.1). Thus:

- as noticed in Section 3.3, $\widetilde{X}$ has the structure of a super graded vector space:

$$\widetilde{X} = W \oplus W^*[1],$$

  with W a $\mathbb{Z}_{\geqslant 0}$-graded vector space generated by fields and ghost fields, while $W^*[1]$ is the corresponding $\mathbb{Z}_{<0}$-graded vector space generated by the antifields and the antighost fields.

- $\widetilde{S}$ is a regular functional on $\widetilde{X}$, which solves the master equation:

$$\{\widetilde{S}, \widetilde{S}\} = 0.$$

Finally, Ψ denotes a gauge-fixing fermion.

To analyze the effects of the gauge-fixing procedure on the classical BRST cohomology complex, we separately analyze the cochain spaces and the coboundary operator.

BRST cochains

The space of cochains of degree $i \in \mathbb{Z}$ for the classical BRST cohomology complex is defined as follows:

$$\mathcal{C}^i(\widetilde{X}, d_{\widetilde{S}}) = [Sym_{\mathcal{O}_{X_0}}(\widetilde{X})]^i.$$

Notice that this notion was introduced in Definition 27 in terms of the structure sheaf of the graded variety N. However, thanks to the assumptions on N, this definition can be restated using the extended configuration space $\widetilde{X}$: thus the cochains of degree i are polynomials of degree i in the generators of the graded vector space $\widetilde{X}$, where the grading on the space of polynomials is the one naturally induced by the grading of $\widetilde{X}$. We also recall that generators of odd degree have to be considered Grassmannian variables.

Applying the gauge-fixing procedure to the extended configuration space $\widetilde{X}$ simply consists of restricting $\widetilde{X}$ to W: therefore, after gauge fixing, the space of cochains simply restricts to the set of cochain depending only on the generators of W. Moreover, in the case in which W is a $\mathbb{Z}_{\geqslant 0}$-graded vector space, then we have only one-sided cohomology complex, i.e., the cochain spaces are defined only in degree $i \geqslant 0$. More explicitly:

$$\mathcal{C}^i(\widetilde{X}_\Psi, d_{\widetilde{S}_\Psi}) = [Sym_{\mathcal{O}_{X_0}}(W)]^i.$$

BRST coboundary operator

The coboundary operator $d_{\widetilde{S}}$ for the classical BRST cohomology is a coboundary operator of degree 1 acting on cochains as follows:

$$\begin{array}{rccc} d_{\widetilde{S}}: & \mathcal{C}^i(\widetilde{X}, d_{\widetilde{S}}) & \longrightarrow & \mathcal{C}^{i+1}(\widetilde{X}, d_{\widetilde{S}}) \\ & \varphi & \mapsto & d_{\widetilde{S}}(\varphi) := \left\{\widetilde{S}, \varphi\right\} . \end{array}$$

We want to know what happens if we impose the gauge-fixing condition on the operator $d_{\widetilde{S}}$, i.e., we want to analyze the properties of the operator

$$d_{\widetilde{S}}|_\Psi = \left\{\widetilde{S}, -\right\}|_{\Sigma_\Psi},$$

where Σ_Ψ is the Lagrangian submanifold of $W \oplus W^*[1]$ defined by the gauge-fixing conditions $\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}$, (see Section 3.3).

Similarly to what already noticed for the operator $d_{\widetilde{S}}$, due to the properties of the Poisson bracket, it immediately follows that also the operator $d_{\widetilde{S}}|_\Psi$ is a linear derivation of degree 1. It still needs to be checked whether, after the gauge-fixing procedure has been performed, the operator $d_{\widetilde{S}}|_\Psi$ defines a coboundary operator, i.e., if it satisfies $(d_{\widetilde{S}}|_\Psi)^2 = 0$.

Proposition 8
Given any field $\Phi_A \in W$, it holds

$$d^2_{\widetilde{S}}|_\Psi(\Phi_A) = \left[\sum_k (-1)^{\epsilon(\Phi_A)(1+\epsilon(\Phi_k))} \frac{\partial \widetilde{S}_\Psi}{\partial \Phi_k} \frac{\partial^2 \widetilde{S}}{\partial \Phi_k^* \partial \Phi_A^*} \right] \Bigg|_{\Sigma_\Psi} .$$

Proof. Using the explicit description of the Poisson bracket given in (3.7), the action of the operator $d_{\widetilde{S}}|_\Psi$ on a fixed field Φ_A can be written as follows:

$$d_{\widetilde{S}}|_\Psi(\Phi_A) = \left\{ \widetilde{S}, \Phi_A \right\} \Big|_{\Sigma_\Psi} = - \left. \frac{\partial \widetilde{S}}{\partial \Phi_A^*} \right|_{\Sigma_\Psi} .$$

Then we notice that the two following equalities hold:

(A) $\sum_k (-1)^{\epsilon(\Phi_A)\epsilon(\Phi_k)} \frac{\partial \widetilde{S}}{\partial \Phi_k^*} \frac{\partial^2 \widetilde{S}}{\partial \Phi_k \partial \Phi_A^*} = \sum_k (-1)^{\epsilon(\Phi_A)(\epsilon(\Phi_k)+1)} \frac{\partial \widetilde{S}}{\partial \Phi_k} \frac{\partial^2 \widetilde{S}}{\partial \Phi_k^* \partial \Phi_A^*}$;

(B) $\frac{\partial \widetilde{S}_\Psi}{\partial \Phi_k} = \frac{\partial \widetilde{S}}{\partial \Phi_k} + \sum_j \frac{\partial \widetilde{S}}{\partial \Phi_j^*} \frac{\partial^2 \Psi}{\partial \Phi_k \partial \Phi_j}$.

Equality (A) can be determined by explicitly computing the expression

$$\frac{\partial}{\partial \Phi_A} \{ \widetilde{S}, \widetilde{S} \},$$

which is equal to zero, due to the fact that $\widetilde{S}$ is supposed to be a solution of the classical master equation. Regarding equality (B), to obtain it we have to recall that the gauge-fixed action $\widetilde{S}_\Psi$, defined as

$$\widetilde{S}_\Psi = \widetilde{S} \left(\Phi_j, \Phi_j^* = \frac{\partial \Psi}{\partial \Phi_j} \right),$$

depends on the fields Φ_j not only explicitly but also implicitly through the gauge-fixing fermion Ψ. Thus we have:

$$\begin{aligned} d^2_{\widetilde{S}}|_\Psi(\Phi_A) &= \left\{ \widetilde{S}, d_{\widetilde{S}}|_\Psi(\Phi_A) \right\} \Big|_{\Sigma_\Psi} \\ &= \sum_k (-1)^{\epsilon(\Phi_k)\epsilon(\Phi_A)} \frac{\partial \widetilde{S}}{\partial \Phi_k^*} \left[\frac{\partial^2 \widetilde{S}}{\partial \Phi_k \partial \Phi_A^*} + \sum_j \frac{\partial^2 \widetilde{S}}{\partial \Phi_j^* \partial \Phi_A^*} \frac{\partial^2 \Psi}{\partial \Phi_k \partial \Phi_j} \right] \Bigg|_{\Sigma_\Psi} . \end{aligned}$$

By using equality (A) for rearranging the first summand and equality (B) on the second, the statement can be immediately deduced. □

Corollary 1
The operator $d_{\widetilde{S}}|_{\Psi}$ defines a coboundary operator on shell, that is, when the equations of motion for the gauge-fixed action $\widetilde{S}_{\Psi}$ are satisfied. Explicitly when

$$\frac{\partial \widetilde{S}_{\Psi}}{\partial \Phi_A} = 0, \qquad \forall\ \Phi_A \in W.$$

Note: as seen in the above corollary, the gauge-fixed coboundary operator $d_{\widetilde{S}}|_{\Psi}$ always defines a coboundary operator when we consider the gauge-fixed theory on shell. However, depending on the explicit form of the theory, the condition of the operator $d_{\widetilde{S}}|_{\Psi}$ defining a coboundary operator, might be satisfied also for the theory off shell. This is the case for example if the extended action $\widetilde{S}$ turn out to be linear in the antifields and antighost fields: this is what happens in the model we are going to consider in the second part of the thesis.

Therefore, under suitable hypotheses on the theory to be on shell or on extended action $\widetilde{S}$ having a certain dependence on the antifields and antighost fields, it is possible to introduce a new cohomology theory with the operator $d_{\widetilde{S}}|_{\Psi}$ as coboundary operator. This new cohomology theory is known in the literature as the *(gauge-fixed) BRST cohomology.*

Definition 34. *Let $(\widetilde{X}, \widetilde{S})$ be an extended theory with $\widetilde{X} \simeq W \oplus W^*[1]$, where $\widetilde{X}$ is a super graded vector space and $\widetilde{S}$ is a solution of the classical master equation, and let Ψ be a gauge-fixing fermion. Then the* (gauge-fixed) BRST cohomology complex $(\mathcal{C}^\bullet, d_{\widetilde{S}}|_{\Psi})$ *is defined as follows:*

- *the vector space $\mathcal{C}^k(W, d_{\widetilde{S}})$ of cochains of degree $k \in \mathbb{Z}$ is the homogeneous component of degree k in the graded algebra of regular functions on W, $\mathcal{O}_W$:*

$$\mathcal{C}^k(W, d_{\widetilde{S}}|_{\Psi}) = [\mathcal{O}_W]^k \ ;$$

- *the coboundary operator is defined as*

$$d_{\widetilde{S}}|_{\Psi} := \{\widetilde{S}, -\}|_{\Sigma_{\Psi}}$$

  with Σ_{Ψ} the Lagrangian submanifold of $W \oplus W^*[1]$ defined by the gauge-fixing conditions $\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}$.

Remark 19
In the physics literature, the coboundary operator $d_{\widetilde{S}}$ is also known as the

classical BRST transformation. Therefore, an equivalent way to say that $\widetilde{S}$ solves the classical master equation is to say that $\widetilde{S}$ *is invariant under the classical BRST transformation*:

$$d_{\widetilde{S}}(\widetilde{S}) = \{\widetilde{S}, \widetilde{S}\} = 0.$$

Analogously, the coboundary operator of the (gauge-fixed) BRST cohomology $d_{\widetilde{S}}|_\Psi$ is known also as the *BRST transformation*.
Using this vocabulary, we can say that $d_{\widetilde{S}}|_\Psi$ is the residual BRST symmetry, which is still present after the gauge-fixing process has been performed.

3.6 Gauge-fixing auxiliary fields

The aim of this section is to introduce the technical devise of *trivial pairs*. In Section 3.3 we have seen that the gauge-fixing process is based on the introduction of a gauge-fixing fermion. However, there are cases in which this procedure is not directly applicable, since it is not possible to define a gauge-fixing fermion for the theory. In this section we explain which kind of problems might occur if we want to define a gauge-fixing fermion, and what can be done to solve them. This construction was first discovered by Batalin and Vilkovisky [7], [8]. Another useful reference is [34].

As usual, $(\widetilde{X}, \widetilde{S})$ denotes an extended theory where the extended configuration space $\widetilde{X}$ has the structure of super graded vector space, with

$$\widetilde{X} = W \oplus W^*[1],$$

for a certain $\mathbb{Z}_{\geqslant 0}$-graded vector space W, whilst $\widetilde{S}$ is a solution of the classical master equation on $\widetilde{X}$. We also recall that W is generated by fields and ghost fields, whereas $W^*[1]$ describes the content of $\widetilde{X}$ in antifields and antighost fields.

As stated in Definition 32, a gauge-fixing fermion Ψ is a Grassmannian function with ghost degree -1 that depends only on fields and ghost fields, that is, $\Psi \in [\mathcal{O}_W]^{-1}$.
It is evident that, in the case in which W is a $\mathbb{Z}_{\geqslant 0}$-graded vector space, e.g., in the case in which the extended theory $(\widetilde{X}, \widetilde{S})$ is obtained by applying the BV construction (see Section 4.1), it is impossible to define a gauge-fixing fermion since there are no fields with negative ghost degree. Therefore, *auxiliary fields* of negative ghost degree have to be introduced.

Goal: we want to find a method to further enlarge the extended configuration space $\widetilde{X}$ without changing the corresponding BRST cohomology. This goal is reached using *trivial pairs*.

Definition 35. *A* trivial pair *is a pair of fields* (B, h) *such that their ghost degrees and parities satisfy the following relations:*

$$deg(h) = deg(B) + 1; \qquad \epsilon(h) = \epsilon(B) + 1 \ (mod\ 2)\ . \tag{3.12}$$

Note: given a trivial pair (B, h), as a consequence of the conditions imposed on the ghost degree and the parity of these two fields, the ghost degree and parity of the corresponding antifields (B^*, h^*) should satisfy the following relations:

$$deg(h^*) = deg(B^*) - 1; \qquad \epsilon(h^*) = \epsilon(B^*) + 1 \ (mod\ 2). \tag{3.13}$$

Definition 36. *Given an extended theory* $(\widetilde{X}, \widetilde{S})$ *and a trivial pair* (B, h)*, the corresponding* total theory (X_{tot}, S_{tot}) *is a pair with*

– X_{tot} *the* total configuration space*, obtained as extension of* $\widetilde{X}$ *by the fields* (B, h) *and their corresponding antifields* (B^*, h^*)*:*

$$X_{tot} = \widetilde{X} \oplus \langle B \rangle \oplus \langle h \rangle \oplus \langle B^* \rangle \oplus \langle h^* \rangle;$$

– S_{tot} *the* total action *defined by*

$$S_{tot} = \widetilde{S} + S_{aux}, \qquad where \quad S_{aux} = hB^*.$$

Remark 20
The pair (X_{tot}, S_{tot}) satisfies the following properties:

- ▶ Due to the fact that the BV formalism always requires the introduction of all antifields corresponding to the extra fields added to the configuration space, also the total configuration space X_{tot} has a decomposition similar to the one of $\widetilde{X}$:

$$X_{tot} = Y \oplus Y^*[1],$$

  but this time Y and $Y^*[1]$ are two $\mathbb{Z}$-graded vector spaces, with Y describing the fields and ghost fields content of X_{tot} whilst $Y^*[1]$ describes the antifields and antighost fields content.

- The total configuration space X_{tot} is naturally endowed with a graded Poisson structure obtained by extending the one on $\widetilde{X}$, as follows:
  - ▷ $\{f, B\} = \{f, B^*\} = \{f, h\} = \{f, h^*\} = 0$, for each element $f \in \widetilde{X}$;
  - ▷ $\{B, B^*\} = \{h, h^*\} = 1$;
  - ▷ $\{B, h\} = \{B^*, h\} = \{B, h^*\} = \{B^*, h^*\} = 0$.
- From conditions (3.12), (3.13) we deduce that S_{aux} has ghost degree 0 and even parity as well as S_{tot}. Therefore, S_{tot} satisfies all conditions imposed on the ghost degree and the parity of an action.
- S_{tot} is a solution of the classical master equation on the total configuration space X_{tot}: this immediately follows from the fact that $\widetilde{S}$ is a solution of the classical master equation that does not depend on the fields (B, h) or on their corresponding antifields (B^*, h^*) and that in the bracket of S_{aux} with itself, we do not have any combinations of a field with its corresponding antifield.

Therefore, given an extended theory $(\widetilde{X}, \widetilde{S})$ and a trivial pair (B, h), the corresponding total theory (X_{tot}, S_{tot}) is a new pair satisfying exactly all properties already satisfied by $(\widetilde{X}, \widetilde{S})$, the only difference is that X_{tot} may also contain negatively graded fields: indeed, the ghost degree of the field B is a free parameter and we can also decide to take it in $\mathbb{Z}_{<0}$. However, we notice that, once the ghost degree of B has been fixed, also the ghost degree and the parity of h, as well as of the corresponding antifields B^*, h^*, are determined by conditions (3.12), (3.13).

Thus it appears that our goal has been achieved: by enlarging our extended theory with the introduction of a trivial pair (B, h) such that $\deg(B) = -1$, it is possible to define a gauge-fixing fermion Ψ. In this case any linear polynomial in B with coefficients in $\mathcal{O}_{X_0}$ would be a suitable gauge-fixing fermion, satisfying all the required properties of being a function of the fields with total degree -1. However, in addition to the conditions already imposed by its definition, a gauge-fixing fermion is forced to satisfy further conditions, as a consequence of requiring the gauge-fixed action to describe a well-defined physical theory.
In what follows, we are going to briefly present these physical aspects. To have a more detailed description of these concepts coming from quantum field theory and perturbation theory we refer to [56] while, for a physical point of view on the gauge-fixing procedure, we refer to [34].

Definition 37. *Let (X_0, S_0) be a gauge theory and let φ be a field in X_0 with components φ_i, $i = 1, \ldots, n$. The field φ is called a* gauge field *if the following conditions are satisfied:*

- *there exists at least one stationary point $\widetilde{\varphi}$ for S_0;*
- *in a neighborhood of the stationary point $\widetilde{\varphi}$ the following identities hold:*

$$\sum_{i=1}^{n} \frac{\partial S_0}{\partial \varphi_i} \cdot R^i_\alpha = 0, \tag{3.14}$$

  with $\alpha = 1, \ldots, r_1$, $r_1 < n$, and R^i_α a regular function on X_0, for any value of i and α. These identities are known as Noether identities*.*

Notation: given a gauge theory (X_0, S_0), by Definition 37 of a gauge field, there exists at least one stationary point $\widetilde{\varphi}$ for the initial action S_0. By $\widetilde{z}$ we denote a point in the total configuration space X_{tot} whose coordinates φ_i, corresponding to the initial fields in X_0, coincide with the ones of the point $\widetilde{\varphi}$, whereas all the other components, corresponding to the ghost fields and to the antifields introduced in X_{tot}, are set to be zero.

Definition 38. *Let (X_0, S_0) be a gauge theory and let $X_{tot} = Y \oplus Y^*[1]$ be the corresponding total configuration space. Given a function S_{tot} in $\mathcal{O}_{X_{tot}}$ solving the master equation as well as a gauge-fixing fermion ψ, $S_{tot}|_\psi$ is the function obtained by applying the gauge-fixing procedure to S_{tot}. Then the* partition function of the extended theory *is defined as follows:*

$$\mathcal{Z}_\psi = \int_Y e^{\frac{i}{\hbar} S_{tot}|_\psi} d\Phi_1 \ldots d\Phi_N,$$

where the domain of integration Y is the space of all the fields $\Phi_1 \ldots \Phi_N$.

The partition function Z is the starting point for the quantization of the theory: for example, the S-matrix is constructed from the partition function Z. Therefore, for reasons coming from quantum field theory, we have to impose an extra condition on the gauge-fixed action $S_{tot}|_\psi$: it has to be a *proper solution* of the classical master equation, as explained in the following definition.

Definition 39. *A solution S_{tot} of the classical master equation on X_{tot} is called a* proper solution *if the Hessian of S_{tot} has maximal possible rank at the stationary point $\widetilde{z}$.*

This requirement is related to the presence of gauge symmetries in the action S_{tot}. Roughly speaking, with this requirement we are ensuring that the gauge-fixing conditions imposed with the gauge-fixing procedure are enough to remove all the invariances of the action S_{tot}. The request that the action appearing in the partition function does not have any residual invariance is necessary to obtain finite quantities from the computation of the partition function.

Therefore, up to now, $X_{tot} = Y \oplus Y^*[1]$ is a total configuration space obtained as an extension of a certain graded vector space $\widetilde{X}$ via the introduction of a certain number of trivial pairs. Our purpose is to determine these trivial pairs and the graded space Y, so that X_{tot} and $S_{tot}|_\Psi$ satisfy the above requirement.

As we are going to explain, the number of trivial pairs that need to be introduced depends on the gauge theory considered, more precisely, on its *level of reducibility*.

Definition 40. *Let (X_0, S_0) be a gauge invariant theory and let $(\widetilde{X}, \widetilde{S})$ be the corresponding minimally-extended theory, with*

$$\widetilde{X} = W \oplus W^*[1],$$

for a certain $\mathbb{Z}_{\geqslant 0}$-graded vector space W. A theory (X_0, S_0) is said to be:

- irreducible *if $W = W_0 \oplus W_1$, i.e., if W only has homogeneous components of degree 0 and 1;*
- reducible with level of reducibility $L = k - 1$ *if* $W = \bigoplus_{i=0}^{k} W_k$.

The procedure of adding the auxiliary fields to the extended configuration space was discovered by Batalin and Vilkovisky, who first presented this procedure for irreducible theories [7] and then extended the construction to reducible theories of any level of reducibility L [8]. They found the following theorem.

Theorem 4. *Let (X_0, S_0) be a gauge theory, with level of reducibility L. Then the minimal number of trivial pairs that have to be introduced to ensure the possibility of defining a suitable gauge-fixing fermion is $(L+1)(L+2)/2$. More precisely:*

$$\forall i \in \mathbb{N},\ 0 \leqslant i \leqslant L, \quad \textit{exactly } i+1 \textit{ trivial pairs have to be introduced.}$$

Let $\{(B_i^j, h_i^j)\}$, $i = 0, \dots, L$, $j = 1, \dots, i+1$, be this collection of trivial pairs. Then the ghost degree and the parity of the fields B_i^j and h_i^j have to satisfy the *following relations:*

$$\begin{cases} deg(B_i^j) = j - i - 2 & \text{if } j \text{ is odd} \\ deg(B_i^j) = i - j + 1 & \text{if } j \text{ is even} \end{cases} \qquad \begin{cases} deg(h_i^j) = j - i - 1 \\ deg(h_i^j) = i - j + 2 \end{cases} \tag{3.15}$$

$$\epsilon(B_i^j) = i + 1 \pmod 2; \qquad \epsilon(h_i^j) = i \pmod 2.$$

In the cited paper this theorem was not explicitly proved, but it was deduced as a generalization of what was observed in the particular cases of $L = 1, 2$. In Appendix A we justify the necessity of introducing this collection of auxiliary fields and we explain why there exists a relation between the level of reducibility of the theory and the number of trivial pairs that needs to be introduced.

The aim of the remaining part of this section is to prove that, even though we had to introduce the extra fields B_i^j, h_i^j to be able to implement the gauge-fixing procedure, these are irrelevant from a cohomological point of view. To prove this, we need to introduce the notion of *contractible pair*.

Definition 41. *Given a cohomology theory with generators $x_1, \dots x_n, B, h$ and coboundary operator δ, a* contractible pair *is defined by two generators B and h satisfying the following conditions:*

1. $\delta(B) = h$;
2. $\delta(h) = 0$;
3. *the coboundary operator δ, applied to the other generators $x_1, \dots x_n$, does not depend on the generators B and h.*

Remark 21
Let $(\widetilde{X}, \widetilde{S})$ be an extended theory, Ψ be a gauge-fixing fermion, and (B, h) be a trivial pair. If the extended theory $(\widetilde{X}, \widetilde{S})$ satisfies the conditions to have the operator $d_{\widetilde{S}}|_\Psi$ defining a coboundary operator (see Proposition 8), then the pair (B, h) is a contractible pair for the gauge-fixed BRST cohomology complex $(\mathcal{C}^\bullet(X_{tot}, d_{S_{tot}}|_\Psi), d_{S_{tot}}|_\Psi)$, where the operator $d_{S_{tot}}|_\Psi$ coincides with the operator $d_{\widetilde{S}}|_\Psi$ on generators in $\widetilde{X}$, while on the generator B and h acts as follows:

- $d_{S_{tot}}|_\Psi(B) = \{S_{tot}, B\}|_{\Sigma_\Psi} = \{h(B^*), B\}|_{\Sigma_\Psi} = h$;
- $d_{S_{tot}}|_\Psi(h) = \{S_{tot}, h\}|_{\Sigma_\Psi} = 0$.

We note that, since the total action S_{tot} is linear in the antifield B^*, the action of the operator $d_{S_{tot}}|_\Psi$ on the field B is independent of the gauge-fixing fermion. For the action on the generator h, since the total action does not contain any terms depending on the antifields h^*, the gauge-fixed BRST-coboundary operator is zero when it is computed on the field h.
Thus each trivial pair (B^i_j, h^i_j) defines a contractible pair for the gauge-fixed BRST cohomology.

As we are going to prove in Theorem 5 ([3]), the trivial pairs (B^i_j, h^i_j) are trivial not only from the point of view of the physical theory but also from the point of view of the corresponding cohomology theory. However, before stating the theorem, we introduce some notation.

Notation:

- X denotes a graded vector space over $\mathbb{K}$,

$$X := \langle x_1, \dots, x_n, B, h\rangle_{\mathbb{K}},$$

  while $(\mathcal{C}^\bullet(X, \delta_X), \delta_X)$ is a cohomology complex with the space of cochains defined as

$$\mathcal{C}^k(X, \delta_X) = [Pol_{\mathbb{K}}(X)]^k, \qquad k \in \mathbb{Z},$$

  where the grading on the space of polynomials is the one naturally induced by the grading of X.

- $\widetilde{X}$ denotes the graded vector space over $\mathbb{K}$ given by

$$\widetilde{X} := \langle x_1, \dots, x_n\rangle_{\mathbb{K}}.$$

  Then the pair $(\mathcal{C}^\bullet(\widetilde{X}, \delta_{\widetilde{X}}), \delta_{\widetilde{X}})$ is given by

$$\mathcal{C}^k(\widetilde{X}, \delta_{\widetilde{X}}) = [Pol_{\mathbb{K}}(\widetilde{X})]^k, \qquad k \in \mathbb{Z},$$

  while $\delta_{\widetilde{X}} = \delta_X|_{\widetilde{X}}$.

- Due to how the spaces $\mathcal{C}^k(X, \delta_X)$ and $\mathcal{C}^k(\widetilde{X}, \delta_{\widetilde{X}})$ are defined, a generic cochain φ in $\mathcal{C}^k(X, \delta_X)$ can be written in a unique way as

$$\varphi = \varphi_N + \varphi_0,$$

  with φ_N a polynomial in $\mathrm{Pol}_{\mathbb{K}}(X) \setminus \mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$, i.e., a polynomial such that each summand depends on at least one field between B and h, while φ_0 is an element in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$.

► The coboundary operator δ can then be seen as sum of two operators

$$\delta = \widetilde{\delta} + \delta_N,$$

where $\widetilde{\delta}$ acts on terms in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$, while the operator δ_N acts on terms in $\mathrm{Pol}_{\mathbb{K}}(X) \setminus \mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$.

Theorem 5. *In the above notation, if (B, h) is a contractible pair, then:*

(1.) the pair $(\mathcal{C}^{\bullet}(\widetilde{X}, \delta_{\widetilde{X}}), \delta_{\widetilde{X}})$ defines a cohomology complex;

(2.) the following isomorphism holds at the level of cohomology groups:

$$\mathcal{H}^k(X, \delta_X) \simeq \mathcal{H}^k(\widetilde{X}, \delta_{\widetilde{X}}), \qquad \forall k \in \mathbb{Z}.$$

Proof. We start by proving statement (1.). To conclude that $(\mathcal{C}^{\bullet}(\widetilde{X}, \delta_{\widetilde{X}}), \delta_{\widetilde{X}})$ defines a cohomology complex, we only have to prove that $\delta_{\widetilde{X}}$ defines a coboundary operator on $\mathcal{C}^{\bullet}(\widetilde{X}, \delta_{\widetilde{X}})$.
Due to the definition of a contractible pair, applying the coboundary operator δ to an element which depends explicitly on the generators B and h we find something that is either zero or a polynomial in the generators B and h: thus

$$\delta_N(\mathrm{Pol}_{\mathbb{K}}(X) \setminus \mathrm{Pol}_{\mathbb{K}}(\widetilde{X})) \subseteq \Big(\mathrm{Pol}_{\mathbb{K}}(X) \setminus \mathrm{Pol}_{\mathbb{K}}(\widetilde{X}) \ \cup \ \{0\}\Big). \tag{3.16}$$

On the other hand, the operator δ applied on the generators in $\widetilde{X}$ does not involve the generators B and h. Therefore,

$$\widetilde{\delta}(\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})) \subseteq \mathrm{Pol}_{\mathbb{K}}(\widetilde{X}). \tag{3.17}$$

Using the previous observations and the fact that δ is a coboundary operator, it is immediate that both $\widetilde{\delta}$ and δ_N are coboundary operators:

$$0 = \delta^2 = (\widetilde{\delta} + \delta_N) \circ (\widetilde{\delta} + \delta_N) = (\widetilde{\delta})^2 + (\delta_N)^2,$$

which implies

$$(\widetilde{\delta})^2 = 0; \qquad\qquad (\delta_N)^2 = 0.$$

since $(\widetilde{\delta})^2$ takes values in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$ whereas $(\delta_N)^2$ has always image that is equal to 0 or is a polynomial depending on the generators B and h. Hence, both $\widetilde{\delta}$ and δ_N are coboundary operators, the first one on the set of cochains defined by $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$, and the second acting on the set of cochains depending explicitly on the generators B and h (together with the zero). This concludes the proof of statement (1.).

We now want to prove the following isomorphism:

$$\frac{\mathrm{Ker}(\delta^k)}{\mathrm{Im}(\delta^{k-1})} \simeq \frac{\mathrm{Ker}(\widetilde{\delta}^k)}{\mathrm{Im}(\widetilde{\delta}^{k-1})}.$$

For this, we first prove the following three isomorphisms:

1. $\mathrm{Ker}(\delta^k) \simeq \mathrm{Ker}(\widetilde{\delta}^k) \oplus \mathrm{Ker}(\delta^k_N), \quad \forall k \in \mathbb{Z};$
2. $\mathrm{Im}(\delta^k) \simeq \mathrm{Im}(\widetilde{\delta}^k) \oplus \mathrm{Im}(\delta^k_N), \quad \forall k \in \mathbb{Z};$
3. $\mathrm{Ker}(\delta^k_N) \simeq \mathrm{Im}(\delta^k_N), \quad \forall k \in \mathbb{Z}.$

To prove the first isomorphism it is enough to note that

$$\delta^k(\varphi) = 0 \Longleftrightarrow \delta^k_N(\varphi_N) + \widetilde{\delta}^k(\varphi_0) = 0 \Longleftrightarrow \delta^k_N(\varphi_N) = 0 \text{ and } \widetilde{\delta}^k(\varphi_0) = 0,$$

which follows from the fact that, while $\widetilde{\delta}^k(\varphi_0)$ depends only on the generators in $\widetilde{X}$, $\delta^k_N(\varphi_N)$ is either zero or depends also on the generators B and h. Thus we conclude that, at the level of sets,

$$\mathrm{Ker}(\delta^k) = \mathrm{Ker}(\widetilde{\delta}^k) \sqcup \mathrm{Ker}(\delta^k_N).$$

Moreover, it is immediate to check that also the linear structures are compatible. Thus the first isomorphism is proved.

To prove the second isomorphism, let us consider a generic element ψ in $\mathrm{Im}(\delta^k)$. Each cochain in $\mathrm{Pol}_{\mathbb{K}}(X)$ can be written as

$$\psi = \psi_0 + \psi_N$$

with ψ_0 in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$ while ψ_N is a polynomial depending explicitly on the generators B and h. Since ψ is an element in $\mathrm{Im}(\delta^k)$, there exists a cochain φ, with $\varphi = \varphi_0 + \varphi_N$, such that:

$$\delta(\varphi) = \delta_N(\varphi_N) + \delta_0(\varphi_0) = \psi = \psi_N + \psi_0.$$

Using similar arguments as in Equation (3.16) and (3.17), we conclude that

$$\delta_N(\varphi_N) = \psi_N \qquad\qquad \delta_0(\varphi_0) = \psi_0.$$

Thus the second isomorphism follows immediately.

To prove the third isomorphism we need to introduce two new operators:

- let ρ be the $\mathbb{K}$-linear operator on $\mathrm{Pol}_{\mathbb{K}}(X)$, defined on the generators as follows:
$$\begin{cases} \rho(B) = 0 \ ; \\ \rho(h) = -B \ ; \\ \rho|_{\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})} = \delta|_{\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})}. \end{cases}$$
Since the coboundary operator δ is a graded derivation, also ρ is supposed to be a graded derivation:
$$\rho(xy) = \rho(x)y + (-1)^{deg(x)}x\rho(y), \qquad \forall x, y \in \mathrm{Pol}_{\mathbb{K}}(X).$$
In homotopy theory, a function ρ satisfying these properties is known as a *contracting homotopy.*

- Let N be the $\mathbb{K}$-linear operator on $\mathrm{Pol}_{\mathbb{K}}(X)$ defined by the following expression:
$$N = B\frac{\partial}{\partial B} + h\frac{\partial}{\partial h}.$$

A generic cochain φ in $\mathrm{Pol}_{\mathbb{K}}(X)$ can be seen as a sum of terms homogeneous with respect to the total degree of the variables B and h appearing in the term itself. More precisely:
$$\varphi = \sum_{j\geq 0} a_j,$$

with a_j the sum of all monomials appearing in φ, with coefficients φ_i in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$, and with total degree j in the variables B and h:

$$a_j = \sum_{i=0}^{j} \varphi_i(x_1, \ldots, x_n)B^i h^{j-i}.$$

We check that the operators δ, ρ and N satisfy the following two properties:

Property 1
The operator N satisfies the following equation, for each term a_j of order j in the generators B and h:
$$N(a_j) = ja_j.$$

Property 2
The operators δ, ρ and N satisfy the following relation:

$$N = \delta \circ \rho + \rho \circ \delta. \tag{3.18}$$

Indeed let a_j be a generic term of order j in the variables B and h.
Then:

$$\begin{aligned} N(a_j) &= \textstyle\sum_{i=0}^{j} \varphi_i \ (B\frac{\partial}{\partial B} + h\frac{\partial}{\partial h})(B^i h^{j-i}) \\ &= \textstyle\sum_{i=0}^{j} \varphi_i \ (iB^i h^{j-i} + (j-i)B^i h^{j-i}) \\ &= \textstyle\sum_{i=0}^{j} \varphi_i \ jB^i h^{j-i} \\ &= ja_j, \end{aligned}$$

and the first property is verified.
In order to establish the second property, note that, since all the operators considered are linear, instead of verifying it on a generic cochain φ in $\mathrm{Pol}_{\mathbb{K}}(X)$, it is enough to prove that Equation (3.18) holds for all terms a_j that are of order j in the variables B and h. To prove this, we proceed by induction on the order j.

Let us consider j equal to 0: thus a term a_j of order 0 is simply a polynomial in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$. By definition, the operators ρ and δ coincide when they are computed on polynomials depending only on the generators in $\widetilde{X}$. Moreover, as already noticed, the image of $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$ under the operator δ is a subset of $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$. Finally, once again by hypothesis, δ is a coboundary operator, so that

$$(\delta \circ \rho + \rho \circ \delta)(a_0) = (\delta \circ \delta + \delta \circ \delta)(a_0) = 0.$$

On the other hand, using the first property of the operator N we have

$$N(a_0) = 0.$$

Thus Property 2 holds for $j = 0$. Let us suppose that it holds up to order j and let us prove it for $j + 1$. Then, let a_{j+1} be a generic element of order $j + 1$ in the variables B and h,

$$a_{j+1} = \sum_{i=0}^{j+1} \varphi_i B^i h^{j+1-i} = \sum_{i=0}^{j} \varphi_i B^i h^{j-i} \cdot h + \varphi_{j+1} B^{j+1} = \widetilde{a}_j \cdot h + \widetilde{b}_j B,$$

where we are defining $\widetilde{a}_j$ as the summation $\sum_{i=0}^{j} \varphi_i B^i h^{j-i}$ appearing in the previous expression, while $\widetilde{b}_j$ denotes the term $\varphi_{j+1} B^j$.

Using the induction hypothesis, we compute

$$\begin{aligned}
&(\delta \circ \rho + \rho \circ \delta)(a_{j+1}) \\
&\quad = (\delta \circ \rho + \rho \circ \delta)(\widetilde{a}_j) \cdot h + (-1)^{2deg(\widetilde{a}_j)}\ \widetilde{a}_j \cdot (\delta \circ \rho + \rho \circ \delta)(h) \\
&\qquad + (\delta \circ \rho + \rho \circ \delta)(\widetilde{b}_j) \cdot B + (-1)^{2deg(\widetilde{b}_j)}\ \widetilde{b}_j \cdot (\delta \circ \rho + \rho \circ \delta)(B) \\
&\quad = N(\widetilde{a}_j) \cdot h + \widetilde{a}_j \cdot N(h) + N(\widetilde{b}_j) \cdot B + \widetilde{b}_j \cdot N(B) \\
&\quad = j\widetilde{a}_j \cdot h + \widetilde{a}_j \cdot h + j\widetilde{b}_j \cdot B + \widetilde{b}_j \cdot B \\
&\quad = (j+1) \textstyle\sum_{i=0}^{j} \varphi_i B^i h^{j+1-i} + (j+1)\varphi_{j+1} B^{j+1} \\
&\quad = N(a_{j+1}).
\end{aligned}$$

Thus also Property 2 is proved.
To conclude the proof of the theorem, we still have to show that $\mathrm{Ker}(\delta_N^k)$ is isomorphic to $\mathrm{Im}(\delta_N^k)$, $\forall k \in \mathbb{Z}$. Since we already noticed that δ_N^k is a coboundary operator, the inclusion

$$\mathrm{Ker}(\delta_N^k) \supseteq \mathrm{Im}(\delta_N^k)$$

is immediate. To prove the reverse inclusion, we use the operator N and the properties proved above: let φ be a generic cochain in $\mathrm{Ker}(\delta_N^k)$. Therefore, by definition, φ is a polynomial in $\mathrm{Pol}_{\mathbb{K}}(X)$ such that each summand in it depends explicitly on the variables B and h. Hence:

$$\delta_N(\varphi) = \sum_{j>0} \delta_N(a_j) = 0. \tag{3.19}$$

Given a term a_j of degree j, it follows that $\delta(a_j)$ is either a term of degree j in the variables B and h or $\delta(a_j) = 0$, which is the case when a_j is equal to fh^j with f an element in $\mathrm{Ker}(\delta)$. So, by using this observation on the degree, from Equation (3.19) it follows that $\delta_N(a_j) = 0$, for all values of j.
Thus it remains to prove that there exists a polynomial ψ in $\mathrm{Pol}_{\mathbb{K}}(X)$ but not in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$ such that $\delta_N(\psi) = \varphi$. Let us start by considering the summands a_j, with $j > 0$. Since $\delta_N(a_j) = 0$, then:

$$a_j = \frac{1}{j} N\left(a_j\right) = \frac{1}{j}\left[\delta \circ \rho + \rho \circ \delta\right]\left(a_j\right) = \frac{1}{j}\left[\delta \circ \rho\right]\left(a_j\right) = \delta\left(\rho\left(\frac{1}{j} a_j\right)\right).$$

Let us denote the term $\rho(\frac{1}{j}a_j)$ by b_j. This term is a polynomial of total degree j in the variables B and h: in fact, as already noticed, the operator ρ does not change the total degree in these variables unless the term considered is in $\mathrm{Ker}(\rho)$. On the other hand, here we are considering terms of degree at least 1 in B and h: therefore, the only possibility to have

$$\rho\left(\frac{1}{j}a_j\right) = 0$$

is that $a_j = fB$, with f a polynomial in $\mathrm{Pol}_{\mathbb{K}}(\widetilde{X})$ that belongs to $\mathrm{Ker}(\delta)$. A short computation shows that fB as above does not satisfy the condition $\delta(fB) = 0$, as required from the beginning. Therefore, each term b_j is not zero and has total degree j in the variables B and h.
To conclude, we use the following equalities:

$$\varphi = \sum_{j>0} a_j = \sum_{j>0} \delta\left(\frac{1}{j}\rho(a_j)\right) = \delta\left(\sum_{j>0}\frac{1}{j}\rho(a_j)\right).$$

Then, the element

$$\psi = \sum_{j>0}\frac{1}{j}\rho(a_j)$$

satisfies the properties that each term in ψ depends on the variables B and h, and $\delta_N(\psi) = \varphi$. So we conclude that, at the level of sets, $\mathrm{Ker}(\delta_N)$ and $\mathrm{Im}(\delta_N)$ coincide. On the other hand, their isomorphism as vector spaces follows easily. This concludes the proof of the third isomorphism and so of the theorem. □

With the proof of this theorem the aim of this section has been reached: we described a method to further enlarge the extended theory $(\widetilde{X}, \widetilde{S})$ to a total theory (X_{tot}, S_{tot}) via the introduction of trivial pairs, so that a gauge-fixing procedure can be performed without causing any changes at the level of the corresponding BRST cohomology groups. With this we also conclude the chapter, whose content will be used in Chapter 5, where it will be applied to our model of interest.

Chapter 4

Construction of extended varieties

The main purpose of this chapter is to present a method to construct an extended variety $(N, \widetilde{S})$ for an initial gauge theory (X_0, S_0). While in Chapter 3 we gave a general presentation of the BV formalism, with the goal of stating the fundamental notion of (gauge-fixed) BRST cohomology, in this chapter we adopt a more technical point of view, giving a detailed description of a method to construct such a kind of extended varieties for an initial gauge theory. The techniques presented in this chapter will play a fundamental role in the remainder of this thesis, since they will be explicitly applied to a $U(2)$-gauge theory, with the goal of determining the corresponding BRST cohomology groups.

- In Section 4.1 we start by explaining how to construct an extended configuration space defined by gauge theories with initial configuration space given by a nonsingular algebraic variety. This method is based on the construction of a Tate resolution for the pertinent Jacobian ring. This resolution is obtained by applying the Tate algorithm, which, for completeness, is briefly recalled in Appendix B.

- In Section 4.2 an algorithm is presented that determines an extended action, defined on the extended configuration space, that solves the classical master equation. This algorithm is applicable to a class of gauge theories, whose Jacobian ring presents a Tate resolution satisfying certain properties.

- To conclude, in Section 4.3 we discuss the gauge equivalence of extended actions on the same extended configuration space.

The content of this chapter has been inspired by the construction presented in [28]. However, making a comparison between these two constructions, the algorithm described in Section 4.2 for constructing the extended action $\widetilde{S}$ associated to a Tate resolution of $J(S_0)$ presents some differences. Thanks to these differences, we are able to give a more explicit description of the linear terms in the ghost fields appearing in $\widetilde{S}$. This allows us to select, among the whole set of generators introduced with the Tate resolution, the only ones that play an active role for the BRST cohomology.

4.1 The extended configuration space

Throughout this section, by (X_0, S_0) we denote a gauge theory where:

- X_0 is the initial configuration space, which is a real vector space and hence can be seen as a nonsingular affine variety;
- S_0 is a regular function on X_0 that solves the classical master equation.

We now recall the fundamental notion of a *Jacobian ring*.

Definition 42. *In the above notation, let $S_0 \in \mathcal{O}_{X_0}$. The* Jacobian ring $J(S_0)$ of S_0 *is defined to be the quotient $J(S_0) = \mathcal{O}_{X_0}/\operatorname{Im}(\delta)$, that is, the cokernel of the map:*

$$\begin{array}{cccc} \delta: & T_{X_0} & \longrightarrow & \mathcal{O}_{X_0} \\ & \xi & \mapsto & \xi(S_0)\ , \end{array}$$

Note: in the case we are considering, that is, when a global system of coordinates can be fixed on the algebraic variety X_0, the Jacobian ring is suitable for a more explicit description. Let $\{x_1, \ldots, x_m\}$ be a global system of coordinates on X_0, then the Jacobian ring $J(S_0)$ can be described as follows:

$$J(S_0) = \frac{\mathcal{O}_{X_0}}{\langle \partial_1 S_0, \cdots, \partial_m S_0 \rangle}. \tag{4.1}$$

More explicitly, in this case the Jacobian ring $J(S_0)$ is the quotient of the ring $\mathcal{O}_{X_0}$ with respect to the ideal generated by the partial derivatives of the action S_0, computed with respect to the given system of coordinates.

Let (A, δ) be a *Tate resolution* of the Jacobian ring $J(S_0)$ (see Appendix B), where (A, δ) is a differential graded commutative algebra over the ring $\mathcal{O}_{X_0}$. More explicitly,

$$A = \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{W}_T^*)$$

for some graded $\mathcal{O}_{X_0}$-module $\mathcal{W}_T^* = \oplus_{j \leq -1}[\mathcal{W}_T^*]^j$, with $[\mathcal{W}_T^*]^j$ locally free and finitely-generated modules over $\mathcal{O}_{X_0}$.

Remark 22
For our purposes we introduce a specific condition on the resolution: we ask that the resolution (A, δ), with $A = \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{W}_T^*)$, is such that $[\mathcal{W}_T^*]^{-1} = T_{X_0}$. This means that we are fixing the first step in the algorithm. From a physical point of view, this indicates that we are introducing exactly m antifields of ghost degree -1 in the model. The reason why we have to fix the content of the extended configuration space in degree -1 has to be found in the BV formalism: one of the requirements imposed by this formalism is that for each field in the initial configuration space, as well as for each ghost field that is inserted in the extended configuration space, a corresponding antifield or antighost field, respectively, has to be introduced. Therefore, since the configuration space is given by a vector space of dimension m, there are exactly m fields of ghost degree 0. To enforce that each of these fields has a corresponding antifield, we need to introduce exactly m antifields of ghost degree -1.

Under this hypothesis, the Tate resolution can be rewritten as

$$\mathcal{W}_T^* = T_{X_0}[1] \oplus \mathcal{E}_T^*[1], \tag{4.2}$$

where, given a graded module $\mathcal{E}_T = \oplus_{i \geqslant 1} \mathcal{E}_T^i$, with the notation $\mathcal{E}_T^*$ we denote the graded module with finitely-generated homogeneous components defined as $[\mathcal{E}_T^*]^j = [\mathcal{E}_T]^{-j}$, for $j \leq -1$.

Remark 23
As explained in Appendix B, the generators introduced in a Tate resolution are determined by the generators of certain cohomology groups $H^{-n}(A^{-n}, \delta)$.
It is immediate to see that, among the whole set of generators of these cohomology groups there are always the ones determined by considering relations of linear dependence involving the generators of higher degree introduced in the previous step.
More explicitly, let $\{\gamma_j^*\}_{j=1,\ldots,m_j}$ be the collection of generators introduced in the step $-n$ of the Tate resolution. To determine the generators that have to be

introduced in degree $-n-1$ we analyze the generators of the cohomology group $H^{-n}(A^{-n}, \delta)$. However, among all the generators of this cohomology group we can identify a subset of them. Indeed, if there exists a collection of elements $\{r_j\}_{j=1,\dots,m_j}$ of the ring R (not all equals to zero) such that

$$r_1\delta(\gamma_1^*) + r_2\delta(\gamma_2^*) + \dots + r_{m_j}\delta(\gamma_{m_j}^*) = 0,$$

then the element

$$\xi = r_1\gamma_1^* + r_2\gamma_2^* + \dots + r_{m_j}\gamma_{m_j}^*$$

certainly belongs to $\mathrm{Ker}(\delta_{-n}^{-n})$ and, since ξ depends on elements which have been introduced in the step $-n$, there is no possibility to see it as an element of $\mathrm{Im}(\delta_{-n+1}^{-n})$. Thus all the elements ξ of this type defines a generator of $H^{-n}(A^{-n}, d)$ and so each of them determines a generator in the Tate resolution.

Notation: In what follows, we call *generators of type* β a collection of generators in the Tate resolution inductively defined as follows:

- All the generators $\{x_i^*\}_{i=1,\dots,m} \subseteq [\mathcal{W}_T^*]^{-1} = T_{X_0}[1]$, which are the antifields associated to the initial fields $\{x_i\}_{i=1,\dots,m}$, are of type β by definition.

- The generators of type β in degree $-q$, collectively denoted by $\{\beta_j^{*,(-q)}\}_{j\in J}$, are inductively determined by the generators of type β of degree $-q+1$.
  A generator $\gamma_j^{*,(-q)} \in [\mathcal{W}_T^*[1]]^{-q}$ in the Tate resolution is called a *generator of type* β if there exists a collection of elements $\{r_j\}_{j=1,\dots,m_j}$ of the ring R such that

  $$\delta(\gamma_j^{*,(-q)}) = r_1\beta_1^{*,(-q+1)} + r_2\beta_2^{*,(-q+1)} + \dots + r_{m_j}\beta_{m_j}^{*,(-q+1)}$$

  with $\beta_1^{*,(-q+1)}, \beta_2^{*,(-q+1)}, \dots, \beta_{m_j}^{*,(-q+1)}$, generators of type β of degree $-q+1$. Thus for this generator $\gamma_j^{*,(-q)}$ the notation $\beta_j^{*,(-q)}$ will be used.

For convenience, in what follows we distinguish the generators $\{\beta_i^{*,(-2)}\}$ from the others of type β by denoting them by $\{C_i^*\}$. Finally, by

$$\mathcal{W}^* = T_{X_0}[1] \oplus \mathcal{E}^*[1], \tag{4.3}$$

we denote the negatively-graded module over $\mathcal{O}_{X_0}$ with finitely-generated homogeneous components obtained by selecting among all the generators determined by the Tate resolution only the ones of type β.

Note: as it will be explained in Chapter 6, it seems that, at least in the context of $U(n)$-matix models, the graded module $\mathcal{W}^*$ usually contains generators of degree $j < 0$ only up to a finite degree while, on the contrary, in most cases this property is not satisfied by the full Tate resolution $\mathcal{W}_T^*$, which may be a $\mathbb{Z}_{<0}$-graded module with homogeneous components of any negative degree.

The extended configuration space $\widetilde{\mathbf{X}}$

Given the graded module $\mathcal{W}^*$ defined in (4.3), let us denote the corresponding graded variety by V, i.e.,

$$V = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E})),$$

where the graded module $\mathcal{E}$ has been obtained from the module $\mathcal{E}_T$ by selecting the ghost fields corresponding to the type β generators in $\mathcal{E}_T^*$. Thus it holds that $\mathcal{E}$ is generated by the dual of the generators in $\mathcal{E}^*$ or, in other words, we have that $\mathcal{E}^*$ is determined by the antighost fields corresponding to the ghost fields in $\mathcal{E}$.
This V is the graded variety underlying the extended variety that we are going to construct. Note that by applying Tate's algorithm we have determined the antifields and antighost fields that we need to introduce in the theory, while in the extended variety also the corresponding fields and ghost fields will play a role. As already noticed in (3.6), the requirement in the BV formalism that for each field and ghost field in the extended configuration space there should be a corresponding antifield or antighost field, respectively, forces the extended configuration space to be a $\mathbb{Z}$-graded vector space of the form

$$\widetilde{X} = W \oplus W^*[1] \ ,$$

where W is a $\mathbb{Z}_{\geqslant 0}$-graded vector space, which describes the field content of the extended configuration space, while $W^*[1]$ is a $\mathbb{Z}_{<0}$-graded vector space that describes the corresponding antifield content.
Therefore, the extended configuration space $\widetilde{X}$ determined by the type β generators of the Tate resolution (4.2) is

$$\widetilde{X} = \mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus X_0 \oplus \mathcal{E} \ ,$$

with X_0 the initial configuration space.

Thus using a Tate resolution of the Jacobian ring $J(S_0)$ we immediately construct the corresponding extended configuration space $\widetilde{X}$. Note that, besides the requirement in deg -1 as explained above (Remark 22), no further conditions have to be enforced on the resolution itself. Therefore, different extended configuration spaces might be defined, depending on the Tate resolution considered.

Remark 24
To construct the extended configuration space $\widetilde{X}$ we have first selected a collection of generators among all the generators determined by a Tate resolution. This may allow us to have a space $\widetilde{X}$ that contains only a finite number of ghost/antighost fields. Making a comparison with the method presented in [28], we see that the authors, in order to construct a BV variety for the initial gauge theory (X_0, S_0), consider the full Tate resolution $\mathcal{W}_T^*$, without making any selection among the generators. Explicitly, they set:

$$N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E}_T^*[1] \oplus T_{X_0}[1] \oplus \mathcal{E}_T)) \ .$$

As explained in more detail in Section 4.2, this choice ensures the existence of an extended action that solves the classical master equation on $\mathcal{O}_N$ without having to impose any further conditions on the initial gauge theory (X_0, S_0), which is simply assumed to have X_0 a nonsingular algebraic variety. However, due to the potentially infinite number of generators in $\mathcal{W}_T^*$, defining such a $\widetilde{S}$ may be difficult to realize.
For this reason we concentrate on a particular type of generators in $\mathcal{W}_T^*$ that, in the case in which the initial gauge theory satisfies an extra condition, are enough to construct an extended action in $\mathcal{O}_{\widetilde{X}}$. Finally, we notice that the generators of type β can be determined also without having to compute the complete Tate resolution, which may be a difficult computation, but simply using their inductive definition. However, the fact of selecting a family of generators has a drawback, loosing the property of uniqueness up to gauge equivalence of the extended variety $(N, \widetilde{S})$, as discussed in Section 4.3.

Remark 25
As explained in more detail in Appendix B, a Tate resolution is obtained via an algorithm in which, at each step, the algebra under consideration is enlarged by introducing new variables. These new variables have a grading, determined by the step of the algorithm in which these variables are introduced, and they might be either real of Grassmannian, once again depending on the parity of the step in the algorithm. From a physical point of view, this procedure of inserting new variables coincides with extending the configuration space though the introduction of antighost fields. Moreover, also the ghost degree and the

parity of these ghost fields are determined by Tate's algorithm. In other words, the characteristics of a ghost field, such as its ghost degree and its parity, are determined by an algebraic geometric construction.

In the physics literature (as, for example, in [34]), another method to construct an extended configuration space is often presented, which is based on the computation of the rank of a sequence of matrices. However, even though this alternative method is based on a simpler construction, involving only the computation of the rank of a sequence of matrices, instead of the construction of a Tate resolution, the negative aspects of this approach are mainly two:

1. With this method, only one among the whole family of equivalent extended configuration spaces can be constructed and not necessary the one obtained is the minimal extension;
2. To construct the extended action $\widetilde{S}$ the only consequential method is the one described by the algorithm presented in the following section and based on the Tate resolution. Otherwise, a more intuitive procedure can be used, based on trails and errors, to determine a solution for the classical master equation on the extended configuration space by simply adding to the initial action terms involving the ghost fields, with the aim of eliminating the reminder in the computation of the Poisson brackets on the approximate action with itself.

These are the reasons why in Chapter 5, in order to compute the extended configuration space for our model of interest, we prefer to use the method explained above, based on the identification of a family of generators in a Tate resolution for the Jacobian ring $J(S_0)$.

4.2 Construction of the extended action

To conclude the construction of the extended variety $(N, \widetilde{S})$ associated to the initial gauge invariant theory (X_0, S_0), we need to define the new action $\widetilde{S}$. Up to now we have only determined the extended configuration space $\widetilde{X}$, with

$$\widetilde{X} = \mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus X_0 \oplus \mathcal{E} \ ,$$

for $\mathcal{E}$ an $\mathcal{O}_{X_0}$-module with homogeneous components that are locally free and finitely generated, obtained from a suitable Tate resolution (A, δ) of the Jacobian ring $J(S_0)$.

In the algorithm we are going to present, we will explain how to construct an extended action $\widetilde{S}$, which solves the classical master equation on $\mathcal{O}_N$. This method is applicable only if the type β generators that have been selected in the Tate resolution satisfy an additional condition. However, when applicable, it may determine an extended action through a finite number of steps.

Before starting we recall that the initial configuration space X_0 is, by hypothesis, a real affine variety. Moreover, on it we have fixed a global system of coordinate $\{x_1, \ldots, x_m\}$. Therefore, as basis for the tangent space $T_{X_0}[1]$ we can use $x_i^* = -\partial_i$, with $i = 1, \ldots, m$:

$$T_{X_0}[1] = \langle x_1^*, \ldots, x_m^* \rangle, \qquad m = dim_{\mathbb{R}}(X_0).$$

Furthermore, $\{\beta_i^*\}$ is a fixed basis of $\mathcal{E}^*$, which contains all the type β generators of the Tate resolution, while $\{\beta_i\}$ is the dual basis of $\mathcal{E}$. Finally, we have set

$$N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1])).$$

It is possible to define a Poisson structure on $\mathcal{O}_N$, as we are going to explain.

The Poisson structure on $\mathcal{O}_N$

To completely determine the Poisson structure on $\mathcal{O}_N$ it is enough to impose the following conditions on the way in which the Poisson bracket acts on the generators of $\mathcal{O}_N$:

$$\begin{aligned}
&\blacktriangleright \quad \big\{f, g\big\} = 0 \;; \\
&\blacktriangleright \quad \big\{f, x_i^*\big\} = \partial_i f \;; \\
&\blacktriangleright \quad \big\{f, \beta_i\big\} = \big\{f, \beta_i^*\big\} = 0 \;; \\
&\blacktriangleright \quad \big\{x_i^*, \beta_j\big\} = \big\{x_i^*, \beta_j^*\big\} = \big\{x_i^*, x_j^*\big\} = 0 \;; \\
&\blacktriangleright \quad \big\{\beta_i, \beta_j\big\} = \big\{\beta_i^*, \beta_j^*\big\} = 0 \;; \\
&\blacktriangleright \quad \big\{\beta_i, \beta_j^*\big\} = \delta_{ij} \;;
\end{aligned} \tag{4.4}$$

where f and g are regular functions in $\mathcal{O}_{X_0}$, x_i^* is an antifield generating $T_{X_0}[1]$, β_i is a ghost field of degree strictly positive and, finally, β_i^* is some generator of any ghost degree strictly less than -1.

Then, by imposing bilinearity being graded Poisson, the bracket defined above can be uniquely extended to a Poisson bracket structure on the graded algebra $\mathcal{O}_N$. Note that, from the last condition stated in the list above, the Poisson bracket computed on a pair of generators does not vanish only when the pair considered is composed of a generator together with the corresponding dual element in the dual basis. Looking at the second condition, we see that this holds also when we consider the Poisson bracket of any antifield x_i^* together with a field x_j.

In the following construction, the extended action will be given by a sequence of approximations, obtained by adding terms with increasing degree in the ghosts, namely in the positively graded generators. Thus we need to introduce a notation that allows to better identify the components of an element in $\mathcal{O}_N$.
Since the graded algebra $\mathcal{O}_N$ can be seen as given by the following tensor product

$$\mathcal{O}_N \cong \mathrm{Sym}_{\mathcal{O}_{X_0}}(T^*_{X_0}[1] \oplus \mathcal{E}^*[1]) \otimes_{\mathcal{O}_{X_0}} \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E}) \ , \tag{4.5}$$

then a generic element φ in $\mathcal{O}_N$ can be written as follows:

$$\varphi = \sum_{i \in I} \varphi_{n,i} \cdot \varphi_{p,i} \ , \tag{4.6}$$

where:

- $\varphi_{n,i}$, i in I, is a polynomial in the antighosts $\left\{\beta_j^*\right\}_{j \in J}$, the antifields $\{x_a^*\}$, and the fields $\{x_a\}$, with $a = 1, \ldots, m$, i.e., in the generators of non-positive ghost degree;
- $\varphi_{p,i}$, i in I, is a monomial in the ghost fields $\{\beta_j\}_{j \in J}$, i.e., in the generators of strictly positive ghost degree. Moreover, we require that $\varphi_{p,i} \neq \varphi_{p,j}$ for $i \neq j$.

Definition 43. *Given a generic element φ in $\mathcal{O}_N$,*

$$\varphi = \sum_{i \in I} \varphi_{n,i} \cdot \varphi_{p,i} \ ,$$

with the terminology positive degree*, denoted by $deg_p(\varphi)$, we indicate the minimum degree of φ in the positive generators.*
More explicitly, we define

$$deg_p(\varphi) = \min_{i \in I} \ deg(\varphi_{p,i}) \ .$$

By $deg_n(\varphi)$ *we denote the* negative degree *of an element* φ*, defined as*

$$deg_n(\varphi) = \max_{i \in I} \ deg(\varphi_{n,i}) \ .$$

Finally, given a summand φ_i*,* $deg(\varphi_i)$ *denotes the* degree *of* φ_i*, defined as:*

$$deg(\varphi_i) = deg_p(\varphi_i) + deg_n(\varphi_i) \ .$$

Property 3
From the definition we deduce the following properties for the positive and the negative degree of a generic element φ in $\mathcal{O}_N$:

1. $deg_p(\varphi) \geqslant 0$;
2. $deg_n(\varphi) \leqslant 0$;
3. $deg(\varphi_{n,i} \cdot \varphi_{p,i}) = deg_p(\varphi_{p,i}) + deg_n(\varphi_{n,i})$;
4. $deg(\varphi_i) \leqslant deg_p(\varphi_i)$;
5. $deg(\varphi_i) \geqslant deg_n(\varphi_i)$;
6. given a pair of elements φ and ψ in $\mathcal{O}_N$, both given by only one monomial, either $\varphi\psi = 0$ or

$$deg_p(\varphi\psi) \geqslant deg_p(\varphi) + deg_p(\psi) \quad \text{and} \quad deg_n(\varphi\psi) \leqslant deg_n(\varphi) + deg_n(\psi) \ .$$

In the construction, an important role will be played by the ideal $F^q\mathcal{O}_N$, with $q \geqslant 0$, which has already been introduced in (3.3) (see Section 3.1), for a $\mathbb{Z}_{\geqslant 0}$-graded module. Here we extend this definition to the case of a $\mathbb{Z}$-graded module, using the notation introduced above:

$$F^q\mathcal{O}_N = \{\varphi \in \mathcal{O}_N : \ deg_p(\varphi) \geqslant q\} \cup \{0\} \, .$$

In particular, for $q = 0$, $F^q\mathcal{O}_N$ coincides with the full algebra $\mathcal{O}_N$.

Remark 26
Note that, for all $q \geqslant 0$, $F^q\mathcal{O}_N$ is closed with respect to the sum. Moreover, it also defines an ideal over $\mathcal{O}_N$: this is a consequence of Properties (1) and (6) of the positive degree.

The last object we need to introduce before being able to state the algorithm for the construction of the extended action is the following collection of modules over $\mathcal{O}_{X_0}$ and ideals over $\mathcal{O}_N$:

- $I_N^{(1)}$ denotes the set of all the elements in $\mathcal{O}_N$ that are linear in the positive generators. More explicitly, referring to the notation introduced in (4.6), we write:

$$I_N^{(1)} = \Big\{\varphi \in \mathcal{O}_N : \varphi = \sum_{j \in J} \varphi_{n,j}\beta_j, \text{ with } \varphi_{n,j} \in \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])\Big\} \cup \{0\}$$

  where β_j, $j \in J$, indicates any β-type generator of positive ghost degree. Note that $I_N^{(1)}$ is closed with respect to the sum and is a module over $\mathcal{O}_{X_0}$, viz. on the algebra of regular functions on the initial algebraic variety X_0, $\mathcal{O}_{X_0} = \mathrm{Pol}_{\mathbb{K}}(x_i)$.
  Moreover, $I_N^{(1)}$ turns out to be closed also with respect to the product of elements in $\mathcal{O}_N/F^1\mathcal{O}_N$ (which are elements of non-positive degree).

- We denote the powers of the module $I_N^{(1)}$, for $q \geqslant 2$, by $I_N^{(q)}$. More precisely: $I_N^{(2)} = I_N^{(1)} \cdot I_N^{(1)}$, with

$$I_N^{(2)} = \Big\{\varphi = \sum_{j,k \in J} \varphi_{n,jk} \cdot \beta_j \beta_k \Big\} \cup \{0\}\,.$$

  More generally, we denote the module over $\mathcal{O}_{X_0}$ given by the elements in $\mathcal{O}_N$ that are q-linear in the positively graded generators by $I_N^{(q)}$, with $q \geqslant 2$. Note also that $I_N^{(q)}$ is not only a module over $\mathcal{O}_{X_0}$: it is also closed with respect to the product with elements $\mathcal{O}_N/F^1\mathcal{O}_N$, as already noticed for $I_N^{(1)}$.

- We denote the set of all the elements in $\mathcal{O}_N$ that are at least q-linear in the positively graded generators, with $q \geqslant 1$, by $I_N^{\geqslant q}$. Then:

$$I_N^{\geqslant q} = \bigcup_{s \geqslant q} I_N^{(s)}\,.$$

  Contrary to what happens for $I_N^{(q)}$, $I_N^{\geqslant q}$ does not only have the structure of a module over $\mathcal{O}_{X_0}$ but it defines an ideal over $\mathcal{O}_N$: given an element φ in $I_N^{\geqslant q}$ and a generic element ψ in $\mathcal{O}_N$, the product $\varphi \cdot \psi$ is either 0 or is at least q-linear in the generators of positive ghost degree.

Note: As already mentioned, the construction of the extended action has similarities with what presented in [28]. However, the notation we have introduced is slightly different. The main reason for our choice is that our aim is to make their construction more explicit, in order to be able to give a more precise description of the dependence of the extended action with respect to the generators.

Before explaining the algorithm, we state some technical lemmas on the Poisson structure on the extended configuration space, which will be used in what follows. The statements of the following lemmas are similar to the ones in Proposition 4.1, Lemma 4.6 and Lemma 4.3 in [28], whereas the proofs are presented in somewhat more detail, using the notation introduced above, in Appendix C.

Notation: by S_{lin} we denote the following expression:

$$S_{lin} = S_0 + \sum_{k\in K} \delta(C_k^*)C_k + \sum_{j\in J} \delta(\beta_j^*)\beta_j .$$

Lemma 4
The canonical isomorphism

$$\mathcal{O}_N \cong \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1]) \otimes_{\mathcal{O}_{X_0}} \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E})$$

identifies, modulo $F^1\mathcal{O}_N$, the operator $\{S_{lin},\ \}$ with the operator $\delta \otimes Id$, where δ is the coboundary operator given by the Tate resolution (A, δ), restricted to act only on generators of type β.
More explicitly, for any element $\varphi = \sum_i \varphi_{n,i} \otimes \varphi_{p,i}$ in $\mathcal{O}_N$,

$$\{S_{lin}, \varphi\} = \sum_i \delta(\varphi_{n,i}) \otimes Id(\varphi_{p,i}) \qquad (mod\ F^1\mathcal{O}_N).$$

Lemma 5
Let q be an integer $q \geqslant 0$. Then the following properties hold:

1. $\left\{F^q\mathcal{O}_N^0, \mathcal{O}_N^0\right\} \subseteq F^q\mathcal{O}_N^1$;
2. $\left\{F^q\mathcal{O}_N^0, F^q\mathcal{O}_N^0\right\} \subseteq F^{q+1}\mathcal{O}_N^1$.

Lemma 6
For any integer $q \geqslant 0$ the following inclusion holds:

$$\left\{I_N^{\geqslant 2} \cap \mathcal{O}_N^0, F^q\mathcal{O}_N\right\} \subseteq F^{q+1}\mathcal{O}_N.$$

An important role in the following construction will be played by a cohomology complex we denote by $\mathcal{G}_{q,r}^\bullet$, with q, $r \in \mathbb{N}_0$ fixed, $r \leqslant q$, which is introduced in the following definition.

Definition 44. *Let q, r be two fixed values in $\mathbb{N}_0$, with $r \leqslant q$. The pair $(\mathcal{G}^{\bullet}_{q,r}, d)$ denotes a collection of sets and a graded map on them, defined as follows:*

- *for j in $\mathbb{Z}$, $j \leqslant q$,*
$$\mathcal{G}^j_{q,r} = \pi_q(F^q\mathcal{O}^j_N \cap I^{(r)}_N),$$
with π_q the canonical projection $\pi_q : F^q\mathcal{O}_N \longrightarrow F^q\mathcal{O}_N/F^{q+1}\mathcal{O}_N$.
More explicitly:
$$\mathcal{G}^j_{q,r} = \Big\{\varphi \in \mathcal{O}^j_N,\ \varphi = \sum_i \varphi_{n,i}\varphi_{p,i} :\ deg(\varphi_{p,i}) = q\ \forall i \ and\ \varphi_{p,i} = \beta_{j_1}\cdots\beta_{j_r}\Big\}.$$
In words, the elements in $\mathcal{G}^j_{q,r}$ are all the elements of $\mathcal{O}_N$ with total degree j, positive degree q and which are r-linear in the positively-graded generators;

- *the graded map $d = \{d^j\}_{j\leqslant p}$ is defined as follows:*
$$d^j : \mathcal{G}^j_{q,r} \longrightarrow \mathcal{G}^{j+1}_{q,r}\ ,$$
with $\varphi = \sum_i \varphi_{n,i}\varphi_{p,i}$, for φ in $\mathcal{G}^j_{q,r}$,
$$d(\varphi) = (\delta \otimes Id)(\varphi) = \sum_i \delta(\varphi_{n,i})\varphi_{p,i}\ ,$$
where δ is the coboundary operator given by the Tate resolution fixed at the beginning of the algorithm and restricted to the type β generators.

Lemma 7
The pair $(\mathcal{G}^{\bullet}_{q,r}, d)$ introduced in Definition 44 defines a cochain complex.

For completeness, the proof of this Lemma has been included in Appendix C. Summarizing, we are in the following setting:

- S_0 is a regular function in $\mathcal{O}_{X_0}$ that solves the classical master equation on $\mathcal{O}_{X_0}$, with X_0 a real affine variety.

- (A, δ) is a Tate resolution of the Jacobian ring $J(S_0)$ on the ring $\mathcal{O}_{X_0}$, with
$$A = \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*_T[1])$$
for some graded $\mathcal{O}_{X_0}$-module $\mathcal{E}^*_T = \oplus_{j\leq -1}[\mathcal{E}^*_T]^j$, with finitely-generated homogeneous components.

- N is the (-1)-symplectic variety with support X_0 defined as follows:

$$N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1])) ,$$

where $\mathcal{E}^*[1]$ is obtained from $\mathcal{E}_T^*[1]$ by selecting the generators of type β, while, analogously, $\mathcal{E}$ is obtained by considering only the generators that are dual of generators of type β.

In this setting we can state the following theorem on the existence of an extended action $\widetilde{S} \in \mathcal{O}_N$, which solves the classical master equation on $\mathcal{O}_N$.

Theorem 6. *Let (X_0, S_0), (A, δ), N and $\mathcal{G}_{q,r}$ be as defined above. If the cohomology complex $(\mathcal{G}_{q,r}, d)$ is such that:*

$$H^j(\mathcal{G}_{q,r}, d) = 0, \qquad \text{for } j \leqslant q, \tag{4.7}$$

then there exists a function $\widetilde{S} \in \Gamma(X_0, \mathcal{O}_N^0)$ such that:

1. $\widetilde{S}|_{X_0} = S_0$;
2. *$\widetilde{S}$ is a solution of the classical master equation on $\mathcal{O}_N$, i.e., $\{\widetilde{S}, \widetilde{S}\} = 0$;*
3. $\widetilde{S} \equiv S_0 + \sum_{k\in K} \delta(C_k^*)C_k + \sum_{j\in J} \delta(\beta_j^*)\beta_j \quad \text{mod } I_N^{\geqslant 2}$,
   with $I_N^{\geqslant 2}$ the ideal generated by the terms in $\mathcal{O}_N$ which are at least quadratic in the positively-graded generators.

Notation: the symbol $\equiv$ is used, instead of the usual $=$, to indicate an equality that only holds at the level of the quotient.

To conclude, we can say that, given a pair (X_0, S_0), with X_0 the configuration space given by a nonsingular algebraic variety and S_0 a regular function on X_0 that solves the classical master equation, using a Tate resolution (A, δ) of the Jacobian ring $J(S_0)$ with

$$A = T_{X_0}[1] \oplus \mathcal{E}_T^*[1]$$

for some graded $\mathcal{O}_{X_0}$-module $\mathcal{E}_T$, if the cohomology complex $(\mathcal{G}_{q,r}, d)$ satisfies condition (4.7), then it is possible to define an extended variety $(N, \widetilde{S})$ associated to the Tate resolution (A, δ) as follows:

- $N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]))$,
  where $\mathcal{E}^*$ is obtained by selecting only the generators of type β.

► $\widetilde{S} \equiv S_0 + \sum_{k\in K} \delta(C_k^*)C_k + \sum_{j\in J} \delta(\beta_j^*)\beta_j \mod I_N^{\geqslant 2}$,
as constructed in Theorem 6.

Notice that, for how it has been constructed, the extended variety $(N, \widetilde{S})$ does not depend only on the initial gauge theory (X_0, S_0) but also on the Tate resolution (A, δ) of the Jacobian ring $J(S_0)$. Therefore, in general different Tate resolutions may determine different extended varieties associated to the same initial gauge theory.

In the rest of this section, we present the proof of Theorem 6 in the form of an algorithm, since it will be used in the second part of this thesis to find the extended action for the matrix model that we want to analyze.

Proof. The extended action $\widetilde{S}$ will be constructed step by step through approximations. More precisely, we prove by induction that for each $q \geqslant 1$ there is an $\widetilde{S}_{\leq q} \in \Gamma(X_0, \mathcal{O}_N^0)$ such that the following conditions are satisfied:

1. $\widetilde{S}_{\leq q} \equiv S_0 + \sum_{k\in K} \delta(C_k^*)C_k + \sum_{j\in J} \delta(\beta_j^*)\beta_j \quad (\mathrm{mod}\ I_N^{\geqslant 2})$,
2. $\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\} \in I_N^{\geqslant 2} \cap F^{q+1}\mathcal{O}_N^0$,
3. $\widetilde{S}_{\leq q+1} \equiv \widetilde{S}_{\leq q}, \quad (\mathrm{mod}\ F^{q+1}\mathcal{O}_N^0)$.

