Master's thesis Mathematics

Real dimensional spaces in noncommutative geometry

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Abstract

Given $z \in (0, \infty)$ Connes and Marcolli showed in [9] that it is possible to construct a space of dimension z. Such spaces are given by semifinite spectral triples, a generalisation of spectral triples which in turn are a generalisation of manifolds. We will investigate these z-dimensional spaces and give applications in physics to dimensional regularisation and zeta regularisation. For these applications the product of semifinite spectral triples is necessary. We will prove in the general case that the product of two semifinite spectral triples again yields a semifinite triple. Also we will establish that these products preserve finite summability and regularity.

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Introduction

As the title suggests, this thesis deals with z-dimensional spaces where z is a positive real number. At first thought it might seem absurd that it is possible for spaces to have a dimension which is not an integer, because the spaces one comes across in linear algebra or differential geometry all have integer dimension. But people who are acquainted with fractals or with noncommutative geometry know that it is possible to define a notion of dimension which extends the dimension with whom everybody is familiar and which attains noninteger values. The framework which we shall use to define these spaces is noncommutative geometry [6]. Geometric spaces in noncommutative geometry are different from spaces in for example linear algebra, topology or differential geometry. In noncommutative geometry spaces are not given by a collection of points, but they are described by a spectral triple: a tuple consisting of an algebra, a Hilbert space and a self-adjoint operator. The motivation to look at such spectral triples comes from the Gelfand duality [16] which relates topology and operator algebras. This duality consists of an isomorphism between compact Hausdorff topological spaces and commutative unital C^*-algebras. For manifolds there exists an analogous construction, namely it is possible to construct a spectral triple. If you start with a manifold, the algebra of the triple is commutative. The converse is also true if the algebra of the triple is commutative and the triple satisfies some other technical conditions, then the triple corresponds to a compact manifold [7]. Therefore generalising from commutative algebras to noncommutative algebras leads to a new kind of geometry: Noncommutative Geometry. This is precisely the geometry we are interested in and with the tools from noncommutative geometry we are able to actually define the spaces we are looking for.

Before I started with this project I had no background in noncommutative geometry. When I first heard about noncommutative geometry and in particular about this topic, I became interested in it. I wanted to know how this machinery works and that it is actually useful in applications. I spent more than half a year on this project and I enjoyed it. I hope this text will give the reader insight in noncommutative geometry and in particular in these z-dimensional spaces. But also I hope that this text convinces the reader that such spaces are more than abstract nonsense and that they actually have useful applications.

This thesis roughly consists of three parts. We start with reviewing the basics from functional analysis, we will show how noncommutative geometry is a generalisation of differential geometry and give an overview of the concepts of noncommutative geometry which will be used in the rest of the thesis. In particular we will explain how the notion of dimension and the smooth structure of a manifold can be extended to spectral triples: the noncommutative manifolds. In the second part we will consider the generalisation of spectral triples to semifinite spectral triples. An important tool which will be needed when we look at the applications is the product of semifinite spectral triples. When we have all the prerequisites we will eventually construct the spaces of real dimension, this construction is based on the work of Connes and Marcolli in [9]. We will investigate these spaces in detail and actually prove that they are z-dimensional. In the last part we will give applications in physics of these spaces. We will describe how the spaces can be used in dimensional regularisation [21] and zeta function regularisation [20]. These are procedures which are commonly used in physics to deal with divergent integrals.

This text is aimed at master students in mathematics who have a good background in functional analysis. The results from functional analysis which will be used are stated in the preliminaries, but familiarity with it is convenient. Knowledge of noncommutative geometry is not necessary, that theory will gradually be recalled.

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1 Preliminaries

In this section we will give a concise overview of the mathematical tools we will use during this thesis and we will also introduce the notion which will be used. Most results will be stated without proof, but we will give references. We start with the basics from measure theory. We continue with functional analysis and finally we will combine the theory of traces on von Neumann algebras with measure theory.

1.1 Measure theory

Now we will recall some notions from measure theory which we will need later on. The books [3, 4] give together a very good overview of measure theory.

Definition 1.1. Suppose X is a set. A *σ-algebra* on X is $\Sigma \subset \mathcal{P}(X)$ is a class of subsets of X, which satisfies the properties

1. $X, \emptyset \in \Sigma$;
2. if $A, B \in \Sigma$, then $A \cap B,\ A \cup B,\ A \setminus B \in X$;
3. if $(A_n)_{n\in\mathbb{N}} \subset \Sigma$, then $\cup_{n\in\mathbb{N}} A_n \in \Sigma$,

If Σ is a σ-algebra on a set X, we call (X, Σ) a *measurable space*. If (X, τ) is a topological space, the smallest σ-algebra Σ which contains τ is called the *Borel σ-algebra* of X, we denote this σ-algebra by $\mathcal{B}(X)$ (if it is clear which topology is involved).

Definition 1.2. Suppose Σ is a σ-algebra on X. We call $f : X \to \mathbb{C}$ a *simple function* if there exists $A_1, \dots, A_n \in \Sigma$ and $a_1, \dots, a_n \in \mathbb{C}$, such that $f = \sum_{j=1}^{n} a_n 1_{A_j}$. Note that we can always select the $A_1, \dots, A_n$ as disjoint sets.
We call f *measurable* if $f^{-1}(U) \in \Sigma$ for all $U \in \mathcal{B}(\mathbb{C})$. More generally for measurable spaces (X_1, Σ_1) and (X_2, Σ_2) we call $f : X_1 \to X_2$ a *measurable function* if $f^{-1}(B) \in \Sigma_1$ for all $B \in \Sigma_2$. We call f *Borel measurable* if it is measurable with respect to the Borel σ-algebras on X_1 and X_2.

Theorem 1.3. [3, Thm. 2.1.5] *Suppose f, g, f_n for $n \in \mathbb{N}$ are measurable functions with respect to (X, Σ) and suppose c, d are scalars. Then*

(i) if $\varphi : \mathbb{C} \to \mathbb{C}$ is Borel-measurable, then $\varphi \circ f$ is Σ-measurable;

(ii) the functions $cf + dg$ and fg are measurable with respect to Σ;

(iii) if $g(x) \neq 0$ for all x, then the function f/g is measurable with respect to Σ;

(iv) if for all x there exists a finite limit $f_0(x) := \lim_{n\to\infty} f_n(x)$, then the function f_0 is measurable with respect to Σ;

(v) if the functions $\sup_n f_n(x)$ and $\inf_n f_n(x)$ are finite for all x, then they are measurable with respect to Σ.

Definition 1.4. Suppose (X, Σ) is a measurable space. A *measure* on (X, Σ) is a map $\mu : \Sigma \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$ for each disjoint sequence $(E_n)_n \subset \Sigma$. In this case (X, Σ, μ) is called a *measure space*.
We call μ a *finite measure* or (X, Σ, μ) a *finite measure space* if $\mu(X) < \infty$ and we call μ *σ-finite* if there exist a sequence $(E_n)_n \subset \Sigma$ such that $\mu(E_n) < \infty$ for all n and $\bigcup_n E_n = X$.

Remark 1.5. From a measure one can construct an integral, we will describe that procedure here. Suppose (X, Σ, μ) is a measure space. For a simple function $f = \sum_{n=1}^{N} a_n 1_{A_n}$ we define $\int_X f\, d\mu := \sum_{n=1}^{N} a_n \mu(A_n)$. One can check that this definition is independent of the chosen representation of f.
Suppose $f : X \to [0, \infty)$ is a measurable function, then we define

$$\int_X f\, d\mu := \sup\Big\{ \int_X g\, d\mu \ :\ g \leq f \text{ and } g \text{ is simple}\Big\}.$$

Now if f is measurable, we can split $f = f^+ - f^-$ where the function f^+, f^- are nonnegative. If $\int_X f^+ \, d\mu < \infty$ or $\int_X f^- \, d\mu < \infty$, we define

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

If both $\int_X f^+ \, d\mu < \infty$ and $\int_X f^- \, d\mu < \infty$, we call f *integrable*. We denote the integrable functions by $\mathcal{L}^1(X, \Sigma, \mu)$ or sometimes simply by $\mathcal{L}^1(X)$ if no confusion is possible. We also define for $1 < p < \infty$ the space

$$\mathcal{L}^p(X, \Sigma, \mu) := \{f : X \to \mathbb{C} \, : \, f \text{ is } \Sigma\text{-measurable}, \int_X |f|^p \, d\mu < \infty\}.$$

We will deal with the case $p = \infty$ later.

Definition 1.6. Note that if f, g are functions on X and $\mu(\{x \in X \, : \, f(x) \neq g(x)\}) = 0$, then $\int_X f \, d\mu = \int_X g \, d\mu$. Let $\mathcal{N}(X, \Sigma, \mu) := \{f \, : \, f \text{ is measurable } \int_X f \, d\mu = 0\}$. Define the quotient spaces

$$L^p(X, \Sigma, \mu) := \mathcal{L}^p(X, \Sigma, \mu)/\mathcal{N}.$$

On such a space there is a norm $\| \cdot \|_p : L^p(X) \to \mathbb{C}$, $\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}$. The case when $p = \infty$ is a little more tricky. By definition $f \in \mathcal{L}^\infty(X, \Sigma, \mu)$ if there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $f 1_{X \setminus N}$ is a bounded function.

$$\|f\|_\infty := \inf \left\{ \sup_{x \in X} |f 1_{X \setminus N}(x)| \, : \, N \in \Sigma, \, \mu(N) = 0 \right\}$$

The quantity $\|f\|_\infty$ is the essential supremum of $|f|$. Again

$$L^\infty(X, \Sigma, \mu) := \mathcal{L}^\infty(X, \Sigma, \mu)/\mathcal{N}.$$

We will not distinguish between $\mathcal{L}^p(X)$ and $L^p(X)$, although in the second space we are working with equivalence classes instead of functions. We say a property P holds almost everywhere if there exists a set N such that $\mu(N) = 0$ and P holds on $X \setminus N$. For example $f = g$ almost everywhere if there exists N such that $\mu(N) = 0$ and $f(x) = g(x)$ for all $x \notin N$.

Proposition 1.7. *The vector space $(L^p(X), \| \cdot \|_p)$ is a Banach space. If $p = 2$, it is a Hilbert-space (cf. Definition 1.14) with inner product given by $\langle f, g \rangle := \int_X f \overline{g} \, d\mu$.*
The following inequality holds (it is known as Hölder's inequality*). If $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(X)$ and $g \in L^q(X)$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Theorem 1.8 (Monotone convergence theorem)**.** . *Let $(f_n)_n$ be a sequence of measurable functions such that $f_n \geq 0$ for all n and $f_n \leq f_{n+1}$ almost everywhere, then*

$$\limsup_{n \to \infty} \int_X f \, d\mu = \int_X \limsup_{n \to \infty} f_n \, d\mu \leq \infty.$$

Theorem 1.9 (Dominated convergence theorem)**.** *Let $(f_n)_n \subset \mathcal{L}^1(X)$ and suppose there exists $h \in \mathcal{L}^1(X)$ such that $|f_n| \leq h$ almost everywhere and $f := \lim_{n \to \infty} f_n$ exists almost everywhere, then*

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Theorem 1.10 (Fubini's theorem)**.** *Suppose for $i = 1, 2$ the tuples (X_i, Σ_i, μ_i) are measure spaces. Let $X := X_1 \times X_2$ and Σ be the σ-algebra generated by $\Sigma_1 \times \Sigma_2$. Then there exists a unique measure μ on Σ such that $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all $A_i \in \Sigma_i$. Furthermore if f is a non-negative Σ-measurable function, then*

$$\int_{X_1} \int_{X_2} f \, d\mu_2 \, d\mu_1 = \int_{X_2} \int_{X_1} f \, d\mu_1 \, d\mu_2 = \int_X f \, d\mu \leq \infty. \tag{1.1}$$

If $f \in \mathcal{L}^1(X, \Sigma, \mu)$ then (1.1) *also holds for f and $\int_X f \, d\mu \in \mathbb{R}$.*

1.2 Functional analysis

In this subsection we will start with Hilbert spaces and bounded operators on such spaces. Then we will review the unbounded operators including the spectral theorems and functional calculus and we will finish with von Neumann algebras. We will follow the books [11], [26], [30].

Notation 1.11. Let V be a vector space and $S \subset V$ a subset. Denote by

$$\operatorname{span}(S) := \Big\{ \sum_{i=1}^{n} \lambda_i s_i \, : \, n \in \mathbb{N},\, \lambda_i \in \mathbb{C},\, s_i \in S \Big\}$$

the *linear span* of S.

Definition 1.12. Let E be a vector space. $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ is called an *inner product* if

1. $x \mapsto \langle x, a \rangle$ is linear for all $a \in E$;
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in E$;
3. $\langle x, x \rangle \in [0, \infty)$ for all $x \in E$.
4. $\langle x, x \rangle = 0$ if and only if $x = 0$.

We call E an *inner product space* if there exists an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ on E. Note that we adopt the convention from mathematics: the inner product is linear in the first entry and conjugate linear in the second entry.

Remark 1.13. We can define a norm on an inner product space by $\|x\| := \sqrt{\langle x, x \rangle}$. So in particular inner product spaces are normed vector spaces.

Definition 1.14. A *Hilbert space* is an inner product space which is complete relative to the topology induced by its inner product norm. We usually denote a Hilbert space by $\mathcal{H}$ or $\mathcal{K}$. If we want to be more specific we use $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ to denote the inner product.

We know from linear algebra that finite dimensional vector spaces always have an (orthonormal) basis. We can also define a basis for Hilbert spaces.

Definition 1.15. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. We call a subset $\{e_j : j \in J\} \subset \mathcal{H}$ *orthonormal* if $\langle e_i, e_j \rangle = \delta_{i,j}$ for all $i, j \in J$. Furthermore if the linear span $\operatorname{span}(\{e_j : j \in J\})$ is dense in $\mathcal{H}$, we call the set $\{e_j : j \in J\}$ an *orthonormal basis*. If $\mathcal{H}$ admits a countable orthonormal basis we call this space *separable*.

Lemma 1.16. *Every Hilbert space $\mathcal{H}$ admits an orthonormal basis and if a set $\mathcal{E} \subset \mathcal{H}$ is orthonormal then there exists an orthonormal basis $\mathcal{E}' \supset \mathcal{E}$.*

Proof. The first statement follows from the second one by picking a vector $h \in \mathcal{H}$ of norm 1 and extending $\{h\}$ to an orthonormal basis. The second statement can be proved using Zorn's lemma. ⊠

The inner product behaves as expected with orthonormal bases.

Lemma 1.17. *If $\mathcal{H}$ is a separable Hilbert space with orthonormal basis $(e_n)_n$, $h \in \mathcal{H}$ and $(c_n)_n \subset \mathbb{C}$ such that $\sum_n |c_n|^2 < \infty$, then $\sum_{n=0}^{\infty} c_n e_n \in \mathcal{H}$ and the equality*

$$\sum_{n=0}^{\infty} c_n \langle e_n, h \rangle = \Big\langle \sum_{n=0}^{\infty} c_n e_n, h \Big\rangle$$

holds.

Proof. From the assumptions it follows $\lim_{N\to\infty}\sum_{n=N}^{\infty}|c_n|^2 = 0$ and $\lim_{N\to\infty}\|\sum_{n=N}^{\infty}c_ne_n\| = 0$. For $N\in\mathbb{N}$ it holds

$$\begin{aligned}\Big|\langle\sum_{n=0}^{\infty}c_ne_n,h\rangle - \sum_{n=0}^{\infty}c_n\langle e_n,h\rangle\Big| &= \Big|\langle\sum_{n=N}^{\infty}c_ne_n,h\rangle - \sum_{n=N}^{\infty}c_n\langle e_n,h\rangle\Big|\\ &\leq \Big|\langle\sum_{n=N}^{\infty}c_ne_n,h\rangle\Big| + \Big|\sum_{n=N}^{\infty}c_n\langle e_n,h\rangle\Big|\\ &\leq \Big\|\sum_{n=N}^{\infty}c_ne_n\Big\|\|h\| + \Big(\sum_{n=N}^{\infty}|c_n|^2\Big)^{-\frac{1}{2}}\Big(\sum_{n=N}^{\infty}|\langle e_n,h\rangle|^2\Big)^{-\frac{1}{2}}\\ &\leq \|h\|\Big(\Big\|\sum_{n=N}^{\infty}c_ne_n\Big\| + \Big(\sum_{n=N}^{\infty}|c_n|^2\Big)^{-\frac{1}{2}}\Big),\end{aligned}$$

which tends to 0 as $N\to\infty$. ⊠

Now we will review operators between Hilbert spaces, usually we will only consider operators on a single Hilbert space. We start with the bounded operators because they are a lot easier to work with, later we will continue with the unbounded ones. In this section $\mathcal{H}$ and $\mathcal{K}$ will denote Hilbert spaces.