$q = 1$

Let us set

$$\widetilde{S}_{\leq 1} = S_0 + \sum_{k\in K} \delta(C_k^*)C_k + \sum_{j\in J} \delta(\beta_j^*)\beta_j \ .$$

Then we have to verify that $\widetilde{S}_{\leq 1}$ satisfies the three conditions required in the above list. However, in this particular case we have to check only condition 2., since 1. holds automatically and 3. does not apply to this case.
To prove 2., we use the properties of the Poisson bracket, listed in Definition 20, together with the fact that $deg(S_0) = 0$ and that S_0 is a solution for the classical master equation, i.e. $\{S_0, S_0\} = 0$.
Thus we obtain the following expression:

$$\begin{aligned}\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\} &= 2\textstyle\sum_k \{S_0\ , \delta(C_k^*)C_k\} + \sum_{k,l} \{\delta(C_k^*)C_k, \delta(C_l^*)C_l\} \\ &\quad + \textstyle\sum_{i,j} \{\delta(\beta_i^*)\beta_i, \delta(\beta_j^*)\beta_j\} \ .\end{aligned} \tag{4.8}$$

The second sum in the previous equality can be rewritten as follows, using the properties of the Poisson bracket stated in Definition 20 and in Remark 7:

$$\left\{\delta(C_i^*)C_i, \delta(C_j^*)C_j\right\} = -C_i \left\{\delta(C_i^*), \delta(C_j^*)\right\} C_j \equiv 0 \quad (mod\ I_N^{(2)}) \ .$$

Note that the summand which is neglected in the last equality is precisely an element of $I_N^{(2)}$: in fact, it contains exactly two generators with positive degree. This is due to the fact that, in view of the definition of the coboundary operator δ, the element $\delta(C_i^*)$ would never depend on the positively-graded generators. Thus the same statement is true if we consider $\left\{\delta(C_i^*), \delta(C_j^*)\right\}$.
Analogously, also the third sum in Equation (4.8) can be neglected when we consider the expression modulo $I_N^{(2)}$.
Instead, for what concerns the first sum, recalling that $\delta(C_j^*)$ is a combination with coefficients in $\mathcal{O}_{X_0}$ of the antifields x_i^* and that

$$\{S_0, x_a^*\} = \partial_a S_0 = \delta(x_a^*) \ ,$$

the expression in (4.8) becomes the following:

$$\left\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\right\} \equiv 2\sum_j \left\{S_0, \delta(C_j^*)C_j\right\} = 2\sum_j \delta(\delta(C_j^*))C_j = 0 \qquad (mod\ I_N^{(2)}) \ ,$$

where the last equality follows from the fact that δ is a coboundary operator.
Finally, the sum

$$2\sum_j \left\{S_0, \delta(\beta_j^*)\beta_j\right\}$$

did not appear in Equation (4.8) because, due to the definition of the generators β_j^*, the terms $\delta(\beta_j^*)$ do not depend on the antifields x_i^*, which are the only generators which may give a nonzero contribution when considered in a Poisson bracket together with the initial action S_0.

To be able to conclude that $\widetilde{S}_{\leq 1}$ satisfies condition 2. we only have to note that, for $q = 1$, $F^{q+1}\mathcal{O}_N \cap I_N^{\gtrless 2}$ coincides with $I_N^{\gtrless 2}$: in fact, $I_N^{\gtrless 2}$ is contained in $F^2\mathcal{O}_N$, because each element that is at least quadratic in the positively-graded generators necessarily has positive degree equal to at least 2. Thus $\widetilde{S}_{\leq 1}$ satisfies all three required properties.

The induction step

Let us define

$$\widetilde{S}_{\leq q+1} = \widetilde{S}_{\leq q} + \nu$$

with $\nu \in \Gamma(X_0, I_N^{\geqslant 2} \cap F^{q+1}\mathcal{O}_N^0)$ to be determined. More explicitly, ν is an element of total degree 0, at least quadratic in the fields, and of positive degree at least $q+1$.

By definition, and using the induction hypothesis, we deduce that $\widetilde{S}_{\leq q+1}$ satisfies conditions 1. and 3. Then we only have to prove that $\widetilde{S}_{\leq q+1}$ also satisfies condition 2. To arrive to this conclusion we start by noticing that:

$$\left\{\widetilde{S}_{\leq q}, \nu\right\} \equiv \left\{\widetilde{S}_{lin}, \nu\right\} \qquad (mod\ F^{q+2}\mathcal{O}_N) \ ,$$

where

$$\widetilde{S}_{lin} = S_0 + \sum_{k \in K} \delta(C_k^*) C_k + \sum_{j \in J} \delta(\beta_j^*) \beta_j \ .$$

Indeed, using the first condition in the induction hypothesis, we deduce that $\widetilde{S}_{\leq q} - \widetilde{S}_{lin}$ consists of summands that are at least quadratic in the positively-graded generators. Moreover, all summands in the action $\widetilde{S}_{\leq q}$ are of total degree 0. This implies that $\widetilde{S}_{\leq q} - \widetilde{S}_{lin}$ belongs to $I_N^{\geqslant 2} \cap \mathcal{O}_N^0$.
Since, by hypothesis, the positive degree of ν is at least $q+1$, we can apply Lemma 6 and deduce that

$$\left\{\widetilde{S}_{\leq q} - \widetilde{S}_{lin}, \nu\right\} \in F^{q+2}\mathcal{O}_N \ .$$

In Lemma 4, we proved that $\{S_{lin}, -\}$, coincides with the operator $\delta \otimes Id$, modulo $F^1\mathcal{O}_N$, where the operator δ is the coboundary operator of the initial Tate resolution, restricted to act only on generators of type β.
Therefore, it holds that

$$\left\{S_{lin}, \nu\right\} \equiv (\delta \otimes Id)\nu \quad (mod\ F^{q+2}\mathcal{O}_N) \ .$$

Furthermore, by applying Lemma 5, we deduce that:

$$\left\{\nu, \nu\right\} \equiv 0 \ .$$

Thus:

$$\left\{\widetilde{S}_{\leq q+1}, \widetilde{S}_{\leq q+1}\right\} \equiv \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\} + 2(\delta \otimes Id)\nu \quad (mod\ F^{q+2}\mathcal{O}_N) \ .$$

On the other hand, from the Jacobi identity we obtain

$$\left\{\widetilde{S}_{\leq q}, \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}\right\} = 0 \ .$$

Once again using the results stated in Lemma 5 and in Lemma 6, we have

$$\begin{aligned} 0 = \left\{\widetilde{S}_{\leq q}, \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}\right\} &\equiv \left\{\widetilde{S}_{lin}, \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}\right\} && (mod\ F^{q+2}\mathcal{O}_N) \\ &\equiv \delta\left(\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}_n\right)\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}_p && (mod\ F^{q+2}\mathcal{O}_N)\ . \end{aligned} \tag{4.9}$$

Indeed, for the induction hypothesis, we have that the term $\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}$ belongs to $F^{q+1}\mathcal{O}_N$. Since, as already noted, $\widetilde{S}_{\leq q} - S_{lin}$ belongs to $I_N^{\geqslant 2} \cap \mathcal{O}_N^0$, by applying Lemma 6, we justify the first step.
As to the second step, as proved in Lemma 4, the operator $\Phi = \left\{S_{lin}, -\right\}$ coincides with $\delta \otimes Id$ modulo $F^1\mathcal{O}_N$. Then, using the fact that $\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}$ belongs to $F^{q+1}\mathcal{O}_N$, we would have

$$\left\{\widetilde{S}_{lin}, \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}\right\} \equiv \left[\delta\left(\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}_n\right) + \alpha\right]\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}_p\ ,$$

with α an element in $F^1\mathcal{O}_N$ and so with $\alpha \cdot \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}_p$ an element in $F^{q+2}\mathcal{O}_N$. So also the second step in (4.9) is justified.
Thus

$$(\delta \otimes Id)\left(\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}\right) \equiv 0 \qquad (mod\ F^{q+2}\mathcal{O}_N)\ . \tag{4.10}$$

Once again for the induction hypothesis, we know that $\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}$ belongs to $I_N^{\geqslant 2}$ and so, if we consider it modulo $F^{q+2}\mathcal{O}_N$, it can be written as a sum of elements in $I^{(r)}$ with $2 \leqslant r \leqslant q+1$. Therefore, $\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}$ mod $F^{q+2}\mathcal{O}_N$ is a cocycle of degree 1 in $\bigoplus_{r=2}^{q+1} \mathcal{G}^1_{q+1,r}$.
More precisely,

$$\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\} \equiv \varphi_2 + \varphi_3 + \cdots + \varphi_{q+1} \qquad (mod\ F^{q+2}\mathcal{O}_N)\ ,$$

with φ_r an element in $\mathcal{G}^1_{q+1,r}$. Moreover, since the coboundary operator $d = \delta \otimes Id$ preserves the number of positively-graded generators, from (4.10) we draw the conclusion that each of the cochains φ_r appearing in the decomposition of $\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\}$ is also a cocycle in the corresponding cohomology complex $\mathcal{G}^1_{q+1,r}$.
Thus

$$\left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\} \in \bigoplus_{r=2}^{q+1} \mathrm{Ker}(\mathcal{G}^1_{q+1,r}, d) \qquad (mod\ F^{q+2}\mathcal{O}_N)\ .$$

Since by hypothesis we know that the cohomology groups defined by the cohomology complex $(\mathcal{G}^1_{q,r}, d)$ with $0 \leq r \leq q$ vanish in degree $j < q$, having fixed

$q \geqslant 1$, we deduce that $\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\}$ is not only a cocycle but also a coboundary.

Therefore, for any r, with $2 \leqslant r \leqslant q+1$, there exists an element ψ_r in $\mathcal{G}^0_{q+1,r}$ such that $(\delta \otimes Id)(\psi_r) = \varphi_r$. So the element

$$\widetilde{\nu} := -\frac{1}{2}(\psi_2 + \psi_3 + \cdots + \psi_{q+1}) \tag{4.11}$$

is an element in $\bigoplus_{r=2}^{q+1} \mathcal{G}^0_{q+1,r}$ such that

$$(\delta \otimes Id)(\widetilde{\nu}) \equiv -\tfrac{1}{2}\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\} \quad (mod\ F^{q+2}\mathcal{O}_N)\ .$$

So we define ν to be an element in $\Gamma(X_0, F^{q+2}\mathcal{O}^0_N \cap I_N^{\geqslant 2})$ such that $\pi_{q+1}(\nu) = \widetilde{\nu}$. This lift ν exists locally, since π_{q+1} is a surjective map, but also globally, since X_0 is an affine variety.

Thus defining

$$\widetilde{S}_{\leq q+1} := \widetilde{S}_{\leq q} + \nu$$

and using the induction hypothesis, we have

$$\{\widetilde{S}_{\leq q+1}, \widetilde{S}_{\leq q+1}\} \equiv \{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\} + 2(\delta \otimes Id)(\nu) \equiv 0 \quad (mod\ F^{q+2}\mathcal{O}_N)\ .$$

Therefore, $\widetilde{S}_{\leq q+1}$ is a solution for the classical master equation modulo $F^{q+2}\mathcal{O}_N$. To conclude the induction step, we still have to prove that $\widetilde{S}_{\leq q+1}$ is a solution to the classical master equation modulo $I_N^{\geqslant 2}$.

Let us recall that:

- $\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\} \equiv 0$ $(mod\ I_N^{\geqslant 2})$ by the inductive hypothesis;
- Since $\widetilde{S}_{\leq q} \equiv S_0 \bmod I_N^{\geqslant 1}$, then $\{\widetilde{S}_{\leq q}, \nu\} \equiv \{S_0, \nu\} \bmod I_N^{\geqslant 2}$.

  This implication follows from noticing that, if we would consider the Poisson bracket between ν and any other summand in $S_{\leq q}$ different from S_0, we would compute the Poisson bracket between an element that is at least bilinear in the positively-graded generators while the other is at least linear in this kind of generators. Therefore, the result of this computation would once again be a term at least bilinear in the positively-graded generators, i.e., it would belong to $I_N^{\geqslant 2}$.
  On the other hand, since $\nu \in I_N^{\geqslant 2}$, $\{S_0, \nu\} \subseteq I_N^{\geqslant 2}$, i.e.,

$$\{S_0, \nu\} \equiv 0 \quad (mod\ I_N^{\geqslant 2}) .$$

This is due to the fact that, when we consider the Poisson bracket on a pair in which one of the two elements is the initial action S_0, i.e., a function depending only on the variables $\{x_a\}$, $a = 1, \ldots, m$, the only possibility to find a summand that is non-zero occurs when a component ν_{ij} depends on the antifields x_a^*. On the other hand, the dependence of ν on the positively-graded generators remains unchanged. Therefore, if ν is an element that is at least quadratic in the positively-graded generator, the same can be ensured for $\{S_0, \nu\}$.

- The fact that by hypothesis $\nu \in I_N^{\geqslant 2}$, allows to conclude that also

$$\{\nu, \nu\} \in I_N^{\geqslant 3} \subseteq I_N^{\geqslant 2} .$$

Indeed, since the term on which we are computing the Poisson bracket is bilinear in the positively-graded generators by hypothesis, also after having computed the Poisson bracket in the previous expression there will always appear at least three positively-graded generators.

Thus we deduce that

$$\{\widetilde{S}_{\leq q+1}, \widetilde{S}_{\leq q+1}\} \equiv 0 \quad (mod\ I_N^{\geqslant 2}) .$$

This completes the induction step.

□

Remark 27
We would like to emphasize that the requirement that the cohomology complex $(\mathcal{G}_{q,r}, d)$ satisfies condition (4.7) is only a sufficient condition to guarantee the existence of an extended action. Indeed, this hypothesis has been used in (4.11) to ensure the possibility of defining the element $\widetilde{\nu}$. However, in order to be able to define $\widetilde{\nu}$ it is not necessary to have that every cocycle of degree j in the cohomology complex $(\mathcal{G}_{q,r}, d)$, with $j \leqslant q$, is also a coboundary element but it is enough to have that this property holds for the cocycle

$$\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\} \in \bigoplus_{r=2}^{q+1} \mathrm{Ker}(\mathcal{G}^1_{q+1,r}, d) \qquad (mod\ F^{q+2}\mathcal{O}_N) ,$$

for any $q \geqslant 1$.

Remark 28
Even though the proof explained above has many similarities with the one already presented in [28], it also has a major difference. Indeed, while we restrict to consider a selection of the generators determined by the Tate resolution, that is, the family of generators of type β, they construct a BV variety associated to the gauge theory (X_0, S_0) using all the generators given by the Tate resolution (A, δ), which may not be finitely generated (as kindly pointed out to the author by Felder). This choice has several consequences:

- Considering all the generators determined by the resolution (A, δ), no further hypothesis on the cohomology complex $(\mathcal{G}_{q,r}, d)$ has to be assumed to guarantee the existence of the extended action $\widetilde{S}$. Indeed, the vanishing of the cohomology groups $H^j(\mathcal{G}_{q,r}, d)$ for $j \leqslant q$ is a direct consequence of the Tate resolution defining an acyclic complex.

- The BV variety associated to an initial gauge theory (X_0, S_0) and constructed using all the (potentially infinite) generators of the Tate resolution is uniquely determined, up to gauge equivalences, by the initial gauge theory (X_0, S_0). This statement does not hold anymore in such a general form in the case in which we restrict to the generators of type β, as will be discussed in more detail in Section 4.3

On the other hand, when applicable, the construction presented in the above proof has a positive aspect, consisting in the possibility of completely determining the extended action $\widetilde{S}$, instead of having only an approximate solution. This is a consequence of the fact that, if we consider a minimal Tate resolution, that is, a resolution in which only the necessary generators have been introduced, then the family of type β generators may be a finite family so that the algorithm for defining the extended action $\widetilde{S}$ may end after a finite number of steps. Moreover, to determine these generators of type β it is not necessary to determine the complete Tate resolution of $J(S_0)$, which may be a tough computation, but we can concentrate on the generators β_i^*, characterized by the fact that

$$\delta(\beta_i^{*,(-q)}) = \sum_a f_{a,i}\beta_a^{*,(-q+1)}, \quad f_{a,i} \in \mathcal{O}_{X_0}, \qquad \text{with } \sum_a f_{a,i}\delta(\beta_a^{*,(-q+1)}) = 0$$

and inductively defined starting with the generators $\{x_i^*\} \subseteq T_{X_0}[1]$.

4.2.1 Summarizing the BV algorithm

In the previous section we have proved, in a constructive way, the existence of an action $\widetilde{S}$ that can be seen as an extension of the initial action S_0 and that is solution of the classical master equation on the extended configuration space $\widetilde{X}$. Since in the following sections we will apply all these techniques to a particular example in order to determine its minimal extended variety, we here briefly summarize the steps of the algorithm.

Let (X_0, S_0) be a pair consisting of a configuration space X_0 given by an affine variety and of a function S_0 that is regular on X_0 and that solves the classical master equation. Let (A, δ) be a Tate resolution of the Jacobian ring $J(S_0)$, defined by the initial action S_0, such that

$$A = T_{X_0}[1] \oplus \mathcal{E}_T^*[1] \ ,$$

for some $\mathbb{Z}_{>0}$-graded $\mathcal{O}_{X_0}$-module $\mathcal{E}_T$. Finally, let $\widetilde{X}$ be the extended configuration space defined by considering the generators of type β in the Tate resolution (A, δ), defined as

$$\widetilde{X} = \mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus X_0 \oplus \mathcal{E} \ .$$

The algorithm to construct an action $\widetilde{S}$ that is a regular function on $\widetilde{X}$ and solves the classical master equation on $\mathcal{O}_N$, with

$$N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1] \oplus \mathcal{E})) \ ,$$

is given by the following steps:

1. Using the Tate resolution, define the approximation of the extended action $\widetilde{S}$ which is linear in the positively-graded generators as follows:

   $$\widetilde{S}_{\leq 1} = S_0 + \sum_i \delta(C_i^*) C_i + \sum_{j \in J} \delta(\beta_j^*) \beta_j \ ,$$

   where C_i is a basis of $[\mathcal{E}]^1$ and C_i^* is its dual, while $\{\beta_j^*\}$ are the generators of the graded module $\mathcal{E}^*[1]$, with $\{\beta_j\}$ their dual generators in $\mathcal{E}$.

2. Compute the bracket $\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\}$.

3. If $\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\} = 0$, then $\widetilde{S}_{\leq 1}$ is a solution of the classical master equation on $\mathcal{O}_N$ and the algorithm stops; if $\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\}$ is not zero, then we continue with the algorithm.

4. To obtain the approximation of the action of degree 2 in the positively-graded generators, introduce a generic element ν, with

$$\nu \in \Gamma(X_0, I_N^{\geqslant 2} \cap F^2\mathcal{O}_N) \ .$$

More explicitly, ν is a generic regular function in $\mathcal{O}_N$ with total degree 0, at least quadratic and of degree 2 in the fields.

5. Determine ν by imposing the following condition:

$$2(\delta \otimes Id)(\nu) + \left\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\right\} \equiv 0 \qquad (mod\ F^3\mathcal{O}_N) \ .$$

6. Define $\widetilde{S}_2 = \widetilde{S}_{\leq 1} + \nu$.
If $\left\{\widetilde{S}_2, \widetilde{S}_2\right\} = 0$, then the algorithm finishes and we define $\widetilde{S} := \widetilde{S}_2$; otherwise, we continue with the algorithm in order to construct an approximation of the action $\widetilde{S}$ up to degree 3 in the fields, et cetera.

The generic step

- ▶ To construct an approximation of the action $\widetilde{S}$ up to degree $q+1$ in the fields, consider the approximation $\widetilde{S}_{\leqslant q}$, obtained in the previous step of the algorithm.

- ▶ Introduce a generic element ν with

$$\nu \in \Gamma(X_0, I_N^{\geqslant 2} \cap F^{q+1}\mathcal{O}_N) \ .$$

That is to say, ν is a generic regular function in $\mathcal{O}_N$ that is of total degree 0 and that is at least bilinear and of degree $q+1$ in the fields.

- ▶ Determine ν by imposing the following condition:

$$2(\delta \otimes Id)(\nu) + \left\{\widetilde{S}_{\leq q}, \widetilde{S}_{\leq q}\right\} \equiv 0 \quad \mathrm{mod}\ F^{q+2}\mathcal{O}_N \ .$$

- ▶ Define $\widetilde{S}_{\leq q+1} = \widetilde{S}_{\leq q} + \nu$.

- ▶ Compute $\left\{\widetilde{S}_{q+1}, \widetilde{S}_{q+1}\right\}$.

If this quantity is zero, then the approximate action $\widetilde{S}_{q+1}$ is an exact solution to the classical master equation.
If $\left\{\widetilde{S}_{\leq q+1}, \widetilde{S}_{\leq q+1}\right\} \neq 0$, repeat the previous step for degree $q+2$.

To summarize, in this section we discussed a method to construct an extended theory $(\widetilde{X}, \widetilde{S})$ for a given gauge theory (X_0, S_0). More precisely we have seen that, to start with this procedure, we first have to fix a suitable Tate resolution (A, δ) for the Jacobian ring $J(S_0)$ (see Remark 22). Then:

- The extended configuration space $\widetilde{X}$ is defined as

$$\widetilde{X} = \mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus X_0 \oplus \mathcal{E} \ ,$$

  where $\mathcal{E}$ is determined by considering the generators of type β in a $\mathbb{Z}_{>0}$-graded $\mathcal{O}_{X_0}$-module $\mathcal{E}_T$ with locally free and finitely generated homogeneous components, such that the Tate resolution can be rewritten as:

$$A = \mathcal{W}^*[1] = T_{X_0}[1] \oplus \mathcal{E}_T^*[1] \ ,$$

  for $T_{X_0}[1]$ the shifted tangent space of X_0.

- The extended action $\widetilde{S}$ is constructed by applying the algorithm explained above to a linear approximation $\widetilde{S}_{\leq 1}$, with

$$\widetilde{S}_{\leq 1} = S_0 + \sum_{i \in I} \delta(C_i^*) C_i + \sum_{j \in J} \delta(\beta_j^*) \beta_j \ ,$$

  where $\{C_i\}$ is a fixed basis for the homogeneous component of degree 1 of the graded module $\mathcal{E}$, while $\{C_i^*\}$ denotes the corresponding dual basis of $[\mathcal{E}^*[1]]^{-2}$. Moreover, $\{\beta_i^*\}$ denotes a collection of generators of type β, which forms a basis of the graded module $\mathcal{E}^*[1]$, with dual basis $\{\beta_i\}$, for the corresponding graded module $\mathcal{E}$.

Thus both the constructions of the extended configuration space $\widetilde{X}$ and of the extended action $\widetilde{S}$ are based on the fixed Tate resolution. However, apart from the first step of the Tate algorithm, which is determined by the field content of X_0 as already noticed in Remark 22, all the other steps of the Tate algorithm are free. In other words, even though the natural choice for the Tate resolution would be to take the minimal Tate resolution leading to $\widetilde{X}$ as the minimally extended configuration space, other choices are possible, such as fixing the number of ghost fields also in higher degrees. However, imposing other conditions on the Tate resolution might force us to introduce more ghost fields of higher degree than the ones strictly necessary.

4.3 Gauge equivalence of extended actions

In the previous sections, given an initial gauge theory (X_0, S_0), we have seen how to construct a corresponding extended theory $(\widetilde{X}, \widetilde{S})$. More precisely we started with:

- An initial gauge theory (X_0, S_0), where the initial configuration space is given by an affine variety;
- A Tate resolution (A, δ) of the Jacobian ring $J(S_0)$ on $\mathcal{O}_{X_0}$, that is, the ring of regular function on X_0. Moreover, the resolution is required to be such that:
$$A = \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{W})$$
for some graded $\mathcal{O}_{X_0}$-module $\mathcal{W} = \oplus_{j\leq -1}\mathcal{W}^j$, with the $\mathcal{W}^j$ locally free modules over $\mathcal{O}_{X_0}$ and $\mathcal{W}^{-1} = T_{X_0}$.

Using these data it is possible to construct an extended variety $(N, \widetilde{S})$ associated to the resolution (A, δ) by defining:

- $N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1] \oplus \mathcal{E}))$,
- $\widetilde{S}$ is the action obtained by applying the algorithm described in Section 4.2.

In this section we discuss when two extended actions $\widetilde{S}_1$, $\widetilde{S}_2$, both belonging to the same $\mathcal{O}_N$, can be considered equivalent. The first step that needs to be taken is to introduce a notion of *gauge equivalence* of actions.

Definition 45. *Let $N = (X_0, \mathcal{O}_N)$ be a (-1)-symplectic variety and let $\mathfrak{g}_N$ be the Lie algebra $\mathfrak{g}_N = \mathcal{O}_N^{-1} \cap I_N^{\geqslant 2}$, with $I_N^{\geqslant 2}$ the 2-power of the ideal I_N generated by elements of positive degree. Then the group of Poisson automorphisms*

$$G(N) = exp(ad(\mathfrak{g}(N))) \ ,$$

with $\mathfrak{g}(N) = \Gamma(X_0, \mathfrak{g}_N)$, is called the group of gauge equivalences.

Theorem 7. *Let $(N, \widetilde{S})$ be a extended variety associated to a Tate resolution (A, δ), corresponding to a gauge theory (X_0, S_0). Suppose that the cohomology complex $(\mathcal{G}_{q,r}, d)$ has vanishing cohomology groups $H^j(\mathcal{G}_{q,r}, d)$ for $j \leqslant q$. Then, if $\widetilde{S}'$ is another regular function on N, $\widetilde{S}' \in \mathcal{O}_N$, such that:*

- $\widetilde{S}' \equiv \widetilde{S} \equiv S_0 + \sum_i \delta(C_i^*)C_i + \sum_{j\in J} \delta(\beta_j^*)\beta_j \quad (mod\ I_N^{\geqslant 2})$;

▶ $\widetilde{S}'$ *is a solution of the classical master equation on* N*,*

then there exists a gauge equivalence $g \in G(N)$ *such that:*

$$\widetilde{S}' = g \cdot \widetilde{S} \ .$$

Proof. Since, by hypothesis, $\widetilde{S}$ and $\widetilde{S}'$ coincide up to terms which are at least quadratic in the positively-graded generators, there exist $p \geqslant q \geqslant 2$ such that

$$\widetilde{S}' \equiv \widetilde{S} \mod (F^p\mathcal{O}_N \cap I_N^{\geqslant q}) + F^{p+1}\mathcal{O}_N \ . \tag{4.12}$$

To prove the statement of the theorem, we show that, for any q, it is possible to define a gauge equivalence, that is, an element $g \in \mathcal{O}_N^{-1} \cap I_N^{\geqslant 2}$ such that

$$g \cdot \widetilde{S}' - \widetilde{S} \in \Gamma(X_0, (F^p\mathcal{O}_N \cap I_N^{\geqslant q+1}) + F^{p+1}\mathcal{O}_N) \ .$$

Then, by taking $q = p$, since it holds that $F^p\mathcal{O}_N \subseteq I_N^{\geqslant p}$, we would be able to conclude that, if there exists a $p \geqslant 2$ such that

$$\widetilde{S}' \equiv \widetilde{S} \mod F^p\mathcal{O}_N \ ,$$

then there exists a gauge equivalence $g \in \mathcal{O}_N^{-1} \cap I_N^{\geqslant 2}$ such that

$$g \cdot \widetilde{S}' - \widetilde{S} \in \Gamma(X_0, F^{p+1}\mathcal{O}_N) \ ,$$

which would conclude the prove of the theorem.
So let us suppose that there exist $p \geqslant q \geqslant 2$ such that $\widetilde{S}'$ and $\widetilde{S}$ satisfy condition (4.12). Equivalently, we are assuming that

$$\widetilde{S}' - \widetilde{S} = \nu \mod F^{p+1}\mathcal{O}_N \ , \qquad \text{with} \quad \nu \in \bigoplus_{r=q}^{p} \Gamma(X_0, \mathcal{G}^0_{p,r}) \ .$$

Since both $\widetilde{S}'$ and $\widetilde{S}$ are solution of the classical master equation, it holds that:

$$0 = \left\{\widetilde{S}' + \widetilde{S},\ \widetilde{S}' - \widetilde{S}\right\} = \left\{\widetilde{S}' + \widetilde{S}, \nu\right\} = 2(\delta \otimes Id)\nu \mod F^{p+1}\mathcal{O}_N.$$

Thus ν is a sum of cocycle in the cohomology complex $\mathcal{C}^0(\mathcal{G}_{p,r}, \delta \otimes Id)$, with $r = q, \dots, p$. By hypothesis, since $p \geqslant 2$, all the cohomology groups $\mathcal{H}^0(\mathcal{G}_{p,r}, \delta \otimes Id)$ vanish, for any $r = q, \dots, p$. Thus there exist elements

$$u_r \in \Gamma(X_0, \mathcal{G}^{-1}_{p,r}), \quad \text{such that} \ \ (\delta \otimes Id)(u_q + \dots + u_p) = \nu \ . \tag{4.13}$$

Therefore, if we denote $u = u_q + \cdots + u_p$, then we have that

$$\nu \equiv \{\widetilde{S}', u\} \mod F^{p+1}\mathcal{O}_N .$$

Since u has odd degree, it holds that $\{u, \widetilde{S}'\} = -\{\widetilde{S}', u\}$. Hence:

$$\nu + \{u, \widetilde{S}'\} \in \Gamma(X_0, F^{p+1}\mathcal{O}_N) .$$

Let $g = exp(ad_u)$. Then

$$\begin{aligned} g \cdot \widetilde{S}' - \widetilde{S} &= g \cdot \widetilde{S}' - \widetilde{S}' + \nu = \nu + \{u, \widetilde{S}'\} + \tfrac{1}{2}\{u, \{u, \widetilde{S}'\}\} + \dots \\ &\equiv \tfrac{1}{2}\{u, \{u, \widetilde{S}'\}\} + \dots \qquad \mod F^{p+1}\mathcal{O}_N . \end{aligned}$$

To conclude the proof we have to show that $g \cdot \widetilde{S}' - \widetilde{S}$ belongs also to $I_N^{\geqslant q+1}$. However, since $\{u, \widetilde{S}'\} \in I^{\geqslant q}$ by construction as well as $u \in I_N^{\geqslant 2}$, it immediately follows that

$$\{u, \{u, \widetilde{S}'\}\} \in I_N^{\geqslant q+1} .$$

With a similar argument we deduce that all the other terms in the last sum of the previous equation are elements in $I_N^{\geqslant q+1}$. Therefore,

$$g \cdot \widetilde{S}' - \widetilde{S} \in \Gamma(X_0, F^p\mathcal{O}_N \cap I_N^{\geqslant q+1}) ,$$

as required. □

Remark 29

Once again, the requirement that the cohomology groups $H^j(\mathcal{G}_{q,r}, d)$ vanish for $j \leqslant q$ is a sufficient but not necessary condition to conclude that two regular functions $\widetilde{S}$, $\widetilde{S}'$ in $\mathcal{O}_N$, both solution of the classical master equation are gauge equivalent. Indeed the vanishing of these cohomology groups is used only in (4.13) to ensure the possibility of defining the elements u_r. However, to draw this conclusion it is not necessary to have that each cocycle in the cohomology complex $(\mathcal{G}_{q,r}, d)$ is also a coboundary element: indeed it is enough that this property holds for elements of the type $\widetilde{S}' - \widetilde{S}$ (mod $F^{p+1}\mathcal{O}_N$), for any $p \geqslant 2$, with $\widetilde{S}$, $\widetilde{S}'$ as above.

To conclude, in this chapter we have presented a method to associate to an initial gauge theory (X_0, S_0) an extended theory $(\widetilde{X}, \widetilde{S})$. We have also seen that a fundamental role in the construction of such an extended theory is played by the choice of a suitable Tate resolution of the Jacobian ring $J(S_0)$ on $\mathcal{O}_{X_0}$ or, more precisely, by the selection of the family of generators of type β in this resolution. Therefore, the extended variety $(N, \widetilde{S})$ as well as the induced BRST cohomology complex may depend on the initial choice of Tate resolution. However, the natural choice for the Tate resolution is the *minimal Tate resolution*, that is, the one obtained by introducing the minimal number of generators. Indeed, if the hope is that the BRST cohomology complex is able to detect some information about the initial gauge theory, it would be more natural to search this information in the cohomology complex determined by an extended variety constructed using only the generators that are required by the presence of the initial gauge symmetry, without adding any extra generator.

What presented in this chapter will play a fundamental role in the second part of this thesis, where this construction is applied to our model of interest.

Part II

Matrix Models, BV formalism and Noncommutative Geometry

Chapter 5

Extended varieties for a matrix model

The ultimate goal of this chapter is to explicitly describe the (gauge-fixed) BRST cohomology complex of a $U(2)$-gauge invariant matrix model (X_0, S_0), naturally induced by a finite spectral triple on the algebra $M_2(\mathbb{C})$ (see Section 2.3). Our analysis consists of the following steps:

- Section 5.1: by applying the construction explained in Chapter 4, we determine the minimally extended theory $(\widetilde{X}, \widetilde{S})$, corresponding to the initial gauge theory (X_0, S_0).
- Section 5.2: following the procedure explained in Chapter 3, we first construct the *classical BRST cohomology* for the model, then we apply to it the *gauge-fixing procedure* and, finally, we describe in detail the corresponding (gauge-fixed) BRST cohomology complex.
- Section 5.3: a generalized notion of *Lie algebra cohomology* is introduced in order to understand the relations between the BRST cohomology complex and the space of ghost fields, which emerges to be endowed with a Lie algebra structure, in each fixed ghost degree. This different approach allows to detect a *double complex structure*, which was not visible at the level of BRST cohomology but appears to be evident for the corresponding generalized Lie algebra cohomology.

We determine all the BRST cohomology groups for our model of interest, collecting explicit computations in Appendix D.

The motivations that led us to consider a matrix model of low dimension was the possibility of simplifying the computations involved in this analysis. Indeed, going from the case of a $U(2)$-gauge invariant matrix model to the more general setting of a $U(n)$-matrix model, with $n > 2$, is not only a formal step but it demands a clear understanding of the underlying gauge structure. Some first steps in this direction have already been done and will be presented in Chapter 6.

Even though all this analysis is performed on a simple example, this setting is surprisingly rich and it gives already interesting insights for the analysis of matrix models of higher order. Indeed, in the general setting of a $U(n)$-gauge invariant matrix model we expect to be able to perform a similar construction, arriving to a reformulation of the BRST-cohomology complex in terms of a generalized Lie algebra complex, where we expect to find a *multicomplex structure*.

For completeness we note that the matrix models had also been treated using orthogonal polynomials to directly compute the corresponding path integral (see [31]).

5.1 Matrix models and gauge invariance

The main goal of this section is to apply what has been explained in Chapter 4 for a generic gauge theory to the $U(2)$-gauge invariant matrix model induced by a finite-dimensional spectral triple on the algebra $M_2(\mathbb{C})$ (see Section 2.3), and to determine its minimal extended variety.

First of all, let us recall our matrix model for a generic degree $n \in \mathbb{N}$.

The configuration space $\mathbf{X_0}$

The configuration space X_0 (at fixed n) is defined as

$$X_0 = \{A \in M_n(\mathbb{C}) : A^* = A\}\,.$$

Note: this space has the structure of an affine variety, since it can be seen as the zero locus of the polynomials that impose the condition of being complex conjugate with respect to the diagonal on the components of the matrices.

More precisely, let us consider the isomorphism $M_n(\mathbb{C}) \cong \mathbb{A}^{n^2}_{\mathbb{C}}$, where $\mathbb{A}^{n^2}_{\mathbb{C}}$ is

a complex affine space of coordinates y_k, with $k = 1, \dots, n^2$. The correspondence between the coordinates on the affine space and the components x_{ij}, with $i, j = 1, \dots, n$, of a generic matrix in $M_n(\mathbb{C})$ is given by $x_{ij} = y_{(i-1)n+j}$.
Then X_0 can be described as follows as the zero locus of a finite number of polynomials:

$$X_0 \cong V(\{f_{ij}\}_{i,j=1}^{n}), \qquad f_{ij} \in Pol_{\mathbb{C}}(y_1, \dots, y_{n^2})$$

with

$$f_{ij} = \bar{x}_{ij} - \bar{x}_{ji} = \bar{y}_{(i-1)n+j} - \bar{y}_{(j-1)n+i} \ ,$$

where $\bar{x}_{ij}$ in turn denotes the complex conjugate of the variable x_{ij}. On the other hand, X_0 can equivalently be described as follows.
Let $\{\sigma_1, \cdots, \sigma_{n^2}\}$ be a $\mathbb{R}$-basis of X_0. Then:

$$X_0 \cong \mathbb{A}_{\mathbb{R}}^{n^2} \ ,$$

which is clearly a nonsingular algebraic variety.
Then:

- ► Since X_0 is a nonsingular affine variety, all constructions presented in Chapters 3, 4 can be applied. Moreover, a global system of coordinates can be fixed on X_0, which can equivalently be seen as the following real vector space:

  $$X_0 = \langle M_1, \dots, M_{n^2} \rangle \cong \mathbb{R}^{n^2},$$

  where M_a are the real variables corresponding to the coordinates on X_0.

- ► The ring of regular functions on X_0 is given by

  $$\mathcal{O}_{X_0} = Pol_{\mathbb{R}}(M_a) \ .$$

The action $\mathbf{S_0}$

Let a gauge group $\mathcal{G} = U(n)$ act on the space X_0 by conjugation:

$$A \to UAU^*, \qquad A \in X_0, \ U \in U(n) \ .$$

Thus an action S_0 on the space X_0 is a regular function on X_0, $S_0 : X_0 \to \mathbb{R}$ that is invariant under the action of the gauge group:

$$S_0(UAU^*) = S_0(A), \qquad \forall A \in X_0, \ \forall U \in U(n) \ . \tag{5.1}$$

Therefore, S_0 can be seen as a polynomial in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ that satisfies the equations describing the condition in (5.1) in terms of components of the matrices.

Let us denote the set of all regular functions that satisfy condition (5.1) by $Inv(X_0, U(n))$. Although an action $S_0 \in Inv(X_0, U(n))$ would be the most general possibility for a well-defined action on the space X_0, in the matrix model which we are considering, we will restrict ourselves to actions of a more specific form. The reason for this choice is that, since the matrix model was initially derived from a finite spectral triple (see Section 2.3), the action functional S_0 on the configuration space X_0 was taken to be the spectral action of the spectral triple from which we started. Thus the action for our model takes the following form:

$$S_0[A] = Tr(f(A)) \ , \qquad\qquad A \in X_0 \ , \tag{5.2}$$

where $f \in Pol_{\mathbb{R}}(x_i)$, with $i = 1, \ldots, n^2$, is a polynomial in n^2 real variables with real coefficients.

Note: from the defining properties of a trace, we deduce that a spectral action S_0 of the form (5.2) is invariant under the action of the gauge group.

The next step is to determine the corresponding Jacobian ring and, more precisely, to compute the number of its independent generators.

Recall that:

$$J(S_0) = \frac{\mathcal{O}_{X_0}}{\langle \partial_1 S_0, \cdots, \partial_m S_0 \rangle} \ .$$

We already know that, in our particular case, $m = n^2$ and that the ring of regular functions is simply given by $\mathcal{O}_{X_0} = Pol_{\mathbb{R}}(M_a)$, with $a = 1, \cdots, n^2$. Therefore, what we still need to analyze is the ideal M generated by the partial derivatives of the action S_0: more precisely, we need to compute the number of independent generators of M as a free module over $\mathcal{O}_{X_0}$.
To proceed with this computation we need to have a more explicit expression of the initial action S_0 in terms of the coordinates $x_1, \ldots, x_n$. Therefore, we analyze the consequence of the condition (5.1) of the action being gauge invariant, which forces the action S_0 to have a particular form, as precisely stated in the following proposition.

Proposition 9
Let $X_0 = \{A \in M_n(\mathbb{C}) : A^* = A\}$ with the gauge group $\mathcal{G} = U(n)$ acting on X_0

by conjugation. Then the invariant polynomials are:

$$Inv(X_0, \mathcal{G}) \simeq Sym(Pol_{\mathbb{R}}(\lambda_1, \cdots, \lambda_n))$$

where $Sym(Pol_{\mathbb{R}}(\lambda_1, \cdots, \lambda_n))$ denotes the set of all symmetric polynomials in n independent real variables $\lambda_1, \cdots, \lambda_n$, which represent the set of eigenvalues of a generic element in X_0.
Equivalently:

$$Inv(X_0, \mathcal{G}) = Pol_{\mathbb{R}}(a_1, \cdots, a_n)$$

with $a_1, \cdots, a_n$ the elementary polynomials corresponding to the n real variables $\lambda_1, \cdots, \lambda_n$.

Proof. Since each matrix $A \in X_0$ is diagonalizable with a unitary matrix and since the action S_0 is invariant under the adjoint action of $U(n)$, S_0 can depend only on the eigenvalues of the matrices in X_0, which we denote by n real variables $\lambda_1, \cdots, \lambda_n$. Then $Inv(X_0, \mathcal{G}) \subseteq Pol_{\mathbb{R}}(\lambda_1, \cdots, \lambda_n)$.

The collection of eigenvalues of a matrix is well defined only up to the order so S_0 is well defined only if it is a symmetric polynomial. Therefore, the set of regular functions that are invariant under the action of the gauge group is precisely given by the set of symmetric polynomials in the eigenvalues of a matrix in the base space X_0. More precisely:

$$Inv(X_0, \mathcal{G}) = \mathrm{Sym}(Pol_{\mathbb{R}}(\lambda_1, \cdots, \lambda_n)).$$

From the theory of symmetric polynomials, each symmetric polynomial can be seen as a polynomial expansion in the elementary polynomials $a_1, \cdots, a_n$ where:

$$\begin{aligned} a_1 &= \lambda_1 + \cdots + \lambda_n \\ a_2 &= \textstyle\sum_{i<j} \lambda_i \lambda_j \\ &\vdots \\ a_n &= \lambda_1 \cdot \lambda_2 \cdots \lambda_n. \end{aligned}$$

Then we conclude that $Inv(X_0, \mathcal{G}) = Pol_{\mathbb{R}}(a_1, \cdots, a_n)$. □

5.1.1 A matrix model of degree 2

In the following we apply our formalism to the particular case of a matrix model of degree $n = 2$.

The action $\mathbf{S}_0$

Let X_0 be the set of all 2×2 matrices A such that $A^* = A$ and let us consider as real basis for X_0 the one composed by the *Pauli matrices* $\{\sigma_a\}_{a=1,\ldots,4}$, given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{5.3}$$

Let M_a, for $a = 1, \cdots, 4$ be the variables that describe the components of a matrix over the basis, i.e. $\{M_a\}_{a=1,\cdots,4}$ is the dual basis corresponding to $\{\sigma_a\}_{a=1,\ldots,4}$. Then the variables λ_1, λ_2 that denote the two eigenvalues of a matrix $A \in X_0$ can be written as:

$$\lambda_i = M_4 \pm \sqrt{M_1^2 + M_2^2 + M_3^2} .$$

The elementary polynomials in this case are:

$$\begin{aligned} a_1 &= \lambda_1 + \lambda_2 = 2M_4 ; \\ a_2 &= \lambda_1 \cdot \lambda_2 = M_4^2 - (M_1^2 + M_2^2 + M_3^2) . \end{aligned}$$

Thus for $n = 2$ the generic form for an action S_0 invariant under the action of the gauge group $\mathcal{G} = U(2)$ is:

$$S_0 = \sum_{k=0}^{r} (M_1^2 + M_2^2 + M_3^2)^k \; g_k(M_4) \tag{5.4}$$

with $r \in \mathbb{N}$ and $g_k(M_4) \in Pol_{\mathbb{R}}(M_4)$.

Remark 30
In Equation (5.4) we have determined the form of the most general $U(2)$-gauge invariant action that we can consider on the given configuration space. Even though often the action considered for this kind of matrix model is of the form (5.2), already at this level of generality there are linear relations over the ring $Pol_{\mathbb{R}}(M_a)$ between the partial derivatives of a generic element S_0 in

$Inv(X_0, U(2))$ with respect to the variables M_a on X_0, with $a = 1, \ldots, 4$. More precisely, we note that the following relations hold:

$$\begin{cases} M_1(\partial_{M_2} S_0) = M_2(\partial_{M_1} S_0) \\ M_1(\partial_{M_3} S_0) = M_3(\partial_{M_1} S_0) \\ M_2(\partial_{M_3} S_0) = M_3(\partial_{M_2} S_0) \ . \end{cases} \tag{5.5}$$

Since a spectral action S_0 is an element in $Inv(X_0, U(2))$, the relations (5.5) will also hold.

Anyhow, we can also find a more explicit description of a generic action of the form $S_0[A] = Tr(f(A))$. This explicit formula for S_0 will be useful in the computation of the Jacobian ring.

Let us first introduce some notation. Let f be an element in $Pol_{\mathbb{R}}(\xi)$,

$$f = \sum_{i=0}^{r} \mu_i \xi^i, \qquad \text{with } \mu_i \in \mathbb{R} \ . \tag{5.6}$$

Then, given a matrix $A \in X_0$, the spectral action S_0 has the following form:

$$S_0[A] = Tr(f(A)) = Tr\left(\sum_{i=0}^{r} \mu_i A^i\right) = \sum_{i=0}^{r} \mu_i Tr(A^i) \ .$$

We are interested in finding a recursive formula to determine the trace of powers of a matrix $A \in M_0$.
Write:

$$A^i = x_i\sigma_1 + y_i\sigma_2 + z_i\sigma_3 + w_i\sigma_4 \ ,$$

where x_i, y_i, z_i, w_i are real variables that describe the coefficients of the i-th power of the matrix A on the basis $\{\sigma_a\}_{a=1,\ldots,4}$. The following proposition will give a recursive formula to determine x_i, y_i, z_i, w_i as functions of the initial values x_1, y_1, z_1, w_1.

Proposition 10
An action S_0 on the space X_0 given by $S_0[A] = Tr(f(A))$ can be written as function of the variables x_1, y_1, z_1, w_1 as follows:

$$\begin{aligned} Tr(f(A)) &= 2\left[\sum_{a=0}^{\lfloor r/2 \rfloor} \mu_{2a} \left(\sum_{s=0}^{a} \binom{2a}{2s} (x_1^2 + y_1^2 + z_1^2)^{a-s} w_1^{2s}\right)\right. \\ &\quad \left. + \sum_{a=0}^{\lceil r/2 \rceil - 1} \mu_{2a+1} \left(\sum_{s=0}^{a} \binom{2a+1}{2s+1} (x_1^2 + y_1^2 + z_1^2)^{a-s} w_1^{2s+1}\right)\right] \ . \end{aligned} \tag{5.7}$$

Proof. Note that $Tr(A^i) = 2w_i$, since all other Pauli matrices have trace zero. We aim for an explicit expression for the variables w_i, seen as function of the initial values x_1, y_1, z_1, w_1.
In view of the relationships existing among products of Pauli matrices, we determine the following equalities:

$$\begin{cases} x_n = (x_1 w_{n-1} + w_1 x_{n-1}) + i(y_1 z_{n-1} - z_1 y_{n-1}) \\ y_n = (y_1 w_{n-1} + w_1 y_{n-1}) + i(z_1 x_{n-1} - x_1 z_{n-1}) \\ z_n = (z_1 w_{n-1} + w_1 z_{n-1}) + i(x_1 y_{n-1} - y_1 x_{n-1}) \\ w_n = x_1 x_{n-1} + y_1 y_{n-1} + z_1 z_{n-1} + w_1 w_{n-1} \end{cases} \tag{5.8}$$

for any $n \in \mathbb{N}_0$.
By induction on $n \in \mathbb{N}$, it can be proved that the purely imaginary summands appearing in (5.8) are zeros, i.e.,

$$x_1 y_{n-1} - y_1 x_{n-1} = x_1 z_{n-1} - z_1 x_{n-1} = y_1 z_{n-1} - z_1 y_{n-1} = 0 \ .$$

Once again by induction, it is possible to prove that there exist a real sequence $\{\alpha_n\}$ such that:

$$\begin{cases} x_n = x_1 \alpha_n \\ y_n = y_1 \alpha_n \\ z_n = z_1 \alpha_n \end{cases}$$

with

$$\alpha_1 = 1, \qquad \alpha_n = w_{n-1} + w_1 \alpha_{n-1} \ .$$

Completely determining the components of the matrix A^i over the basis B is equivalent to finding the solution for the following recursive formula:

$$\begin{bmatrix} \alpha_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} w_1 & 1 \\ T & w_1 \end{bmatrix} \begin{bmatrix} \alpha_n \\ w_n \end{bmatrix} = \begin{bmatrix} w_1 & 1 \\ T & w_1 \end{bmatrix}^n \begin{bmatrix} 1 \\ w_1 \end{bmatrix}$$

where we have introduced the notation $T = x_1^2 + y_1^2 + z_1^2$.
Equivalently:

$$\begin{cases} \alpha_{n+1} = H_n + w_1 K_n \\ w_{n+1} = T K_n + w_1 H_n \end{cases} \tag{5.9}$$

for two suitable functions H_n and K_n, satisfying the recursive relations:

$$\begin{cases} K_{n+1} = H_n + w_1 K_n \\ H_{n+1} = w_1 H_n + T K_n \ , \end{cases} \tag{5.10}$$

with $K_1 = 1$ and $H_1 = w_1$.
Therefore, to conclude we only have to solve the following recursion relation for the sequence of polynomials $\{H_n\} \subseteq Pol_{\mathbb{R}}(x_1, y_1, z_1, w_1)$:

$$\begin{cases} H_r = w_1 H_{r-1} + T\left(\sum_{s=0}^{r-4} {w_1}^s H_{r-s-2}\right) + 2T{w_1}^{r-2} \\ H_1 = w_1 \end{cases}$$

with $r \in \mathbb{N}$, $r \geqslant 2$.

By induction, it is possible to prove that the following two identities hold for all r in $\mathbb{N}$, with $r \geqslant 1$, depending on the parity of r:

$$\begin{cases} H_{2k} = \sum_{s=0}^{k} \binom{2k}{2s} T^{k-s} {w_1}^{2s} \\ H_{2k+1} = \sum_{s=0}^{k} \binom{2k+1}{2s+1} T^{k-s} {w_1}^{2s+1} \ . \end{cases} \tag{5.11}$$

A direct computation shows that the two formulas hold for $k = 1$. Then, suppose that they hold for k and let us check that, considering the case for r even, the formula holds for $r = 2(k+1)$ (the case for r odd can be proved in an analogous way). Using the recursive definition of H_r and the induction hypothesis, after some manipulations on the indices, we arrive at the following expression:

$$\begin{aligned} H_{2(k+1)} &= \sum_{a=1}^{k-1} \left[\binom{2k+1}{2a-1} + \sum_{p=0}^{2a} \binom{2(k-a)+p}{p}\right] T^{k+1-a} {w_1}^{2a} \\ &\quad + \binom{2(k+1)}{2k} T {w_1}^{2k} + w_1^{2(k+1)} + T^{k+1}. \end{aligned}$$

To conclude, it only remains to show that the following identity among binomial coefficients holds:

$$\binom{2k+1}{2a-1} + \sum_{p=0}^{2a} \binom{2(k-a)+p}{p} = \binom{2(k+1)}{2a}. \tag{5.12}$$

However, the formula in (5.12) follows immediately by using Tartaglia's identity together with the following formula, which holds $\forall j \in \mathbb{N}$, $j \geqslant 0$:

$$\sum_{p=0}^{b} \binom{j+p}{p} = \binom{j+b+1}{b}.$$

This allows us to conclude the induction step of the proof and state that the formulas (5.11) hold, for all even r in $\mathbb{N}$. As said, analogously the formula can be proved for r odd.

Thus from (5.10) and (5.11) we deduce that

$$\begin{cases} K_{2r} = \sum_{s=0}^{r-1} \binom{2r}{2s+1} T^{r-s-1} {w_1}^{2s+1} \\ K_{2r+1} = \sum_{s=0}^{r} \binom{2r+1}{2s} T^{r-s} {w_1}^{2s} \end{cases} \qquad \forall r \geqslant 0.$$

In view of (5.9), we conclude that for all r in $\mathbb{N}_0$,

$$\begin{cases} \alpha_{2r} = \sum_{s=0}^{r-1} \binom{2r}{2s+1} T^{r-s-1} {w_1}^{2s+1} \\ \alpha_{2r+1} = \sum_{s=0}^{r} \binom{2r+1}{2s} T^{r-s} {w_1}^{2s} \end{cases}$$

and

$$\begin{cases} w_{2r} = \sum_{s=0}^{r} \binom{2r}{2s} T^{r-s} {w_1}^{2s} \\ w_{2r+1} = \sum_{s=0}^{r} \binom{2r+1}{2s+1} T^{r-s} {w_1}^{2s+1} \end{cases} .$$

Having found an explicit expression of α_n and w_n for any value $n \geqslant 0$ we have determined all the components, over the fixed basis, of a generic n-th power of a matrix $A \in X$. In particular we recall that $Tr(A^n) = 2w_n$.

Therefore, given a polynomial f in $Pol_{\mathbb{R}}(\xi)$, with $f = \sum_{i=0}^{r} \mu_i \xi^i$ and $\mu_i \in \mathbb{R}$, the action $S_0[A] = Tr(f(A))$ can be written as:

$$\begin{aligned} Tr(f(A)) &= 2 \Big[\sum_{a=0}^{\lfloor r/2 \rfloor} \mu_{2a} \left(\sum_{s=0}^{a} \binom{2a}{2s} T^{a-s} w_1^{2s} \right) \\ &\quad + \sum_{a=0}^{\lceil r/2 \rceil -1} \mu_{2a+1} \left(\sum_{s=0}^{a} \binom{2a+1}{2s+1} T^{a-s} w_1^{2s+1} \right) \Big] . \end{aligned}$$

□

Remark 31
Using the fields $M_1, \ldots, M_4$, the expression in the previous proposition can be rewritten as follows:

$$\begin{aligned} S_0 = \quad & Tr(f(A)) = 2 \Big[\sum_{a=0}^{\lfloor r/2 \rfloor} \mu_{2a} \left(\sum_{s=0}^{a} \binom{2a}{2s} (M_1^2 + M_2^2 + M_3^2)^{a-s} M_4^{2s} \right) \\ & + \sum_{a=0}^{\lceil r/2 \rceil -1} \mu_{2a+1} \left(\sum_{s=0}^{a} \binom{2a+1}{2s+1} (M_1^2 + M_2^2 + M_3^2)^{a-s} M_4^{2s+1} \right) \Big] . \end{aligned} \tag{5.13}$$

This reconfirms that S_0 is in particular a element in $Inv(X_0, U(2))$, which has been explicitly determined in Equation (5.4).

This computation of the explicit form of the action for our $U(2)$-matrix model is used in the following section to determine the minimal Tate resolution for the Jacobian ring $J(S_0)$ and then the minimal extended variety associated to the model.

5.1.2 Minimal extended variety for a U(2)-matrix model

The aim of this section is to apply the algorithm described in Chapter 4 to the example of a matrix model of degree $n = 2$, in order to determine the corresponding extended theory $(\widetilde{X}, \widetilde{S})$: first, we explicitly compute the generators of type β of the minimal Tate resolution for the Jacobian ring, which are used to construct the extended configuration space $\widetilde{X}$, obtained from the initial configuration space X_0 by introducing the minimal number of ghost fields. Then we apply the algorithm explained in Section 4.2 to determine the most general solution of the classical master equation on $\widetilde{X}$, under the conditions of being linear in the antifields, being of at most degree 2 in the ghost fields, and having as underlying ring $\mathcal{O}_{X_0} = \mathrm{Pol}(M_a)$, ring of regular functions on X_0. Therefore, by the end of this section we determine the minimal extended variety associated to the initial gauge theory.

Recall that the gauge invariant action S_0 takes the following form:

$$S_0 = \sum_{k=0}^{r} \; (M_1^2 + M_2^2 + M_3^2)^k \; g_k(M_4) \; ,$$

with $g_k(M_4)$ a polynomial in $\mathrm{Pol}_{\mathbb{R}}(M_4)$ for all k in $\mathbb{N}$.

Let us start by determining the collection of generators of type β in the minimal Tate resolution for the Jacobian ring $J(S_0)$, which is suitable for the construction of a corresponding extended variety. We apply the algorithm explained in Appendix B step by step to the model. As already noticed in Chapter 4, since our goal is to analyze the BRST cohomology complex for our matrix model, we are not interested in determining a complete Tate resolution for $J(S_0)$ but we are going to determine only the generators on which the extended action will depend, that is, the generators of type β denoted in Section 4.2 by $\{\beta_j^*\}$.

Step 0

We define A^0 to be the ring $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Then we have the following exact sequence of finitely-generated $\mathrm{Pol}_{\mathbb{R}}(M_a)$-modules:

$$\mathrm{Pol}_{\mathbb{R}}(M_a) \xrightarrow{\pi} J(S_0) = \frac{\mathrm{Pol}_{\mathbb{R}}(M_a)}{\langle \partial_1 S_0, \partial_2 S_0, \partial_3 S_0, \partial_4 S_0 \rangle} \to 0 \tag{5.14}$$

where the map π is the projection on the quotient.

Step −1

In this step we introduce antisymmetric variables of degree -1 to define A^{-1}. As already noticed in Remark 22, to construct a Tate resolution suitable to define an extended variety with support (X_0, S_0), we are forced to require that the resolution is a graded algebra $\mathcal{W} = \oplus_{j \leq -1} \mathcal{W}^j$ such that $\mathcal{W}^{-1} = T_{X_0}$.

So we define the algebra A^{-1} as the extension of A^0 by the adjunction of variables of degree -1 such that they correspond to $\partial_1 S_0, \ldots, \partial_4 S_0$. Since these variables will describe the antifields M_a^*, corresponding to the fields M_a, with $a = 1, \ldots, 4$, for simplicity we already use this notation.
So we have

$$A^{-1} = \mathrm{Pol}(M_a)\langle M_1^*, \ldots, M_4^* \rangle, \qquad \text{with } \delta_{-1}^{-1}(M_j^*) = \partial_j(S_0) \ . \tag{5.15}$$

Then we extend the exact sequence above as follows:

$$\mathrm{Pol}(M_a)\langle M_1^*, \ldots, M_4^* \rangle \xrightarrow{\delta_{-1}^{-1}} \mathrm{Pol}(M_a) \xrightarrow{\pi} J(S_0) \to 0 \ .$$

Step −2

In this step we introduce symmetric variables of degree -2. To determine the minimal number of variables that need to be introduced, we have to compute the cohomology group $H^{-1}(A^{-1})$. By definition:

$$H^{-1}(A^{-1}) = \frac{\mathrm{Ker}(\delta_{-1}^{-1})}{\mathrm{Im}(\delta_{-2}^{-1})} \ .$$

Let us start computing $\mathrm{Ker}(\delta_{-1}^{-1})$:

$$\begin{aligned} &\delta_{-1}^{-1}(f_1 M_1^* + f_2 M_2^* + f_3 M_3^* + f_4 M_4^*) \\ &\qquad = f_1 \cdot \partial_1 S_0 + f_2 \cdot \partial_2 S_0 + f_3 \cdot \partial_3 S_0 + f_4 \cdot \partial_4 S_0 = 0 \ , \end{aligned} \tag{5.16}$$

with $f_1, \ldots, f_4 \in \mathrm{Pol}(M_a)$.
In the previous expression we used the fact that the Tate coboundary operator δ is linear and homogeneous with respect to elements of the base ring $\mathrm{Pol}_{\mathbb{R}}(M_a)$.
To solve Equation (5.16), we use the explicit form of the action S_0 given in (5.13) and note that, if we write

$$\varphi = \sum_{k=1}^{r} 2k(M_1^2 + M_2^2 + M_3^2)^{k-1} \; g_k(M_4) \; , \tag{5.17}$$

where g_k are polynomials in M_4, then we have

$$\partial_1 S_0 = M_1 \varphi, \qquad \partial_2 S_0 = M_2 \varphi, \qquad \partial_3 S_0 = M_3 \varphi \; .$$

Let A and B be coprime polynomials in $\mathrm{Pol}(M_a)$ such that:

$$\varphi = AD, \qquad \partial_4(S_0) = BD,$$

with $D := GCD(\varphi, \partial_4(S_0))$. Thus the condition which appears in Equation (5.16) can be rewritten as follows:

$$f_1 M_1 A + f_2 M_2 A + f_3 M_3 A + f_4 B = 0.$$

Since A and B are coprime, there are two possibilities for the coefficients f_1, f_2, f_3, f_4 to solve the previous equation:

$$1. \begin{cases} M_1 f_1 + M_2 f_2 + M_3 f_3 = 0 \\ f_4 = 0 \end{cases} \qquad 2. \begin{cases} M_1 f_1 + M_2 f_2 + M_3 f_3 = -BQ \\ f_4 = AQ \end{cases}$$

with $Q \in Pol_{\mathbb{R}}(M_a)$, $Q \neq 0$.

For the first possibility, the most general solution is given by

$$\begin{cases} f_1 = +M_2 P + M_3 R \\ f_2 = -M_1 P + M_3 S \\ f_3 = -M_1 R - M_2 S \end{cases}$$

with $P, R, S \in Pol_{\mathbb{R}}(M_a)$. Therefore, as independent generators of this family of solutions over $Pol_{\mathbb{R}}(M_a)$ we consider the following elements $\beta_1, \beta_2, \beta_3 \in A^{-1}$:

$$\begin{cases} \beta_1 = +M_3 M_2^* - M_2 M_3^* \\ \beta_2 = -M_3 M_1^* + M_1 M_3^* \\ \beta_3 = +M_2 M_1^* - M_1 M_2^* \; . \end{cases} \tag{5.18}$$

For the second possibility we obtain the following independent generators for the space $\mathrm{Ker}(\delta_{-1}^{-1})$ seen as a module on $\mathrm{Pol}(M_a)$:

$$\begin{cases} \beta_4 = BM_1^* - M_1AM_4^* \\ \beta_5 = BM_2^* - M_2AM_4^* \\ \beta_6 = BM_3^* - M_3AM_4^* \ . \end{cases} \tag{5.19}$$

Note that the generators β_1, β_2 and β_3 depend only on the antifields M_1^*, M_2^* and M_3^* while the other three involve also the antifield M_4^*.

Next we consider the space $\mathrm{Im}(\delta_{-2}^{-1})$. Let ψ be a generic element in A_{-2}^{-1}, say of the form:

$$\psi = f_{12}M_1^*M_2^* + f_{13}M_1^*M_3^* + f_{14}M_1^*M_4^* + f_{23}M_2^*M_3^* + f_{24}M_2^*M_4^* + f_{34}M_3^*M_4^*$$

with $f_{12}, \ldots, f_{34} \in Pol_{\mathbb{R}}(M_a)$. Thus:

$$\begin{aligned} &\delta_{-2}^{-1}(\psi) \\ &\quad = M_1^*D(-f_{12}M_2A - f_{13}M_3A - f_{14}B) + M_2^*D(f_{12}M_1A - f_{23}M_3A - f_{24}B) \\ &\qquad + M_3^*D(f_{13}M_1A + f_{23}M_2A - f_{34}B) + M_4^*AD(f_{14}M_1 + f_{24}M_2 + f_{34}M_3). \end{aligned}$$

Therefore, as independent generators for the space $\mathrm{Im}(\delta_{-2}^{-1})$ seen as a module on $\mathrm{Pol}(M_a)$, we consider:

$$\begin{cases} \gamma_1 = (M_3AD)M_2^* - (M_2AD)M_3^* \\ \gamma_2 = (M_1AD)M_3^* - (M_3AD)M_1^* \\ \gamma_3 = (M_2AD)M_1^* - (M_1AD)M_2^* \\ \gamma_4 = (BD)M_1^* - (M_1AD)M_4^* \\ \gamma_5 = (BD)M_2^* - (M_2AD)M_4^* \\ \gamma_6 = (BD)M_3^* - (M_3AD)M_4^* \ . \end{cases} \tag{5.20}$$

Recalling the expression found in (5.18) and (5.19) for the generators of $\mathrm{Ker}(\delta_{-1}^{-1})$, we have that:

$$\begin{cases} \gamma_1 = AD\beta_1 \\ \gamma_2 = AD\beta_2 \\ \gamma_3 = AD\beta_3 \end{cases} \qquad \begin{cases} \gamma_4 = D\beta_4 \\ \gamma_5 = D\beta_5 \\ \gamma_6 = D\beta_6 \ . \end{cases} \tag{5.21}$$

To conclude:

$$H^{-1}(A) = \frac{\mathrm{Ker}(\delta_{-1}^{-1})}{\mathrm{Im}(\delta_{-2}^{-1})} = \frac{\langle \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \rangle}{\langle AD\beta_1, AD\beta_2, AD\beta_3, D\beta_4, D\beta_5, D\beta_6 \rangle} \ . \tag{5.22}$$

At this point, depending on the explicit form of the action and on the explicit form of the polynomials φ and $\partial_4 S_0$, there are two possibilities which will give two different Tate resolutions for the Jacobian ring. The first possibility is that the polynomials φ and $\partial_4 S_0$ are coprime and so $D = 1$, while the second is that $D \notin \mathbb{R}$. We are going to analyze these two cases separately.

φ and $\partial_4 S_0$ are coprime

Under the hypothesis that φ and $\partial_4 S_0$ are coprime as polynomials in $Pol_{\mathbb{R}}(M_a)$, the cohomology group $H^{-1}(A)$ is simply given by

$$H^{-1}(A) = \frac{\mathrm{Ker}(\delta_{-1}^{-1})}{\mathrm{Im}(\delta_{-2}^{-1})} = \frac{\langle \beta_1, \beta_2, \beta_3 \rangle}{\langle A\beta_1, A\beta_2, A\beta_3 \rangle} \ ,$$

since in this case the generators $\beta_4, \beta_5, \beta_6$ of $\mathrm{Ker}(\delta_{-1}^{-1})$ coincide with the generators $\gamma_4, \gamma_5, \gamma_6$ of $\mathrm{Im}(\delta_{-2}^{-1})$. Thus for the algorithm we need to introduce three symmetric variables with degree -2: since these variables will describe the antifields C_a^*, corresponding to the ghost fields C_a, with $a = 1, 2, 3$, for simplicity we start using this notation already now. So we define the algebra A^{-2} as the extension of A^{-1} by the adjunction of variables C_1^*, C_2^*, C_3^* of degree -2 corresponding to $\beta_1, \beta_2, \beta_3$. More explicitly:

$$A^{-2} = \mathrm{Pol}(M_a)\langle M_1^*, M_2^*, M_3^*, M_4^*, C_1^*, C_2^*, C_3^* \rangle \ , \tag{5.23}$$

with

$$\begin{cases} \delta_{-2}^{-2}(C_1^*) = M_3 M_2^* - M_2 M_3^* \\ \delta_{-2}^{-2}(C_2^*) = M_1 M_3^* - M_3 M_1^* \\ \delta_{-2}^{-2}(C_3^*) = M_2 M_1^* - M_1 M_2^* \ . \end{cases}$$

Step -3 (coprime case)

In this step we introduce antisymmetric variables of degree -3. To determine the minimum number of variables of this degree we have to introduce we need to compute the following cohomology group:

$$H^{-2}(A^{-2}) = \frac{\mathrm{Ker}(\delta_{-2}^{-2})}{\mathrm{Im}(\delta_{-3}^{-2})} \ .$$

As we explained in Chapter 4, since we are interested in analyzing the BRST cohomology groups of our model, it is not necessary to determine a full Tate resolution for the Jacobian ring $J(S_0)$ but we can restrict ourselves to determine the elements in $\mathrm{Ker}(\delta_{-2}^{-2})$ given by generators depending only on variables of degree -2 introduced in the previous step: more precisely, instead of computing the cohomology group $H^{-2}(A^{-2})$, it is enough to finding out the relations of linear dependence between the elements $\delta(C_i^*)$ over the ring $Pol_{\mathbb{R}}(M_a)$. Let us consider an element $\xi \in A_{-2}^{-2}$ of the following form:

$$\xi = f_1 C_1^* + f_2 C_2^* + f_3 C_3^* \ ,$$

with $f_1, f_2, f_3 \in Pol_{\mathbb{R}}(M_a)$.
Then:

$$\delta_{-2}^{-2}(\xi) = M_1^*(-f_2 M_3 + f_3 M_2) + M_2^*(f_1 M_3 - f_3 M_1) + M_3^*(f_2 M_1 - f_1 M_2) \ .$$

We want to determine the polynomials f_1, f_2, f_3 for which $d_{-2}^{-2}(\xi)$ vanishes. Since the variables M_1^*, M_2^*, M_3^* are independent, for the previous expression to vanish the only possibility is that all coefficients are zero:

$$\delta_{-2}^{-2}(\xi) = 0 \quad \Leftrightarrow \quad \begin{cases} f_3 M_2 - f_2 M_3 = 0 \\ f_1 M_3 - f_3 M_1 = 0 \\ f_2 M_1 - f_1 M_2 = 0 \ . \end{cases}$$

Thus as generator of $\mathrm{Ker}(\delta_{-2}^{-2})$ as a module on the ring $\mathrm{Pol}_{\mathbb{R}}(M_a)$ we consider the element

$$\xi = M_1 C_1^* + M_2 C_2^* + M_3 C_3^* \ .$$

Since this kind of relations of dependence among the elements $\delta(C_i^*)$ completely determine the relevant part of the cohomology group $H^{-2}(A^2)$, we conclude that at this step of the algorithm we need to introduce a new antisymmetric variable E^* of degree -3 such that:

$$\delta_{-3}^{-3}(E^*) = M_1 C_1^* + M_2 C_2^* + M_3 C_3^* \ .$$

Thus the algebra A^{-3} is defined as follows:

$$A^{-3} = \mathrm{Pol}(M_a)\langle M_1^*, M_2^*, M_3^*, M_4^*, C_1^*, C_2^*, C_3^*, E^*\rangle \ . \tag{5.24}$$

In the next step, the algorithm would require determining the relations of linear dependence existing among the quantities obtained by applying the coboundary operator δ to the variables introduced in the previous step, i.e. in this particular case among the variables of degree -3. Thus we would have to determine a polynomial $f \in Pol_{\mathbb{R}}(M_a)$ such that

$$f\delta_{-3}^{-3}(E^*) = (fM_1)C_1^* + (fM_2)C_2^* + (fM_3)C_3^* = 0 \ .$$

Since C_1^*, C_2^* and C_3^* have to be considered independent variables, the only solution would be $f = 0$. This implies that, among the generators of $Ker(\delta_{-3}^{-3})$ there are no more generators of type β, which are the only ones we are interested in determining to construct the extended theory $(\widetilde{X}, \widetilde{S})$.
Hence it is straightforward to conclude that we can stop at step -3 the computation of the part of the Tate resolution which will be used in the construction of the extended action $\widetilde{S}$.

In conclusion, if the initial action S_0 is given by an element in $Pol_{\mathbb{R}}(M_a)$ such that the polynomials φ, defined in equation (5.17), and $\partial_4 S_0$ are coprime, then the part of the graded variety N, defined starting from the minimal Tate resolution for the Jacobian ring and which will determined the BRST cohomology cochains, is given by

$$N = (X_0, \mathrm{Sym}_{Pol_{\mathbb{R}}(M_a)}(\mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus \mathcal{E})) \ ,$$

where $\mathcal{E}^*[1]$ is the negatively graded module over $Pol_{\mathbb{R}}(M_a)$ defined as

$$\mathcal{E}^*[1] = \langle E^* \rangle_{-3} \oplus \langle C_1^*, C_2^*, C_3^* \rangle_{-2} \ ,$$

while $T_{X_0}[1]$ is the shifted tangent space, which is generated, as module over $Pol_{\mathbb{R}}(M_a)$, by the antifields M_a^*, that is,

$$T_{X_0}[1] = \langle M_1^*, M_2^*, M_3^*, M_4^* \rangle_{-1}$$

and, finally, $\mathcal{E}$ is the positively-graded module over $Pol_{\mathbb{R}}(M_a)$, dual to $\mathcal{E}^*$:

$$\mathcal{E} = \langle C_1, C_2, C_3 \rangle_1 \oplus \langle E \rangle_2 \ .$$

Now we have to analyze the case in which the polynomials φ and $\partial_4 S_0$ are such that $D := GCD(\varphi, \partial_4 S_0) \notin \mathbb{R}$.

φ and $\partial_4 S_0$ are not coprime

Under the hypothesis that the polynomials φ and $\partial_4 S_0$ are not coprime, the cohomology group $H^{-1}(A)$ in (5.22) has six independent generators. Then, for

Tate's algorithm, we need to introduce six symmetric independent variables of degree -1. Since these variables will be the antifields corresponding to the ghost fields C_i, we use for them the notation C_i^*. Thus we define the algebra A^{-2} as the extension of A^{-1} by the adjunction of variables $C_1^*, \ldots, C_6^*$ of degree -2 such that they correspond to $\beta_1, , \ldots, \beta_6$.

More explicitly:

$$A^{-2} = \mathrm{Pol}(M_a)\langle M_1^*, \ldots, M_4^*, C_1^*, \ldots, C_6^*\rangle \ , \tag{5.25}$$

with

$$\begin{cases} \delta_{-2}^{-2}(C_1^*) = \beta_1 = M_3M_2^* - M_2M_3^* \\ \delta_{-2}^{-2}(C_2^*) = \beta_2 = M_1M_3^* - M_3M_1^* \\ \delta_{-2}^{-2}(C_3^*) = \beta_3 = M_2M_1^* - M_1M_2^* \\ \delta_{-2}^{-2}(C_4^*) = \beta_4 = BM_1^* - M_1AM_4^* \\ \delta_{-2}^{-2}(C_5^*) = \beta_5 = BM_2^* - M_2AM_4^* \\ \delta_{-2}^{-2}(C_6^*) = \beta_6 = BM_3^* - M_3AM_4^* \ . \end{cases} \tag{5.26}$$

Step -3 (non-coprime case)

To determine the number of antisymmetric variables of degree -3 we need to introduce in order to have a resolution of the Jacobian ring $J(S_0)$, we would have to analyze the cohomology group $H^{-2}(A)$. However, as already explained in Chapter 4, we can restrict ourselves to determine the linear relations with coefficients in $Pol_{\mathbb{R}}(M_a)$ existing among the elements $\delta(C_1^*), \ldots, \delta(C_6^*)$, for $C_1^*, \ldots, C_6^*$ the elements introduced in the previous step.

Let ξ be an element in A_{-2}^{-2}:

$$\xi = f_1C_1^* + f_2C_2^* + \cdots + f_6C_6^* \ ,$$

with $f_1, f_2, \ldots, f_6 \in Pol_{\mathbb{R}}(M_a)$. Then

$$\begin{aligned}\delta_{-2}^{-2}(\xi) = {} & M_1^*(f_3M_2 - f_2M_3 + f_4B) + M_2^*(f_1M_3 - f_3M_1 + f_5B) \\ & + M_3^*(-f_1M_2 + f_2M_1 + f_6B) + M_4^*(-AM_1f_4 - AM_2f_5 - AM_3f_6) \ .\end{aligned}$$

We want to determine the independent solutions for which $\delta_{-2}^{-2}(\xi)$ vanishes. Since the variables $M_1^*, \ldots, M_4^*$ are independent

$$\delta_{-2}^{-2}(\xi) = 0 \quad \Leftrightarrow \quad \begin{cases} f_1 = +M_1 P - BS \\ f_2 = +M_2 P + BR \\ f_3 = +M_3 P - BQ \\ f_4 = +M_2 Q + M_3 R \\ f_5 = -M_1 Q + M_3 S \\ f_6 = -M_1 R - M_2 S \ , \end{cases}$$

where P, Q, R, S are polynomials belonging to $Pol_{\mathbb{R}}(M_a)$.
We choose the following elements in A_{-2}^{-2} as independent generators of the part of $\mathrm{Ker}(\delta_{-2}^{-2})$ that we are interested in determining:

$$\begin{cases} \alpha_1 = M_1 C_1^* + M_2 C_2^* + M_3 C_3^* \\ \alpha_2 = -BC_1^* + M_3 C_5^* - M_2 C_6^* \\ \alpha_3 = -BC_2^* - M_3 C_4^* + M_1 C_6^* \\ \alpha_4 = -BC_3^* + M_2 C_4^* - M_1 C_5^* \ . \end{cases} \tag{5.27}$$

Therefore, at this step of the algorithm, we need to introduce four antisymmetric variables of degree -3. Thus we define the algebra A^{-3} to be the extension of A^{-2} by the adjunction of variables $E_1^*, E_2^*, E_3^*, E_4^*$ of degree -3 such that they correspond to $\alpha_1, \ldots, \alpha_4$.

More explicitly:

$$A^{-3} = \mathrm{Pol}(M_a)\langle M_1^*, \ldots, M_4^*, C_1^*, \ldots, C_6^*, E_1^*, \ldots, E_4^* \rangle \ , \tag{5.28}$$

with

$$\begin{cases} \delta_{-3}^{-3}(E_1^*) = \alpha_1 \\ \delta_{-3}^{-3}(E_2^*) = \alpha_2 \\ \delta_{-3}^{-3}(E_3^*) = \alpha_3 \\ \delta_{-3}^{-3}(E_4^*) = \alpha_4 \ . \end{cases} \tag{5.29}$$

Step -4 (non-coprime case)

To conclude, we need to examine if, in order to determine the generators of type β of a resolution for the Jacobian ring $J(S_0)$ as required to construct an extended variety, we need to introduce also variables of degree -4.

Let us consider the cohomology group $H^{-3}(A) = \mathrm{Ker}(\delta^{-3}_{-3})/\,\mathrm{Im}(\delta^{-3}_{-4})$.
Given an element $\xi \in A^{-3}_{-3}$, with

$$\xi = g_1 E_1^* + g_2 E_2^* + g_3 E_3^* + g_4 E_4^* \ ,$$

for some $g_1, g_2, g_3, g_4 \in Pol_{\mathbb{R}}(M_a)$, we what to determine for which choice of polynomials g_1, g_2, g_3, g_4 the cochain ξ is an element of $\mathrm{Ker}(\delta^{-3}_{-3})$.

Explicitly:

$$\begin{aligned} \delta^{-3}_{-3}(\xi) = \quad & g_1(M_1C_1^* + M_2C_2^* + M_3C_3^*) + g_2(-BC_1^* + M_3C_5^* - M_2C_6^*) \\ & + g_3(-BC_2^* - M_3C_4^* + M_1C_6^*) + g_4(-BC_3^* + M_2C_4^* - M_1C_5^*) \ , \end{aligned}$$

and so

$$\delta^{-3}_{-3}(\xi) = 0 \quad \Leftrightarrow \quad \begin{cases} g_1 M_1 = g_2 B \\ g_1 M_2 = g_3 B \\ g_1 M_3 = g_4 B \\ M_2 g_4 - M_3 g_3 = 0 \\ M_3 g_2 - M_1 g_4 = 0 \\ M_1 g_3 - M_2 g_2 = 0 \ . \end{cases}$$

Thus as a generator for the cohomology group $H^{-3}(A)$ we consider the cohomology class represented by the element

$$\xi = BE_1^* + M_1E_2^* + M_2E_3^* + M_3E_4^* \ .$$

Therefore, among the generators of $\mathrm{Ker}(\delta^{-3}_{-3})$, there is only one generator of type β. This implies that, at this step of the algorithm, we need to introduce only one symmetric variable K^* of degree -4 such that $\delta^{-4}_{-4}(K^*) = \xi$.

Therefore, the part of resolution for the Jacobian ring in which we are interested is given by the algebra A^{-4} with:

$$A^{-4} = \mathrm{Pol}(M_a)\langle M_1^*, \ldots, M_4^*, C_1^*, \ldots, C_6^*, E_1^*, E_2^*, \ldots, E_4^*, K^* \rangle \ . \tag{5.30}$$

In fact, analogously to what we have observed in degree -3 for the coprime case, also here, when we arrive to have only one generator of type β in a certain degree, then the construction of these generators stops at that point. Indeed, one can prove that the cohomology group $H^{-4}(A^{-4})$ does not contain any generator given by a linear combination of generators introduced in step -3. Thus we do not need to introduce any other variables of higher degree.

Summarizing, given an element S_0 in $Pol_{\mathbb{R}}(M_a)$ as initial action, supposed to be invariant under the adjoint action of the unitary group $U(2)$, we have two possible kinds of resolution for the corresponding Jacobian ring, depending on the relations which link the polynomials φ and $\partial_4 S_0$, with φ defined in (5.17). These two possibilities are the following:

1. If the polynomials φ and $\partial_4 S_0$ are coprime, then the collection of generators of type β in the minimal Tate resolution for the Jacobian ring is obtain by adding to the ring $Pol_{\mathbb{R}}(M_a)$:
   - 4 antisymmetric antifields $M_1^*, \cdots, M_4^*$, with degree -1;
   - 3 symmetric antighosts $C_1^*, \cdots, C_3^*$, with degree -2;
   - 1 antisymmetric anti-ghosts for ghosts E^* with degree -3.
2. If the polynomials φ and $\partial_4 S_0$ are not coprime, i.e. if $D := GCD(\varphi, \partial_4 S_0)$ is not an element of $\mathbb{R}$, then the family of generators of type β given by the minimal resolution for the Jacobian ring is obtained by adding to the ring $Pol_{\mathbb{R}}(M_a)$:
   - 4 antisymmetric antifields $M_1^*, \cdots, M_4^*$, with degree -1;
   - 6 symmetric antighosts $C_1^*, \cdots, C_6^*$, with degree -2;
   - 4 antisymmetric anti-ghosts for ghosts $E_1^*, \ldots, E_4^*$, with degree -3;
   - 1 symmetric anti-ghosts for ghosts K^*, with degree -4.

Remark 32
The construction that we have just described holds for a generic action with $S_0 = Tr(f(A))$, where $f \in Pol_{\mathbb{R}}(\xi)$ with $deg(f) > 1$. In fact, in the case in which $deg(f) = 1$, the resolution of the Jacobian ring will be different: let us consider this case separately.

If $deg(f) = 1$, then $f(\xi) = \alpha_1 \xi + \alpha_0$, for some $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_1 \neq 0$.
Therefore,

$$S_0(A) = Tr(f(A)) = Tr(\alpha_0 Id + \alpha_1 A) = 2\alpha_0 + 2\alpha_1 M_4 \ ,$$

and

$$\partial_1 S_0 = \partial_2 S_0 = \partial_3 S_0 = 0 \qquad\qquad \partial_4 S_0 = 2\alpha_1 \ .$$

Hence the ideal generated by the partial derivatives of the action S_0 contains an invertible element, and so,

$$\langle \partial_1 S_0, \partial_2 S_0, \partial_3 S_0, \partial_4 S_0 \rangle \cong \mathbb{R} \ .$$

To conclude, if we are considering an initial action given by a polynomial of degree 1, we find that the corresponding Jacobian ring is trivial, i.e., $J(S_0) \cong \{0\}$.

The graded variety

Since, depending on the explicit form of the initial action S_0, there are two possible families of generators of type β given by two minimal Tate resolutions for the Jacobian ring, also for the corresponding graded variety there are two possibilities.

In the first case:

$$N_1 = (X_0, \mathrm{Sym}_{\mathrm{Pol}_{\mathbb{R}}(M_a)}(\mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus \mathcal{E})) \ , \tag{5.31}$$

with $\mathcal{E}^*[1]$, $T_{X_0}[1]$ and $\mathcal{E}$ given the the following finitely generated graded module over $Pol_{\mathbb{R}}(M_a)$:

- $\mathcal{E}^*[1] = \langle E^* \rangle_{-3} \oplus \langle C_1^*, C_2^*, C_3^* \rangle_{-2}$;
- $T_{X_0}[1] = \langle M_1^*, M_2^*, M_3^*, M_4^* \rangle_{-1}$;
- $\mathcal{E} = \langle C_1, C_2, C_3 \rangle_1 \oplus \langle E \rangle_2$.

In the second case,

$$N_2 = (X_0, \mathrm{Sym}_{\mathrm{Pol}_{\mathbb{R}}(M_a)}(\mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus \mathcal{E})) \ , \tag{5.32}$$

where the module $T_{X_0}[1]$ is defined as above while the graded modules $\mathcal{E}^*$ and $\mathcal{E}$ are defined as follows:

- $\mathcal{E}^*[1] = \langle K^* \rangle_{-4} \oplus \langle E_1^*, \ldots, E_4^* \rangle_{-3} \oplus \langle C_1^*, \cdots, C_6^* \rangle_{-2}$;
- $\mathcal{E} = \langle C_1, \cdots, C_6 \rangle_1 \oplus \langle E_1, \ldots, E_4 \rangle_2 \oplus \langle K \rangle_3$.

The extended action

To conclude the construction of the extended variety associated to the minimal Tate resolution of the Jacobian ring for the matrix model of degree $n = 2$, we still have to determine the extended action, obtained by starting with the linear approximation in the positively-graded generators, which is denoted by $\widetilde{S}_{\leq 1}$, as described in Section 4.2.
Since we have found two possible sets of generators, depending on the initial action S_0 having the polynomials $\partial_4 S_0$ and φ coprime or not coprime, also for the extended action we will find two different actions for the two possible graded varieties.

In the first (coprime) case, the approximation $\widetilde{S}_{\leq 1}$ of the extended action is given by the following expression:

$$\begin{aligned}\widetilde{S}_{\leq 1,1} &= S_0 + C_1 \cdot \delta_{-2}^{-2}(C_1^*) + C_2 \cdot \delta_{-2}^{-2}(C_2^*) + C_3 \cdot \delta_{-2}^{-2}(C_3^*) + E \cdot \delta_{-3}^{-3}(E^*) \\ &= S_0 + M_1^*(-M_3C_2 + M_2C_3) + M_2^*(M_3C_1 - M_1C_3) \\ &\quad +M_3^*(-M_2C_1 + M_1C_2) + C_1^*(M_1E) + C_2^*(M_2E) + C_3^*(M_3E)\ .\end{aligned} \tag{5.33}$$

In the second (non-coprime) case, the approximation $\widetilde{S}_{\leq 1}$ of the extended action is:

$$\begin{aligned}\widetilde{S}_{\leq 1,2} &= S_0 + C_1 \cdot \delta_{-2}^{-2}(C_1^*) + C_2 \cdot \delta_{-2}^{-2}(C_2^*) + \cdots + C_6 \cdot \delta_{-2}^{-2}(C_6^*) \\ &\quad + E_1 \cdot \delta_{-3}^{-3}(E_1^*) + E_2 \cdot \delta_{-3}^{-3}(E_2^*) + \cdots + E_4 \cdot \delta_{-3}^{-3}(E_4^*) + K \cdot \delta_{-4}^{-4}(K^*) \\ &= S_0 + M_1^*(-M_3C_2 + M_2C_3 + BC_4) + M_2^*(M_3C_1 - M_1C_3 + BC_5) \\ &\quad + M_3^*(-M_2C_1 + M_1C_2 + BC_6) + M_4^*(-M_1AC_4 - M_2AC_5 - M_3AC_6) \\ &\quad + C_1^*(M_1E_1 - BE_2) + C_2^*(M_2E_1 - BE_3) + C_3^*(M_3E_1 - BE_4) \\ &\quad + C_4^*(-M_3E_3 + M_2E_4) + C_5^*(M_3E_2 - M_1E_4) + C_6^*(-M_2E_2 + M_1E_3) \\ &\quad + E_1^*K + E_2^*M_1K + E_3^*M_2K + E_4^*M_3K\ .\end{aligned}$$

In this section, we conclude the construction of the minimal extended variety associated to the matrix model we are considering for degree $n = 2$ and for an initial action S_0, without additional symmetries in the variables M_a, besides the ones required to be invariant under the action of the gauge group $U(2)$ on the configuration space. More explicitly, we concentrate on the coprime case. This is the most generic case, whereas the non-coprime case refers to a situation in which the initial action satisfies further conditions beyond the necessary ones defining a $U(2)$-gauge invariant action on M_0.

Since we focus on the first case, we simplify the notation: in what follows N indicates N_1, while the $\widetilde{S}_{\leq 1}$ is used instead of $\widetilde{S}_{\leq 1,1}$.

Theorem 8. *Let (X_0, S_0) be a pair consisting of a configuration space $X_0 \simeq \mathbb{A}^4_{\mathbb{R}}$ and an action functional S_0 on X_0 that is a solution of the classical master equation. Suppose that S_0 is as in (5.4), i.e.,*

$$S_0 = \sum_{k=0}^{r} (M_1^2 + M_2^2 + M_3^2)^k \; g_k(M_4),$$

with $r \in \mathbb{N}$, and $g_k(M_4)$ an element in $Pol_{\mathbb{R}}(M_4)$, for each value of k.
If the polynomials $\partial_4 S_0$ and φ, with φ defined by (5.17), are coprime, then the most general solution of the classical master equation on the extended configuration space $\widetilde{X}$ that is linear in the antifields, of at most degree 2 in the ghost fields and has coefficients in the ring $Pol_{\mathbb{R}}(M_a)$ obtained by applying the BV algorithm is the following:

$$\begin{aligned}\widetilde{S} = \;& S_0 + M_1^*(-\mu M_3 C_2 + \omega M_2 C_3) + M_2^*(\lambda M_3 C_1 - \omega M_1 C_3) \\ & + M_3^*(-\lambda M_2 C_1 + \mu M_1 C_2) + C_1^*(\alpha\mu\omega M_1 E + \mu M_1 M_3 T C_1 C_2 \\ & - \omega M_1 M_2 T C_1 C_3 + \tfrac{\omega\mu}{\lambda}(1 + M_1^2 T) C_2 C_3) + C_2^*(\alpha\lambda\omega M_2 E \\ & + \lambda M_2 M_3 T C_1 C_2 + \tfrac{\lambda\omega}{\mu}(-1 - M_2^2 T) C_1 C_3 + \omega M_1 M_2 T C_2 C_3) \\ & + C_3^*(\alpha\lambda\mu M_3 E + \tfrac{\mu\lambda}{\omega}(1 + M_3^2 T) C_1 C_2 - \lambda M_2 M_3 T C_1 C_3 \\ & + \mu M_1 M_3 T C_2 C_3)\end{aligned} \tag{5.34}$$

with $\alpha, \lambda, \mu, \omega \in \mathbb{R} \setminus \{0\}$ and with $T \in Pol_{\mathbb{R}}(M_a)$.

Proof. We use the algorithm explained in Section 4.2: we start considering $\widetilde{S}_{\leq 1}$, i.e., the approximation of the extended action that is linear in the positively-graded generators, which was explicitly written in (5.33). Then, step by step, we construct approximations of higher degree in the fields up to the point in which the approximate action will turn out to be an exact solution of the classical master equation for the extended configuration space.

1. Let us start considering the approximation of the extended action that is

linear in the positively-graded generators, given by:

$$\begin{aligned}\widetilde{S}_{\leq 1} &= S_0 + M_1^*(-\mu M_3 C_2 + \omega M_2 C_3) + M_2^*(\lambda M_3 C_1 - \omega M_1 C_3)\\ &\quad + M_3^*(-\lambda M_2 C_1 + \mu M_1 C_2) + C_1^*(\alpha\mu\omega M_1 E) + C_2^*(\alpha\lambda\omega M_2 E)\\ &\quad + C_3^*(\alpha\lambda\mu M_3 E)\ .\end{aligned}$$

2. We explicitly compute the quantity $\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\}$:
   - ▶ Using the properties of the Poisson bracket (see Definition 20) and the parity of the variables considered, we deduce:

     $$\begin{aligned}&\{S_0, M_1^*(-\mu M_3 C_2 + \omega M_2 C_3)\} = -(-\mu M_3 C_2 + \omega M_2 C_3) M_1 \varphi\ ,\\ &\{S_0, M_2^*(\lambda M_3 C_1 - \omega M_1 C_3)\} = -(\lambda M_3 C_1 - \omega M_1 C_3) M_2 \varphi\ ,\\ &\{S_0, M_3^*(-\lambda M_2 C_1 + \mu M_1 C_2)\} = -(-\lambda M_2 C_1 + \mu M_1 C_2) M_3 \varphi\ .\end{aligned}$$

     The sum of the previous terms turns out to be zero.
   - ▶ Since the initial action is a function of the variables M_a only, with $a = 1, \ldots, 4$, the other terms involving the initial action, that is, the summands

     $$\{S_0, C_1^*(\alpha\mu\omega M_1 E)\}\,, \quad \{S_0, C_2^*(\alpha\lambda\omega M_2 E)\}\,, \quad \{S_0, C_3^*(\alpha\lambda\mu M_3 E)\}\,,$$

     are zero. Therefore, we conclude that $\{S_0, \widetilde{S}_{\leq 1}\} = 0$.

   Hence to compute the quantity $\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\}$, we have to consider only the terms that do not involve the initial action. A computation yields the following expression:

   $$\begin{aligned}&\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\}\\ &\quad = 2\lambda\mu(-M_2 M_1^* + M_1 M_2^*) C_1 C_2 + 2\lambda\omega(-M_3 M_1^* + M_1 M_3^*) C_1 C_3\\ &\quad +2\mu\omega(-M_3 M_2^* + M_2 M_3^*) C_2 C_3 + 2\alpha\mu\omega(\omega M_2 C_3 - \mu M_3 C_2) C_1^* E\\ &\quad +2\alpha\lambda\omega(\lambda M_3 C_1 - \omega M_1 C_3) C_2^* E + 2\alpha\lambda\mu(\mu M_1 C_2 - \lambda M_2 C_1) C_3^* E\ .\end{aligned} \tag{5.35}$$

3. Since $\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\} \neq 0$, to obtain the approximation of the action that is of degree 2 in the fields, we need to introduce a generic element ν, with

   $$\nu \in \Gamma(X_0, I_N^{\geqslant 2} \cap F^2 \mathcal{O}_N)\ ,$$

viz. a generic regular function in $\mathcal{O}_N$ with total degree 0 that is at least bilinear and of degree 2 in the fields. We start considering an element of degree 2 in the fields: thus the generic form for such an element ν is

$$\nu = \sum_{i<j,\ k<l} h_{ij}^{kl} M_i^* M_j^* C_k C_l + \sum_{i<l} g_k^{il} C_k^* C_i C_l$$

where the indices i and j are from 1 to 4 while the others change between 1 and 3; h_{ij}^{kl} and g_k^{il} are elements in $\mathrm{Pol}_{\mathbb{R}}(M_a)$.

4. To determine ν, we have to impose the following condition:

$$2(\delta \otimes Id)\nu + \left\{\widetilde{S}_{\leq 1}, \widetilde{S}_{\leq 1}\right\} \equiv 0 \quad \mod F^3\mathcal{O}_N \ .$$

Thus we need to compute $(\delta \otimes Id)\nu$. We immediately see that we will not obtain any useful contribution from the summands present in the first sum in ν: in fact, by letting the operator δ acting on the antifields M_i^*, we would obtain terms containing the partial derivatives of the initial action, but these kind of terms are not present in (5.35), which is the quantity that we need to compensate. Therefore, we concentrate our attention only on the terms contained in the second sum of ν and set $h_{ij}^{kl} = 0$.
Thus we have to determine the polynomials g_k^{il}, for $k, i, l = 1, 2, 3$ with $i < l$ in such a way that the following equalities are satisfied:

$$\begin{array}{l}
M_1^* C_1 C_2 : -2\lambda\mu M_2 - 2\mu g_2^{12} M_3 + 2\omega g_3^{12} M_2 = 0 \\
M_2^* C_1 C_2 : +2\lambda\mu M_1 + 2\lambda g_1^{12} M_3 - 2\omega g_3^{12} M_1 = 0 \\
M_3^* C_1 C_2 : -2\lambda g_1^{12} M_2 + 2\mu g_2^{12} M_1 = 0 \\
\\
M_1^* C_1 C_3 : -2\lambda\omega M_3 - 2\mu g_2^{13} M_3 + 2\omega g_3^{13} M_2 = 0 \\
M_2^* C_1 C_3 : +2\lambda g_1^{13} M_3 - 2\omega g_3^{13} M_1 = 0 \\
M_3^* C_1 C_3 : +2\lambda\omega M_1 - 2\lambda g_1^{13} M_2 + 2\mu g_2^{13} M_1 = 0 \\
\\
M_1^* C_2 C_3 : -2\mu g_2^{23} M_3 + 2\omega g_3^{23} M_2 = 0 \\
M_2^* C_2 C_3 : -2\mu\omega M_3 + 2\lambda g_1^{23} M_3 - 2\omega g_3^{23} M_1 = 0 \\
M_3^* C_2 C_3 : +2\mu\omega M_2 - 2\lambda g_1^{23} M_2 + 2\mu g_2^{23} M_1 = 0 \ .
\end{array} \tag{5.36}$$

Hence:

$$\begin{array}{lll}
g_1^{12} = \mu M_1 Q & g_2^{12} = \lambda M_2 Q & g_3^{12} = \frac{\mu\lambda}{\omega}(1 + M_3 Q) \\
g_1^{13} = \omega M_1 P & g_2^{13} = \frac{\lambda\omega}{\mu}(-1 + M_2 P) & g_3^{13} = \lambda M_3 P \\
g_1^{23} = \frac{\omega\mu}{\lambda}(1 + M_1 R) & g_2^{23} = \omega M_2 R & g_3^{23} = \mu M_3 R \ ,
\end{array} \tag{5.37}$$

with $Q, P, R \in \mathrm{Pol}_{\mathbb{R}}(M_a)$.
Therefore, an approximate action that solves the classical master equation up to terms with degree 2 in the fields is given by

$$\begin{aligned}\widetilde{S}_{\leq 2} = \quad & \widetilde{S}_{\leq 1} + C_1^*(\mu M_1 Q C_1 C_2 + \omega M_1 P C_1 C_3 + \tfrac{\omega\mu}{\lambda}(1 + M_1 R) C_2 C_3) \\ & + C_2^*(\lambda M_2 Q C_1 C_2 + \tfrac{\lambda\omega}{\mu}(-1 + M_2 P) C_1 C_3 + \omega M_2 R C_2 C_3) \\ & + C_3^*(\tfrac{\mu\lambda}{\omega}(1 + M_3 Q) C_1 C_2 + \lambda M_3 P C_1 C_3 + \mu M_3 R C_2 C_3)\ .\end{aligned}$$

5. In the previous expression, no explicit conditions were imposed on the polynomials $P, Q, R \in \mathrm{Pol}_{\mathbb{R}}(M_a)$. Therefore, we compute the quantity $\left\{\widetilde{S}_{\leq 2}, \widetilde{S}_{\leq 2}\right\}$, and check if it is possible to choose these polynomials in a suitable way to convert the approximate solution $\widetilde{S}_{\leq 2}$ in an exact solution to the classical master equation.
Since $\widetilde{S}_{\leq 2}$ is an approximate solution up to degree 2 in the fields, the only terms in the expression $\left\{\widetilde{S}_{\leq 2}, \widetilde{S}_{\leq 2}\right\}$ that are not necessarily zero are the one of degree greater or equal to 3 in the fields. After some computations, the conditions that need to be imposed on $\widetilde{S}_{\leq 2}$ to have an exact solution turn out to be the following:

- to cancel the terms of the type $C_i^* C_j C_k$, with $i, j, k = 1, 2, 3,\ j < k$:

$$\begin{cases} M_2 Q + M_3 P = 0 \\ M_1 Q - M_3 R = 0 \\ M_1 P + M_2 R = 0 \end{cases} ; \tag{5.38}$$

- to cancel the term $C_i^* C_1 C_2 C_3$, with $i = 1, 2, 3$:

$$M_2 \frac{\partial Q}{\partial M_1} - M_1 \frac{\partial Q}{\partial M_2} + M_3 \frac{\partial R}{\partial M_2} - M_1 \frac{\partial P}{\partial M_3} - M_2 \frac{\partial R}{\partial M_3} + M_3 \frac{\partial P}{\partial M_1} = 0\ . \tag{5.39}$$

Therefore, choosing

$$P = -M_2 T, \qquad Q = M_3 T, \qquad R = M_1 T,$$

with T a polynomial in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, all the conditions are satisfied. Thus:

$$\begin{aligned}\widetilde{S}_{gen} = \quad & S_0 + M_1^*(-\mu M_3 C_2 + \omega M_2 C_3) + M_2^*(\lambda M_3 C_1 - \omega M_1 C_3) \\ & + M_3^*(-\lambda M_2 C_1 + \mu M_1 C_2) + C_1^*(\alpha\mu\omega M_1 E + \mu M_1 M_3 T C_1 C_2 \\ & - \omega M_1 M_2 T C_1 C_3 + \tfrac{\omega\mu}{\lambda}(1 + M_1^2 T) C_2 C_3) + C_2^*(\alpha\lambda\omega M_2 E \\ & + \lambda M_2 M_3 T C_1 C_2 + \tfrac{\lambda\omega}{\mu}(-1 - M_2^2 T) C_1 C_3 + \omega M_1 M_2 T C_2 C_3) \\ & + C_3^*(\alpha\lambda\mu M_3 E + \tfrac{\mu\lambda}{\omega}(1 + M_3^2 T) C_1 C_2 - \lambda M_2 M_3 T C_1 C_3 \\ & + \mu M_1 M_3 T C_2 C_3)\end{aligned}$$

is a solution to the classical master equation. Moreover, $\widetilde{S}_{gen}$ is the most general solution to the classical master equation on the extended configuration of degree 2 in the fields that can be obtained from the linear approximation $\widetilde{S}_{\leq 1}$.

□

To conclude, since we found both the extended configuration space and the corresponding action, we have defined an extended variety associated to the minimal Tate resolution of a matrix model of degree $n = 2$, in the case of a generic action.

5.2 The BRST cohomology for the U(2)-matrix model

The purpose of this section is to explicitly describe the BRST cohomology complex for the $U(2)$-matrix model that we are analyzing.
To achieve this goal, after having constructed the minimal extended variety $(N, \widetilde{S})$ associated to the model in the previous section, we apply the techniques presented in Chapter 3. We first define the *classical BRST cohomology complex* induced by $(N, \widetilde{S})$. Then we carry out the *gauge fixing procedure* to determine the *(gauge-fixed) BRST cohomology complex* associated to the model. Finally, the corresponding cohomology groups are determined. The detailed computation of these groups are collected in Appendix D.

5.2.1 The classical BRST cohomology

In what follows, we concentrate on the matrix model of degree $n = 2$ and we explicitly describe the classical BRST complex associated to the extended variety we found in the previous section. Let us start by summing up the structure and the notation:

- (X_0, S_0) is a pair in which the configuration space X_0 is defined as follows

$$X_0 = \{A \in M_2(\mathbb{C}) : A^* = A\} ,$$

while the initial action functional, which is a solution of the classical master equation on X_0, can be written as the following polynomial:

$$S_0 = \sum_{k=0}^{r} (M_1^2 + M_2^2 + M_3^2)^k \; g_k(M_4) ,$$

with $r \in \mathbb{N}$ and $g_k(M_4)$ a polynomial in $Pol_{\mathbb{R}}(M_4)$, for each value of k.

- $(N, \widetilde{S})$ is an extended variety determined by the minimal Tate resolution for (X_0, S_0). We denote the extended configuration space corresponding to the extended variety defined in Equation (5.31) by $\widetilde{X}$, that is, $\widetilde{X}$ is given by the following graded vector space:

$$\widetilde{X} = \langle E^* \rangle \oplus \langle C_1^*, .., C_3^* \rangle \oplus \langle M_1^*, .., M_4^* \rangle \oplus \langle M_1, .., M_4 \rangle \oplus \langle C_1, .., C_3 \rangle \oplus \langle E \rangle . \tag{5.40}$$

We denote the extended action by $\widetilde{S}$, which solves the classical master equation for $\widetilde{X}$. Since in what follows we proceed with an explicit computation, to simplify it, we suppose that the extended action takes the following form:

$$\begin{aligned} \widetilde{S} = \;& S_0 + M_1^*(-M_3C_2 + M_2C_3) + M_2^*(M_3C_1 - M_1C_3) \\ & +M_3^*(-M_2C_1 + M_1C_2) + C_1^*(M_1E + C_2C_3) \\ & +C_2^*(M_2E - C_1C_3) + C_3^*(M_3E + C_1C_2) , \end{aligned} \tag{5.41}$$

where we have chosen the polynomial T in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ to be zero and the real coefficients λ, μ, ω, α to be equal to 1.

Before applying Definition 27 to this context, we recall that in our model we have the following identities:

- $\mathcal{O}_{X_0} \simeq \mathrm{Pol}_{\mathbb{R}}(M_a)$, with $a = 1, \ldots, 4$;

- $T_{X_0}[1] \simeq \langle M_1^*, \ldots, M_4^* \rangle$,
recalling that the antifields M_a^*, corresponding to the fields M_a, have, by construction, ghost degree -1;

- $\mathcal{E}$ is a graded $\mathcal{O}_{X_0}$-module,

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \quad \text{where} \quad \begin{cases} \mathcal{E}_1 = \langle C_1, C_2, C_3 \rangle \\ \mathcal{E}_2 = \langle E \rangle \ ; \end{cases}$$

- $\mathcal{E}^*[1]$ is a graded $\mathcal{O}_{X_0}$-module obtained from $\mathcal{E}$ by changing the positivity of the grading and shifting it by -1: more explicitly,

$$\mathcal{E}^*[1] = \mathcal{E}^*[1]_1 \oplus \mathcal{E}^*[1]_2 \quad \text{where} \quad \begin{cases} \mathcal{E}^*[1]_1 = \langle C_1^*, C_2^*, C_3^* \rangle \\ \mathcal{E}^*[1]_2 = \langle E^* \rangle \ . \end{cases}$$

Therefore, the classical BRST complex takes the form described in the following definition.

Definition 46. *Let $(\widetilde{X}, \widetilde{S})$ be the extended variety constructed in (5.40), (5.41) for the matrix model of degree $n = 2$. Then the classical BRST complex associated to it is given by $(C^{\bullet}(\widetilde{X}, d_{\widetilde{S}}), d_{\widetilde{S}})$, where:*

- *The vector space of cochains of degree k, with $k \in \mathbb{Z}$, is the homogeneous component of degree k in the following graded algebra:*

$$\mathcal{O}_N = \widehat{\mathrm{Sym}}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]) \ .$$

  More explicitly, in this context we have

$$C^{\,k}(\widetilde{X}, d_{\widetilde{S}}) = [\mathcal{O}_N]^k = [\mathrm{Sym}_{\mathrm{Pol}_{\mathbb{R}}(M_a)}(\langle M_a^* \rangle \oplus \langle C_i^* \rangle \oplus \langle E^* \rangle \oplus \langle C_i \rangle \oplus \langle E \rangle)]^k,$$

  $a = 1, \ldots, 4$, $i = 1, \ldots, 3$.

- *The coboundary operator is defined as $d_{\widetilde{S}} = \left\{\widetilde{S}, -\right\}$ and it acts on the generators as follows:*

  ▷ *on the generator E^* of degree -3:*

$$d_{\widetilde{S}}(E^*) = M_1 C_1^* + M_2 C_2^* + M_3 C_3^* \ ; \tag{5.42}$$

  ▷ *on the generators C_1^*, C_2^*, C_3^* of degree -2:*

$$\begin{cases} d_{\widetilde{S}}(C_1^*) = (M_3 M_2^* - M_2 M_3^*) + (C_2^* C_3 - C_3^* C_2) \\ d_{\widetilde{S}}(C_2^*) = (-M_3 M_1^* + M_1 M_3^*) + (-C_1^* C_3 + C_3^* C_1) \\ d_{\widetilde{S}}(C_3^*) = (M_2 M_1^* - M_1 M_2^*) + (C_1^* C_2 - C_2^* C_1) \ ; \end{cases} \tag{5.43}$$

▷ *on the antifields M_1^*, M_2^*, M_3^*, M_4^* of ghost degree -1:*

$$\begin{cases} d_{\widetilde{S}}(M_1^*) = \partial_{M_1} S_0 - M_2^* C_3 + M_3^* C_2 + C_1^* E \\ d_{\widetilde{S}}(M_2^*) = \partial_{M_2} S_0 + M_1^* C_3 - M_3^* C_1 + C_2^* E \\ d_{\widetilde{S}}(M_3^*) = \partial_{M_3} S_0 - M_1^* C_2 + M_2^* C_1 + C_3^* E \\ d_{\widetilde{S}}(M_4^*) = \partial_{M_4} S_0 \ ; \end{cases} \tag{5.44}$$

▷ *on the initial fields M_1, M_2, M_3, M_4 of ghost degree 0:*

$$\begin{cases} d_{\widetilde{S}}(M_1) = -M_3 C_2 + M_2 C_3 \\ d_{\widetilde{S}}(M_2) = M_3 C_1 - M_1 C_3 \\ d_{\widetilde{S}}(M_3) = -M_2 C_1 + M_1 C_2 \\ d_{\widetilde{S}}(M_4) = 0 \ ; \end{cases} \tag{5.45}$$

▷ *on the generators C_1, C_2, C_3 of ghost degree 1:*

$$\begin{cases} d_{\widetilde{S}}(C_1) = C_2 C_3 + M_1 E \\ d_{\widetilde{S}}(C_2) = -C_1 C_3 + M_2 E \\ d_{\widetilde{S}}(C_3) = C_1 C_2 + M_3 E \ ; \end{cases} \tag{5.46}$$

▷ *finally, on the ghost field E of ghost degree 2 the coboundary operator acts trivially:*

$$d_{\widetilde{S}}(E) = 0. \tag{5.47}$$

Note: to extend the action of the coboundary operator $d_{\widetilde{S}}$ from the generators to a generic cochain in $C^{\,k}(\widetilde{X}, d_{\widetilde{S}})$ it is enough to recall that $d_{\widetilde{S}}$ acts as a graded derivation.

Remark 33
The coboundary operator $d_{\widetilde{S}}$ is known in the physics literature as *the classical BRST differential*. This terminology has been introduced to distinguish the transformation described above from what is known as *the BRST transformation*, that is to say, the classical BRST transformation *after* a gauge-fixing procedure has been implemented.

5.2.2 The gauge-fixing process

Having defined the classical BRST cohomology complex corresponding to the extended variety $(N, \widetilde{S})$ in the previous section, in order to construct the (gauge-fixed) BRST cohomology complex, we still have to implement the gauge-fixing procedure: indeed, the aim of this section is to apply the gauge-fixing process, presented in a general setting in Section 3.6, to our matrix model. As the physical theory with which we start the procedure, we take the extended variety $(N, \widetilde{S})$ computed in Section 5.1.2, with $\widetilde{S}$ as in (5.41).

First note that, according to Definition 40, our model describes a reducible theory with level of reducibility 1. Therefore, to carry out the gauge-fixing procedure, we need to introduce three types of trivial pairs, one at the level $i = 0$ and two at the level $i = 1$.
To be more precise, let us start considering the level $i = 0$. As already noticed, at level of reducibility $i = 0$ we have to introduce only one type of trivial pair. The trivial pairs introduced at level $i = 0$ correspond to the ghost fields C_i: since we have three ghost fields C_i, we have to introduce three trivial pairs, each of them corresponding to one among the ghost fields C_i. Thus we introduce the following extra fields:

$$(B_1, h_1) \qquad (B_2, h_2) \qquad (B_3, h_3) \ ,$$

with

$$\begin{cases} deg(B_1) = deg(B_2) = deg(B_3) = -1 \\ \epsilon(B_1) = \epsilon(B_2) = \epsilon(B_3) = 1 \end{cases} \begin{cases} deg(h_1) = deg(h_2) = deg(h_3) = 0 \\ \epsilon(h_1) = \epsilon(h_2) = \epsilon(h_3) = 0 \ . \end{cases}$$

For $i = 1$, we know already that we have to introduce two different trivial pairs. The trivial pairs introduced at level $i = 1$ correspond to the ghost fields of ghost degree 2. Therefore, since in our model we have only one ghost field with ghost degree equals to 2, namely the ghost field E, for each type of trivial pair we have to introduce only one couple of fields $(B_i^j,\, h_i^j)$.
We simplify notation by using the letters B and h to denote the extra fields introduced at level $i = 0$, while for the trivial pair at level $i = 1$ we will use the letters A and k, respectively. This choice allows us to distinguish the trivial pairs introduced at level $i = 0$ from the ones introduced at level $i = 1$, clarifying the distinction between real and Grassmannian fields.
Thus for $i = 1$ we introduce the following trivial pairs:

$$(A_1, k_1) \qquad (A_2, k_2),$$

with

$$\begin{cases} deg(A_1) = -2 \\ \epsilon(A_1) = 0 \end{cases} \quad \begin{cases} deg(k_1) = -1 \\ \epsilon(k_1) = 1 \end{cases} \quad \begin{cases} deg(A_2) = 0 \\ \epsilon(A_2) = 0 \end{cases} \quad \begin{cases} deg(k_2) = 1 \\ \epsilon(k_2) = 1 \ . \end{cases}$$

Therefore, the total action for the matrix model is the following:

$$S_{tot} = \widetilde{S} + S_{aux} \tag{5.48}$$

with

$$S_{aux} = B_1^*\ h_1 + B_2^*\ h_2 + B_3^*\ h_3 + A_1^*\ k_1 + A_2^*\ k_2\ . \tag{5.49}$$

Considering the conditions imposed on the parity and on the ghost degree of a gauge-fixing fermion, i.e., the requests of being a regular function in the ghost and antighost fields of total degree -1 and Grassmannian parity, the most generic gauge-fixing fermion for the model has the following form:

$$\begin{aligned}\Psi\ &= \sum_a f_a B_a + \sum_i g_i C_i A_1 + l k_2 A_1 + m k_1 + \sum_{k,i} p_{k,i} (B_1)^k E^{k-1} C_i \\ &+ \sum_k q_k (B_1)^{k+1} E^{k-1} C_1 C_2 C_3\ ,\end{aligned}$$

where $a, i = 1, 2, 3$, while $k \in \mathbb{N}$ and f_a, g_i, l, m, $p_{k,i}$, q_k are elements of $\mathrm{Pol}_{\mathbb{R}}(M_a, h_i, A_1)$.

As discussed in Section 3.6, if we want to ensure that the gauge-fixed action is a proper solution of the classical master equation, further conditions need to be imposed on the gauge-fixing fermion, mainly involving its second-order derivatives (see Remark 51 in Appendix A).
Another criterion for choosing the gauge-fixing fermion is to simplify the computation that we want to do. In what follows our main goal is to compute the BRST cohomology groups of our matrix model for degree $n = 2$. Thus the aim in choosing a gauge-fixing fermion is to make the BRST-coboundary operator as simple as possible. However, since in the extended action $\widetilde{S}$ of the model the antifields and antighost fields appear only in degree 1, the explicit form of the gauge-fixing fermion will never enter the definition of the gauge-fixed coboundary operator. For this reason, we do not go into details in analyzing the best possible choice for the gauge-fixing fermion.

Note that the introduction of the trivial pairs, which are necessary to carry out the gauge-fixing procedure, changes the classical BRST cohomology complex for our model. Indeed we have introduced extra fields (B_i, h_i) and (A_j, k_j). Therefore, the theory we are now considering is given by a pair (X_{tot}, S_{tot}), obtained from the minimally extended pair $(\widetilde{X}, \widetilde{S})$ by the inclusion of the extra fields. That is:

$$\begin{aligned}X_{tot}\ &= \langle E^* \rangle \oplus \langle C_1^*, .., C_3^*, A_1, k_2^* \rangle \oplus \langle M_1^*, .., M_4^*, B_1, .., B_3, h_1^*, .., h_3^*, k_1, A_2^* \rangle \\ &\oplus \langle M_1, .., M_4, B_1^*, .., B_3^*, h_1, .., h_3, k_1^*, A_2 \rangle \oplus \langle C_1, .., C_3, A_1^*, k_2 \rangle \oplus \langle E \rangle,\end{aligned}$$

where the different homogeneous components of this graded vector space are ordered by increasing degree, from -3 up to 2.
As already seen in (5.48) and (5.49), the total action also involves the extra fields. Therefore, the classical BRST cohomology complex defined by the theory extended by the trivial pairs is different from the one found in Section 5.2 for the extended variety $(N, \widetilde{S})$. More precisely:

- the vector space of cochains of degree m, with $m \in \mathbb{Z}$, is

$$C^{\,m}(X_{tot}, d_{S_{tot}}) = [\mathrm{Sym}_{\mathrm{Pol}_{\mathbb{R}}(M_a)}(X_{tot})]^m;$$

- the coboundary operator is defined as $d_{S_{tot}} = \{S_{tot}, -\}$.

Therefore, to the expression found in Equation (5.42)-(5.47), for the coboundary operator one needs to include the value of the coboundary on the new generators:

$$\begin{cases} d_{S,tot}(B_i) = h_i \\ d_{S,tot}(h_i) = 0 \end{cases} \begin{cases} d_{S,tot}(h_i^*) = B_i^* \\ d_{S,tot}(B_i^*) = 0 \end{cases} \begin{cases} d_{S,tot}(A_j) = k_j \\ d_{S,tot}(k_j) = 0 \end{cases} \begin{cases} d_{S,tot}(k_j^*) = A_j^* \\ d_{S,tot}(A_j^*) = 0 \;. \end{cases} \tag{5.50}$$

This is not yet the cohomology complex that we are interested in: in fact, from a physical point of view, it is more interesting to consider the BRST complex, i.e., the complex obtained by applying the gauge-fixing condition to the classical BRST complex. Indeed, this is the complex which we will compute and analyze in the following section.

5.2.3 The gauge-fixed BRST cohomology

The purpose of this section is to determine the (gauge-fixed) BRST cohomology complex for the $U(2)$-matrix model in which we are interested. The construction of this cohomology complex was already presented in the general setting in Section 3.5.
In few words, we may say that the gauge-fixed BRST cohomology complex is simply the complex obtained by imposing the gauge-fixing conditions on this extended version of the classical BRST complex. More precisely:

- The vector spaces of cochains only involves the cochains defined using fields: given the total configuration space X_{tot}, we know that it is endowed with a super vector space structure, that is to say,

$$X_{tot} = W_{tot} \oplus W^*_{tot}[1] \;,$$

where W_{tot} is a $\mathbb{Z}$-graded vector space. The graded vector space W_{tot} describes the field content of the total configuration space, while $W^*_{tot}[1]$ gives the corresponding antifields. To apply the gauge-fixing procedure to the total configuration space simply means to restrict the total configuration space X_{tot} to its field content W_{tot}. Thus the cochains we consider for the gauge-fixed BRST cohomology are only the ones obtained using the generators of W_{tot} and so they are only functions of the fields.

Schematically, by applying the gauge-fixing procedure at the level of cochains, we have

$$\mathcal{C}^\bullet(X_{tot}, d_{S_{tot}}) \rightsquigarrow \mathcal{C}^\bullet(W_{tot}, d_{S_{tot,\Psi}}) \ ,$$

where

$$C^{\,m}(W_{tot}, d_{S_{tot,\Psi}}) = [\mathrm{Sym}_{\mathrm{Pol}_{\mathbb{R}}(M_a)}(\langle C_w\rangle \oplus \langle E\rangle \oplus \langle B_i\rangle \oplus \langle h_i\rangle \oplus \langle A_j\rangle \oplus \langle k_j\rangle)]^m \tag{5.51}$$

with $a = 1, \ldots, 4$, $w = 1, \ldots, 3$, $i = 1, \ldots, 3$ and $j = 1, 2$.

Note: since, in order to be able to proceed with the gauge-fixing process, we were forced to introduce extra fields with negative ghost degree, also considering only cochains in the fields, nevertheless we still have a double-sided complex, that is to say, we have cochains of degree m for any value of m in $\mathbb{Z}$.

► To obtain the corresponding coboundary operator, first of all we concentrate only on establishing the behavior of this operator on the fields and ghost fields, since the antifields and antighost fields do not play any role as generators of the cochains in this cohomology complex. To reach this purpose, we start with what we found for the coboundary operator of the classical BRST complex and then we apply the gauge-fixing condition (3.8).
So, the coboundary operator $d_{S_{tot,\Psi}}$ on which we are interested is defined as

$$d_{S_{tot,\Psi}} = \{S_{tot}, -\}\,|_{\Sigma_\Psi}$$

and it can be seen as a linear and graded derivation which acts as follows on the generators:

▷ on the initial fields M_1, M_2, M_3, M_4 of ghost degree 0:

$$\begin{cases} d_{S_{tot,\Psi}}(M_1) = -M_3C_2 + M_2C_3 \\ d_{S_{tot,\Psi}}(M_2) = +M_3C_1 - M_1C_3 \\ d_{S_{tot,\Psi}}(M_3) = -M_2C_1 + M_1C_2 \\ d_{S_{tot,\Psi}}(M_4) = 0 \ ; \end{cases} \tag{5.52}$$

▷ on the ghost fields C_1, C_2, C_3 of ghost degree 1:

$$\begin{cases} d_{S_{tot,\Psi}}(C_1) = C_2C_3 + M_1E \\ d_{S_{tot,\Psi}}(C_2) = -C_1C_3 + M_2E \\ d_{S_{tot,\Psi}}(C_3) = C_1C_2 + M_3E \ ; \end{cases} \tag{5.53}$$

▷ on the ghost field E of ghost degree 2 the coboundary operator acts trivially:

$$d_{S_{tot,\Psi}}(E) = 0 \ ; \tag{5.54}$$

▷ on the auxiliary fields B_1, B_2, B_3 of ghost degree -1:

$$\begin{cases} d_{S_{tot,\Psi}}(B_1) = h_1 \\ d_{S_{tot,\Psi}}(B_2) = h_2 \\ d_{S_{tot,\Psi}}(B_3) = h_3 \ ; \end{cases} \tag{5.55}$$

▷ on the auxiliary fields h_1, h_2, h_3 of ghost degree 0:

$$\begin{cases} d_{S_{tot,\Psi}}(h_1) = 0 \\ d_{S_{tot,\Psi}}(h_2) = 0 \\ d_{S_{tot,\Psi}}(h_3) = 0 \ ; \end{cases} \tag{5.56}$$

▷ on the second type of auxiliary fields, that is, on the auxiliary fields A_1, A_2, the coboudary operator acts as follows:

$$\begin{cases} d_{S_{tot,\Psi}}(A_1) = k_1 \\ d_{S_{tot,\Psi}}(A_2) = k_2 \ ; \end{cases} \tag{5.57}$$

▷ finally, for the auxiliary fields k_1, k_2 we have

$$\begin{cases} d_{S_{tot,\Psi}}(k_1) = 0 \\ d_{S_{tot,\Psi}}(k_2) = 0 \ . \end{cases} \tag{5.58}$$

Remark 34
Looking at the BRST cohomology complex described above, note that it does not depend on the gauge-fixing fermion chosen for the gauge-fixing process: this is due to the linearity of the extended action with respect to the antifields and antighost fields. In fact, the cohomology complex may depend on the gauge-fixing fermion only at the level of the coboundary operator. However, linearity of the action in the antifields and antighost fields implies that the BRST coboundary operator sends fields and ghost fields to fields and ghost

fields (rather than antifields and antighost fields), respectively.
Recall that the gauge-fixing fermion is used to substitute the antifields and the antighost fields by an expression depending only on fields and ghost fields, given by the derivative of the gauge-fixing fermion:

$$\varphi_i^* \rightsquigarrow \frac{\partial \Psi}{\partial \varphi_i}.$$

Therefore, since no antifields or antighost fields appear in the expression of the coboundary operator for the fields, it follows that also the gauge-fixing fermion will not appear. Therefore, for our matrix model of degree $n = 2$, the BRST cohomology groups neither depend on the gauge-fixing fermion chosen for the gauge-fixing process, nor does the gauge-fixing fermion play any role at the level of the BRST cohomology complex.

Looking more closely at the behavior of the coboundary operator over the extra fields, it is immediately clear that the extra fields, which we introduced in order to the be able to proceed with the gauge-fixing procedure, describe contractible pairs from the point of view of the cohomology complex.
More precisely, let us consider a cohomology complex

$$(C^{\,\bullet}(W, d_{\widetilde{S}}), d_{\widetilde{S}}),$$

where:

- ► the space of cochains of degree k, with k in $\mathbb{N}_0$, is given by:

  $$C^{\,k}(W, d_{\widetilde{S}}) = [\mathrm{Sym}_{\mathrm{Pol}_{\mathbb{R}}(M_a)}(\langle C_i \rangle \oplus \langle E \rangle)]^k$$

  where $a = 1, \ldots, 4$ while $i = 1, \ldots, 3$;

- ► the coboundary operator $d_{\widetilde{S}}$ is a graded derivation which acts on the generators M_a, C_i and E exactly as done by the coboundary operator $d_{S_{tot,\Psi}}$:

  $$d_{\widetilde{S}}|_W = d_{S_{tot,\Psi}}|_W.$$

Then we have the following proposition.

Proposition 11
Let $(\mathcal{C}^\bullet(W_{tot}, d_{S_{tot,\Psi}}), d_{S_{tot,\Psi}})$ be the cohomology complex introduced in Equation (5.51) and (5.52)-(5.58) while $(C^{\,\bullet}(W, d_{\widetilde{S}}), d_{\widetilde{S}})$, denotes the cohomology complex just introduced. Then these two cohomology complexes are quasi-isomorphic, i.e.,

$$H^j(W_{tot}, d_{S_{tot,\Psi}}) \simeq H^j(W, d_{\widetilde{S}}), \qquad \forall k \in \mathbb{Z}.$$

Proof. This follows immediately by applying Theorem 5 to the contractible pairs defined by the following pairs of extra fields: (B_1, h_1), (B_2, h_2), (B_3, h_3), (A_1, k_1), (A_2, k_2). □

To conclude, the only fields and ghost fields that play a role in the (gauge-fixed) BRST cohomology complex are:

- the initial fields M_a, $a = 1, \ldots, 4$, with ghost degree $\deg(M_a) = 0$ and parity $\epsilon(M_a) = 0$;
- the ghost fields C_i, $i = 1, \ldots, 3$, with ghost degree $\deg(C_i) = 1$ and parity $\epsilon(C_i) = 1$;
- the ghost field E, with ghost degree $\deg(E) = 2$ and parity $\epsilon(E) = 0$.

Thus to compute the cohomology groups defined by the gauge-fixed BRST cohomology complex we can equivalently compute the cohomology groups defined by the cohomology complex $(C^\bullet(W, d_{\widetilde{S}}), d_{\widetilde{S}})$. The next section is devoted to the analysis of these cohomology groups.

5.2.4 Computation of the BRST cohomology groups

First of all, recall the notation:

- $Z^j(W, d_{\widetilde{S}}) = \mathrm{Ker}(d^j_{\widetilde{S}})$ denotes the space of cocycles of degree j,
- $B^j(W, d_{\widetilde{S}}) = \mathrm{Im}(d^{j-1}_{\widetilde{S}})$ denotes the space of coboundaries of degree j,
- $H^j(W, d_{\widetilde{S}})$ denotes the gauge-fixed BRST cohomology group of degree j,

all of them referring to the cohomology complex $(C^\bullet(W, d_{\widetilde{S}}), d_{\widetilde{S}})$.
Before starting with the explicit computation of the cohomology groups, let us first of all make the following observation: since in the model there is only one ghost of positive ghost degree and of even parity, namely the ghost E, to compute the BRST cohomology groups $H^j(W, d_{\widetilde{S}})$ of the model we apparently need to compute this group for each non-negative value of j. However, as more precisely stated in the following proposition, we can concentrate on only few of them.

Proposition 12
Let j be an element in $\mathbb{N}$, $j \geqslant 4$. Then:

- if j is odd, then $H^j(W, d_{\widetilde{S}}) \simeq H^3(W, d_{\widetilde{S}})$;

► if j is even, then $H^j(W, d_{\widetilde{S}}) \simeq H^4(W, d_{\widetilde{S}})$.

Proof. To prove the statement above we start noticing that isomorphisms are present already at the level of cocycle and coboundary spaces.
More precisely, given $j \geqslant 4$, the following isomorphisms hold:

$$\begin{cases} Z^j(W, d_{\widetilde{S}}) \simeq Z^2(W, d_{\widetilde{S}}) \cdot E^i & \text{if } j = 2i+2 \\ Z^j(W, d_{\widetilde{S}}) \simeq Z^3(W, d_{\widetilde{S}}) \cdot E^i & \text{if } j = 2i+3 \end{cases}$$
$$\begin{cases} B^j(W, d_{\widetilde{S}}) \simeq B^4(W, d_{\widetilde{S}}) \cdot E^i & \text{if } j = 2i+4 \\ B^j(W, d_{\widetilde{S}}) \simeq B^3(W, d_{\widetilde{S}}) \cdot E^i & \text{if } j = 2i+3 \ . \end{cases} \tag{5.59}$$

To prove the previous statements, let us start considering the case in which $j = 2i+2$. Then a generic cochain φ of ghost degree j takes the following form:

$$\varphi = [fE + g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3]E^i,$$

where $f, g_{12}, g_{13}, g_{23}$ are elements in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Moreover, the polynomials f and g_{ij} are determined by the cochain φ itself.
Then the following equivalences follow immediately:

$$\begin{aligned} \varphi \in Z^j(W, d_{\widetilde{S}}) \quad & \Leftrightarrow d_{\widetilde{S}}(\varphi) = 0 \\ & \Leftrightarrow d_{\widetilde{S}}(fE + g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3) \cdot E^i \\ & \quad\ + [fE + g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3] \cdot d_{\widetilde{S}}(E^i) = 0 \\ & \Leftrightarrow [fE + g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3] \in Z^2(W, d_{\widetilde{S}}) \ , \end{aligned}$$

where we simply used the fact that $d_{\widetilde{S}}$ is a derivation which sends the generator E to zero. Then we deduce that, seen as sets, $Z^j(W, d_{\widetilde{S}})$ and $Z^2(W, d_{\widetilde{S}}) \cdot E^i$ coincide. Moreover, all other conditions to have an isomorphism follow immediately. Thus we conclude that:

$$Z^j(W, d_{\widetilde{S}}) \simeq Z^2(W, d_{\widetilde{S}}) \cdot E^i \qquad \text{for } j = 2i+2.$$

Proceeding in an analogous way, it is possible to prove also the second isomorphism concerning the space of cocycles $Z^j(W, d_{\widetilde{S}})$, for $j = 2i+3$.

Now we still have to prove the following isomorphism among spaces of coboundaries:

$$B^j(W, d_{\widetilde{S}}) \simeq B^4(W, d_{\widetilde{S}}) \cdot E^i \qquad \text{for } j = 2i+4.$$

Let ψ be an element in $B^j(W, d_{\widetilde{S}})$: then by definition there exists an element φ in $\mathcal{C}^{j-1}(W, d_{\widetilde{S}})$ such that $\psi = d_{\widetilde{S}}(\varphi)$. However, a generic cochain φ in $\mathcal{C}^k(W, d_{\widetilde{S}})$, with $k = 2i + 3$, is necessarily of the following form:

$$\varphi = [fC_1C_2C_3 + (g_1C_1 + g_2C_2 + g_3C_3)E] \cdot E^i$$

for some polynomials f, g_i in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, uniquely determined by the cochain φ itself. Thus:

$$\psi = d_{\widetilde{S}}(\varphi) = d_{\widetilde{S}}([fC_1C_2C_3 + (g_1C_1 + g_2C_2 + g_3C_3)E]) \cdot E^i.$$

So we conclude that each element in $B^j(W, d_{\widetilde{S}})$ can be seen in a unique way as an element of $B^4(W, d_{\widetilde{S}}) \cdot E^i$ and, conversely, each element of $B^4(W, d_{\widetilde{S}}) \cdot E^i$ determines a unique element of $B^j(W, d_{\widetilde{S}})$. Thus $B^j(W, d_{\widetilde{S}})$ and $B^4(W, d_{\widetilde{S}}) \cdot E^i$ coincide as sets and their algebraic structures turns out to be isomorphic. Once again, an analogous proof can be given for the case of coboundaries of degree $j = 2i + 3$.

Using the isomorphisms listed in (5.59), the statements in the proposition immediately follow:

- if $j = 2i + 3$, then

$$H^j(W, d_{\widetilde{S}}) = \frac{Z^j(W, d_{\widetilde{S}})}{B^j(W, d_{\widetilde{S}})} \simeq \frac{Z^3(W, d_{\widetilde{S}})}{B^3(W, d_{\widetilde{S}})} = H^3(W, d_{\widetilde{S}}) \ ;$$

- if $j = 2i + 4$, then

$$H^j(W, d_{\widetilde{S}}) = \frac{Z^j(W, d_{\widetilde{S}})}{B^j(W, d_{\widetilde{S}})} \simeq \frac{Z^4(W, d_{\widetilde{S}})}{B^4(W, d_{\widetilde{S}})} = H^4(W, d_{\widetilde{S}}) \ .$$

□

Therefore, for a full description of the BRST cohomology groups of the model it is enough to determine the cohomology groups $H^j(W, d_{\widetilde{S}})$ for $j = 0, \ldots, 4$. The computation of these cohomology groups is done in full detail in Appendix D. In the following theorem we summarize the obtained results.

Theorem 9. *The gauge-fixed BRST cohomology groups determined by the extended variety $(N, \widetilde{S})$, with N introduced in (5.31), while $\widetilde{S}$ is given in (5.41), and associated to a $U(2)$-matrix model are the following cohomology groups:*

- $H^0(W, d_{\widetilde{S}}) = \left\{\sum_{k=0}^{r} g_k(M_4)(M_1^2 + M_2^2 + M_3^2)^k,\ r \in \mathbb{N}_0,\ g_k \in \mathrm{Pol}_{\mathbb{R}}(M_4)\right\}$;
- $H^1(W, d_{\widetilde{S}}) = 0$;
- $H^2(W, d_{\widetilde{S}}) = K \oplus \mathrm{Pol}_{\mathbb{R}}(M_4)E$;
- $H^{2k}(W, d_{\widetilde{S}}) = \mathrm{Pol}_{\mathbb{R}}(M_4)E^k, \quad$ *for* $k \geq 2$;
- $H^{2k+1}(W, d_{\widetilde{S}}) = 0, \quad$ *for* $k \geq 1$;

with

$$K := \{f(M_1C_2C_3 - M_2C_1C_3 + M_3C_1C_2),\ f \in \mathrm{Pol}_{\mathbb{R}}(M_a)\}\,.$$

Proof. For the proof of this theorem we refer to Appendix D. □

5.3 Relation between BRST cohomology and Lie algebra cohomology

The main purpose of this section is to describe the BRST cohomology complex for the matrix model of degree $n = 2$ from a different perspective, by relating it to a generalization of a suitable Lie algebra cohomology complex.
More precisely, by the introduction of the new notions of a *generalized Lie algebra complex* and of a *symmetric cochain complex*, we are going to prove that the set of the generators for the BRST cohomology can be split into two parts, separating the real (bosonic) ghost field from the Grassmannian (fermionic) ones: the full BRST cohomology complex can then be described as a *shifted double complex* in this generalized Lie algebra cohomology setting, where the part concerning the real ghost fields is described using the notion of symmetric cochain complex, while the part regarding the Grassmannian ghost fields involves the generalized Lie algebra complex.
Having described this shifted double complex structure, its properties are analyzed and the relations with the BRST-cohomology complex are determined also at the level of the corresponding cohomology groups.

We mention that there have been earlier attempts to relate BRST cohomology complex to Lie algebra cohomology complex (see [43]). However, while in this thesis the BRST cohomology complex is our starting point and then the corresponding Lie algebra complex is defined in such a way as to coincide with what had already been constructed, in [43] the construction of the Lie algebra is the starting point, which induces the BRST operator.

Lie algebra cohomology was first introduced by Chevalley and Eilenberg [18]: for this reason it is also known as Chevalley-Eilenberg cohomology. Other classical references are [41] and [15]. In recent years, several attempts to generalize the usual notion of Lie algebra cohomology have been made: in this direction there is the work of Dubois-Violette and Landi, who adapt the notion of Lie algebra cohomology to the context of Hopf algebras [25].

5.3.1 Generalized Lie algebra cohomology

The main goal of this section is the introduction of a new notion of *generalized Lie algebra cohomology*. We first introduce the notion of a *module of degree p*, with $p \in \mathbb{Z}$. Then we define the *symmetric* and *the antisymmetric cohomology of a Lie algebra* $\mathfrak{g}$ over a module of degree p.
These two definitions are generalizations of the standard notion of Lie algebra cohomology in two directions: first of all, we include also the possibility of defining a cochain complex using symmetric maps on the Lie algebra, while the usual definition only considers alternating maps as cochains. Moreover, we generalize the cohomology theory considering as cochains maps that take value in a module of degree p, with $p \in \mathbb{Z}$.

In what follows, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 47. *Let* $\mathfrak{h}$ *be a vector space over the field* $\mathbb{K}$. *Then an* $\mathfrak{h}$-module of degree p *($p \in \mathbb{Z}$) is a pair*

$$(\{V_i\}_{i=1}^{n}, \{\alpha_i\}_{i=1}^{n-p}) \ ,$$

where

- ► $\{V_i\}_{i=1}^{n}$, $n \geqslant p$ *is a collection of* $\mathbb{K}$*-vector spaces,* $V = \bigoplus_{i=1}^{n} V_i$*;*
- ► $\{\alpha_i\}_{i=1}^{n-p}$ *is a collection of linear maps,*

$$\alpha_i : \mathfrak{h} \longrightarrow \mathrm{Lin}(V_i, V_{i+p}) \ .$$

Given a Lie algebra $\mathfrak{h}$ and a module of degree p over it, it is possible to introduce two notions of Lie algebra cohomology on the $\mathfrak{h}$-module, as stated in the next two definitions. However, since in one of the two situations we are not considering the Lie structure of the algebra, that definition will be stated in full generality where $\mathfrak{h}$ is simply a vector space over $\mathbb{K}$.

Definition 48. *Let $\mathfrak{h}$ be a vector space over the field $\mathbb{K}$ and let (V_i, α_i) be a $\mathfrak{h}$-module of degree p such that, for all $i = 1, \ldots, n-p$, the following condition is satisfied:*

$$\alpha_{i+p}(e_1) \circ \alpha_i(e_2) = -\alpha_{i+p}(e_2) \circ \alpha_i(e_1) \quad \forall e_1, e_2 \in \mathfrak{h}. \tag{5.60}$$

The symmetric cochain complex *is given by a pair $(\mathcal{C}^j_{sym}, \delta^j_{sym})$ where:*

- $\mathcal{C}^j_{sym}$ *denotes the set of all the symmetric cochains of order j, with $j \geqslant 0$. By definition,*

  $$\mathcal{C}^j_{sym}(\mathfrak{h}, V) := \bigoplus_{i=1}^{n} \mathcal{C}^{j,i}_{sym}(\mathfrak{h}, V_i) \ ,$$

  with

  $\mathcal{C}^{j,i}_{sym}(\mathfrak{h}, V_i) := \mathrm{Sym}^j(\mathfrak{h}, V_i) = \{\varphi : \mathfrak{h} \times \cdots \times \mathfrak{h} \to V_i,$ *j-linear symmetric map*$\}$.

  In particular, $\mathrm{Sym}^0(\mathfrak{h}, V_i) = V_i$.

- δ^j_{sym} *is the coboundary operator. By definition,*

  $$\delta^j_{sym} := \bigoplus_{i=1}^{n-p} \delta^{j,i}_{sym} \ ,$$

  with $\delta^{j,i}_{sym}$ a linear operator of degree 1,

  $$\delta^{j,i}_{sym} : \mathcal{C}^{j,i}_{sym}(\mathfrak{h}, V_i) \to \mathcal{C}^{j+1,i+p}_{sym}(\mathfrak{h}, V_{i+p}) \ ,$$

  such that, given $\varphi \in \mathcal{C}^{j,i}_{sym}(\mathfrak{h}, V_i)$ and $e_1, \ldots, e_{j+1} \in \mathfrak{h}$,

  $$\delta^{j,i}_{sym}(\varphi)[e_1, \ldots, e_{j+1}] := \sum_{n=1}^{j+1} \alpha_i(e_n)[\varphi(e_1, \ldots, \hat{e}_n, \ldots, e_{j+1})].$$

Remark 35
It is straightforward that, for any allowed value of i and j, both $\mathcal{C}^{j,i}_{sym}(\mathfrak{h}, V_i)$ and $\mathcal{C}^j_{sym}(\mathfrak{h}, V)$ have a natural structure of a vector space. Moreover, the operator $\delta^{j,i}_{sym}$ is well defined since it clearly sends j-linear and symmetric maps on $\mathfrak{h}$ to $(j+1)$-linear and symmetric maps on $\mathfrak{h}$.

Proposition 13
Let $\mathfrak{h}$ be a vector space over the field $\mathbb{K}$ and let (V_i, α_i) be an $\mathfrak{h}$-module of degree p such that the condition (5.60) is satisfied for all $i = 1, \ldots, n-p$. Then the pair $(\mathcal{C}^j_{sym}(\mathfrak{h}, V), \delta^j_{sym})$ defined above is a cochain complex.

Proof. In view of the previous remark, the only condition that needs to be checked is that δ^j_{sym} is a coboundary operator, i.e.,

$$\delta^{j+1}_{sym} \circ \delta^j_{sym} = 0 \ ,$$

for all $j \geqslant 0$. Equivalently, we can check that

$$\delta^{j+1,i+p}_{sym} \circ \delta^{j,i}_{sym} = 0 \ ,$$

for all $j \geqslant 0$ and for all $i = 1, \ldots, n-p$. It is immediate that the composition of maps in which we are interested is automatically zero if $i \geq n - 2p + 1$. Therefore, we can restrict ourselves to the case in which $i = 1, \ldots, n - 2p$.

Let φ be a fixed element in $\mathcal{C}^{j,i}_{sym}(\mathfrak{h}, V_i)$ and let $e_1, \ldots, e_{j+2}$ be fixed elements of $\mathfrak{h}$. Thus:

$$\begin{aligned}
&\delta^{j+1,i+p}_{sym}(\delta^{j,i}_{sym}(\varphi))[e_1, \ldots, e_{j+2}] \\
&= \textstyle\sum_{l<m}(\alpha_{i+p}(e_l) \circ \alpha_i(e_m))[\varphi(e_1, .., \hat{e}_l, .., \hat{e}_m, .., e_{j+2})] \\
&+ \textstyle\sum_{l>m}(\alpha_{i+p}(e_l) \circ \alpha_i(e_m))[\varphi(e_1, .., \hat{e}_m, .., \hat{e}_l, .., e_{j+2})] \\
&= \textstyle\sum_{l<m}(\alpha_{i+p}(e_l) \circ \alpha_i(e_m) + \alpha_{i+p}(e_m) \circ \alpha_i(e_l))[\varphi(e_1, .., \hat{e}_l, \hat{e}_m, .., e_{j+2})] = 0 \ ,
\end{aligned}$$

where, in the previous expression, we have used the linearity of the maps α_i and the hypothesis (5.60). Therefore, we conclude that the operator δ^j_{sym} is a coboundary operator and the pair $(\mathcal{C}^\bullet_{sym}(\mathfrak{h}, V_i), \delta^\bullet_{sym})$ gives a cochain complex. □

As announced before, given a Lie algebra and a module of order p over it, it is possible to introduce also another type of cochain complex, defined by antilinear maps, as stated in the following definition.

Definition 49. *Let $\mathfrak{h}$ be a Lie algebra over $\mathbb{K}$ and let (V_i, α_i), $i = 1, \ldots, n$, be a module of order p on $\mathfrak{h}$, with $p \in \mathbb{Z}$. Moreover, let $\{\beta_i\}$, $i = 1, \ldots, n-p$, be a collection of linear maps with $\beta_i : V_i \to V_{i+p}$ such that the two following conditions are satisfied:*

1. *the following diagram commutes for all $i = 1, \ldots, n-p$ and for all $e \in \mathfrak{h}$:*

$$\begin{array}{ccc} V_i & \xrightarrow{\alpha_i(e)} & V_{i+p} \\ \Big\downarrow{\scriptstyle \beta_i} & & \Big\downarrow{\scriptstyle \beta_{i+p}} \\ V_{i+p} & \xrightarrow{\alpha_{i+p}(e)} & V_{i+2p} \end{array}$$

In other words, for all $i = 1, \ldots, n-p$ and for all elements e in $\mathfrak{h}$ one has:

$$\alpha_{i+p}(e) \circ \beta_i - \beta_{i+p} \circ \alpha_i(e) = 0 \ ;$$

2. *for all $e_1, e_2 \in \mathfrak{h}$ and for all $i = 1, \ldots, n-p$,*

$$\alpha_{i+p}(e_1) \circ \alpha_i(e_2) - \alpha_{i+p}(e_2) \circ \alpha_i(e_1) = \beta_{i+p}(\alpha_i([e_1, e_2])) \tag{5.61}$$

   where $[-,-]$ denotes the Lie algebra bracket on $\mathfrak{h}$.

Then the generalized Lie algebra cochain complex *of $\mathfrak{h}$ over the module of order p (V_i, α_i) is given by a pair $(\mathcal{C}^j_{Lie}, \delta^j_{Lie})$, where:*

- ► $\mathcal{C}^j_{Lie}(\mathfrak{h}, V)$ *is the set of all the Lie algebra cochains of order j, with $j \geqslant 0$. By definition:*

$$\mathcal{C}^j_{Lie}(\mathfrak{h}, V) := \bigoplus_{i=1}^{n} \mathcal{C}^{j,i}_{Lie}(\mathfrak{h}, V_i) \ ,$$

  with
  $\mathcal{C}^{j,i}_{Lie}(\mathfrak{h}, V_i) := \mathrm{Alt}^j(\mathfrak{h}, V_i) = \{\varphi : \mathfrak{h} \times \cdots \times \mathfrak{h} \to V_i,\ j\text{-linear alternating map}\}$.
  In particular, $\mathrm{Alt}^0(\mathfrak{h}, V_i) = V_i$. Moreover, $\mathcal{C}^j_{Lie} \simeq \{0\}$, for $j > \dim_{\mathbb{K}}(\mathfrak{h})$.

- ► δ^j_{Lie} *is the coboundary operator. By definition,*

$$\delta^j_{Lie} := \bigoplus_{i=1}^{n-p} \delta^{j,i}_{Lie} \ ,$$

  with $\delta^{j,i}_{Lie}$ a linear operator of degree 1,

$$\delta^{j,i}_{Lie} : \mathcal{C}^{j,i}_{Lie}(\mathfrak{h}, V_i) \to \mathcal{C}^{j+1,i+p}_{Lie}(\mathfrak{h}, V_{i+p}) \ ,$$

  such that, given $\varphi \in \mathcal{C}^{j,i}_{Lie}(\mathfrak{h}, V_i)$ and $e_1, \ldots, e_{j+1} \in \mathfrak{h}$,

$$\begin{aligned} \delta^{j,i}_{Lie}(\varphi)[e_1, .., e_{j+1}] \quad &:= \textstyle\sum_{r=1}^{j+1} (-1)^{r+1} \alpha_i(e_r)[\varphi(e_1, .., \hat{e}_r, .., e_{j+1})] \\ &\quad + \textstyle\sum_{r<s} (-1)^{r+s+1} \beta_i(\varphi([e_r, e_s], .., \hat{e}_r, \hat{e}_s, .., e_{j+1})) \ . \end{aligned}$$

Remark 36
The definition of the algebra cochain complex introduced above can be seen as a generalization of the usual notion of Lie algebra cochain complex, as introduced by Chevalley and Eilenberg [18] and later developed by Hochschild and Serre [41]. In fact, it is possible to recover the usual definition of a Lie algebra cochain complex by considering $p = 0$ and $n = 1$.
Therefore, with this choice:

- the module of order p given by the pair (V_i, α_i), with $i = 1$ turns out to be simply a module $V_1 := M$ over the Lie algebra $\mathfrak{h}$;
- the corresponding map $\alpha_1 := \alpha$ is then a linear map from $\mathfrak{h}$ with value in $\mathrm{Lin}(M, M)$;
- there is only one map β: to recover the usual definition for the Lie algebra complex we need to consider the map β given by the identity on the module M.

It is straightforward to check that the first condition required in Definition 49 turns out to be trivial in this setting while, imposing the second condition, we are requiring the map α to be a Lie algebra homomorphism between $\mathfrak{h}$ and $\mathrm{Lin}(M, M)$. With these choices, the cochain complex introduced in Definition 49 coincides with the standard definition of a Lie algebra cochain complex.
At this point we would like also to report that, after the discovery of Chevalley and Eilenberg, the notion of Lie algebra cohomology has already been investigated in the context of Lie superalgebras first by Tanaka (see [54]) and then by Scheunert and Zhang (see [50]). Indeed in their generalization they also took into account the possibility of considering symmetric maps defined on the Lie superalgebra. In our definitions above we generalize not only the type of functions considered but also the target space, introducing the notion of module of degree p.