Definition 1.18. A linear map $T:\mathcal{H}\to\mathcal{K}$ is a *bounded operator* if *the norm* of T

$$\|T\| := \sup\{\|Th\| \,:\, h\in\mathcal{H},\ \|h\|=1\} < \infty.$$

By $B(\mathcal{H},\mathcal{K})$ we denote the bounded operators from $\mathcal{H}$ in $\mathcal{K}$. We use the shorthand notation $B(\mathcal{H}) := B(\mathcal{H},\mathcal{H})$.

Proposition 1.19. [11, Prop. II.2.1] *If $T:\mathcal{H}\to\mathcal{K}$ is a linear map, then the following are equivalent*

(i) $T\in B(\mathcal{H},\mathcal{K})$;

(ii) T is continuous.

Definition 1.20. We call $T\in B(\mathcal{H},\mathcal{K})$ invertible if there exists $T^{-1}\in B(\mathcal{K},\mathcal{H})$ such that $TT^{-1} = I_{\mathcal{K}}$ and $T^{-1}T = I_{\mathcal{H}}$. The *resolvent set* of T is the set $\rho(T) := \{\lambda\in\mathbb{C} : T-\lambda \text{ is invertible}\}$. The *spectrum* of T equals $\sigma(T) := \mathbb{C}\setminus\rho(T)$. The function

$$R:\rho(T)\to B(\mathcal{H}),\qquad \lambda\mapsto(T-\lambda)^{-1}$$

is called the *resolvent* of T. We define the spectral radius of T by $r(T) := \sup\{|\lambda| : \lambda\in\sigma(T)\}$.

Theorem 1.21. [27, Thm. VI.5, VI.6] *If $T\in B(\mathcal{H})$, then $\sigma(T)$ is compact and nonempty, $r(T) = \lim_{n\to\infty}\|T^n\|^{1/n}$ and the* resolvent identity

$$(T-\lambda)^{-1} - (T-\mu)^{-1} = (\mu-\lambda)(T-\mu)^{-1}(T-\lambda)^{-1}\qquad \textit{for } \lambda,\mu\in\rho(T)$$

holds. In particular $(T-\lambda)^{-1}$ and $(T-\mu)^{-1}$ commute for $\lambda,\mu\in\rho(T)$.

Theorem 1.22. [26, Thm. 3.2.3] *For every $T\in B(\mathcal{H},\mathcal{K})$ there exists an operator $T^*\in B(\mathcal{K},\mathcal{H})$ such that*

$$\langle Th,k\rangle = \langle h,T^*k\rangle\qquad \textit{for all } h\in\mathcal{H},\ k\in\mathcal{K}.$$

Definition 1.23. We call the operator T^* corresponding to T obtained from Theorem 1.22 the *adjoint* of T. If $T\in B(\mathcal{H})$ and $T=T^*$, then T is called *self-adjoint.* If $TT^* = T^*T$, then T is called *normal.*

Self-adjoint and normal operators play a very important role, because for these operators one can construct a very import tool, the so called functional calculus. We will tell more about that later. First some elementary results of the definition.

Theorem 1.24. [26, Thm. 3.2.3] *If $T \in B(\mathcal{H})$, then $\|T\| = \|T^*\| = \|TT^*\|^{1/2}$ (this implies that $B(\mathcal{H})$ is a C^*-algebra, but more about that later). The map $T \mapsto T^*$ is conjugate linear, $(T^*)^* = T$, $(TS)^* = S^*T^*$ and if T is invertible then $(T^*)^{-1} = (T^{-1})^*$*

Definition 1.25. An operator $T \in B(\mathcal{H}, \mathcal{K})$ is called a *finite rank operator* if the space $T(\mathcal{H}) \subset \mathcal{K}$ is finite dimensional.

Theorem 1.26. [11, Thm. II.4.4] *Let $T \in B(\mathcal{H})$. Denote $B := \{h \in \mathcal{H} : \|h\| \leq 1\}$ the closed unit ball in $\mathcal{H}$. Then the following are equivalent:*

(i) $T(B) \subset \mathcal{H}$ is compact;

(ii) T is the norm-limit of finite rank operators;

(iii) T^* is the norm-limit of finite rank operator.

Definition 1.27. An operator $T \in B(\mathcal{H})$ which satisfies one of the conditions (and hence all) of the previous theorem is called a *compact operator*. We denote the compact operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$.

Lemma 1.28. *$\mathcal{K}(\mathcal{H}) \subset B(\mathcal{H})$ is a two-sided ideal.*

Proof. The only non trivial fact we have to check is that if $K \in \mathcal{K}(\mathcal{H})$ and $T \in B(\mathcal{H})$ then TK and KT are compact. We establish the first, the second is analogous. Suppose $(K_n)_n$ is a sequence of finite rank operators such that $\|K - K_n\| \to 0$. Then TK_n is of finite rank and $\|TK - TK_n\| \leq \|T\| \, \|K_n - K\| \to 0$. So TK is compact. ⊠

We obtain directly from this lemma and Theorems 1.21 and 1.24 the following results.

Corollary 1.29. *If for some $\lambda \in \rho(T)$ the operator $(T - \lambda)^{-1}$ is compact then for any $\mu \in \rho(T)$ the operator $(T - \mu)^{-1}$ is compact. If T is self-adjoint for the spectral radius it holds that $r(T) = \|T\|$.*

We will now state the spectral theorem for compact self-adjoint operators. Later we will give also a spectral theorem unbounded normal operators (cf. Theorem 1.70).

Theorem 1.30 (Spectral theorem)**.** [11, Thm. II.5.1] *If $T \in B(\mathcal{H})$ is a compact self-adjoint operator, then T has at most a countable number of distinct eigenvalues, say $\{\lambda_n,\ n \in \mathbb{N}\}$. Then $\lambda_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Denote by $P_n \in B(\mathcal{H})$ the projection onto $\ker(T - \lambda_n)$, which is a finite dimensional subspace. Then $P_nP_m = P_mP_n = 0$ if $n \neq m$ and*

$$T = \sum_{n=0}^{\infty} \lambda_n P_n,$$

where the series is norm-convergent in $B(\mathcal{H})$.

Now we will give an short overview of von Neumann algebras after that we will continue with the unbounded operators.

Definition 1.31. Let $\mathcal{H}$ be a Hilbert space and $S \subset B(\mathcal{H})$ be a subset. The *commutant* of S, denoted by S', is defined as

$$S' := \{b \in B(\mathcal{H}) : ab = ba \text{ for all } a \in S\}.$$

On $B(\mathcal{H})$ one can define a lot of different topologies, the most common one is the norm topology, but among others we have the weak and strong operator topology.

Definition 1.32. The *weak operator topology* (WOT) is the weakest topology on $B(\mathcal{H})$ such that $T \mapsto \langle Tx, y\rangle$ is continuous for all $x, y \in \mathcal{H}$. The *strong operator topology* (SOT) is the weakest topology on $B(\mathcal{H})$ such that $T \mapsto \|Tx\|$ is continuous for all $x \in \mathcal{H}$.
So a net $(a_i)_i \subset B(\mathcal{H})$ converges SOT to $a \in B(\mathcal{H})$ if and only if $\|(a_i - a)x\| \to 0$ for all $x \in \mathcal{H}$. Similary, a net $(a_i)_i \subset B(\mathcal{H})$ converges WOT to $a \in B(\mathcal{H})$ if and only if $\langle (a_i - a)x, y\rangle \to 0$ for all $x, y \in \mathcal{H}$.

Remark 1.33. Note that the WOT is weaker than the SOT which in turn is weaker than the norm-topology. So if $a_i \to a$ in norm, then it converges to a in SOT as well. And if $a_i \to a$ in SOT then it converges also to a in WOT.

The double commutant theorem of von Neumann relates the algebraic notion of the commutant with the analytic notion of a topology to each other.

Theorem 1.34 (Double commutant theorem, von Neumann)**.** [23] *Suppose $\mathcal{M} \subset B(\mathcal{H})$ is a unital $*$-subalgebra, then the following are equivalent:*

(i) $\mathcal{M}'' = \mathcal{M}$*;*

(ii) $\mathcal{M}$ *is closed in the weak operator topology;*

(iii) $\mathcal{M}$ *is closed in the strong operator topology.*

Definition 1.35. A unital $*$-subalgebra $\mathcal{M} \subset B(\mathcal{H})$ is called a *von Neumann algebra* if it satisfies one and therefore all conditions of the double commutant theorem. The algebra $\mathcal{M} \cap \mathcal{M}'$ is called the *center* of $\mathcal{M}$. If the center equals $\mathbb{C}$, then $\mathcal{M}$ is called a *factor*.

Now we consider the projections of $\mathcal{M}$. The projections say a lot about the von Neumann algebra, so we will elaborate on this subject. We will follow [30].

Definition 1.36. A *projection* $p \in \mathcal{M}$ is an element satisfying $p = p^* = p^2$. Let $\mathcal{P}(\mathcal{M})$ denote the projections of $\mathcal{M}$. If $p \in \mathcal{M} \cap \mathcal{M}'$, it is called a *central projection*.
Define the partial ordering $\leq$ on the projections by $p \leq q$ if $p\mathcal{H} \subset q\mathcal{H}$. We can define an equivalence relation on the projections by $p \sim q$ if there exists $u \in \mathcal{M}$ such that $p = u^*u$ and $q = uu^*$. These two together define a partial ordering $\preccurlyeq$ on the projections by $p \preccurlyeq q$ if and only if there exists a projection p' such that $p \sim p' \leq q$.
A projection p is said to be *finite* if $p \sim q \leq p$ implies $p = q$. Otherwise p is called *infinite*. A projection p is *purely infinite* if $q \leq p$ and q is finite implies $q = 0$. p is called *abelian* if the algebra $p\mathcal{M}p$ is abelian. The projection p is *minimal* if $q \leq p$ implies $q = 0$ or $q = p$.

Observe that in a factor the projection 1 is the only nonzero central projection.

Definition 1.37. A von Neumann algebra $\mathcal{M}$ is called *finite, infinite, purely infinite* if the projection 1 has this property.

Definition 1.38. A von Neumann algebra $\mathcal{M}$ is of *type I* if for every central projection p there exists a nonzero projection q such that $q \leq p$.
If $\mathcal{M}$ does not admit any finite projections, then $\mathcal{M}$ is of *type III*. Equivalently we could have demanded that 1 is purely infinite.
If $\mathcal{M}$ has no nonzero abelian projections and for every nonzero central projection p there exists q nonzero finite projection such that $q \leq p$, then $\mathcal{M}$ is of *type II*. If $\mathcal{M}$ is of type II and finite then it is of *type II_1*. If $\mathcal{M}$ is of type II and has no nonzero finite projection, then it is of *type II_∞*.

Theorem 1.39. [30, Thm. V.2.19] *Let $\mathcal{M}$ be a von Neumann algebra. Then there is a unique direct sum decomposition*

$$\mathcal{M} = \mathcal{M}_I \oplus \mathcal{M}_{II_1} \oplus \mathcal{M}_{II_\infty} \oplus \mathcal{M}_{III} \tag{1.2}$$

Definition 1.40. Suppose $\mathcal{M}$ is decomposed as in (1.2). If $\mathcal{M}_{III} = 0$, the algebra $\mathcal{M}$ is called *semifinite*.

Definition 1.41. A *trace* τ on a von Neumann algebra $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ which satisfies:

- $\tau(a + b) = \tau(a) + \tau(b) \qquad$ for all $a, b \in \mathcal{M}_+$;
- $\tau(\lambda a) = \lambda\tau(a) \qquad$ for all $a \in \mathcal{M}_+, \lambda \geq 0$;
- $\tau(aa^*) = \tau(a^*a) \qquad$ for all $a \in \mathcal{M}$.

If $\tau(I) = \infty$, we call τ an *infinite trace.* If for all nonzero $x \in \mathcal{M}_+$ there exists a nonzero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$, the trace τ is called *semifinite.* If $\tau(a) = 0$ implies $a = 0$, τ is called a *faithful trace.* If $\tau(a) = \lim \tau(a_i)$ for each increasing net $(a_i)_i \subset \mathcal{M}_+$ with $\lim a_i = a$ in the strong operator topology, τ is called *normal.*

Example 1.42. The trace Tr on $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is a faithful normal trace. Tr is semifinite and it is finite if and only if $\mathcal{H}$ is finite dimensional.
The linear mapping $f \mapsto \int_{(a,b)} f(x)\,dx$ yields a trace on the von Neumann algebra $L^\infty(a, b)$. This trace is finite if and only if a and b are finite. It is semifinite, normal (because of the monotone convergence theorem) and faithful.

Theorem 1.43. *For a von Neumann algebra $\mathcal{M}$ the following are equivalent*

(i) $\mathcal{M}$ is semifinite;

(ii) $\mathcal{M}$ admits a semifinite faithful normal trace.

Proof. See [30, Theorem V.2.15]. ⊠

Lemma 1.44. *Let $\mathcal{M}$ be a type I factor. Let p be a nonzero minimal projection in $\mathcal{M}$ and let τ be a faithful (semi-)finite trace on $\mathcal{M}$. Then for all projections $q \in \mathcal{M}$ we have $\tau(q) \in \{n\,\tau(p) : n \in \mathbb{N}\} \cup \{\infty\}$.*

Proof. Suppose $q \in \mathcal{P}(\mathcal{M})$. We have three cases:

1. $\tau(q) = \infty$ then we are done.
2. $\tau(q) = 0$ then we are done as well.
3. $0 < \tau(q) < \infty$. In this case there exists $m \in \mathbb{N}$ such that $m\tau(p) > \tau(q)$. If $q \sim p$, then by definition there exists a $u \in \mathcal{M}$ such that $q = u^*u$ and $p = uu^*$ hence $\tau(q) = \tau(u^*u) = \tau(uu^*) = \tau(p)$. So suppose $p \prec q$. Let $q_1 \in \mathcal{P}(\mathcal{M})$ such that $q \sim q_1 > p$. Then since $q_1 - p > 0$ and p are orthogonal
$$\tau(q) = \tau(q_1) = \tau(q_1 - p + p) = \tau(q_1 - p) + \tau(p).$$
Thus $\tau(q_1 - p) = \tau(q) - \tau(p)$. In particular $(m - 1)\tau(p) > \tau(q_1)$. Now q_1 is again in case 2 or 3. If we are in case 2, then $\tau(q_1) = 0$ and hence $\tau(q) = \tau(p)$. If we are in case 3, we can apply the same reasoning to q_1, we obtain q_2 such that $\tau(q_2) = \tau(q_1) - \tau(p) = \tau(q) - 2\tau(p)$. Repeating this argument at most m times gives a sequence of projection $q, q_1, \ldots, q_m$ such that $\tau(q_j) = \tau(q) - j\tau(p)$. By the choice of m there exists an i such that $0 \leq \tau(q_i) < \tau(p)$. However $0 < \tau(q_i) < \tau(p)$ is not possible, because of minimality of p. Hence $\tau(q_i) = 0$ and $\tau(q) = i\tau(p)$.

This concludes the proof. ⊠

Lemma 1.45. *If $\mathcal{M}$ is a finite von Neumann algebra, then $\mathcal{M}$ admits a finite normal trace. In particular if $\mathcal{M}$ is a factor all semifinite normal traces on $\mathcal{M}$ are finite.*

Proof. The first statement is immediate from [30, Thm. V.2.4]. The other assertion follows from the first one and the fact that any two traces on a semifinite factor are proportional ([30, Cor V.2.32]). ⊠

A trace on a von Neumann algebra can be considered as a noncommutative integral. For integrals we have the Hölder inequality and Fubini's theorem. We therefore expect that such results generalise to traces, such a generalisation is true if one assumes some regularity conditions on the trace.

Proposition 1.46. [31, Thm. IX.2.13] *If $\mathcal{M}$ is a semifinite von Neumann algebra with a faithful, semifinite, normal trace τ, one can define a measure topology on $\mathcal{M}$ (consult [31, §IX.2] for the details). Denote the completion of $\mathcal{M}$ in this topology by $\mathfrak{M}(\mathcal{M})$. For $x \in \mathfrak{M}(\mathcal{M})$ put $\|x\|_p := \tau(|x|^p)^{1/p}$. Then the following holds*

$$\|ax\|_1 \leq \|a\| \, \|x\|_1 \qquad \text{for } x \in \mathfrak{M}(\mathcal{M}), \|x\|_1 < \infty \text{ and } a \in \mathcal{M}.$$

In particular since $\mathcal{M} \subset \mathfrak{M}(\mathcal{M})$ for all $x, y \in \mathcal{M}$ it holds that

$$\tau(|xy|) \leq \|x\| \, \tau(|y|). \tag{1.3}$$

A lot more can be said about projections and traces in von Neumann algebras, but here we only state the results we need. Now we will continue with unbounded operators. The difficulty with these operators is that they are no longer continuous and that they are not defined on the whole space.