Proposition 14
Let $\mathfrak{h}$, (V_i, α_i), $i = 1, \ldots, n$ and $\{\beta_i\}$, $i = 1, \ldots, n - p$, be as in Definition 49. Then the pair $(\mathcal{C}^j_{Lie}(\mathfrak{h}, V), \delta^j_{Lie})$, with $j = 0, \ldots, \dim_{\mathbb{K}}(\mathfrak{h})$ is a cochain complex.

Proof. We need to check that δ^j_{Lie} is well defined as an operator from $\mathcal{C}^j_{Lie}$ to $\mathcal{C}^{j+1}_{Lie}$ and that it satisfies the coboundary condition:

$$\delta^{j+1}_{Lie} \circ \delta^j_{Lie} = 0.$$

Since the maps α_i and β_i are linear, δ^j_{Lie} is linear and well defined, sending antisymmetric maps in antisymmetric maps. To prove that it gives a coboundary operator, we have to prove that it is a differential: let us fix a generic element φ in $\mathcal{C}^j_{Lie}(\mathfrak{h}, V)$ and let $e_1, \ldots, e_{j+2}$ be generic elements in $\mathfrak{h}$.

Then the following identity holds:

$$\begin{aligned}
&\delta^{j+1,i+p}_{Lie}\left(\delta^{j,i}_{Lie}(\varphi)\right)\Big|_{[e_1,\ldots,e_{j+2}]} \\
&\quad = \textstyle\sum_{l=1}^{j+2}(-1)^{l+1}\ \alpha_{i+p}(e_l)\left[\delta^{j,i}_{Lie}(\varphi)(e_1,\ldots,\hat{e}_l,\ldots,e_{j+2})\right] \\
&\quad + \textstyle\sum_{m<n}(-1)^{m+n+1}\beta_{i+p}\left[\delta^{j,i}_{Lie}(\varphi)([e_m,e_n],\ldots,\hat{e}_m,\hat{e}_n,\ldots,e_{j+2})\right].
\end{aligned} \tag{5.62}$$

To have a simpler notation we denote by $[A]$ the first sum present in Equation (5.62) while the second sum will be denoted with the symbol $[B]$:

$$\begin{aligned}
&[A] := \textstyle\sum_{l=1}^{j+2}(-1)^{l+1}\ \alpha_{i+p}(e_l)\left[\delta^{j,i}_{Lie}(\varphi)(e_1,\ldots,\hat{e}_l,\ldots,e_{j+2})\right]; \\
&[B] := \textstyle\sum_{m<n}(-1)^{m+n+1}\beta_{i+p}\left[\delta^{j,i}_{Lie}(\varphi)([e_m,e_n],\ldots,\hat{e}_m,\hat{e}_n,\ldots,e_{j+2})\right].
\end{aligned}$$

Let l be a fixed index: by $[A]_l$ we denotes the summand in $[A]$ corresponding to this index l. Analogously, given a fix pair of indices m and n, by $[B]_{m,n}(\beta_i)$ we denote the part of the sum $[B]$ involving the map β_i.
Therefore, considering the sum on all possible indices m and n, relabeling the indices and using the linearity of the maps β_i, β_{i+p} and φ, together with the antisymmetry of the Lie brackets, the previous expression can be rewritten as

$$\begin{aligned}
&\textstyle\sum_{m,n}([B]_{m,n}(\beta_i)) \\
&\qquad = \textstyle\sum_{z<m<n}(-1)^{z+m+n}\beta_{i+p}\left(\beta_i\left[\varphi(w,\hat{e}_z,\ldots\hat{e}_m,\ldots,\hat{e}_n,\ldots)\right]\right) \\
&\qquad + \textstyle\sum_{u<v<m<n}(-1)^{m+n+u+v}\beta_{i+p}\left(\beta_i\left[\varphi([e_u,e_v],[e_m,e_n],\ldots)\right.\right. \\
&\qquad - \left.\left.\varphi([e_u,e_v],[e_m,e_n],\ldots)\right]\right),
\end{aligned} \tag{5.63}$$

where we write

$$w := [[e_m,e_n],e_z] + [[e_n,e_z],e_m] + [[e_z,e_m],e_n].$$

Since w is zero by the Jacobi identity and φ is a linear map by definition, it follows that the first sum in (5.63) is zero, whereas the second is trivially zero. Therefore, to conclude the computation, we only need to take into account the terms $[A]_l$ and $[B]_{m,n}(\alpha_i)$. Changing the labeling of the indices and using linearity of all maps, the sum of $[A]_l$ for all the possible indices l together with the sum of $[B]_{m,n}(\alpha_i)$ for all the pairs of indices m and n can be rewritten as follows:

$$
\begin{aligned}
&[A] + [B] \\
&\quad = \textstyle\sum_{r<l} (-1)^{r+l} \left[\alpha_{i+p}(e_l) \circ \alpha_i(e_r) - \alpha_{i+p}(e_r) \circ \alpha_i(e_l)\right. \\
&\qquad \left.-\beta_{i+p}(\alpha_i([e_r, e_l]))\right] [\varphi(\hat{e}_r, \ldots, \hat{e}_l)] \\
&\qquad + \textstyle\sum_{s<t<l} (-1)^{s+t+l} \left(\alpha_{i+p}(e_l) \circ \beta_i - \beta_{i+p} \circ \alpha_i(e_l)\right) [\varphi([e_s, e_t], \hat{e}_s, \hat{e}_t, \hat{e}_l)] \\
&\qquad + \textstyle\sum_{s<l<t} (-1)^{s+t+l} \left(-\alpha_{i+p}(e_l) \circ \beta_i + \beta_{i+p} \circ \alpha_i(e_l)\right) [\varphi([e_s, e_t], \hat{e}_s, \hat{e}_l, \hat{e}_t)] \\
&\qquad + \textstyle\sum_{l<s<t} (-1)^{s+t+l} \left(\alpha_{i+p}(e_l) \circ \beta_i - \beta_{i+p} \circ \alpha_i(e_l)\right) [\varphi([e_s, e_t], \hat{e}_l, \hat{e}_s, \hat{e}_t)] \,.
\end{aligned}
$$

Using the relations imposed among the maps β_i and α_i in the definition of a generalized Lie algebra cochain complex, we finally conclude that the sum in the previous expression is equal to zero.
Thus for all cochains φ,

$$\delta_{Lie}^{j+1,i+p}\left(\delta_{Lie}^{j,i}(\varphi)\right) = 0.$$

Therefore, the operator δ_{Lie} defines a coboundary operator and the pair $(\mathcal{C}_{Lie}^{\bullet}, \delta_{Lie}^{\bullet})$ is a cohomology complex. □

5.3.2 Relation between BRST cochain complex and generalized Lie algebra cochain complex

First of all, let us introduce some notation:

- $\mathfrak{g}$ denotes the Lie algebra generated on $\mathbb{R}$ by the matrices $i\sigma_1$, $i\sigma_2$ and $i\sigma_3$, where σ_1, σ_2, σ_3 are the Pauli matrices listed in Equation (5.3), seen as the dual of the ghost fields C_1, C_2, C_3. Therefore, as Lie algebra, $\mathfrak{g} \simeq su(2)$.
- $\mathfrak{h}$ denotes the Lie algebra generated on $\mathbb{R}$ by τ, defined as the dual of the ghost field E. Thus $\mathfrak{h} \simeq u(1)$.

Now we want to introduce the structure of a module of order $p = 1$ over $\mathfrak{h}$. This kind of structure is given by a pair $(\{V_i\}_{i=1}^{n}, \{\alpha_i\}_{i=1}^{n-1})$ satisfying some conditions, stated in Definition 47.

Let us then consider:

- as collection of vector spaces $\{V_i\}_{i=1}^4$ the cochain spaces of the Lie algebra cohomology defined by the Lie algebra $\mathfrak{g}$ over the module $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$V_i = \mathcal{C}^{i-1}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$$

where $i = 1, \dots, 4$. Here we are using the standard notion of Lie algebra cohomology. The $\mathfrak{g}$-module structure of $\mathrm{Pol}_{\mathbb{R}}(M_a)$ is given by a map

$$\omega : \mathfrak{g} \to Lin(\mathrm{Pol}_{\mathbb{R}}(M_a)),$$

defined as follows, for an element x in $\mathfrak{g}$, with $x = \sum_i x_i \sigma_i$, and a generic polynomial f in $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$\begin{aligned} \omega(x)(f) = \quad & (\partial_{M_1} f)(-M_3 x_2 + M_2 x_3) + (\partial_{M_2} f)(M_3 x_1 - M_1 x_3) \\ & +(\partial_{M_3} f)(-M_2 x_1 + M_1 x_2). \end{aligned} \tag{5.64}$$

The linear maps $\{\alpha_i\}_{i=1}^3$ are defined as follows:

- $\alpha_1 : \mathfrak{h} \longrightarrow \mathrm{Lin}\left(\mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)),\ \ \mathrm{Pol}_{\mathbb{R}}(M_a)\right)$.

Let $A\tau$ be a generic element of $\mathfrak{h}$, with $A \in \mathbb{R}$, and let φ be a generic cochain of degree 1, $\varphi \in \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$. Then:

$$\varphi = f_1 C_1 + f_2 C_2 + f_3 C_3,$$

with f_1, f_2 and f_3 polynomials in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Thus the linear map α_1 is defined as follows:

$$\alpha_1(A\tau)(\varphi) := (f_1 M_1 + f_2 M_2 + f_3 M_3)A.$$

Equivalently, we can define the map $\alpha_1(A\tau)$ on the generators C_i as follows:

$$\begin{cases} \alpha(A\tau)(C_1) = M_1 A \\ \alpha(A\tau)(C_2) = M_2 A \\ \alpha(A\tau)(C_3) = M_3 A \ . \end{cases}$$

Extending this definition by linearity on $\mathrm{Pol}_{\mathbb{R}}(M_a)$, we obtain the definition of α_1 on the full space $\mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$.

► $\alpha_2 : \mathfrak{h} \longrightarrow \text{Lin}\left(\mathcal{C}^2_{Lie}(\mathfrak{g}, \text{Pol}_{\mathbb{R}}(M_a)),\ \mathcal{C}^1_{Lie}(\mathfrak{g}, \text{Pol}_{\mathbb{R}}(M_a))\right).$

Let $A\tau$ be a generic element in $\mathfrak{h}$ and let φ be a generic cochain of degree 2,

$$\varphi = f_{12}C_1C_2 + f_{13}C_1C_3 + f_{23}C_2C_3 \tag{5.65}$$

with f_{12}, f_{13}, f_{23} in $\text{Pol}_{\mathbb{R}}(M_a)$. We then define the map α_2 as follows:

$$\alpha_2(A\tau)(\varphi) := \sum_{i<j} f_{ij}(M_iC_j - M_jC_i)A,$$

with $i, j = 1, 2, 3$.

Equivalently, the map $\alpha_2(A\tau)$ can be obtained by extending the map $\alpha_1(A\tau)$ on pairs of generators by requiring that it acts as a graded derivation, i.e.,

$$(\alpha_2(A\tau))(C_iC_j) = \alpha_1(A\tau)|_{C_i}C_j - C_i\alpha_1(A\tau)|_{C_j}.$$

► $\alpha_3 : \mathfrak{h} \longrightarrow \text{Lin}\left(\mathcal{C}^3_{Lie}(\mathfrak{g}, \text{Pol}_{\mathbb{R}}(M_a)),\ \mathcal{C}^2_{Lie}(\mathfrak{g}, \text{Pol}_{\mathbb{R}}(M_a))\right).$

Given a generic element $A\tau$ in $\mathfrak{h}$ and a generic cochain φ of ghost degree 3, $\varphi = fC_1C_2C_3$, with f in $\text{Pol}_{\mathbb{R}}(M_a)$, we define the map α_3 as follows:

$$\alpha_3(A\tau)(\varphi) := f(M_1C_2C_3 - M_2C_1C_3 + M_3C_1C_2)A.$$

Once again, also the map $\alpha_3(A\tau)$ can be obtained from $\alpha_1(A\tau)$ requiring that it acts as a graded derivation:

$$(\alpha_3(A\tau))(C_1C_2C_3) = \alpha_1(A\tau)|_{C_1}C_2C_3 - C_1\alpha_1(A\tau)|_{C_2}C_3 + C_1C_2\alpha_1(A\tau)|_{C_3}.$$

A comparison with the definition of the coboundary operator for the BRST cohomology complex of the model suggests that the maps α_i are defined so as to correspond to the part of the action of the coboundary operator on the generators C_i involving the ghost field E.

Proposition 15
The pair $(\{V_i\}_{i=1}^4, \{\alpha_i\}_{i=1}^3)$ introduced above defines a module structure of order $p = 1$ on $\mathfrak{h}$.

Proof. It is immediate that the maps α_i defined above are well defined and $\mathfrak{h}$-linear maps. Thus we have only to check that

$$\alpha_{i+1}(A\tau) \circ \alpha_i(B\tau) + \alpha_{i+1}(B\tau) \circ \alpha_i(A\tau) = 0,$$

for each pair of elements $A\tau$, $B\tau$ in $\mathfrak{h}$ and for $i = 1, 2$. These conditions can be immediately verified by a direct computation. □

Remark 37
The explicit computation in the proof of the previous proposition yields stronger conditions than those necessary for a module structure. In fact, the following two conditions turn out to be satisfied for all pairs of elements $A\tau$, $B\tau$ in $\mathfrak{h}$ and for all φ in $\mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$, ψ in $\mathcal{C}^3_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$:

$$\alpha_1(A\tau)(\alpha_2(B\tau)|_{\varphi}) = 0 \; ; \qquad\qquad \alpha_2(A\tau)(\alpha_3(B\tau)|_{\psi}) = 0.$$

Therefore, the part of the BRST coboundary operator acting on the generators C_i that involves only the ghost field E, already it turns out to be a differential operator.

Notation:

- $(\mathcal{C}^\bullet(W, d_{\widetilde{S}}), d_{\widetilde{S}})$ denotes the BRST cohomology complex for the matrix model of degree $n = 2$, as defined in Section 3.5;
- $\mathcal{C}^\bullet_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$ indicates the Lie algebra cohomology complex of $\mathfrak{g}$ over the module $\mathrm{Pol}_{\mathbb{R}}(M_a)$, where the $\mathfrak{g}$-module structure of $\mathrm{Pol}_{\mathbb{R}}(M_a)$ is given by the linear map ω defined in (5.64);
- $\mathcal{C}^\bullet_{sym}(\mathfrak{h}, \mathcal{C}^\bullet_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$ is the symmetric cohomology complex of the Lie algebra $\mathfrak{h}$ over $\mathcal{C}^\bullet_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$, seen as a $\mathfrak{h}$-module of order $p = 1$ with the structure described in Proposition 15.

Theorem 10. *In the above notation, there is a one to one correspondence, both at the level of cochain spaces and of coboundary operators, between the BRST cohomology complex and the double cohomology complex* $\mathcal{C}^\bullet_{sym}(\mathfrak{h}, \mathcal{C}^\bullet_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$. *More precisely we have:*

1. $\mathcal{C}^k(W, d_{\widetilde{S}}) = \bigoplus_{2i+j=k} \mathcal{C}^i_{sym}(\mathfrak{h}, \mathcal{C}^j_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$, *with* i *in* $\mathbb{N}_0$ *and* $j = 0, \ldots, 3$;
2. $d_{\widetilde{S}} = \bigoplus_{2i+j=k} d^{i,j}_{\widetilde{S}}$, *with* i *in* $\mathbb{N}_0$ *and* $j = 0, \ldots, 3$;

$$d^{i,j}_{\widetilde{S}} := \delta^{i,j}_{sym} \oplus (Id_{\mathfrak{h}^i} \otimes \tfrac{1}{j+1}\delta^{j}_{Lie})$$

$$\delta^{i,j}_{sym} \equiv Id_{\mathfrak{h}^i} \otimes \alpha_j : \mathcal{C}^i_{sym}(\mathfrak{h}, \mathcal{C}^j_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \to \mathcal{C}^{i+1}_{sym}(\mathfrak{h}, \mathcal{C}^{j-1}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))).$$

Remark 38
Before proving the theorem we notice that, with this approach, the different aspects that characterize the ghost fields, such as their parity or their ghost degree, have a natural translation in terms of properties of the double complex. More precisely:

- The fact that the ghost fields C_i have ghost degree 1 while the ghost field E has ghost degree 2 is reflected in the different weight given to the indices of the two complexes in the setting of the generalized Lie algebra complex. More explicitly, the reason why the sum $2i + j = k$ appears in part 1. of the theorem is that the index i refers to the symmetric cohomology complex of the Lie algebra $\mathfrak{h}$, generated by the dual of the ghost field E, whose ghost degree is 2, while the index j refers to the Lie algebra cohomology complex defined by the Lie algebra $\mathfrak{g}$, generated by the dual of the generators C_i, whose ghost degree is 1.

- The different parities of the generators, i.e., the fact that, while the ghost fields C_i are given by Grassmannian variables, the ghost field E is a real variable, are related to the type of cohomology considered for the double complex. More precisely, for the part involving the algebra $\mathfrak{g}$, we consider the Lie algebra cohomology while, for the part concerning the Lie algebra $\mathfrak{h}$, we consider the symmetric cohomology complex: therefore, in the first case we have cochains defined as antilinear maps while in the second case the cochains are given by symmetric maps. This difference reflects the properties of the variables C_i to be antisymmetric, whereas the variable E commutes with all the other generators.

Proof. As far as the cochains are concerned, the correspondence stated in 1. follows immediately once we notice that the condition $2i + j = k$ is due to the fact that the ghost field E has ghost degree 2 while the ghost fields C_i have ghost degree 1. Second, the reason why for the Lie algebra $\mathfrak{h}$ we consider the symmetric cohomology, while for the Lie algebra $\mathfrak{g}$ the Lie algebra cohomology consist of the ghost field E is real, while the C_i are Grassmannian ghost fields. Looking at the cochains as maps, this implies that the part involving the ghost fields C_i gives an antisymmetric map while, considering the part involving the ghost field E, we have a symmetric map. Thus both ghost degree and parity can be translated in a precise condition on the corresponding cochain complex in the generalized Lie algebra cohomology.

To check the condition for the coboundary operator, let us explicitly write the coboundary operator δ_{Lie} for our particular case. The standard definition of a Lie algebra cohomology can be recovered from our definition of generalized Lie algebra cohomology by considering $p = 0$ and $n = 1$. Therefore, instead of a module of a generic order p given by a pair (V_i, α_i), we are simply considering a module M over the Lie algebra $\mathfrak{g}$ and ω is a linear map from $\mathfrak{g}$ with values in $\mathrm{Lin}(M, M)$. The map β in this case is simply the identity on the module M.

- $\delta^0_{Lie} : \mathcal{C}^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \to \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$.

  Given a generic polynomial f in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ and an element x in $\mathfrak{g}$, then:

$$\begin{aligned} \delta^0_{Lie}(f)[x] &= \omega(x)[f] \\ &= [(\partial_{M_1} f)(-M_3C_2 + M_2C_3) + (\partial_{M_2} f)(M_3C_1 - M_1C_3) \\ &\quad + (\partial_{M_3} f)(-M_2C_1 + M_1C_2)]\big|_x . \end{aligned}$$

- $\delta^1_{Lie} : \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \to \mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$.

  Given a pair of elements x, y in $\mathfrak{g}$ and a cochain φ in $\mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$,

$$\varphi = f_1C_1 + f_2C_2 + f_3C_3,$$

  for some polynomials f_1, f_2, f_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, we have:

$$\begin{aligned} &\delta^1_{Lie}(\varphi)|_{(x,y)} \\ &= \omega(x)[\varphi(y)] - \omega(y)[\varphi(x)] + \varphi([x,y]) \\ &= 2\textstyle\sum_{i=1}^3 [(\partial_{M_1} f_i)(-M_3C_2 + M_2C_3)C_i + (\partial_{M_2} f_i)(M_3C_1 - M_1C_3)C_i \\ &\ +(\partial_{M_3} f_i)(-M_2C_1 + M_1C_2)C_i + 2\,(f_1C_2C_3 - f_2C_1C_3 + f_3C_1C_2)]\big|_{(x,y)} . \end{aligned}$$

- $\delta^2_{Lie} : \mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \to \mathcal{C}^3_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$.

  Given x, y, z in $\mathfrak{g}$ and φ in $\mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$,

$$\varphi = g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3$$

with g_{12}, g_{13}, g_{23} in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, then

$$\begin{aligned}
&\delta^2_{Lie}(\varphi)[x,y,z] \\
&\quad = \omega(x)[\varphi(y,z)] - \omega(y)[\varphi(x,z)] + \omega(z)[\varphi(x,y)] + \varphi([x,y],z) \\
&\qquad -\varphi([x,z],y) + \varphi([y,z],x) \\
&\quad = 3\left[\textstyle\sum_{i<j}((\partial_{M_1}g_{ij})(-M_3C_2 + M_2C_3)C_iC_j + (\partial_{M_2}g_{ij})(M_3C_1\right. \\
&\qquad -M_1C_3)C_iC_j + \; (\partial_{M_3}g_{ij})(-M_2C_1 + M_1C_2)C_iC_j)]\big|_{(x,y,z)}\,.
\end{aligned}$$

► $\delta^3_{Lie} : \mathcal{C}^3_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \to \{0\}\,.$

Thus the map δ^3_{Lie} is necessarily zero.

By direct comparison, it is then possible to check that also the statement 2. regarding the relation among the coboundary operators holds: in fact $\delta^\bullet_{sym}$ describes the part of the BRST coboundary operator involving the generator E, while the operator $\delta^\bullet_{Lie}$ contains the part of the BRST coboundary operator involving the generators C_i. Thus the statement is proved. □

Remark 39

It is natural to conjecture the emergence of an analogous structure also for the case of an $U(n)$-matrix model: indeed, in this general context, where we have to introduce ghost fields of ghost degree higher than 2 (see Section 6), we expect to find a multi-complex structure, where the generalized Lie algebra complex and the symmetric complex alternate, as well as the parity of the ghost fields alternates from degree to degree. Moreover, to determine the degree of a cochain complex, we also expect to find a weighted sum of the indices of the different complexes, which takes into account the ghost degree of the generators considered.

Now that we have described the relation between the BRST cohomology complex of a $U(2)$-matrix model and a particular complex in the generalized Lie algebra cohomology, we want to determine the effect of this relation at the level of cochain complexes on the level of the corresponding cohomology groups. But first, we look more closely the generalized Lie algebra cohomology complex introduced above: as proved in the following proposition, in fact there are precise relations connecting the two different coboundary operators involved in its definition.
To simplify the notation, we use the symbol $\delta^{i,j}$ to denote the coboundary operator $\delta^{i,j}_{sym}$, while the operator $Id_{\mathfrak{h}^i} \otimes \frac{1}{j+1}\delta^j_{Lie}$ is denoted by the symbol $d^{i,j}$.

Proposition 16
Let $\left\{\mathcal{C}^{\bullet}_{sym}(\mathfrak{h}, \mathcal{C}^{\bullet}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))), \delta^{\bullet} \oplus d^{\bullet}\right\}$ be the cochain complex described in Theorem 10. Then the coboundary operators satisfy the following relations for all $k \geq 0$:

1. $\delta^{k,1} \circ d^{k,0} = 0$,
2. $d^{k+1,1} \circ \delta^{k,2} = -\delta^{k,3} \circ d^{k,2}$,
3. $d^{k+1,2} \circ \delta^{k,3} = 0$,
4. $d^{k+1,0} \circ \delta^{k,1} = -\delta^{k,2} \circ d^{k,1}$.

The previous relations can be summarized as follows:

$$d^{k+1,i-1} \circ \delta^{k,i} = -\delta^{k,i+1} \circ d^{k,i}$$

with $k \geq 0$ and $i = 0, \ldots, 3$, recalling that by definition $\delta^{k,0} = 0$ for all non-negative values of k and $d^{k,i} = 0$ for negative values of i and if $i > 3$.

Proof. Recalling the correspondence in Theorem 10 between the coboundary operators, we have that:

$$\begin{aligned} d^{2k}_{\widetilde{S}} &= d^{k,0} \oplus [\delta^{k-1,2} \oplus d^{k,2}] \\ d^{2k+1}_{\widetilde{S}} &= [\delta^{k,1} \oplus d^{k,1}] \oplus \delta^{k-1,3}. \end{aligned}$$

Since $d^{\bullet}_{\widetilde{S}}$, $\delta^{\bullet}$ and $d^{\bullet}$ are all coboundary operators, we deduce that

$$d^{2k+1}_{\widetilde{S}} \circ d^{2k}_{\widetilde{S}} = \left[\delta^{k,1} \circ d^{k,0}\right] \oplus \left[d^{k,1} \circ \delta^{k-1,2} \oplus \delta^{k-1,3} \circ d^{k-1,2}\right] = 0.$$

Therefore, since the first term in the previous equation takes values in $\mathcal{C}^{k+1,0}$ while the second part takes values in $\mathcal{C}^{k,2}$, we deduce that the relations 1. and 2. hold. By a similar computation, this time considering the composition $d^{2k+2}_{\widetilde{S}} \circ d^{2k+1}_{\widetilde{S}}$, we find that also the relations 3. and 4. among the coboundary operators are satisfied. □

Definition 50. *Let $\mathcal{C}^{i,j}$ be a family of cochain spaces, denoted using two indices, and let d and δ be two coboundary operators with:*

$$d^{i,j} : \mathcal{C}^{i,j} \longrightarrow \mathcal{C}^{i,j+1} \qquad \delta^{i,j} : \mathcal{C}^{i,j} \longrightarrow \mathcal{C}^{i+1,j}.$$

Then the structure defined by $(\mathcal{C}^{\bullet,\bullet}, d, \delta)$ is called a double cochain complex *if the following conditions are satisfied, for each pair of indices (i, j):*

1. $d^{i,j+1} \circ d^{i,j} = 0$;
2. $\delta^{i+1,j} \circ \delta^{i,j} = 0$;
3. $d^{i+1,j} \circ \delta^{i,j} + \delta^{i,j+1} \circ d^{i,j} = 0$.

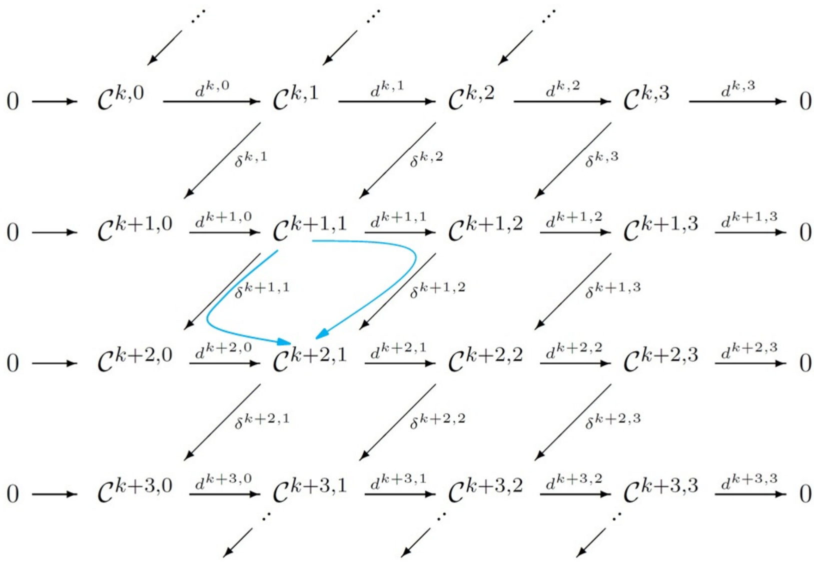


Figure 5.1: In this figure it is depicted the *shifted double complex structure* that appears describing the BRST cohomology complex in terms of the generalized Lie algebra cohomology complex. The reason why there is a shift in the first indices has to be searched in the different ghost degree of the generators: indeed, while the first index in the double complex refers to the dual of the ghost field E, which is of ghost degree 2, the second index describes the contribution coming from the ghost fields C_i, which have ghost degree 1.

Remark 40
Comparing Definition 50 and the properties of the coboundary operators proved in Proposition 16, it is immediately clear that the complex we are analyzing can be seen as a particular kind of double complex: in fact, the structure we consider satisfies the conditions required for a double complex up to a shift in the indices.

For this reason in the following we refer to it using the terminology *shifted double complex* (see Figure 5.1).

Remark 41
To describe the BRST cohomology complex for the model of degree $n = 2$ it was enough to consider the symmetric complex over a module structure given by a standard Lie algebra cohomology complex: we did not need to consider a generalized Lie algebra cohomology complex. This is due to the fact that the minimal resolution for the model in this case leads to the introduction only of Grassmannian fields of ghost degree 1. In the case in which the extended configuration space contains Grassmannian ghost fields of higher ghost degree we would have been forced to consider a generalized Lie algebra complex over a module structure given by a suitable symmetric complex.
Furthermore, for a resolution with ghost fields of maximum ghost degree 2, we have to consider a shifted double complex to rewrite the BRST cohomology group in terms of this generalized notion of Lie algebra complex. However, if we had a resolution containing generators with degree higher than 2, we would find a multicomplex, where the order of the multicomplex would coincide with the maximum ghost degree appearing in the minimal resolution.

5.3.3 Relation between the cohomology groups

The aim of this section is to identify what the relationship we just indicated at the level of cochain complex between the BRST cohomology theory and a shifted double complex defined by a suitable generalization of a Lie algebra complex implies at the level of the corresponding cohomology groups. The notation use in this section is the same that have been already introduced in the previous section.

Theorem 11. *Let $(\mathcal{C}^\bullet(W, d_{\widetilde{S}}), d_{\widetilde{S}})$ be the BRST cohomology complex for the matrix model of degree $n = 2$ while $\mathcal{C}^\bullet_{sym}(\mathfrak{h}, \mathcal{C}^\bullet_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$ is the double complex analyzed in Theorem 10. Then there exist the following relations at the level of cohomology groups:*

- $H^0(W, d_{\widetilde{S}}) \simeq H^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$;
- $H^1(W, d_{\widetilde{S}}) \simeq H^0_{sym}(\mathfrak{h}, H^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$;
- $H^2(W, d_{\widetilde{S}}) \simeq H^0_{sym}(\mathfrak{h}, Z^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \oplus H^1_{sym}(\mathfrak{h}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$;
- $H^{2k+1}(W, d_{\widetilde{S}}) \simeq H^k_{sym}(\mathfrak{h}, \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \{0\}$, *for* $k \geq 1$;

- $H^{2k}(W, d_{\widetilde{S}}) \simeq H^k_{sym}(\mathfrak{h}, \mathcal{Z}^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))), \quad \textit{for } k \geq 2.$

Before starting with the proof of the theorem, we want to introduce a simpler notation for the double cochain complex and for its coboundary operators:

- In what follows we indicate the cochain space
$$\mathcal{C}^j_{sym}(\mathfrak{h}, \mathcal{C}^i_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$$
with the symbol $\mathcal{C}^{j,i}$.

- We denote the operator
$$Id_{\mathfrak{h}^j} \otimes \frac{1}{i+1}\delta^i_{Lie} \ ,$$
where δ^i_{Lie} is the coboundary operator of the Lie algebra cohomology complex $\mathcal{C}^{\bullet}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$, by $d^{j,i}$. Thus the operator $d^{j,i}$ acts on the cochain space $\mathcal{C}^{j,i}$ and takes values in $\mathcal{C}^{j,i+1}$.

- We denote the operator
$$\delta^{j,i}_{sym},$$
which is the coboundary operator corresponding to the cohomology complex $\mathcal{C}^{j,i}$, by $\delta^{j,i}$. Thus the operator $\delta^{j,i}$ is defined on the space $\mathcal{C}^{j,i}$ and takes values in $\mathcal{C}^{j+1,i-1}$.

Proof. We prove the theorem degree by degree.

Degree 0

When we consider the statement of Theorem 10 for cochains of ghost degree 0, we have

- $\mathcal{C}^0(W, d_{\widetilde{S}}) = \mathcal{C}^{0,0} \simeq \mathcal{C}^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \simeq \mathrm{Pol}_{\mathbb{R}}(M_a);$
- $d_{\widetilde{S}} = d^{0,0} = \delta^0_{Lie}.$

Therefore, it is immediate to deduce that

$$H^0(W, d_{\widetilde{S}}) = \mathrm{Ker}(d_{\widetilde{S}}) = \mathrm{Ker}(\delta^0_{Lie}) = H^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)).$$

Degree 1

A generic cochain φ of ghost degree 1 can be written as

$$\varphi = f_1 C_1 + f_2 C_2 + f_3 C_3,$$

for some polynomials $f_1,\ f_2,\ f_3$ in $\mathrm{Pol}_{\mathbb{R}}(M_a)$.
We recall the conditions that φ needs to satisfy in order to be an element in $Z^1(W, d_{\widetilde{S}})$ (see Appendix D):

(1) $M_3(\partial_{M_1} f_1) - M_1(\partial_{M_3} f_1) + M_3(\partial_{M_2} f_2) - M_2(\partial_{M_3} f_2) + f_3 = 0$;

(2) $-M_2(\partial_{M_1} f_1) + M_1(\partial_{M_2} f_1) + M_3(\partial_{M_2} f_3) - M_2(\partial_{M_3} f_3) - f_2 = 0$;

(3) $-M_2(\partial_{M_1} f_2) + M_1(\partial_{M_2} f_2) - M_3(\partial_{M_1} f_3) + M_1(\partial_{M_3} f_3) + f_1 = 0$;

(4) $M_1 f_1 + M_2 f_2 + M_3 f_3 = 0$.

For ghost degree 1, we have:

$$\mathcal{C}^1(W, d_{\widetilde{S}}) = \mathcal{C}^{0,1} \quad \text{and} \qquad d^1_{\widetilde{S}} = \delta^{0,1} \oplus d^{0,1}.$$

Let us separately consider the two spaces $\mathrm{Ker}(d^{0,1})$ and $\mathrm{Ker}(\delta^{0,1})$. Then, given a generic cochain φ as above,

$$\begin{aligned} d^{0,1}(\varphi) & \\ &= [M_3(\partial_{M_1} f_1) - M_1(\partial_{M_3} f_1) + M_3(\partial_{M_2} f_2) - M_2(\partial_{M_3} f_2) + f_3]\, C_1 C_2 \\ &+ [-M_2(\partial_{M_1} f_1) + M_1(\partial_{M_2} f_1) + M_3(\partial_{M_2} f_3) - M_2(\partial_{M_3} f_3) - f_2]\, C_1 C_3 \\ &+ [-M_2(\partial_{M_1} f_2) + M_1(\partial_{M_2} f_2) - M_3(\partial_{M_1} f_3) + M_1(\partial_{M_3} f_3) + f_1]\, C_2 C_3. \end{aligned}$$

Therefore, it follows immediately that imposing the condition that the cochain φ be an element in $\mathrm{Ker}(d^{0,1})$, is equivalent to imposing the conditions (1), (2), (3), listed above. On the other hand,

$$\delta^{0,1}(\varphi) = (f_1 M_1 + f_2 M_2 + f_3 M_3) E,$$

so, being an element in $\mathrm{Ker}(\delta^{0,1})$ is equivalent to satisfy the condition (4). Thus

$$\mathrm{Ker}(d^1_S) = \mathrm{Ker}(\delta^{0,1}) \cap \mathrm{Ker}(d^{0,1}). \tag{5.66}$$

Since d^0_S coincides with the operator $d^{0,0}$, one deduces the following identities:

$$\begin{aligned} H^1(W, d_{\widetilde{S}}) &= \frac{\mathrm{Ker}(\delta^{0,1}) \cap \mathrm{Ker}(d^{0,1})}{\mathrm{Im}(d^{0,0})} = \mathrm{Ker}(\delta^{0,1}) \cap \frac{\mathrm{Ker}(d^{0,1})}{\mathrm{Im}(d^{0,0})} \\ &= \mathrm{Ker}(\delta^{0,1}) \cap H^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) = H^0_{sym}(\mathfrak{h}, H^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))). \end{aligned}$$

Degree 2

Before starting to consider the cohomology group $H^2(W, d_{\widetilde{S}})$, let us recall how the BRST cochain of ghost degree 2 and the corresponding coboundary operator is described using the generalized Lie algebra cohomology complex:

- $\mathcal{C}^2(W, d_{\widetilde{S}}) = \mathcal{C}^{0,2} \oplus \mathcal{C}^{1,0} \simeq \mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \oplus \mathcal{C}^1_{sym}(\mathfrak{h}, \mathrm{Pol}_{\mathbb{R}}(M_a))$;
- $d^2_{\widetilde{S}} = \left[\delta^{0,2} \oplus d^{0,2}\right] \oplus \left[\delta^{1,0} \oplus d^{1,0}\right]$.

Moreover, we have already computed explicitly the BRST cohomology group of degree 2, viz.

$$H^2(W, d_{\widetilde{S}}) = K \oplus \mathrm{Pol}_{\mathbb{R}}(M_4)E,$$

with

$$K := \{f(M_1C_2C_3 - M_2C_1C_3 + M_3C_1C_2),\ f \in \mathrm{Pol}_{\mathbb{R}}(M_a)\}\,.$$

Therefore, for degree 2 we have to prove the following relation between the cohomology groups:

$$H^2(W, d_{\widetilde{S}}) \simeq H^0_{sym}(\mathfrak{h}, Z^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \oplus H^1_{sym}(\mathfrak{h}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))).$$

To achieve this, we are going to show that the following isomorphisms hold:

1. $H^0_{sym}(\mathfrak{h}, Z^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \cong K$;
2. $H^1_{sym}(\mathfrak{h}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \cong \mathrm{Pol}_{\mathbb{R}}(M_4)E$.

Let us start considering the first isomorphism. We want to prove that

$$H^0_{sym}(\mathfrak{h}, Z^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \mathrm{Ker}(\delta^{0,2}) \cap \mathrm{Ker}(d^{0,2}) = K.$$

Let φ be a generic cochain in $\mathcal{C}^{0,2}$, that is,

$$\varphi = g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3,$$

with g_{12}, g_{13}, g_{23} elements in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Then, for φ to be an element of the intersection of $\mathrm{Ker}(\delta^{0,2})$ with $\mathrm{Ker}(d^{0,2})$, the following identities need to be satisfied:

1. $-M_2g_{12} - M_3g_{13} = 0$;
2. $M_1g_{12} - M_3g_{23} = 0$;

3. $M_1 g_{13} + M_2 g_{23} = 0$;

4. $M_2(\partial_{M_1} g_{12}) - M_1(\partial_{M_2} g_{12}) + M_3(\partial_{M_1} g_{13}) - M_1(\partial_{M_3} g_{13}) + M_3(\partial_{M_2} g_{23})$
$- M_2(\partial_{M_3} g_{23}) = 0$.

The first three follow from imposing that φ be a cocycle with respect to the coboundary operator $\delta^{0,2}$, while the last one follows by requiring that φ is an element in $\mathrm{Ker}(d^{0,2})$. The first three conditions imply that the polynomials g_{12}, g_{13}, g_{23} are of the following form, for a certain polynomial P in $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$g_{12} = M_3 P \qquad\qquad g_{13} = -M_2 P \qquad\qquad g_{23} = M_1 P.$$

It is immediate that, with this choice of the polynomials g_{12}, g_{13}, g_{23}, also the last condition is verified. Therefore, we conclude that

$$\mathrm{Ker}(\delta^{0,2}) \subseteq \mathrm{Ker}(d^{0,2}).$$

Thus we have that the cochain φ in $\mathrm{Ker}(\delta^{0,2}) \cap \mathrm{Ker}(d^{0,2})$ necessarily takes the following form:

$$\varphi = P(M_3 C_1 C_2 - M_2 C_1 C_3 + M_1 C_2 C_3),$$

i.e., φ is an element of K. With the first isomorphism proved, let us consider the second. We want to prove that

$$H^1_{sym}(\mathfrak{h}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}(M_a)})) = \frac{\mathrm{Ker}(\delta^{1,0}) \cap \mathrm{Ker}(d^{1,0})}{\mathrm{Im}(\delta^{0,1})} = \mathrm{Pol}_{\mathbb{R}}(M_4)E. \tag{5.67}$$

Since the map $\delta^{0,1}$ is the zero map, $\mathrm{Ker}(\delta^{1,0})$ coincides with $\mathcal{C}^{1,0}$. On the other hand, it can be easily proved that it holds the following isomorphism:

$$\mathrm{Ker}(d^{1,0}) \simeq Ker(d^{0,0}) \cdot E.$$

More explicitly we have that

$$\mathrm{Ker}(d^{1,0}) = \Big\{ fE : f = \sum_{k=0}^{r} g_k(M_4)(M_1^2 + M_2^2 + M_3^2)^k, \quad r \in \mathbb{N}_0, g_k \in \mathrm{Pol}_{\mathbb{R}}(M_4) \Big\}.$$

Concerning the coboundary, we have

$$\mathrm{Im}(\delta^{0,1}) = \{(M_1 f_1 + M_2 f_2 + M_3 f_3)E,\ f_i \in \mathrm{Pol}_{\mathbb{R}}(M_a)\}.$$

Given a generic cochain φ belonging to $\mathrm{Ker}(d^{1,0}) \subseteq \mathcal{C}^{1,0}$, with $\varphi = fE$ for a certain polynomial f in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, we have

$$\varphi = [g_1 M_1 + g_2 M_2 + g_3 M_3 + g_0]E$$

for suitable polynomials g_1, g_2, g_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ and g_0 in $\mathrm{Pol}_{\mathbb{R}}(M_4)$. Thus we see that

$$\varphi \equiv g_0(M_4)E \qquad\qquad \mathrm{mod}\ \ \mathrm{Im}(\delta^{0,1}_{sym}).$$

Hence the identity in (5.67) holds, and the proof is concluded.

Degree $2k+1$

As proved in Theorem 10, the space of BRST cochains of ghost degree $2k+1$ with $k \geq 1$ can be seen as the following direct sum:

$$\mathcal{C}^{2k+1}(W, d_{\widetilde{S}}) = \mathcal{C}^{k,1} \oplus \mathcal{C}^{k-1,3}.$$

In Section 5.2.4 and Appendix D we explicitly computed the BRST cohomology groups for odd degree $2k+1$ with $k \geq 1$, finding that all these cohomology groups are trivial. Therefore, to prove the statement in the theorem it is enough to prove that:

1. $H^k_{sym}(\mathfrak{h}, \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \{0\}$;
2. $H^{k-1}_{sym}(\mathfrak{h}, \mathcal{C}^3_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \{0\}$.

To check the first identity, we prove that

$$\mathrm{Ker}(\delta^{k,1}) = \mathrm{Im}(\delta^{k-1,2}).$$

Let φ be a cochain in $\mathcal{C}^{k,1}$, with

$$\varphi = [f_1 C_1 + f_2 C_2 + f_3 C_3]E^k,$$

for some polynomials f_1, f_2, f_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. For φ to be a cocycle in $\mathrm{Ker}(\delta^{k,1})$, the polynomials f_i need to satisfy

$$f_1 M_1 + f_2 M_2 + f_3 M_3 = 0,$$

which forces them to be of the following form, for some elements Q, R and S in $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$f_1 = M_2 Q + M_3 R \qquad f_2 = -M_2 Q + M_3 S \qquad f_3 = -M_1 R - M_2 S.$$

On the other hand, it can be checked that the cochain ψ, defined by

$$\psi = [QC_1C_2 + RC_1C_3 + SC_2C_3]E^{k-1}$$

satisfies $\delta^{k,1}(\psi) = \varphi$.
We conclude that, for all $k \geq 1$,

$$\mathrm{Ker}(\delta^{k,1}) = \mathrm{Im}(\delta^{k-1,2}).$$

Hence $H^k_{sym}(\mathfrak{h}, \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$ is trivial.

As to the second identity that needs to be proved, let us show that the map $\delta^{k-1,3}$ is injective for all $k \geq 1$. Let φ be a cochain in $\mathcal{C}^{k-1,3}$, i.e.,

$$\varphi = [fC_1C_2C_3]E^{k-1}$$

with f a polynomial in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Then φ is an element in $\mathrm{Ker}(\delta^{k-1,3})$ if

$$[fM_1C_2C_3 - fM_2C_1C_3 + fM_3C_2C_3]E^k = 0.$$

This implies that $f = 0$ and hence that φ is the zero cochain. Therefore,

$$\mathrm{Ker}(\delta^{k-1,3}) = \{0\}\,, \qquad \forall k \geq 1. \tag{5.68}$$

Thus we conclude that, for all $k \geq 1$,

$$\begin{aligned} &H^{2k+1}(X, d_S) \\ &\quad \simeq H^k_{sym}(\mathfrak{h}, \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \oplus H^{k-1}_{sym}(\mathfrak{h}, \mathcal{C}^3_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \{0\}\,. \end{aligned}$$

Degree $2k$

We now prove the following identity:

$$H^{2k}(W, d_{\widetilde{S}}) = H^k_{sym}(\mathfrak{g}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}(M_a)})).$$

First of all, note that by definition the map $\delta^{k,0}$ is the zero map.
Therefore:

$$H^k_{sym}(\mathfrak{g}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}(M_a)})) = \frac{\mathrm{Ker}(d^{k,0})}{\mathrm{Im}(\delta^{k-1,1})}.$$

It is immediate that, for $k \geq 1$, we have the following identities:

$$\begin{aligned}
\mathrm{Ker}(d^{k,0}) &= Z^0(X, d_S) \cdot E^k \\
&= \left\{ \begin{array}{l} \left[g_0(M_4) + \sum_{j=1}^{s}(M_1^2 + M_2^2 + M_3^2)^j g_j(M_4)\right] E^k, \\ \text{with } g_i \in \mathrm{Pol}_{\mathbb{R}}(M_4) \end{array} \right\}
\end{aligned}$$

$$\mathrm{Im}(\delta^{k-1,1}) = \left\{[M_1 f_1 + M_2 f_2 + M_3 f_3] \cdot E^k, f_1, f_2, f_3 \in \mathrm{Pol}_{\mathbb{R}}(M_a)\right\}.$$

A generic cocycle φ in $\mathrm{Ker}(d^{k,0})$ described by a polynomial depending explicitly on the variables M_1, M_2, M_3 can be seen as a coboundary element: more precisely, in the notation used in the expression above, if we consider a cocycle φ described by a polynomial with $s \geq 1$, then, as representative of an element in the cohomology group, it is equivalent to the cocycle described by a polynomial depending only on the variable M_4, namely the polynomial $g_0(M_4)$. Therefore, we obtain

$$H^k_{sym}(\mathfrak{h}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \mathrm{Pol}_{\mathbb{R}}(M_4) \cdot E^k.$$

This expression coincides with what we found for a BRST cohomology group of degree $2k$. This proves the statement for one type of cochains of ghost degree $2k$: in fact, for ghost degree $2k$ a generic cochain is an element belonging to the direct sum

$$\mathcal{C}^{2k}(W, d_{\widetilde{S}}) = \mathcal{C}^{k,0} \oplus \mathcal{C}^{k-1,2}.$$

Up to now we have considered only cochains of ghost degree $2k$ belonging to the first summand. It would be possible, from the point of view of the ghost degree, that a contribution arises in the cohomology group from the second summand. The reason why this contribution is not visible is that, for all $k > 1$,

$$H^{k-1}(\mathfrak{h}, \mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \{0\}. \tag{5.69}$$

In fact, by a computation similar to the one in degree 2, one checks that:

$$\begin{aligned}
\mathrm{Ker}(\delta^{k-1,2}) &\simeq \mathrm{Ker}(\delta^{0,2}) \cdot E^{k-1} \\
&= \{P\,[M_3C_1C_2 - M_2C_1C_3 + M_1C_2C_3]\,, P \in \mathrm{Pol}_{\mathbb{R}}(M_a)\} \cdot E^{k-1}.
\end{aligned}$$

Since

$$\mathrm{Im}(\delta^{k-2,3}) = \{P\,[M_3C_1C_2 - M_2C_1C_3 + M_1C_2C_3]\,, P \in \mathrm{Pol}_{\mathbb{R}}(M_a)\} \cdot E^{k-1},$$

the identity (5.69) follows immediately, for all $k \geq 2$. This explains why in the cohomology group $H^{2k}(W, d_{\widetilde{S}})$ $(k \geq 2)$ there are only contributions from the first kind of cochains of ghost degree $2k$. □

Using the computations and observations in the proof of the previous theorem one deduces exactness of a sequence of cochain spaces, as stated in the following Corollary.

Corollary 2
Let $\{\mathcal{C}^j_{sym}(\mathfrak{h}, \mathcal{C}^i_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))\} = \{\mathcal{C}^{j,i}\}$, with $j \geq 0$ and $i = 0, \ldots, 3$, be the set cochain spaces whose structure is described in Theorem 10.
Then, for all $k \geq 2$, the following sequence is exact:

$$0 \to \mathcal{C}^{k-2,3} \xrightarrow{\delta^{k-2,3}} \mathcal{C}^{k-1,2} \xrightarrow{\delta^{k-1,2}} \mathcal{C}^{k,1} \xrightarrow{\delta^{k,1}} \widetilde{\mathcal{C}}^{k+1,0} \to 0, \tag{5.70}$$

where

$$\widetilde{\mathcal{C}}^{k+1,0} = \mathcal{C}^{k+1}_{sym}(\mathfrak{h}, \mathcal{C}_{Lie}(\mathfrak{g}, W)),$$

with $W := \mathrm{Pol}_{\mathbb{R}}(M_a)) \setminus \mathrm{Pol}_{\mathbb{R}}(M_4)$.
Moreover, restricting the coboundary operators to cochains with coefficients in the space of cocycles for the standard Lie algebra cohomology of $\mathfrak{g}$, we obtain another exact sequence:

$$\begin{aligned} &0 \to \mathcal{C}^{k-2}_{sym}(\mathfrak{h}, Z^3_{Lie}) \xrightarrow{\delta^{k-2,3}} \mathcal{C}^{k-1}_{sym}(\mathfrak{h}, Z^2_{Lie}) \xrightarrow{\delta^{k-1,2}} \mathcal{C}^{k}_{sym}(\mathfrak{h}, Z^1_{Lie}) \ldots \\ &\ldots \xrightarrow{\delta^{k,1}} \widetilde{\mathcal{C}}^{k+1}_{sym}(\mathfrak{h}, \widetilde{W}) \to 0, \end{aligned} \tag{5.71}$$

where we use the following notation:

- ► $\mathcal{C}^j_{sym}(\mathfrak{h}, Z^i_{Lie}) := \mathcal{C}^j_{sym}(\mathfrak{h}, Z^i_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$;
- ► $\widetilde{\mathcal{C}}^{k+1}_{sym}(\mathfrak{h}, \widetilde{W}) = \mathcal{C}^{k+1}_{sym}(\mathfrak{h}, \mathcal{C}^0_{Lie}(\mathfrak{g}, \widetilde{W}))$;
- ► $\widetilde{W}$ is such that $Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) = \mathrm{Pol}_{\mathbb{R}}(M_4) \oplus \widetilde{W}$:

$$\widetilde{W} = \left\{ \sum_{j=1}^{r} (M_1^2 + M_2^2 + M_3^2)^j g_j(M_4), \quad g_j(M_4) \in \mathrm{Pol}_{\mathbb{R}}(M_4), \quad r \in \mathbb{N} \right\}.$$

Proof. As shown in (5.68), the coboundary operator $\delta^{k-2,3}$ is an injective map already when it is applied to the space $\mathcal{C}^{k-2,3}$, without having to restrict its domain to the cocycles on $Z^3_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$. Therefore, both sequences are exact at the first element.
To prove exactness of the first sequence at the second element, we have to prove that $\mathrm{Ker}(\delta^{k-1,2})$ coincides with $\mathrm{Im}(\delta^{k-2,3})$. Indeed, in (5.69) we stated that the

cohomology group $H^{k-1}_{sym}(\mathfrak{h}, \mathcal{C}^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$ is trivial: therefore, we conclude that the first sequence is exact also in the second element.
Continuing with the proof of the exactness of the first sequence, now we have to show that $\mathrm{Im}(\delta^{k-1,2})$ coincides with $\mathrm{Ker}(\delta^{k,1})$. Indeed, since the cohomology group $H^{k}_{sym}(\mathfrak{h}, \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$ is trivial it follows that the first sequence is exact also in its third element. Finally, the map $\delta^{k,1}$ has image

$$\begin{aligned}\mathrm{Im}(\delta^{k-1,1}) &= \left\{[M_1 f_1 + M_2 f_2 + M_3 f_3] \cdot E^k, f_1, f_2, f_3 \in \mathrm{Pol}_{\mathbb{R}}(M_a)\right\} \\ &\simeq [\mathrm{Pol}_{\mathbb{R}}(M_a) \backslash \mathrm{Pol}_{\mathbb{R}}(M_4)] E^k,\end{aligned}$$

which coincides with $\mathcal{C}^{k+1}_{sym}(\mathfrak{h}, W)$. Thus the map $\delta^{k,1}$ is surjective and the first sequence is exact.

We still have to prove exactness of sequence (5.71). This is a consequence of the exactness of sequence (5.70): (5.71) can be obtained from (5.70) via the restriction of the domains and codomains of the coboundary operators $\delta^{\bullet,\bullet}$ to cochain spaces with coefficients in the cocycle spaces $Z^{\bullet}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$.
Therefore, to conclude the proof we only need to prove that the sequence (5.71) is well defined.

Let φ be a cochain in $\mathcal{C}^{i}_{sym}(\mathfrak{h}, Z^{j}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$, with $i \geq 0$ and $j = 1, 2, 3$: we want to check if $\delta^{i,j}(\varphi)$ is an element in $\mathcal{C}^{i+1}_{sym}(\mathfrak{h}, Z^{j-1}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$. By hypothesis, φ is of the following form:

$$\varphi = \psi \cdot E^i, \qquad \text{with } \psi \in Z^{j}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)).$$

To conclude that also the second sequence is well defined, we show that

$$\delta^{i,j}(\varphi) = \chi \cdot E^i, \qquad \text{for some} \qquad \chi \in Z^{j-1}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)) \cdot E.$$

Let us define $\chi := \delta^{0,j}(\psi)$. Then we only have to show that χ is an element of $\mathrm{Ker}(d^{1,j-1})$. Recall what we proved in Proposition 16, which implies

$$d^{i+1,j-1}(\delta^{i,j}(\varphi)) = d^{1,j-1}(\delta^{0,j}(\psi)) \cdot E^{i+1} = -\delta^{0,j+1}(d^{0,j}(\psi)) \cdot E^{i+1} = 0,$$

since ψ in an element in $Z^{j}_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$. Therefore, the second sequence is well defined and hence exact. □

Before we come to the shifted double complex and its properties, we prove the following Lemma, which will be used in the next section.

Lemma 8
Let $d_S^\bullet$, $d^\bullet$, $\delta^\bullet$ and $\{\mathcal{C}^{k,i}\}$, $k \in \mathbb{N}_0$, $i = 0,\dots,3$, be the same coboundary operators and the same collection of cochain spaces as in Theorem 11. For each $k \geq 1$, we have

$$\mathrm{Ker}(d_S^{2k}) = \mathrm{Ker}(d^{k,0}) \oplus \left[\mathrm{Im}(d^{k-1,1}) + \mathrm{Ker}(\delta^{k-1,2})\right].$$

Proof. For the cocycle space $\mathrm{Ker}(d_S^{2k})$, let us start considering a generic cochain φ of ghost degree $2k$: this can be written as sum of two cochains $\varphi_{k,0}$ and $\varphi_{k-1,2}$, which belong to the cochain spaces $\mathcal{C}^{k,0}$ and $\mathcal{C}^{k-1,2}$ respectively. Thus

$$d_S^{2k}(\varphi) = (\delta^{k-1,2} + d^{k-1,2})(\varphi_{k-1,2}) + d^{k,0}(\varphi_{k,0}).$$

Since $d^{k,0}(\varphi_{k,0})$ and $\delta^{k-1,2}(\varphi_{k-1,2})$ are elements in $\mathcal{C}^{k,1}$ while $d^{k-1,2}(\varphi_{k-1,2})$ belongs to $\mathcal{C}^{k-1,3}$, to have that φ is a cocycle element is equivalent to imposing

- $d^{k,0}(\varphi_{k,0}) = -\delta^{k-1,2}(\varphi_{k-1,2})$;
- $d^{k-1,2}(\varphi_{k-1,2}) = 0$.

Now consider the intersection of $\mathrm{Im}(d^{k,0})$ together with $\mathrm{Im}(\delta^{k-1,2})$. A generic coboundary element α in $\mathrm{Im}(d^{k,0})$ takes the following form, for some polynomial f in $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$\begin{aligned}\alpha = \;& \{[M_3(\partial_{M_2} f) - M_2(\partial_{M_3} f)]\, C_1 + [M_1(\partial_{M_3} f) - M_3(\partial_{M_1} f)]\, C_2 \\ & + [M_2(\partial_{M_1} f) - M_1(\partial_{M_2} f)]\, C_3\}\, E^k.\end{aligned}$$

A generic element β in $\mathrm{Im}(\delta^{k-1,2})$ has the following form, for some polynomials g_{12}, g_{13} and g_{23} in $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$\beta = [(-g_{12}M_2 - g_{13}M_3)\, C_1 + (g_{12}M_1 - g_{23}M_3)\, C_2 + (g_{13}M_1 - g_{23}M_2)\, C_3]\, E^k.$$

Therefore, to have a cochain that is a coboundary with respect to both maps $\delta^{k-1,2}$ and $d^{k,0}$, the polynomials f, g_{12}, g_{13} and g_{23} need to satisfy

$$g_{12} = M_1 Q + \partial_{M_3} f\,, \qquad g_{13} = -M_2 Q - \partial_{M_2} f\,, \qquad g_{23} = M_3 Q + \partial_{M_1} f.$$

with Q a polynomial in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Hence, for $\delta^{k-1,2}(\varphi_{k-1,2})$ to coincide with $-d^{k.0}(\varphi_{k,0})$, we need

$$\begin{aligned}\varphi_{k-1,2} = \;& [(M_1 Q + \partial_{M_3} f)\, C_1 C_2 - (M_2 Q + \partial_{M_2} f)\, C_1 C_3 \\ & + (M_3 Q + \partial_{M_1} f)\, C_2 C_3]\, E^{k-1}\,,\end{aligned}$$

where Q and f are generic polynomials in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Therefore, there exist some polynomials A_1, A_2, A_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, as well as a polynomial A_0 in $\mathrm{Pol}_{\mathbb{R}}(M_4)$, such that f can be rewritten as

$$f = -M_1 A_1 - M_2 A_2 - M_3 A_3 + A_0.$$

Let us define

$$\widetilde{Q} = -\frac{\partial A_1}{\partial M_1} - \frac{\partial A_2}{\partial M_2} - \frac{\partial A_3}{\partial M_3} \ .$$

Then $\varphi_{k-1,2} = \psi + \chi$, with:

- ψ an element in $\mathrm{Im}(d^{k-1,1})$,

  $$\psi = [(M_1\widetilde{Q} + \partial_{M_3} f)C_1C_2 - (M_2\widetilde{Q} + \partial_{M_2} f)C_1C_3 + (M_3\widetilde{Q} + \partial_{M_1} f)C_2C_3]E^{k-1}.$$

- $\chi = \left(Q - \widetilde{Q}\right)[-M_3C_1C_2 + M_2C_1C_3 - M_1C_2C_3]\,E^{k-1}$, which belongs to $\mathrm{Ker}(\delta^{k-1,2})$.

In fact one can check that:

► $\mathrm{Im}(d^{k-1,1})$

$$\begin{aligned} = \{ & [(M_3\partial_{M_1}g_1 - M_1\partial_{M_3}g_1 + M_3\partial_{M_2}g_2 - M_2\partial_{M_3}g_2 - g_3)\,C_1C_2 + \\ & (-M_2\partial_{M_1}g_1 + M_1\partial_{M_2}g_1 + M_3\partial_{M_2}g_3 - M_2\partial_{M_3}g_3 + g_2)\,C_1C_3 + \\ & (-M_2\partial_{M_1}g_2 + M_1\partial_{M_2}g_2 - M_3\partial_{M_1}g_3 + M_1\partial_{M_3}g_3 - g_1)\,C_2C_3]\,E^{k-1} \\ & \text{with } g_1, g_2, g_3 \in \mathrm{Pol}_{\mathbb{R}}(M_a)\} \ ; \end{aligned}$$

► $\mathrm{Ker}(\delta^{k-1,2})$

$$= \left\{ P\,[-M_3C_1C_2 + M_2C_1C_3 - M_1C_2C_3]\,E^{k-1}, \text{ with } P \in \mathrm{Pol}_{\mathbb{R}}(M_a) \right\}.$$

Finally, we conclude that

$$\mathrm{Ker}(d_S^{2k}) = \mathrm{Ker}(d^{k,0}) \oplus \left[\mathrm{Im}(d^{k-1,1}) + \mathrm{Ker}(\delta^{k-1,2})\right].$$

□

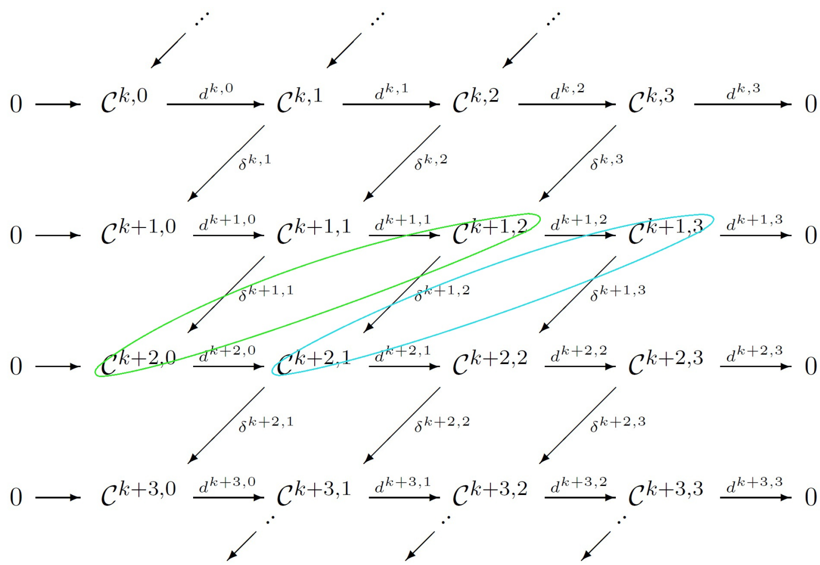


Figure 5.2: The shifted double complex: the two cochain spaces whose direct sum gives the BRST-cochain space of degree $2(k+2)$ are enclosed in the green oval, while the two cochain spaces that determine the BRST-cochain space of degree $2(k+2)+1$ are enclosed in the blue oval.

5.3.4 The shifted double complex

In the previous sections we found several properties of the shifted double complex structure corresponding to the matrix model. The main goal of this section is to use these properties to give a more intrinsic explanation of the relations existing between the BRST cohomology groups and the corresponding generalized Lie algebra cohomology groups and to investigate how the relations found at the level of cochains spaces and coboundary operators can be reinterpreted at the level of cohomology groups.

Let us start by displaying the structure of the shifted double complex in a diagram (see Figure 5.2). In the figure we decided to emphasize the fact that one cochain space in the BRST cohomology is obtained as a direct sum of two cochain spaces in the shifted double complex.

In the following list we enumerate all the properties of the bicomplex

$$\left\{\mathcal{C}^{i,j}, \delta^{i,j} \oplus d^{i,j}\right\}$$

with $i \geq 0$, $j = 0, \ldots, 3$, known up to now.

Property 4
Let $\left\{\mathcal{C}^{i,j}, \delta^{i,j} \oplus d^{i,j}\right\}$ be the cochain complex whose structure was defined in Theorem 10. This satisfies the following properties:

(1) The operator $d^{k,\bullet}$ is a coboundary operator, i.e., for all $k \geq 0$ and $i = 0, 1, 2$,

$$d^{k,i+1} \circ d^{k,i} = 0.$$

(2) The operator $\delta^{\bullet}$ is a coboundary operator, that is, for all $k \geq 0$, $i = 0, 1, 2$,

$$\delta^{k+1,i} \circ \delta^{k,i+1} = 0.$$

(3) The composition for the two operators $d^{\bullet}$ and $\delta^{\bullet}$ satisfies the following relation, for all $k \geq 0$ and $i = 0, \ldots, 3$:

$$d^{k+1,i-1} \circ \delta^{k,i} = -\delta^{k,i+1} \circ d^{k,i}.$$

We recall that by definition $\delta^{k,0} = 0$, for all non negative value of k and $d^{k,i} = 0$ for negative values of i and if $i > 3$.

(4) The operator $\delta^{k,3}$ is injective for all non negative value of k.

(5) As proved in Lemma 8, for all $k \geq 1$, the following identity holds:

$$\mathrm{Ker}(d_S^{2k}) = \mathrm{Ker}(d^{k,0}) \oplus \left[\mathrm{Im}(d^{k-1,1}) + \mathrm{Ker}(\delta^{k-1,2})\right].$$

(6) The diagonals in the diagram displayed above are exact sequences. More precisely, using Corollary 2, the following sequence is exact, for all $k \geq 0$:

$$\mathcal{C}^{k,3} \xrightarrow{\delta^{k,3}} \mathcal{C}^{k+1,2} \xrightarrow{\delta^{k+1,2}} \mathcal{C}^{k+2,1} \xrightarrow{\delta^{k+2,1}} \mathcal{C}^{k+3,0}.$$

We are interested in the cohomology groups defined by the cochain spaces:

$$\begin{cases} \mathcal{C}^{2k} = \mathcal{C}^{k,0} \oplus \mathcal{C}^{k-1,2} \; ; \\ \mathcal{C}^{2k+1} = \mathcal{C}^{k,1} \oplus \mathcal{C}^{k-1,3} \; , \end{cases}$$

where

$$\mathcal{C}^{i,j} = \mathcal{C}^i_{sym}(\mathfrak{h}, \mathcal{C}^j_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))).$$

The coboundary operator is given by:

$$\begin{cases} d_S^{2k} = d^{k,0} \oplus [\delta^{k-1,2} \oplus d^{k-1,2}] \\ d_S^{2k+1} = [d^{k,1} \oplus \delta^{k,1}] \oplus \delta^{k-1,3} \end{cases},$$

with

$$d^{k,i} = Id_{\mathfrak{h}^k} \otimes \frac{1}{i+1}\delta^i_{Lie}, \qquad \delta^{k,i} = \delta^{k,i}_{sym}.$$

We once again want to find the relations already described in Section 5.3.3 between the BRST cohomology groups and the corresponding generalized Lie algebra cohomology groups in a more intrinsic way, deducing them by the properties listed above for the shifted double complex and its coboundary maps.

Theorem 12. *Let* $\{\mathcal{C}^{i,j}\}$, $i \geq 0$, $j = 0, \dots, 3$ *be a collection of vector spaces and let* $d^{i,j}$ *and* $\delta^{i,j}$ *be two linear and differential operators such that*

$$d^{i,j} : \mathcal{C}^{i,j} \longrightarrow \mathcal{C}^{i,j+1}, \qquad \delta^{i,j} : \mathcal{C}^{i,j} \longrightarrow \mathcal{C}^{i+1,j-1}.$$

Suppose that the cochain spaces $\mathcal{C}^{i,j}$ *and the coboundary operators satisfy the properties (1)–(6) listed above. Let us also introduce the following objects:*

- $\{\mathcal{C}^m\}_{m\in\mathbb{N}_0}$ *is a collection of vector spaces with*

$$\mathcal{C}^{2k} = \mathcal{C}^{k,0} \oplus \mathcal{C}^{k-1,2} \quad \text{and} \quad \mathcal{C}^{2k+1} = \mathcal{C}^{k,1} \oplus \mathcal{C}^{k-1,3};$$

- d_S^m, *with* m *in* $\mathbb{N}_0$, *is an operator of degree* 1, $d_S^m : \mathcal{C}^m \to \mathcal{C}^{m+1}$, *with*

$$d_S^{2k} = d^{k,0} \oplus [\delta^{k-1,2} \oplus d^{k-1,2}] \qquad d_S^{2k+1} = [d^{k,1} \oplus \delta^{k,1}] \oplus \delta^{k-1,3}.$$

  The operators $d^{i,j}$ *and* $\delta^{i,j}$ *are defined to be the zero map when* $i < 0$ *or* $j \geq 4$ *or* $j < 0$.

Then the pair $(\mathcal{C}^m, d_S^m)$ *defines a cochain complex, and its cohomology groups satisfy the following isomorphisms:*

- $H^{2k}(d_S) \simeq \dfrac{\mathrm{Ker}(d^{k,0}) \cap \mathrm{Ker}(\delta^{k,0})}{\mathrm{Im}(\delta^{k-1,1})} \oplus \dfrac{\mathrm{Ker}(\delta^{k-1,2}) \cap \mathrm{Ker}(d^{k-1,2})}{\mathrm{Im}(\delta^{k-2,3})}$, $\forall k \geqslant 2$;

► $H^{2k+1}(d_S) \simeq \dfrac{\mathrm{Ker}(\delta^{k,1})}{\mathrm{Im}(\delta^{k-1,2})} = \{0\}$, $\forall k \geqslant 1$.

In particular, for $k = 0, 1$ we have isomorphisms

► $H^0(d_S) \simeq \mathrm{Ker}(d^{0,0})$;

► $H^1(d_S) \simeq \dfrac{\mathrm{Ker}(d^{0,1}) \cap \mathrm{Ker}(\delta^{0,1})}{\mathrm{Im}(d^{0,0})}$;

► $H^2(d_S) \simeq \dfrac{\mathrm{Ker}(\delta^{1,0}) \cap \mathrm{Ker}(d^{1,0})}{\mathrm{Im}(\delta^{0,1})} \oplus \left[\mathrm{Ker}(\delta^{0,2}) \cap \mathrm{Ker}(d^{0,2})\right]$.

Proof. In order to conclude that the pair $(\mathcal{C}^m, d_S^m)$ defines a cochain complex we only have to prove that the operator d_S is a differential: in fact, the spaces $\mathcal{C}^m$ are vector spaces and the map d_S by definition is a linear derivation of degree 1 on the spaces $\mathcal{C}^m$. The fact that d_S defines a differential operator can be deduced from the properties (1), (2), (3). We now consider the composition $d_S^{2k+1} \circ d_S^{2k}$, but a similar computation can be done starting from cochains of odd degree.

$$\begin{aligned} d_S^2 &= [(d^{k,1} + \delta^{k,1}) + \delta^{k-1,3}] \circ [d^{k,0} + (\delta^{k-1,2} + d^{k-1,2})] \\ &= [d^{k,1} \circ d^{k,0}] + [\delta^{k,1} \circ d^{k,0}] + [d^{k,1} \circ \delta^{k-1,2}] + [\delta^{k,1} \circ \delta^{k-1,2}] \\ &\quad + [\delta^{k-1,3} \circ d^{k-1,2}] \\ &= [-\delta^{k-1,3} \circ d^{k-1,2}] + [\delta^{k-1,3} \circ d^{k-1,2}] = 0 \ . \end{aligned}$$

Thus the pair $(\mathcal{C}^m, d_S^m)$ defines a cochain complex.
Before starting with the computation of the cohomology groups defined by the cochain complex $(\mathcal{C}^m, d_S^m)$, from Property (6), we deduce that

$$\mathrm{Ker}(\delta^{k+1,2}) = \mathrm{Im}(\delta^{k,3}) \ , \qquad \mathrm{Ker}(\delta^{k+1,1}) = \mathrm{Im}(\delta^{k,2}) \ , \qquad \forall k \geq 0.$$

We first consider the cohomology group $H^{2k}(d_S)$ for an even degree $2k$, and we start by computing the cocycle space defined by $\mathrm{Ker}(d_S^{2k})$.

$\mathbf{Ker}(d_S^{2k})$

Let φ be a generic cochain of degree $2k$ in $\mathcal{C}^{2k}$. Using Property (5), we know that for $k \geq 1$,

$$\mathrm{Ker}(d_S^{2k}) = \mathrm{Ker}(d^{k,0}) \oplus \left[\mathrm{Im}(d^{k-1,1}) + \mathrm{Ker}(\delta^{k-1,2})\right].$$

However, if $k = 0$, then a generic cochain of degree 0 is necessarily an element of $\mathcal{C}^{k,0}$, and the coboundary operator d_S^0 coincides with $d^{0,0}$. Therefore,

$$\mathrm{Ker}(d_S^0) = \mathrm{Ker}(d^{0,0}).$$

$\mathbf{Im}(d_S^{2k-1})$

Let us consider the coboundary operator d_S for degree $2k-1$, with $k \geq 2$. Then

$$d_S^{2k-1} = [d^{k-1,1} \oplus \delta^{k-1,1}] \oplus \delta^{k-2,3}.$$

Therefore, since the coboundary maps $d^{k-1,1}$ and $\delta^{k-2,3}$ take values in $\mathcal{C}^{k-1,2}$, while the map $\delta^{k-1,1}$ takes values in $\mathcal{C}^{k,0}$, we have

$$\mathrm{Im}(d_S^{2k-1}) = \mathrm{Im}(\delta^{k-1,1}) \oplus [Im(d^{k-1,1}) + \mathrm{Im}(\delta^{k-2,3})].$$

However, in case of $k = 1$ the coboundary operator $\delta^{k-2,3}$ does not appear. Therefore,

$$\mathrm{Im}(d_S^1) = \mathrm{Im}(\delta^{0,1}) \oplus Im(d^{0,1}).$$

Thus we deduce that a cohomology group $H^{2k}(d_S)$ for an even degree $2k$ with $k \geq 2$ satisfies

$$\begin{aligned} H^{2k}(d_S) &= \frac{\mathrm{Ker}(d^{k,0})}{\mathrm{Im}(\delta^{k-1,1})} \oplus \frac{\mathrm{Im}(d^{k-1,1}) + \mathrm{Ker}(\delta^{k-1,2})}{\mathrm{Im}(d^{k-1,1}) + \mathrm{Im}(\delta^{k-2,3})} \\ &= \frac{\mathrm{Ker}(d^{k,0}) \cap \mathrm{Ker}(\delta^{k,0})}{\mathrm{Im}(\delta^{k-1,1})} \oplus \frac{\mathrm{Ker}(\delta^{k-1,2}) \cap \mathrm{Ker}(d^{k-1,2})}{\mathrm{Im}(\delta^{k-2,3})} \ , \end{aligned}$$

where the last equality follows from the fact that the coboundary operator $\delta^{k,0}$ is the zero map and so the space $\mathrm{Ker}(d^{k,0})$ coincides with $\mathrm{Ker}(d^{k,0}) \cap \mathrm{Ker}(\delta^{k,0})$. Moreover, using (3) for $i = 3$ and exactness of the sequence in (6), it follows that $\mathrm{Ker}(\delta^{k-1,2})$ is a subset of $\mathrm{Ker}(d^{k-1,2})$.

Finally, for the particular cases $k = 1$ and $k = 0$, we have

- $H^2(d_S) = \dfrac{\mathrm{Ker}(\delta^{1,0}) \cap \mathrm{Ker}(d^{1,0})}{\mathrm{Im}(\delta^{0,1})} \oplus \left[\mathrm{Ker}(d^{0,2}) \cap \mathrm{Ker}(\delta^{0,2})\right];$
- $H^0(d_S) = \mathrm{Ker}(d^{0,0}).$

Let us consider a cohomology group of odd degree $2k+1$, with $k \geq 1$.

$\mathbf{Ker}(d_S^{2k+1})$

Let φ be a generic cochain of degree $2k+1$. Then this is the sum of two cochains $\varphi_{k,1}$ and $\varphi_{k-1,3}$, which are elements in $\mathcal{C}^{k,1}$ and $\mathcal{C}^{k-1,3}$, respectively. Applying the coboundary operator d_S^{2k+1} to φ, we find

$$d_S^{2k+1}(\varphi) = (d^{k,1} \oplus \delta^{k,1})(\varphi_{k,1}) \oplus \delta^{k-1,3}(\varphi_{k-1,3}).$$

Since $d^{k,1}(\varphi_{k,1})$ and $\delta^{k-1,3}(\varphi_{k-1,3})$ are elements in $\mathcal{C}^{k,2}$ while $\delta^{k,1}(\varphi_{k,1})$ belongs to $\mathcal{C}^{k+1,0}$, in order that φ be a cocycle, we impose

- $d^{k,1}(\varphi_{k,1}) = -\delta^{k-1,3}(\varphi_{k-1,3})$;
- $\delta^{k,1}(\varphi_{k,1}) = 0$.

Moreover, using Property (6), i.e., exactness of the sequence, we deduce that $\mathrm{Ker}(\delta^{k,1})$ coincides with $\mathrm{Im}(\delta^{k-1,2})$. Therefore, $\varphi_{k,1}$ is a coboundary element with respect to the coboundary operator $\delta^{k-1,2}$: let then α be a preimage of $\varphi_{k,1}$ in $\mathcal{C}^{k-1,2}$.

Using Property (3), we then deduce that

$$\delta^{k-1,3}(\varphi_{k-1,3}) = -d^{k,1}(\delta^{k-1,2}(\alpha)) = \delta^{k-1,3}(d^{k-1,2}(\alpha)).$$

Finally, using injectivity of the coboundary operator $\delta^{k,3}$ for all non-negative value of k, i.e. Property (4), we deduce that $\varphi_{k-1,3}$ is an element of $\mathrm{Im}(d^{k-1,2})$. Thus

$$\mathrm{Ker}(d_S^{2k+1}) = \mathrm{Im}(\delta^{k-1,2}) \oplus \mathrm{Im}(d^{k-1,2}).$$

In case of $k = 0$, it is immediate that, in order for a cochain $\varphi_{0,1}$ to be a cocycle, it is not enough to impose that $\varphi_{0,1}$ belongs to $\mathrm{Ker}(\delta^{0,1})$; we also have to impose that $\varphi_{0,1}$ belongs to $\mathrm{Ker}(d^{0,1})$. Therefore,

$$\mathrm{Ker}(d_S^1) = \mathrm{Ker}(\delta^{0,1}) \cap \mathrm{Ker}(d^{0,1}).$$

$\mathbf{Im}(d_S^{2k})$

The coboundary operator d_S^{2k}, with $k \geq 1$, can be written as a sum of coboundary operators $\delta^\bullet$ and $d^\bullet$:

$$d_S^{2k} = d^{k,0} \oplus [\delta^{k-1,2} \oplus d^{k-1,2}].$$

Both operators $\delta^{k-1,2}$ and $d^{k,0}$ take values in $\mathcal{C}^{k,1}$ while the image of the map $d^{k-1,2}$ is in $\mathcal{C}^{k-1,3}$. Moreover, using Property (3) for $i = 0$ as well as Property (6), we deduce that

$$Im(d^{k,0}) \subseteq \mathrm{Ker}(\delta^{k,1}) = \mathrm{Im}(\delta^{k-1,2}).$$

Therefore,

$$\mathrm{Im}(d_S^{2k}) = \mathrm{Im}(\delta^{k-1,2}) \oplus \mathrm{Im}(d^{k-1,2}).$$

In the case $k = 0$, we immediately see that $\mathrm{Im}(d_S^0)$ coincides with $\mathrm{Im}(d^{0,0})$.

To conclude, a cohomology group $H^{2k+1}(d_S)$ of odd degree $2k+1$, for $k \geq 1$, satisfies the following identity:

$$H^{2k+1}(d_S) = \frac{\mathrm{Ker}(\delta^{k,1}) \oplus \mathrm{Im}(d^{k-1,2})}{\mathrm{Im}(\delta^{k-1,2}) \oplus \mathrm{Im}(d^{k-1,2})} = \frac{\mathrm{Im}(\delta^{k-1,2})}{\mathrm{Im}(\delta^{k-1,2})} = \{0\}\,.$$

Finally, if $k = 0$, we have

$$H^1(d_S) = \frac{\mathrm{Ker}(\delta^{0,1}) \cap \mathrm{Ker}(d^{0,1})}{\mathrm{Im}(d^{0,0})}.$$

□

Applying the previous theorem to the shifted double complex corresponding to the BRST cohomology complex of our matrix model, we finally arrive at an explicit relationship between the BRST cohomology groups and the generalized Lie algebra cohomology groups, which agrees with the one found earlier in Section 5.3.3 through an explicit computation.

Theorem 13. *Let the shifted double complex* $(\mathcal{C}^{i,j}, d^{i,j} \oplus \delta^{i,j})$ *with* $i \geq 0$ *and* $j = 0, \ldots, 3$ *be given by*

$$\mathcal{C}^{i,j} = \mathcal{C}^i_{sym}(\mathfrak{h}, \mathcal{C}^j_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$$

and let $(\mathcal{C}^\bullet(W, d_{\widetilde{S}}), d_{\widetilde{S}})$ *be the BRST cohomology complex for the* $U(2)$*-matrix model. Then the following isomorphisms among the corresponding cohomology groups hold, for all* $k \geq 0$*:*

- $H^{2k}(X, d_S) \cong H^k_{sym}(\mathfrak{h}, Z^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \oplus H^{k-1}_{sym}(\mathfrak{h}, Z^2_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$;
- $H^{2k+1}(X, d_S) \cong H^k_{sym}(\mathfrak{h}, \mathcal{C}^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) = \{0\}$.

In particular,

- ▶ $H^0(X, d_S) \cong H^0_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))$;
- ▶ $H^1(X, d_S) \cong H^0_{sym}(\mathfrak{h}, H^1_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a)))$.

Proof. The statement of the theorem follows immediately by applying Theorem 12 to the shifted double complex $(\mathcal{C}^{i,j}, d^{i,j} \oplus \delta^{i,j})$, $i \geq 0$, $j = 0, \ldots, 3$, defined by

$$\mathcal{C}^{i,j} = \mathcal{C}^i_{sym}(\mathfrak{h}, \mathcal{C}^j_{Lie}(\mathfrak{g}, \mathrm{Pol}_{\mathbb{R}}(M_a))) \ ,$$

Indeed with this choice, the complex $(\mathcal{C}^m, d_S)$, $m \in \mathbb{N}_0$, coincides with the BRST cohomology complex. □

Chapter 6

Towards extended varieties for $U(n)$-matrix models

In this chapter we present an approach to the BV analysis of $U(n)$-gauge invariant matrix models, for any degree $n \in \mathbb{N}$, using the methods which have been explained in Chapter 4 and have been applied to a $U(2)$-model in Chapter 5. It is our purpose to determine the minimal extended variety associated to a $U(n)$-matrix model.

First of all, let us recall some notation. The starting point of the construction is a $U(n)$-gauge invariant matrix model, which is defined by a pair (X_0, S_0) where:

- X_0 is the *initial configuration space*, which is defined to be the nonsingular algebraic variety given by $n \times n$ self-adjoint matrices:

  $$X_0 = \{A \in M_n(\mathbb{C}) : A^+ = A\}.$$

  By fixing a real basis, $X_0 \simeq \mathbb{A}_{\mathbb{R}}^{n^2}$, where $\mathbb{A}_{\mathbb{R}}^{n^2}$ denotes the real n^2-dimensional affine space. Let $\{x_i\}$, with $i = 1, \ldots, n^2$ be the collection of real variables which form a coordinate system on X_0. So the configuration space X_0 can also be seen as the real vector space generated by this collection of variables $\{x_i\}$:

  $$X_0 = \langle x_1, \ldots, x_{n^2} \rangle_{\mathbb{R}}.$$

From the point of view of the physical model, these $\{x_i\}$ are the *initial fields*. They are characterized by the fact that they have ghost degree 0 and even parity:

$$deg(x_i) = 0, \qquad \epsilon(x_i) = 0, \qquad \forall i.$$

- S_0 is the *initial action*. It is defined as a functional on X_0:

$$S_0 : X_0 \longrightarrow \mathbb{R}.$$

Moreover, since we are considering a $U(n)$-gauge invariant matrix model, S_0 is supposed to be invariant under the adjoint action of the gauge group $\mathcal{G} \simeq U(n)$. This condition can be rewritten more explicitly by requiring

$$S_0[UAU^*] = S_0[A]$$

for each $A \in X_0$ and $U \in U(n)$.
In Proposition 9 we proved that this condition can be restated in terms of the eigenvalues of the matrices in X_0. More precisely, let $\lambda_1 \ldots, \lambda_n$ be (real) eigenvalues of the matrices in X_0. In particular $\lambda_1, \ldots, \lambda_n$ can be thought as polynomials in the variables $\{x_i\}$:

$$\lambda_1, \ldots, \lambda_n \in \mathrm{Pol}_{\mathbb{R}}(x_i).$$

In the quoted proposition we proved that

$$S_0 \in \mathrm{Pol}_{\mathbb{R}}(R_1, \ldots, R_n) \ ,$$

where $R_1, \ldots, R_n$ are the symmetric polynomials generated by the variables $\lambda_1, \ldots, \lambda_n$.

To determine the minimal extended variety associated to the initial theory (X_0, S_0) we have to compute the minimal Tate resolution of the Jacobian ring $J(S_0)$, with

$$J(S_0) = \frac{\mathrm{Pol}_{\mathbb{R}}(x_i)}{\langle \partial_1 S_0, \ldots, \partial_{n^2} S_0 \rangle},$$

where $\mathrm{Pol}_{\mathbb{R}}(x_i)$ is the ring of regular functions on X_0 while $\partial_i S_0$ are the partial derivatives of the initial action S_0 taken with respect to the variables $x_1, \ldots, x_{n^2}$.

We recall that the minimal Tate resolution of the Jacobian ring is used to determine the minimally-extended configuration space $\widetilde{X}$ associated to a gauge theory (X_0, S_0). More precisely, the antifield and antighost field content of $\widetilde{X}$ is determined using the Tate resolution while the fields and ghost field content is obtained by imposing the physical condition that to each antifield and antighost field there should be an associated field and ghost field satisfying a certain relation with respect to the ghost degree and the parity.

As already noted in Remark 22, in order to have a Tate resolution suitable to define the minimally-extended configuration space $\widetilde{X}$ for a gauge theory, the number of generators that need to be introduced in degree -1 is already fixed. In fact, the generators of ghost degree -1 describe the antifields corresponding to the initial fields x_i, with $i = 1, \dots, n^2$. Since we have n^2 fields x_i, in degree -1 we have exactly n^2 generators, which will be denoted by x_i^*, with $i = 1, \dots, n^2$.

Step −1

From what explained above, in the notation of Appendix B we have

$$A^{-1} = \mathrm{Pol}_{\mathbb{R}}(x_i)\langle x_1^*, \dots, x_{n^2}^* \rangle.$$

In other words, A^{-1} is obtained by extending the ring $\mathrm{Pol}_{\mathbb{R}}(x_i)$ by adding the variables $x_1^*, \dots, x_{n^2}^*$, that are supposed to be Grassmannian variables. Thus

$$deg(x_i^*) = -1, \qquad \epsilon(x_i) = 1, \qquad \forall i = 1, \dots, n^2.$$

As to the coboundary operator δ associated to the Tate resolution, up to now we have

$$\delta_0 = 0 \quad \text{on } \mathrm{Pol}_{\mathbb{R}}(x_i), \qquad \delta_{-1}(x_i^*) = \partial_i S_0, \qquad \forall i = 1, \dots, n^2 \ ,$$

where, we denote the operator δ by δ_i when it is acting on an element of degree i. This action of δ is extended to the whole space A^{-1} by requiring that it acts as a graded derivation.

Step −2

To establish the number of antighosts of degree -2 that need to be introduced in the model, we have to determine a set of generators for the following cohomology group:

$$H^{-1}(A^{-1}) = \frac{\mathrm{Ker}(\delta_{-1}^{-1})}{\mathrm{Im}(\delta_{-2}^{-1})} \ .$$

In the above expression, δ_{-1}^{-1} denotes the operator δ as defined at step -1 and considered acting on elements of degree -1, while δ_{-2}^{-1} denotes once again the operator δ as defined at step -1 but this time it is considered acting on elements of degree -2.

Let φ be a generic element in A^{-1}:

$$\varphi = F_1 x_1^* + F_2 x_2^* + \cdots + F_{n^2} x_{n^2}^*,$$

with $F_1, \ldots, F_{n^2}$ polynomials in $\mathrm{Pol}_{\mathbb{R}}(x_i)$.
We want to determine the conditions that need to be imposed on these polynomials $F_1, \ldots, F_{n^2}$ for φ to be an element in $\mathrm{Ker}(\delta_{-1}^{-1})$. Explicitly, we want to solve the following equation:

$$\delta_{-1}^{-1}(\varphi) = F_1 \cdot \partial_1 S_0 + F_2 \cdot \partial_2 S_0 + \cdots + F_{n^2} \cdot \partial_{n^2} S_0 = 0. \tag{6.1}$$

In other words we are interested in determining the relations of linear dependence in the ring $\mathrm{Pol}_{\mathbb{R}}(x_i)$ that exist among the partial derivatives of the action S_0.

However, using the module structure of $\mathrm{Ker}(\delta_{-1}^{-1})$ we can restrict ourselves to determining the relations of dependence involving pairs of generators x_j^*, x_k^*. So we are interested in finding the solution of the following equation:

$$\delta_{-1}^{-1}(F_j x_j^* + F_k x_k^*) = F_j \cdot \partial_j S_0 + F_k \cdot \partial_k S_0 = 0. \tag{6.2}$$

To solve this equation, let us define:

$$D_{jk} = GCD\left(\partial_j S_0, \partial_k S_0\right) \in \mathrm{Pol}_{\mathbb{R}}(x_i)\ .$$

Then there exist two polynomials A_{jk} and B_{jk} in $\mathrm{Pol}_{\mathbb{R}}(x_i)$ such that

$$\partial_j S_0 = A_{jk} D_{jk} \qquad \partial_k S_0 = B_{jk} D_{jk}.$$

A solution for Equation (6.2) is then of the following type:

$$\psi_{jk} = -B_{jk} x_j^* + A_{jk} x_k^*.$$

So, using the notation introduced above, we have the following collection of generators for the space $\mathrm{Ker}(\delta_{-1}^{-1})$:

$$V = \left\{\psi_{jk} = -B_{jk} x_j^* + A_{jk} x_k^* :\ j,k = 1, \ldots, n^2,\ j < k\right\}.$$

As already noted, taking the module generated by the element ψ_{jk} we obtain also the whole collection of generators of $\mathrm{Ker}(\delta_{-1}^{-1})$ depending on s generators

x_j^*, with $s = 3, \dots, n^2$.
This is not the only possible type of solution to Equation (6.2): in fact, the dependence of S_0 on the variables x_i is not "direct" but we defined S_0 as a polynomial in the symmetric polynomials R_a, with $a = 1, \dots, n$, which are the ones which depend on the fields x_i. Thus the partial derivatives of S_0 are given by

$$\partial_j S_0 = \sum_{a=1}^{n} \frac{\partial S_0}{\partial R_a} \frac{\partial R_a}{\partial x_j}.$$

As a consequence, (6.1) can be rewritten as follows:

$$\delta_{-1}^{-1}(\varphi) = \sum_{a=1}^{n} \frac{\partial S_0}{\partial R_a} \left(F_1 \frac{\partial R_a}{\partial x_1} + \dots + F_{n^2} \frac{\partial R_a}{\partial x_{n^2}} \right) = 0. \tag{6.3}$$

One possibility to solve this equation is to set all factors that multiply the partial derivatives of the action S_0 taken with respect to the symmetric polynomials R_a to zero: in other words, we are looking for polynomials $F_1, \dots, F_{n^2}$ in $\mathrm{Pol}_{\mathbb{R}}(x_i)$ such that

$$F_1 \frac{\partial R_a}{\partial x_1} + \dots + F_{n^2} \frac{\partial R_a}{\partial x_{n^2}} = 0. \tag{6.4}$$

Of course, this condition needs to be imposed only for the values of the index a for which $\partial S_0 / \partial R_a \neq 0$. Let $\widetilde{A}$ denote the subset of $\{1, \dots, n\}$ containing all the indices that identify symmetric polynomials R_a which appear in the action S_0:

$$\widetilde{A} = \left\{ j \in \mathbb{N} : 1 \leqslant j \leqslant n, \frac{\partial S_0}{\partial R_j} \neq 0 \right\}, \qquad |\widetilde{A}| = m.$$

The conditions stated in Equation (6.4) define a system of m equations with n^2 variables. To solve this system we consider the matrix defined by the parameters $\partial R_a / \partial x_i$:

$$N = \begin{bmatrix} \frac{\partial R_{a_1}}{\partial x_1} & \frac{\partial R_{a_1}}{\partial x_2} & \dots & \frac{\partial R_{a_1}}{\partial x_{n^2}} \\ \vdots & & & \vdots \\ \frac{\partial R_{a_m}}{\partial x_1} & \frac{\partial R_{a_m}}{\partial x_2} & \dots & \frac{\partial R_{a_m}}{\partial x_{n^2}} \end{bmatrix}.$$

In the generic case, the matrix N has maximal rank, namely: $rank(N) = m$. Note that N might have not maximal rank only if either the columns or the rows of N are linear dependent, which would imply the existence of a linear dependence among the partial derivatives of the polynomials R_a with respect to the variables x_i.

Assumption: we assume that each $m \times m$ submatrix of N has maximal rank.

Under this assumption any m-tuple taken in the set of indices $i = 1, \dots, n^2$ identifies an $m \times m$ invertible submatrix of N.
Thus for any m-tuple $\{i_1, i_2, \dots, i_m\} \subseteq \{1, \dots, n^2\}$ the homogeneous system has only one solution, namely the trivial solution obtained by setting all polynomials $F_{i_1}, \dots, F_{i_m}$ to 0.

If we consider an $m+1$-tuple, it would determine a system with a 1-dimensional space of solutions generated (as a module over $\mathrm{Pol}_{\mathbb{R}}(x_i)$) by an element $\varphi_{i_1,\dots,i_{m+1}}$. Any other solution of (6.4) that corresponds to an element φ that depends on more than $m+1$ generators x_i^* can be written as a combination with coefficients in $\mathrm{Pol}_{\mathbb{R}}(x_i)$ of these "basic generators" $\varphi_{i_1,\dots,i_{m+1}}$.

In case that

$$GCD\left(\frac{\partial R_a}{\partial x_1}, \dots, \frac{\partial R_a}{\partial x_{n^2}}\right) = 1, \qquad \forall a \in \widetilde{A}, \tag{6.5}$$

these basic generators $\varphi_{i_1,\dots,i_m}$ would have the following form:

$$\begin{aligned}\varphi_{i_1 \dots i_{m+1}} = \; & \left[\textstyle\sum_{\sigma \in S_{m+1} \backslash i_1} (-1)^{\sigma+1} \frac{\partial R_{a_1}}{\partial x_{\sigma(i_2)}} \cdot \frac{\partial R_{a_2}}{\partial x_{\sigma(i_3)}} \cdots \frac{\partial R_{a_m}}{\partial x_{\sigma(i_{m+1})}}\right] x_{i_1}^* \\ & + \dots + \left[\textstyle\sum_{\sigma \in S_{m+1} \backslash i_{m+1}} (-1)^{\sigma+1} \frac{\partial R_{a_1}}{\partial x_{\sigma(i_1)}} \cdot \frac{\partial R_{a_2}}{\partial x_{\sigma(i_2)}} \cdots \frac{\partial R_{a_m}}{\partial x_{\sigma(i_m)}}\right] x_{i_m}^* \;, \end{aligned} \tag{6.6}$$

where by S_{m+1} we denote the group of permutation of the indices $i_1, \dots, i_{m+1}$, while by $S_{m+1} \backslash i_k$ we denote the subgroup of S_{m+1} that leaves the index i_k invariant.

To justify the expression (6.6), we first consider what happens for the index a_1. In this case, the solutions to the first equation in the system (6.4) are linear combinations on $\mathrm{Pol}_{\mathbb{R}}(x_i)$ of the solutions

$$\varphi_{i_1 i_2} = \frac{\partial R_{a_1}}{\partial x_{i_2}} x_{i_1}^* - \frac{\partial R_{a_1}}{\partial x_{i_1}} x_{i_2}^*,$$

with $i_1, i_2 = 1, \dots, n^2$, $i_1 < i_2$.
Let Sol_{a_1} be the space of solutions for the first equation in the system. Then

$$Sol_{a_1} = \langle \varphi_{i_1 i_2} : i_1, i_2 = 1, \dots, n^2,\ i_1 < i_2 \rangle.$$

Now we consider the second equation in the system, identified by the index a_2. From the hypothesis on the rank of the matrix N we have that, to solve both the first and the second equation, we have to consider elements φ that depend at least on three generators $x^*_{i_1}$, $x^*_{i_2}$, $x^*_{i_3}$. Then, since the partial derivatives of the polynomials R_a are supposed to be coprime, we have

$$Sol_{a_1} \cap Sol_{a_2} = \langle \varphi_{i_1 i_2 i_3} : i_1, i_2, i_3 = 1, \ldots, n^2,\ i_1 < i_2 < i_3 \rangle \ ,$$

where

$$\begin{aligned} \varphi_{i_1 i_2 i_3} = & \left[-\frac{\partial R_{a_1}}{\partial x_{i_2}} \cdot \frac{\partial R_{a_2}}{\partial x_{i_3}} + \frac{\partial R_{a_1}}{\partial x_{i_3}} \cdot \frac{\partial R_{a_2}}{\partial x_{i_2}} \right] x^*_{i_1} + \left[+ \frac{\partial R_{a_1}}{\partial x_{i_3}} \cdot \frac{\partial R_{a_2}}{\partial x_{i_1}} - \frac{\partial R_{a_1}}{\partial x_{i_1}} \cdot \frac{\partial R_{a_2}}{\partial x_{i_3}} \right] x^*_{i_2} \\ & + \left[-\frac{\partial R_{a_1}}{\partial x_{i_2}} \cdot \frac{\partial R_{a_2}}{\partial x_{i_1}} + \frac{\partial R_{a_1}}{\partial x_{i_1}} \cdot \frac{\partial R_{a_2}}{\partial x_{i_2}} \right] x^*_{i_3} . \end{aligned}$$

To obtain (6.6), we have to iterate the process explained above and take care of the sign of the permutations of the indices.
In the case in which the condition (6.5) is not satisfied, the generators $\varphi_{i_1,\ldots,i_{m+1}}$ can still be defined but we would get a much more complicated formula to describe them.

Thus we conclude that $\mathrm{Ker}(\delta^{-1}_{-1})$ can be seen as the module generated by the direct sum of two modules that describe the two possible kinds of solutions to (6.1):

$$\mathrm{Ker}(\delta^{-1}_{-1}) = \langle \langle \psi_{jk} = -B_{jk} x^*_j + A_{jk} x^*_k \rangle \oplus \langle \varphi_{i_1,\ldots,i_{m+1}} \rangle \rangle, \tag{6.7}$$

where $1 \leqslant j < k \leqslant n^2$ and $1 \leqslant i_1 < \cdots < i_{m+1} \leqslant n^2$.

Thus $\mathrm{Ker}(\delta^{-1}_{-1})$ has:

- $\binom{n^2}{2}$ generators of the first type,

- $\binom{n^2}{m+1}$ generators of the second type,

where m, $1 \leqslant m \leqslant n$, is the number of symmetric polynomials R_a that appear in the definition of the initial action S_0.

Note: some attention should be given to the symmetric polynomial R_1, defined as the sum of all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a generic matrix in X_0.

Now, in the case in which we pick a basis for X_0 composed of self-adjoint matrices σ_i, $i = 1, \dots, n^2$, where all but for example σ_{n^2} have trace equal to zero, then

$$R_1 = \sum \lambda_1 + \cdots + \lambda_n = x_{n^2},$$

where x_{n^2} is the real variable that is dual to the base element σ_{n^2}.
In fact, the relationship between the symmetric polynomials R_a defined by the eigenvalues of a generic matrix A in X_0 and the trace of powers of it is given by

$$\begin{aligned} R_l = \quad & -\tfrac{Tr(A^l)}{l} + \textstyle\sum_{i+j=l} \tfrac{Tr(A^i)\cdot Tr(A^j)}{i\cdot j} - \textstyle\sum_{i+j+k=l} \tfrac{Tr(A^i)\cdot Tr(A^j)\cdot Tr(A^k)}{i\cdot j\cdot k} \\ & + \cdots + (-1)^l \tfrac{Tr(A)^l}{l!}. \end{aligned}$$

Thus with this choice for a basis we have $R_1 = x_{n^2}$.
Therefore,

$$\frac{\partial R_1}{\partial x_1} = \cdots = \frac{\partial R_1}{\partial x_{n^2-1}} = 0 \qquad \frac{\partial R_1}{\partial x_{n^2}} = 1.$$

Hence the condition (6.4) would already be satisfied for $a = 1$ by only setting $F_{n^2} = 0$. With this choice of basis we would have that, if $1 \in \widetilde{A}$, then the second type of generators in $\mathrm{Ker}(\delta^{-1}_{-1})$ cannot depend on the generator $x^*_{n^2}$ and the value $a = 1$ should not be considered in solving the system, which would turn out to be a system of $m-1$ equation. Thus the generators of the second type would be determined by m-tuple of indices $1 \leqslant i_1 < \cdots < i_m < n^2$.

Now we have to determine a set of generators for the module $\mathrm{Im}(\delta^{-1}_{-2})$, which by definition is the module of images via the operator δ^{-1} of elements in A^{-1} of degree -2. Let χ be a generic element in A^{-1} of degree -2, i.e.,

$$\chi = \sum_{j<k} G_{jk} x^*_j x^*_k,$$

with G_{jk} a polynomial in $\mathrm{Pol}_{\mathbb{R}}(x_i)$ for each j, $k = 1, \dots, n^2$, $j < k$.
Then

$$\delta^{-1}_{-2}(\chi) = \sum_{j<k} G_{jk} \left(\partial_j S_0 \cdot x^*_k - \partial_k S_0 \cdot x^*_j\right).$$

Hence $\mathrm{Im}(\delta^{-1}_{-2})$ can be seen as the module generated on the ring $\mathrm{Pol}_{\mathbb{R}}(x_i)$ by the following $\binom{n^2}{2}$-generators ξ_{jk}, with j, $k = 1, \dots, n^2$, $j < k$:

$$\mathrm{Im}(\delta^{-1}_{-2}) = \langle \xi_{jk} = (-\partial_k S_0) x^*_j + (\partial_j S_0) x^*_k \rangle.$$

By looking at the description of $\mathrm{Ker}(\delta_{-1}^{-1})$ in terms of its generators given in (6.7), we immediately see that $\mathrm{Im}(\delta_{-1}^{-1})$ is contained in the first summand of the decomposition of $\mathrm{Ker}(\delta_{-1}^{-1})$. Moreover, given a generator $\psi_{jk} \in \mathrm{Ker}(\delta_{-1}^{-1})$, the following equivalence holds:

$$\psi_{jk} \in \mathrm{Im}(\delta_{-2}^{-1}) \Leftrightarrow D_{jk} = GCD(\partial_j S_0, \partial_k S_0) = 1.$$

Assumption: we assume that

$$D_{jk} = GCD(\partial_j S_0,\ \partial_k S_0) = 1, \qquad \forall j, k = 1, \ldots, n^2. \tag{6.8}$$

Under this hypothesis,

$$H^{-1}(A^{-1}) = \langle \varphi_{i_1,\ldots,i_{m+1}} \rangle.$$

Thus we have to introduce $\binom{n^2}{m+1}$ real generators of degree -2 or, that is, we have to extend the configuration space by the introduction of $r_2 = \binom{n^2}{m+1}$ antighost fields C_k^*.

Therefore,

$$A^{-2} = \mathrm{Pol}_{\mathbb{R}}(x_i)(\langle x_1^*, \ldots, x_{n^2}^* \rangle_{-1} \oplus \langle C_1^*, \ldots, C_{r_2}^* \rangle_{-2}).$$

In the following computation it might be useful to keep track of the generator $\varphi_{i_1 \ldots i_{m+1}}$ of degree -1 that determines a generator C_α^*. For this reason we also use the notation $C^*_{i_1 \ldots i_{m+1}}$ for the generator of degree -2 determined by $\varphi_{i_1 \ldots i_{m+1}}$. However, being a somewhat heavy notation, we use it only when absolutely necessary.

The coboundary operator δ^{-1} acts on A^{-2} as an operator δ^{-2} by requiring that it is a graded derivation such that it acts as follows on the generators of degree -2:

$$\delta_{-2}^{-2}(C^*_{i_1 \ldots i_{m+1}}) = \varphi_{i_1 \ldots i_{m+1}}.$$

Step -3

To determine how many generators of degree -3 have to be introduced in this step of the algorithm, we analyze the cohomology group

$$H^{-2}(A^{-2}) = \frac{\mathrm{Ker}(\delta_{-2}^{-2})}{\mathrm{Im}(\delta_{-3}^{-2})}.$$

As already explained in Chapter 4, it may be possible that to determine the part of the Tate resolution that plays a role in the construction of an extended variety, instead of having to compute the cohomology group $H^{-2}(A^{-2})$, to solve this step of the algorithm we may just compute the linear relations between the quantities $\delta(C^*_\alpha)$, where C^*_α are the generators of degree -2 introduced in the previous step.

Let $\chi \in A^{-2}$ be a generic linear combination with coefficients in $\mathrm{Pol}_{\mathbb{R}}(x_i)$ of the generators C^*_α, where $\alpha = (i_1 \dots i_{m+1})$:

$$\chi = \sum_I G_\alpha C^*_\alpha,$$

with $I = \{\alpha = (i_1, \dots, i_{m+1}) : 1 \leqslant i_1 < \dots < i_{m+1} \leqslant n^2\}$. Thus we have to solve the following equation:

$$\begin{aligned} \delta^{-2}_{-2}(\chi) &= \textstyle\sum_I G_\alpha \cdot \varphi_{i_1 \dots i_{m+1}} \\ &= \textstyle\sum_I G_\alpha \cdot \left[F^\alpha_{i_1} x^*_{i_1} + \dots + F^\alpha_{i_{m+1}} x^*_{i_{m+1}} \right] = 0 \ . \end{aligned} \tag{6.9}$$

We recall that all the generators $x^*_1, \dots, x^*_{n^2}$ are supposed to be independent variables and that the polynomials $F^\alpha_{i_k}$ have been determined in the previous step of the algorithm. Once again, using the fact that $\mathrm{Ker}(\delta^{-2}_{-2})$ has the structure of a module, it is enough to determine the generators that involve the minimum number of variables C^*_α.
By looking at (6.9) we see that, in order to find a non-trivial solution we have to consider at least $m+2$ generators C^*_α such that each pair of these generators has m out of $m+1$ indices in common.
More precisely, the generators of $\mathrm{Ker}(\delta^{-2}_{-2})$ are of the following type:

$$\xi_{i_1 \dots i_{m+2}} = \sum_{j=1}^{m+2} (-1)^{j-1} \left[G_{i_1 \dots \hat{i}_j \dots i_{m+2}} C^*_{i_1 \dots \hat{i}_j \dots i_{m+2}} \right],$$

with $1 \leqslant i_1 < \dots < i_{m+2} \leqslant n^2$, where $G_{i_1 \dots \hat{i}_j \dots i_{m+2}}$ are suitable polynomials in $\mathrm{Pol}_{\mathbb{R}}(x_i)$.

A more explicit description of the generators of $\mathrm{Ker}(\delta^{-1}_{-1})$ can be obtained under the assumption that all partial derivatives of the symmetric polynomials R_a, taken with respect to the variables x_i, are coprime (see the condition of Equation (6.5)). Under this assumption, the generators $\xi_{i_1 \dots i_{m+2}}$ take the following form:

$$\xi_{i_1\dots i_{m+2}} = \sum_{j=1}^{m+2} (-1)^{j-1} \left[\sum_{a\in\widetilde{A}} \frac{\partial R_a}{\partial x_{i_j}} \cdot C^*_{i_1\dots\hat{i}_j\dots i_{m+2}} \right].$$

Moreover, it is possible to check that these generators do not belong to $\mathrm{Im}(\delta^{-2}_{-3})$. Thus the set of generators of $H^{-2}(A^{-2})$ in which we are interested is

$$\{\xi_{i_1\dots i_{m+2}} : 1 \leqslant i_1 < \cdots < i_{m+2} \leqslant n^2\} \subseteq H^{-2}(A^{-2}).$$

This implies that we have to enlarge the graded algebra A^{-2} by the introduction of $\binom{n^2}{m+2}$ Grassmannian generators of degree -3. Thus the algebra A^{-3} is

$$A^{-3} = Sym_{\mathrm{Pol}_{\mathbb{R}}(x_i)}(\langle x_1^*, \dots, x_{n^2}^*\rangle_{-1} \oplus \langle C_1^*, \dots, C_{r_2}^*\rangle_{-2} \oplus \langle E_1^*, \dots, E_{r_3}^*\rangle_{-3})\ ,$$

with $r_2 = \binom{n^2}{m+1}$, $r_3 = \binom{n^2}{m+2}$, and m the number of symmetric polynomials R_a appearing in the expression of the initial action S_0, $1 \leqslant m \leqslant n$.

The behavior of the algorithm in further steps for lower degrees can be deduced from what we have observed in degree -3. In fact, in each step we are assuming that we introduce the minimal number of new generators that correspond to a specific subset of generators for the cohomology group $H^{-k}(A^{-k})$. From the construction presented in Chapter 4, we deduce that we can restrict ourselves to analyzing linear combinations of the generators introduced in the previous step with coefficients in $\mathrm{Pol}_{\mathbb{R}}(x_i)$. Thus we reduce to solving equations similar to (6.9), with the only difference that, instead of the generators x_i^*, these now involve generators of lower degree.

Remark 42

Under the hypothesis of introducing only the minimal number of variables at each step of the algorithm and of concentrating only on the generators of type β, we can also ensure that the algorithm itself will end after a finite number of steps. In fact, if we are not introducing any unnecessary variables, to conclude the algorithm we only have to eliminate all initial symmetries of the theory. These symmetries are represented by the relations existing among the partial derivatives of the action S_0, or equivalently, they are given by the elements in the module $\mathrm{Ker}(\delta^{-1}_{-1})$.

In the above construction, we assumed that we only have the second type of generators for the module $\mathrm{Ker}(\delta^{-1}_{-1})$, viz. the ones determined by linear relations among the partial derivatives of the symmetric polynomials R_a. Under this

hypothesis, a set of generators for this module is determined by the $(m+1)$-tuples of indices $i_1, \dots, i_{m+1}$.
Above, we discussed how the set of elements $\varphi_{i_1 \dots i_{m+1}}$, $1 \leqslant i_1 < \cdots < i_{m+1} \leqslant n^2$, defines a basis of $\mathrm{Ker}(\delta_{-1}^{-1})$ as a module. However, when we introduce the new variables $C^*_{i_1 \dots i_{m+1}}$ subject to

$$\delta_{-2}^{-2}(C^*_{i_1 \dots i_{m+1}}) = \varphi_{i_1 \dots i_{m+1}},$$

we are eliminating only the generators $\varphi_{i_1 \dots i_{m+1}}$ but not the other elements in $\mathrm{Ker}(\delta_{-1}^{-1})$ obtained as linear combinations of them on the ring $\mathrm{Pol}_{\mathbb{R}}(x_i)$.
In other words, we are eliminating only the symmetries involving $m+1$ partial derivatives of the polynomials R_a: all relations that involve more than $m+1$ partial derivatives are still present and need to be eliminated, too. This is exactly what we do in the step -3: in fact, we have seen that the generators $\xi_{i_1 \dots i_{m+2}}$ are determined by $(m+2)$-tuples of indices and so they involve $(m+2)$ partial derivatives of the polynomials R_a.
To conclude, at each step of the algorithm we are increasing the number of partial derivatives of R_a considered. Thus the algorithm will finish after a finite number of steps. More precisely, it will finish as soon as we introduce a variable to compensate the residual relation existing among all the derivatives of the polynomials R_a taken with respect to the variables x_i, with $i = 1, \dots, n^2$.

To conclude we state the result obtained by the previous computation.

Proposition 17
In the notation summarized above, let us assume that:

- $GCD\left(\frac{\partial S_0}{\partial x_j}, \frac{\partial S_0}{\partial x_k}\right) = 1, \quad \forall j, k = 1, \dots, n^2.$
- the matrix N, with

$$N = \begin{bmatrix} \frac{\partial R_{a_1}}{\partial x_1} & \frac{\partial R_{a_1}}{\partial x_2} & \cdots & \frac{\partial R_{a_1}}{\partial x_{n^2}} \\ \vdots & & & \vdots \\ \frac{\partial R_{a_m}}{\partial x_1} & \frac{\partial R_{a_m}}{\partial x_2} & \cdots & \frac{\partial R_{a_m}}{\partial x_{n^2}} \end{bmatrix}$$

  has maximal rank and each $m \times m$ submatrix of N has non-zero determinant.

Then the minimally extended configuration space $\widetilde{X}$ is defined as

$$\widetilde{X} = W \oplus W^*[1],$$

where

$$W^*[1] = \langle x_1^*, .., x_{n^2}^* \rangle_{-1} \oplus \langle C_1^*, .., C_{r_2}^* \rangle_{-2} \oplus \langle E_1^*, .., E_{r_3}^* \rangle_{-3} \oplus \cdots \oplus \langle L^* \rangle_{-(n^2-m+1)}.$$

Thus $W^*[1]$ is a graded vector space such that its homogeneous components are generated by the following variables:

- in degree -1: the basis is composed of n^2 Grassmannian variables $x_1^*, \ldots, x_{n^2}^*$ which describe the antifields corresponding to the initial fields $x_1, \ldots, x_{n^2}$;
- in degree $-k$ with $-2 \geqslant -k \geqslant -(n^2 - m + 1)$: the basis is composed of $\binom{n^2}{m+k-1}$ variables. These variables are real if $-k$ is even while they are Grassmannian if $-k$ is odd.

Thus the total number of ghosts that need to be add to the theory is

$$\sum_{j=m+1}^{n^2} \binom{n^2}{j}.$$

Remark 43
As we have already underlined in the previous construction, there is a particular choice for the coordinate system x_i that could help in introducing even fewer ghosts: in fact, if we fix a real basis of X_0 given by n^2 self-adjoint matrices $\sigma_1, \ldots, \sigma_{n^2}$ such that only σ_{n^2} has nonvanishing trace, then we have seen that the second type of generators of $\mathrm{Ker}(\delta_{-1}^{-1})$ cannot depend on the antifields $x_{n^2}^*$.

Therefore, in this case the number of generators of degree -2 is

$$\begin{cases} r_2 = \binom{n^2-1}{m} & \text{if } \frac{\partial S_0}{\partial R_1} \neq 0 \ ; \\ r_2 = \binom{n^2-1}{m+1} & \text{if } \frac{\partial S_0}{\partial R_1} = 0 \ . \end{cases}$$

Therefore, making this particular choice for the basis of X_0, the number of ghosts that need to be added to the model is:

$$\begin{cases} \sum_{j=m}^{n^2-1} \binom{n^2-1}{j} & \text{if } \frac{\partial S_0}{\partial R_1} \neq 0 \ ; \\ \sum_{j=m+1}^{n^2-1} \binom{n^2-1}{j} & \text{if } \frac{\partial S_0}{\partial R_1} = 0 \ . \end{cases} \tag{6.10}$$

To conclude, let us make a comparison with what we explicitly found for the case $n = 2$ in Chapter 5. In that case, we established that the minimal number

of ghost that we had to introduce was $3 + 1 = 4$.
This coincides exactly with what we have just stated. In fact, in doing the explicit computation for the $U(2)$-model we fixed as basis for X_0 the one given by the Pauli matrices. Thus we were in the case in which only one of the matrices in the basis of X_0 had nonzero trace.
Moreover, we analyzed the case in which the action S_0 depends on both symmetric polynomials R_1 and R_2, so that, in agreement with (6.10), the minimal number of ghosts is obtained by imposing $n = m = 2$.

Finally, we recall that in the explicit analysis of the $U(2)$-matrix model we distinguished two cases (see Section 5.1.2): in fact, in the notation used in that context, the partial derivatives of the initial action S_0 with respect to the variables M_1, M_2, M_3, M_4 satisfied

$$\partial_1 S_0 = M_1 \varphi \qquad \partial_2 S_0 = M_2 \varphi \qquad \partial_3 S_0 = M_3 \varphi \ ,$$

for $\varphi \in \mathrm{Pol}_{\mathbb{R}}(M_a)$.
Then there were two possibilities for the minimal Tate resolution of the Jacobian ring $J(S_0)$: in one of them we were assuming that φ and $\partial_4 S_0$ were coprime polynomials, while for the second we assumed that $GCD(\varphi, \partial_4 S_0) \notin \mathbb{R}$.
It is immediate that the condition imposed in the first case is equivalent to the requirement that all the partial derivatives of S_0 taken with respect to the variables $M_1, \ldots, M_4$ are coprime. Therefore, through direct computation in the case of $n = 2$, we identified the same condition as the one found in (6.8) for the general case.

Remark 44
In Proposition 17 we determined the minimally-extended configuration space $\widetilde{X}$ for a $U(n)$-gauge invariant matrix model. However, this was possible only under some hypotheses concerning the partial derivatives of the action S_0 and of the symmetric polynomials R_a with respect to the variables x_i. These conditions were introduced in step -2 of the algorithm, where we computed the module $\mathrm{Ker}(\delta_{-1}^{-1})$. In fact, at that point we found that there were two possible kinds of generators for $\mathrm{Ker}(\delta_{-1}^{-1})$:

- ► $\psi_{jk} = \frac{1}{D_{jk}}[-(\partial_k S_0) \cdot x_j^* + (\partial_j S_0) \cdot x_k^*]$
  with $D_{jk} = GCD(\partial_j S_0, \partial_k S_0)$ and $1 \leqslant i < j \leqslant n^2$;
- ► $\varphi_{i_1 \ldots i_{m+1}}$, with $1 \leqslant i_1 < \cdots < i_{m+1} \leqslant n^2$, which involves the partial derivatives of the polynomials R_a (see (6.6)).

Note that, while the generators of the first type are related to the way in which the action S_0 depends on the coordinate system, the second kind of generators involves the partial derivatives of the polynomials R_a.
The fundamental difference among these two types of generators is due to the fact that any $U(n)$-gauge invariant action S_0 is forced to depend on these polynomials R_a: therefore, the relations that involve these R_a are in some sense the "essential" ones. In other words, since this second type of generators involve the polynomials R_a related to $U(n)$-gauge invariance, they are the only ones induced by the gauge structure. Even though it is possible that relations appear also at the level of partial derivatives of the action S_0, these additional relations are not required from the model itself. For this reason, we can restrict our analysis to the case in which only these "fundamental" generators appear, i.e., the generators $\varphi_{i_1 \ldots i_{m+1}}$ involving the partial derivatives of the polynomials R_a.

For the future:
The introduction of the minimally extended configuration space $\widetilde{X}$ for a generic $U(n)$-gauge invariant matrix model is only the first step towards constructing an extended variety for this kind of theory. To continue with the analysis of a generic $U(n)$-gauge invariant matrix model, the next steps would be the following:

- First we have to determine the extended action $\widetilde{S}$, which plays a fundamental role in the definition of the coboundary operator for the BRST cohomology complex. Since we have determined the extended configuration space by using a Tate resolution for the Jacobian ring, the corresponding linear approximation of the action can be immediately deduced. Thus we have all we need to apply the algorithm explained in Section 4.2. However, up to know it still needs to be determined if it is possible also for the generic case (as it was for the case $n = 2$) to determine an extended action on this extended configuration space considering only generators of type β (see Remark 27).

- Once this minimal extended variety has been constructed, in order to arrive at the corresponding BRST cohomology complex, a gauge-fixing procedure needs to be implemented. To determine the type and the number of trivial pairs necessary to obtain a proper gauge-fixed action it is enough to know the level of reducibility of the theory and then follow the procedure presented in Section 3.3. Note that the construction done in this chapter actually determines the level of reducibility of these theories. Indeed, we have found that in the generic case a $U(n)$-gauge invariant matrix model is a gauge theory with level of reducibility $L = n^2 - m - 1$.

Therefore, since the minimally-extended configuration space has been determined, we have all we need to continue with the construction of the BRST-cohomology complex, following the general algorithm of Chapter 5.
However, the application of the algorithm to determine a BV action in this general setting is not immediate. It would be interesting to analyze the BRST cohomology complex also for a generic $U(n)$-gauge invariant matrix model, which we leave for future research.

Chapter 7

Noncommutative geometry and the BV approach

The aim of this chapter is to reconsider the material discussed so far from the point of view of noncommutative geometry. More precisely, in Chapter 5 we applied the BV construction on a $U(2)$-gauge invariant matrix model, arriving at a corresponding minimal extended variety and an explicit computation of the BRST cohomology. The starting point for all this was the pair (X_0, S_0), composed of an initial configuration space X_0 and a gauge invariant action S_0. This pair was first introduced in Section 2.3 as the gauge theory naturally derived from a finite-dimensional spectral triple on the algebra $M_2(\mathbb{C})$. Thus this matrix model was originally defined in the noncommutative geometry setting. The main goal of this chapter is to also consider the BV construction from the point of view of noncommutative geometry, restricting ourselves to the $U(2)$-matrix model already analyzed.

- ▶ The aim of Section 7.1 is to introduce what we call a *BV-spectral triple* for the $U(2)$-gauge invariant matrix model, corresponding to the minimally extended theory $(\widetilde{X}, \widetilde{S})$ already determined.
  This BV-spectral triple is composed of an algebra, a Hilbert space, and a self-adjoint operator on this Hilbert space, and is defined in such a way that all fields and ghost fields that appear in the extended configuration space are components of the vectors of the Hilbert space in question, all antifields and antighost fields are components of the operator and, finally, the fermionic action corresponding to this operator coincides with the BV action of the

minimally extended theory. Thus with the introduction of a BV-spectral triple we collect all elements constituting the extended theory in the structure of a spectral triple. At the end of the section we also describe how the properties of $(\widetilde{X}, \widetilde{S})$, such as the distinction between fields and antifields and their parity, are translated into the language of noncommutative geometry, giving a geometric interpretation of them.

- In Section 7.2 we include in the construction also the trivial pairs: more precisely, we introduce the *BV-auxiliary spectral triple* corresponding to the $U(2)$-gauge invariant matrix model. The direct sum of the BV-spectral triple together with the BV-auxiliary spectral triple will give a noncommutative geometric description of the pair $(\widetilde{X}_{tot}, \widetilde{S}_{tot})$ for the $U(2)$-gauge invariant matrix model.

7.1 The BV-spectral triple

In this section an important role will be played by the device given by *real spectral triples*, which have already been treated in Chapter 2, (refer to Definition 3). Before proceeding, we quickly recall from Chapter 5 the notation used for our matrix model of degree $n = 2$:

- (X_0, S_0) denotes the *initial theory*;
- X_0 is the *initial configuration space*. It has the structure of a real vector space, namely
$$X_0 = \langle M_1, M_2, M_3, M_4 \rangle_{\mathbb{R}}$$
with M_a, $a = 1, \ldots, 4$ the initial real fields. Here, $\langle\ \rangle_{\mathbb{R}}$ denotes the real vector space generated by a set of elements;
- S_0 is the *initial action*. This is a regular function on X_0 taking real values. Moreover, S_0 is gauge invariant under the adjoint action of the unitary group $U(2)$. As already seen in (5.4), S_0 has to be of the form
$$S_0 = \sum_{k=0}^{r} (M_1^2 + M_2^2 + M_3^2)^k \; g_k(M_4),$$
with $g_k(M_4) \in \mathrm{Pol}_{\mathbb{R}}(M_4)$ for any value of k.
- By $(\widetilde{X}, \widetilde{S})$ we denote the minimally *extended theory*, corresponding to the initial theory (X_0, S_0);

- $\widetilde{X}$ is the *extended configuration space*. This has the structure of a graded vector space with homogeneous components $\widetilde{X}_k$ of degree $k = -3, \ldots, 2$. More explicitly,

$$\begin{aligned} \widetilde{X} = \quad & \langle E^* \rangle_{\mathbb{R}} \oplus \langle C_1^*, C_2^*, C_3^* \rangle_{\mathbb{R}} \oplus \langle M_1^*, M_2^*, M_3^*, M_4^* \rangle_{\mathbb{R}} \\ & \oplus \langle M_1, M_2, M_3, M_4 \rangle_{\mathbb{R}} \oplus \langle C_1, C_2, C_3 \rangle_{\mathbb{R}} \oplus \langle E \rangle_{\mathbb{R}} \ , \end{aligned} \tag{7.1}$$

  with:

  - E^*: Grassmannian variable with ghost degree -3;
  - C_1^*, C_2^*, C_3^*: real variables with ghost degree -2;
  - $M_1^*, M_2^*, M_3^*, M_4^*$: Grassmannian variables with ghost degree -1;
  - M_1, M_2, M_3, M_4: real variables with ghost degree 0;
  - C_1, C_2, C_3: Grassmannian variables with ghost degree 1;
  - E: real variable with ghost degree 2.

- $\widetilde{S}$ is the *extended action*, defined as the sum of the initial action S_0 and the *BV action* S_{BV}. For our theory of interest, S_{BV} is given by

$$\begin{aligned} S_{BV} = \quad & M_1^*(M_2C_3 - M_3C_2) + M_2^*(M_3C_1 - M_1C_3) + M_3^*(M_1C_2 - M_2C_1) \\ & + C_1^*(M_1E + C_2C_3) + C_2^*(M_2E - C_1C_3) + C_3^*(M_3E + C_1C_2). \end{aligned} \tag{7.2}$$

In the remaining of this section we are going to construct a new spectral triple, denoted by $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, which corresponds to the extended theory $(\widetilde{X}, \widetilde{S})$. The extended theory $(\widetilde{X}, \widetilde{S})$ determines only the Hilbert space $\mathcal{H}_{BV}$, the real structure J_{BV} and the linear operator D_{BV}. For this reason these are the first elements which will be introduced, after which the algebra $\mathcal{A}_{BV}$ is discussed.

The extended Hilbert space H_{BV}

Let $\mathcal{H}_{BV}$ be the following Hilbert space:

$$\mathcal{H}_{BV} = \mathcal{H}_M \oplus \mathcal{H}_C = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

The inner product structure on $\mathcal{H}_{BV}$ is naturally defined by the Hilbert-Schmidt inner product on each summand $M_2(\mathbb{C})$ in $\mathcal{H}_{BV}$:

$$\begin{array}{rccc} \langle \ \ , \ \ \rangle : & \mathcal{H}_{BV} \times \mathcal{H}_{BV} & \longrightarrow & \mathbb{C} \\ & \Big((A_1, A_2), (B_1, B_2)\Big) & \mapsto & Tr(A_1^* B_1) + Tr(A_2^* B_2) \end{array}$$

with A_1, B_1 matrices in $\mathcal{H}_M \simeq M_2(\mathbb{C})$ and A_2, B_2 matrices in $\mathcal{H}_C \simeq M_2(\mathbb{C})$. At the level of vectors in $\mathbb{C}^8$, this inner product can be rewritten as:

$$\langle \varphi, \psi \rangle = \sum_{a=1}^{3} \overline{m}_{a,\varphi} m_{a,\psi} + \bar{e}_\varphi e_\psi + \sum_{j=1}^{4} \bar{c}_{j,\varphi} c_{j,\psi} \ ,$$

with $m_{a,\varphi}$, e_φ, $c_{j,\varphi}$ the components of the vector φ and $m_{a,\psi}$, e_ψ, $c_{j,\psi}$ components of the vector ψ, while by $^{-}$ we denote complex conjugation.

The reason we introduce the notation $\mathcal{H}_M$ and $\mathcal{H}_C$ for the two summands appearing in the Hilbert space $\mathcal{H}_{BV}$ is that, as we will see in the following, the first summand contains the fields M_a, $a = 1, 2, 3$, as well as the ghost field E, while the second summand incorporates the ghost fields C_j, $j = 1, \ldots, 4$.
The fact that in the summand of the Hilbert space that describes the ghost fields C_j one also has a fourth ghost field C_4 that does not appear in the extended configuration space $\widetilde{X}$, should not be considered a problem. In fact, we will see that, from the definition of the operator D_{BV}, this ghost will not enter the fermionic action and hence it decouples.

Taking the Pauli matrices (see (5.3)) as basis for the space of matrices $M_2(\mathbb{C})$, from the point of view of its vector space structure, $\mathcal{H}_{BV}$ can be described as

$$\mathcal{H}_{BV} \simeq \langle m_1, m_2, m_3, e \rangle \oplus \langle c_1, c_2, c_3, c_4 \rangle,$$

with m_a, $a = 1, 2, 3$, e, and c_j, $j = 1, \ldots, 4$, complex variables describing the components of a generic matrix A in $M_2(\mathbb{C})$ with respect to the ordered basis. So a generic element φ in $\mathcal{H}_{BV}$ can be seen as a vector in $\mathbb{C}^8$, i.e.,

$$\varphi = [m_1,\ m_2,\ m_3,\ e,\ c_1,\ c_2,\ c_3,\ c_4]^T \ .$$

The Hilbert space $\mathcal{H}_{BV}$ has been defined as the direct sum of two complex Hilbert spaces. However, $\mathcal{H}_{BV}$ has another decomposition:

$$\mathcal{H}_{BV} = \mathcal{H}_{BV,f} \oplus i \cdot \mathcal{H}_{BV,f} \ ,$$

with

$$\begin{aligned} \mathcal{H}_{BV,f} &= [i \cdot su(2) \oplus u(1)] \oplus i \cdot u(2) \\ &\simeq \langle M_1, M_2, M_3, iE \rangle_{\mathbb{R}} \oplus \langle C_1, C_2, C_3, C_4 \rangle_{\mathbb{R}} \ . \end{aligned} \tag{7.3}$$

In (7.3), by M_a and C_j, with $a = 1, 2, 3$ and $j = 1, \ldots, 4$ we denoted the real part of the complex variables m_a and c_j, respectively, while E is the imaginary part of the complex variable e. This choice of notation is due to the fact that, as we will prove in the following theorem, these variables coincide with the fields and ghost fields generating the positively graded part of the extended configuration space $\widetilde{X}$.

The real structure J_{BV}

We want to define a real structure J_{BV} on the Hilbert space $\mathcal{H}_{BV}$. For a function J_{BV} to be a real structure, certain conditions need to be satisfied involving also the elements of the algebra in the spectral triple. Because up to now we have not introduced the algebra $\mathcal{A}_{BV}$ yet, we simply define J_{BV} as the following antilinear isometry on $\mathcal{H}_{BV}$:

$$\begin{array}{cccc} J_{BV}: & \mathcal{H}_{BV} & \longrightarrow & \mathcal{H}_{BV} \\ & (A_1, A_2) & \mapsto & J_{BV}(A_1, A_2) := i \cdot (A_1^*, A_2^*) \ , \end{array} \tag{7.4}$$

where by A^* we denote the adjoint of the matrix A in $M_2(\mathbb{C})$. Recalling that the Pauli matrices are self-adjoint, the definition of J_{BV} can be rewritten as follows (when we regard the elements of $\mathcal{H}_{BV}$ as vectors):

$$\begin{array}{cccc} J_{BV}: & \mathcal{H}_{BV} & \longrightarrow & \mathcal{H}_{BV} \\ & \varphi & \mapsto & J_{BV}(\varphi) = i \cdot (\bar{\varphi}) := i \cdot [\bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{e}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4]^T \ . \end{array}$$

It is immediate to check that J_{BV} is:

- ▶ conjugate linear;
- ▶ an antilinear isometry on H_{BV}.

The next step is to introduce the operator D_{BV}.

The linear operator D_{BV}

Another element that needs to be introduced is a linear self-adjoint operator D_{BV} acting on the Hilbert space $\mathcal{H}_{BV}$. Its explicit definition as 8×8 matrix acting on $\mathcal{H}_{BV} \simeq \mathbb{C}^8$ is given in Figure 7.1 (a more intrinsic definition of D_{BV} will be deduced in the proof of Proposition 18 below).

The operator D_{BV} depends on the real variables M_a^*, $a = 1, 2, 3$ and C_j^*, $j = 1, 2, 3$. The notation used for these variables has been chosen with purpose: we will prove that upon inserting all antifields in the linear operator

$$
\begin{bmatrix}
0 & 0 & 0 & C_1^* & 0 & +iM_3^* & -iM_2^* & 0 \\
0 & 0 & 0 & C_2^* & -iM_3^* & 0 & +iM_1^* & 0 \\
0 & 0 & 0 & C_3^* & +iM_2^* & -iM_1^* & 0 & 0 \\
C_1^* & C_2^* & C_3^* & 0 & 0 & 0 & 0 & 0 \\
0 & +iM_3^* & -iM_2^* & 0 & 0 & +iC_3^* & -iC_2^* & 0 \\
-iM_3^* & 0 & +iM_1^* & 0 & -iC_3^* & 0 & +iC_1^* & 0 \\
+iM_2^* & -iM_1^* & 0 & 0 & +iC_2^* & -iC_1^* & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 7.1: The BV operator D_{BV}

D_{BV}, the corresponding fermionic action yields the BV action S_{BV}. We would like to underline that for the moment both the family of variables M_a^* and C_j^* are considered to be real, while in our model M_a^* represent the antifields corresponding to the initial fields M_a and so, algebraically they are Grassmannian variables. However, their parity will be deduced by their behavior on the vectors in the Hilbert space $\mathcal{H}_{BV}$. Thus the parity of these variables is not imposed as an arbitrary choice for obtaining the structure that we already know but it will be a consequence of the construction itself (see Theorem 14).

The reason why the operator D_{BV} in Figure 7.1 has been divided into four submatrices 4×4 is that these different parts have different behavior with respect to the real structure J_{BV}. Thus we introduce the following notation to distinguish the different parts of D_{BV}:

- By $D_{1,\ off}$ we denote the 8×8 matrix given by the off-diagonal part of D_{BV}, namely the two blocks framed with green bracket, which we denote here by R and R^*. On the diagonal $D_{1,\ off}$ has two 4×4 blocks full of zeros. More explicitly:

$$D_{1,\ off} = \left[\begin{array}{c|c} \mathbf{0} & R \\ \hline R^* & \mathbf{0} \end{array}\right]$$

- $D_{1,\ diag}$ is an 8×8 matrix with, in the right bottom corner, the 4×4 matrix S that in Figure 7.1 has been printed in blue. More explicitly:

$$D_{1,\ diag} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & S \end{array}\right]$$

- D_2 is an 8×8 matrix with, in the left top corner, the 4×4 matrix T that in Figure 7.1 has been printed in red. More explicitly:

$$D_2 = \left[\begin{array}{c|c} T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]$$

Thus in the above notation:

$$D_{BV} = D_{1,\ diag} + D_{1,\ off} + D_2.$$

So, up to now, we have introduced a triple $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ composed of a finite-dimensional Hilbert space $\mathcal{H}_{BV}$, a self-adjoint linear operator D_{BV} on $\mathcal{H}_{BV}$, and an antilinear isometry J_{BV}. The last element which needs to be introduced to define a spectral triple is the algebra $\mathcal{A}_{BV}$.

The extended algebra $\mathcal{A}_{BV}$

We put:

$$\mathcal{A}_{BV} = M_2(\mathbb{C}).$$

Note that this is exactly the algebra considered in the initial spectral triple (2.8).

The action of $\mathcal{A}_{BV}$ on $\mathcal{H}_{BV}$ is the following natural one:

$$\begin{array}{ccccc} \mathcal{A}_{BV} & \times & \mathcal{H}_{BV} & \longrightarrow & \mathcal{H}_{BV} \\ A & , & (X_1, X_2) & \mapsto & (AX_1, AX_2). \end{array}$$

Let D_1 denote the sum $D_{1,\ diag} + D_{1,\ off}$. In what follows, we separately analyze the two triples defined by the two operators D_1 and D_2, whose sum defines the operator D_{BV} introduced above. In the two lemmas below we show that both the operators, D_1 and D_2, can be seen as part of a real spectral triple. However, these two real spectral triples differ in their KO-dimensions. We recall that the notion of *KO-dimension* is strictly related to the presence of a real structure on a spectral triple: the precise definition of *KO-dimension* has been explained in Chapter 2 (see Definition 3), when also the concepts of *real structure* and *real spectral triple* were first presented.

Lemma 9
In the notation above, $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1, J_{BV})$ is a real spectral triple of KO-dimension $1 \in \mathbb{Z}/8$.

Proof. To prove that $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1, J_{BV})$ defines a real spectral triple we start by noting that the following properties holds:

- $\mathcal{A}_{BV}$ is a unital and involutive algebra, where the involution is given by taking the adjoint of a matrix;
- $\mathcal{A}_{BV}$ is represented as linear and bounded operators on $\mathcal{H}_{BV}$;
- D_1 is a linear and self-adjoint operator on $\mathcal{H}_{BV}$ (since the variables M_a^*, $a = 1, 2, 3$ and C_j^*, with $j = 1, 2, 3$ are real variables).

Thus $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1)$ is a spectral triple. Moreover, we have already noticed that J_{BV} defines an antilinear isometry on $\mathcal{H}_{BV}$. Thus to conclude the proof of the statement, the only conditions that still need to be verified are the following:

(1) $J^2 = Id$;

(2) $J_{BV} D_1 = -D_1 J_{BV}$;

(3) $\left[A,\ J_{BV} B^* J_{BV}^{-1}\right] = 0, \quad$ for all A, B in $\mathcal{A}_{BV}$;

(4) $\left[\left[D_1, A\right],\ J_{BV} B^* J_{BV}^{-1}\right] = 0, \quad$ for all A, B in $\mathcal{A}_{BV}$.

Conditions (1), (2), (3) can be checked by a direct computation, which is omitted here.
To check condition (4) there are two possibilities: the first is by a direct computation while the second uses the Krajewski diagrams, introduced in Section 2.2. Here we prefer to adopt this second approach. Let us construct the Krajewski diagram of the spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1, J_{BV})$.

- Since $\mathcal{A}_{BV} = M_2(\mathbb{C})$, we start by writing the labels of the diagram obtaining:

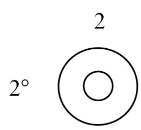
  2

  2°

- The second passage is to consider the Hilbert space $\mathcal{H}_{BV}$ and to determine its decomposition into irreducible representations of $\mathcal{A}_{BV} \otimes \mathcal{A}^{\circ}_{BV}$. We obtain

$$\mathcal{H}_{BV} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) = \mathbb{C}^2 \otimes \mathbb{C}^{2\circ} \otimes V\ ,$$

  with V a vector space of dimension 2. The presence of V is due to the fact that the representation in question has multiplicity 2.
  The decomposition of the Hilbert space determines the circles in the diagram: in our case we have to insert two circles in the unique node $(2, 2^\circ)$, indicating the multiplicity 2 of the representation.



- Finally, we draw the lines that describe the action of the operator D_1. Since $D_{1,\ off}$ interchanges the two summands $\mathcal{H}_M$ and $\mathcal{H}_C$, we add a line to the diagram connecting a circle with the other one: it is drawn in green since it represents the operator $D_{1,\ off}$. Concerning the line in blue connecting

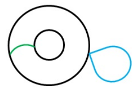


Figure 7.2: Krajewski diagram for the real spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1, J_{BV})$, where the algebra considered is $\mathcal{A}_{BV} \simeq M_2(\mathbb{C})$.

a circle with itself, it represents the operator $D_{1,\ diag}$, which acts on $\mathcal{H}_{BV}$ sending the summand $\mathcal{H}_C$ in itself.

The diagram thus obtained is a Krajewski diagram that is trivially symmetric with respect to the diagonal and which has only lines connecting a vertex with itself. Therefore, from Theorem 1, we deduce that this diagram represents a real spectral triple, and also the first-order condition for the operator D_1 is satisfied. Moreover, note that this also gives an alternative proof of condition (3).

□

Lemma 10
In the notation above, $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_2, J_{BV})$ is a real spectral triple of KO-dimension $7 \in \mathbb{Z}/8$.

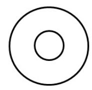
Proof. Since the triple we are considering in this lemma has the same algebra, the same Hilbert space and the same real structure as the spectral triple in Lemma 9, to prove the statement we only check the conditions involving the operator D_2. These are:

(1) $J_{BV} D_2 = +D_2 J_{BV}$;

(2) $\big[[D_2, A],\ J_{BV} B^* J_{BV}^{-1}\big] = 0, \quad$ for all A, B in $\mathcal{A}_{BV}$.

For the first condition, this can be checked by a direct computation.
The last condition that needs to be checked is the first-order condition for the operator D_2, that is condition (2). Again there are two possible ways to verify if this condition is satisfied: the first possibility is by a direct computation while the second uses the Krajewski diagram. This second way is the one that we follow.
Let us construct the Krajewski diagram corresponding to the triple we are analyzing:

- The corresponding Krajewski diagram is the same as before, at the level of labels and nodes:



- The differences with the proof of Lemma 9 appears when we insert in the diagram the lines corresponding to the action of the operator on the Hilbert space. Looking at Figure 7.1, we deduce that the operator D_2 sends the summand $\mathcal{H}_M$ to itself. Therefore, in the diagram we draw a line connecting one of the two circles with itself. The color picked for the line is the same as the block that identifies the component of D_2 in D_{BV} in Figure 7.1.

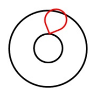


Figure 7.3: Krajewski diagram for the real spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_2, J_{BV})$, where the algebra considered is $\mathcal{A}_{BV} \simeq M_2(\mathbb{C})$.

From Theorem 1 we draw the conclusion that the triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_2, J_{BV})$ has a real structure. Its KO-dimension is $7 \in \mathbb{Z}/8$, since the real structure is such that $J_{BV}^2 = Id$ and it commutes with the operator D_2. □

Remark 45
From Lemmas 9 and 10 we deduce that the BV operator D_{BV} can be seen as the sum of two operators D_1 and D_2 such that D_1 anticommutes with the isometry J_{BV}, while D_2 commutes. Therefore, making a comparison with the possibilities listed in Definition 3, we see that a spectral triple defined by D_{BV} does not have a well-defined KO-dimension, or, in other words, it has a different KO-dimension depending on the summand considered. More precisely, the behavior of the summand D_1 describes a real structure of KO-dimension $1 \in \mathbb{Z}/8$, while the isometry J_{BV} considered with the summand D_2 defines a real structure of KO-dimension $7 \in \mathbb{Z}/8$. As explained in what follows, this changing in the

KO-dimension distinguishes between the bosonic and the fermionic part in the Hilbert space and in the components of the operator.

The reason we have introduced all these objects was to translate the BV construction obtained for our matrix model into the language of noncommutative geometry. In the following theorem we prove that this goal has been achieved with $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, which will be proved to be a real spectral triple, and by the corresponding fermionic action. Thus we will refer to this spectral triple as the *BV-spectral triple* for our matrix model.

Theorem 14. *Let $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ be the triple introduced above. Then:*

(1) $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ is a real spectral triple (with mixed KO-dimension);

(2) the expression

$$\mathcal{U}_{D_1}(\varphi, \psi) = \langle J_{BV}(\varphi), D_1\psi \rangle \qquad \forall \varphi, \psi \in \mathcal{H}_{BV,f}$$

defines an antisymmetric bilinear form on $\mathcal{H}_{BV,f}$;

(3) the expression

$$\mathcal{U}_{D_2}(\varphi, \psi) = \langle J_{BV}(\varphi), D_2\psi \rangle \qquad \forall \varphi, \psi \in \mathcal{H}_{BV,f}$$

defines a symmetric bilinear form on $\mathcal{H}_{BV,f}$;

(4) the trilinear pairing

$$\mathcal{U}_{D_{BV}}(\varphi, \psi) = \langle J_{BV}(\varphi), D_{BV}\psi \rangle$$

between D_{BV}, φ, ψ is gauge invariant under the adjoint action of the unitary group of $\mathcal{A}_{BV}$, namely

$$\mathcal{U}_{D_{BV}}(\varphi, \psi) = \mathcal{U}_{D_{BV,u}}(Ad(u)\varphi, Ad(u)\psi)$$

with $D_{BV,u} = Ad(u)D_{BV}Ad(u^*)$ and $Ad(u) = uJuJ^{-1}$;

(5) the fermionic action corresponding to the operator D_{BV} coincides with the BV action S_{BV} in Equation (7.2) of the matrix model. More precisely:

$$S_{BV} = \frac{1}{2}\langle J_{BV}(\varphi), D_{BV}\varphi \rangle, \qquad \text{with } \varphi \in \mathcal{H}_{BV,f},$$

under the following conditions:

- *the operator* $D_{1,\ off}$ *depends only on Grassmannian variables;*
- *the operators* $D_{1,\ diag}$ *and* D_2 *depend only on real variables;*
- *given*

$$\mathcal{H}_{BV,f} = [i \cdot su(2) \oplus u(1)] \oplus i \cdot u(2) \ ,$$

the generators of the summand $[i \cdot su(2) \oplus u(1)]$ *are real variables while the generators of* $u(2)$ *are Grassmannian variables.*

Proof. (1) First of all, we note that $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV})$ defines a spectral triple: in fact all the conditions on the algebra and the Hilbert space necessary to be elements of a spectral triple have already been checked in Lemma 9, where both the algebra and the Hilbert space were the same as the ones considered now. Moreover, looking at the explicit definition of the operator D_{BV} as 8×8 matrix given in Figure 7.1, it is immediate to conclude that D_{BV} is a self-adjoint operator. Thus we conclude that $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV})$ is a spectral triple.

To prove that it defines also a real spectral triple we have to consider the real structure J_{BV}. As already noted in Remark 45, a spectral triple with operator the BV operator D_{BV} cannot be real with respect to the usual definition recalled in Definition 3. However, if we do not fix the KO-dimension, then the BV-spectral triple is still real in the sense that it satisfies the following conditions:

(a) J_{BV} is an antilinear isometry on $\mathcal{H}_{BV}$ such that $J^2 = Id$;

(b) $J_{BV}(D_1 + D_2) = (-D_1 + D_2)J_{BV}$;

(c) $\big[A, J_{BV} B^* J_{BV}^{-1}\big] = 0$, for all A, B in $\mathcal{A}_{BV}$;

(d) $\big[\big[D_{BV}, A\big], J_{BV} B^* J_{BV}^{-1}\big] = 0$, for all A, B in $\mathcal{A}_{BV}$.

The only property which is different from the ones required in the usual definition of a real spectral triple is that the operator D_{BV} does not fully commute or anticommute with the real structure; it behaves differently for the different summands, so that it is not possible to define its KO-dimension. Conditions (a) and (c) have already been checked in the proof of Lemma 9 while, due to the linearity of the bracket, conditions (b) and (d) can be immediately deduced from Lemmas 9 and 10.
Thus we draw the conclusion that $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ defines a real spectral triple with mixed KO-dimension. This concludes the proof of statement (1). For completeness, we draw the Krajewski diagram also for the BV-spectral triple.

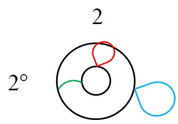


Figure 7.4: In this picture is depicted the Krajewski diagram for the BV-spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, where the operator D_{BV} is defined as $D_{BV} = D_1 + D_2$. While the red loop represents the action of the operator D_2, the blue loop and the green arch describe the behavior of the operator D_1.

(2) To prove (2) we start by noting that, due to the linearity of D_1 and the linearity of the inner product in the second component, also $\mathcal{U}_{D_1}$ is linear in the second component. Moreover, since the real form J_{BV} is antilinear and the same holds for the first component of the inner product, $\mathcal{U}_{D_1}$ turns out to be linear also in the first component and so it defines a bilinear form. Finally, property

$$\mathcal{U}_{D_1}(\varphi, \psi) = -\, \mathcal{U}_{D_1}(\psi, \varphi) \ ,$$

for any φ, ψ in $\mathcal{H}_{BV,f}$, is consequence of the following facts:

▷ D_1 is a self-adjoint operator;

▷ D_1 anticommutes with J_{BV};

▷ $J_{BV}^2 = Id$;

▷ J_{BV} is an antilinear isometry.