Definition 1.47. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a *linear operator* is a pair $(T, \mathrm{Dom}(T))$ where $\mathrm{Dom}(T) \subset \mathcal{H}$ is a linear space and $T : \mathrm{Dom}(T) \to \mathcal{K}$ is a linear operator. We call $\mathrm{Dom}(T)$ the *domain* of T. We say $(T, \mathrm{Dom}(T))$ is *densely defined* if $\mathrm{Dom}(T) \subset \mathcal{H}$ is dense. We say $(S, \mathrm{Dom}(S))$ is an *extension* of T if $\mathrm{Dom}(T) \subset \mathrm{Dom}(S)$ and $Sx = Tx$ for all $x \in \mathrm{Dom}(T)$, we denote this by $T \subset S$. Thus $T = S$ is by definition $T \subset S$ and $T \supset S$, thus for two operators to be equal it is necessary that the domains coincide. We say $(T, \mathrm{Dom}(T))$ is bounded if there exists a constant $c > 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in \mathrm{Dom}(T)$.

Note that if $(T, \mathrm{Dom}(T))$ is a bounded operator we can continuously extend T to a bounded operator on $\overline{\mathrm{Dom}(T)}$. It has become more difficult to take the sum of two operators, the product or the inverse, because we have to deal with the domains. Outside of the domains the operators are undefined.

Definition 1.48. Suppose $(S, \mathrm{Dom}(S))$ and $(T, \mathrm{Dom}(T))$ are linear operators, then

$$(S + T, \mathrm{Dom}(S) \cap \mathrm{Dom}(T)), \qquad (ST, \mathrm{Dom}(T) \cap T^{-1}(\mathrm{Dom}(S))), \qquad (T^{-1}, T(\mathrm{Dom}(T)))$$

are linear operators. These are the *sum*, *product* and *inverse* of S and T.

Note that it is *not* necessary that the sum or product of two densely defined operators is again densely defined. It is even possible that if $(S, \mathrm{Dom}(S))$ and $(T, \mathrm{Dom}(T))$ are densely defined, but $\mathrm{Dom}(S) \cap \mathrm{Dom}(T) = \{0\}$.
Unbounded operators are no longer continuous (cf. Proposition 1.19), but there is a class of operators which still have a proper behaviour with respect to limits. These operators are the closed operators.

Definition 1.49. An operator $(T, \mathrm{Dom}(T)) : \mathcal{H} \to \mathcal{K}$ is called *closed* if its *graph* $\mathcal{G}(T)$ defined by,

$$\mathcal{G}(T) := \{x \oplus Tx \, : \, x \in \mathrm{Dom}(T)\} \subset \mathcal{H} \oplus \mathcal{K},$$

is closed in $\mathcal{H} \oplus \mathcal{K}$. We call an operator T *closable* if there exists a closed extension of T. One can show that an operator is closable if and only if the closure $\overline{\mathcal{G}(T)}$ is again a graph of some operator. We call the (unique) operator which has graph $\overline{\mathcal{G}(T)}$, the *closure* of T. We denoted the closure by $\overline{T}$.

Note that the closed graph theorem implies that a closed operator with domain the whole Hilbert space is necessarily a bounded operator. The following lemma explains the controlled behaviour of limits.

Lemma 1.50. *For an operator $(T, \mathrm{Dom}(T))$, the following are equivalent:*

(i) *T is closed;*

(ii) *If $(x_n)_n \subset \mathrm{Dom}(T)$, $x_n \to x$ and $Tx_n \to y$ for some $x \in \mathcal{H}$ and $y \in \mathcal{K}$, then $x \in \mathrm{Dom}(T)$ and $Tx = y$.*

We also want to construct an adjoint operator, again this is more difficult than in the bounded case. In this construction we apply the Riesz representation theorem, for this it is necessary that T is densely defined.

Definition 1.51. Suppose $(T, \mathrm{Dom}(T))$ is a densely defined operator. We put

$$\mathrm{Dom}(T^*) := \{x \in \mathcal{K} \,:\, y \mapsto \langle Ty, x\rangle \text{ is a bounded functional on } \mathrm{Dom}(T)\}.$$

Because $\mathrm{Dom}(T) \subset \mathcal{H}$ dense, the map $\varphi : y \mapsto \langle Ty, x\rangle$ extends to a continuous functional on $\mathcal{H}$. By the Riesz representation theorem there exists a unique vector $z \in \mathcal{K}$ such that $\varphi(y) = \langle y, z\rangle$. We define $T^*(x) := z$, then $(T^*, \mathrm{Dom}(T^*))$ is a linear operator.

Remark 1.52. From the definitions it follows that if $S \subset T$, then $T^* \subset S^*$ and

$$S^* + T^* \subset (S + T)^*, \qquad T^*S^* \subset (ST)^*,$$

provided that $S, T, S + T$ and ST are all densely defined.

Proposition 1.53. [11, Prop. X.1.6] *If $(T, \mathrm{Dom}(T)) : \mathcal{H} \to \mathcal{K}$ is a densely defined operator, then*

(i) *T^* is closed;*

(ii) *T^* is densely defined if and only if T is closable;*

(iii) *If T is closable, then $\overline{T} = T^{**}$.*

From now on we will only consider densely defined operators from $\mathcal{H}$ in $\mathcal{H}$, unless stated otherwise.

Definition 1.54. An operator $(T, \mathrm{Dom}(T))$ is called *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \qquad \text{for all } x, y, \in \mathrm{Dom}(T).$$

If $T = T^*$, thus in particular it is required that $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$, then $(T, \mathrm{Dom}(T))$ is called *self-adjoint.* If T is symmetric we say T is *essentially self-adjoint* if T^* is self-adjoint. An operator T is called normal if T is closed, densely defined and $N^*N = NN^*$.

Note that if T is densely defined and symmetric, then T^* is the closure of T. So an equivalent definition of essentially self-adjointness would be that T has a unique self-adjoint extension.

Definition 1.55. Suppose T is an (unbounded) operator. Denote $\mathrm{Dom}^\infty(T) := \bigcap_{n=1}^\infty \mathrm{Dom}(T^n)$. If $x \in \mathrm{Dom}^\infty(T)$ and there exists a constant $B > 0$ (dependent on x) such that $\|T^n x\| \leq B^n$ for all n, then x is called a *bounded vector.* If there exists a constant $C > 0$ such that $\|T^n x\| \leq C^n n!$ for all n, then x is called an *analytic vector.* We denote all bounded vectors of T by $\mathrm{Dom}^b(T)$ and all analytic vectors by $\mathrm{Dom}^a(T)$. It is clear that $\mathrm{Dom}^b(T)$ and $\mathrm{Dom}^a(T)$ are linear subspaces and $\mathrm{Dom}^b(T) \subset \mathrm{Dom}^a(T)$.

Proposition 1.56. *Suppose T is an unbounded operator on $\mathcal{H}$ with domain $\mathrm{Dom}(T)$. The following holds:*

(i) if T is self-adjoint, then $\mathrm{Dom}^b(T) \subset \mathcal{H}$ *dense;*

(ii) (Nelson's theorem) if T is symmetric and $\mathrm{Dom}^a(T) \subset \mathcal{H}$ *is dense, then T is essentially self-adjoint;*

(iii) if T is closed and symmetric, then T is self-adjoint if and only if $\mathrm{Dom}^a(T) \subset \mathcal{H}$ *dense.*

Proof. Item (i) is [29, Lemma 7.13]. Item (ii) is Nelson's theorem, [29, Theorem 7.16]. Assertion (iii) follows directly from (i) and (ii) with the observation that $\mathrm{Dom}^b(T) \subset \mathrm{Dom}^a(T)$. ⊠

Definition 1.57. If $(T, \mathrm{Dom}(T)) : \mathcal{H} \to \mathcal{K}$ is a linear operator, we call T *boundedly invertible* if there exists a bounded linear operator $S : \mathcal{K} \to \mathcal{H}$ such that $TS = 1$ and $ST \subset 1$.

Proposition 1.58. [11, Prop. X.1.15] *Let $(T, \mathrm{Dom}(T))$ be a linear operator. Then*

(i) T is boundedly invertible if and only if $\ker(T) = \{0\}$, $\mathrm{ran}(T) = \mathcal{K}$ *and T is closed;*

(ii) If T is boundedly invertible, the inverse is unique. We denote this inverse by T^{-1}.

As we did before for bounded operators, we can introduce the spectrum and resolvent set for unbounded operators.

Definition 1.59. For a linear operator $(T, \mathrm{Dom}(T)) : \mathcal{H} \to \mathcal{K}$ we define the *resolvent set* $\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is boundedly invertible }\}$. We define the *spectrum* as before $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Theorem 1.60. [11, Prop. X.1.15] *If $(T, \mathrm{Dom}(T))$ is a closed symmetric operator, then precisely one of the following possibilities occurs*

(i) $\sigma(T) = \mathbb{C}$*;*

(ii) $\sigma(T) = \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geq 0\}$*;*

(iii) $\sigma(T) = \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \leq 0\}$*;*

(iv) $\sigma(T) \subset \mathbb{R}$.

Furthermore (iv) holds if and only if $(T, \mathrm{Dom}(T))$ is self-adjoint.

The spectrum of operators will be of importance later when we state the spectral theorems.

Theorem 1.61. [27, Thm. VIII.2] *Let $(T, \mathrm{Dom}(T))$ be a closed densely defined linear operator. Then $\rho(T) \subset \mathbb{C}$ open subset and*

$$\rho(T) \to B(\mathcal{H}), \qquad \lambda \mapsto (\lambda - T)^{-1}$$

is an analytic operator valued function. Furthermore the first resolvent identity

$$(T - \lambda)^{-1} - (T - \mu)^{-1} = (\mu - \lambda)(T - \mu)^{-1}(T - \lambda)^{-1} \qquad \text{for } \lambda, \mu \in \rho(T) \tag{1.4}$$

holds. In particular $\{(\lambda - T)^{-1} : \lambda \in \rho(T)\}$ is a commuting family of bounded operators.

Note that the difference with the above theorem and Theorem 1.21 is that now we do not assume that T is bounded.
If a separable Hilbert space admits an orthonormal basis consisting of eigenvectors of a linear operator D, then it is easy to check wether D is self-adjoint and if the resolvent is compact.

Theorem 1.62. *Suppose $\mathcal{H}$ is a separable Hilbert space and $D : \mathrm{Dom}(D) \to \mathcal{H}$ is an (unbounded) linear operator, such that $\mathcal{H}$ has an orthonormal basis $(e_n)_n$ consisting of eigenvectors of D with corresponding eigenvalues $(\lambda_n)_n$. If $\lambda_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then D is self-adjoint on the domain* $\mathrm{Dom}(D) := \{h = \sum_n a_n e_n \in \mathcal{H} : \sum_n |a_n|^2 < \infty \text{ and } \sum_n \lambda_n^2 |a_n|^2 < \infty\}$.
If in addition $\lim_{n \to \infty} |\lambda_n| = \infty$, then $\sigma(D) = \{\lambda_n : n \in \mathbb{N}\}$ and for $\alpha \in \rho(D)$ the operator $(\alpha - D)^{-1}$ is compact.

Proof. For $h \in \mathcal{H}$ we write $a_n := \langle h, e_n \rangle$, then $h = \sum_n a_n e_n$ in $\|\cdot\|_2$-norm. Observe that for $h = \sum_n a_n e_n \in \mathrm{Dom}(D)$ we have $Dh = \sum_n \lambda_n a_n e_n$, because $\sum_n \lambda_n^2 |a_n|^2 < \infty$ it follows that $Dh \in \mathcal{H}$. Since $e_n \in \mathrm{Dom}(D)$ for all $n \in \mathbb{N}$, $\mathrm{span}(\{e_n : n \in \mathbb{N}\}) \subset \mathrm{Dom}(D)$. The vector space $\mathrm{span}(\{e_n : n \in \mathbb{N}\})$ is dense in $\mathcal{H}$, so $\mathrm{Dom}(D) \subset \mathcal{H}$ dense. Therefore D is densely defined.
Now suppose $f, g \in \mathrm{Dom}(D)$. Write $f = \sum_n a_n e_n$ and $g = \sum_n b_n e_n$. Then because $\lambda_n \in \mathbb{R}$

$$\langle Df, g \rangle = \sum_{n,m} \lambda_n a_n \overline{b_m} \langle e_n, e_m \rangle = \sum_n \lambda_n a_n \overline{b_n} \langle e_n, e_n \rangle = \sum_{n,m} a_n \overline{\lambda_m b_m} \langle e_n, e_m \rangle = \langle f, Dg \rangle.$$

The first and the last equalities hold, because of Lemma 1.17. This shows D is symmetric.
Now we will show $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$. It is trivial that $\mathrm{Dom}(D) \subset \mathrm{Dom}(D^*)$, so we have to establish the converse inclusion. Suppose $h \notin \mathrm{Dom}(D)$. Then $\sum_n |a_n|^2 < \infty$, but $\sum_n \lambda_n^2 |a_n|^2 = \infty$. Put $f_N := \sum_{n=1}^N \lambda_n a_n e_n$. Since f_N is given as a finite linear combination of e_n it is obvious that $f_N \in \mathrm{Dom}(D)$. Now

$$\langle Df_N, h \rangle = \sum_{m=1}^N \langle \lambda_n^2 a_m e_m, \sum_{n=1}^\infty a_n e_n \rangle = \sum_{m=1}^N \sum_{n=1}^\infty \lambda_m^2 a_m \overline{a_n} \langle e_m, e_n \rangle = \sum_{m=1}^N \lambda_m^2 |a_m|^2.$$

Since h is not in the domain of D^*, this becomes arbitrarily large as $N \to \infty$. Hence the linear map $\mathrm{Dom}(D) \to \mathbb{C}$, $f \mapsto \langle Af, h \rangle$ cannot be bounded. This implies $h \notin \mathrm{Dom}(D^*)$, thus $\mathrm{Dom}(D) \supset \mathrm{Dom}(D^*)$ and $(D, \mathrm{Dom}(D))$ is self-adjoint.
Assume $\lim_{n\to\infty} |\lambda_n| = \infty$. Fix $\alpha \in \mathbb{C} \setminus \{\lambda_n : n \in \mathbb{N}\}$. Define $T : \mathcal{H} \to \mathrm{Dom}(D)$ and for $n \in \mathbb{N}$ the maps $T_n : \mathcal{H} \to \mathrm{Dom}(D)$ on the orthonormal basis by $T(e_m) := \frac{1}{\alpha - \lambda_m} e_m$ and

$$T_n(e_m) := \begin{cases} \frac{1}{\alpha-\lambda_m} e_m & \text{if } m \le n \\ 0 & \text{if } m > n \end{cases}$$

and extend them linearly to $\mathcal{H}$. Then all maps T_n have finite rank and map into $\mathrm{Dom}(D)$. To show that T maps into $\mathrm{Dom}(D)$ we use the fact that $\lim_{n\to\infty} |\lambda_n| = \infty$, so $\{\lambda_n : n \in \mathbb{N}\}$ has no accumulation points and therefore this set is closed. Let $\delta > 0$ be such that $B_\delta(\alpha) \cap \{\lambda_n : n \in \mathbb{N}\} = \emptyset$, then $\frac{1}{|\alpha - \lambda_n|} < \frac{1}{\delta}$ for all n. Furthermore it holds that $\lim_{n\to\infty} \left|\frac{\lambda_n}{\alpha - \lambda_n}\right| = 1$, thus there exists $C > 0$ such that $\left|\frac{\lambda_n}{\alpha-\lambda_n}\right| < C$ for all n. Suppose $f \in \mathcal{H}$, write $f = \sum_n a_n e_n$, then $Tf = \sum_n \frac{a_n}{\alpha - \lambda_n} e_n$. So

$$\sum_n \left|\frac{a_n}{\alpha - \lambda_n}\right|^2 \le \frac{1}{\delta^2} \sum_n |a_n|^2 < \infty$$
$$\sum_n \left|\frac{a_n}{\alpha - \lambda_n}\right|^2 \lambda_n^2 \le C \sum_n |a_n|^2 < \infty.$$

Hence $Tf \in \mathrm{Dom}(D)$.
We will show that T is the norm-limit of the finite rank operators T_n. This will imply that T is compact. Since $\lim_{n\to\infty} |\lambda_n| = \infty$ and $|\alpha - \lambda_n| > \delta$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \frac{1}{|\alpha-\lambda_n|} = 0$. Let $\varepsilon > 0$, pick $M \in \mathbb{N}$ such that for all $m > M$ it holds $|\frac{1}{\alpha - \lambda_m}| < \varepsilon$. Let $f \in \mathcal{H}$ with $\|f\| = 1$, write $f = \sum_n a_n e_n$, then for $m > M$ we have

$$\|Tf - T_m f\| = \Big\| \sum_{n=m+1}^\infty \frac{1}{\alpha - \lambda_n} a_n e_n \Big\| < \varepsilon \Big\| \sum_{n=m+1}^\infty a_n e_n \Big\| \le \varepsilon \|f\| = \varepsilon.$$

So $\lim_{m\to\infty} \|T - T_m\| = 0$, hence T is compact. Since $T(\alpha - D)e_n = (D - \alpha)Te_n = e_n$ and $T(\mathcal{H}) \subset \mathrm{Dom}(D)$ we obtain $T(\alpha - D) \subset 1$ and $(\alpha - D)T = 1$. So $\alpha \in \rho(D)$. Clearly $\{\lambda_n : n \in \mathbb{N}\} \subset \sigma(D)$. Thus we have $\sigma(D) = \{\lambda_n : n \in \mathbb{N}\}$ and $(\alpha - D)^{-1}$ is compact for all $\alpha \in \rho(D)$. ⊠

The converse of this theorem does not need to hold, it is possible that a self-adjoint operator D does not have any eigenvectors. However a generalisation is true, this will be the spectral theorem see Theorem 1.70.