Thus we conclude that (2) holds.

(3) For statement (3), bilinearity of $\mathcal{U}_{D_2}$ can be ensured in a way completely analogous to the one follows for the form $\mathcal{U}_{D_1}$. So it still needs to be checked that $\mathcal{U}_{D_2}$ is also symmetric, i.e., given two generic elements φ and ψ in $\mathcal{H}_{BV,f}$,

$$\mathcal{U}_{D_2}(\varphi, \psi) = \mathcal{U}_{D_2}(\psi, \varphi).$$

This property is a consequence of the facts already noticed in (2). The reason why $\mathcal{U}_{D_1}$ is an antisymmetric bilinear form, while $\mathcal{U}_{D_2}$ is symmetric, lies in the fact that, while D_1 *anti*commutes with the real structure J_{BV}, D_2 commutes with it.

(4) To prove gauge invariance of the bilinear form $\mathcal{U}_{D_{BV}}$ claimed in statement (4), we start by noticing that for a fixed a unitary element u, the real structure J_{BV} commutes with $Ad(u) = uJ_{BV}uJ_{BV}^{-1}$, that is:

$$J_{BV}Ad(u) = Ad(u)J_{BV}.$$

This property of the real structure can be proved using the fact that $J_{BV}^2 = Id$, together with the commutativity of the left and right action of the algebra, i.e., the condition

$$\left[A, J_{BV}B^* J_{BV}^{-1}\right] = 0 \ , \qquad \forall A, B \in \mathcal{A}_{BV} \ ,$$

which has to be applied considering $A = u = B^*$.
Therefore, using the property that the real structure J_{BV} commutes with $Ad(u)$ together with the fact that $Ad(u)$ is a unitary operator satisfying $Ad(u)^* = Ad(u^*)$, we deduce the following series of equalities:

$$\begin{aligned} \mathcal{U}_{D_{BV,u}}(Ad(u)\varphi, Ad(u)\psi) \ &= \langle J_{BV}(Ad(u)\varphi), Ad(u)D_{BV}Ad(u^*)Ad(u)\psi\rangle \\ &= \langle Ad(u)J_{BV}(\varphi), Ad(u)D_{BV}\psi\rangle \\ &= \langle J_{BV}(\varphi), D_{BV}\psi\rangle = \mathcal{U}_{D_{BV}}(\varphi, \psi). \end{aligned} \tag{7.5}$$

This implies that the bilinear form $\mathcal{U}_{D_{BV}}$ is gauge invariant under the action of the unitary group of $\mathcal{A}_{BV}$.

(5) The last statement that needs to be proved asserts that the BV action coincides with the fermionic action corresponding to the operator D_{BV}. This can be proved by an explicit computation, considering a generic element φ in $\mathcal{H}_{BV,f}$:

$$\varphi = [M_1,\ M_2,\ M_3,\ iE,\ C_1,\ C_2,\ C_3,\ C_4]^T$$

with M_a, $a = 1, 2, 3$, E and C_j, $j = 1, \dots, 4$ real variables. Using the explicit definition of D_{BV} given in Figure 7.1 and recalling the parity imposed on the variables M_a, M_a^*, $a = 1, 2, 3$, E and C_j, C_j^*, $j = 1, \dots, 4$ and the corresponding (anti)commuting relations, we conclude that:

$$\frac{1}{2}\langle J_{BV}(\varphi), D_{BV}\varphi\rangle = S_{BV},$$

where S_{BV} has the form found in (7.2). Finally, we observe that the ghost field C_4 does not appear in the action as it should be.

□

Remark 46
The reason for the appearing of a mixed KO-dimension in the BV-spectral triple we have constructed lies in the particular behavior of the real structure with respect to the operator D_{BV}, that is to say, in the fact that the real structure anticommutes with the component D_1 of the operator while it commutes with the component D_2.
However, this should not be interpreted as the signal that, from a topological point of view, the BV-spectral triple we are studying is given by the disjoint union of two noncommutative manifolds. More precisely, given two possibly real spectral triples

$$(\mathcal{A}_1, \mathcal{H}_1, D_1, (J_1)), \qquad (\mathcal{A}_2, \mathcal{H}_2, D_2, (J_2))$$

a third one can be constructed as the direct sum of them:

$$(\mathcal{A}_1 \oplus \mathcal{A}_2, \mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus D_2, (J_1 \oplus J_2)).$$

Viewing the spectral triples as noncommutative manifolds, when we consider the direct sum of two of them, from a topological point of view we are considering the disjoint union of two manifolds.
However, the appearance of a mixed KO-dimension in our BV-spectral triple is not related to the operation of taking the direct sum of two spectral triples. In fact, our BV-spectral triple is not obtained as a direct sum of two other spectral triples, which is immediately clear if we consider the algebra $\mathcal{A}_{BV}$, which is not a direct sum of two algebras.
Therefore, our mixed KO-dimension has a different origin and detects different aspects of the construction: as we are going to explain, it detects the difference in *parity*, such as bosonic or fermionic parity, in the components of the BV-spectral triple.

Before continuing, we would like to make some remarks on the structure on this BV-spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, comparing it with the extended gauge theory $(\widetilde{X}, \widetilde{S})$: the goal is to understand how the elements which compose the extended pair $(\widetilde{X}, \widetilde{S})$ and their properties are formulated in terms of the BV-spectral triples and its structure. Our hope is that, by identifying these relations between the extended theory and the corresponding BV-spectral triple, we will get some ideas on how to formulate the BV construction in general within noncommutative geometry.

Properties of the BV-spectral triple

Making a comparison between the extended pair $(\widetilde{X}, \widetilde{S})$ and the corresponding BV-spectral triple, we discover the following properties:

- The different roles played by fields and ghost fields on one hand and antifields and antighost fields on the other clearly appear in the structure of the spectral triple:
  - the antifields M_a^* and the antighost fields C_j^* appear as the components of the operator D_{BV};
  - the fields M_a and the ghost fields C_j, E are the components of the vectors in the subspace $\mathcal{H}_{BV,f}$.
- The parities of all fields, antifields, ghosts and antighosts are a consequence of the structure of the spectral triple. More precisely, the choice of parity for the ghost fields and the antighost fields done in Theorem 14 is the only one for which the following conditions are satisfied:
  - there are non-trivial contributions to the fermionic action coming both for the part depending on D_1 and the part involving D_2;
  - the product of two real variables as well as the product of two Grassmannian variables gives a real variable, whereas the product of two variables with different parities gives always a Grassmannian variable.

  Requiring that there is a non-trivial contribution coming from the antisymmetric bilinear form $\mathcal{U}_{D_1}$, we have to introduce Grassmannian variables as components of the vectors φ in $\mathcal{H}_{BV,f}$. However, to have a non-trivial contribution coming from the symmetric bilinear form $\mathcal{U}_{D_2}$, the vectors in $\mathcal{H}_{BV,f}$ should also have real components.

  In other words, the initial symmetry or antisymmetry of the bilinear forms $\mathcal{U}_{D_1}$ and $\mathcal{U}_{D_2}$ and the conditions on the behavior of the parity with respect to the product jointly force us to impose that:
  - M_a, E, and C_j^*, with $a = 1, 2, 3$, and $j = 1, \ldots, 4$ are real variables;
  - M_a^*, and C_j, with $a = 1, 2, 3$, and $j = 1, \ldots, 4$ are Grassmannian variables.

  These conditions coincide with the distinction between bosonic and fermionic fields we had in the extended configuration space $\widetilde{X}$. We would like to stress that the parities of the fields, ghost fields, antifields, and antighost fields in the BV-spectral triple has not been imposed arbitrarily, but are a consequence of the structure itself.

► The BV-spectral triple can be divided into a bosonic and a fermionic part. Moreover, it is the KO-dimension that gives this distinction between bosons and fermions. More precisely:

- $(\mathcal{A}_{BV}, \mathcal{H}_M, T, J_{BV})$ is a *bosonic spectral triple* since it contains only bosonic terms both in the Hilbert space $\mathcal{H}_M$ and in the operator T;
- $(\mathcal{A}_{BV}, \mathcal{H}_M \oplus \mathcal{H}_C, D_1, J_{BV})$ is a *fermionic spectral triple*: all elements which appear in this spectral triple are either fermionic or they act on fermionic elements.

Recall the notation T for the non-trivial 4×4 block insides the 8×8 matrix D_2. From what we have proved in Lemmas 9 and 10 we deduce that:

▷ the bosonic spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_M, T, J_{BV})$ is a real spectral triple with KO-dimension $7 \in \mathbb{Z}/8$;

▷ the fermionic spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_M \oplus \mathcal{H}_C, D_1, J_{BV})$ is a real spectral triple with KO-dimension $1 \in \mathbb{Z}/8$.

Thus the property that the BV-spectral triple does not have a well-defined KO-dimension can be interpreted as a consequence of the fact that it contains real and Grassmannian variables both in the operator and in the Hilbert space. The KO-dimension then distinguishes these two parts.

Up to now, there is still a component of the BV-spectral triple which has not been analyzed, namely the algebra $\mathcal{A}_{BV}$: for this reason, to conclude the description of the BV-spectral triple, we want to focus on the algebra $\mathcal{A}_{BV}$. In fact, in defining the BV-spectral triple, as an algebra we simply consider the one we already had in the initial spectral triple

$$(\mathcal{A}, \mathcal{H}, D) = (M_2(\mathbb{C}), \mathbb{C}^2, D) \ ,$$

which induced the gauge theory (X_0, S_0) from which we started the whole construction. In the following proposition we prove that this choice was optimal.

Proposition 18
Let $\mathcal{H}_{BV}$, D_{BV} and J_{BV} be as above. Let $\widetilde{\mathcal{A}}$ denote the algebra $\mathcal{L}(\mathcal{H}_{BV})$ defined by the bounded linear operators on the Hilbert space $\mathcal{H}_{BV}$. Then the largest

subalgebra $\mathcal{A}_F \subseteq \widetilde{\mathcal{A}}$ for which $(\mathcal{A}_F, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ defines a real spectral triple, i.e. for which the first-order condition holds, is

$$\mathcal{A}_F = M_2(\mathbb{C}) = \mathcal{A}_{BV}.$$

Proof. To prove the statement we use once again Krajewski diagrams, which have been described in Section 2.2. We first recall that, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, a diagram composed by labeled nodes connected by lines can be constructed to describe this triple as follows:

- the labels are determined by the decomposition of the algebra $\mathcal{A}$;
- the nodes are related to the Hilbert space $\mathcal{H}$;
- the lines connecting the nodes are established by the operator D.

Moreover, as asserted in Theorem 1, there is a one-to-one correspondence between, on one hand, spectral triples endowed with a real structure satisfying all the required conditions to define a real spectral triple and, on the other hand, particular kinds of diagrams satisfying the following conditions:

1. the diagram is symmetric with respect to the diagonal;
2. the lines connecting two different nodes are only horizontal or vertical while a node might be connected with itself with a loop.

The strategy used to prove the statement is to show how the largest algebra for which the corresponding Krajewski diagram satisfies conditions 1., 2. is $\mathcal{A}_F = M_2(\mathbb{C})$.

We start by noting that, due to the way in which the real structure J_{BV} acts on $\mathcal{H}_{BV}$, namely sending the summands $\mathcal{H}_M$ and $\mathcal{H}_C$ in themselves, the largest algebra we can consider is

$$\widetilde{\mathcal{A}} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

Thus the Krajewski diagram corresponding to the choice $(\widetilde{\mathcal{A}}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ is the one represented in Figure 7.5 where the colors used to draw the lines representing the behavior of the operator D_{BV} are once again the ones used in Figure 7.1 to designate the different blocks inside the matrix which defines D_{BV}.

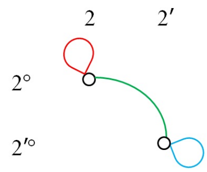
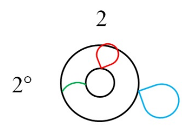


Figure 7.5: In this picture is depicted the Krajewski diagram for the spectral triple $(\widetilde{\mathcal{A}}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, where as algebra we consider $\widetilde{\mathcal{A}} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. This is the largest algebra that allows to complete the triple $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ to a spectral triple. However, since in this diagram there is a diagonal line connecting the two nodes, we draw the conclusion that this spectral triple does not satisfy all the required conditions to be a real spectral triple.

From the Krajewski diagram above, by applying Theorem 1 we immediately deduce that the spectral triple $(\widetilde{\mathcal{A}}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ does not satisfy the necessary conditions to be a real spectral triple. More precisely, since the diagram is symmetric with respect to the diagonal, the real structure J_{BV} is an antilinear isometry with $J_{BV}^2 = \pm Id$. However, since there is a line that connects two different nodes and which is neither vertical or horizontal, the first-order condition is not satisfied.
Therefore, to complete the triple $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ to a real spectral triple, we should consider an algebra $\mathcal{A}_F$ for which we no longer have this line diagonally connecting the two nodes.
Looking more closely at the structure of $\mathcal{H}_{BV}$ and J_{BV}, we see that the way in which the real structure is defined forces the nodes to be both on the diagonal: for this reason, in order to verify both the conditions required to have a Krajewski diagram corresponding to a real spectral triple, we are forced to require that the two nodes coincide or, in other words, that we have multiplicity 2 for the same node. Thus the green line would no longer be diagonal, but it would connect a node with itself, obtaining the following diagram:



Thus the largest algebra which completes the triple $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ to a real spectral triple and, in particular, for which the operator D_{BV} satisfies the first-order condition is $\mathcal{A}_F = M_2(\mathbb{C}) = \mathcal{A}_{BV}$. □

We stress that the line of the proof presented above is not the only one possible: the same conclusion can be drawn from a direct computation. For completeness we briefly present this approach now.

Direct proof of Proposition 18

To simplify the computation, we first rewrite the operator D_{BV} defined in Figure 7.1 as function of two matrices α and β in $M_2(\mathbb{C})$, which are themselves functions of the antifields M^*_a, $a = 1, 2, 3$ and of the antighost fields C^*_j, $j = 1, 2, 3$.
Let α and β be defined as follows on the basis given by the Pauli matrices:

$$\begin{cases} \alpha = \frac{1}{2}\left[(-C^*_1)\sigma_1 + (-C^*_2)\sigma_2 + (-C^*_3)\sigma_3\right] \\ \beta = \frac{1}{2}\left[(-M^*_1)\sigma_1 + (-M^*_2)\sigma_2 + (-M^*_3)\sigma_3\right] . \end{cases}$$

Given a matrix A in $M_2(\mathbb{C})$, we define two operators $Ab(A)$ and $Ad(A)$ on $M_2(\mathbb{C})$ by

$$\begin{aligned} Ab(A) : \quad M_2(\mathbb{C}) &\longrightarrow M_2(\mathbb{C}) \\ Y &\mapsto Ab(A)(Y) = AY + YA, \end{aligned}$$

$$\begin{aligned} Ad(A) : \quad M_2(\mathbb{C}) &\longrightarrow M_2(\mathbb{C}) \\ Y &\mapsto Ad(A)(Y) = AY - YA. \end{aligned}$$

Using these operators and the matrices α and β, the operator D_{BV} can be rewritten as follows:

$$D_{BV} = \left[\begin{array}{c|c} \boldsymbol{Ab(\alpha)} & \boldsymbol{Ad(\beta)} \\ \hline \boldsymbol{Ad(\beta)} & \boldsymbol{Ad(\alpha)} \end{array}\right]$$

Let (P, Q) and $(\widetilde{P}, \widetilde{Q})$ be two generic elements in $\widetilde{\mathcal{A}}$ and let (φ, ψ) be a generic element in $\mathcal{H}_{BV}$, with $\varphi \in \mathcal{H}_M$ and $\psi \in \mathcal{H}_C$. From the definition of the real structure J_{BV} on $\mathcal{H}_{BV}$, we find that the left and the right actions of the algebra $\widetilde{\mathcal{A}}$ on $\mathcal{H}_{BV}$ are as follows:

$$\begin{array}{rcl} \widetilde{\mathcal{A}} \times \mathcal{H}_{BV} & \longrightarrow & \mathcal{H}_{BV} \\ (P,Q),(\varphi,\psi) & \mapsto & \begin{pmatrix} P\cdot\varphi \\ Q\cdot\psi \end{pmatrix} \end{array} \qquad \begin{array}{rcl} \mathcal{H}_{BV} \times \widetilde{\mathcal{A}} & \longrightarrow & \mathcal{H}_{BV} \\ (\varphi,\psi)(P,Q) & \mapsto & \begin{pmatrix} \varphi\cdot P \\ \psi\cdot Q \end{pmatrix} \end{array} .$$

To verify under which conditions on the algebra $\widetilde{\mathcal{A}}$ the first-order condition holds, we have to compute the following expressions:

(1) $D_{BV}(P,Q)(\widetilde{P},\widetilde{Q})^\circ|_{(\varphi,\psi)}$;

(2) $-(P,Q)D_{BV}(\widetilde{P},\widetilde{Q})^\circ|_{(\varphi,\psi)}$;

(3) $-(\widetilde{P},\widetilde{Q})^\circ D_{BV}(P,Q)|_{(\varphi,\psi)}$;

(4) $(\widetilde{P},\widetilde{Q})^\circ(P,Q)D_{BV}|_{(\varphi,\psi)}$.

By a direct computation one can check the following identities:

$$(1) = \begin{bmatrix} -\alpha P\varphi\widetilde{P} - P\varphi\widetilde{P}\alpha + \beta Q\psi\widetilde{Q} - Q\psi\widetilde{Q}\beta \\ \beta P\varphi\widetilde{P} - P\varphi\widetilde{P}\beta + \alpha Q\psi\widetilde{Q} - Q\psi\widetilde{Q}\alpha \end{bmatrix}$$

$$(2) = \begin{bmatrix} P\alpha\varphi\widetilde{P} + P\varphi\widetilde{P}\alpha - P\beta\psi\widetilde{Q} + P\psi\widetilde{Q}\beta \\ -Q\beta\varphi\widetilde{P} + Q\varphi\widetilde{P}\beta - Q\alpha\psi\widetilde{Q} + Q\psi\widetilde{Q}\alpha \end{bmatrix}$$

$$(3) = \begin{bmatrix} \alpha P\varphi\widetilde{P} + P\varphi\alpha\widetilde{P} - \beta Q\psi\widetilde{P} + Q\psi\beta\widetilde{P} \\ -\beta P\varphi\widetilde{Q} + P\varphi\beta\widetilde{P} - \alpha Q\psi\widetilde{Q} + Q\psi\alpha\widetilde{Q} \end{bmatrix}$$

$$(4) = \begin{bmatrix} -P\alpha\varphi\widetilde{P} - P\varphi\alpha\widetilde{P} + P\beta\psi\widetilde{P} - P\psi\beta\widetilde{P} \\ Q\beta\varphi\widetilde{Q} - Q\varphi\beta\widetilde{Q} + Q\alpha\psi\widetilde{Q} - Q\psi\alpha\widetilde{Q} \end{bmatrix} .$$

Therefore, by imposing that $(1)+(2)+(3)+(4)=0$, we find the following conditions:

$$\begin{cases} (-\beta Q\psi + P\alpha\psi)(\widetilde{P}-\widetilde{Q}) + (P-Q)(\psi\widetilde{Q}\beta - \psi\alpha\widetilde{P}) = 0 \ ; \\ (\beta P\varphi - Q\alpha\varphi)(\widetilde{P}-\widetilde{Q}) + (P-Q)(\varphi\beta\widetilde{Q} - \varphi\widetilde{P}\beta) \ . \end{cases} \tag{7.6}$$

Since the conditions imposed in Equation (7.6) have to be satisfied by every matrix P, Q, $\widetilde{P}$, $\widetilde{Q}$, φ, ψ in $M_2(\mathbb{C})$, this imposes

$$\begin{cases} P = Q \\ \widetilde{P} = \widetilde{Q} \ . \end{cases} \tag{7.7}$$

In other words, the algebra $\mathcal{A}_F$, which was the largest algebra for which the first-order condition is satisfied, cannot coincide with $\widetilde{\mathcal{A}}$ and so it cannot consist of two summands $M_2(\mathbb{C})$ that act independently on the two copies of $M_2(\mathbb{C})$ which form the Hilbert space $\mathcal{H}_{BV}$. In fact, from conditions (7.7) we deduce that

$$\mathcal{A}_F = M_2(\mathbb{C}) = \mathcal{A}_{BV}.$$

Thus we conclude that the algebra $\mathcal{A}_{BV} = M_2(\mathbb{C})$ is the largest for which, given $\mathcal{H}_{BV}$, D_{BV}, J_{BV} as above, the first-order condition holds.

To conclude the presentation of the BV-spectral triple, we emphasize another property of it: making a comparison between the spectral triple which defines the initial gauge theory (X_0, S_0), namely

$$(\mathcal{A}, \mathcal{H}, D) = (M_2(\mathbb{C}), \mathbb{C}^2, D),$$

where D is a self-adjoint 2×2 matrix, on the one hand, and the BV-spectral triple which describes the minimally extended gauge theory $(\widetilde{X}, \widetilde{S})$, on the other hand, i.e.,

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) = (M_2(\mathbb{C}), M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), D_{BV}, J_{BV}),$$

we immediately note that the BV construction does not induce any change in the algebra, but rather acts only on the Hilbert space and on the operator D. Moreover, it forces one to introduce a real structure. This observation might be useful when one tries to generalize this construction of the BV-spectral triples to other gauge theories.

7.2 The BV-auxiliary spectral triple

The aim of this section is to introduce the notion of a *BV-auxiliary spectral triple* for the $U(2)$-gauge invariant matrix model. In the previous section we formulated the minimally extended theory $(\widetilde{X}, \widetilde{S})$, which was obtained by applying the BV construction, in a real spectral triple whose fermionic action coincides with the BV action S_{BV}. As explained in Section 3.6, once the pair $(\widetilde{X}, \widetilde{S})$ has been determined, the next step towards defining the BRST-cohomology complex is the gauge-fixing procedure. However, to define a suitable gauge fixing fermion (Section 5.2.2), extra fields need to be added and this is done through the introduction of so-called trivial pairs. The goal of this section is to translate this construction to the setting of noncommutative geometry and consider the trivial pairs in the context of *BV-auxiliary spectral triples.*

We start by recalling the notation already used in Section 5.2.2.

▶ Let $(\widetilde{X}, \widetilde{S})$ be the *extended theory* with $\widetilde{X}$ the *extended configuration space* and $\widetilde{S}$ the *extended action*. More precisely $\widetilde{X}$ has the structure of a graded vector space:

$$\widetilde{X} = W \oplus W^*[1] \ ,$$

where:

- W is a $\mathbb{Z}_{\geqslant 0}$-graded vector space describing the content in fields and ghost fields of the extended configuration space $\widetilde{X}$;
- $W^*[1]$ is a $\mathbb{Z}_{<0}$-graded vector space describing the correspondent antifields and antighost fields.

Explicitly,

$$W = \langle M_1, M_2, M_3, M_4 \rangle \oplus \langle C_1, C_2, C_3 \rangle \oplus \langle E \rangle \ ,$$

with:

- M_a, $a = 1, \ldots, 4$: fields with $deg(M_a) = 0$ and $\epsilon(M_a) = 0$;
- C_j, $j = 1, 2, 3$: ghost fields with $deg(C_j) = 1$ and $\epsilon(C_j) = 1$;
- E: ghost field with $deg(E) = 2$ and $\epsilon(E) = 0$.

On the other hand,

$$W^*[1] = \langle M_1^*, M_2^*, M_3^*, M_4^* \rangle \oplus \langle C_1^*, C_2^*, C_3^* \rangle \oplus \langle E^* \rangle \ ,$$

with:

- M_a^*, $a = 1, \ldots, 4$: antifields with $deg(M_a^*) = -1$ and $\epsilon(M_a^*) = 1$;
- C_j^*, $j = 1, 2, 3$: antighost fields with $deg(C_j^*) = -2$ and $\epsilon(C_j^*) = 0$;
- E^*: antighost field with $deg(E^*) = -3$ and $\epsilon(E^*) = 1$.

▶ In Section 5.2.2 we explained that, in order to have a proper gauge fixed action, we have to introduce the following extra fields:

- B_1, B_2, B_3 are fields with $deg(B_j) = -1$ and $\epsilon(B_j) = 1$ for $j = 1, 2, 3$;
- h_1, h_2, h_3 are fields with $deg(h_j) = 0$ and $\epsilon(h_j) = 0$ for $j = 1, 2, 3$. Together with the fields B_j, they define three trivial pairs:

$$(B_1, h_1) \qquad (B_2, h_2) \qquad (B_3, h_3).$$

  These are introduced since in the extended configuration space $\widetilde{X}$ there are three ghost fields C_j with ghost degree 1.
- A_1, A_2: they are fields with $deg(A_1) = -2$, $\epsilon(A_1) = 0$ and $deg(A_2) = 0$, $\epsilon(A_2) = 0$;
- k_1, k_2: they are fields with $deg(k_1) = -1$, $\epsilon(k_1) = 1$ and $deg(k_2) = 1$, $\epsilon(k_2) = 1$. Together with the fields A_1 and A_2, they define two trivial pairs:

$$(A_1, k_1) \qquad (A_2, k_2).$$

  These trivial pairs are introduced because in the extended configuration space there is one ghost field E with ghost degree 2.

▶ For all extra fields introduced now we also have to include the corresponding antifields, which are listed here:

- B_1^*, B_2^*, B_3^* are antifields with $deg(B_j^*) = 0$ and $\epsilon(B_j^*) = 0$ for $j = 1, 2, 3$;
- h_1^*, h_2^*, h_3^* are antifields with $deg(h_j^*) = -1$ and $\epsilon(h_j^*) = 1$ for $j = 1, 2, 3$;
- A_1^*, A_2^*: they are antifields with $deg(A_1^*) = 1$, $\epsilon(A_1^*) = 1$ and $deg(A_2^*) = -1$, $\epsilon(A_2^*) = 0$;
- k_1^*, k_2^*: they are antifields with $deg(k_1^*) = 0$, $\epsilon(k_1^*) = 0$ and $deg(k_2^*) = -2$, $\epsilon(k_2^*) = 0$.

▶ To the action $\widetilde{S}$ a summand S_{aux} needs to be added; it depends on the trivial pairs and is defined as follows:

$$S_{aux} = B_1^* h_1 + B_2^* h_2 + B_3^* h_3 + A_1^* k_1 + A_2^* k_2. \tag{7.8}$$

The main goal of this section is the introduction of the notion of *BV-auxiliary spectral triple*. The idea is that a BV-auxiliary spectral triple describes the auxiliary parts which have been introduced in the extended theory $(\widetilde{X}, \widetilde{S})$ to make it suitable for the gauge-fixing procedure. More precisely, the BV-auxiliary spectral triple

$$(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux})$$

is supposed to describe the trivial pairs and the auxiliary action S_{aux}. Our aim is to find a structure similar to the one already discovered for the BV-spectral triple:

- The fields B_j, h_j, A_l and k_l, with $j = 1, 2, 3$ and $l = 1, 2$ are variables which describe the components of the vectors in a Hilbert space.
- The corresponding antifields B_j^*, h_j^*, A_l^* and k_l^* are components of the linear operator D_{aux}. To be more precise, since the antifields h_j^* and k_l^* do not appear in the auxiliary action, we expect the operator D_{aux} to depend only on the antifields B_j^* and A_l^*.
- The action S_{aux} is obtained from a fermionic action associated to a certain operator D_{aux}. Note that the action S_{aux} is not bilinear in the fields, as it was in the BV action S_{BV}. Thus a different construction is needed, but once again the action will be defined by an inner product depending on the operator D_{aux} and computable on vectors of the Hilbert space.
- The BV-auxiliary spectral triple is a real spectral triple and the KO-dimension distinguishes the bosonic and the fermionic part of the triple.
- The algebra is the largest algebra such that the first-order condition is satisfied.

The Hilbert space H_{aux}

The Hilbert space is supposed to describe the field content. Thus we expect it to be related to the fields B_j, h_j, A_l and k_l, with $j = 1, 2, 3$ and $l = 1, 2$. However, the auxiliary action S_{aux} depends only on the fields h_j and k_l. Thus only these fields will appear as variables that describe the components of vectors in the Hilbert space. So we have

$$\mathcal{H}_{aux} = H_h \oplus H_k \ ,$$

with

$$\mathcal{H}_h = M_2(\mathbb{C}) \qquad \mathcal{H}_k = \mathbb{C}^2.$$

The first summand $\mathcal{H}_h$ is related to the fields h_j while the second $\mathcal{H}_k$ is related to the fields k_l. If we take the Pauli matrices as a basis for $M_2(\mathbb{C})$, then

$$\mathcal{H}_{aux} \simeq \mathbb{C}^6 \ ,$$

and a generic vector χ in $\mathcal{H}_{aux}$ is of the following form:

$$\chi = [\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \psi_1,\ \psi_2]^T \ ,$$

with φ_a, $a = 1, \ldots, 4$ and ψ_1, ψ_2 complex variables.

Let $\mathcal{H}_{aux,f}$ be the following subspace:

$$\mathcal{H}_{aux,f} = u(2) \oplus i[u(1) \oplus u(1)].$$

Then a generic element in $\mathcal{H}_{aux,f}$ has the following form:

$$\chi = [ih_1,\ ih_2,\ ih_3,\ ih_4,\ k_1,\ k_2]^T,$$

with h_j and k_l, $j = 1, \ldots, 4$, $l = 1, 2$ real variables.
The notation has been chosen on purpose: in fact, the real variables h_j and k_l which here represent the real components of a vector in $\mathcal{H}_{aux,f}$ will play the role of the auxiliary fields of our matrix model.

The real structure J_{aux}

The real structure of the BV-auxiliary spectral triple has a definition similar to the real structure J_{BV} introduced for the BV-spectral triple:

$$\begin{array}{rccl} J_{aux}: & \mathcal{H}_h \oplus \mathcal{H}_k & \longrightarrow & \mathcal{H}_h \oplus \mathcal{H}_k \\ & (T, V) & \mapsto & J_{aux}(\varphi, \psi) := (i \cdot T^*, i \cdot \overline{V}) \ , \end{array}$$

with $T \in \mathcal{H}_h = M_2(\mathbb{C})$ and $V \in \mathbb{C}^2$. Here T^* is the adjoint of the matrix T, while $\overline{V}$ is the complex conjugate of the vector V, considered componentwise. Therefore, if we identify the Hilbert space $\mathcal{H}_{aux}$ with $\mathbb{C}^6$, then the action of the real structure J_{aux} on a generic vector χ in $\mathcal{H}_{aux}$ can be written as follows:

$$J_{aux}(\chi) = [\overline{\varphi}_1,\ \overline{\varphi}_2,\ \overline{\varphi}_3,\ \overline{\varphi}_4,\ \overline{\psi}_1,\ \overline{\psi}_2]^T.$$

$$\begin{bmatrix} \frac{1}{2}(+B_1^*+B_2^*+B_3^*) & 0 & 0 & \frac{1}{2}(+B_1^*-B_2^*-B_3^*) \\ 0 & \frac{1}{2}(+B_1^*+B_2^*+B_3^*) & 0 & \frac{1}{2}(-B_1^*+B_2^*-B_3^*) \\ 0 & 0 & \frac{1}{2}(B_1^*+B_2^*+B_3^*) & \frac{1}{2}(-B_1^*-B_2^*+B_3^*) \\ \frac{1}{2}(+B_1^*-B_2^*-B_3^*) & \frac{1}{2}(-B_1^*+B_2^*-B_3^*) & \frac{1}{2}(-B_1^*-B_2^*+B_3^*) & \frac{1}{2}(+B_1^*+B_2^*+B_3^*) \end{bmatrix}$$

Figure 7.6: The matrix P

$$\begin{bmatrix} -\frac{i}{3}A_1^* & -\frac{i}{3}A_1^* & -\frac{i}{3}A_1^* & 0 \\ -\frac{i}{3}A_2^* & -\frac{i}{3}A_2^* & -\frac{i}{3}A_2^* & 0 \end{bmatrix}$$

Figure 7.7: The matrix Q

The operator D_{aux}

We represent the operator D_{aux} as a 6×6 matrix which acts on $\mathcal{H}_{aux} \simeq \mathbb{C}^6$. The operator D_{aux} is defined as the sum of two operators:

$$D_{aux} = D_{diag} + D_{off}$$

where:

$$D_{diag} = \left[\begin{array}{c|c} P & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \qquad D_{off} = \left[\begin{array}{c|c} \mathbf{0} & Q^* \\ \hline Q & \mathbf{0} \end{array}\right]$$

with the matrices P and Q explicitly defined in Figure 7.6 and Figure 7.7 respectively.

The components of the matrix P are supposed to be real variables, so that the operator D_{aux} is a self-adjoint operator on $\mathcal{H}_{aux}$. Once again, the choice of notation for the components of these matrices is not incidental: in fact they

will play the role of the antifields A_l^* and B_a^*, with $l = 1,2$ and $a = 1,2,3$. Their fermionic and bosonic parity will be derived below, as a consequence of the conditions imposed to have a non-trivial fermionic action corresponding to the operator D_{aux}.
Note that the matrix P, and so also the operator D_{diag} depends only on the antifields B_a^*, with $a = 1,2,3$, while the matrix Q and hence also the operator D_{off} depends only on the antifields A_1^*, A_2^*.

The algebra $\mathcal{A}_{aux}$

We pose

$$\mathcal{A}_{aux} = \mathbb{C}.$$

In Proposition 21 we will prove that, given $\mathcal{H}_{aux}$, J_{aux} and D_{aux} as defined above, $\mathbb{C}$ is the largest algebra such that the first-order condition is satisfied.

Proposition 19
Let $\mathcal{A}_{aux}$, $\mathcal{H}_{aux}$, D_{aux} and J_{aux} be as defined above. Then:

(1) $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{diag}, J_{aux})$ defines a real spectral triple with KO-dimension $7 \in \mathbb{Z}/8$;

(2) $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{off}, J_{aux})$ defines a real spectral triple with KO-dimension $1 \in \mathbb{Z}/8$.

Proof. We start by proving (1): it is immediate that the algebra $\mathcal{A}_{aux} = \mathbb{C}$ is an involutive unital algebra which can be represented as bounded operators on $\mathcal{H}_{aux}$. Moreover, we have already noted that, since B_j^*, $j = 1,2,3$ are real variables, the operator D_{diag} is self-adjoint. As to J_{aux}, the way in which it acts on the components of the vectors in $\mathcal{H}_{aux}$ coincides with the action of J_{BV}. Hence it follows that also J_{aux} is an antilinear isometry such that $J_{aux}^2 = Id$. Finally, since the algebra is simply $\mathcal{A}_{aux} \simeq \mathbb{C}$, the operator D_{diag} automatically satisfies the first-order condition. Therefore, to finish the proof of statement (1), we only have to check that

$$J_{aux} D_{diag} = + \, D_{diag} J_{aux}.$$

This condition can be checked by an explicit computation, which completes the proof of the first statement.
To prove (2), we start by noting that the only element of this spectral triple that differs with respect to the one analyzed in the previous point is the operator D_{off}. Thus we have to check only if the conditions regarding the operator in a

real spectral triple are satisfied. Using the hypothesis that the variables A_1^*, A_2^* are real, it follows immediately that D_{off} is a self-adjoint operator. Moreover D_{off} automatically satisfies the first-order condition because the algebra $\mathcal{A}_{aux}$ is $\mathbb{C}$. Thus the last condition which has to be checked is that D_{off} and J_{aux} anticommute:

$$J_{aux}D_{off} = -\, D_{off}J_{aux}.$$

Also this equality can be easily verified by an explicit computation, which proves statement (2). □

The last element that needs to be introduced to complete the definition of the BV-auxiliary spectral triple is the fermionic action. This is what we are going to do in the following proposition. More precisely, the fermionic action corresponding to the operators D_{diag} and D_{off} will be defined using two linear forms $\mathcal{L}_{D_{diag}}$, $\mathcal{L}_{D_{off}}$ instead of a bilinear form $\mathcal{U}$, as done for the BV action. This is a consequence of the fact that the auxiliary action S_{aux} is only linear (rather than quadratic) in the fields. However, the definition of the linear form $\mathcal{L}$ will be similar to the definition of the bilinear form $\mathcal{U}$.

Proposition 20
In the notation introduced above,

(1) The expression

$$\begin{array}{rccl} \mathcal{L}_{D_{aux}}: & \mathcal{H}_{aux} & \longrightarrow & \mathbb{C} \\ & \chi & \mapsto & \langle J_{aux}(\underline{1}), D_{aux}(\chi)\rangle + \langle J_{aux}(\chi), D_{aux}(\underline{1})\rangle \end{array}$$

defines a linear form on $\mathcal{H}_{aux}$, where $\underline{1}$ denotes the vector in $\mathbb{C}^6$ with all its components equal to unity.

(2) The bilinear paring $\mathcal{L}_{D_{aux}}(\chi)$ between χ and D_{aux} is gauge invariant under the adjoint action of the unitary group of $\mathcal{A}_{aux}$:

$$\mathcal{L}_{D^{\lambda}_{aux}}(Ad(\lambda)(\chi)) = \mathcal{L}_{D_{aux}}(\chi), \qquad \forall \lambda \in \mathcal{U}(\mathcal{A}_{aux}),\ \forall \chi \in \mathcal{H}_{aux}\ ,$$

with $D^{\lambda}_{aux} = Ad(\lambda)D_{aux}Ad(\lambda^*)$.

Proof. Linearity of $\mathcal{L}_{D_{aux}}$ is consequence of the properties of the inner product, the operator D_{aux}, and the real structure J. In fact, the first summand appearing in the definition of $\mathcal{L}_{D_{aux}}$ is linear because the first component of the inner product and the real structure J are supposed to be conjugate linear while the linearity of the second summand is due to the linearity of the operator D_{aux} and of the second component of the inner product. This proves (1).

To prove (2), we first note that, since the algebra is $\mathcal{A}_{aux} = \mathbb{C}$, the corresponding unitary group is $U(1)$. Let λ be a generic element in $U(1)$, namely λ belongs to $\mathbb{C}$ with $|\lambda| = 1$. Then

$$\begin{aligned}&\mathcal{L}_{D^{\lambda}_{aux}}(Ad(\lambda)(\chi))\\&\quad = \langle J_{aux}(\underline{1}), \lambda D_{aux}\bar{\lambda}(\lambda\chi\bar{\lambda})\rangle + \langle J_{aux}(\lambda\chi\bar{\lambda}), \lambda D_{aux}\bar{\lambda}(\underline{1})\rangle\\&\quad = \mathcal{L}_{D_{aux}}(\chi)\ .\end{aligned}$$

The previous equality is an immediate consequence of the linearity of $\mathcal{L}_{D_{aux}}$, together with the fact that λ is unimodular. □

Theorem 15. *In the above notation, $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux})$ defines a real spectral triple (with mixed KO-dimension). Moreover, the fermionic action defined by the linear form $\mathcal{L}_{D_{aux}}$ coincides with the auxiliary action S_{aux}:*

$$S_{aux} = \frac{1}{2}\left[\langle J_{aux}(\underline{1}), D_{aux}(\chi)\rangle + \langle J_{aux}(\chi), D_{aux}(\underline{1})\rangle\right]\ ,$$

where $\chi \in \mathcal{H}_{aux,f}$, under the following hypothesis:

- *the components in D_{diag} are real variables;*
- *the components in D_{off} are Grassmannian variables;*
- *given $\mathcal{H}_{aux,f} \simeq u(2) \oplus i[u(1) \oplus u(1)]$, the generators of $u(2)$ are real variables, while the generators of $u(1) \oplus u(1)$ are Grassmannian variables.*

Proof. We start by noting that $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux})$ is a spectral triple. In fact, the conditions which need to be satisfied by the algebra and the Hilbert space in order to define a spectral triple have already been verified in Proposition 19. Moreover, looking at the definition of the operator $D_{aux} = D_{diag} + D_{off}$ as the 6×6 matrix given in Figure 7.6 and 7.7, it follows immediately that D_{aux} is a self-adjoint operator on $\mathcal{H}_{aux}$. Thus $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux})$ is a spectral triple.
Concerning J_{aux}, we have already proved that it defines an antilinear isometry on $\mathcal{H}_{aux}$ such that $J^2 = Id$.
To conclude that $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux})$ defines a real spectral triple we prove the following identities:

(1) $\left[a, Jb^*J^*\right] = 0, \qquad$ for all $a, b \in \mathcal{A}_{aux}$;

(2) $J(D_{diag} + D_{off}) = (+D_{diag} - D_{off})J$;

(3) $\big[[D_{aux}, a], Jb^*J\big] = 0 \qquad$ for all $a, b \in \mathcal{A}_{aux}$.

However, condition (2) can be immediately deduced by what we proved separately for the operators D_{diag} and D_{off} in Proposition 19. Conditions (1) and (3) are trivially true, since we are considering $\mathcal{A}_{aux} = \mathbb{C}$. Thus we conclude that $(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{diag}+D_{off}, J_{aux})$ is a real spectral triple. From the fact that the operator D_{aux} does not fully commute or anticommute with the real structure J_{aux} we deduce that this real spectral triple has a mixed KO-dimension.

To prove the second part of the statement, let χ be a generic vector in $\mathcal{H}_{aux,f}$:

$$\chi = \big[ih_1,\ ih_2,\ ih_3,\ ih_4,\ k_1,\ k_2\big]^T$$

with h_j, $j = 1, \ldots, 4$ the real variables which generate the summand $u(2)$ in $\mathcal{H}_{aux,f}$ while k_1, k_2 are the two Grassmannian generators of the summand $i[u(1) \oplus u(1)]$.

With an explicit computation one can check that the following holds:

- $$\begin{aligned}\langle J_{aux}(\underline{1}), D_{aux}(\chi)\rangle = \;& B_1^*h_1 + B_2^*h_2 + B_3^*h_3 + A_1^*k_1 + A_2^*k_2 \\ & + \tfrac{1}{3}(-i)(A_1^* + A_2^*)[h_1 + h_2 + h_3] \ ;\end{aligned}$$
- $$\begin{aligned}\langle J_{aux}(\chi), D_{aux}(\underline{1})\rangle = \;& h_1B_1^* + h_2B_2^* + h_3B_3^* - k_1A_1^* - k_2A_2^* \\ & + [h_1 + h_2 + h_3]\tfrac{i}{3}(A_1^* + A_2^*) \ .\end{aligned}$$

Therefore, if we compute the sum of the previous two terms considering the hypothesis done on the parity of the variables, we find the following expression:

$$\frac{1}{2}\left[\langle J_{aux}(\underline{1}), D_{aux}(\chi)\rangle + \langle J_{aux}(\chi), D_{aux}(\underline{1})\rangle\right] = B_1^*h_1 + B_2^*h_2 + B_3^*h_3 + A_1^*k_1 + A_2^*k_2.$$

Making a comparison between what we have just found and the explicit definition of S_{aux} recalled in Equation (7.8), we see that they coincide. □

Remark 47
We want to remark that also for the BV-auxiliary spectral triple we have a similar structure to the one already found for the BV-spectral triple:

- The fields that appear in the auxiliary action S_{aux} are components of the vectors in the subspace $\mathcal{H}_{aux,f}$ of the Hilbert space $\mathcal{H}$.

- ▶ The antifields on which the action S_{aux} depends are components of the operator D_{aux}.

- ▶ The change of the KO-dimension distinguishes the "bosonic spectral triple" from the "fermionic spectral triple".
  We explain this statement by noticing that the spectral triple of KO-dimension $7 \in \mathbb{Z}/8$ with operator D_{diag}, which has been described in Proposition 19, can be thought of as a spectral triple defined only on the first summand of $\mathcal{H}_{aux}$, namely on $\mathcal{H}_h$. Thus:

  $$(\mathcal{A}_{aux}, \mathcal{H}_h, D_{diag}, J_{aux})$$

  is a real spectral triple depending only on bosonic fields and antifields. Therefore, as we have already seen in the construction of the BV-spectral triple, the KO-dimension $7 \in \mathbb{Z}/8$ identifies the bosonic part of the spectral triple. On the other hand, if we consider the other spectral triple described in Proposition 19, namely

  $$(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{diag}, J_{aux}) \ ,$$

  we see that it depends on the fermionic fields and antifields. Once again, the KO-dimension $1 \in \mathbb{Z}/8$ identifies the fermionic part of the spectral triple.

- ▶ The parity of some of the fields and antifields is a consequence of the structure of the real spectral triple.
  Indeed, if we suppose that all variables considered, namely the fields h_j, $j = 1, \ldots, 4$ and k_1, k_2 and the antifields B_l^*, $l = 1, 2, 3$, as well as A_1^*, A_2^*, are real variables, then in the linear spectral action $\mathcal{L}_{D_{aux}}$ we would not have any contribution coming from the operator D_{off}. This is due to the fact that D_{off} anticommutes with the real structure J_{aux}. Therefore, to have a non-trivial contribution from D_{off} we have to assign parity 1 to some of the variables involved. There are two possibilities:

  (1) If we want to keep the summand $\frac{1}{3}(-i)(A_1^* + A_2^*)[h_1 + h_2 + h_3]$ we have to assume that A_1^*, A_2^*, h_1, h_2, h_3 are all Grassmannian variables.

  (2) If we want to keep the summand $A_1^* k_1 + A_2^* k_2$ we have to assume that A_1^*, A_2^*, k_1, k_2 are Grassmannian variables.

  Thus in any case the antifields A_1^*, A_2^* are described by Grassmannian variables. Before further analyzing these two possibilities, we note that, to avoid the eventuality of not having summands depending on the operator D_{diag} in the linear fermionic action, either the antifields B_l^*, $l = 1, 2, 3$ or the fields

h_j, $j = 1, \dots, 4$ have to be defined by real variables. In fact, if this would not be the case, the summands

$$B_1^* h_1 + B_2^* h_2 + B_3^* h_3$$

coming from the term $\langle J(\underline{1}), D_{diag}(\chi)\rangle$ would be canceled by the summands

$$h_1 B_1^* + h_2 B_2^* + h_3 B_3^*$$

from the term $\langle J(\chi), D_{diag}(\underline{1})\rangle$.
In other words, $\mathcal{L}_{D_{diag}}(\chi) = 0$ if we assume that D_{diag} and χ depend on Grassmannian variables. Furthermore we also note that the possibilities (1) and (2) cannot both be valid. In fact, from what we have just noticed about the part of the fermionic action determined by the operator D_{diag}, the fact that the fields h_l are supposed to be fermionic would imply that the antifields B_l^* are bosonic. With these assumptions we would deduce that:

- both summands $\mathcal{H}_h$ and $\mathcal{H}_k$ in $\mathcal{H}_{aux}$ have components given by Grassmannian variables;
- the matrix P which defines a block in D_{off} depends on the Grassmannian components A_l^* and acts on the vectors in $\mathcal{H}_h$, sending them to vectors in $\mathcal{H}_h$.

These would contradict the fact that an operator depending on Grassmannian variables sends Grassmannian variables in variables with parity 0. Thus only one of the possibilities (1) and (2) can hold. Moreover, from what we have just noticed about the action of the block P, we deduce that possibility (2) should hold. In fact, the only case in which a matrix with Grassmannian components acts preserving the parity of the vectors is when the vectors depend on real variables.
Thus we conclude that:

▷ In order to have a contribution to the fermionic action coming from the operator D_{off} we have to assume that one of the possibilities (1) and (2) is satisfied; for both of them the antifields A_1^*, A_2^* are supposed to be defined by Grassmannian variables.

▷ The fields h_l, $l = 1, 2, 3$ are real because of the way the block P, which depends on the antifields A_1^*, A_2^*, acts on $\mathcal{H}_h$, i.e., keeping the parity of the vectors on which it acts.

▷ Among the two possibilities (1) and (2) which would ensure a contribution to the fermionic action coming from D_{off}, this selects the second. Thus the fields k_1, k_2 are necessarily fermionic fields.

Even though the structure of the spectral triple forces some of the variables and generators to have a fixed parity, we still have the freedom to determine the parity of the fields B_l^*, $l = 1, 2, 3$. However, the BV-auxiliary spectral triple has no function on its own: it has been introduced as the "auxiliary part" for the BV-spectral triple. Thus the parity of the antifields B_l^* is determined by the structure of the BV-spectral triple, as explicitly explained in Remark 49.

Before discussing the relations between the BV-spectral triple and the corresponding BV-auxiliary spectral triple, we want to justify the choice for the algebra $\mathcal{A}_{aux}$. In fact, we did not have any restriction on the algebra coming from the model: since we were looking for a spectral triple with (linear) fermionic action coinciding with the auxiliary action S_{aux}, we had conditions on the Hilbert space and on the operator coming from the dependence of the action on fields and antifields, respectively, but none on the algebra.

Proposition 21
Let $(\mathcal{H}_{aux}, D_{aux}, J_{aux})$ be as defined above. Then the algebra $\mathcal{A}_{aux} \simeq \mathbb{C}$ is the largest $*$-algebra that represents faithfully on $\mathcal{H}_{aux}$ such that

$$(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux})$$

defines a real spectral triple.

Proof. To prove the statement we use Krajewski diagrams, which have been presented in Section 2.2.1. First of all, we recall that

$$\mathcal{H}_{aux} \simeq M_2(\mathbb{C}) \oplus \mathbb{C}^2.$$

Moreover, we recall that also the real structure imposes conditions on the algebra $\mathcal{A}_{aux}$. In fact, to have a real spectral triple, we have to impose that the real structure sends the left action of the algebra to its right action. This implies that the largest algebra that respects the action of the real structure on the Hilbert space $\mathcal{H}_{aux}$ is

$$\widetilde{\mathcal{A}} \simeq M_2(\mathbb{C}) \oplus \mathbb{C}^2.$$

So $\mathcal{A}_{aux} \subseteq \widetilde{\mathcal{A}}$. Let us start by checking if $\widetilde{\mathcal{A}}$ satisfies the required conditions to make the triple $(\mathcal{H}_{aux}, D_{aux}, J_{aux})$ into a real spectral triple. To check the conditions we draw the Krajewski diagram corresponding to $(\widetilde{\mathcal{A}}, \mathcal{H}_{aux}, D_{aux}, J_{aux})$.

The first step to construct a Krajewski diagram is to determine the labels for the coordinates. This is done by looking at the algebra and to consider its decomposition as direct sum of matrix algebras: the labels for the diagram are the dimensions of the matrix algebras appearing in this decomposition.
In our case, the algebra $\widetilde{\mathcal{A}}$ is already written as a sum of matrix algebras and so the labels are immediately determined:

$$\{2, 1, 1'\},$$

where $1'$ represents the second copy of $\mathbb{C}$ appearing in the decomposition of $\widetilde{\mathcal{A}}$.
Once the labels are known, we insert the nodes in the diagram by looking at the Hilbert space $\mathcal{H}_{aux}$ and at its decomposition as a sum of right and left irreducible representations of $\widetilde{\mathcal{A}}$:

$$\mathcal{H}_{aux} \simeq M_2(\mathbb{C}) \oplus \mathbb{C}^2 \simeq \left[\mathbb{C}^2 \otimes \mathbb{C}^{2\circ}\right] \oplus \left[\mathbb{C} \otimes \mathbb{C}^{\circ}\right] \otimes V \ ,$$

with V a vector space of dimension 2. In the previous decomposition we used $M_2(\mathbb{C}) \simeq \mathbb{C}^2 \otimes \mathbb{C}^{2\circ}$ and $\mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C}^{\circ}$, while the presence of the vector space V is due to the multiplicity 2 with which the summand $\mathbb{C}$ appears.
Therefore, looking at this decomposition, the diagram contains three nodes:

▷ one at coordinates $(2, 2^{\circ})$, which represents the summand $\mathbb{C}^2 \otimes \mathbb{C}^{2\circ}$;

▷ one at coordinates $(1, 1^{\circ})$, which represents the first summand $\mathbb{C} \otimes \mathbb{C}^{\circ}$;

▷ one at coordinates $(1', 1'^{\circ})$, which represents the second summand $\mathbb{C} \otimes \mathbb{C}^{\circ}$.

Finally, the edges are determined by the operator D_{aux} and its decomposition in matrices acting on the summands in the decomposition of the Hilbert space:

$$D_{aux} = D_{22^{\circ}} \oplus \left[D_{22^{\circ},11^{\circ}} \oplus D_{22^{\circ},1'1'^{\circ}}\right] \oplus \left[D_{22^{\circ},11^{\circ}} \oplus D_{22^{\circ},1'1'^{\circ}}\right]^*$$

with

▷ $D_{22^{\circ}} : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})$, $D_{22^{\circ}} = Q$, where Q is the 4×4 matrix in Figure 7.7;

▷ $D_{22^{\circ},11^{\circ}} : M_2(\mathbb{C}) \longrightarrow \mathbb{C}$, with $D_{22^{\circ},11^{\circ}}$ a vector with 4 components given by the first row of the matrix P in Figure 7.6;

▷ $D_{22^{\circ},1'1'^{\circ}} : M_2(\mathbb{C}) \longrightarrow \mathbb{C}'$, with $\mathbb{C}'$ the second copy of $\mathbb{C}$ in $\mathcal{H}_{aux}$ and with $D_{22^{\circ},1'1'^{\circ}}$ a vector with 4 components given by the second row of the matrix P in Figure 7.6;

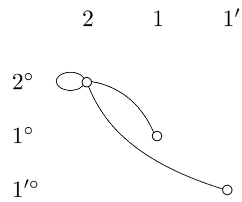


Figure 7.8: Krajewski diagram for the triple $(\mathcal{H}_{aux}, D_{aux}, J_{aux})$ considered with the algebra $\widetilde{\mathcal{A}} \simeq M_2(\mathbb{C}) \oplus \mathbb{C}^2$.

▷ the last summand is given by the adjoints of the operators $D_{22^\circ,11^\circ}$ and $D_{22^\circ,1'1'^\circ}$, which are defined, respectively, by the first and the second column of the matrix P^*.

Thus there will be three edges in the diagram:

▷ one is a loop with the node at coordinates $(2, 2^\circ)$ as base point: it represents the component D_{22°;

▷ one connects the node $(1, 1^\circ)$ with the node at coordinates $(2, 2^\circ)$: it represents the component $D_{22^\circ,11^\circ}$. This edge represents also the operator $D^*_{22^\circ,11^\circ} = D_{11^\circ,22^\circ}$: due to the fact that it is a non-orientated edge, it can be thought of as an edge connecting the node $(1, 1^\circ)$ to the node $(2, 2^\circ)$ and so it describes the operator $D_{11^\circ,22^\circ}$;

▷ one connects the nodes $(2, 2^\circ)$ and $(1', 1'^{,\circ})$: it describes the operators $D_{22^\circ,1'1'^\circ}$ and $D_{11^\circ,22^\circ}$.

So the Krajewski diagram corresponding to $(\widetilde{\mathcal{A}}, \mathcal{H}_{aux}, D_{aux}, J_{aux})$ is the one presented in Figure 7.8.
Recalling Theorem 1, we immediately deduce that, together with the algebra $\widetilde{\mathcal{A}}$ the triple $(\mathcal{H}_{aux}, D_{aux}, J_{aux})$ does not define a real spectral triple. The problem is that there are edges running diagonally: for the operator D_{aux} to satisfy the first-order condition as well as to commute-anticommute with the real structure J_{aux}, the edges representing the operator D_{aux} have to run either horizontally or vertically, or be loops.

Thus the algebra $\mathcal{A}_{aux}$ we are searching for is a proper subalgebra of $\widetilde{\mathcal{A}}$. The only way to solve this problem and do not have diagonal edges is to put all the nodes in the same position, so that the edges that now connect different nodes become loops with a multiple node as base point. Thus the algebra $\mathcal{A}_{aux}$ is necessarily $\mathcal{A}_{aux} \simeq \mathbb{C}$. □

For completeness, in Figure 7.9 we draw the Krajewski diagram of the BV-auxiliary spectral triple

$$(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux}) \simeq (\mathbb{C}, M_2(\mathbb{C}) \oplus \mathbb{C}^2, D_{aux}, J_{aux}).$$

Since the algebra $\mathcal{A}_{aux}$ is simply given by the algebra of complex numbers, the set of labels contains only one element: $\{1\}$.
Therefore, looking at the Hilbert space, we have to draw a node of multiplicity 6 at coordinates $(1, 1^\circ)$, since

$$\mathcal{H}_{aux} \simeq [\mathbb{C} \otimes \mathbb{C}^\circ] \otimes V \ ,$$

where V is a vector space of dimension 6.
To obtain a more comprehensible image, in Figure 7.9 we separately draw the six nodes that represent the multiplicity 6 of the representation. However, each edge connecting two of these nodes should be interpreted as a loop connecting different circles representing the different multiplicities, but all in the same point with the same coordinates $(1, 1^\circ)$. In other words, the figure has only one node with multiplicity 6, and 22 loops. In particular, this Krajewski diagram satisfies all requirements define a real spectral triple. Moreover, we order the nodes from left to right following the order of the components of the vectors in $\mathcal{H}_{aux}$: given a generic vector χ in $\mathcal{H}_{aux}$ with

$$\chi = [\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \psi_1,\ \psi_2]^T \ ,$$

the first four nodes represent the components φ_1 to φ_4, while the last two represent the components ψ_1 and ψ_2. We also use different colors to identify the different components in the operator D_{aux}. More precisely, the correspondence between colors of the edges and components in the matrix is given in the following list:

- green $\rightsquigarrow \frac{1}{2}(B_1^* - B_2^* - B_3^*)$;
- orange $\rightsquigarrow \frac{1}{2}(-B_1^* + B_2^* - B_3^*)$;
- red $\rightsquigarrow \frac{1}{2}(-B_1^* - B_2^* + B_3^*)$;
- blue $\rightsquigarrow \frac{1}{2}(B_1^* + B_2^* + B_3^*)$;

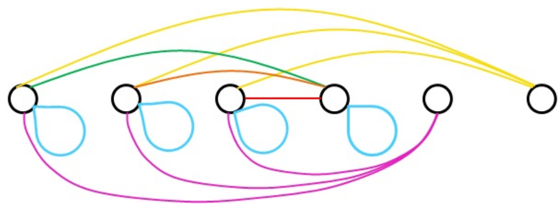


Figure 7.9: Krajewski diagram for the BV-auxiliary spectral triple given by $(\mathbb{C}, M_2(\mathbb{C}) \oplus \mathbb{C}^2, D_{aux}, J_{aux})$. Since the algebra is simply the algebra of complex numbers, all the nodes have the same coordinates, but they have been drawn separately in order to obtain a clearer picture. Thus all the edges in the figure, even if they connect different circles, have to be regarded as loops.

- purple $\rightsquigarrow \frac{1}{3}iA_1^*$ and $-\frac{1}{3}iA_1^*$;
- yellow $\rightsquigarrow \frac{1}{3}iA_2^*$ and $-\frac{1}{3}iA_2^*$.

Remark 48
The BV-auxiliary spectral triple gives a description of the trivial pairs in terms of noncommutative geometry. The reason for the introduction of the trivial pairs was to define a gauge-fixing fermion, which was necessary to carry out a gauge-fixing procedure and eliminate the antifields from the configuration space. In other words, trivial pairs do not add anything to the model and in particular they do not change the corresponding cohomology complex. For this reason it is not unexpected that the algebra on which this BV-auxiliary spectral triple is defined is the most trivial one, that is, $\mathbb{C}$. In fact, recalling how the model was introduced in Section 2.3, the algebra enters the definition of the spectral action, namely through the initial action S_0. When we extended the spectral triple to obtain the BV-spectral triple associated to the initial one, the algebra remained unchanged. Therefore, from the point of view of the cohomology the fact that the algebra of the BV-auxiliary spectral triple is trivial reflects of the "triviality"of the trivial pairs.

Remark 49
If we look at the relations between the BV-spectral triple and the corresponding BV-auxiliary spectral triple, we also identify a relationship between the fields in the Hilbert space $\mathcal{H}_{BV}$ and the elements in $\mathcal{H}_{aux}$. More precisely, the BV-

spectral triple for the $U(2)$-matrix model was given by

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \simeq (M_2(\mathbb{C}), M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), D_{BV}, J_{BV}).$$

The BV action S_{BV} coincides with the (bilinear) fermionic action associated to the operator D_{BV}, computed on vectors belonging to the Hilbert space:

$$\mathcal{H}_{BV,f} \simeq [i \cdot su(2) \oplus u(1)] \oplus i \cdot u(2),$$

where a generic vector has the following form:

$$\varphi = [M_1,\ M_2,\ M_3,\ iE,\ C_1,\ C_2,\ C_3,\ C_4]^T\ .$$

The BV-auxiliary spectral triple is defined by

$$(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux}) \simeq (\mathbb{C}, M_2(\mathbb{C}) \oplus \mathbb{C}^2, D_{aux}, J_{aux}).$$

The auxiliary action S_{aux} coincides with the (linear) fermionic action associated to the operator D_{aux}, computed on the vectors of the effective Hilbert space $\mathcal{H}_{aux,f}$, with

$$\mathcal{H}_{aux,f} \simeq u(2) \oplus i[u(1) \oplus u(1)],$$

where a generic vector has the following form:

$$\chi = [ih_1,\ ih_2,\ ih_3,\ ih_4,\ k_1,\ k_2]^T.$$

When we discussed the method to determine the number of gauge-fixing auxiliary fields that have to be introduced (see Section 3.6), we explained that, for a first-level reducible theory like the one which we are considering, the number of trivial pairs of type (B_j, h_j) coincides with the number of ghost fields C_i with degree 1, while the number of trivial pairs (A_j, k_j) is twice the number of ghost fields E with ghost number 2.
In the above construction with the spectral triple we see that this relation is still visible. However, it is the auxiliary fields h_j and k_j that seem to be related to the fields C_i and E. Moreover, the parities of C_i and h_j on one hand and the one of E and k_j on the other hand are opposite compared to the subspaces in $\mathcal{H}_{BV,f}$ and $\mathcal{H}_{aux,f}$ to which they belong: in fact, while the fields C_i are the components of vectors in $iu(2)$, the fields h_j represent the components of vectors in $u(2)$ and, analogously, while E is the real variable such that iE generates $u(1)$, the variables k_1 and k_2 are the components of $i[u(1) \oplus u(1)]$ with respect to the base $(1, 1)$.
Therefore, also in this setting the relation between the ghost fields in the extended configuration space and the trivial pairs is still clear.

To summarize: in this chapter we reached the goal of reformulating what we obtained by applying the BV approach to a matrix model induced by a finite-dimensional spectral triple in the setting of noncommutative geometry. More precisely, we introduced *BV-spectral triples* and *BV-auxiliary spectral triples*, which describe, respectively, the extended theory and the trivial pairs constructed by using the BV approach on a $U(2)$-gauge invariant theory.
With this approach we were able to give a "geometrical interpretation" (in the sense of noncommutative geometry) to the elements of the extended theory, considering the fields as the components of vectors in the Hilbert space and the antifields as the components of the operator, while the appearance of a mixed KO-dimension was related to the presence of bosonic and fermionic elements in the spectral triple.

A possible general approach to BV-spectral triples

In this chapter we have presented an approach to the problem of describing the BV construction for gauge theories in the setting of noncommutative geometry. Even though the solution presented was restricted to the case of a $U(2)$-gauge invariant matrix model, what we have found seems to suggest a possible way to face the problem in a more general setting.
As an initial gauge theory (X_0, S_0), let us consider a theory naturally derived by a finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (cf.. Section 2.3) and let the initial action S_0 be the spectral action, given on the initial gauge fields φ by

$$S_0[\varphi] = Tr(f(\varphi)),$$

with f a polynomial function.

Applying the BV formalism, an extended theory $(\widetilde{X}, \widetilde{S})$ may then be obtained from the initial one by introducing ghost fields, antifields and antighost fields. To reach the goal of formulating this extended theory in the setting of noncommutative geometry, a possible approach would be to investigate if it is possible to include all information contained in the extended theory $(\widetilde{X}, \widetilde{S})$ in a BV-spectral triple

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}).$$

The property which characterizes the BV-spectral triple would be the fact that the fermionic action associated to the operator D_{BV} coincides with the BV-action of the extended theory $(\widetilde{X}, \widetilde{S})$.
Moreover, the Hilbert space $\mathcal{H}_{BV}$ is expected to contain all gauge fields and

ghost fields of any ghost degree appearing in the extended configuration space $\widetilde{X}$, while the components of the operator D_{BV} are all antifields and antighost fields appearing in the extended action $\widetilde{S}$.
However, to have this kind of construction a more general notion of *multilinear fermionic action* has to be introduced for gauge theory of higher degree. This would be required in the case in which the BV-action has order greater than 2 in the ghost fields.
Thus it is conceivable that the translation of the BV procedure in the setting of noncommutative geometry gives a procedure in which the corresponding BV-spectral triple is obtained through the extension of the Hilbert space and the definition of a suitable fermionic action, leaving the algebra unchanged.

In case that, to carry out a gauge-fixing procedure, the introduction of the trivial pairs is required, it might be that this procedure can be seen in the noncommutative geometric setting as the operation of taking the direct sum of the BV-spectral triple with a BV-auxiliary spectral triple

$$(\mathcal{A}_{aux}, \mathcal{H}_{aux}, D_{aux}, J_{aux}).$$

The main property which would characterize this BV-auxiliary spectral triple is that the linear fermionic action corresponding to the operator D_{aux} coincides with the auxiliary action S_{aux} depending on the fields in the trivial pairs:

$$S_{aux} = \langle J(\underline{1}), D_{aux}\psi\rangle$$

for a certain vector $\underline{1} \in \mathcal{H}_{aux}$. Moreover, in the BV-auxiliary spectral triple the auxiliary fields are expected to appear in the Hilbert space $\mathcal{H}_{aux}$ while the antifields are the components of the operator D_{aux}.
As to the algebra, since we are introducing something that would not cause any change at the level of the corresponding cohomology complex, $\mathcal{A}_{aux}$ is still expected to be $\mathbb{C}$.

It remains to be seen whether this proposal is correct by considering other examples, such as $U(n)$-matrix models for $n > 2$, or other gauge theories defined by other types of finite dimensional spectral triples. That would also suggest the way to introduce a general notion of *BV-spectral triple* and *BV-auxiliary spectral triple* arriving at a “noncommutative-geometric interpretation” of the BV construction.

Chapter 8

Conclusions and Outlook

In this thesis we gave a mathematical description of the BRST construction and of the corresponding cohomology groups, in the context of a particular matrix model derived from a 0-dimensional noncommutative manifold. More precisely, we analyzed the BRST construction for a model given by the following finite spectral triple on the algebra $M_n(\mathbb{C})$:

$$(M_n(\mathbb{C}), \mathbb{C}^n, D)$$

where D is an hermitian matrix. This spectral triple defines a $U(n)$-gauge invariant physical theory (X_0, S_0).
Although this kind of model describes an extremely simple spacetime composed of only one point and without time, this setting turns out to be a good and unexpectedly rich context for a geometrical investigation of the BRST construction. First we presented a possible approach (inspired by the construction first described by Felder and Kazhdan in [28]) to construct extended varieties for a pair (X_0, S_0), where X_0 is the configuration space of a physical model, given by a nonsingular algebraic variety. Then we concentrated on the gauge theory defined by the model in the case $n = 2$.
For $U(2)$-gauge invariant matrix models we determined the minimal BV extension $(\widetilde{X}, \widetilde{S})$ of the theory, obtained by the introduction of the minimal number of ghost fields. Once the pair $(\widetilde{X}, \widetilde{S})$ was determined we carried out the gauge-fixing procedure, arriving at the explicit BRST cohomology complex for our matrix model. We computed all BRST cohomology groups: this explicit computation is interesting not only from a mathematical point of view but also

from a physical one. In fact, in the 4-dimensional case, the BRST cohomology groups describe properties of the physical theory, as the set of observables or the renormalizability of the theory.
Moreover, we found that the space of ghost fields gives rise to the structure of a Lie algebra. After the introduction of a *generalized notion of Lie algebra cohomology*, we established a relation between the BRST cohomology and a generalized Lie algebra cohomology and we explicitly determined how this relation at the level of the cohomology complex translates to the level of cohomology groups. Finally, we discovered a double complex structure and we described how this structure determines the properties of the corresponding BRST cohomology groups.
With this approach to the BRST complex, we were able to determine the role played by the different kinds of ghost fields and to translate the physical properties of these ghosts, such as their ghost degree and their parity, in terms of properties of the double complex structure.

Finally, we presented a possible approach to the analysis of the general case of a $U(n)$-gauge invariant theory, obtained from a finite spectral triple on the algebra $M_n(\mathbb{C})$, for n an element in $\mathbb{N}$. Although this step of going from a 2×2 matrix model to $n \times n$ matrix model could appear to simply be formal, it turned out to require the knowledge of how the presence of a $U(n)$-gauge symmetry forces the initial action to be. Moreover, it appears that there is a strong connection between action of the gauge group $U(n)$ on the configuration space and the minimal number of ghost fields that need to be introduced to obtain an extended theory $(\widetilde{X}, \widetilde{S})$. Up to now, we have been able to determine the minimally extended configuration space $\widetilde{X}$, while further analysis has to be carried out to construct the extended action $\widetilde{S}$. Once the pair $(\widetilde{X}, \widetilde{S})$ has been determined, it would be suitable for an analysis with the techniques already developed, such as our generalized Lie algebra cohomology.

A second direction which we followed in this thesis was to try to incorporate the BRST process into the setting of noncommutative geometry. More precisely, since our matrix model was defined by the pair (X_0, S_0) obtained from a finite spectral triple, we aimed at rewriting also its minimal BRST extension $(\widetilde{X}, \widetilde{S})$ in the language of noncommutative geometry. We reached this goal, once again for a matrix model with degree $n = 2$, with the introduction of a so-called *BV-spectral triple*. With this approach, all the physical properties of the ghost fields, such as their bosonic or fermionic character, have a natural translation

in terms of the spectral triple itself.
Moreover, we identified an intriguing structure in the BV-spectral triple for the $U(2)$-matrix model, with all fields and ghost fields appearing as components of vectors in a Hilbert space, while all the antifields and antighost fields are components of the “Dirac” operator D. Furthermore, we noted that at the level of this spectral triple, the BRST construction does not involve any change of the algebra: it only affects the Hilbert space and the “Dirac” operator D as well as forcing the introduction of a real structure. These observations might be useful for generalizing this construction to other gauge theories.
Finally, we also incorporated the notion of the trivial pairs into the setting of noncommutative geometry. This is a first step towards a different geometric interpretation for the BRST-cohomology complex.

Further steps

To conclude, two interesting directions in which this research could be further developed are the following:

- first of all, having already described in full detail the BRST construction of a matrix model with a $U(2)$-gauge symmetry, it seems possible to arrive at a rigorous and clear understanding of the BRST construction for matrix model with $U(n)$-gauge symmetry, for any natural number n. Preliminary results suggest that what has been discovered for the case $n = 2$ could be extended also to the general case.

  It seems that, using the formulation of the BRST cohomology complex via the notion of generalized Lie algebra cohomology, a multi-complex structure will appear. With this kind of approach we can obtain a detailed description of the contributions given by the different types of ghost fields to the BRST cohomology complex, arriving at a complete understanding of the role played by these extra fields in the construction.

- A second direction in which this research might be continued is in introducing the BRST construction, which until now has been developed in an algebraic geometry context, in the setting of noncommutative geometry. As mentioned above, encouraging results have already been found, involving not only the BRST construction but also other procedures, as for example the introduction of auxiliary fields, obtaining in this way a geometrical interpretation for them in the context of noncommutative geometry.

The integration of the gauge fixing procedure in the noncommutative geometry framework is the first step needed to insert in this setting also the BRST cohomology complex. This may lead to a comparison of the BRST cohomology with other cohomological theories that are naturally defined in the noncommutative geometry setting, as for example cyclic and Hochschild cohomology. This could give a different point of view for studying the BRST cohomology and might suggest a way to investigate the BV construction also at the physically relevant level of 4-dimensional gauge theories.

- Finally, it would be of interest to search for a direct way to obtain the BV-spectral triple associated to a gauge theory induced by a (finite-dimensional) spectral triple, without having to pass through the BV construction. To reach this goal, a deeper understanding of the relations that exist between the initial spectral triple, the corresponding BV-spectral triple and the BV-auxiliary spectral triple is still needed.

Appendix A

Auxiliary fields for $L = 1$ gauge theories

The content presented in this appendix refers to Section 3.6 and in particular to Theorem 4, which we restate here for convenience.

Theorem 4. *Let (X_0, S_0) be a gauge theory, with level of reducibility L. Then the minimal number of trivial pairs that have to be introduced to ensure the possibility of defining a suitable gauge-fixing fermion is $(L+1)(L+2)/2$. More precisely:*

$$\forall i \in \mathbb{N},\ 0 \leqslant i \leqslant L, \quad \textit{exactly } i+1 \textit{ trivial pairs have to be introduced.}$$

Let $\{(B_i^j, h_i^j)\}$, $i = 0, \ldots, L$, $j = 1, \ldots, i+1$, be this collection of trivial pairs. Then the ghost degree and the parity of the fields B_i^j and h_i^j have to satisfy the following relations:

$$\begin{cases} deg(B_i^j) = j - i - 2 & \textit{if } j \textit{ is odd} \\ deg(B_i^j) = i - j + 1 & \textit{if } j \textit{ is even} \end{cases} \quad \begin{cases} deg(h_i^j) = j - i - 1 \\ deg(h_i^j) = i - j + 2 \end{cases} \tag{A.1}$$

$$\epsilon(B_i^j) = i + 1 \quad (mod\ 2); \qquad \epsilon(h_i^j) = i \quad (mod\ 2).$$

The main purpose of this appendix is to justify the necessity of introducing this collection of auxiliary fields and to explain why there exists a relation between the level of reducibility of the theory and the number of trivial pairs that needs to be introduced. To do this, we concentrate on gauge theories with level of

reducibility $L = 1$. There are two reasons for this choice: first, this construction was used in this thesis exactly in this context (see Section 5.2.2); second, going from the case of a theory with level of reducibility $L = 1$ to a generic reducible theory with level L, the notation immediately becomes more complicated without adding anything essential to the argument. However, we state the theorem for the general case, since there are interesting models for which this construction is needed (see Chapter 6).
The main ideas which we are going to describe have already been presented in [8]. However, we are going to make these ideas (as well as their background) more explicit, including all necessary computations.