1.3 Spectral theory

Here we will describe the importance of the spectrum of an operator. We will state the spectral theorem and the functional calculus of self-adjoint operators.

Definition 1.63. Suppose (X, Σ) is a measurable space and $\mathcal{H}$ a Hilbert space. A mapping $E : \Sigma \to B(\mathcal{H})$ is called a *spectral measure* if it satisfies the following conditions:

(i) $E(A)$ is a projection for each $A \in \Sigma$;

(ii) $E(\emptyset) = 0$, $E(X) = 1$;

(iii) $E(A \cap B) = E(A)E(B)$, for all $A, B \in \Sigma$;

(iv) if $(A_n)_n \subset \Sigma$ is a sequence of disjoint measurable sets, then

$$E\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} E(A_n).$$

Notation 1.64. If $E : \mathcal{B}(X) \to \mathcal{B}(\mathcal{H})$ is a spectral measure, each pair $x, y \in \mathcal{H}$ defines a finite complex measure on $\mathcal{B}(X)$ by $E_{x,y}(A) := \langle P(A)x, y\rangle$. That $E_{x,y}$ is a finite complex measure can directly be verified from the properties of a spectral measure.

With a spectral measure we can integrate functions to obtain operators. The following proposition gives the precise relation for bounded functions.

Proposition 1.65. [11, Prop. IX.1.10] *If $E : \Sigma \to B(\mathcal{H})$ is a spectral measure and $\varphi : X \to \mathbb{C}$ is a bounded Σ-measurable function, then there exists a unique normal operator $T \in B(\mathcal{H})$ such that if $\varepsilon > 0$ and $\{A_1, \ldots, A_n\}$ is a Σ-partition of X with the property that for all k*

$$\sup\{|\varphi(x) - \varphi(x')| : x, x' \in A_k\} < \varepsilon,$$

it holds that for all $x_k \in A_k$

$$\Big\|T - \sum_{k=1}^{n} \varphi(x_k)E(A_k)\Big\| < \varepsilon.$$

Notation 1.66. The operator T of Proposition 1.65 obtained from E and φ is called *the integral of φ with respect to E*. We denote $T := \int \varphi \, dE$.

With some efforts it is possible to extend this result to unbounded operators. The idea is to write an unbounded function as a sum of bounded functions and apply the construction on these bounded functions. One only needs to specify the domain of the operator.

Definition 1.67. Suppose $\varphi : X \to \mathbb{C}$ is a Σ-measurable function and E a spectral measure. For $n \in \mathbb{N}$ define $A_n := \{x \in X : n - 1 \le \varphi(x) < n\}$. Then $\varphi_n := 1_{A_n}\varphi$ is bounded and E_n defined by $E_n(A) := E(A \cap A_n)$ is a spectral measure. Hence the operator $\int \varphi_n \, dE_n$ is well-defined. Let

$$\mathcal{D}_\varphi := \Big\{h \in \mathcal{H} : \sum_{n=1}^{\infty} \Big\|\Big(\int \varphi_n \, dE_n\Big)E_n(A_n)h\Big\|^2 < \infty\Big\}$$

and for $h \in \mathcal{D}_\varphi$ put

$$Th := \sum_{n=1}^{\infty} \Big(\int \varphi_n \, dE_n\Big)E_n(A_n)h.$$

For each $h \in \mathcal{D}_\varphi$ his series converges in norm. It appears that T is a normal operator with domain $\mathcal{D}_\varphi$. Again we call T the *integral of φ with respect to E*. We also denote $T := \int \varphi \, dE$.

In this thesis we will often use the functional calculus for unbounded self-adjoint operators. We will state here the properties for normal operators [11, Theorem 4.7, 4.10, 4.11], this is a little bit more general than we will need.

Theorem 1.68. [11, Thm. X.4.7] *If Σ is a σ-algebra of Ω, $E : \Sigma \to \mathcal{P}(\mathcal{H})$ is a spectral measure and $\varphi : \Omega \to \mathbb{C}$ is Σ-measurable. Then the following holds*

(i) *The operator* $\int \varphi \, dE$ *is normal, thus in particular closed;*

(ii) *The operator* $\int \varphi \, dE$ *has domain* $\mathrm{Dom}\left(\int \varphi \, dE\right) = \left\{h \in \mathcal{H} \, : \, \int |\varphi|^2 \, dE_{h,h} < \infty\right\}$*;*

(iii) $\left\langle \left(\int \varphi \, dE\right) h, g \right\rangle = \int \varphi \, dE_{h,g}$*;*

(iv) $\left\| \left(\int \varphi \, dE\right) h \right\|^2 = \int |\varphi|^2 \, dE_{h,h}$.

Theorem 1.69. [11, Thm. X.4.10] *Under the same assumptions as in Theorem 1.68. Denote by $\overline{\Sigma}$ the collection of all Σ-measurable functions $\varphi : \Omega \to \mathbb{C}$ and consider the map $\rho : \overline{\Sigma} \to \mathcal{C}(\mathcal{H})$ of measurable functions into the closed operators given by $\rho(\varphi) := \int \varphi \, dE$. Then for $\varphi, \psi \in \overline{\Sigma}$*

(i) $\rho(\varphi)^* = \rho(\overline{\varphi})$*;*

(ii) $\rho(\varphi\psi) \supset \rho(\varphi)\rho(\psi), \qquad \mathrm{Dom}(\rho(\varphi)\rho(\psi)) = \mathrm{Dom}(\rho(\psi)) \cap \mathrm{Dom}(\rho(\varphi\psi))$*;*

(iii) *If ψ is bounded,* $\rho(\varphi)\rho(\psi) = \rho(\psi)\rho(\varphi) = \rho(\varphi\psi)$*;*

(iv) $\rho(\varphi)^*\rho(\varphi) = \rho(|\varphi|^2)$.

Theorem 1.70 (Spectral theorem)**.** [11, Thm. X. 4.11] *If N is a normal operator on $\mathcal{H}$, then there exists a unique spectral measure $E : \mathcal{B}(\mathbb{C}) \to \mathcal{P}(\mathcal{H})$ such that*

(i) $N = \int z \, dE(z)$*;*

(ii) $E(B) = 0$ *if* $B \cap \sigma(N) = \emptyset$*;*

(iii) *if $U \subset \mathbb{C}$ open and $U \cap \sigma(N) \neq \emptyset$, then $E(U) \neq 0$;*

(iv) *if $A \in B(\mathcal{H})$ such that $AN \subset NA$, then $A(\int \varphi \, dE) \subset (\int \varphi \, dE)A$ for all φ Borel measurable.*

The relation between N and its spectral measure E is sometimes expressed by "assume N has spectral decomposition $N = \int z \, dE$" or "suppose N has spectral measure E".
For unbounded operators which can be diagonalized as in Theorem 1.62 the functional calculus (for unbounded operators) has an appealing form. This is the content of the next theorem

Theorem 1.71. *Let $(D, \mathrm{Dom}(D))$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and let $(e_n)_n$ be an orthonormal basis consisting of eigenvectors for D with corresponding eigenvalues $(\lambda_n)_n$. Then for $\varphi \in C(\sigma(D), \mathbb{C})$ the functional calculus is given by*

$$\varphi(D)f = \sum_n \varphi(\lambda_n)\langle f, e_n\rangle e_n, \qquad (f \in \mathrm{Dom}(\varphi(D)),$$

where $\mathrm{Dom}(\varphi(D)) := \{f \in \mathcal{H} \, : \, \sum_n |\varphi(\lambda_n)\langle f, e_n\rangle|^2 < \infty\}$.

Proof. By assumption the operator D is given by $Df = \sum_n \lambda_n \langle f, e_n\rangle e_n$, for $f \in \mathrm{Dom}(D) = \{f \in \mathcal{H} \, : \, \sum_n |\lambda_n \langle f, e_n\rangle|^2 < \infty\}$. Another way to write D is by using spectral measures. Observe that $\sigma(D) = \{\lambda_n \, : \, n \in \mathbb{N}\}$. Denote by E_n the projection on e_n. Define $E(\Omega) := \sum\{E_n \, : \, \lambda_n \in \Omega\}$, a sum of orthogonal projections, then E is a spectral measure for D. We write $D = \int \lambda \, dE(\lambda)$ and $\varphi(D) = \int \varphi \, dE$.

Observe $f \in D_\varphi$ if and only if $\int |\varphi|^2 \, dE_{f,f} < \infty$. Since $\sigma(D)$ is countable

$$\begin{aligned}\int_{\sigma(D)} |\varphi|^2 \, dE_{f,f} &= \sum_{n\in\mathbb{N}} |\varphi(\lambda_n)|^2 E_{f,f}(\lambda_n) \\ &= \sum_{n\in\mathbb{N}} |\varphi(\lambda_n)|^2 \langle E_n f, f\rangle \\ &= \sum_{n\in\mathbb{N}} |\varphi(\lambda_n)|^2 \langle \langle f, e_n\rangle e_n, f\rangle \\ &= \sum_{n} |\varphi(\lambda_n)\langle f, e_n\rangle|^2.\end{aligned}$$

Hence $\mathrm{Dom}(\varphi(D)) := \{f \in \mathcal{H} : \sum_n |\varphi(\lambda_n)\langle f, e_n\rangle|^2 < \infty\}$. Using lemma 1.17 we have for $f \in D_\varphi$ and $h \in \mathcal{H}$ the following equalities

$$\begin{aligned}\Big\langle \sum_n \varphi(\lambda_n)\langle f, e_n\rangle e_n, h\Big\rangle &= \sum_{n\in\mathbb{N}} \varphi(\lambda_n)\langle\langle f, e_n\rangle e_n, h\rangle \\ &= \sum_{n\in\mathbb{N}} \varphi(\lambda_n) E_{f,h}(\lambda_n) \\ &= \int \varphi \, dE_{f,h}.\end{aligned}$$

Thus for all $h \in \mathcal{H}$

$$\langle \varphi(D)f, h\rangle = \int \varphi \, dE_{f,h} = \Big\langle \sum_n \varphi(\lambda_n)\langle f, e_n\rangle e_n, h\Big\rangle$$

and we conclude $\varphi(D)f = \sum_n \varphi(\lambda_n)\langle f, e_n\rangle e_n$. ⊠

We cannot expect that an unbounded operator is an element of a von Neumann algebra. But we can introduce a notion which is similar. For a von Neumann algebra $\mathcal{M}$ we have by definition $\mathcal{M}'' = \mathcal{M}$. So observe $x \in \mathcal{M}$, if and only if $\{x\}'' \subset \mathcal{M}$.

Definition 1.72. If S is an unbounded operator on $\mathcal{H}$ and $T \in B(\mathcal{H})$ we say that S and T *commute* if $TS \subset ST$. Denote by $\{S\}' := \{T \in B(\mathcal{H}) : T \text{ commutes with } S\}$. Then $\{S\}'$ is an algebra in $B(\mathcal{H})$. If S is densely defined and closed, then $\{S\}'$ is closed.
Put $W^*(S) := (\{S\}' \cap \{S^*\}')'$. If S is bounded, then $W^*(S)$ is precisely the von Neumann algebra generated by S. We say S is *affiliated* with a von Neumann algebra $\mathcal{M}$ if $W^*(S) \subset \mathcal{M}$. Or equivalently if $\{S\}' \cap \{S^*\}' \supset \mathcal{M}'$.

For self-adjoint operators it is possible to prove the spectral theorem via the Cayley transform. This transform maps (unbounded) self-adjoint operators to unitary operators and vice-versa [26, Prop. 5.2.5]. One can then prove the spectral theorem for unbounded self-adjoint operators via the spectral theorem of bounded unitary operators. However the spectral theorem as stated above for normal operators cannot be proved via this method. But we have another reason to be interested in the Cayley transform, see the lemmas below. We are only interested in self-adjoint operators, so we will not define it for symmetric operators.

Definition 1.73. Let $(T, \mathrm{Dom}(T))$ be a self-adjoint operator in $\mathcal{H}$. The *Cayley transform* of T is defined as the operator $\kappa(T) := (T - i)(T + i)^{-1}$.

Lemma 1.74. [26, Lemma 5.2.8] *Suppose* $(T, \mathrm{Dom}(T))$ *is an unbounded self-adjoint operator on* $\mathcal{H}$*, then* T *affiliated to* $\mathcal{N}$ *if and only* $\kappa(T) \in \mathcal{N}$.

We have another equivalent formulation when an operator is affiliated to a von Neumann algebra.

Lemma 1.75. *Suppose $(T, \mathrm{Dom}(T))$ is an (unbounded) self-adjoint operator on $\mathcal{H}$ with spectral decomposition $T = \int \lambda \, dE$ and suppose $\mathcal{N} \subset B(\mathcal{H})$ is a von Neumann algebra. Then T is affiliated to $\mathcal{N}$ if and only if $E(A) \in \mathcal{N}$ for all $A \in \mathcal{B}(\sigma(T))$.*

Proof. First assume T is affiliated with $\mathcal{N}$. Then $E(A) = 1_A(T)$ and by the spectral theorem if T and B commute, so do $1_A(T)$ and B. Thus $\{T\}' \subset \{E(A) : A \in \mathcal{B}(\sigma(T))\}'$. Hence

$$\mathcal{N} = \mathcal{N}'' \supset \{T\}'' \supset \{E(A) : A \in \mathcal{B}(\sigma(T))\}'' \supset \{E(A) : A \in \mathcal{B}(\sigma(T))\}.$$

For the converse implication we will use lemma 1.74, thus that a self-adjoint operator T is affiliated to $\mathcal{N}$ if and only if the Cayley transform $\kappa(T) \in \mathcal{N}$. Observe

$$\kappa(T) = \int f \, dE, \qquad \text{where } f(x) = \frac{i-x}{i+x}.$$

Then $f(\sigma(T)) \subset \{z \in \mathbb{C} : |z| = 1\}$, f is continuous and f is bounded. Write $f = \sum_{j=0}^{3} i^j f^{(j)}$, where each function $f^{(j)}$ is positive. There exists sequences of simple functions $(f_n^{(j)})_n$ such that $f_n^{(j)} \uparrow f^{(j)}$ as $n \to \infty$. By the monotone convergence theorem we have

$$\left\| \Big(\int f^{(j)} \, dE \Big) x - \Big(\int f_n^{(j)} \, dE \Big) x \right\|^2 = \int |f^{(j)} - f_n^{(j)}|^2 \, dE_{x,x} \to 0. \tag{1.5}$$

Because $f_n^{(j)}$ are simple functions the operators $\int f_n^{(j)} \, dE$ are linear combinations of the spectral projections $E(A)$. By assumption each projection $E(A)$ commutes with all $B \in \mathcal{N}'$. Thus the operators $\int f_n^{(j)} \, dE$ commute with all $B \in \mathcal{N}'$. Since $\kappa(T)$ is bounded, $\mathrm{Dom}(\kappa(T)) = \mathcal{H}$. Now let $x \in \mathcal{H}$ and $\varepsilon > 0$. Let J be large such that (1.5) $< \varepsilon$ for all $j \geq J$, then

$$\begin{aligned}
&\left\| \Big(\int f^{(j)} \, dE \Big) Bx - B\Big(\int f^{(j)} \, dE \Big) x \right\| \\
&\quad \leq \left\| \Big(\int f^{(j)} \, dE \Big) Bx - \Big(\int f_n^{(j)} \, dE \Big) Bx \right\| + \left\| \Big(\int f_n^{(j)} \, dE \Big) Bx - B\Big(\int f_n^{(j)} \, dE \Big) x \right\| \\
&\qquad + \left\| B\Big(\int f_n^{(j)} \, dE \Big) x - B\Big(\int f^{(j)} \, dE \Big) x \right\| \\
&\quad \leq \varepsilon + 0 + \|B\| \left\| \Big(\int f_n^{(j)} \, dE \Big) x - \Big(\int f^{(j)} \, dE \Big) x \right\| \\
&\quad \leq \varepsilon(1 + \|B\|).
\end{aligned}$$

So $\Big(\int f^{(j)} \, dE \Big) Bx = B\Big(\int f_n^{(j)} \, dE \Big) x$. Thus $\kappa(T) = \sum_{j=0}^{3} i^j f^{(j)}(T)$ commutes with $\mathcal{N}'$. Hence $\kappa(T) \in \mathcal{N}'' = \mathcal{N}$ and T is affiliated with $\mathcal{N}$. ⊠

We introduce some notation.

Notation 1.76. Let $\mathcal{L}(X, \Sigma, E)$ be the Σ-measurable functions. Denote by

$$\mathcal{N} := \Big\{ f \in L(X, \Sigma, E) : \int f \, dE = 0 \Big\},$$

the *E-null functions*. Then we can define the equivalence classes of measurable functions $L(X, \Sigma, E) := \mathcal{L}(X, \Sigma, E)/\mathcal{N}$.