We have already noticed that the necessity of introducing these trivial pairs lies in the requirement of having a proper gauge-fixed action. However, in order to be able to check if a solution of the classical master equation is proper we first have to determine what the maximal possible rank is for the Hessian matrix at the stationary point. (see [8, Appendix]).

Notation: by X_{tot} we denote the total configuration space, which is endowed with a super graded vector space structure:

$$X_{tot} = Y \oplus Y^*[1],$$

with Y a $\mathbb{Z}$-graded vector space. Then Φ_A, $A = 1, \ldots, N$, denotes a generic field in Y and Φ^*_A, $A = 1, \ldots, N$, denotes the corresponding antifield, which is a generic element in $Y^*[1]$, while we use the collective notation $z_a = (\Phi_A, \Phi^*_A)$, with $a = 1, \ldots, 2N$ for the fields and antifields in the total configuration space.

Remark 50
Let F and G be two functions defined on the space X_{tot}. As seen in (3.7), using the notation introduced in the BV formalism, the Poisson bracket defined on the space of regular functions $\mathcal{O}_{X_{tot}}$ can be expressed as follows:

$$\{F, G\} = \sum_i (-1)^{\epsilon(\varphi_i^*)(\epsilon(G)+1)} \frac{\partial F}{\partial \varphi_i} \frac{\partial G}{\partial \varphi_i^*} - (-1)^{\epsilon(\varphi_i)(\epsilon(G)+1)} \frac{\partial F}{\partial \varphi_i^*} \frac{\partial G}{\partial \varphi_i} \ ,$$

or, equivalently, as

$$\{F, G\} = \sum_{a,b=1}^{2N} \frac{\partial F}{\partial z_a} \zeta^{ab} \frac{\partial G}{\partial z_b}, \tag{A.2}$$

where ζ^{ab} is an invertible $2N \times 2N$ matrix of the following form:

$$\zeta^{ab} = \begin{bmatrix} 0 & (-1)^{(\epsilon(\Phi_A)+1)(\epsilon(G)+1)}\delta^{AB} \\ -(-1)^{\epsilon(\Phi_A)(\epsilon(G)+1)}\delta^{AB} & 0 \end{bmatrix}.$$

Proposition 22
In the above notation, let S_{tot} be a solution of the classical master equation on $X_{tot} = Y \oplus Y^*[1]$. Then the maximal possible rank of the Hessian of S_{tot} at the stationary point coincides with the number of fields in Y.

Proof. Since S_{tot} is a solution of the classical master equation, we have

$$\{S_{tot}, S_{tot}\} = 0,$$

where $\{-,-\}$ denotes the Poisson structure defined on the total configuration space. Using the explicit description of the Poisson structure noticed in Remark 50, the classical master equation can be rewritten as follows:

$$\{S_{tot}, S_{tot}\} = \sum_{a,b=1}^{2N} \frac{\partial S_{tot}}{\partial z_a}\zeta^{ab}\frac{\partial S_{tot}}{\partial z_b} = 0. \tag{A.3}$$

By taking the derivative of (A.3) with respect to a field z_c, with $c = 1, \ldots, 2N$, and defining a matrix

$$\mathcal{R}^a_c = \sum_{b=1}^{2N} \zeta^{ab}\frac{\partial^2 S_{tot}}{\partial z^b \partial z^c},$$

one can prove the following identity:

$$\sum_{a=1}^{2N} \frac{\partial S_{tot}}{\partial z_a}\mathcal{R}^a_c = 0. \tag{A.4}$$

Note that the matrix $\mathcal{R}^a_c$ we just introduced is simply given by the Hessian of the function S_{tot} multiplied by an invertible matrix ζ.
Moreover, Equation (A.4) is the Noether identity stated for the total action S_{tot}: we deduce that any solution of the classical master equation is automatically gauge invariant.
By a simple computation one also checks that the matrix $\mathcal{R}^a_c$ does not have maximal rank. More precisely, let us take the derivative of Equation (A.4):

$$\frac{\partial}{\partial z_b}\left(\sum_{a=1}^{2N} \frac{\partial S_{tot}}{\partial z_a}\mathcal{R}^a_c\right) = \sum_{a=1}^{2N}\left(\frac{\partial^2 S_{tot}}{\partial z_b \partial z_a}\right)\mathcal{R}^a_c + \sum_{a=1}^{2N}\frac{\partial S_{tot}}{\partial z_a}\left(\frac{\partial}{\partial z_b}\mathcal{R}^a_c\right) = 0. \tag{A.5}$$

Furthermore, if we evaluate Equation (A.5) at the stationary point $\widetilde{z}$, the second summand comes out to be zero, since $\frac{\partial S_{tot}}{\partial z_a}|_{\widetilde{z}} = 0$. Then

$$\sum_{a=1}^{2N} \left(\frac{\partial^2 S_{tot}}{\partial z_b \partial z_a} \right) \mathcal{R}^a_c \bigg|_{\widetilde{z}} = 0.$$

Recalling that the matrix $\mathcal{R}^a_c$ is defined as the product of the Hessian of S_{tot} and an invertible matrix ζ^{ab}, the previous equation implies that the matrix $\mathcal{R}^a_c$ evaluated at the stationary point $\widetilde{z}$ has no maximal rank,

$$\sum_{a=1}^{2N} \left(\zeta^{db} \frac{\partial^2 S_{tot}}{\partial z_b \partial z_a} \right) \mathcal{R}^a_c \bigg|_{\widetilde{z}} = \sum_{a=1}^{2N} \mathcal{R}^d_a \mathcal{R}^a_c|_{\widetilde{z}} = 0.$$

Using Sylvester's rank inequality we may deduce an additional condition on $rank(\mathcal{R}^a_c)|_{\widetilde{z}}$. Given two generic matrices A and B with dimensions $m \times n$ and $n \times k$, respectively, Sylvester's rank inequality asserts that

$$rank(A) + rank(B) - n \leqslant rank(AB).$$

Applying this inequality to our case, we have $A = \mathcal{R}^d_a|_{\widetilde{z}}$, $B = \mathcal{R}^a_b|_{\widetilde{z}}$, $n = 2N$, and $AB = 0$. Therefore, we deduce that

$$rank(\mathcal{R})|_{\widetilde{z}} \leqslant N.$$

Recalling once again that the matrix $\mathcal{R}$ coincides with the Hessian of the action S_{tot} up to the product with an invertible matrix, we deduce that the same condition just found on $rank(\mathcal{R})|_{\widetilde{z}}$ holds also for the Hessian of S_{tot} evaluated at the stationary point $\widetilde{z}$. That is to say, the following inequality holds:

$$rank \left(\frac{\partial^2 S_{tot}}{\partial z^a \partial z^b} \right) \bigg|_{\widetilde{z}} \leqslant N.$$

To conclude, we proved that the maximal rank of the Hessian of a function S_{tot} that is a solution of the classical master equation, evaluated at the stationary point $\widetilde{z}$ is N, i.e., the number of fields in Y. □

The aim of the following analysis is to prove that the collection of auxiliary fields described in Theorem 4 is the minimal set necessary to satisfy the condition of the action to be a proper solution also after the gauge-fixing procedure has been performed. As said, we restrict to the case of a reducible theory (X_0, S_0) with level of reducibility $L = 1$. Thus:

- Since (X_0, S_0) describes a gauge invariant theory, the Noether identities are satisfied in a neighborhood of the stationary point $\widetilde{\varphi}$:

$$\sum_{i=1}^{n} \frac{\partial S_0}{\partial \varphi_i} \mathcal{R}^i_\alpha = 0,$$

with $i = 1, \ldots, n$, $\alpha = 1, \ldots, r_1$, $r_1 < n$. There is a relation between the level of reducibility of a theory and the rank of a particular sequence of matrices, as better explained in [34]. Thus we deduce that, due to the assumption on the reducibility of the theory, the rank of the matrix $\mathcal{R}^i_\alpha$ is not maximal, as it is strictly less than r_1. Then we assume that

$$rank(R^i_\alpha) = r_1 - r_2,$$

for a certain $r_2 > 0$; because the rank of the matrix $\mathcal{R}^i_\alpha$ is not maximal, there exists a matrix $\mathcal{Z}^\alpha_\beta$, with $\beta = 1, \ldots, r_2$ such that:

$$\sum_{\alpha=1}^{r_1} \mathcal{R}^i_\alpha \mathcal{Z}^\alpha_\beta = 0.$$

Moreover, since the theory is supposed to be reducible with level of reducibility $L = 1$, the matrix $\mathcal{Z}^\alpha_\beta$ has maximal rank, i.e.,

$$rank(\mathcal{Z}^\alpha_\beta) = r_2.$$

- $(\widetilde{X}, \widetilde{S})$ is the minimally extended theory corresponding to the initial gauge theory (X_0, S_0). As discussed in detail in [34], an alternative method to determine the extended configuration space relates the number of ghost fields to be introduced to the rank of the matrices $\mathcal{R}^i_\alpha$ and $\mathcal{Z}^\alpha_\beta$, so that $\widetilde{X}$ is given by

$$\widetilde{X} = W \oplus W^*[1],$$

with $W = \langle \varphi_1, \ldots, \varphi_n \rangle_0 \oplus \langle C_1, \ldots, C_{r_1} \rangle_1 \oplus \langle E_1, \ldots, E_{r_2} \rangle_2$.
Consequently:

$$W^*[1] = \langle \varphi^*_1, \ldots, \varphi^*_n \rangle_{-1} \oplus \langle C^*_1, \ldots, C^*_{r_1} \rangle_{-2} \oplus \langle E^*_1, \ldots, E^*_{r_2} \rangle_{-3}.$$

As to the extended action $\widetilde{S}$, it is possible to prove that the coefficients appearing together with the linear terms in the antifields are determined by the matrices $\mathcal{R}^i_\alpha$ and $\mathcal{Z}^\alpha_\beta$. Explicitly, we have

$$\widetilde{S} = S_0 + \sum_{i=1}^{n} \sum_{\alpha=1}^{r_1} \varphi^*_i \mathcal{R}^i_\alpha C_\alpha + \sum_{\alpha=1}^{r_1} \sum_{\beta=1}^{r_2} C^*_\alpha \mathcal{Z}^\alpha_\beta E_\beta + \ldots, \qquad \text{(A.6)}$$

where all terms that appear in the extended action and are linear in the antifields have been listed.

- By $\widetilde{z}$ we denote the stationary point for the extended action $\widetilde{S}$. In coordinates:
$$\begin{cases} \varphi_i|_{\widetilde{z}} = \widetilde{\varphi} \\ E^*_l|_{\widetilde{z}} = C^*_j|_{\widetilde{z}} = \varphi^*_i|_{\widetilde{z}} = C_i|_{\widetilde{z}} = E_l|_{\widetilde{z}} = 0 \end{cases}$$
for any value of i, j, l.

Using the notation introduced above, we restate and prove Theorem 4 in the particular case of a reducible theory of level $L = 1$.

Proposition 23
Let $(\widetilde{X}, \widetilde{S})$ be the extended theory corresponding to a reducible theory with level of reducibility $L = 1$, as described above. Then the extended action $\widetilde{S}$ is a proper solution on $\widetilde{X}$. Let (X_{tot}, S_{tot}) be the extended theory obtained by further enlarging $\widetilde{X}$ with the following extra fields:

- a trivial pair (B_j, h_j), with $j = 1, \ldots, r_1$ such that:
$$\begin{cases} deg(B_j) = -1 \\ \epsilon(B_j) = 1 \end{cases} \qquad \begin{cases} deg(h_j) = 0 \\ \epsilon(h_j) = 0 \ ; \end{cases} \tag{A.7}$$

- two trivial pairs (A_l, k_l) and $(\widetilde{A}_l, \widetilde{k}_l)$, with $l = 1, \ldots, r_2$ such that:
$$\begin{cases} deg(A_l) = -2 \\ \epsilon(A_l) = 0 \end{cases} \qquad \begin{cases} deg(k_l) = -1 \\ \epsilon(k_l) = 1 \end{cases}$$
$$\begin{cases} deg(\widetilde{A}_l) = 0 \\ \epsilon(\widetilde{A}_l) = 0 \end{cases} \qquad \begin{cases} deg(\widetilde{k}_l) = 1 \\ \epsilon(\widetilde{k}_l) = 0 \end{cases} \tag{A.8}$$

and by adding to $\widetilde{S}$ the summands

$$S_{aux} = \sum_{j=1}^{r_1} B^*_j h_j + \sum_{l=1}^{r_2} A^*_l k_l + \sum_{l=1}^{r_2} \widetilde{A}^*_l \widetilde{k}_l,$$

Then the gauge-fixed total action

$$S_{tot}|_\psi = \widetilde{S}|_\psi + S_{aux}|_\psi$$

is a proper solution of the classical master equation on the space of fields $\widetilde{Y}$, with $X_{tot} = \widetilde{Y} \oplus \widetilde{Y}^*[1]$.

Proof. In order to show that the extended action $\widetilde{S}$ is a proper solution when it is considered on the extended configuration space $\widetilde{X}$, we have to show that its Hessian $\left[\frac{\partial^2 \widetilde{S}}{\partial z_a \partial z_b}\right]$, with z_a, z_b any possible pair of fields, ghosts, antifields and antighost fields, has maximal rank when it is evaluated at the stationary point $\widetilde{z}$.
Since the ghost fields, the antifields, and the antighost fields evaluated at $\widetilde{z}$ are all zero, we deduce that the only non-trivial contributions to the Hessian come from the terms of the action $\widetilde{S}$ that are at most quadratic in ghosts, antifields and antighosts. Thus the only terms of $\widetilde{S}$ we need to take into account are the one explicitly listed in Equation (A.6).
By an explicit computation, one checks that the Hessian of $\widetilde{S}$ evaluated at $\widetilde{z}$ is given by the following matrix:

	φ_j	φ_j^*	C_j	C_j^*	E_j	E_j^*
φ_i	$\frac{\partial^2 S_0}{\partial \varphi_i \partial \varphi_j}\Big\vert_{\widetilde{\varphi}}$					
φ_i^*			$\mathcal{R}_j^i\vert_{\widetilde{\varphi}}$			
C_i		$\mathcal{R}_j^i\vert_{\widetilde{\varphi}}$				
C_i^*					$\mathcal{Z}_j^i\vert_{\widetilde{\varphi}}$	
E_i				$\mathcal{Z}_j^i\vert_{\widetilde{\varphi}}$		
E_i^*						

Thus the rank of the previous matrix turns out to be

$$rank\left(\frac{\partial^2 \widetilde{S}}{\partial z_a \partial z_b}\bigg|_{\widetilde{z}}\right) = rank\left(\frac{\partial^2 S_0}{\partial \varphi_i \partial \varphi_j}\bigg|_{\widetilde{\varphi}}\right) + 2\ rank\left(\mathcal{R}_j^i|_{\widetilde{\varphi}}\right) + 2\ rank\left(\mathcal{Z}_j^i|_{\widetilde{\varphi}}\right).$$

Using the following hypothesis on the rank of the previous matrices:

$$rank\left(\frac{\partial^2 S_0}{\partial \varphi_i \partial \varphi_j}\bigg|_{\widetilde{\varphi}}\right) = n - r_1 + r_2, \quad rank\left(\mathcal{R}_j^i|_{\widetilde{\varphi}}\right) = r_1 - r_2, \quad rank\left(\mathcal{Z}_j^i|_{\widetilde{\varphi}}\right) = r_2,$$

we obtain that the rank of the Hessian evaluated at the stationary point is

$$rank\left(\frac{\partial^2\widetilde{S}}{\partial z_a \partial z_b}\bigg|_{\widetilde{z}}\right) = n + r_1 + r_2.$$

In $\widetilde{X}$ we have exactly n fields φ_i, r_1 ghost fields C_j and r_2 ghost fields E_k, which sum up to a total of $n + r_1 + r_2$ fields and ghost fields. Since this quantity coincides with the rank computed before, we conclude that $\widetilde{S}$ is a proper solution of the classical master equation on the extended configuration space $\widetilde{X}$.

To prove the second part of the proposition, we have to show that, defining the total configuration space X_{tot} as obtained by adding to $\widetilde{X}$ the extra fields listed in Equation (A.7), (A.8) and the corresponding antifields, the gauge-fixed total action

$$S_{tot}|_{\psi} = \widetilde{S}|_{\psi} + S_{aux}|_{\psi}$$

is a proper solution of the classical master equation on $\widetilde{Y}$, with

$$X_{tot} = \widetilde{Y} \oplus \widetilde{Y}^*[1].$$

More precisely:

- $\widetilde{Y} = \langle A_l\rangle \oplus \langle B_j, k_l\rangle \oplus \langle \varphi_i, h_j, \widetilde{A}_l\rangle \oplus \langle C_j, \widetilde{k}_l\rangle \oplus \langle E_l\rangle$, with $i = 1,\dots,n$, $j = 1,\dots,r_1$ and $l = 1,\dots,r_2$. So the total number of fields in the theory is: $n + 3r_1 + 5r_2$.

- The total action, up to terms that are of higher order than quadratic in the antifields, ghost fields and antighosts, takes the following form:

$$S_{tot} \simeq S_0 + \sum_{i=1}^{n}\sum_{\alpha=1}^{r_1} \varphi_i^* \mathcal{R}^i_\alpha C_\alpha + \sum_{\alpha=1}^{r_1}\sum_{\beta=1}^{r_2} C^*_\alpha \mathcal{Z}^\alpha_\beta E_\beta + \sum_{j=1}^{r_1} B_j^* h_j + \sum_{l=1}^{r_2} A_l^* k_l + \sum_{l=1}^{r_2} \widetilde{A}_l^* \widetilde{k}_l.$$

- The approximation of the gauge-fixing fermion takes then the following form:

$$\begin{aligned}\psi \;&= \textstyle\sum_{t=1}^{r_1} B_t\left[f_t^0 + \sum_{j=1}^{r_1} f_t^{1,j} h_j + \sum_{y=1}^{r_2} p_{ty}\widetilde{A}_y\right]\\ &\quad + \textstyle\sum_{l,w=1}^{r_2}\sum_{s=1}^{r_1} A_l\left[g^0_{ls} C_s + m_{lw}\widetilde{k}_w\right] + \sum_{y=1}^{r_2} k_y\left[l_y^0 + \sum_{w=1}^{r_2} q_{yw}\widetilde{A}_w\right],\end{aligned}$$

  with f_t^0, $f_t^{1,j}$, g^0_{ls}, p_{ty}, m_{yw}, l_y^0 and q_{yw} in $\mathrm{Pol}_{\mathbb{R}}(\varphi_i)$.

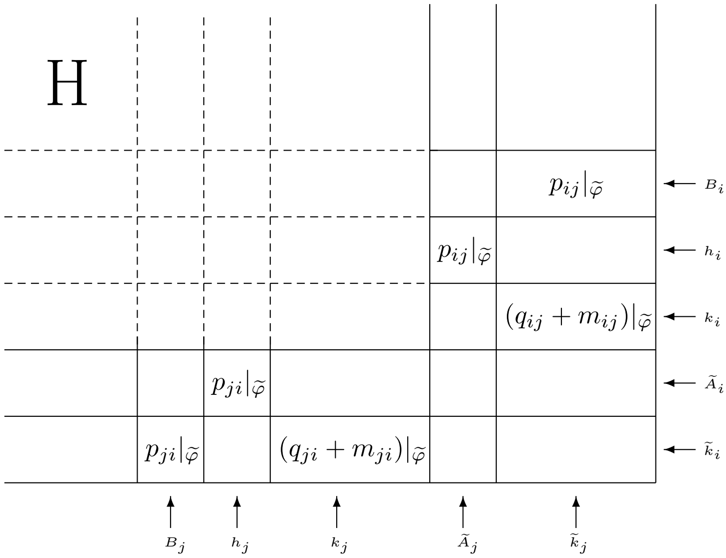


Figure A.1: Matrix $\widetilde{H}$: Hessian for the gauge-fixed total action $S_{tot}|_\psi$ in (A.9) We denoted by H the matrix explicitly described in Figure A.2: even though in the figure it appears only in the left-top corner, actually the matrix H fills the whole area among dotted lines.

Thus the gauge-fixed action turns out to be the following:

$$\begin{aligned} S_{tot}|_\psi \simeq \quad & S_0 + \textstyle\sum_{i=1}^{n} \sum_{\alpha,t=1}^{r_1} B_t \frac{\partial f_t^0}{\partial \varphi_i} \mathcal{R}_\alpha^i C_\alpha + \sum_{\alpha=1}^{r_1} \sum_{\beta,l=1}^{r_2} A_l g_{l\alpha} \mathcal{Z}_\beta^\alpha E_\beta \\ & + \textstyle\sum_{j=1}^{r_1} \left[f_j^0 + \sum_{t=1}^{r_1} f_j^{1,t} h_t + \sum_{y=1}^{r_2} p_{jy} \widetilde{A}_y \right] h_j \\ & + \textstyle\sum_{s=1}^{r_1} \sum_{l=1}^{r_2} \left[g_{ls}^0 C_s + m_{lw} \widetilde{k}_w \right] k_l + \sum_{l,y=1}^{r_2} \left[\sum_{t=1}^{r_1} B_t p_{tl} + k_y q_{yl} \right] \widetilde{k}_l . \end{aligned} \tag{A.9}$$

By a direct computation one can check that the Hessian of the gauge-fixed action $S_{tot}|_\psi$ evaluated at the stationary point $\widetilde{z}$ is the one described in Figure A.1.

Up to now the polynomials f_t^0, $f_t^{1,j}$, g_{ls}^0, p_{ty}, m_{yw}, l_y^0 and q_{yw} in $\mathrm{Pol}_{\mathbb{R}}(\varphi_i)$ are completely free: there are no more conditions than the one on the total degree which forced them to depend only on the fields φ_i.
Conditions will be imposed on them in order to have that the Hessian has

$\left.\frac{\partial^2 S_0}{\partial \varphi_i \partial \varphi_j}\right\|_{\widetilde{\varphi}}$				$\left.\frac{\partial f_j^0}{\partial \varphi_i}\right\|_{\widetilde{\varphi}}$		
			$\sum_l \left.\left(\frac{\partial f_j^0}{\partial \varphi_l} \mathcal{R}_i^l\right)\right\|_{\widetilde{\varphi}}$			$g_{ij}\|_{\widetilde{\varphi}}$
					$\sum_l \left.\left(g_{jl} \mathcal{Z}_i^l\right)\right\|_{\widetilde{\varphi}}$	
	$\sum_l \left.\left(\frac{\partial f_i^0}{\partial \varphi_l} \mathcal{R}_j^l\right)\right\|_{\widetilde{\varphi}}$					
$\left.\frac{\partial f_i^0}{\partial \varphi_j}\right\|_{\widetilde{\varphi}}$				$f_j^{1,i}\|_{\widetilde{\varphi}}$		
		$\sum_l \left.\left(g_{il} \mathcal{Z}_j^l\right)\right\|_{\widetilde{\varphi}}$				
	g_{ij}					
↑ φ_j	↑ C_j	↑ E_j	↑ B_j	↑ h_j	↑ A_j	↑ k_j

Figure A.2: The matrix H. In the rows we have taken the derivatives with respect to the fields ordered from the top to the bottom, in the same order as the one used for the columns, by left to right and with index i.

maximal rank. More precisely, we require the following properties:

1. $rank\left(\sum_l \left(\frac{\partial f_i^0}{\partial \varphi_l}\mathcal{R}_j^l\right)\Big|_{\widetilde{\varphi}}\right) = r_1 - r_2;$

2. $rank\left(\frac{\partial f_i^0}{\partial \varphi_j}\Big|_{\widetilde{\varphi}}\right) = r_1 - r_2;$

3. $rank\left(\sum_l \left(g_{il}\mathcal{Z}_j^l\right)\Big|_{\widetilde{\varphi}}\right) = r_2;$

4. $rank\left(g_{ij}|_{\widetilde{\varphi}}\right) = r_2;$

5. $rank(p_{ij}|_{\widetilde{\varphi}}) = r_2.$

Under these hypothesis we have

$$\begin{aligned} & rank(\widetilde{H}) \\ & = rank\left(\frac{\partial^2 S_0}{\partial \varphi_i \partial \varphi_j}\Big|_{\widetilde{\varphi}}\right) + 2\ rank\left(\sum_l \left(\frac{\partial f_i^0}{\partial \varphi_l}\mathcal{R}_j^l\right)\Big|_{\widetilde{\varphi}}\right) + 2\ rank\left(\frac{\partial f_i^0}{\partial \varphi_j}\Big|_{\widetilde{\varphi}}\right) \\ & \quad +2\ rank\left(\sum_l \left(g_{il}\mathcal{Z}_j^l\right)\Big|_{\widetilde{\varphi}}\right) + 2\ rank\left(g_{ij}|_{\widetilde{\varphi}}\right) + 4 rank(p_{ij}|_{\widetilde{\varphi}}) = n + 3r_1 + 5r_2. \end{aligned}$$

Thus we conclude that the rank of the Hessian $\widetilde{H}$ evaluated at the stationary point coincides with the number of fields in the theory. Therefore, the gauge-fixed total action $S_{tot}|_\psi$ is a proper solution for the classical master equation on the space of fields $\widetilde{Y}$. □

Remark 51
As explicitly shown in the proof of Proposition 22, in the case of a reducible theory with level of reducibility $L = 1$, to have a gauge-fixed action that is also a proper solution we did not only have to extend the space of fields by introducing three types of trivial pairs with suitable ghost degrees, but we also had to impose explicit conditions on the polynomials appearing in the gauge-fixing fermion. Since these conditions are necessary to be able to draw the desired conclusion, we rewrite them here, seeing them as conditions on the second derivatives of the gauge-fixing fermion evaluated at the stationary point.

1. $rank\Big(\sum_l \frac{\partial^2 \psi}{\partial B_i \partial \varphi_l}\mathcal{R}_j^l\Big|_{\widetilde{\varphi}}\Big) = r_1 - r_2;$

2. $rank\left(\frac{\partial^2\psi}{\partial B_i \partial \varphi_l}\right) = r_1 - r_2;$

3. $rank\left(\sum_l \frac{\partial^2\psi}{\partial A_i \partial C_l} \mathcal{Z}_j^l \Big|_{\widetilde{\varphi}} \right) = r_2;$

4. $rank\left(\frac{\partial^2\psi}{\partial A_i \partial C_l} \Big|_{\widetilde{\varphi}} \right) = r_2.$

Note that we are not forced to impose any conditions on the polynomials q_{ij} and m_{ij}, that is to say, on the second derivative of the gauge-fixing fermion with respect to the fields $\widetilde{k}_j$ and the fields B_j or the fields k_j.

To conclude, we briefly explain why the extra fields listed in Equation (A.7), (A.8) are the minimal set of extra fields that need to be introduced to have a gauge-fixed total action which is a proper solution on the space of fields $\widetilde{Y}$.

Remark 52
The central observation to prove that the extra fields (A.7), (A.8) form a minimal set is that the only terms that might contribute to the Hessian of the gauge-fixed action are the ones in ψ which are at most quadratic in the ghost fields.
Moreover, since the action $\widetilde{S}$ depends explicitly both on the antifields φ_i^* and on the antighost fields C_j^*, if the gauge-fixing fermion would not depend on φ_i or on C_j, it would be eliminated the contribution to the Hessian given by the quadratic terms:

$$\sum_{i=1}^{n} \sum_{\alpha=1}^{r_1} \varphi_i^* \mathcal{R}_\alpha^i C_\alpha \quad \text{and} \quad \sum_{\alpha=1}^{r_1} \sum_{\beta=1}^{r_2} C_\alpha^* \mathcal{Z}_\beta^\alpha E_\beta.$$

This would lower the rank of the Hessian: thus necessarily, the gauge fixing fermion needs to depend explicitly both on the fields φ_i and the ghosts C_j.
Thus:

- ▶ To be able to define suitable summands in Ψ depending on the fields φ_i, we have to introduce the extra fields B_j, with $j = 1, \ldots, r_1$ and $deg(B_j) = -1$.
- ▶ To be able to define suitable summands in Ψ depending on the ghost fields C_i, we have to introduce the extra fields A_l, with $l = 1, \ldots, r_2$ and $deg(A_l) = -2$.

Notice that we are taking exactly r_1 extra fields B_j and r_2 extra fields A_l because r_1 and r_2 are the minimal numbers of fields that allows us to maximize the rank of the corresponding contribution to the Hessian matrix. Therefore, after having introduced the trivial pairs defined by the extra fields B_j and A_l, the gauge-fixed action $S_{tot}|_\Psi$ is given by the following expression:

$$\begin{aligned} S_{tot}|_\psi \simeq \quad & S_0 + \textstyle\sum_{i=1}^{n}\sum_{\alpha,t=1}^{r_1} B_t \frac{\partial f_t^0}{\partial \varphi_i}\mathcal{R}_\alpha^i C_\alpha + \sum_{\alpha=1}^{r_1}\sum_{\beta,l=1}^{r_2} A_l g_{l\alpha}\mathcal{Z}_\beta^\alpha E_\beta \\ & + \textstyle\sum_{j=1}^{r_1}\left[f_j^0 + \sum_{t=1}^{r_1} f_j^{1,t}h_t\right]h_j + + \sum_{s=1}^{r_1}\sum_{l=1}^{r_2}\left[g_{ls}^0 C_s\right]k_l. \end{aligned} \tag{A.10}$$

Then the Hessian defined by this gauge-fixed total action $S_{tot}|_\Psi$ coincides with the matrix H in Figure A.2. With some computations one can check that the maximal rank of H evaluated at the stationary point $\widetilde{z}$ is the following:

$$\begin{aligned} & rank\left(\frac{\partial^2 S_{tot}|_\psi}{\partial z_a \partial z_b}\Big|_{\widetilde{z}}\right) \\ & \quad = rank\left(\frac{\partial^2 S_0}{\partial\varphi_i\partial\varphi_j}\Big|_{\widetilde{\varphi}}\right) + 2\ rank\left(\textstyle\sum_l\left(\frac{\partial f_i^0}{\partial\varphi_l}\mathcal{R}_j^l\right)\Big|_{\widetilde{\varphi}}\right) + 2\ rank\left(\frac{\partial f_i^0}{\partial\varphi_j}\Big|_{\widetilde{\varphi}}\right) \\ & \quad +2\ rank\left(\textstyle\sum_l\left(g_{il}\mathcal{Z}_j^l\right)\big|_{\widetilde{\varphi}}\right) + 2\ rank\left(g_{ij}|_{\widetilde{\varphi}}\right) = n + 3r_1 + r_2. \end{aligned}$$

Comparing the rank of the Hessian with the number of fields, we see that the difference is precisely $2r_1$: thus we have to introduce another trivial pair composed by r_2 fields in order to have as new contribution for the Hessian another matrix of rank r_2. This is the reason why we introduce the last trivial pair $(\widetilde{A}_l, \widetilde{k}_l)$, with $l = 1, \dots, r_1$. Note that, up to now, we are not imposing any condition on the ghost degree of the fields $\widetilde{A}_l$ and so we have that:

$$\begin{cases} deg(\widetilde{A}_l) = m \\ \epsilon(\widetilde{A}_l) \equiv m \pmod 2 \end{cases} \qquad \begin{cases} deg(\widetilde{k}_l) = m+1 \\ \epsilon(\widetilde{k}_l) \equiv m+1 \pmod 2. \end{cases}$$

for a certain $m \in \mathbb{Z}$. If we simply added a new trivial summand to the action, it would contribute a matrix of rank r_2 to the Hessian (under suitable hypothesis on the coefficients). Therefore, the rank of the Hessian H would increase by $2r_2$, obtaining

$$rank(H) = n + 3r_1 + 3r_2.$$

Once again, the rank of the Hessian does not coincide with the number of fields since, adding also this new trivial pair, the total number of fields in the theory would be $n + 3r_1 + 5r_2$. For this reason we use the possibility of fixing the ghost degree $-m$ of this new trivial pair, in such a way that these extra fields $(\widetilde{A}_l, \widetilde{k}_l)$ contribute to the Hessian not only through the trivial term but also in combination with other terms already present in the theory. Hence the number of fields remain unchanged but the rank of the Hessian increase. This is the reason for fixing the ghost degree $m = 0$. Thus the trivial pairs (A.7), (A.8) form a minimal set for which the Hessian of the gauge-fixed total action, evaluated in the singular point, has maximal rank.

Appendix B

Tate's algorithm

The purpose of this appendix is to briefly recall Tate's algorithm [55]. The importance of this algorithm in the context of this thesis is that the Tate resolution is used as a first step in the construction of an extended variety corresponding to a gauge invariant theory (X_0, S_0). The theoretical procedure for constructing an extended variety given a Tate resolution was presented in Chapter 4, while in Chapter 5 this construction was explicitly applied to our model of interest.
Another important aspect related to Tate's algorithm is that it gives a mathematical interpretation of physical concepts such as the ghost degree and the parity of the ghost fields, introduced to deal with the gauge symmetries of the physical system.
Even though in this thesis Tate's algorithm is used in a precise context, i.e., is applied to the Jacobian ring determined as quotient of the structure sheaf $\mathcal{O}_{X_0}$ corresponding to the initial configuration space over the ideal generated by the partial derivatives of the initial action S_0, in this appendix we review this algorithm in a general context.

Let R be a commutative Noetherian ring with unit element and let M be an ideal in R: Tate's algorithm is a canonical procedure for constructing a free resolution of R/M that is a differential R-algebra. Before we start describing the algorithm, we recall some definitions and we introduce some notation.

Definition 51. *Given a commutative Noetherian ring R with unit element, a* differential R-algebra *is an associative differential graded commutative algebra*

$$A = \bigoplus_{i \in \mathbb{Z}_{\leq 0}} A_i$$

over R with homogeneous components A_i of finite rank as R-module.
More explicitly, A is an associative algebra equipped with a map $\delta : A \to A$, satisfying the following axioms:

1. *A is graded over $\mathbb{Z}_{\leq 0}$, i.e., $A = \bigoplus_{i \in \mathbb{Z}_{\leq 0}} A_i$, the direct sum of R-modules A_i such that $A_i A_j \subseteq A_{i+j}$;*
2. *A has a unit element $1 \in A_0$ such that $A_0 = R \cdot 1$;*
3. *for all $i \in \mathbb{Z}_{\leq 0}$, A_i is a finitely generated R-module;*
4. *A is a graded algebra, in that*

   $$x \cdot y = (-1)^{deg(x)deg(y)} \; y \cdot x \ ,$$

   where $deg(x) = i$ for $x \in A_i$. In particular, we notice that if i is odd, then $x^2 = 0$, for all $x \in A_i$;
5. *the map δ is a skew derivation of degree 1, i.e., the following properties hold:*
   - $\delta = \{\delta_i\}_{i \in \mathbb{Z}_{<0}}$ *where* $\delta_i : A_i \to A_{i+1}$;
   - $\delta^2 = 0$;
   - $\delta(x \cdot y) = (\delta x) \cdot y + (-1)^{deg(x)} x \cdot (\delta y)$.

Definition 52. *Let A, $A^{'}$ be two differential R-algebras with derivations δ_A and $\delta_{A'}$ and unit elements 1_A and $1_{A'}$ respectively. A* homomorphism of differential R-algebras *from A to $A^{'}$ is a map $\varphi : A \to A^{'}$ such that:*

1. *φ is R-linear;*
2. *$\varphi(A_i) \subseteq A^{'}_i$, for all $i \in \mathbb{Z}_{\leq 0}$;*
3. *$\varphi(1_A) = 1_{A'}$;*
4. *$\forall a \in A$, $\varphi(\delta_A(a)) = \delta_{A'}(\varphi(a))$.*

A consequence of the previous definition is that, given a homomorphism of differential R-algebras φ with

$$\varphi = \{\varphi_i\}_{i\in\mathbb{Z}_{\leq 0}}, \qquad \varphi_i : A_i \to A_i^{'},$$

then $\varphi_0 = Id_R$. This fact follows from conditions (1) and (3) in the definition of a homomorphism of differential R-algebras and from the fact that $A_0 = R \cdot 1_A$ and $A_0^{'} = R \cdot 1_{A'}$.

Definition 53. *A differential R-algebra A is a* differential R-subalgebra *of a differential R-algebra $A^{'}$ if the following conditions hold:*

1. $A \subseteq A^{'}$;
2. *the inclusion map $i : A \hookrightarrow A^{'}$ is a homomorphism of differential R-algebras.*

An equivalent way to describe a differential R-algebra is to see it as a complex of finitely generated R-modules together with a coboundary operator δ:

$$\cdots \xrightarrow{\delta_{-n-1}} A_{-n} \xrightarrow{\delta_{-n}} A_{-n+1} \xrightarrow{\delta_{-n+1}} \cdots \xrightarrow{\delta_{-2}} A_{-1} \xrightarrow{\delta_{-1}} A_0 \cong R \xrightarrow{\delta_0} 0. \qquad \text{(B.1)}$$

This equivalent description allows us to speak about cocycles and coboundaries for a given differential R-algebra A.

Definition 54. *Let A be a differential R-algebra. We define:*

- $Z = \bigoplus_{i\in\mathbb{Z}_{\leq 0}} Z_i$, *where* $Z_i = \mathrm{Ker}(\delta_i) \subseteq A_i$. *The elements in Z are called* cocycles;
- $B = \bigoplus_{i\in\mathbb{Z}_{\leq 0}} B_i$, *where* $B_i = \mathrm{Im}(\delta_{i-1}) \subseteq A_i$. *The elements in B are called* coboundaries.

Using the properties required for the derivation δ of the differential R-algebra A, one deduces the following inclusions:

- $Z_i \cdot Z_j \subseteq Z_{i+j}$;
- $B_i \cdot Z_j \subseteq B_{i+j}$;
- $Z_i \cdot B_j \subseteq B_{i+j}$.

Therefore, Z is a graded subalgebra of A while B is a homogeneous two-sided ideal in Z. Moreover, given these inclusions, the cohomology algebra of A is well defined.

Definition 55. *The* cohomology algebra $H(A)$ *of a given differential R-algebra A is defined as follows:*

$$H(A) = \bigoplus_{k \in \mathbb{Z}_{\leq 0}} H^k(A),$$

where:

$$H^k(A) = \frac{Z_k}{B_k} = \frac{\mathrm{Ker}(\delta_k)}{\mathrm{Im}(\delta_{k-1})}. \tag{B.2}$$

Definition 56. *A differential R-algebra A is said to be:*

- acyclic *if $H(A) = H^0(A)$, i.e. if $H^k(A) = 0$, $\forall k < 0$;*
- free *if each homogeneous component A_i of A is a free R-module.*

In (B.1), we have by definition, $\delta_0 = 0$. Therefore, we deduce that $Z_0 = A_0$.

Moreover:

- If A is a free differential R-algebra, then $A_0 = R \cdot 1_A \cong R$. So $B_0 := M$ is an ideal in $Z_0 = R$ and $H^0(A) = Z_0/B_0 \cong R/M$.
- If A is free and acyclic, it gives a free resolution of the R-module R/M, i.e. the following sequence is exact and the A_i are free R-modules:

$$\cdots \xrightarrow{\delta_{-3}} A_{-2} \xrightarrow{\delta_{-2}} A_{-1} \xrightarrow{\delta_{-1}} R \xrightarrow{\pi} R/M \to 0 \ , \tag{B.3}$$

where π is the canonical projection map.

The purpose of Tate's algorithm is to construct a free and acyclic resolution of R/M, where R is a given ring as above and M is an ideal in R. An important role in proving Tate's theorem will be played by the process of adjoining a variable in a differential R-algebra, as explained in the following section.

B.1 The process of adjoining a variable

The process of adjoining a variable is the fundamental tool on which the construction of a Tate resolution is based. A particularly clear exposition can be found in [55]. Here, we quickly recall the main points.

Let A be a differential R-algebra as above and let $p \in \mathbb{Z}_{\leq 0}$. We describe a canonical procedure for constructing an extended differential R-algebra $\widetilde{A}$, $\widetilde{A} \supseteq A$, such that the following conditions hold:

(a) $\widetilde{A}_i = A_i, \quad \forall i > p$;

(b) $B_{p+1}(\widetilde{A}) = B_{p+1}(A) + R\tau$, for a fixed element $\tau \in Z_{p+1}(A)$.

Since the procedure is quite different for the case of odd and even p, we discuss the two cases separately.

p odd. Let $\widetilde{T}$ be a free R-module generated by one element, $\widetilde{T} := \langle T \rangle$. Define

$$A\widetilde{T} = \{aT, a \in A\} \ , \qquad \widetilde{A} = A \oplus A\widetilde{T} \ .$$

We equip $\widetilde{A}$ with a graded structure by giving to T the degree p:

$$\widetilde{A} = \bigoplus_{i \in \mathbb{Z}_{\leq 0}} \widetilde{A}_i \ , \quad \text{where} \begin{cases} \widetilde{A}_i = A_i, & \text{for } i > p \\ \widetilde{A}_i = A_i \oplus A_{i-p}\widetilde{T} & \text{for } i \leq p. \end{cases}$$

So $\widetilde{A}$ is a graded R-module.

In order to provide $\widetilde{A}$ with a differential R-algebra structure, we need to define an associative and graded commutative product over $\widetilde{A}$ and extend the derivation δ_A of A to a derivation over $\widetilde{A}$.

Product: since T has odd degree we have

$$T^{\,2} = 0 \quad \text{and} \quad T \cdot a = (-1)^i a \cdot T, \quad \text{for } a \in A_i.$$

Therefore, given two elements $f + gT$ and $h + kT$ in $\widetilde{A}$, we define:

$$(f + gT) \cdot (h + kT) := f \cdot h + (g \cdot h + f \cdot k)T.$$

Since $f, g, h, k \in A$, we are using the product already defined between these elements to define a product in $\widetilde{A}$. Moreover, since the product in A is associative and graded commutative, ones easily verifies that also the product defined in $\widetilde{A}$ has the same properties.

Derivation: we extend the derivation δ_A to a derivation defined over $\widetilde{A}$ by imposing that $\delta_{\widetilde{A}}(T) = \tau$. More explicitly, given an element $f + gT \in A_i$, with $i \leq p$, we define

$$\delta_{\widetilde{A}}(f + gT) := \delta_A(f) + \delta_A(g)T + (-1)^{i-p}\, g\tau.$$

If $i > p$, then by definition $\widetilde{A}_i = A_i$ and so $\delta_{\widetilde{A}}|_{\widetilde{A}_i} = \delta_A|_{A_i}$, $\forall i > p$.
Using the fact that δ_A is a graded derivation of degree 1, one shows that the same holds also for $\delta_{\widetilde{A}}$.

With the product and the derivation defined above, $\widetilde{A}$ is a differential R-algebra. Moreover, condition (a) is satisfied by definition and condition (b) follows from the definition of $\delta_{\widetilde{A}}$. This concludes the procedure for odd degree p.

p even. In order to simplify the explanation of the procedure for the case in which p is even, we assume that the ring R contains a subfield of characteristic 0. In the context in which we applied Tate's algorithm, this hypothesis was always satisfied.

Let $\widetilde{A}$ denote the ring of polynomials in one commuting variable T with coefficients in A, $\widetilde{A} = A[T]$. We equip $\widetilde{A}$ with a graded structure by assigning degree kp to T^k. Therefore,

$$\widetilde{A}_i = A_i + A_{i-p}T + A_{i-2p}T^2 + \dots$$

Notice that the previous sum is always finite since, by hypothesis, the differential R-algebra A is graded over $\mathbb{Z}_{\leq 0}$.

Product: this is the usual product defined on the ring of polynomials.

Derivation: we extend the derivation defined over A to a derivation over $\widetilde{A}$ by requiring that

$$\delta_{\widetilde{A}}(T^k) = \tau \cdot kT^{k-1},$$

which uniquely determines the derivation. One can check that the product and the derivation have the required properties and that also in this case the conditions (a) and (b) are satisfied.

Notice that the differential R-algebra $\widetilde{A}$ obtained from the differential R-algebra A following the previous procedure is completely determined by the degree p and the element $\tau \in Z_{p+1}(A)$. Therefore, we introduce the following notation:

$$\widetilde{A} = A\langle T \rangle, \quad \text{with } \delta(T) = \tau.$$

and we will say that $\widetilde{A}$ *is obtained from* A *by the adjunction of a variable* T *of degree* p *corresponding to* τ.

Now we have everything we need to state Tate's theorem [55].

Theorem 16. *Let* R *be a commutative Noetherian ring with unit element and let* M *be an ideal in* R*. Then there exists a free acyclic differential* R*-algebra* A *such that* $H^0(A) = R/M$*.*

Proof. We obtain A as the union of an ascending chain of differential R-algebras

$$A^0 \subseteq A^{-1} \subseteq A^{-2} \subseteq \cdots \tag{B.4}$$

Notice that each A^k will be differential R-algebras: therefore, each of them will have a graded structure. So, by the notation A_i^j we mean the homogeneous component of degree i for the $j-$th element in the ascending chain (B.4). For the pertinent derivation, we use the notation δ_i^j: analogously to what we do for the differential R-algebras A^k, also in this case the index j will indicate that we are considering the differential R-algebra A^j in the chain (B.4) while the index i will indicate the component of the derivation that acts over the i-th homogeneous component inside A^j.

Step 0

We define A^0 to be the ring R itself, which can be equipped with a differential R-algebra structure in the following way:

$$A^0 = \bigoplus_{i \in \mathbb{Z}_{\leq 0}} A_i^0 , \quad \text{where} \quad \begin{cases} A_0^0 = R \\ A_i^0 = 0 \quad \forall i < 0. \end{cases}$$

The derivation δ is defined as the zero map in each degree:

$$\delta^0 = \{\delta_i^0\}, \quad \text{with } \delta_i^0 = 0, \quad \forall i \in \mathbb{Z}_{\leq 0}.$$

Therefore, we have the following exact sequence of finitely generated R-modules:

$$A_0^0 = R \xrightarrow{\pi} R/M \to 0 \tag{B.5}$$

where the map π is the projection on the quotient.

Step −1

Since M in an ideal in a Noetherian ring, it is finitely generated: let $\tau_1, \ldots, \tau_n$ be a set of generators for M as an R-module. Since by construction the derivation δ^0 is zero on R, we can see these elements $\tau_1, \ldots, \tau_n$ in R as 0-cocycles. Therefore, we define the differential R-algebra A^{-1} as the extension of A^0 by the adjunction of variables $T_1, \ldots, T_n$ of degree -1 such that they correspond to $\tau_1, , \ldots, \tau_n$:

$$A^{-1} = R\langle T_1, \ldots, T_n\rangle, \qquad \text{with} \qquad \delta_{-1}^{-1}(T_j) = \tau_j \ . \tag{B.6}$$

Then A^{-1} is a graded algebra where:

$$\begin{aligned}
&A_0^{-1} = A_0^0 = R \\
&A_{-1}^{-1} = \{r_1 T_1 + r_2 T_2 + \cdots + r_n T_n, \ \text{with } r_1, r_2, \ldots, r_n \in R\} \\
&A_{-2}^{-1} = \{\textstyle\sum_{i<j} r_{ij} T_i T_j, \ \text{with } i, j = 1, \ldots, n \text{ and } r_{ij} \in R\} \\
&\vdots \\
&A_{-n}^{-1} = \{r T_1 T_2 \ldots T_n, \ \text{with } r \in R\} \\
&A_{-n-1}^{-1} = A_{-n-k}^{1} = 0 \ , \qquad\qquad \forall k \in \mathbb{N}
\end{aligned}$$

Thus in A^{-1} there are terms with degree at least $-n$, since the variables T_j are of odd degree. For the derivation map, we have that $\delta_0^{-1} = \delta_0^0 = 0$ while on A_{-1}^{-1} it is defined by requiring that $\delta_{-1}^{-1}(T_j) = \tau_j$.

We can also extend the exact sequence (B.5), adding a part of degree -1. In fact:

$$\mathrm{Im}(\delta_{-1}^{-1}) = \{r_1\tau_1 + r_2\tau_2 + \cdots + r_n\tau_n\} = M,$$

since $\tau_1, \ldots \tau_n$ are generators of M as module over R. So, since $\mathrm{Ker}(\pi) = M$, we conclude that the following sequence is exact:

$$A_{-1}^{-1} \xrightarrow{\delta_{-1}^{-1}} A_0^0 = R \xrightarrow{\pi} R/M \to 0 \ . \tag{B.7}$$

Moreover:

$$H^0(A^{-1}) = \frac{\mathrm{Ker}(\delta_0^{-1})}{\mathrm{Im}(\delta_{-1}^{-1})} = \frac{\mathrm{Ker}(\delta_0^{0})}{\mathrm{Im}(\delta_{-1}^{-1})} = \frac{R}{M} \ .$$

Step −2

Now we want to extend the sequence (B.7) keeping it exact, adjoining elements of degree −2. In order to do this, we first consider the cohomology group $H^{-1}(A^{-1})$. By definition:

$$H^{-1}(A^{-1}) = \frac{\mathrm{Ker}(\delta_{-1}^{-1})}{\mathrm{Im}(\delta_{-2}^{-1})}.$$

Since $H^{-1}(A^{-1}) \subseteq A_{-1}^{-1}$, which is finitely generated as R-module, we can find a finite number of generators for $H^{-1}(A^{-1})$. Let $\sigma_1, \dots, \sigma_m$ in $\mathrm{Ker}(\delta_{-1}^{-1})$ be −1-cocycles whose cohomology classes $s_1, \dots, s_m$ generate $H^{-1}(A^{-1})$,

$$H^{-1}(A^{-1}) = \langle s_1, \dots, s_m \rangle.$$

Then we define A^{-2} as the differential R-algebra obtained from A^{-1} by adjoining variables $S_1, \dots, S_m$ of degree −2 and such that $\delta_{-2}^{-2}(S_i) = \sigma_i$.

$$A^{-2} = A^{-1}\langle S_1, \dots, S_n \rangle, \qquad \text{with} \qquad \delta_{-2}^{-2}(S_j) = \sigma_j \ . \tag{B.8}$$

Then A^{-2} is a graded algebra with:

$A_0^{-2} = A_0^{-1} = A_0^0 = R$

$A_{-1}^{-2} = A_{-1}^{-1} = \{r_1T_1 + r_2T_2 + \cdots + r_nT_n, \text{ with } r_1, r_2, \dots, r_n \in R\}$

$A_{-2}^{-2} = \{\sum_{i<j} r_{ij}T_iT_j, \text{ with } r_{ij} \in R\} \oplus \{r_1S_1 + r_2S_2 + \cdots + r_mS_n, r_i \in R\}$

$\vdots$

$A_{-k}^{-2} = \bigoplus_{w=0}^{\lfloor \frac{k}{2} \rfloor} \{\sum_{i_1 \dots j_w} r_{i_1 \dots j_w} T_{i_1} \dots T_{i_{k-2w}} S_{j_1} \dots S_{j_w}, r_{i_1 \dots j_w} \in R\}.$

In the sum appearing in the expression of A_{-k}^{-2}, each summand corresponding to a value of w such that $k - 2w > n$ is automatically zero, since the variables T_j are antisymmetric. On the other hand, since the variables S_i are of even degree, in A^{-2} there are elements of each degree $k \in \mathbb{Z}_{\leq 0}$.

For the cohomology groups of A^{-2}, since $\delta_0^{-2} = \delta_0^0$ and $\delta_{-1}^{-2} = \delta_{-1}^{-1}$, then we have $H^0(A^{-2}) = H^0(A^{-1}) = R/M$. One verifies that $H^{-1}(A^{-2}) = 0$. Therefore, the following sequence is exact:

$$A^{-2}_{-2} \xrightarrow{\delta^{-2}_{-2}} A^{-1}_{-1} \xrightarrow{\delta^{-1}_{-1}} A^0_0 = R \xrightarrow{\pi} R/M \to 0. \tag{B.9}$$

To conclude the proof, we proceed by induction.

Let us suppose that we have defined a differential R-algebra A^{-k} such that the following sequence of finitely generated modules over R is exact:

$$A^{-k}_{-k} \xrightarrow{\delta^{-k}_{-k}} \dots \xrightarrow{\delta^{-2}_{-2}} A^{-1}_{-1} \xrightarrow{\delta^{-1}_{-1}} A^0_0 = R \xrightarrow{\pi} R/M \to 0. \tag{B.10}$$

Moreover, we suppose that A^{-k} gives a free and acyclic resolution of R/M up to degree $-k+1$: explicitly, we are assuming that

$$\begin{cases} H^0(A^{-k}) = R/M \\ H^{-j}(A^{-k}) = 0 \qquad \forall j = 1, \dots, k-1. \end{cases}$$

Step $-(k+1)$

Now we want to define a differential R-algebra A^{-k-1} such that, when adding the module A^{-k-1}_{-k-1} to the sequence (B.10), we obtain a free and acyclic resolution of R/M up to degree $-k$.

In order to do this, we first consider the cohomology group $H^{-k}(A^{-k})$. This is a finitely generated module over R. Therefore, let $u_1 \dots u_{n_k}$ be $(-k)$-cocycles such that their corresponding cohomology classes generate $H^{-k}(A^{-k})$.

Then we define the differential R-algebra A^{-k-1} as

$$A^{-k-1} = A^{-k}\langle U_1, \dots, U_{n_k}\rangle, \quad \text{with} \quad \delta^{-k-1}_{-k-1}(U_j) = u_j \ . \tag{B.11}$$

Using the fact that by construction $\delta^{-k-1}_{-i} = \delta^{-k}_{-i}$, $\forall i < k+1$ and using the induction hypothesis, we conclude that:

$$\begin{cases} H^0(A^{-k-1}) = H^0(A^{-k}) = R/M; \\ H^{-j}(A^{-k-1}) = H^{-j}(A^{-k}) = 0 \qquad \forall j = 1, \dots, k-1. \end{cases}$$

For the cohomology group of degree $-k$, one has

$$H^{-k}(A^{-k-1}) = \frac{H^{-k}(A^{-k})}{(Ru_1 + \cdots + Ru_{n_k})}.$$

Since $u_1, \cdots, u_{n_k}$ were chosen to be generators of $H^{-k}(A^{-k})$ as an R-module, we conclude that A^{-k-1} gives a free and acyclic resolution of R/M up to degree $-k$.

Therefore, the algebra $A = \bigcup_{k=0}^{\infty} A^{-k}$ is a free and acyclic resolution of R/M such that $H^0(A) = R/M$. □

To conclude, we state a proposition that may be useful when we are applying the Tate algorithm: indeed, to explicitly compute the generators of the cohomology group $H^{-n}(A^{-n})$ not all the possible elements of degree $-n$ have to be considered. As already noticed, in the differential R-algebra A^{-n} coming from Tate's algorithm, the elements of degree $-n$ are given both by the variables of degree $-n$ introduced at the $-n$th step of the algorithm and by appropriate products of variables with degree $-m > -n$, introduced in previous steps of the procedure.
As precisely stated in the following proposition, the terms of this second type do not give any contribution to the cohomology group $H^{-n}(A^{-n})$ and so there is no need to analyze them when applying the Tate algorithm.

Proposition 24
Let R be a ring as in the hypothesis of Tate's theorem and let M be an ideal in R. Let A^{-n} be the differential R-algebra defined by Tate's algorithm at step $-n$. Then:

$$\frac{\mathrm{Ker}(\delta_{-n}^{-n+1})}{\mathrm{Im}(\delta_{-n-1}^{-n+1})} = 0. \tag{B.12}$$

Proof. Since δ^{-n+1} is a coboundary operator we already know that

$$\mathrm{Ker}(\delta_{-n}^{-n+1}) \supseteq \mathrm{Im}(\delta_{-n-1}^{-n+1}).$$

If we prove the reverse inclusion, (B.12) holds. Let w be a generic element in $\mathrm{Ker}(\delta_{-n}^{-n+1})$. It can be written as a finite sum of elements

$$w_a = r_a U_{a_1}^{p_1} \ldots U_{a_k}^{p_k},$$

where $U_{a_j}^{p_j}$ is a variable of degree $i_j > -n$ and $\sum_{j=1}^{k} i_j = -n$. Note that, if the degree i_j is odd, then necessarily the exponent is $p_j = 1$.

Then, imposing that w is an element of $\mathrm{Ker}(\delta_{-n}^{-n+1})$ is equivalent to requiring that $\delta_{-n}^{-n+1}(w_a) = 0$, $\forall a$. This follows from the fact that, since

$$\delta_{-n}^{-n+1}(w_a) = r_a \cdot \delta(U_{a_1}^{p_1})U_{a_2}^{p_2} \dots U_{a_k}^{p_k} + \dots + (-1)^{-n+i_k} r_a \cdot U_{a_1}^{p_1} \dots U_{a_{k-1}}^{p_{k-1}} \delta(U_{a_k}^{p_k}),$$

there exist no a and b such that a summand in $\delta_{-n}^{-n+1}(w_a)$ cancels a summand in $\delta_{-n}^{-n+1}(w_b)$. Indeed, this situation would be possible only if one of the following two situations would occur:

1. There were to exist a variable $U_{a_j}^{p_j}$ in w_a and a variable $U_{b_m}^{q_m}$ in w_b, such that $\delta(U_{a_j}^{p_j}) = f\delta(U_{b_m}^{q_m})$ for an element $f \in R$. Moreover, all the other factors appearing in w_a next to $U_{a_j}^{p_j}$ have to coincide with the factors appearing in w_b, next to the variable $U_{b_m}^{q_m}$.

2. There were to exist a variable $U_{a_j}^{p_j}$ in w_a and a variable $U_{b_m}^{q_m}$ in w_b, such that $\delta(U_{a_j}^{p_j}) = U_{b_m}^{q_m}$.

For what concerns the first possibility, from the conditions imposed it follows immediately that $\delta(U_{a_j}^{p_j})$ and $\delta(U_{b_m}^{q_m})$ define the same cohomology class in the cohomology group $H^{i_m+1}(A^{i_m+1})$, where i_m is the degree of the variable $U_{b_m}^{q_m}$. Thus the two variables $U_{a_j}^{p_j}$ and $U_{b_m}^{q_m}$ have to coincide, so that the two summands w_a, w_b from which we started are indeed the same summand.
For what concerns the second possibility, from requiring that $\delta(U_{a_j}^{p_j}) = U_{b_m}^{q_m}$, it would follow that $\delta(U_{b_m}^{q_m}) = 0$, since δ is a coboundary operator. However, we know that $\delta(U_{b_m}^{q_m})$ is a generator of the cohomology group $H^{i_m+1}(A^{i_m+1})$, where i_m is the degree of the variable $U_{b_m}^{q_m}$. Since these generators are supposed to be nonzero, we conclude that imposing $\sum_a \delta_{-n}^{-n+1}(w_a) = 0$ is equivalent to requiring $\delta_{-n}^{-n+1}(w_a) = 0$, $\forall a$.
Therefore, we restrict ourselves to the case $w = w_a$. Then, by the independence of the variables $U_{a_1}^{p_1}, \cdots, U_{a_k}^{p_k}$, it follows that:

$$w \in \mathrm{Ker}(\delta_{-n}^{-n+1}) \Leftrightarrow U_{a_j}^{p_j} \in \mathrm{Ker}(\delta_{i_j}^{-n}) \quad \forall j = 1, \cdots k.$$

By hypothesis, we know that the differential R-algebra A^{-n} gives a free and acyclic resolution of R/M up to degree $-n+1$. Explicitly, this means that

$$H^{-m}(A^{-n}) = 0, \quad \forall m < n.$$

Therefore, $U_{a_j}^{p_j} \in \mathrm{Ker}(\delta_{i_j}^{-n}) = \mathrm{Im}(\delta_{i_j-1}^{-n})$, $\forall j = 1, \cdots k$.

Let $V_{a_j}^{p_j} \in A_{i_j-1}^{-n}$ be such that $\delta_{i_j-1}^{-n}(V_{a_j}^{p_j}) = U_{a_j}^{p_j}$, $\forall j = 1, \cdots k$.
Then, defining

$$\begin{aligned}\psi = \quad & \tfrac{1}{k}\left[V_{a_1}^{p_1} \cdot U_{a_2}^{p_2} \cdots U_{a_k}^{p_k} + (-1)^{i_1 p_1}\, U_{a_1}^{p_1} \cdot V_{a_2}^{p_2} \cdots U_{a_k}^{p_k} + \ldots \right.\\ & \left. +(-1)^{-n-1+i_k p_k}\, U_{a_1}^{p_1} \cdot U_{a_2}^{p_2} \cdots V_{a_k}^{p_k}\right],\end{aligned}$$

one sees that $\delta_{-n-1}^{-n+1}(\psi) = w$ and so $w \in \mathrm{Im}(\delta_{-n-1}^{-n+1})$.
This proves that $\mathrm{Ker}(\delta_{-n}^{-n+1}) \subseteq \mathrm{Im}(\delta_{-n-1}^{-n+1})$, and finishes the proof. □

Appendix C

The BV algorithm: further details

In this appendix we present the proofs of the lemmas that have been used in Section 4.2 in the construction of the algorithm for determining the extended action $\widetilde{S}$, given an initial gauge theory (X_0, S_0), together with the generators of type β defined by a Tate resolution (A, δ) of the Jacobian ring $J(S_0)$.
We recall that the generators of type β can be inductively defined as follows:

- All the generators $\{x_i^*\}_{i=1,\dots,m} \subseteq [\mathcal{W}_T^*]^{-1} = T_{X_0}[1]$, which are the antifields associated to the initial fields $\{x_i\}_{i=1,\dots,m}$, are of type β by definition.

- The generators of type β in degree $-q$, collectively denoted by $\{\beta_j^{*,(-q)}\}_{j\in J}$, are inductively determined by the generators of type β of degree $-q+1$.
  A generator $\gamma_j^{*,(-q)} \in [\mathcal{W}_T^*[1]]^{-q}$ in the Tate resolution is called a *generator of type* β if there exists a collection of elements $\{r_j\}_{j=1,\dots,m_j}$ of the ring R such that
  $$\delta(\gamma_j^{*,(-q)}) = r_1\beta_1^{*,(-q+1)} + r_2\beta_2^{*,(-q+1)} + \dots + r_{m_j}\beta_{m_j}^{*,(-q+1)}$$
  with $\beta_1^{*,(-q+1)}, \beta_2^{*,(-q+1)}, \dots, \beta_{m_j}^{*,(-q+1)}$, generators of type β of degree $-q+1$. Thus for this generator $\gamma_j^{*,(-q)}$ the notation $\beta_j^{*,(-q)}$ will be used.

Since in this context the behavior of the generators $\{C_i^*\}$, which are the generators of type β in degree -2, does not present any peculiarities with respect to the other generators of type β of lower degree, in this section for these generators $\{C_i^*\}$ we will keep the notation $\{\beta_i^{*,(-2)}\}$.

Notation:

- In what follows, the symbol $\equiv$ is used, instead of the usual $=$, for identities that only hold at the level of the quotient.
- By S_{lin} we denote the following expression:

$$S_{lin} = S_0 + \sum_{j \in J} \delta(\beta_j^*)\beta_j,$$

  where the terms defined by the generators $\{C_i^*\}$ are in the sum taken over the β generators.

- Finally, the graded variety N was defined as follows:

$$N = (X_0, \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E}^*[1] \oplus T_{X_0}[1] \oplus \mathcal{E}))\ ,$$

  where $\mathcal{E}^*[1]$ denotes the $\mathbb{Z}_{<0}$-graded $\mathcal{O}_{X_0}$-module with finitely generated homogeneous components, whose generators are the selected generators of type β. Concerning $\mathcal{E}$, it is the positively graded module over $\mathcal{O}_{X_0}$ generated by the dual generators of the generators of type β, which describe the ghost fields corresponding to the antighost fields in $\mathcal{E}^*[1]$.

Lemma 4
The canonical isomorphism

$$\mathcal{O}_N \cong \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1]) \otimes_{\mathcal{O}_{X_0}} \mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E})$$

identifies, modulo $F^1\mathcal{O}_N$, the operator $\{S_{lin}, -\}$ with the operator $\delta \otimes Id$, where δ is the coboundary operator given by the Tate resolution (A, δ), restricted to act only on generators of type β.
More explicitly, for any element $\varphi = \sum_i \varphi_{n,i} \otimes \varphi_{p,i}$ in $\mathcal{O}_N$,

$$\{S_{lin}, \varphi\} = \sum_i \delta(\varphi_{n,i}) \otimes Id(\varphi_{p,i}) \qquad (mod\ F^1\mathcal{O}_N).$$

Proof. Let us denote by Φ the operator defined by $\{S_{lin}, -\}$ over $\mathcal{O}_N$:

$$\Phi(\varphi) := \{S_{lin}, \varphi\}\,.$$

To prove the statement, we first show that the two operators Φ and $\delta \otimes Id$ coincide modulo $F^1\mathcal{O}_N$ on the generators x_i, x_i^*, and β_i^* of $\mathcal{O}_{X_0}$, $T_{X_0}[1]$ and $\mathcal{E}^*[1]$, respectively.

Generators x_i

We start considering the generators x_i of ghost degree 0. Recalling the explicit definition of the linear action S_{lin}, we have

$$\Phi(x_i) = \{S_0, x_i\} + \Big\{\sum_j \delta(\beta_j^*)\beta_j, x_i\Big\}.$$

Since $\{S_0, f\} = 0$ for any f in $\mathcal{O}_{X_0}$, the first summand in the previous expression is certainly zero. As to the second summand, since the only possibility in which the Poisson bracket is non-zero occurs when we consider a pair composed of a field together with the corresponding antifield, and since $\delta(\beta_j^*)$ might depend on the antifield x_i^*, we have

$$\Big\{\sum_j \delta(\beta_j^*)\beta_j, x_i\Big\} = \sum_{j\in J}(-1)^{\deg(\beta_j)}\beta_j \left\{\delta(\beta_j^*), x_i\right\} \equiv 0 \;, \qquad (mod\ F^1\mathcal{O}_N),$$

where the last equality follows from noticing that each summand $\beta_j \left\{\delta(\beta_j^*), x_i\right\}$ is an element in $F^1\mathcal{O}_N$: this is a consequence of the fact that $F^1\mathcal{O}_N$ is an ideal over $\mathcal{O}_N$ and that each generator β_j has degree at least 1 so it belongs to $F^1\mathcal{O}_N$ for any value of j.

Noting that $(x_i)_n = x_i$ and recalling how the coboundary operator δ has been defined in the Tate resolution (see the proof of Theorem 16), we have $\delta = 0$ on $\mathcal{O}_{X_0}$ and so it holds that

$$(\delta \otimes Id)(x_i) = \delta(x_i) = 0.$$

Therefore,

$$\Phi(x_i) = (\delta \otimes Id)(x_i) \qquad (mod\ F^1\mathcal{O}_N) \qquad \forall x_i.$$

Generators x_i^*

We now consider the generators x_i^*. Recalling the definition of the Poisson bracket on $\mathcal{O}_N$ (4.4), we deduce:

$$\Phi(x_i^*) = \partial_i S_0 + \sum_j (-1)^{\deg(\beta_j)}\beta_j\big\{\delta(\beta_j^*), x_i^*\big\} \equiv \partial_i S_0, \qquad (mod\ F^1\mathcal{O}_N).$$

Once again, since β_j belongs to $F^1\mathcal{O}_N$ for any possible value of j, and since $F^1\mathcal{O}_N$ is an ideal over $\mathcal{O}_N$, the last sum in the previous expression is an element in $F^1\mathcal{O}_N$.

On the other hand, since $(x_i^*)_n = x_i^*$, we have

$$(\delta \otimes Id)(x_i^*) = \delta(x_i^*) = \partial_i S_0.$$

Thus Φ and $\delta \otimes Id$ coincide modulo $F^1\mathcal{O}_N$, also when they are computed on the antifields x_i^*.

Generators β_i^*

The last type of generators of $\mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])$ we still need to consider are the antighosts β_i^*. We have

$$\Phi(\beta_i^*) = \{S_0, \beta_i^*\} + \Big\{ \sum_j \delta(\beta_j^*)\beta_j, \beta_i^* \Big\} = \sum_j \delta(\beta_j^*)\{\beta_j, \beta_i^*\} = \delta(\beta_i^*),$$

again by definition of the Poisson structure on $\mathcal{O}_N$ in (4.4), together with the observation that the coboundary operator δ can only act on and depend on non-positively graded generators, since it is the coboundary operator of a Tate resolution.

Thus we conclude that the two operators Φ and $\delta \otimes Id$ coincide, modulo $F^1\mathcal{O}_N$, on each generator of $\mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])$.
Moreover, both Φ and $\delta \otimes Id$ are graded derivations on the graded algebra $\mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])$: the operator δ is a graded derivation by definition, while the map Φ is a graded derivation, due to the properties imposed on the Poisson bracket defined on $\mathcal{O}_N$. Indeed, using the property recalled in Remark 7, we have

$$\begin{aligned}\Phi(\varphi\psi) &= \{S_{lin}, \varphi\}\psi + (-1)^{\deg(\varphi)\deg(\psi)}\{S_{lin}, \psi\}\varphi \\ &= \Phi(\varphi)\psi + (-1)^{\deg(\varphi)\deg(\psi)}\Phi(\psi)\varphi,\end{aligned}$$

with φ, $\psi \in \mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])$.
Thus also Φ is a graded derivation. To conclude, since Φ and $\delta \otimes Id$ are graded derivations on the graded algebra $\mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])$ and they coincide modulo $F^1\mathcal{O}_N$ on all the generators of $\mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1])$, they coincide modulo $F^1\mathcal{O}_N$ on the whole

$$\mathrm{Sym}_{\mathcal{O}_{X_0}}(T_{X_0}[1] \oplus \mathcal{E}^*[1]) \simeq \mathcal{O}_N / F^1\mathcal{O}_N,$$

as well as on $\mathcal{O}_N$. $\square$

Note: the reason why we restricted to consider only the generators of type β in the proof of the previous lemma is not related to any specific properties of this kind of generators. Instead, the reason of this choice has to be searched in the way in which the linear action was defined. Indeed, analogously to what done above, it is possible to prove the coincidence of the operator $\{S_{lin}, -\}$ with the operator $\delta \otimes Id$, module $F^1\mathcal{O}_N$, on the whole graded algebra generated by the generators of the Tate resolution considered in S_{lin}. For example, if in the definition of S_{lin} all the generators of the Tate resolution are considered, then the operator $\{S_{lin}, -\}$ coincides with $\delta \otimes Id$, module $F^1\mathcal{O}_N$, over the whole graded algebra A determined by the Tate resolution, without having to restrict the coboundary operator δ. (cf. [28, Proposition 4.1]).

Lemma 5
Let q be an integer $q \geqslant 0$. Then the following properties hold:

1. $\left\{F^q\mathcal{O}_N^0, \mathcal{O}_N^0\right\} \subseteq F^q\mathcal{O}_N^1$;
2. $\left\{F^q\mathcal{O}_N^0, F^q\mathcal{O}_N^0\right\} \subseteq F^{q+1}\mathcal{O}_N^1$.

Proof. Let us start by proving the first property in the case $q = 0$. By definition, $F^0\mathcal{O}_N = \mathcal{O}_N$ and so $F^0\mathcal{O}_N^0 = \mathcal{O}_N^0$. Since the Poisson bracket is a bilinear map of degree 1, we have

$$\left\{\mathcal{O}_N^0, \mathcal{O}_N^0\right\} \subseteq \mathcal{O}_N^1,$$

from which the required inclusion immediately follows for $q = 0$.

Analogously, also for $q > 0$, given φ an element in $F^q\mathcal{O}_N^0$ and ψ a generic element in $\mathcal{O}_N^0$, the corresponding Poisson bracket $\{\varphi, \psi\}$ would give a term of degree 1, namely an element in $\mathcal{O}_N^1$.
Thus we only have to prove that $\{\varphi, \psi\}$ belongs also to $F^q\mathcal{O}_N$. In that case, it would automatically follow that $\{\varphi, \psi\}$ is an element in $F^q\mathcal{O}_N \cap \mathcal{O}_N^1 = F^q\mathcal{O}_N^1$.
In order to conclude that $\{\varphi, \psi\}$ belongs to $F^q\mathcal{O}_N$, by definition we have to show that

$$\deg_p(\{\varphi, \psi\}) \geqslant q.$$

Since φ is an element in $F^q\mathcal{O}_N^0$, it can be written as

$$\varphi = \sum_i \varphi_{n,i}\varphi_{p,i}$$

with

$$\deg_p(\varphi) = \min_i(\deg(\varphi_{p,i})) \geqslant q.$$

Using linearity of the Poisson bracket we can restrict ourselves to elements ψ consisting of only one monomial and elements φ given by the product of one monomial in the positive and one monomial in the negative generators. Thus: $\varphi = \varphi_n \varphi_p$ with

$$\deg(\varphi) = 0 \quad \text{and} \quad \begin{cases} \deg_p(\varphi) = \deg(\varphi_p) = k \geqslant q \\ \deg_n(\varphi) = \deg(\varphi_n) = -k \leqslant -q \end{cases} .$$

Hence:

$$\{\varphi, \psi\} = \varphi_n \{\varphi_p, \psi\} + (-1)^{k \cdot k} \varphi_p \{\varphi_n, \psi\}. \tag{C.1}$$

To prove that the first summand is an element in $F^q\mathcal{O}_N$ we should establish that its positive degree is at least q. Using the definition of positive degree and property (4) in Properties 3, we have

$$\begin{aligned} \deg_p(\varphi_n\{\varphi_p, \psi\}) &= \deg_p(\{\varphi_p, \psi\}) \geqslant \deg(\{\varphi_p, \psi\}) \\ &= \deg(\varphi_p) + \deg(\psi) + 1 = k + 1 \geqslant q + 1. \end{aligned}$$

To conclude that the second summand in (C.1) belongs to $F^q\mathcal{O}_N$ we recall that $F^q\mathcal{O}_N$ is an ideal and that, by definition, φ_p is an element in $F^q\mathcal{O}_N$. This completes the proof of the first inclusion stated in the lemma, for any $q \geqslant 0$.