With the Theorems 1.68, 1.69 and 1.70 the following functional calculus is immediate.

Theorem 1.77. [26, Thm. 5.3.8] *Suppose $(D, \mathrm{Dom}(D))$ is a self-adjoint operator on $\mathcal{H}$ with spectral measure E. Then the map*

$$L(\sigma(D), \mathcal{B}(\sigma(D)), E) \to \{S : S \text{ is affiliated with } W^*(D)\}, \qquad f \mapsto \int f \, dE$$

is an essential $*$-isomorphism. In particular $Id_{\sigma(D)}(D) = D$ and $1(D) = Id_{\mathcal{H}}$.

Definition 1.78. Suppose (S, Σ, ν) is a *finite* measure space, $\mathcal{M} \subset B(\mathcal{H})$ a semifinite von Neumann algebra and ρ a semifinite faithful normal trace on $\mathcal{M}$. We denote

$$L^1(\mathcal{M}, \rho) := \{T : \mathrm{Dom}(T) \to \mathcal{H} \,:\, T \text{ affiliated with } \mathcal{M},\ \rho(|T|) < \infty\}.$$

$\mathcal{L}^1(\mathcal{M}, \rho) := L^1(\mathcal{M}, \rho) \cap \mathcal{M}$, i.e. the bounded trace class operators.
A bounded function $f : S \to \mathcal{L}^1(\mathcal{M}, \rho)$ is $*$*-measurable* if for all $h \in \mathcal{H}$ the functions $f(\cdot)h : S \to \mathcal{H}$ and $f(\cdot)^*h : S \to \mathcal{H}$ are measurable. Define

$$\mathcal{L}_\infty^{so^*}\big(S, \nu, \mathcal{L}^1(\mathcal{M}, \rho)\big) := \{f : S \to \mathcal{L}^1(\mathcal{M}, \rho) \,:\, f \text{ is } \|\cdot\|\text{-bounded , } *\text{-measurable}\}.$$

Proposition 1.79. [1, Lemma 3.10] *Let $\mathcal{M} \subset B(\mathcal{H})$ be a semifinite von Neumann algebra with semifinite faithful normal trace ρ and (S, Σ, ν) a finite measure space. Assume that $f \in \mathcal{L}_\infty^{so^*}(S, \nu; \mathcal{L}^1(\mathcal{M}, \rho))$ and f is uniformly $\mathcal{L}^1(\mathcal{M}, \rho)$-bounded (i.e. there exists $C > 0$ such that $\rho(|f(s)|) < C$ for all $s \in S$), then $\int_S f(s)\, d\nu \in \mathcal{L}^1(\mathcal{N}, \rho)$, the function $\rho(f(\,\cdot\,))$ is measurable with respect to the σ-algebra Σ and*

$$\tau\Big(\int_S f(s)\, d\nu(s)\Big) = \int_S \tau(f(s))\, d\nu(s).$$

1.4 Relations between traces and measures

In this subsection we will dive into the relation between traces and (spectral) measures. We will see that we can combine measure theory and spectral theory. First we will show that the combination of a trace and a spectral measure defines an ordinary measure on the sigma algebra on which the spectral measure is defined. Then we will integrate with respect to that measure. In the second part of this subsection we will give a generalisation of compact operators.

Lemma 1.80. *Let Σ be a σ-algebra on a set Ω. Let $\mathcal{N}$ be a von Neumann algebra and $\tau : \mathcal{N} \to [0, \infty]$ be a normal trace. Suppose $E : \Sigma \to \mathcal{P}(\mathcal{H})$ is a spectral measure such that $E(A) \in \mathcal{N}$ for all $A \in \Sigma$. Then*

$$\mu_{\tau,E}(A) := \tau(E(A)) \tag{1.6}$$

defines a measure on Σ. If τ is a finite trace, then $\mu_{\tau,E}$ is a finite measure.

Proof. First observe that $\mu_{\tau,E}$ is well defined since $E(A) \in \mathcal{N}_+$ for all $A \in \Sigma$, therefore $\mu_{\tau,E}$ maps indeed into $[0, \infty]$. It remains to show we have σ-additivity. Let $(A_n)_n$ be a sequence of disjoint Σ-measurable sets. Then, by definition of $\mu_{\tau,E}$, the fact that E is a spectral measure and that τ is normal, we have

$$\mu_{\tau,E}\Big(\bigcup_n A_n\Big) = \tau\Big(E\Big(\bigcup_n A_n\Big)\Big) = \tau\Big(\sum_n E(A_n)\Big) = \sum_n \tau(E(A_n)) = \sum_n \mu_{\tau,E}(A_n).$$

Hence $\mu_{\tau,E}$ is a measure. ⊠

Recall that if μ is a measure, a function $f : \Omega \to \mathbb{C}$ is called μ-essentially bounded if there exists a measurable set A and a constant C such that $\mu(A) = 0$ and $|f(y)| < C$ for all $y \in \Omega \setminus A$. We can generalise this definition to spectral measures and relate it to general measures.

Definition 1.81. Let Σ be a σ-algebra of Ω, $E : \Sigma \to \mathcal{P}(\mathcal{H})$ be a spectral measure and $f : \Omega \to \mathbb{C}$ be a Σ-measurable function. We call f *E-essentially bounded* if there exists an $A \in \Sigma$ and a constant $C > 0$ such that the projection $E(A) = 0$ and $|f(y)| < C$ for all $y \in \Omega \setminus A$. By $L^\infty(\Omega, E)$ we denote all the E-essentially bounded functions on Ω. We can equip this space with the norm

$$\begin{aligned}\|f\|_{\infty,E} &:= \inf\{\sup_{x \notin A} |f(x)| \,:\, A \in \Sigma,\ E(A) = 0\} \\ &= \inf\{a \in \mathbb{R} \,:\, E(\{y \,:\, |f(y)| > a\}) = 0\}.\end{aligned}$$

Lemma 1.82. *Let Σ be a σ-algebra of Ω, $E : \Sigma \to \mathcal{B}(\mathcal{H})$ be a spectral measure and $f : \Omega \to \mathbb{C}$ be a Σ-measurable function. Then the following are equivalent:*

(i) f *is* E*-essentially bounded;*

(ii) *for all* $x \in \mathcal{H}$ *the function* f *is* $E_{x,x}$*-essentially bounded;*

(iii) $\mathrm{Dom}(\int f\,dE) = \mathcal{H}$.

Proof. Assume (i), then select A and C from the definition of E-essentially boundedness. If $x \in \mathcal{H}$, clearly $E(A)x = 0$. Hence $E_{x,x}(A) = \langle E(A)x, x\rangle = \langle 0, x\rangle = 0$. Hence f is $E_{x,x}$-essentially bounded. So (ii) holds.
Now assume (ii). Let $x \in \mathcal{H}$. Select $A \in \Sigma$ and $C > 0$ such that $E_{x,x}(A) = 0$ and $|f(y)| < C$ for all $y \in \Omega \setminus A$. Then we have

$$\begin{aligned}\int_\Omega |f|^2\,dE_{x,x} &= \int_{\Omega\setminus A} |f|^2\,dE_{x,x} + \int_A |f|^2\,dE_{x,x}\\ &\le \int_{\Omega\setminus A} C^2\,dE_{x,x} + 0\\ &= C^2\langle E(\Omega\setminus A)x, x\rangle\\ &\le C^2\|x\|^2.\end{aligned}$$

Hence $\int_\Omega |f|^2\,dE_{x,x} < \infty$, so $x \in \mathrm{Dom}(\int f\,dE)$. Hence (iii) holds.
Now we will show that (iii) implies assertion (i). According to Theorem 1.68 $\int f\,dE$ is a closed operator. Now if $\mathrm{Dom}(\int f\,dE) = \mathcal{H}$, the closed graph theorem implies that $\int f\,dE$ is a bounded operator. Denote $f_n := f\,1_{\{x\in\Omega\,:\,|f(x)|\le n\}}$. Then $\|f_n\|_{\infty,E} \le n$. Using Theorem 1.77 it follows that the map $L^\infty(\Omega, E) \to \mathcal{B}(\mathcal{H})$, $g \mapsto \int g\,dE$ is norm preserving. Hence for all n we have

$$\|f_n\|_{\infty,E} = \Big\|\int f_n\,dE\Big\| \le \Big\|\int f\,dE\Big\| < \infty.$$

Thus also $\|f\|_{\infty,E} < \infty$. So (i) holds. ⊠

In general from a measure we can construct an integral. In the case the spectral measure is given as the spectral decomposition of a self-adjoint operator we obtain the following link between the functional calculus and integration with respect to the measure $\mu_{\tau,E}$ constructed above (1.6).

Theorem 1.83. *Suppose T is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with spectral decomposition $T = \int_{\sigma(T)} \lambda\,dE$. Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, with a normal trace $\tau : \mathcal{N}_+ \to [0,\infty]$ and assume that T is affiliated with $\mathcal{N}$. If $f : \sigma(T) \to \mathbb{C}$ is a Borel-measurable function such that $f \ge 0$ or $f \in \mathcal{L}^1(\sigma(T), \mathcal{B}(\sigma(T)), \mu_{\tau,E})$ then,*

$$\int_{\sigma(T)} f\,d\mu_{\tau,E} = \tau(f(T)). \tag{1.7}$$

Proof. To prove this theorem we will apply the standard machine of measure theory. So first suppose $f = 1_A$ for some $A \in \mathcal{B}(\sigma(T))$. Then since T is affiliated with $\mathcal{N}$ by Lemma 1.75 we have $E(A) \in \{T\}'' \subset \mathcal{N}$. Because $E(A)$ is a projection it is positive, hence $E(A) \in \mathcal{N}_+$. Then we obtain

$$\int_{\sigma(T)} f\,d\mu_{\tau,E} = \int_{\sigma(T)} 1_A\,d\mu_{\tau,E} = \mu_{\tau,E}(A) = \tau(E(A)) = \tau\Big(\int 1_A\,dE\Big) = \tau(1_A(T)) = \tau(f(T)).$$

Now if f is a simple function, say $f \sum_{n=1}^N \alpha_n A_n$ we obtain by linearity of the integral, the trace and the functional calculus

$$\begin{aligned}\tau(f(T)) &= \tau\Big(\sum_{n=1}^{N} \alpha_n A_n(T)\Big)\\ &= \sum_{n=1}^{N} \alpha_n \tau(A_n(T))\\ &= \sum_{n=1}^{N} \alpha_n \int_{\sigma(T)} 1_{A_n}\, d\mu_{\tau,E}\\ &= \int_{\sigma(T)} \sum_{n=1}^{N} \alpha_n 1_{A_n}\, d\mu_{\tau,E}\\ &= \int_{\sigma(T)} f\, d\mu_{\tau,E}.\end{aligned}$$

If f is a positive measurable function, then there exists a sequence of simple functions $(f_n)_n$ with $f_n \uparrow f$ pointwise. Theorem 1.68 implies that

$$\langle (f_n(T) - f(T))x, y\rangle = \int f_n - f\, dE_{x,y},$$

which tends to 0 as $n \to \infty$ because of the monotone convergence theorem (Theorem 1.8). By construction the operators $f_n(T) \in \mathcal{N}$. Since $\mathcal{N}$ is a von Neumann algebra it is WOT-closed, hence $f(T) \in \mathcal{N}$. Clearly $f(T)$ is positive because f is, so $f(T) \in \mathcal{N}_+$. Using the fact that τ is a normal trace gives $\tau(f_n(T)) \to \tau(f(T))$. Again an application of the monotone convergence theorem yields

$$\tau(f(T)) = \lim_{n\to\infty} \tau(f_n(T)) = \lim_{n\to\infty} \int_{\sigma(T)} f_n\, d\mu_{\tau,E} = \int_{\sigma(T)} \limsup_{n\to\infty} f_n\, d\mu_{\tau,E} = \int_{\sigma(T)} f\, d\mu_{\tau,E}. \tag{1.8}$$

This proves the proposition in the case that f is positive.
If f is not necessarily positive, but $f \in \mathcal{L}^1(\sigma(T), \mathcal{B}(\sigma(T)), \mu_{\tau,E})$, then $\int_{\sigma(T)} |f|\, d\mu_{\tau,E} < \infty$. We split up f in four parts, $f = \mathrm{Re}(f)^+ - \mathrm{Re}(f)^- + i\,\mathrm{Im}(f)^+ - i\,\mathrm{Im}(f)^-$ and apply the previous equality (1.8) to each of the four summands. It follows that $\int_{\sigma(T)} \mathrm{Re}(f)^+\, d\mu_{\tau,E} < \infty$ and for $\mathrm{Re}(f)^-, \mathrm{Im}(f)^+, \mathrm{Im}(f)^-$ as well. By linearity of the integral and trace we obtain $\int_{\sigma(T)} f\, d\mu_{\tau,E} = \tau(f(T))$, as desired. ⊠

The aim of the rest of this section is use spectral measures and the classical trace to give an equivalent formulation of a compact operator. Thereafter it will be generalised to semifinite traces to obtain a generalised notion of a compact operator.

Lemma 1.84. *If $A \in \mathcal{B}(\mathbb{R})$, $f : A \to \mathbb{C}$ is a bounded measurable function and T has the spectral decomposition $T = \int \lambda\, dE$, then*

$$\Big\|\Big(\int_A f\, dE\Big)\, x\Big\| \le \|f\|_\infty\, \|E(A)x\| \le \|f\|_\infty\, \|x\|,$$

for $x \in \mathrm{Dom}(f(T)) := \{x \in \mathcal{H} : \int |f|^2\, dE_{x,x} < \infty\}$.

Proof. By Hahn-Banach, for all $z \in \mathcal{H}$ there exists $z' \in \mathcal{H}$, $\|z'\| = 1$ such that $\|z\| = \langle z, z'\rangle =$

$\sup_{\|y\|=1}\langle z, y\rangle$. Let z be such that $\| \int_A f\, dE\, x\| = \langle \int_A f\, dE, x, z\rangle$. Then we have

$$\begin{aligned}\Big\|\Big(\int_A f\, dE\Big)\, x\Big\| &= \Big\langle \Big(\int_A f\, dE\Big)\, x, z\Big\rangle \\ &= \int_A f\, dE_{x,z} \\ &\leq \|f\|_\infty\, |E_{x,z}(A)| \\ &\leq \|f\|_\infty \sup_{\|y\|=1} |\langle E(A)x, y\rangle| \\ &= \|f\|_\infty\, \|E(A)x\| \\ &\leq \|f\|_\infty\, \|x\|.\end{aligned}$$

The second last equality also follows by Hahn-Banach. ⊠

Proposition 1.85. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let T be an (unbounded) self-adjoint operator with spectral decomposition $T = \int \lambda\, dE$. Then the following are equivalent:*

(i) for all $\lambda \notin \sigma(T)$, $(T-\lambda)^{-1} \in \mathcal{K}(\mathcal{H})$;

(ii) there exists a $\lambda_0 \notin \sigma(T)$ such that $(T-\lambda_0) \in \mathcal{K}(\mathcal{H})$;

(iii) for all $\lambda \in \mathbb{R}$, $\mathrm{Tr}(E([-\lambda, \lambda])) < \infty$.

Proof. Clearly (i) implies (ii). For the converse we use the so called first resolvent formula (1.4). For $\lambda \in \mathbb{R}$, $\lambda \neq \lambda_0$ we then have $(T-\lambda)^{-1} = (T-\lambda_0)^{-1} + (\lambda_0 - \lambda)(T-\lambda_0)^{-1}(T-\lambda)^{-1}$. Since $\mathcal{K}(\mathcal{H})$ is an ideal in $B(\mathcal{H})$ it follows that $(T-\lambda)^{-1} \in \mathcal{K}(\mathcal{H})$.
Now we will show (iii) implies (i). For $n \in \mathbb{N}$ consider $S_n := \int_{[-n,n]} \frac{1}{\lambda-\lambda_0}\, dE$. Let $Q_n : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection of $\mathcal{H}$ onto $S_n(\mathcal{H})$. Then by construction $Q_n(\mathcal{H}) \subset E([-n,n])(\mathcal{H})$. By assumption $\mathrm{Tr}(E([-n,n])) < \infty$, so $\mathrm{Tr}(Q_n) < \infty$. Therefore S_n is a finite rank operator. It remains to show that $(S_n)_n$ converges to $(T-\lambda_0)^{-1}$ in norm.
Observe that for the function $f : \mathbb{R} \to \mathbb{C}$, $\lambda \mapsto \frac{1}{\lambda-\lambda_0}$ we have $\mathrm{Dom}(f(T)) = \mathcal{H}$. This holds because for $A \in \mathcal{B}(\mathbb{R})$, we have $E(A) = E(A \cap \sigma(T))$ and by assumption $\lambda_0 \notin \sigma(T)$ and $\sigma(T)$ is closed. So there exists $\delta > 0$ such that $B_\delta(\lambda_0) \cap \sigma(T) = \emptyset$. But then

$$\int_{\mathbb{R}} \Big|\frac{1}{\lambda-\lambda_0}\Big|^2 dE_{x,x} \leq \frac{1}{\delta^2}\int_{\mathbb{R}} dE_{x,x} = \frac{1}{\delta^2}\langle E(\mathbb{R})x, x\rangle < \infty.$$

So indeed $\mathcal{H} = \mathrm{Dom}(f(T))$. Now let $\varepsilon > 0$. Select $N \in \mathbb{N}$ such that $|\frac{1}{N-\lambda_0}|$, $|\frac{1}{-N-\lambda_0}| < \varepsilon$. Then for $n > N$ and $x \in \mathcal{H}$, we have by Lemma 1.84

$$\begin{aligned}\|S_n x - (T-\lambda_0)^{-1}x\| &= \Big\|\Big(\int_{[-n,n]} \frac{1}{\lambda-\lambda_0}\, dE\Big)\, x - \Big(\int_{\mathbb{R}} \frac{1}{\lambda-\lambda_0}\, dE\Big)\, x\Big\| \\ &= \Big\|\Big(\int_{(-\infty,-n)\cup(n,\infty)} \frac{1}{\lambda-\lambda_0}\, dE\Big)\, x\Big\| \\ &\leq \varepsilon\|x\|. \end{aligned} \tag{1.9}$$

This establishes (i).
Now suppose (i) holds we will show (iii) is true. Because $\sigma(T) \subset \mathbb{R}$, the operator $(T-i)^{-1}$ is compact. So there exists a sequence of finite rank operators $(S_n)_n$ such that S_n converges to $(T-i)^{-1}$ in norm. Since $(T-i)^{-1} = \int_{\mathbb{R}}(\lambda-i)^{-1}\, dE$, for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $\nu_0 > 0$ such that for all $n > N$ and $\nu > \nu_0$ it holds that

$$\Big\|S_n - \int_{[-\nu,\nu]} \frac{1}{\lambda-i}\, dE\Big\| \leq \|S_n - (T-i)^{-1}\| + \Big\|(T-i)^{-1} - \int_{[-\nu,\nu]} \frac{1}{\lambda-i}\, dE\Big\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{1.10}$$

Now suppose that $\mathrm{Tr}(E([-\mu_0,\mu_0])) = \infty$. Let $\varepsilon := \frac{1}{\mu_0^2+1}$ and select corresponding N and ν_0. Note that because $\mathrm{Tr}(E([-\mu_0,\mu_0])) = \infty$, the range of $E([-\mu_0,\mu_0])$ is infinite dimensional. Let $n > N$, since S_n is finite rank there exists $x \in \ker(S_n) \cap E([-\mu_0,\mu_0])\mathcal{H}$, $x \neq 0$. Fix such an element x of norm 1. Then for this x and for $\nu > \max\{\nu_0,\mu_0\}$ the following equality holds

$$\Big(\int_{[-\nu,\nu]} \frac{1}{\lambda - i}\, dE\Big)x = \Big(\int_{[-\mu_0,\mu_0]} \frac{1}{\lambda - i}\, dE\Big)x.$$

This implies that

$$\begin{aligned}
\Big\| S_n x - \big(\int_{[-\nu,\nu]} \frac{1}{\lambda - i}\, dE\big)\, x \Big\| &= \Big\| \big(\int_{[-\mu_0,\mu_0]} \frac{1}{\lambda - i}\, dE\big)\, x \Big\| \\
&= \sup_{\|y\|=1} \Big| \Big\langle \big(\int_{[-\mu_0,\mu_0]} \frac{1}{\lambda - i}\, dE\big)\, x, y \Big\rangle \Big| \\
&\geq \Big| \Big\langle \big(\int_{[-\mu_0,\mu_0]} \frac{1}{\lambda - i}\, dE\big)\, x, x \Big\rangle \Big| \\
&= \Big| \int_{[-\mu_0,\mu_0]} \frac{\lambda + i}{\lambda^2 + 1}\, dE_{x,x} \Big| \\
&\geq \Big| \mathrm{Im}\,\big(\int_{[-\mu_0,\mu_0]} \frac{\lambda + i}{\lambda^2 + 1}\, dE_{x,x}\big) \Big| \\
&= \Big| \int_{[-\mu_0,\mu_0]} \frac{1}{\lambda^2 + 1}\, dE_{x,x} \Big| \\
&> \varepsilon\, |E_{x,x}([-\mu_0,\mu_0])| \\
&= \varepsilon\, \langle E([-\mu_0,\mu_0])x, x \rangle \\
&= \varepsilon.
\end{aligned}$$

But this is a contradiction with equation (1.10). Hence we established (iii). ⊠

A converse of theorem 1.62 holds, but we need to assume that the resolvent is compact.

Proposition 1.86. *Suppose T is a self-adjoint operator and $(T - \lambda)^{-1}$ compact for some $\lambda \notin \sigma(T)$, then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T.*

Proof. Let $\lambda \in \rho(T)$ and assume that the operator $(T - \lambda)^{-1}$ is compact. By [27, Thm. VI.15] the spectrum $\sigma((T-\lambda)^{-1})$ is discrete. Therefore $\sigma(T)$ is discrete. Select $\lambda_0 \in \mathbb{R} \setminus \sigma(T)$. Then by the first resolvent (1.4) $(T-\lambda_0)^{-1}$ is compact and since $\lambda_0 \in \mathbb{R}$ the operator is self-adjoint. Thus by [27, Thm. VI.16] $(T-\lambda_0)^{-1}$ admits an orthonormal basis $(e_n)_n$ consisting of eigenvectors with eigenvalues $(\mu_n)_n$ for $(T - \lambda_0)^{-1}$. But then $(e_n)_n$ is also an orthonormal basis of eigenvectors with eigenvalues $(\mu_n^{-1} + \lambda_0)_n$ for T. ⊠

We can easily generalise Proposition 1.85 to semifinite infinite traces on a type II von Neumann algebra. This will be the content of Theorem 1.90. To obtain such a similar result we first need to introduce some terminology. Note that for a projection $P : \mathcal{H} \to \mathcal{H}$ we have $\mathrm{Tr}(P) = \dim(\mathrm{ran}(P))$, the following is a straightforward generalisation of this fact.

Notation 1.87. Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ and τ be a trace on $\mathcal{M}$. We denote
$\mathcal{P}(\mathcal{M}) := \{P \in \mathcal{M} : P \text{ is a projection}\}$, the *projections*;
$\mathcal{P}_{\mathrm{f}}(\mathcal{M},\tau) := \{P \in \mathcal{P}(\mathcal{M}) : \tau(P) < \infty\}$, the *$\tau$-finite projections*;
$\mathcal{R}(\mathcal{M},\tau) := \mathrm{span}(\mathcal{P}_{\mathrm{f}}(\mathcal{M},\tau))$, the *$\tau$-finite rank operators*;
$\mathcal{K}(\mathcal{M},\tau) := \mathrm{clo}(\mathcal{R}(\mathcal{M},\tau))$, the *$\tau$-compact operators*. Here the closure is taken in the norm topology.

Lemma 1.88. *Let $\mathcal{A}$ be a Banach algebra and $I \subset \mathcal{A}$ an ideal. Then $\overline{I} \subset \mathcal{A}$ is a closed ideal.*

Proof. That $\overline{I}$ is a linear subspace and that it is closed, is by construction. Remains to show that it is an ideal. So let $x \in \overline{I}$ and $a \in \mathcal{A}$. Pick a net $(x_i)_i \subset I$ such that $\|x - x_i\| \to 0$. Then $\|xa - x_i a\| \leq \|x - x_i\|\|a\| \to 0$. But $x_i a \in I \subset \overline{I}$ for all i. By closedness $xa \in \overline{I}$. Similarly for x^* we have $\|x_i^* - x^*\| \to 0$ and $x_i^* \in I \subset \overline{I}$ for all i, by closedness $x^* \in \overline{I}$. ⊠

As a direct corollary of this lemma we obtain.

Corollary 1.89. *Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ and τ be a trace on $\mathcal{M}$, then $\mathcal{K}(\mathcal{M}, \tau)$ is a closed ideal in $B(\mathcal{H})$ in the norm topology. Furthermore $\mathcal{K}(\mathcal{M}, \tau)$ equals the closed ideal generated by the τ-finite projections.*

Theorem 1.90. *Let $\mathcal{M}$ be an infinite, semi-finite von Neumann algebra acting on $\mathcal{H}$, equipped with a normal, faithful trace τ. Suppose T is a self-adjoint $\mathcal{M}$-affiliated (unbounded) operator on $\mathcal{H}$ with spectral decomposition $T = \int \lambda \, dE$. Then the following are equivalent:*

(i) for all $\lambda \notin \sigma(T)$, $(T - \lambda)^{-1} \in \mathcal{K}(\mathcal{M}, \tau)$;

(ii) there exists a $\lambda_0 \notin \sigma(T)$ such that $(T - \lambda_0) \in \mathcal{K}(\mathcal{M}, \tau)$;

(iii) for all $\lambda \in \mathbb{R}$, $\tau(E([-\lambda, \lambda])) < \infty$.

Proof. We can copy most of the proof of Proposition 1.85 by replacing Tr by τ. Equivalence of (i) and (ii) remains the same.
In the proof of (iii) implies (i) it remains to show that the constructed operators S_n are elements of $\mathcal{M}$. But this follows from 1.77, because the function

$$f : \sigma(T) \to \mathbb{C}, \; \lambda \mapsto \frac{1}{\lambda - \lambda_0}$$

is continuous and bounded ($\sigma(T)$ is closed and $\lambda_0 \notin \sigma(T)$). And therefore the functions $f_n := f 1_{[-n,n]}$ are Borel-measurable and bounded. So S_n are bounded normal operators affiliated with $W^*(T)$, hence $S_n \in W^*(T) \subset \mathcal{M}$. Let Q_n be the orthogonal projection of $\mathcal{H}$ on $S_n(\mathcal{H})$. Then by the same argument as before we can show that $\tau(Q_n) < \infty$, so $S_n \in \mathcal{R}(\mathcal{H}, \tau)$. And the estimate (1.9) shows $S_n \to (T - \lambda_0)^{-1}$ in norm.
In the implication (i) to (iii) we again argue by contradiction. Suppose $\tau(E([-\mu_0, \mu_0])) = \infty$. The proof is almost the same as in 1.85, the only extra thing we have to check is that we can find an element $x \in \ker(S_n) \cap E([-\mu_0, \mu_0])$. This can be done. Namely let Q_n be again the projection on the range of S_n. By orthogonality of Q_n and $1 - Q_n$ we obtain

$$\begin{aligned}
\tau(E([-\mu_0, \mu_0])) &= \tau(E([-\mu_0, \mu_0] \wedge Q_n + E([-\mu_0, \mu_0]) \wedge (1 - Q_n)) \\
&= \tau(E([-\mu_0, \mu_0] \wedge Q_n)) + \tau(E([-\mu_0, \mu_0]) \wedge (1 - Q_n)) \\
&\leq \tau(Q_n) + \tau(E([-\mu_0, \mu_0]) \wedge (1 - Q_n)).
\end{aligned}$$

Since $\tau(Q_n) < \infty$ for all n

$$\tau(E([-\mu_0, \mu_0]) \wedge (1 - Q_n)) \geq \tau(E([-\mu_0, \mu_0])) - \tau(Q_n) = \infty.$$

Hence $(E([-\mu_0, \mu_0]) \wedge (1 - Q_n))\mathcal{H} \neq \{0\}$. In other words exists an element $x \in E([-\mu_0, \mu_0])\mathcal{H} \cap \ker(S_n)$, with $x \neq 0$. ⊠

Definition 1.91. If $\mathcal{M}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, with a normal, faithful trace τ and if T is a self-adjoint $\mathcal{M}$-affiliated operator on $\mathcal{H}$ which satisfies one of the conditions of Theorem 1.90, then T is called a *τ-discrete operator.*

2 Essentials from noncommutative geometry

In this section we will give the basics of noncommutative geometry. In the first two subsections we will closely follow the lecture notes by Landsman [22]. If in these subsections of a result no proof is given, the proof can be found in those notes. In this section we will describe how noncommutative geometry is a generalisation of differential geometry. We will use the torus as our guiding example. We choose the torus, because its tangent bundle is trivial which makes the differential geometry easier.

2.1 Differential geometry

In this section we will review the concepts from differential geometry which we will need later on when we consider the spin manifolds, see e.g. [15] for more information on differential geometry. We assume familiarity with manifolds and vector bundles. We would like to define the Dirac operator on a spin manifold. For this we need to define the spin-bundle. We denote $\mathbb{T}^2 := S^1 \times S^1$ for the torus, where S^1 is the circle. Equivalently we can also define $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Example 2.1. We have $T\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2$, i.e. the tangent bundle is trivial.

Proof. Define for $i = 1, 2$ the vector fields $Y_i(x, y) := (x, y) \times e_i$ where $(e_i)_{i=1,2}$ are the standard basis vectors of $\mathbb{R}^2$. Then Y_i are smooth and $Y_i(x + m, y + n) = Y_i(x, y)$ for $m, n \in \mathbb{Z}$. Thus y_i indeed defines a vector field on $\mathbb{T}^2$. In each fiber $T_{(x,y)}\mathbb{T}^2$ the sections $(Y_i)_{i=1,2}(x, y)$ form a complete orthonormal set. Therefore [15, Thm. 1.40] implies $T\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2$. ⊠

Definition 2.2. A *metric* g on a vector bundle E over M is given by a collection of inner products $\{g_p : E_p \times E_p \to \mathbb{C} : p \in M\}$, such that for all smooth section $s, s' \in \Gamma(E)$ the map $p \mapsto g_p(s_p, s'_p)$ is smooth.

Example 2.3. Since the tangent bundle $T\mathbb{T}^2$ is trivial, we can define a metric on the tangent bundle which is independent of the fiber. Namely define the metric $g : T\mathbb{T}^2 \times T\mathbb{T}^2 \to \mathbb{R}$ by

$$g_x : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad g_x(v, w) = v_1 w_1 + v_2 w_2.$$

Note that this is just the standard inner product in each fiber. It is clear that if $s, s' : \mathbb{T}^2 \to T\mathbb{T}^2$ are smooth sections of the tangent bundle, the map $x \mapsto g_x(s_x, s'_x)$ is smooth.

Suppose a group acts on a manifold, if this action satisfies some nice properties the quotient space is again a manifold. These conditions are satisfied for principal fibre bundles

Definition 2.4. Suppose P is a manifold and G a Lie group which acts on P from the right. We let $M := P/G$ with projection $\pi : P \to M$. We say that the pair (P, G) is a *principal G-bundle* if it satisfies the conditions:

(i) G acts free on P (i.e. if $pg = p$ then $g = e$);

(ii) for all $x \in M$ there exists $U \subset M$ open and a bundle morphism $P|_U : \pi^{-1}(U) \to U \times G$ such that $(P|_U(x, h))g = P|_U(x, hg)$, thus $P|_U$ intertwines the action of G on P with the action of G on $U \times G$.

Lemma 2.5. *If (P, G) is a principal fibre bundle then*

(i) the quotient space $M = P/G$ is a manifold;

(ii) the projection $\pi : P \to M$ is G-invariant (i.e. $\pi(pg) = \pi(p)$).

If the group also acts on a vector space under some conditions we can use this to define a vector bundle on the quotient space. This will be the associated vector bundle.

Definition 2.6. Let (P, G) be a principal G-bundle and let G be acting on a vector space V. Again denote $M := P/G$. Then $E := P \times_G V := (P \times V)/G$ is a vector bundle over M with fibers $E_p = V$. The quotient is defined by the right action of G via $(p, v)g := (pg, g^{-1}v)$. E is called the *associated vector bundle* of V to (P, G).

The smooth sections of this vector bundle are related to the smooth G-invariant functions of P to V.

Lemma 2.7. *Given a principal fiber bundle (P, G) it holds that*

$$\Gamma(P \times_G V) \cong C^\infty(P, V)^G \ \big(:= \{f \in C^\infty(P, V) \,:\, f(pg) = g^{-1} f(p) \ \textit{for all}\ p \in P,\ g \in G\}\big).$$

We would like to differentiate sections of vector bundles along vector fields. This is what covariant derivatives do for us.

Definition 2.8. Denote by $\mathfrak{X}(M)$ the smooth vector fields on M. A *covariant derivative* ∇ on a vector bundle E is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \qquad (X, \sigma) \mapsto \nabla_X \sigma,$$

which for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ satisfies:

(i) $\nabla_{fX}\sigma = f\nabla_X \sigma$;

(ii) $\nabla_X(f \cdot \sigma) = \nabla_X(\sigma) \cdot f + \sigma \cdot (Xf)$.

Definition 2.9. A *connection* on a vector bundle E is a linear map

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes_{C^\infty(M)} \Gamma(T^*M)$$

which satisfies the Leibniz rule

$$\nabla(\sigma \cdot f) = (\nabla\sigma) \cdot f + \sigma \otimes df \qquad \text{for all } f \in C^\infty(M), \sigma \in \Gamma(E).$$

Here $d : C^\infty(M) \to \Gamma(T^*M)$ is the exterior derivative.