In order to prove the second inclusion, let φ and ψ be two generic elements in $F^q\mathcal{O}_N^0$. First of all, since the Poisson bracket is a map of degree 1, given two elements φ, ψ of total degree 0, $\{\varphi, \psi\}$ is an element of $\mathcal{O}_N^1$.
Thus we only have to prove that $\{\varphi, \psi\}$ is also an element in $F^{q+1}\mathcal{O}_N$, i.e., that its positive degree is at least $q+1$. To draw this conclusion, we first analyze the case in which either φ or ψ have positive degree strictly greater than q, i.e., the case in which, for example, φ belongs not only to $F^q\mathcal{O}_N$ but also to $F^{q+1}\mathcal{O}_N$. Once again, we can restrict ourselves to elements φ that are monomials both in the positive and in the negative generators. Then:

$$\{\varphi, \psi\} = \varphi_n \{\varphi_p, \psi\} + (-1)^{\deg(\varphi_p) \cdot \deg(\varphi_n)} \varphi_p \{\varphi_n, \psi\} .$$

Using the first inclusion proved above together with the fact that φ is supposed to be an element in $F^{q+1}\mathcal{O}_N$ and that $F^{q+1}\mathcal{O}_N$ is an ideal, we conclude that the first summand in the above expression is an element of $F^{q+1}\mathcal{O}_N$. As to the second summand, it is enough to use the fact that $F^{q+1}\mathcal{O}_N$ is an ideal and that φ_p belongs to $F^{q+1}\mathcal{O}_N$. An analogous argument can be deduced in the case in which ψ has positive degree strictly greater than q.

Therefore, the only possibility that still needs to be considered is the one in which both φ and ψ have positive degree equal to q.
We can restrict ourselves to the case $q > 0$: in fact, if $q = 0$, then φ and ψ are elements whose total degree as well as whose positive degree is zero, i.e, they belong to $\mathcal{O}_{X_0}$. Since, by definition, $\{\varphi, \psi\} = 0$, for any φ, ψ in $\mathcal{O}_{X_0}$, the claimed inclusion straightforwardly follows.
Therefore, we only consider the case $\deg_p(\varphi) = \deg_p(\psi) = q \geqslant 1$. Then:

$$\{\varphi, \psi\} = \varphi_n\{\varphi_p, \psi\} + (-1)^a \varphi_p\{\varphi_n, \psi_n\}\psi_p + (-1)^{a^2+b^2}\varphi_p\{\varphi_n, \psi_p\}\psi_n$$

with $a = \deg(\varphi_p) = -\deg(\varphi_n)$ and $b = \deg(\psi_p) = -\deg(\psi_n)$.

Once more using the fact that $F^{q+1}\mathcal{O}_N$ is an ideal, to conclude the proof it is enough to show that each summand in the previous equation has a factor whose positive degree is greater than or equal to $q+1$. Explicitly, using the properties listed in Properties 3, we have:

- $\deg_p(\{\varphi_p, \psi\}) \geqslant \deg(\{\varphi_p, \psi\}) = \deg(\varphi_p) + \deg(\psi) + 1 = q + 1$;
- $\deg_p(\varphi_p\{\varphi_n, \psi_n\}\psi_p) \geqslant \deg_p(\varphi_p) + \deg_p(\{\varphi_n, \psi_n\}) + \deg_p(\psi_p) \geqslant q + 1$;
- $\deg_p(\varphi_p\{\varphi_n, \psi_p\}\psi_n) \geqslant \deg_p(\varphi_p) + \deg(\varphi_n) + \deg(\psi_p) + 1 = q + 1$.

Thus $\{\varphi, \psi\} \in F^{q+1}\mathcal{O}_N$. □

Note: also in this case the restriction to generators of type β was not needed. However, we decided to state the lemma in this setting since this is the context where it has been used. Anyhow, the previous lemma can be restated considering, instead of the graded algebra $\mathcal{O}_N$ generated only by the type β generators, the whole graded algebra generated by the graded algebra A of the Tate resolution (cf. [28, Lemma 4.6]).
An analogous remark can be done on the following lemma, which has to be compared with [28, Lemma 4.3].

Lemma 6
For any integer $q \geqslant 0$ the following inclusion holds:

$$\left\{I_N^{\geqslant 2} \cap \mathcal{O}_N^0, F^q\mathcal{O}_N\right\} \subseteq F^{q+1}\mathcal{O}_N.$$

Proof. Let φ be a generic element in $I_N^{\geqslant 2} \cap \mathcal{O}_N^0$. Then φ can be written as

$$\sum_{i,j} \varphi_{ij}\beta_i\beta_j,$$

with β_i, β_j generators in $\mathcal{E}$ while φ_{ij} belongs to $\mathrm{Sym}_{\mathcal{O}_{X_0}}(\mathcal{E}^*[1] \oplus T^*_{X_0}[1] \oplus \mathcal{E})$, for any pair of value (i, j). Here we are assuming that φ is an element which is at least bilinear in the positively-graded generators.
We recall once again that, because of the linearity of the Poisson bracket, we can restrict our argument to elements expressed by monomials.

Thus as a generic element in $F^q\mathcal{O}_N$ we can consider ψ, with $\psi = \psi_n\psi_p$ and $\deg(\psi_p) \geqslant q$. Moreover, we also assume that the summands appearing in the following expressions are non-zero: in case they are zero, they automatically belong to the ideal $F^{q+1}\mathcal{O}_N$.

Using the properties of the Poisson bracket, we have the following equality:

$$\begin{aligned} \{\varphi, \psi\} &= \textstyle\sum_{i,j} \varphi_{ij}\beta_i\{\beta_j, \psi_n\}\psi_p + \sum_{i,j}(-1)^b\beta_j\{\varphi_{ij}\beta_i, \psi_n\}\psi_p \\ &\quad + (-1)^a \textstyle\sum_{i,j} \{\varphi_{ij}\beta_i\beta_j, \psi_p\}\psi_n. \end{aligned} \tag{C.2}$$

with $a = \deg(\psi_n) \cdot \deg(\psi_p)$, $b = [\deg(\varphi_{ij}) + \deg(\beta_i)] \cdot \deg(\beta_j)$.
Let us separately analyze the summands belonging to the three sums appearing in the above expression. The goal is to prove that the positive degree of each term is at least $q + 1$.

- For the first sum, using the properties of the positive degree and the hypothesis on ψ_p, we have

$$\deg_p(\varphi_{ij}\beta_i\{\beta_j, \psi_n\}\psi_p) \geqslant \deg_p(\beta_i) + \deg_p(\psi_p) \geqslant 1 + q.$$

- Regarding the second sum we have a similar inequality:

$$\deg_p(\beta_j\{\varphi_{ij}\beta_i, \psi_n\}\psi_p) \geqslant \deg_p(\beta_j) + \deg_p(\psi_p) \geqslant 1 + q.$$

- For the summands in the last term we have

$$\deg_p(\{\varphi_{ij}\beta_i\beta_j, \psi_p\}\psi_n) \geqslant \deg(\{\varphi_{ij}\beta_i\beta_j, \psi_p\}) = q + 1.$$

Therefore, all terms appearing in Equation (C.2) belong to $F^{q+1}\mathcal{O}_N$. □

To conclude, we present the proof of Proposition 7, first stated in Section 4.2 and used once again in the construction of the algorithm for the extended action $\widetilde{S}$.

Proposition 7
The pair $(\mathcal{G}^\bullet_{q,r}, d)$ introduced in Definition 44 is well defined.

Proof. We start the proof of this proposition by noting that, for any value of j with $j \leqslant q$, $\mathcal{G}^j_{q,r}$ has the structure of a vector space: this is an immediate consequence of the way in which the product is defined in the algebra $\mathcal{O}_N$. Moreover, $\mathcal{G}^j_{q,r}$ is not only a vector space but it also has the structure of a module over $\mathcal{O}_{X_0}$. In fact, any element in $\mathcal{O}_{X_0}$ depends only on zero-degree generators and so, when we take the product of an element in $\mathcal{G}^j_{q,r}$ together with an element in $\mathcal{O}_{X_0}$, not only the total degree and the positive degree do not change, but also the number of positively-graded generators appearing remains unchanged.

Now we focus on the map d: we have to prove that it is a 1-degree linear differential on $\mathcal{G}^\bullet_{q,r}$. Linearity immediately follows from the linearity of both the operators δ and Id. To prove that d has degree 1 we show that the following conditions are satisfied, for any element φ in $\mathcal{G}^j_{q,r}$, with j any value $j \leqslant q$:

1. $\deg(d(\varphi)) = j + 1$;
2. $\deg_p(d(\varphi)) = q$;
3. $d(\varphi)$ is r-linear in the positively-graded generators.

The first condition follows from the fact that the coboundary operator δ was defined to be a 1-degree operator while trivially Id has degree 0. The second and the third conditions are also verified, since the operator δ depends only on the generators of non-positive degree: therefore, $d(\varphi)$ not only has the same positive degree of φ but it also has the same number of positively-graded generators of φ. Thus we conclude that d is a well-defined 1-degree operator on $\mathcal{G}^\bullet_{q,r}$.

To conclude that the pair $(\mathcal{G}^\bullet_{q,r}, d)$ defines a cohomology complex, we only have to check that d is a differential, namely that $d^{j+1} \circ d^j \equiv 0$, $\forall j \in \mathbb{Z}$.
Given the properties of the operator δ, for a generic element $\varphi = \varphi_n \varphi_p$ in $\mathcal{G}^j_{q,r}$, we have

$$d(\varphi) = (\delta \otimes Id)(\varphi_n \varphi_p) = \psi \quad \text{with} \quad \begin{cases} \psi_n = \delta(\varphi_n) \\ \psi_p = \varphi_p \ . \end{cases}$$

Therefore,

$$d(d(\varphi)) = (\delta \otimes Id)(\delta(\varphi_n)\varphi_p) = \delta(\delta(\varphi_n))\varphi_p = 0$$

where the last equality follows from the fact that δ is a coboundary operator. So we conclude that d is a differential and that $(\mathcal{G}^\bullet_{q,r}, d)$ defines a cohomology complex. □

Appendix D

The BRST cohomology groups for a $U(2)$-model

In this appendix we present the explicit computation of the BRST cohomology groups for the $U(2)$-matrix model. The notation that is used in what follows is the same as in Section 5.2.4, to which this appendix refers. More precisely, here we explain the computation necessary to prove Theorem 9.

$\mathbf{H^0(W, d_{\tilde{S}})}$

The gauge-fixed BRST complex defines a one-sided cohomology, that is to say, the cochains in this cohomology complex always have non-negative degree. For this reason, the cohomology group of degree 0 coincides with $Z^0(W, d_{\tilde{S}})$, namely with the cocycles of ghost degree 0, and we do not have to take the quotient with respect to a space of coboundary elements.
Let f be a generic element in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Then, in order to be a cocycle, the polynomial f needs to satisfy the following condition:

$$\begin{aligned} d_{\tilde{S}}(f) \quad &= (\partial_{M_1} f)(-M_3 C_2 + M_2 C_3) + (\partial_{M_2} f)(+M_3 C_1 - M_1 C_3) \\ &\quad +(\partial_{M_3} f)(-M_2 C_1 + M_1 C_2) \\ &= [M_3(\partial_{M_2} f) - M_2(\partial_{M_3} f)]C_1 + [-M_3(\partial_{M_1} f) + M_1(\partial_{M_3} f)]C_2 \\ &\quad +[M_2(\partial_{M_1} f) - M_1(\partial_{M_2} f)]C_3 = 0. \end{aligned}$$

Due to the fact that the ghosts C_1, C_2, C_3 are independent variables, imposing the previous condition is equivalent to imposing the following three:

1. $M_3(\partial_{M_2} f) - M_2(\partial_{M_3} f) = 0$;
2. $-M_3(\partial_{M_1} f) + M_1(\partial_{M_3} f) = 0$;
3. $M_2(\partial_{M_1} f) - M_1(\partial_{M_2} f) = 0$.

This requires a polynomial P in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ such that:

$$\partial_{M_1} f = M_1 P \qquad \partial_{M_2} f = M_2 P \qquad \partial_{M_3} f = M_3 P.$$

It follows that f is necessarily of the following form:

$$f = \sum_{k=0}^{r} g_k(M_4)(M_1^2 + M_2^2 + M_3^2)^k,$$

with r a non-negative number.

Note: as the BRST cohomology group of degree 0 coincides with the space of all polynomials that are invariant under the gauge group action, we have confirmed (5.4), stating the most generic form of an action for the matrix model of degree $n = 2$.

To conclude:

$$H^0(W, d_{\widetilde{S}}) = \left\{ \sum_{k=0}^{r} g_k(M_4)(M_1^2 + M_2^2 + M_3^2)^k,\ r \in \mathbb{N}_0,\ g_k \in \mathrm{Pol}_{\mathbb{R}}(M_4) \right\}.$$

$\mathbf{H^1(W, d_{\widetilde{S}})}$

To compute the space $Z^1(W, d_{\widetilde{S}})$, let us start by considering a generic cochain φ of ghost degree 1. There exist some polynomials f_1, f_2, f_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ such that φ can be written in the following way:

$$\varphi = f_1 C_1 + f_2 C_2 + f_3 C_3.$$

Using the fact that the ghost fields C_1, C_2, C_3 and E are supposed to be independent variables, one can check that the following conditions need to be satisfied, for φ to be a cocycle,:

1. $M_3(\partial_{M_1} f_1) - M_1(\partial_{M_3} f_1) + M_3(\partial_{M_2} f_2) - M_2(\partial_{M_3} f_2) + f_3 = 0$;
2. $-M_2(\partial_{M_1} f_1) + M_1(\partial_{M_2} f_1) + M_3(\partial_{M_2} f_3) - M_2(\partial_{M_3} f_3) - f_2 = 0$;

3. $-M_2(\partial_{M_1} f_2) + M_1(\partial_{M_2} f_2) - M_3(\partial_{M_1} f_3) + M_1(\partial_{M_3} f_3) + f_1 = 0$;

4. $M_1 f_1 + M_2 f_2 + M_3 f_3 = 0$.

By deriving the last condition with respect to M_i and substituting the first three conditions, they can be rewritten as:

1. $M_3 \left[\partial_{M_1} f_1 + \partial_{M_2} f_2 + \partial_{M_3} f_3\right] = -2 f_3$;

2. $M_2 \left[\partial_{M_1} f_1 + \partial_{M_2} f_2 + \partial_{M_3} f_3\right] = -2 f_2$;

3. $M_1 \left[\partial_{M_1} f_1 + \partial_{M_2} f_2 + \partial_{M_3} f_3\right] = -2 f_1$.

Using once again the condition 4., we deduce the following equation:

$$(M_1^2 + M_2^2 + M_3^2) \left[\partial_{M_1} f_1 + \partial_{M_2} f_2 + \partial_{M_3} f_3\right] = 0,$$

which implies that:

$$\partial_{M_1} f_1 + \partial_{M_2} f_2 + \partial_{M_3} f_3 = 0.$$

Using this last equation and the relations among the partial derivatives of the polynomials f_1, f_2 and f_3 obtained by deriving condition 4., the first condition can be rewritten as follows:

$$\begin{aligned} 0 &= M_3 \left[\partial_{M_1} f_1 + \partial_{M_2} f_2\right] + \left[-M_1(\partial_{M_3} f_1) - M_2(\partial_{M_3} f_2)\right] + f_3 \\ &= \left[-M_1(\partial_{M_3} f_1) - M_2(\partial_{M_3} f_2) - M_3(\partial_{M_3} f_3)\right] + f_3 \\ &= 2 f_3. \end{aligned}$$

An analogous procedure can be followed also for the other conditions, drawing to the conclusion that:

$$f_1 = 0 \qquad f_2 = 0 \qquad f_3 = 0.$$

Thus we state that:

$$Z^1(W, d_{\widetilde{S}}) = \{0\} \qquad \text{and so} \qquad H^1(W, d_{\widetilde{S}}) = \{0\}\,.$$

$\mathbf{H^2(W, d_{\tilde{S}})}$

To explicitly compute the cohomology group $H^2(W, d_{\widetilde{S}})$, we start considering $Z^2(W, d_{\widetilde{S}})$, namely the space of cocycles of degree 2. Let φ be a generic cochain of ghost degree 2: then, for some polynomials g_{12}, g_{13}, g_{23}, h in $\mathrm{Pol}_{\mathbb{R}}(M_a)$, φ can be written as:

$$\varphi = g_{12}C_1C_2 + g_{13}C_1C_3 + g_{23}C_2C_3 + hE.$$

Recalling once again that the ghost fields C_1, C_2, C_3 and E are supposed to be independent variables, the condition on φ to be a cocycle is equivalent to impose the following equalities:

1. $M_2(\partial_{M_1} g_{12}) - M_1(\partial_{M_2} g_{12}) + M_3(\partial_{M_1} g_{13}) - M_1(\partial_{M_3} g_{13}) + M_3(\partial_{M_2} g_{23})$ $-M_2(\partial_{M_3} g_{23}) = 0;$
2. $-M_2 g_{12} - M_3 g_{13} + (\partial_{M_2} h)M_3 - (\partial_{M_3} h)M_2 = 0;$
3. $M_1 g_{12} - M_3 g_{23} - (\partial_{M_1} h)M_3 + (\partial_{M_3} h)M_1 = 0;$
4. $M_1 g_{13} + M_2 g_{23} + (\partial_{M_1} h)M_2 - (\partial_{M_2} h)M_1 = 0.$

The most general solution for the condition 2. is

$$g_{12} = M_3 P - \partial_{M_3} h, \qquad g_{13} = -M_2 P + \partial_{M_2} h,$$

with P a polynomial in $\mathrm{Pol}_{\mathbb{R}}(M_a)$. Analogously, imposing the condition 3., we find:

$$g_{23} = M_1 P - \partial_{M_1} h.$$

Finally, choosing the polynomials g_{12}, g_{13}, g_{23} as above the conditions 1. and 4. are automatically satisfied.

Thus we deduce that:

$$Z^2(W, d_{\widetilde{S}}) \simeq \left\{ \begin{array}{c} \varphi = (M_3 P - \partial_{M_3} h)\, C_1C_2 + (-M_2 P + \partial_{M_2} h)\, C_1C_3 \\ + (M_1 P - \partial_{M_1} h)\, C_2C_3 + hE : P, h \in \mathrm{Pol}_{\mathbb{R}}(M_a) \end{array} \right\}. \tag{D.1}$$

The previous expression can be rewritten into the following more explicit form:

$$Z^2(W, d_{\widetilde{S}}) = K \oplus B^2(W, d_{\widetilde{S}}) \oplus \mathrm{Pol}_{\mathbb{R}}(M_4)E \tag{D.2}$$

with

$$K := \{f(M_1C_2C_3 - M_2C_1C_3 + M_3C_1C_2),\ f \in \mathrm{Pol}_{\mathbb{R}}(M_a)\}\,.$$

To prove this statement it is enough to show that, given a generic cochain φ in $Z^2(W, d_{\widetilde{S}})$ as described in Equation (D.1), it is possible to uniquely determine a cochain β of degree 1, a polynomial Q in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ and a polynomial h_0 in $\mathrm{Pol}_{\mathbb{R}}(M_4)$ such that

$$\varphi - d_{\widetilde{S}}(\beta) = (M_3Q)C_1C_2 - (M_2Q)C_1C_3 + (M_1Q)C_2C_3 + h_0E.$$

Let us first consider the polynomial h in Equation (D.1): it can be written as

$$h = h_1 + h_0,$$

where h_0 is an element in $\mathrm{Pol}_{\mathbb{R}}(M_4)$ while $h_1 = M_1A_1 + M_2A_2 + M_3A_3$ for suitable polynomials A_1, A_2, A_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$.

Thus the cochain φ defined by h and the polynomial P takes the following form:

$$\begin{aligned}\varphi\ &= (M_3P - \partial_{M_3}h_1)\,C_1C_2 + (-M_2P + \partial_{M_2}h_1)\,C_1C_3\\ &\quad + (M_1P - \partial_{M_1}h_1)\,C_2C_3 + h_1E + h_0E.\end{aligned}$$

Then let β be the following cochain of degree 1:

$$\beta = A_1C_1 + A_2C_2 + A_3C_3,$$

where the polynomials A_1, A_2 and A_3 are exactly the ones appearing in the expression of h_1. Therefore, applying the coboundary operator on β and using the relation between the partial derivative of the polynomials A_1, A_2, A_3 and h_1, we find the following equality:

$$\begin{aligned}d_{\widetilde{S}}(\beta)\ &= [-\partial_{M_3}h_1 + M_3\,(\partial_{M_1}A_1 + \partial_{M_2}A_2 + \partial_{M_3}A_3)]\,C_1C_2\\ &\quad + [\partial_{M_2}h_1 - M_2\,(\partial_{M_1}A_1 + \partial_{M_2}A_2 + \partial_{M_3}A_3)]\,C_1C_3\\ &\quad + [-\partial_{M_1}h_1 + M_1\,(\partial_{M_1}A_1 + \partial_{M_2}A_2 + \partial_{M_3}A_3)]\,C_2C_3\\ &\quad + (M_1A_1 + M_2A_2 + M_3A_3)\,E.\end{aligned}$$

Therefore:

$$\varphi - d_{\widetilde{S}}(\beta) = (M_3Q)C_1C_2 - (M_2Q)C_1C_3 + (M_1Q)C_2C_3 + h_0E$$

where

$$Q := -\left(P + \partial_{M_1} A_1 + \partial_{M_2} A_2 + \partial_{M_3} A_3\right).$$

Thus the equality in (D.2) is proved and the BRST cohomology group of degree 2 is

$$H^2(W, d_{\widetilde{S}}) \simeq \frac{K \oplus B^2(W, d_{\widetilde{S}}) \oplus \mathrm{Pol}_{\mathbb{R}}(M_4)E}{B^2(W, d_{\widetilde{S}})} = K \oplus \mathrm{Pol}_{\mathbb{R}}(M_4)E \ , \tag{D.3}$$

with

$$K := \{f(M_1C_2C_3 - M_2C_1C_3 + M_3C_1C_2),\ f \in \mathrm{Pol}_{\mathbb{R}}(M_a)\}\,.$$

$\mathbf{H^3(W, d_{\widetilde{S}})}$

To compute the BRST cohomology group of degree 3, we start considering the vector space $Z^3(W, d_{\widetilde{S}})$ defined by the cocycles of ghost degree 3.
Let φ be a generic cochain of ghost degree 3: then there exist some polynomials f, g_1, g_2, g_3 such that φ can be written in the following way:

$$\varphi = fC_1C_2C_3 + g_1C_1E + g_2C_2E + g_3C_3E.$$

Due to the independence of the ghost fields C_1, C_2, C_3 and E, one can check that to $\varphi \in \mathrm{Ker}(d^3_{\widetilde{S}})$ amounts to imposing the following conditions:

1. $M_1g_1 + M_2g_2 + M_3g_3 = 0$;
2. $fM_3 + M_3(\partial_{M_1}g_1) - M_1(\partial_{M_3}g_1) - M_2(\partial_{M_3}g_2) + g_3 + M_3(\partial_{M_2}g_2) = 0$;
3. $-fM_2 - M_2(\partial_{M_1}g_1) + M_1(\partial_{M_2}g_1) - g_2 + M_3(\partial_{M_2}g_3) - M_2(\partial_{M_3}g_3) = 0$;
4. $fM_1 + g_1 - M_2(\partial_{M_1}g_2) + M_1(\partial_{M_2}g_2) - M_3(\partial_{M_1}g_3) + M_1(\partial_{M_3}g_3) = 0$.

By deriving with respect to M_i the first condition and substituting it in the others, we obtain the following expressions:

$$\begin{cases} M_1\left(f + \partial_{M_1}g_1 + \partial_{M_2}g_2 + \partial_{M_3}g_3\right) = -2g_1 \\ M_2\left(f + \partial_{M_1}g_1 + \partial_{M_2}g_2 + \partial_{M_3}g_3\right) = -2g_2 \\ M_3\left(f + \partial_{M_1}g_1 + \partial_{M_2}g_2 + \partial_{M_3}g_3\right) = -2g_3. \end{cases}$$

Using once again the first condition, we deduce:

$$f = -\partial_{M_1}g_1 - \partial_{M_2}g_2 - \partial_{M_3}g_3.$$

Therefore, the vector space of cocycles of ghost degree 3 is given by the collection of all cochains φ, with

$$\varphi = fC_1C_2C_3 + g_1C_1E + g_2C_2E + g_3C_3E,$$

where f, g_1, g_2 and g_3 are polynomials belonging to $\mathrm{Pol}_{\mathbb{R}}(M_a)$, such that the following two conditions are satisfied:

$$M_1g_1 + M_2g_2 + M_3g_3 = 0 \qquad\qquad f = -\partial_{M_1}g_1 - \partial_{M_2}g_2 - \partial_{M_3}g_3.$$

The most general solution for the first condition is given by polynomials g_1, g_2 and g_3 that can be written as follows for some polynomials Q, R and S in $\mathrm{Pol}_{\mathbb{R}}(M_a)$:

$$g_1 = M_2Q + M_3R \qquad g_2 = -M_1Q + M_3S \qquad g_3 = -M_1R - M_2S.$$

Therefore:

$$Z^3(W, d_{\widetilde{S}}) \simeq \left\{ \begin{array}{l} \varphi = (M_2Q + M_3R)\, C_1E + (-M_1Q + M_3S)\, C_2E \\ \quad + (-M_1R - M_2S)\, C_3E + (-M_2(\partial_{M_1}Q) + M_1(\partial_{M_2}Q) \\ \quad -M_3(\partial_{M_1}R) + M_1(\partial_{M_3}R) - M_3(\partial_{M_2}S) + M_2(\partial_{M_3}S))\, C_1C_2C_3 \\ \text{with } Q, R, S \in \mathrm{Pol}_{\mathbb{R}}(M_a) \end{array} \right\}.$$

To conclude the computation of the cohomology group of degree 3 we show that a generic cocycle in $Z^3(W, d_{\widetilde{S}})$ can be seen as a coboundary element, so that $H^3(W, d_{\widetilde{S}})$ vanishes.
Let Q, R, S be fixed polynomials in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ and let φ be the corresponding cocycle in $Z^3(W, d_{\widetilde{S}})$ of the form described above. Then, let β be the following cochain of ghost degree 2,

$$\beta = -QC_1C_2 - RC_1C_3 - SC_2C_3.$$

An explicit computation shows that φ coincides with $d_{\widetilde{S}}(\beta)$ and so:

$$Z^3(W, d_{\widetilde{S}}) = B^3(W, d_{\widetilde{S}}).$$

Thus:

$$H^3(W, d_{\widetilde{S}}) = \frac{Z^3(W, d_{\widetilde{S}})}{B^3(W, d_{\widetilde{S}})} = 0.$$

$\mathbf{H^4(W, d_{\widetilde{S}})}$

Before starting with the explicit computation of the BRST cohomology group of degree 4, let us recall that in Proposition 12 we have proved the following isomorphism:

$$Z^4(W, d_{\widetilde{S}}) \simeq Z^2(W, d_{\widetilde{S}}) \cdot E.$$

Therefore, since in the computation of the BRST cohomology group of degree 2 we have already explicitly determined the space $Z^2(W, d_{\widetilde{S}})$, to conclude the computation of $H^4(W, d_{\widetilde{S}})$ we only need to give a description of $B^4(W, d_{\widetilde{S}}) = \mathrm{Im}(d^3_{\widetilde{S}})$.

Let φ be a generic cochain of degree 3. Then there are polynomials f, g_1, g_2, g_3 in $\mathrm{Pol}_{\mathbb{R}}(M_a)$ for which φ can be written in the following form:

$$\varphi = fC_1C_2C_3 + g_1C_1E + g_2C_2E + g_3C_3E.$$

Thus:

$$\begin{aligned} d_{\widetilde{S}}(\varphi) = \quad & (fM_1C_2C_3 - fM_2C_1C_3 + fM_3C_1C_2)E \\ & + \left[(\partial_{M_1}g_1)d_{\widetilde{S}}(M_1) + (\partial_{M_2}g_1)d_{\widetilde{S}}(M_2) + (\partial_{M_3}g_1)d_{\widetilde{S}}(M_3)\right] C_1E \\ & + \left[(\partial_{M_1}g_2)d_{\widetilde{S}}(M_1) + (\partial_{M_2}g_2)d_{\widetilde{S}}(M_2) + (\partial_{M_3}g_2)d_{\widetilde{S}}(M_3)\right] C_2E \\ & + \left[(\partial_{M_1}g_3)d_{\widetilde{S}}(M_1) + (\partial_{M_2}g_3)d_{\widetilde{S}}(M_2) + (\partial_{M_3}g_3)d_{\widetilde{S}}(M_3)\right] C_3E \\ & + g_1(M_1E + C_2C_3)E + g_2(M_2E - C_1C_3)E + g_3(M_3E + C_1C_2)E. \end{aligned}$$

Comparing the expression found for a generic element in $\mathrm{Im}(d^3_{\widetilde{S}})$ with the explicit form of a generic element in the vector space $\mathrm{Im}(d^1_{\widetilde{S}})$, it follows immediately that:

$$B^4(W, d_{\widetilde{S}}) \simeq [K \oplus B^2(W, d_{\widetilde{S}})] \cdot E,$$

with

$$K := \left\{ f(M_1C_2C_3 - M_2C_1C_3 + M_3C_1C_2),\ f \in \mathrm{Pol}_{\mathbb{R}}(M_a) \right\}.$$

Therefore, using what we have already determined in (D.2), we conclude that the BRST cohomology group of degree 4 for our model is

$$\begin{aligned} H^4(W, d_{\widetilde{S}}) \quad & = \frac{Z^2(W, d_{\widetilde{S}})E}{[K \oplus B^2(W, d_{\widetilde{S}})]E} \simeq \frac{\left[K \oplus B^2(W, d_{\widetilde{S}}) \oplus \mathrm{Pol}_{\mathbb{R}}(M_4)E\right] E}{[K \oplus B^2(W, d_{\widetilde{S}})]E} \\ & \simeq \mathrm{Pol}_{\mathbb{R}}(M_4)E^2. \end{aligned} \tag{D.4}$$

To complete the proof for the cohomology groups of degree k with $k > 4$, it is enough to recall Proposition 12, where the following isomorphism are stated:

- if k is odd, then $H^k(W, d_{\widetilde{S}}) \simeq H^3(W, d_{\widetilde{S}})$;
- if k is even, then $H^k(W, d_{\widetilde{S}}) \simeq H^4(W, d_{\widetilde{S}})$.

Using these isomorphisms and the computations done above, the proof of the Theorem is concluded.

Bibliography

[1] M. Alexandrov, M. Kontsevich, A. Schwarz, and O.Zabronsky. The geometry of the master equation and topological quantum field theory. *Int. J. Mod. Phys.*, A 12, (1997), 1405-1430.

[2] V.I. Arnold and A.B. Givental. Symplectic geometry. *Dynamical systems IV*, Encyclopaedia Math. Sci. Vol. 4, Springer, Berlin,, (1990), pp. 1-136.

[3] G. Barnich, F. Brandt, and M. Henneaux. Local BRST cohomology in gauge theories. *Phys. Rep.*, 338, (2000), 439-569.

[4] G. Barnich, F. Brandt, and M. Henneaux. Local BRST cohomology in the antifield formalism I. General theorems. *Commun. Math. Phys.*, 174, (1995), 57-92.

[5] G. Barnich, F. Brandt, and M. Henneaux. Local BRST cohomology in the antifield formalism II. Application to Yang-Mills theory. *Commun. Math. Phys.*, 174, (1995), 93-116.

[6] I.A. Batalin and G.A. Vilkovisky. Feynman rules for reducible gauge theories. *Phys. Lett.*, B120 (1983), 166-170.

[7] I.A. Batalin and G.A. Vilkovisky. Gauge algebra and quantization. *Phys. Lett.*, B 102, (1981), 27-31.

[8] I.A. Batalin and G.A. Vilkovisky. Quantization of gauge theories with linearly dependent generators. *Phys. Rev.*, D28 (1983), 2567-2582, Erratum, D30 (1984), 508.

[9] C.M. Becchi, A. Rouet, and R. Stora. The abelian Higgs-Kibble model. Unitarity of the S operator. *Phys. Lett.*, B 52, (1974), 344-346.

[10] C.M. Becchi, A. Rouet, and R. Stora. Renormalization of gauge theories. *Ann. Phys.*, 98, 2 (1976), 287-321.

[11] C.M. Becchi, A. Rouet, and R. Stora. Renormalization of the abelian Higgs-Kibble model. *Commun. Math. Phys.*, 42, (1975), 127-162.

[12] K. Bering and H. Grosse. On Batalin-Vilkovisky formalism of non-commutative field theories. *European Physical Journal C. Particles and Fields (EPJC)*, 68, 1-2, (2010), 313-324.

[13] D. Bleecker. *Gauge theories and variational principles.* Addison-Wesley, 1981.

[14] L. Bonora and P. Cotta-Ramusino. Some remarks on BRS transformations, anomalies and the cohomology of the Lie algebra of the group of gauge transformations. *Commun. Math. Phys.*, 87, (1983), 589-603.

[15] H. Cartan and S. Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N.J., (1956).

[16] A.H. Chamseddine and A. Connes. The spectral action principle. *Commun. Math. Phys.*, 186, (1997), 731-750.

[17] A.H. Chamseddine and A. Connes. Universal formula for noncommutative geometry actions: unifications of gravity and the standard model. *Phys. Rev. Lett.*, 77, (1996),4868-4871.

[18] C. Chevalley and S. Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 63, (1948), 85-124.

[19] A. Connes. Gravity coupled with matter and the foundation of non-commutative geometry. *Commun. Math. Phys.*, 182, (1996), 155-176.

[20] A. Connes. *Noncommutative geometry.* Academic Press, San Diego, (1994).

[21] A. Connes and M. Marcolli. *Noncommutative geometry, quantum fields and motives.* Amer. Math. Soc., Colloquium Publications 55, (2008).

[22] J.B. Conway. *A course in functional analysis.* Springer, Graduate Texts in Mathematics, Vol. 96, (1990).

[23] K. Costello. *Renormalization and effective field theory.* Amer. Math. Soc., Mathematical Surveys and Monographs, (2011).

[24] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory.* Cambridge University Press, (2014).

[25] M. Dubois-Violette and G. Landi. The Weil algebra of a Hopf algebra - I - A noncommutative framework. *Commun. Math. Phys.*, 326, 3, (2014), 851-874.

[26] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yu-dovina. *Introduction to representation theory.* Amer. Math. Soc., Providence, RI, (2011).

[27] L.D. Faddeev and V.N. Popov. Feynman diagrams for the Yang-Mills field. *Phys. Lett.*, B 25, (1967), 29-30.

[28] G. Felder and D. Kazhdan. The classical master equation. In *Perspectives in Representation Theory, Contemporary Mathematics,* Eds P. Etingof, M. Khovanov, and A. Savage, (2014).

[29] R.P. Feynman and A.R. Hibbs. Quantum mechanics and path integrals. *McGraw-Hill, New York*, (1965).

[30] D. Fiorenza. An introduction to the Batalin-Vilkovisky formalism. *Comptes Rendus des Rencontres Mathmatiques de Glanon*, (2003).

[31] P. Di Francesco, P. Ginsparg, and J. ZinnJustin. 2D gravity and random matrices. *Phys. Rep.*, 254, (1995), 1-133.

[32] A. Fuster, M. Henneaux, and A. Maas. BRST quantization: a short review. *Int. J. Geom. Meth. Mod. Phys.*, 2, (2005), 939-964.

[33] M. Gerstanhaber. The cohomology structure of an associative ring. *Ann. of Math.*, 78, (1963), 267-288.

[34] J. Gomis, J. Pars, and S. Samuel. Antibracket, antifields and gauge-theory quantization. *Phys. Rep.*, 259, (1995), 1-145.

[35] J. Gomis and S. Weinberg. Are nonrenormalizable gauge theories renormalizable? *Nucl. Phys.*, B469, (1996), 473-487.

[36] B.E. Hanlon and G.C. Joshi. BRS symmetry in Connes' non-commutative geometry. *J. Phys. A: Math. Gen.*, 28, 10, (1995), 2889-2904.

[37] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, Graduate Texts in Mathematics, Vol.52 New York, (1977).

[38] M. Henneaux. Hamiltonian form of the path integral for theories with a gauge freedom. *Phys. Rep.*, 126, 1, (1985), 1-66.

[39] M. Henneaux. Lectures on the antifield - BRST formalism for gauge theories. *Nucl. Phys. Proc. Suppl.*, 18 A, (1990), 47-106.

[40] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems.* Princeton University Press, (1992).

[41] G. Hochschild and J-P. Serre. Cohomology of Lie algebras. *Ann. of Math.*, 57, 3, (1953), 591-603.

[42] S. Hollands. Renormalized quantum Yang-Mills fields in curved spacetime. *Ref. Math. Phys.*, 20, (2008), 1033-1172.

[43] J.W. van Holten. The BRST complex and the cohomology of compact Lie algebras. *Nuclear Physics B*, 339, 1, (1990), 158-176.

[44] J. Humphreys. *Introduction to Lie algebras and representation theory.* Springer, Graduate Texts in Mathematics, Vol. 9, (1997).

[45] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, Vol I.* Interscience Publishers, New York-London, 1963.

[46] T. Krajewski. Classification of finite spectral triples. *J. Geom. Phys.*, 28, (1998), 1-30.

[47] S. Lang. *Algebra.* Springer-Verlag, Graduate Texts in Mathematics, Vol. 211, (2002).

[48] Y. I. Manin. *Gauge field theory and complex geometry.* Springer - Verlag, Fundamental Principles of Mathematical Sciences, Vol. 289, Berlin, (1988).

[49] G.K. Pedersen. *Analysis now.* Springer, Graduate Texts in Mathematics, 118, (1989).

[50] M. Scheunert and R.B. Zhang. Cohomology of Lie superalgebras and of their generalizations. *J.Math.Phys.*, 39, (1998), 5024-5061.

[51] A. Schwarz. Geometry of Batalin-Vilkovisky quantization. *Commun.Math.Phys.*, 155, (1993), 249-260.

[52] J. Stasheff. Deformation theory and the Batalin-Vilkovisky master equation. *Math. Phys. Stud.*, 20, (1997), pp. 271-284.

[53] W.D. van Suijlekom. *Noncommutative geometry and particle physics.* Springer, Mathematical Physics Studies, (2014).

[54] J. Tanaka. On homology and cohomology of Lie superalgebras with coefficients in their finite-dimensional-representations. *Proc. Japan Acad. Ser. A Math. Sci.*, 71, 3, (1995), 51-53.

[55] J. Tate. Homology of Noetherian rings and local rings. *Illinois Journal of Mathematics*, 1, (1957), 14-27.

[56] R. Ticciati. *Quantum field theory for mathematicians.* Cambridge University Press, (1999).

[57] I.V. Tyutin. Gauge invariance in field theory and statistical physics in operator formalism. *Preprint of P.N. Lebedev Physical Institute*, 39, (1975).

[58] C.A. Weibel. *An introduction to homological algebra.* Cambridge University Press, (1994).

[59] A. Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Mathematics*, 6, (1971), 329-346.

[60] J. Zinn-Justin. Renormalization of gauge theories. *in Trends in Elementary Particle Theory,*, Eds: H.Rollnik and K. Dietz, Lecture Notes in Physics, Vol 37, Springer-Verlag, Berlin, (1975).

Index

Samenvatting

Wiskunde heeft één eigenschap die haar onderscheidt van alle andere wetenschappen: elke eenmaal bewezen stelling of propositie kan in de toekomst onmogelijk nog worden ontkracht. Immers, meetfouten bestaan niet binnen de wiskunde en er bestaan ook geen experimenten om de theorie te toesten. Aldus kan wiskunde noch worden weerlegd, noch worden ingetrokken; ze staat op zichzelf en kan alleen progressie boeken door voort te bouwen op dezelfde fundamenten die eeuwen geleden gelegd zijn door de oude Grieken, zoals Pythagoras (ca. 570-495 v.Chr.) en Euclides (tussen 367-283 v.Chr.). Hetgeen zij hebben bewezen, eeuwen voor de uitvinding van de rekenmachine of de computer, was destijds waar en is nog steeds waar.

Dit geldt echter niet voor andere wetenschappen, waaronder natuurkunde. Immers, natuurkunde is gebaseerd op theorieën, die experimenteel bevestigd dan wel ontkracht kunnen worden en die in geen geval absoluut waar zijn. Ze worden eenvoudigweg geaccepteerd totdat er een experiment wordt uitgevoerd dat hen ontkracht.

Dit substantiële verschil tussen wiskunde en natuurkunde wordt veroorzaakt doordat er binnen de natuurkunde andere doelen worden gesteld: de drijvende kracht achter natuurkundig onderzoek is de zoektocht naar een wetenschappelijke verklaring van de wereld om ons heen en van de wijze waarop zij werkt. Ondanks dat natuur de mens altijd heeft gefascineerd en dat men voortdurend heeft geprobeerd om waargenomen verschijnselen te verklaren, ligt de oorsprong van de klassieke natuurkunde niet ver in het verleden. Immers, datgene wat wij klassieke natuurkunde noemen is pas door Galileo Galilei (1564-1642) en Isaac Newton (1642-1727) ontwikkeld. Om preciezer te zijn, de natuurkunde, zoals we die tegenwoordig kennen, is ontstaan door de waarnemingen van natuurlijke verschijnselen met het wiskundige formalisme te combineren.

"Filosofie staat beschreven in het grote boek van de natuur, dat onbegrijpelijk is zonder eerst de taal en symbolen waarin het geschreven is, geleerd te hebben. Het boek is geschreven in de taal van de wiskunde, en de symbolen zijn driehoeken, cirkels en andere meetkundige figuren, zonder welke het onmogelijk is één enkel woord te begrijpen; zonder welke men vergeefs dwaalt in een donker doolhof."

(Galileo Galilei, De Keurmeester, Hoofdstuk VI).

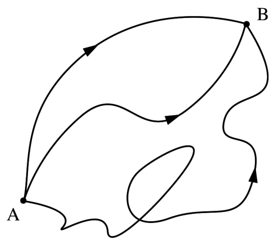
Aldus, wat is klassieke natuurkunde? Een klassiek *natuurkundig systeem* bestaat uit de volgende onderdelen:

- *tijd*,
- *ruimte*,
- *observabelen*, oftewel, grootheden die berekend kunnen worden, zoals de snelheid of de positie van hetgeen we willen beschrijven, zoals objecten of deeltjes,
- *krachten*, zoals de zwaartekracht of the elektromagnetische kracht,
- *symmetrieën*, van bijvoorbeeld de ruimte of de kracht.

Aldus kunnen we nu zeggen dat een klassieke *natuurkundige theorie* uit een stelsel vergelijkingen bestaat dat de tijdevolutie van een natuurkundig systeem voorspelt. Dit was hoe men tegen natuurkunde aankeek ten tijde van Galilei en Newton. Echter, de ontdekking van de quantummechanica maakte het noodzakelijk om het klassieke idee van een natuurkundige theorie aan te passen: de natuurkundigen werden gedwongen om voorgoed de droom van een deterministische theorie voor (sub)atomaire verschijnselen te laten varen en te accepteren dat het gedrag van de natuur op subatomaire schaal slechts gekend kan worden in termen van waarschijnlijkheid. Aldus werd het begrip *quantizatie* geintroduceerd: elke natuurkundige theorie die de natuur op subatomaire schaal beschrijft, dient gequantizeerd te worden.

Door de jaren heen hebben diverse natuurkundigen, zoals Dirac, Feynman, Heisenberg, Pauli, Planck en Schrödinger, gewerkt aan de quantizatie van klassieke theorieën, en hebben daartoe meerdere quantizatieprocedures ontwikkeld. Eén daarvan is gebaseerd op het begrip *padintregraal.*

Zoals eerder opgemerkt is het fundamentele onderliggende principe achter de quantummechanica dat, wanneer we (sub)atomaire deeltjes analyzeren, het



Figuur D.1: De padintegraal maakt de berekening van de kans dat een deeltje in punt A na een bepaalde tijd punt B zal bereiken mogelijk. Teneinde deze berekening te maken, dienen alle mogelijke paden van A naar B die het deeltje kan afleggen in beschouwing genomen te worden, hetgeen de naam *pad*integraal verklaart.

onmogelijk is de trajecten die de deeltjes zullen afleggen, exact te bepalen: gegeven een deeltje in punt A in Figuur D.1, is de enige grootheid die bepaald kan worden, de kans dat het deeltje na een bepaalde hoeveelheid tijd in punt B aangetroffen kan worden. Het concept padintegraal werd ontdekt teneinde deze vraag te beantwoorden. Het idee achter dit concept is het volgende: als we de kans om het deeltje in punt B aan te treffen willen bepalen, kunnen we de kans dat het deeltje een bepaald traject dat A met B verbindt, berekenen, om vervolgens al deze kansen gerelateerd aan de verschillende trajecten op te tellen.

Dit is eenvoudigweg de intuïtie achter de ontdekking van de padintegraal, die in 1948 door Feynman geïntroduceerd werd en om die reden vaak de *Feynmanintegraal* genoemd wordt. De Feynmanintegraal is een zeer belangrijk hulpmiddel voor de studie van de quantumveldentheorie, met name vanwege de volgende twee fundamentele eigenschappen:

- de voorspellingen berekend met de integraal komen uiterst nauwkeurig overeen met de meetresultaten,
- de integraal kan uitgerekend worden met behulp van zeer expliciete formules en duidelijke regels.

We moeten echter opmerken dat de Feynmanintegraal ook zwakheden kent: in het algemeen is de integraal wiskundig niet goed gedefinieerd. Immers, wanneer we een klassieke theorie proberen te quantizeren met gebruik van de Feynmanintegraal, stuiten we op het probleem van de berekening van integralen van het volgende type:

$$\langle g \rangle = \int_{X_0} g e^{-S_0} [d\mu],$$

waar het linkerlid in een bepaald opzicht de grootheid staat die bepaald kan worden door middel van experimenten, terwijl het rechterlid de natuurkundige theorie, die we analyzeren, beschreven staat. Immers,

- $\langle g \rangle$ is de *verwachtingswaarde* van de functionaal g, die de grootheid representeert die bepaald kan worden door middel van experimenten,
- X_0, wordt de *configuratieruimte* genoemd en bevat informatie over de ruimte en over de velden binnen de theorie,
- S_0 is de *actie*, die informatie bevat over de krachten die onderdeel zijn van het natuurkundige systeem dat we bestuderen.

Echter, zoals reeds opgemerkt, is de definitie van deze integraal niet goed gedefinieerd vanuit wiskundig perspectief; wat wordt veroorzaakt doordat de maat $d\mu$ in het algemeen niet goed gedefinieerd kan worden indien de configuratieruimte X_0 oneindig-dimensionaal is.

Hoe dan ook, zelfs als we ons beperken tot eindig-dimensonale theorieën, kunnen we geconfronteerd worden met andere problemen, die gerelateerd zijn aan symmetrieën in het desbetreffende natuurkundige systeem. Deze theorieën, die met symmetrieën uitgerust zijn, worden *ijktheorieën* genoemd. Het belang van een exacte formulering van een quantizatieprocedure voor ijktheorieën wordt onderstreept door het feit dat alle bekende fundamentele interacties in de natuur beheerst worden door ijktheorieën.

Het probleem waar we mee te krijgen zodra we een ijktheorie proberen te quantizeren door middel van de padintegraalaanpak, is dat de Feynmanintegraal gedegenereerd blijkt te zijn zodra we in de ijkrichting integreren: in plaats van een eindige verwachtingswaarde, zoals we zouden verwachten, verkrijgen we een oneindige waarde. Het is dus noodzakelijk om de ijk vast te leggen om overbodige ijkvariabelen te kunnen elimineren. We verliezen echter de ijkinvariantie van de theorie na deze procedure en bovendien lijkt het erop dat we grip op de natuurkundige betekenis van wat we berekenen met de padintegraal verliezen.

Een oplossing voor dit probleem werd in 1967 gevonden door Faddeev en Popov.

Zij introduceerden extra velden in de theorie teneinde lokale symmetrieën op te heffen, opdat de padintegraal berekend kan worden. Inderdaad kan de berekening van een integraal soms vereenvoudigd worden door het domein te vergroten en extra variabelen toe te voegen, bijvoorbeeld in de berekening van de Gaussische integraal:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dx\ dy} = \sqrt{\pi}.$$

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dx\ dy} = \sqrt{\pi}.$$

In geval van een ijktheorie gaan we op soortgelijke wijze te werk: we starten met een initiële ijktheorie (X_0, S_0) en voegen extra velden toe om zowel de configuratieruimte X_0 als de actie S_0 uit te breiden, om op deze wijze een *uitgebreide theorie* $(\widetilde{X}, \widetilde{S})$ te verkrijgen. In plaats van de padintegraal voor de ijktheorie (X_0, S_0), waar we mee startten, beschouwen we dus de integraal voor de uitgebreide theorie $(\widetilde{X}, \widetilde{S})$:

$$\int_{X_0} g e^{-S_0}[d\mu] \rightsquigarrow \int_{\widetilde{X}} g e^{-\widetilde{S}}[d\nu].$$

Vervolgens is het idee om aan de initiële ijktheorie extra velden toe te voegen, die *spookvelden (Engels: ghost fields)* genoemd worden, omdat ze geen werkelijke natuurkundige betekenis hebben.

Om alles samen te vatten, teneinde een ijktheorie (X_0, S_0) te quantizeren, willen we een uitgebreide theorie $(\widetilde{X}, \widetilde{X})$ construeren, waar:

$$\begin{aligned} X_0 &\rightsquigarrow \widetilde{X} = X_0 + \{\text{antivelden, spookvelden en antispookvelden}\} \\ S_0 &\rightsquigarrow \widetilde{S} = S_0 + \{\text{termen die afhangen van antivelden,} \\ &\qquad\qquad \text{spookvelden en antispookvelden.}\} \end{aligned}$$

Zoals gezegd zijn deze spookvelden ingevoerd om de degeneratie in de padintegraal in aanwezigheid van een ijksymmetrie op te lossen. In 1975 ontdekten Becchi, Rouet, Stora, en onafhankelijk van de eerstgenoemden, Tyutin, dat deze spookvelden zich vanuit wiskundig perspectief interessant gedragen: ze blijken zich als voortbrengers van een cohomologiecomplex, het zogeheten *BRST complex*, te gedragen. De fundamentele eigenschap van dit complex is dat het ons in staat stelt om de verloren ijkvrijheid terug te vinden, de mogelijkheid biedt om de ijkinvariante functies van de initiële ijktheorie (X_0, S_0) te bepalen, oftewel de elementen die in de natuurkundige literatuur bekend staan als de *observabelen* van de theorie. Inderdaad valt de cohomologiegroep van graad 0 van het cohomologiecomplex samen met de verzameling van observabelen van de initiële theorie na het vastleggen van de ijk:

$$H^0(\widetilde{X}, \delta_{B,\Psi}) = \{\text{Observabelen van de initiële ijktheorie } (X_0, S_0)\}.$$

De ontdekking van het bestaan van de BRST cohomologie voor ijktheorieën die uitgebreid zijn met spookvelden, maakte duidelijk dat de spookvelden, die oorspronkelijk als gereedschap voor het oplossen van het probleem van het definiëren en berekenen van padintegralen geïntroduceerd zijn, ook een belangrijke rol kunnen spelen als voortbrengers van een cohomologietheorie met natuurkundige relevantie, minstens voor vier-dimensionale theoriën.

Concluderend, de BRST constructie is een procedure die het probleem van oneindige termen in de padintegraal aanpakt, wanneer we een oneindig-dimensionale ijktheorie beschouwen: zelfs wanneer we gedwongen worden om de ijk vast te leggen, is het mogelijk om de ijkinvariantie van de theorie terug te vinden via de cohomologiegroepen van de cohomologietheorie, waarvan de voortbrengers de niet-fysische velden zijn die toegevoegd zijn aan de initiële theorie.

In dit proefschrift hebben we de constructie van de uitgebreide theorie $(\widetilde{X}, \widetilde{S})$ voor een bepaald soort eindig-dimensionale ijktheorieën met een $U(n)$-ijking, ook wel bekend als de *BV construcie*, in detail geanalyseerd. Ook hebben we de BRST cohomologiegroepen voor deze ijktheorieën in detail bestudeerd en vergeleken met een ander soort cohomologiecomplex (het *gegeneraliseerde Lie algebra cohomologiecomplex*). Dankzij deze vergelijking zijn we in staat een dubbele complexstructuur, onzichtbaar binnen het BRST-cohomologieformalisme, waar te nemen.

De reden voor het bestuderen van dit soort ijktheorieën is dat ze op natuurlijke wijze verschijnen zodra we nul-dimensionale niet-commutatieve variëteiten, één van de fundamentele gereedschappen binnen de hendendaagse niet-commutatieve meetkunde, beschouwen.

Dat er een sterke relatie tussen deze wiskundige theorie en ijktheorieën in de natuurkunde bestaat, was al duidelijk in de beginperiode van de niet-commutatieve meetkunde. De grootste prestatie in deze richting is zonder twijfel de beschrijving van het volledige Standaard Model in het kader van de niet-commutatieve meetkunde. Deze relatie tussen niet-commutatieve meetkunde en ijktheorieën moet echter niet toeschreven worden aan één specifiek geval ondanks haar betekenis binnen de natuurkunde. In werkelijkheid kunnen ijktheorieën op natuurlijke wijze afgeleid worden uit *spectraaltripels*, die het belangrijkste technische hulpmiddel binnen de hedendaagse niet-commutatieve meetkunde vormen.

Het is dus redelijk om te proberen andere procedures en technieken, die ontwikkeld zijn voor de analyse van ijktheorieën, binnen het kader van de niet-commutatieve meetkunde in te voeren. In het tweede deel van dit proefschrift zetten we de eerste stap in deze richting door de BV-aanpak van de BRST quantizatie van niet-abelse ijktheorieën binnen het kader van eindig-dimensionale spectraaltripels in te bouwen. Dat de niet-commutatieve meetkunde wellicht nieuwe inzichten biedt in de BRST quantizatieprocedure en de relatie tussen de initiële ijktheorie en haar BRST-comhomologiecomplex helpt te bepalen, vormt de drijvende kracht en de onderliggende hoop achter deze poging.

Zoals eerder gezegd is het werk in dit proefschrift slechts een eerste stap in deze richting, en ongetwijfeld hebben we nog een lange weg te gaan.

"*Echter, ook een reis van duizend mijl begint met een enkele stap.*"

(Laozi, Chinees filosoof, vijfde eeuw voor Christus)

Riassunto

Fra tutte le scienze, la matematica presenta una caratteristica unica, che la distingue da tutte le altre: quando un teorema o una proposizione vengono dimostrati essere veri, non vi è possibilità che un giorno vengano scoperti essere falsi. Nella matematica infatti non vi possono essere errori di misurazione e non vi sono esperimenti con i quali la teoria debba confrontarsi. Per questo motivo la matematica non è soggetta a trasformazioni o smentite, è autosufficiente e può solamente progredire, continuando a costruire sulle stesse basi che vennero fondate molti secoli addietro, ai tempi dell'antica Grecia, da matematici del calibro di Pitagora (570-495 a.C.) ed Euclide (367-283 a.C.). Quello che loro dimostrarono molto prima dell'avvento della calcolatrice e del computer è vero oggi come lo era all'epoca.

Lo stesso non lo si può però dire di altre scienze, una fra tutte la fisica. Essa infatti si basa su teorie, le quali possono essere suffragate o confutate dagli esperimenti e che, in ogni caso, non sono "vere" in assoluto ma vengono semplicemente accettate, fino a quando non vi sarà un esperimento a smentirle.

La ragione di questa profonda differenza fra fisica e matematica è che la fisica si è da sempre prefissata altri obiettivi: essa infatti ha come motivazione di base la ricerca di una spiegazione scientifica del mondo che ci circonda e del suo funzionamento. Nonostante l'Uomo sia sempre stato affascinato dalla Natura e dai suoi meccanismi e abbia sempre tentato di fornire spiegazioni per i fenomeni che osservava, tuttavia la nascita della fisica va ricercata in un passato non molto lontano da noi. Infatti è solo con Galileo Galilei (1564-1642) e Isaac Newton (1642-1727) che nasce quella che possiamo definire la Fisica Classica. Più precisamente, è con l'incontro fra l'osservazione dei fenomeni naturali e il formalismo matematico che nasce la fisica così come la conosciamo oggi.

"*La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.*"

(Galileo Galilei, il Saggiatore, Cap. VI).

Ma quindi cos'è quella che chiamiamo Fisica Classica? Classicamente un *sistema fisico* è composto dai seguenti elementi:

- il *tempo*;
- lo *spazio*;
- gli *osservabili*, cioè le quantità che possiamo calcolare, come per esempio la velocità o la posizione di ciò che vogliamo descrivere, siano essi corpi o particelle;
- le *forze*, quali per esempio la forza di attrazione gravitazionale oppure la forza di interazione elettrica;
- le *simmetrie*, le quali possono riguardare lo spazio oppure la forza considerata.

Quindi, classicamente potremmo dire che una *teoria fisica* non è altro che un insieme di equazioni usate per predire l'evoluzione temporale di un sistema fisico. Questa era l'idea di fisica che si aveva ai tempi di Galileo e di Newton. D'altra parte, con la scoperta della Meccanica Quantistica all'inizio del XX secolo, questa nozione classica di teoria fisica ha dovuto essere modificata: i fisici furono constretti a rinunciare definitivamente al sogno di costruire una teoria deterministica per descrivere i fenomeni che avvengono a livello subatomico e hanno dovuto accettare l'idea che il comportamento della Natura a livello delle particelle potesse essere conosciuto solo in termini di probabilità. Venne così introdotto il concetto di *quantizzazione*: ogni teoria fisica, per poter descrivere la natura a livello subatomico, ha bisogno di essere quantizzata.

Negli anni molti fisici, come per esempio Dirac, Feynman, Heisenberg, Pauli, Planck e Schrödinger, studiarono il problema della quantizzazione delle teorie classiche, arrivando a scoprire varie procedure di quantizzazione. Una di esse si basa sulla nozione di *integrale sui cammini*.

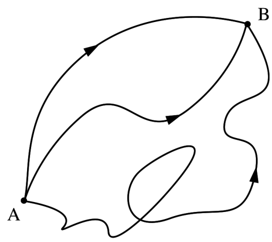


Figura D.2: L'*integrale sui cammini* permette di calcolare la probabilità che una particella posta nel punto A possa raggiungere il punto B dopo un certo lasso di tempo. Per fare questo, tale integrale prende in considerazione tutte le traiettorie da A a B che la particella potrebbe seguire. Questo spiega il nome con cui tale integrale è conosciuto.

Come accennato, l'idea fondamentale alla base della Meccanica Quantistica è che, quando si considerano particelle atomiche o subatomiche non è possibile determinare con esattezza la traiettoria che esse seguiranno: data una particella posta nel punto A in Figura D.2, l'unica cosa che potremo stabilire è quale sia la probabilità che, dopo un certo lasso di tempo, questa particella si trovi nel punto B. Il concetto di integrale sui cammini è nato proprio per rispondere a questa domanda. L'idea è la seguente: se vogliamo stabilire quale sia la probabilità di ritrovare la nostra particella nel punto B si può semplicemente valutare quale sia la probabilità che tale particella percorra ognuno dei cammini che portano da A a B e quindi sommare tutte le probabilità così ottenute.

Questa ovviamente è l'intuizione che fu alla base della scoperta del concetto di integrale sui cammini. La nozione di integrale sui cammini venne introdotta da Feynman nel 1948 e per questo tale integrale è spesso indicato anche con il nome di *integrale di Feynman.* L'integrale di Feynman è uno strumento davvero molto importante nello studio della teoria di campi quantistici, principalmente a causa delle sue seguenti due fondamentali proprietà:

- esso fornisce previsioni estremamente accurate dei risultati ottenuti tramite gli esperimenti;

- esistono regole precise e formule che consentono di calcolare questo tipo di integrali.

Occorre però menzionare il fatto che l'integrale di Feynman presenta anche un difetto: non è ben definito da un punto di vista matematico. Infatti, quando si prova a quantizzare una teoria classica utilizzando gli integrali di Feynman, ci si trova a dover calcolare integrali di questo tipo:

$$\langle g \rangle = \int_{X_0} g e^{-S_0} [d\mu],$$

dove, in un certo senso, abbiamo a sinistra quanto può essere calcolato tramite un esperimento, mentre a destra si trova la teoria fisica che stiamo considerando. Infatti:

- $\langle g \rangle$ è l'*expectation value* del funzionale g e rappresenta la quantità che possiamo determinare tramite gli esperimenti;
- X_0, noto con il nome di *spazio delle configurazioni*, contiene informazioni legate allo spazio e ai campi presenti nella teoria;
- S_0 è l'*azione*, la quale contiene le informazioni sulle forze alle quali è soggetto il sistema fisico che stiamo analizzando.

Come detto però la definizione di questo integrale in generale non è ben posta da un punto di vista matematico. Il problema risiede nel fatto che la misura $d\mu$, nel caso di uno spazio delle configurazioni X_0 infinito-dimensionale, in generale non può essere ben definita.

D'altra parte, anche se ci limitassimo a considerare solo teorie finito-dimensionali, altri problemi sorgerebbero, questa volta legati all'eventuale presenza di simmetrie nel sistema fisico considerato. Le teorie dotate di simmetrie sono note anche con il nome di *teorie di gauge*. L'importanza di avere una formulazione precisa di una procedura per quantizzare teorie di gauge deriva dal fatto che tutte le interazioni fondamentali che compaiono in Natura sono governate da teorie di gauge.

Il problema che si viene a creare nel caso di teorie di gauge è che l'integrale di Feynman risulta essere degenere: quando integriamo nella direzione di gauge, invece di ottenere un expectation value finito, come ci aspetteremmo, otteniamo invece un valore infinito. Occorrebbe quindi fissare il gauge, eliminando così le variabili in eccessio. D'altra parte, così facendo la teoria perderebbe la sua invarianza di gauge e vi sarebbe meno controllo su quello che è il significato fisico dell'integrale sui cammini che stiamo calcolando.

Una soluzione a questo problema venne trovata da Faddeev e Popov nel 1967. Essi suggerirono di introdurre nella teoria dei nuovi campi da usare per eliminare le simmetrie locali ed essere così in grado di calcolare l'integrale sui cammini. Qualche volta infatti, per semplificare il calcolo di un integrale, può essere utile allargare il dominio ed aggiungere delle variabili extra, come si fa per esempio per il calcolo dell'integrale Gaussiano:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dx\ dy} = \sqrt{\pi}.$$

Nel nostro caso si procede in modo analogo: data una teoria di gauge iniziale (X_0, S_0), vengono inseriti dei campi extra, usati per estendere sia lo spazio delle configurazioni X_0 che l'azione S_0, ottenendo cosí una *teoria estesa* $(\widetilde{X}, \widetilde{S})$. Quindi, invece di considerare l'integrale sui cammini per la teoria di gauge iniziale (X_0, S_0), consideriamo l'integrale dato dalla teoria estesa $(\widetilde{X}, \widetilde{S})$:

$$\int_{X_0} ge^{-S_0}[d\mu] \rightsquigarrow \int_{\widetilde{X}} ge^{-\widetilde{S}}[d\nu].$$

L'idea é quindi quella di inserire nella teoria di gauge iniziale dei campi extra: tali campi, dal momento che non sono campi fisici, vengono anche detti *campi fantasma*.
Per riassumere quindi, per poter quantizzare una teoria di gauge (X_0, S_0) usando il metodo dato dall'integrale di Feynman, vogliamo costruire una teoria estesa $(\widetilde{X}, \widetilde{S})$ dove:

$$\begin{aligned} X_0 &\rightsquigarrow \widetilde{X} = X_0 + \{\text{anticampi, campi fantasma e anticampi fantasma}\} \\ S_0 &\rightsquigarrow \widetilde{S} = S_0 + \text{termini che coinvolgono gli anticampi e i campi fantasma.} \end{aligned}$$

Come detto, questi campi fantasma vennero inizialmente introdotti per risolvere la degenerazione dell'integrale sui cammini in presenza di simmetrie di gauge. D'altra parte, nel 1975, Becchi, Rouet, Stora e, indipendentemente, Tyutin scoprirono che questi campi fantasma svolgevano un interessante ruolo da un punto di vista matematico: essi infatti potevano essere visti come i generatori di un complesso di coomologia, noto con il nome di *complesso BRST*. La caratteristica fondamentale di tale complesso è quella di consentire di recuperare la simmetria di gauge persa, permettendo di determinare le funzioni invarianti sotto trasformazioni di gauge della teoria iniziale (X_0, S_0), cioè gli elementi che in fisica sono detti gli *osservabili* della teoria. Infatti, dopo aver fissato il gauge, si ha che il gruppo di coomologia di grado 0 del complesso di coomologia così ottenuto coincide con l'insieme degli osservabili della teoria iniziale:

$$H^0(\widetilde{X}, \delta_{B,\Psi}) = \{\text{Osservabili della teoria di gauge iniziale } (X_0, S_0)\}.$$

La scoperta dell'esistenza della coomologia BRST per teorie di gauge ha reso evidente come questi campi fantasma, inizialmente introdotti come strumento per risolvere il problema di definire e calcolare gli integrali di cammino per teorie di gauge, giochino un ruolo significativo come generatori di un complesso di coomologia il quale presenta rilevanza fisica, almeno nel caso di teorie 4-dimensionali.
In conclusione quindi la costruzione BRST è una procedura che permette di risolvere il problema di avere termini infiniti all'interno dell'integrale sui cammini quando consideriamo teorie di gauge infinito-dimensionali: pur essendo costretti a fissare un gauge per la teoria, ugualmente riusciamo a riottenere la simmetria di gauge tramite i gruppi di coomologia i cui generatori sono dei campi non fisici che vengono aggiunti alla teoria iniziale.

In questa tesi abbiamo analizzato in dettaglio come effettuare la costruzione della teoria estesa $(\widetilde{X}, \widetilde{S})$, nota anche con il nome di *costruzione* BV, per un particolare tipo di teorie di gauge finito-dimensionali con gruppo di gauge $U(2)$. Inoltre, abbiamo anche analizzato in dettaglio i gruppi di coomologia BRST per tali teorie di gauge, mettendoli in relazione con un altro tipo di coomologia (*la coomologia generalizzata di algebre di Lie*). Dal confronto con quest'altro tipo di complesso di coomologia siamo stati in grado di evidenziare una struttura di doppio complesso, la quale non era visibile a livello della coomologia BRST.

La motivazione che ci ha spinti ad analizzare questo tipo di teorie di gauge è che tali teorie compaiono in modo del tutto naturale quando consideriamo varietà non commutative 0-dimensionali, le quali rappresentano uno degli strumenti fondamentali della geometria non commutativa contemporanea.

Fin da quando la geometria non commutativa venne scoperta, è sempre stata chiara la presenza di una forte relazione fra questa teoria matematica e le teorie di gauge in fisica. Senza dubbio il più grande traguardo raggiunto in questa direzione è stato quello di riuscire a descrivere l'intero Modello Standard nel contesto della geometria non commutativa. D'altra parte la connessione esistente fra geometria non commutativa e teorie di gauge non deve essere attribuita solo a un caso specifico, nonostante la sua rilevanza in ambito fisico.

Infatti, la stretta connessione fra geometria non commutativa e teorie di gauge risiede nel fatto che teorie di gauge vengono naturalmente indotte da *triple spettrali*, le quali sono un oggetto fondamentale nello studio della geometria non commutativa contemporanea.

Alla luce della presenza di questo stretto legame fra geometria non commutativa e teorie di gauge, risulta quindi ragionevole il voler provare ad inserire nel contesto matematico dato dalla geometria non commutativa anche altre procedure e tecniche sviluppate per l'analisi delle teorie di gauge. Per questo motivo, nella seconda parte della tesi, abbiamo provato a compiere un primo passo in questa direzione, provando ad inserire la costruzione BV per la quantizzazione BRST di teorie di gauge non-abeliane nel contesto dato dalla geometria non commutativa, tramite l'utilizzo di triple spettrali finito-dimensionali. La ragione che ci ha spinto a precorrere questa strada è stata la speranza che la geometria non commutativa potesse fornire un nuovo punto di vista sulla procedura di quantizzazione BRST, permettendo di comprendere meglio le relazioni fra la teoria di gauge iniziale e il corrispondente complesso di coomologia BRST.

Come detto, quanto presente in questa tesi non è che un primo passo in questa direzione e sicuramente molta altra strada deve ancora essere percorsa.

"*Ma anche un viaggio di mille miglia inizia con un passo*".

(Laozi, filosofo cinese, V sec. a.C.)

Acknowledgements

Now that I have arrived at the end of this journey, I have fully realized how much this experience has changed me. In these few lines, I really would like to try to thank all the persons that have helped me, everyone in his own specific way, to reach this important goal.
Since now, I apologize for not explicitly mention everybody but I will always have warm memories of you all.

I would like to start thanking who made all this possible: I owe heartfelt thanks to my supervisors Walter D. van Suijlekom and Klaas Landsman, who first believed in me and gave me the possibility of starting this wonderful experience that has been my Ph.D. Thanks for everything you have taught me over these years, for your helpfulness, patience and kindness.

Many thanks also to Matilde Marcolli, not only for her hospitality during my visit at the California Institute of Technology but also for her time, for many inspiring talks and for having shared with me ideas and intuitions that I hope I will have the possibility of developing in future.
I would also like to thank Gianni Landi, for his hospitality and kindness during the months I spent in Trieste, at the final stages of writing this thesis. Many thanks for many inspiring talks and for having shown me an interesting point of view to look to my project.

Gratitude and utmost respect are owed to the members of the reading committee, for their patience and for the time they spent on my work. In particular, I owe a debt of gratitude to Prof. Felder, for his kindness when some years ago he gave me the draft of his article, offering me the possibility to work on my project from a very useful and fruitful point of view. Moreover, I would like to thank him for his accurate and precise remarks that helped in improving the final version of this thesis.

I really would like to thank all the persons that I have met throughout these years during conferences and workshops, who listened with patience and interest when I explained my project: your questions and remarks helped me in taking the steps up to this point.

Heartfelt thanks also to all the persons that, maybe in a less direct but anyhow essential way, helped me in achieving such an important goal: my friends and my family. I think that the first that have to be mentioned are my friends from university: with you I discovered what is my dream and it has been with you that I started fulfilling it. Thanks a lot for your encouragement and friendship. I can always count on you, independently of the number of kilometers or oceans that divide us.
I really would like to warmly thank all the Ph.D. students, actual and former, for the kindness with which you welcomed me when I first arrived here. Thanks for many amusing moments together, for the concerts and the trips, for the dinners and for having taught me how to properly celebrate Sinterklaas, thanks to whom helped me moving or watering my plants when I was away, and of course thanks to whom helped me with the translation of the summary in this thesis! I will keep a wonderful memory of all of you, with the hope there will be other occasions to have again many adventures together!
A special thanks to Geert, for many nice lunches together and for his support, I will never forget it!
Many thanks to all my friends in Milan for the happy and unforgettable moments we had together and for having made me always feeling your friendship each time I came back.

I really would like to find the words to thank my family for all they have been doing for me during these years but I am afraid it would not be possible within few lines. Thus I simply say thanks for your love and for the firm confidence you have always had in my capabilities. Thanks for having supported me and for always being there when I needed.
In the end, I would like to thank Francesco, who during a warm afternoon in the charming Rome, pushed me into taking on the challenge of a Ph.D. abroad, deciding to postpone his projects in order to give me the possibility of making my dreams come true.

Curriculum Vitae

Roberta Anna Iseppi was born on 16th November 1986 in Milan (Italy).
In the period 2000-2005, she attended the high school "Liceo Classico Statale Giosuè Carducci"(Milan, Italy), obtaining her high school diploma in July 2005, with the mark 100/100. During her high school studies she had the opportunity to participate to two international competitions of Latin language translation: the XVIII edition of Certamen Horatianum, Venosa (Potenza, Italy) and the XXV edition of Certamen Ciceronianum, Arpino (Roma, Italy).

In October 2005 she started a Bachelor Degree course in Mathematics at the University of Milan, obtaining her Bachelor's Degree in October 2008, with the mark 110/110, discussing the thesis "Cremona transformations".
From October 2008 until July 2010 she attended a Master Degree course in Mathematics at the University of Milan. In July 2010 she obtained her Master's Degree with the mark 110/110 cum laude, discussing the thesis "F-theory and elliptic fibrations", under the supervision of Prof. dr. Bert van Geemen and Dr. S. Cacciatori.

On September 2010 she was appointed as Ph.D. Student at the Radboud Universiteit Nijmegen (The Netherlands), where she has been working on non-commutative geometry and its application to quantum field theory, under the supervision of Dr. Walter D. van Suijlekom and Prof. dr. Klaas Landsman.
During her Ph.D. studies, Roberta had the opportunity to take part to many conferences and workshops, which gave her the possibility of traveling to several cities, among all Bangkok, Luxembourg city, Paris, Berlin.
Moreover, in the period January-March 2014, she had been invited to participate in the Visiting Student Researcher program under the Mathematics option at the California Institute of Technology (United States), supervised by Prof. dr. Matilde Marcolli.

In the period September-October 2014, she was offered the opportunity to participate to the Trimester Program: "Non-commutative Geometry and its Applications", hosted by the Hausdorff Center for Mathematics, in Bonn (Germany).
At the final stages of the writing of this thesis, from February until June 2015, Roberta has been hosted by Prof. dr. Giovanni Landi, at the University of Trieste (Italy).

In occasion of the "50th Dutch Mathematical Congress", April 2014, Roberta has been awarded of the Philips Mathematics Prize.
Currently, she is working as postdoc at the Max Planck Institute for Mathematics in Bonn (Germany).