If we have an associated vector bundle one can look at covariant derivatives in another way which appears to be equivalent (cf. Lemma 2.11).

Definition 2.10. Suppose (P, G) is a principal G-bundle, let $M := P/G$. An *Ehresmann connection* on P is a collection of maps $\{h_p : T_{\pi(p)}M \to T_pP \,:\, p \in P\}$ which satisfies the conditions

(i) $\pi'(h_p(\sigma)) = \sigma$, for all $\sigma \in T_pP$ (here π' is the derivative of π);

(ii) $h_{pg} = R'_g \circ h_p$ for all $p \in P$, $g \in G$ (here R_g is the map $p \mapsto pg$);

(iii) the vector field $p \mapsto h_p(X)$ is smooth for all $X \in \mathfrak{X}(M)$.

Lemma 2.11. *Suppose $E := P \times_G V$ is an associated vector bundle over the manifold M and $(h_p)_{p\in P}$ is an Ehresmann connection on P. Then the map*

$$\nabla_X \sigma(p) := h_p(X_{\pi(p)})\sigma(p) \qquad \sigma \in C^\infty(P, V)^G \cong \Gamma(P \times_G V)$$

defines a covariant derivative on E.

There is also a converse to this construction. Given a covariant derivative, you can construct an Ehresmann connection.

Remark 2.12. Suppose $E := P \times_G V$ is an associated vector bundle over the manifold M and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a covariant derivative. Then $\tilde{\nabla}$ is a connection where $\tilde{\nabla}(\sigma) := \nabla.\sigma \in \Gamma(E) \otimes T^*M$. I.e. $\tilde{\nabla}(\sigma)(X) = \nabla_X\sigma$ for $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$.
Assume we work locally such that $P|_U \cong U \times G$ and $(P \times_G V)|_U \cong U \times V$. Then $\tilde{\nabla} - d$ is linear, thus because we are working locally there exists a smooth T_eG-valued function A such that on U we have $\nabla - d = A$. Now define

$$h_{x,g}\sigma := \sigma - \langle A(x), \sigma\rangle.$$

This defines an Ehresmann connection on P.

A special connection on the tangent space is given by the Levi-Civita connection.

Theorem 2.13. *For each manifold M with metric g there exists a unique covariant derivative ∇ on the tangent bundle TM, called the* Levi-Civita connection*, which satisfies the following two requirements.*

(i) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$*, for all* $X, Y, Z \in \mathfrak{X}(M)$*;*

(ii) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$*, for all* $X, Y \in \mathfrak{X}(M)$*.*

If a connection ∇ satisfies condition (i) it is called *metric*, if it satisfies condition (ii) the connection is *torsion free*.

Example 2.14. For $X, Y \in \Gamma(T\mathbb{T}^2)$, write $X = x_1\partial_1 + x_2\partial_2, Y = y_1\partial_1 + y_2\partial_2$. Consider the vector field $\nabla_X Y := \sum_{i=1}^2 X(y_i)\partial_i$, then $\nabla : (X,Y) \mapsto \nabla_X Y$ is the Levi-Civita connection of the torus.

Proof. Because the Levi-Civita connection exists and is unique, it is sufficient to check that ∇ is a connection and that it satisfies the conditions of the Levi-Civita connection.
It is immediate that ∇ is bilinear. Suppose X, Y, Z are vector fields on $\mathbb{T}^2$ and $f : \mathbb{T}^2 \to \mathbb{R}$ a smooth map. Then

$$\begin{aligned}
\nabla_{fX} Y &= \sum_i fX(y_i)\partial_i = f\sum_i X(y_i)\partial_i = f\nabla_X Y;\\
\nabla_X(fY) &= \sum_i X(fy_i)\partial_i = \sum_i (X(f)y_i\partial_i + fX(y_i)\partial_i) = X(f)Y + f\nabla_X Y;\\
Xg(Y,Z) &= \sum_i x_i\partial_i(y_1z_1 + y_2z_2) = \sum_{i,j} x_i\partial_i(y_j)z_j + x_iy_j\partial_i(z_j)\\
&= \sum_i X(y_i)z_i + y_iX(z_i) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z);\\
[X,Y] &= \sum_{i,j} x_i\partial_i(y_j\partial_j) - y_i\partial_i(x_j\partial_j) = \sum_{i,j} x_i\partial_i(y_j)\partial_j - y_i\partial_i(x_j)\partial_j + \sum_{i,j} x_iy_j\partial_i\partial_j - y_ix_j\partial_i\partial_j\\
&= \sum_{i,j} x_i\partial_i(y_j)\partial_j - y_i\partial_i(x_j)\partial_j = \nabla_X Y - \nabla_Y X.
\end{aligned}$$

The first two computations show that ∇ is a connection, the third shows it is metric and the fourth shows it is torsion free. Hence ∇ is the Levi-Civita connection. ⊠

2.2 Towards noncommutative geometry

Here we will describe the spin manifolds and how these give rise to a spectral triple. We start with some algebraic objects: Clifford algebras and spin groups. With these we will construct the spinor bundle so that we can define a Hilbert space associated to the manifold M on which the Dirac operator can act.

Definition 2.15. Suppose V is a vector space. The *tensor algebra* $T(V) := \bigoplus_{n=0}^\infty V^{\otimes^n}$. The product of elementary tensors is given by concatenation, to be explicit

$$(v_1 \otimes \dots v_m) \cdot (w_1 \otimes \dots w_n) := v_1 \otimes \dots v_m \otimes w_1 \otimes \dots w_n.$$

This product can be extended with distributivity to arbitrary elements of $T(V)$.

Definition 2.16. Let $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the bilinear map given by the standard inner product (i.e. $g(v,w) := v_1w_1 + \dots v_nw_n$). Denote by $\mathcal{I}_g$ the two-sided ideal generated by elements of the form $v \otimes v - g(v,v)$. Now the algebra $Cl_n^+ := T(\mathbb{R}^n)/\mathcal{I}_g$ is called a *Clifford algebra*. The

multiplication in Cl_n^+ is called the *Clifford multiplication*. There are more kinds of Clifford algebras when one varies the bilinear map g. Since we do not need them, we will not define those algebras.
Let $\mathcal{I}_a$ be the two-sided ideal generated by the elements $v \otimes v$ for $v \in V$. Define the exterior algebra $\bigwedge^\bullet V := T(V)/\mathcal{I}_a$.

It is possible to select a specific set in Cl_n^+ which is a group. This is called the Spin group.

Definition 2.17. Let

$$Spin(n) := \{tv_1 \cdots v_p \,:\, p \text{ is even}, \, t = \pm 1, \, v_i \in \mathbb{R}^n, g(v_i, v_i) = 1\} \subset Cl_n^+.$$

$Spin(n)$ is a group, called the *spin group*. Multiplication in $Spin(n)$ is given by restricting the multiplication of Cl_n^+. Define an anti-automorphism $^!$ of $Spin(n)$ by

$$(tv_1 \cdots v_p)^! := tv_p \cdots v_1.$$

Using this anti-automorphism it is possible to let $Spin(n)$ act on $\mathbb{R}^n$. Again the Clifford multiplication is needed.

Lemma 2.18. *For $s \in Spin(n)$ define a linear map*

$$\lambda(s) : \mathbb{R}^n \to \mathbb{R}^n, \qquad w \mapsto sws^!.$$

It holds that $\lambda(Spin(n)) \subset SO(n)$. This follows from the fact that for a vector v the map $w \mapsto vwv$ is a reflection and the composition of two reflections is in $SO(n)$. In fact the following sequence is exact

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \xrightarrow{\lambda} SO(n) \longrightarrow 1 \, .$$

This map λ is called the double covering *of $SO(n)$.*

Example 2.19. We have $Spin(2) \cong SO(2)$, but also the sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(2) \xrightarrow{\lambda} SO(2) \longrightarrow 1$$

is exact. Here the map $\lambda : Spin(2) \to SO(2)$ is given by $\lambda(u)x := uxu^!$, where the multiplication is in the Clifford algebra and if $(u_1 \cdots u_k)^! := u_k \cdots u_1$.

Proof. We will only show that $Spin(2) \cong SO(2)$. Denote by e_1, e_2 the standard basis of $\mathbb{R}^2$. Then a basis for the Clifford algebra Cl_2^+ is given by 1, e_1, e_2, $e_1 \otimes e_2$. One readily checks that the following map given on the basis vectors extends linearly to an algebra isomorphism φ between Cl_2^+ and $M_2(\mathbb{R})$.

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad\qquad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad e_1 \otimes e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

By definition the spin group is given by

$$\begin{aligned} Spin(2) &= \{tv_1 \cdots v_p : p \text{ even}, \, v_i \in \mathbb{R}^2, \, g(v_i, v_i) = 1 \text{ for } i = 1, \cdots p, \, t = \pm 1\} \\ &= \{tv_1 v_2 : v_i \in \mathbb{R}^2, \, g(v_i, v_i) = 1 \text{ for } i = 1, 2, \, t = \pm 1\} \cup \{\pm 1\}. \end{aligned}$$

This equality holds because of the multiplication in Cl_2^+. We now show that φ restricts to a group isomorphism ψ between $Spin(2)$ and $SO(2)$. Let $vw \in Spin(2)$. Say $v = (v_1, v_2)$, $w = (w_1, w_2)$ with $v_1^2 + v_2^2 = w_1^2 + w_2^2 = 1$. Then a straightforward computation shows

$$\psi(vw) = \begin{pmatrix} v_1 w_1 + v_2 w_2 & -v_1 w_2 + v_2 w_1 \\ v_1 w_2 - v_2 w_1 & v_1 w_1 + v_2 w_2 \end{pmatrix}$$

Then $\det(\psi(vw)) = 1$ and $\psi(vw)\psi(vw)^t = I$. So indeed ψ maps into $SO(2)$. Since φ is an isomorphism, ψ is an injective group homomorphism. So it remains to show surjectivity. Recall

$$SO(2) = \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} : \alpha \in [0, 2\pi) \right\} =: \{r_\alpha : \alpha \in [0, 2\pi)\}.$$

Let $r_\alpha \in SO(2)$ then $\psi((\cos(\alpha), \sin(\alpha))(1, 0)) = r_\alpha$. Hence we have surjectivity. The statement about the exact sequence is Lemma 2.18. ⊠

An oriented manifold is a manifold in which we can choose an orientation, that is we can choose in each fiber a positively oriented basis $(e_p^1, \ldots, e_p^n)$ such that for each i $p \mapsto e_p^i$ varies in a smooth way along the fibers. To be precise, there exists an atlas such that the Jacobians of the transition functions are positive. In the class of oriented manifolds we can select a class with extra structure.

Definition 2.20. Suppose M is a manifold of dimension n. We say M is a *spin manifold* if M is oriented and there exists a principal $Spin(n)$-bundle $Spin(M)$ with an isomorphism

$$Spin(M) \times_{Spin(n)} \mathbb{R}^n \cong TM.$$

We continue by computing the associated $spin(2)$-bundle $Spin(\mathbb{T}^2)$ on the torus. It appears this bundle is again trivial.

Example 2.21. The spin bundle $Spin(\mathbb{T}^2) \cong Spin(2) \times \mathbb{T}^2$.

Proof. The spin bundle on a spin manifold M is defined implicitly by

$$Spin(M) \times_{Spin(n)} \mathbb{R}^n \cong TM,$$

via the isomorphism $[p, v] \mapsto p(v)$. Thus it is sufficient to show that we have an isomorphism

$$(Spin(2) \times \mathbb{T}^2) \times_{Spin(2)} \mathbb{R}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2 = T\mathbb{T}^2.$$

Recall that from the construction of the associated vector bundle it is given that the group $Spin(2)$ acts on $Spin(\mathbb{T}^2) \times \mathbb{R}^2$ by $(p, v)g := (pg, g^{-1}v)$. We define the map

$$\varphi : (Spin(2) \times \mathbb{T}^2) \times_{Spin(2)} \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2, \qquad [(u, p), v] \mapsto (p, u(v)).$$

Since

$$\varphi(((u, p), v)g) = \varphi((ug, p)g^{-1}v) = (p, ug(g^{-1}v)) = (p, u(v)) = \varphi((u, p), v),$$

the map φ is well-defined. It is clear that φ restricted to a fiber is linear a linear map. So φ is a vector bundle morphism.
Because $\varphi((I, p), v) = (p, v)$, φ is surjective. It remains to show that φ is injective. Suppose $\varphi([(u, p), v]) = \varphi([(u', p'), v'])$. Then $p = p'$ and $u(v) = u'(v')$. Put $g = u^{-1}u'$, which lies in $Spin(2)$. From this we obtain

$$((u, p), v)g = ((ug, p), g^{-1}v) = ((uu^{-1}u', p'), u'^{-1}uv) = ((u', p'), v').$$

Hence $[(u, p), v] = [(u', p'), v']$ thus φ is injective. ⊠

The Dirac operator on a spin manifold is a first order partial differential operator on a specific vector bundle. On this bundle one needs to be able to multiply with elements from TM, on the spinor bundle (to be defined later) this is possible. For this multiplication we will use the Fock representation. To deal with (partial) derivations we need connections, the Levi-Civita connection will be used to construct a connection on the spinor bundle.

Lemma 2.22. *Given $n \in \mathbb{N}$, there exists an irreducible faithful representations π_F of Cl_n^+ on the vector space $\bigwedge^\bullet \mathbb{C}^{n/2} \cong \mathbb{C}^{2^{n/2}}$ if n is even and on the vector space $\bigwedge^\bullet \mathbb{C}^{(n-1)/2} \cong \mathbb{C}^{2^{(n-1)/2}}$ if n is odd. This representation is called the* Fock representation.

Restricting this representation to $Spin(n) \subset Cl_n^+$ gives a representation of $Spin(n)$. This one needs no longer to be irreducible. Indeed, if n is even the representation splits in two inequivalent representations.
We will compute the Fock representation for the torus, thus for $Spin(2)$.

Example 2.23. The Fock representation of $Spin(2)$ on $\mathbb{S}_2$ is given on the generators by

$$1 = e_1e_1 = e_2e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad e_1e_2 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3.$$

This representation is reducible.

Proof. We have an (irreducible) representation π_F of Cl_2^+ on $\mathbb{C}^2$ given on the generators by $e_1 \mapsto \sigma_1, e_2 \mapsto \sigma_2, e_1e_2 \mapsto i\sigma_3$, where σ_i are the Pauli matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2 \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \tag{2.1}$$

This is indeed a representation of Cl_2^+, because $\pi_F(e_i^2) = \pi_F(1) = I = \sigma_i^2 = \pi_F(e_i)^2$, thus π_F preserves the relation $vv = g(v,v)$. The representation π_F is irreducible, because if it is not, there would be a vector $v \in \mathbb{C}^2$ and scalars $\lambda_i \in \mathbb{C}$ such that $\sigma_i v = \lambda_i v$ for $i = 1, 2, 3$. But it is easily seen that this is impossible.
The restriction of π_F to $Spin(2) \subset Cl_2^+$ gives a representation. This representation is not irreducible, because it leaves the spaces $\mathbb{C}\binom{1}{0}$ and $\mathbb{C}\binom{0}{1}$ invariant. We also see that σ_1 is a map between these two invariant subspaces, σ_1 is a grading operator. ⊠

Definition 2.24. The *spinor bundle* $\mathcal{S}_n$ is defined as the vector bundle associated to the bundle principal-$Spin(n)$ bundle $Spin(M)$, thus $\mathcal{S}_n := Spin(M) \times_{Spin(n)} \mathbb{S}_n$. Here $Spin(n)$ acts on $\mathbb{S}$ via the Fock representation.
As a special case of lemma 2.7 we have the following isomorphism

$$C^\infty(Spin(M), \mathbb{S}_n)^{Spin(n)} \cong \Gamma(\mathcal{S}_n). \tag{2.2}$$

Now we have every ingredient for the Dirac operator. In the following definition we will give the definition of the Dirac operator on a spin manifold. We will give a local expression. For a more detailed explanation one can consult e.g. [19, 22].

Definition 2.25. Suppose M is an oriented manifold. Given the Levi-Civita connection on TM we can define an Ehresmann connection on $SO(M)$ via Remark 2.12. Denote this connection by $h_p^g : T_{\pi(p)}M \to T_pSO(M)$. Because of the double covering of $SO(n)$ by $Spin(n)$ (cf. Lemma 2.18) we have a projection $\pi : Spin(M) \to SO(M)$. Therefore in each fiber the tangent map

$$\pi_p' : T_p(Spin(M)) \to T_{\pi(p)}(SO(M))$$

is an isomorphism. We can lift the Ehresmann connection to $Spin(M)$ via this map π. Namely define

$$h_p^S := (\pi_p')^{-1} \circ h_{\pi'(p)}^g : T_{\pi(p)}M \to T_pSpin(M).$$

Use the isomorphism (2.2) and lemma 2.11 to obtain a connection on $\mathcal{S}_n$ by

$$\nabla_X^S \sigma(p) := h_p^S(X_{\pi(p)})\sigma(p), \qquad \sigma \in \Gamma(\mathcal{S}_n),\ X \in \Gamma(TM).$$

This connection is called the *spin connection* of M. We now turn to the other object which is needed for the Dirac operator. Recall the Fock representation (Definition 2.22) of the Clifford algebra $\pi_F : Cl_n^+ \to \mathrm{End}(\mathbb{S}_n)$. Locally $TM|_U \cong V \times \mathbb{R}^n$, for some $V \subset \mathbb{R}^n$. Thus in this chart U, if $(x, w) \in V \times \mathbb{R}^n$, we can act with $\pi_F(w)$ on sections $\varphi \in \Gamma(\mathcal{S}_n)$. In particular let $(e_a)_a$ be an orthonormal frame[1] and $\psi \in \Gamma(\mathcal{S}_n)$, then $\pi_f(e_a)\psi(x)$ is well-defined. Note that the elements of a local frame are vector fields themselves. Now we can locally define the Dirac operator by

$$\not{D}\psi(x) := -i\sum_{a=1}^{n} \pi_F(e_a)\nabla^S_{e_a}\psi(x), \qquad (\psi \in \Gamma(\mathcal{S}_n)),$$

where $(e_a)_a$ is an orthonormal frame of TM, π_F is the Fock represenation and ∇^S_a is the spin connection.
We will also briefly sketch the global construction. Globally $\not{D}$ is defined by

$$i\not{D}:\ \mathcal{S}_n \xrightarrow{\nabla^S} \mathcal{S}_n \otimes \Gamma(T^*M) \xrightarrow{\text{flip}\circ\sharp} \Gamma(TM)\otimes\mathcal{S}_n \xrightarrow{c} \mathcal{S}_n\ .$$

Here ∇^S is again the spinor connection on the spinor bundle $\mathcal{S}_n$. For vector spaces the map flip is by

$$\text{flip}: V\otimes W \to W\otimes V, \qquad v\otimes w \mapsto w\otimes v.$$

The function $\sharp$ gives the canonical isomorphism induced by the metric g between T^*M and TM. To be precise $\varphi \mapsto \varphi^\sharp$, where $\varphi^\sharp \in TM$ is the unique element such that $g(\varphi^\sharp, X) = \varphi(X)$ for all $X \in \Gamma(TM)$. c denotes the Clifford multiplication, this is a action of TM on the spinor bundle. Since this involves Clifford modules we will not go into details about this operator.

Finally we can compute the Dirac operator on the torus.

Example 2.26. Since the metric is independent of the coordinates (in each fiber it is the standard inner product of $\mathbb{R}^2$), all the Cristoffel symbols vanish. We choose the orthonormal frame $(e_i)_{i=1,2}$ of $\mathcal{S}_2$, where $e_1(x) = \big(x, \binom{1}{0}\big) \in \mathcal{S}_2 = \mathbb{C}^2$ and $e_2(x) = \big(x, \binom{0}{1}\big) \in \mathcal{S}_2 = \mathbb{C}^2$. Using the Fock representation of Example 2.23 and the fact that in the case of the torus the spinor connection is simply derivation in the direction of the vector field, we can locally write the Dirac operator as

$$\begin{aligned}
\not{D} &: \Gamma(\mathcal{S}_2) \to \Gamma(\mathcal{S}_2);\\
\not{D}\psi(x) &= -i\sum_{j=1,2} \pi_F(x_j)\partial_{x_j}\psi(x)\\
&= -i(\pi_F(e_1)\partial_{x_1}\psi(x) + \pi_F(e_2)\partial_{x_2}\psi(x))\\
&= -i(\sigma_1\partial_{x_1} + \sigma_2\partial_{x_2})\psi(x)\\
&= \begin{pmatrix} 0 & -i\partial_{x_1} - \partial_{x_2} \\ -i\partial_{x_1} + \partial_{x_2} & 0 \end{pmatrix}\begin{pmatrix}\psi_1(x_1,x_2)\\ \psi_2(x_1,x_2)\end{pmatrix}.
\end{aligned}$$

Since every bundle we considered on the torus was trivial, in particular the spinor bundle, this expression for $\not{D}$ holds globally. From now on we will write x and y instead of x_1 and x_2. The Dirac operator was invented because in quantum mechanics one needed first order partial differential operators and not second order. So the Laplacian needed to be reduced to a first order system. Indeed, in our case of the torus $\not{D}$ can be considered as the square root of the Laplacian, since

$$\begin{aligned}
\not{D}^2 &= \begin{pmatrix} 0 & -i\partial_x - \partial_y \\ -i\partial_x + \partial_y & \end{pmatrix}\begin{pmatrix} 0 & -i\partial_x - \partial_y \\ -i\partial_x + \partial_y & \end{pmatrix}\\
&= \begin{pmatrix} (-i\partial_x - \partial_y)(-i\partial_x + \partial_y) & 0 \\ 0 & (-i\partial_x - \partial_y)(-i\partial_x + \partial_y) \end{pmatrix}\\
&= \begin{pmatrix} -\partial_x^2 - \partial_y^2 & 0 \\ 0 & -\partial_x^2 - \partial_y^2 \end{pmatrix}\\
&= \Delta.
\end{aligned}$$

[1] sometimes also called a *vielbein*

The space

$$L^2(\mathcal{S}_n) := \Big\{ f : M \to \mathcal{S}_n \ : \ f(p) \in \mathcal{S}_{np}, \int |f(p)|^2 \, dvol(p) < \infty \Big\}.$$

is a Hilbert space. We will make computations in this Hilbert space for the torus. We introduce the following constants and functions.

Notation 2.27. In this section we will use the following notations. For $n, m \in \mathbb{Z}$ and $\varepsilon = \pm 1$ define

$$\begin{aligned}
\varphi_{m,n}(x,y) &:= e^{2\pi i(mx+ny)};\\
c_{m,n} &:= \frac{m+in}{\sqrt{m^2+n^2}};\\
\psi_{m,n,\varepsilon}(x,y) &:= \begin{pmatrix} \frac{1}{\sqrt{2}}\, e^{2\pi i(mx+ny)} \\ \varepsilon \frac{1}{\sqrt{2}} \frac{m+in}{\sqrt{m^2+n^2}}\, e^{2\pi i(mx+ny)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\, \varphi_{m,n}(x,y) \\ \varepsilon c_{m,n} \frac{1}{\sqrt{2}}\, \varphi_{m,n}(x,y) \end{pmatrix} =: \begin{pmatrix} \psi_{m,n,\varepsilon,1}(x,y) \\ \psi_{m,n,\varepsilon,2}(x,y) \end{pmatrix};\\
\lambda_{m,n,\varepsilon} &:= \varepsilon 2\pi \sqrt{m^2+n^2};\\
\lambda_{m,n} &:= \lambda_{m,n,1}.
\end{aligned}$$

Observe $c_{m,n}\overline{c_{m,n}} = 1$.

Example 2.28. The collection of spinors $\mathcal{E} := \{\psi_{m,n,\varepsilon} : m, n \in \mathbb{Z}, \varepsilon = \pm 1\}$ forms an orthonormal basis of $L^2(\mathbb{T}^2, \mathbb{C}^2)$. Furthermore $\psi_{n,m,\varepsilon}$ where $n, m \in \mathbb{Z}$, $\varepsilon = \pm 1$ are eigenspinors with corresponding eigenvalues $\lambda_{m,n,\varepsilon} := \varepsilon 2\pi\sqrt{m^2+n^2}$.

Proof. Indeed, a direct computation shows that $\not{D}\psi_{m,n,\varepsilon} = \lambda_{m,n,\varepsilon}\psi_{m,n,\varepsilon}$. To find these eigenspinors one can first search for the eigenspinors of $\not{D}^2 = \Delta$ using separation of variables. These eigenspinors give conditions on the eigenspinors of $\not{D}$, because if $\not{D}\psi = \lambda\psi$, then $\not{D}^2\psi = \lambda^2\psi$. The eigenspinors of $\not{D}^2$ appear to be of the form

$$(x,y) \mapsto \begin{pmatrix} ce^{2\pi i(m_1x+n_1y)} \\ de^{2\pi i(m_2x+n_2y)} \end{pmatrix},$$

for some $c, d \in \mathbb{C}$, $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ such that $m_1^2 + n_1^2 = m_2^2 + n_2^2$. Once these are known, one shows that the eigenfunctions of $\not{D}$ should be of the form $\psi_{m,n,\varepsilon}$ and linear combinations thereof. We use the Stone-Weierstrass theorem to show that $A := \mathrm{span}(\mathcal{E})$ is dense in $\Gamma(\mathcal{S}_2) = C^\infty(\mathbb{T}^2, \mathbb{C}^2)$. First consider the linear space $A' := \mathrm{span}\{\varphi_{m,n} : m, n \in \mathbb{Z}\}$. The torus $\mathbb{T}^2$ is compact and Hausdorff, A' is an algebra, it separates points, is closed under complex conjugation and it contains the unit. So the Stone-Weierstrass theorem asserts that A' is dense in $C(\mathbb{T}^2, \mathbb{C})$.
Now observe that $\psi_{m,n,+1} + \psi_{m,n-1} = (\sqrt{2}\varphi_{m,n}, 0)$ and $\psi_{m,n,+1} - \psi_{m,n,-1} = (0, \sqrt{2}\frac{m+in}{\sqrt{m^2+n^2}}\varphi_{m,n})$. Then since A' is dense in $C(\mathbb{T}^2, \mathbb{C})$, A is dense in $C(\mathbb{T}^2, \mathbb{C}^2)$ in the $\|\cdot\|_\infty$-norm. But then also A is dense in $L^2(\mathbb{T}^2, \mathbb{C}^2)$. So $\mathcal{E}$ is a complete system.
It remains to show independence. Recall, for $n \in \mathbb{Z}$ it holds

$$\int_0^1 e^{2\pi inx} dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}.$$

Therefore

$$
\begin{aligned}
&\langle \psi_{m',n',\varepsilon}, \psi_{m',n',\varepsilon'} \rangle \\
&\qquad = \frac{1}{2}\int_0^1 \int_0^1 \varphi_{m,n}\varphi_{-m',-n'} + \varepsilon\varepsilon' c_{m,n}\overline{c_{m',n'}}\, \varphi_{m,n}\varphi_{-m',-n'} dxdy \\
&\qquad = \frac{1}{2}\int_0^1 \int_0^1 \varphi_{m-m',n-n'} dxdy + \frac{1}{2}\varepsilon\varepsilon' c_{m,n}\overline{c_{m',n'}} \int_0^1 \int_0^1 \varphi_{m-m',n-n'} dxdy \\
&\qquad = \frac{1}{2}\left(\delta_{m,m'}\delta_{n,n'} + \varepsilon\varepsilon' c_{m,n}\overline{c_{m',n'}}\, \delta_{m,m'}\delta_{n,n'}\right) \\
&\qquad = \begin{cases} 1 & \text{if } m = m' \text{ and } n = n' \text{ and } \varepsilon = \varepsilon' \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

Thus the set is orthonormal. ⊠

Example 2.29. The spectrum of the Dirac operator $\not{D}$ of $(C^\infty(\mathbb{T}^2), L^2(\mathbb{T}^2, \mathbb{C}^2), \not{D})$ consists of the eigenvalues of $\not{D}$ i.e. $\sigma(\not{D}) = \{\varepsilon 2\pi\sqrt{m^2+n^2} \,:\, \varepsilon = \pm 1,\ m, n \in \mathbb{Z}\}$. For

$$
\mathrm{Dom}(\not{D}) := \Big\{ \sum_{m,n,\varepsilon} d_{m,n,\varepsilon}\psi_{m,n,\varepsilon} \,:\, \sum_{m,n,\varepsilon} |d_{m,n,\varepsilon}|^2 < \infty,\ \sum_{m,n,\varepsilon} \lambda^2_{m,n,\varepsilon}|d_{m,n,\varepsilon}|^2 < \infty \Big\},
$$

the operator $(\not{D}, \mathrm{Dom}(\not{D}))$ is self-adjoint and $\not{D}$ has a compact resolvent.

Proof. Example 2.28 shows the conditions of Theorem 1.62 are satisfied, from which the result follows. ⊠

Definition 2.30. A *spectral triple* consists of a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra, $\mathcal{H}$ is a Hilbert space and D is a self-adjoint (unbounded) operator on $\mathcal{H}$ with compact resolvent, for which there exists a faithful representation $\pi : \mathcal{A} \to B(\mathcal{H})$ of bounded operators such that for all $a \in \mathcal{A}$ the operator $[D, \pi(a)]$ is densely defined and extends to a bounded operator on $\mathcal{H}$. If in addition there exists a grading $\chi : \mathcal{H} \to \mathcal{H}$ such that $\chi D = -D\chi$ and $\chi\pi(a) = \pi(a)\chi$ for all $a \in \mathcal{A}$, then the tuple $(\mathcal{A}, \mathcal{H}, D, \chi)$ is called an *even spectral triple.* If no such grading exists, the triple $(\mathcal{A}, \mathcal{H}, D)$ is called an *odd spectral triple.*

A very important class of spectral triples is given by spin manifolds. Following the construction outlined in this paragraph they yield commutative spectral triples, i.e. triples for which the algebra $\mathcal{A}$ is commutative. It is also possible to go the other way around, given a commutative spectral triple (which satisfies some additional requirements), then there exists a compact manifold which generates this spectral triple [7]. An explicit proof of the fact that the torus admits a spectral triple can be found below (cf. Example 2.34).

Theorem 2.31. *Suppose M is a spin manifold. The previous constructions (Definition 2.24 and 2.25) yield the spinor bundle $\mathcal{S}$ and Dirac operator $\not{D}$. The tuple $(C^\infty(M), L^2(M, \mathcal{S}), \not{D})$ is a spectral triple.*

Proof. See [19, Thm. 11.1]. ⊠

For the torus it appears to be convenient to do computations with vectors of $L^2(\mathbb{T}^2, \mathbb{C}^2)$ of the form $(f, 0)$ and $(0, f)$, where $f : \mathbb{T}^2 \to \mathbb{C}$. Therefore we introduce in addition to Notation 2.27 the following functions

Notation 2.32. For $n, m \in \mathbb{Z}$ define

$$
\begin{aligned}
d_{m,n,1}(x,y) &:= \begin{pmatrix} e^{2\pi i(mx+ny)} \\ 0 \end{pmatrix}; \\
d_{m,n,-1}(x,y) &:= \begin{pmatrix} 0 \\ e^{2\pi i(mx+ny)} \end{pmatrix}; \\
e_{m,n}(x,y) &:= \pi(\varphi_{m,n}).
\end{aligned}
$$

Remark 2.33. Note that the set $\{d_{m,n,\varepsilon}\ :\ m,n\in\mathbb{Z},\ \varepsilon=\pm 1\}$ is an orthonormal basis of $L^2(\mathbb{T}^2,\mathbb{C}^2)$. Indeed, clearly the spinors $d_{m,n,\varepsilon}$ are orthonormal. And the set is complete, because $\psi_{m,n,\varepsilon}=\frac{1}{\sqrt{2}}d_{m,n,1}+\frac{1}{\sqrt{2}}\varepsilon c_{m,n}d_{m,n,-1}$. Furthermore

$$\begin{aligned}
e_{k,l}d_{m,n,\varepsilon}&=d_{k+m,l+n,\varepsilon};\\
\not{D}(d_{m,n,1})&=\begin{pmatrix}0 & -i\partial_x-\partial_y\\ -i\partial_x+\partial_y\end{pmatrix}\begin{pmatrix}e^{2\pi i(mx+ny)}\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 2\pi(m+in)e^{2\pi i(mx+ny)} & 0\end{pmatrix}\\
&=2\pi(m+in)d_{m,n,-1};\\
\not{D}(d_{m,n,-1})&=\begin{pmatrix}0 & -i\partial_x-\partial_y\\ -i\partial_x+\partial_y & 0\end{pmatrix}\begin{pmatrix}0\\ e^{2\pi i(mx+ny)}\end{pmatrix}=\begin{pmatrix}2\pi(m-in)e^{2\pi i(mx+ny)}\\ 0\end{pmatrix}\\
&=2\pi(m-in)d_{m,n,1}.
\end{aligned}$$

So $\not{D}(d_{m,n,\varepsilon})=2\pi(m+\varepsilon in)d_{m,n,-\varepsilon}$. In particular $\not{D}^2(d_{m,n,\varepsilon})=4\pi^2(m^2+n^2)d_{m,n,\varepsilon}$, so the vectors $d_{m,n,\varepsilon}$ are eigenvectors for $\not{D}^2$. Also the vectors $\{d_{m,n,\varepsilon}\ :\ m,n\in\mathbb{Z},\ \varepsilon=\pm1\}$ induce an orthogonal decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_{-1}$, where $\mathcal{H}_\varepsilon$ equals the Hilbert space generated by $\{d_{m,n,\varepsilon}\ :\ m,n\in\mathbb{Z}\}$.

Example 2.34. For $\mathcal{A}:=C^\infty(\mathbb{T}^2)$, the triple $(\mathcal{A},L^2(\mathbb{T}^2,\mathbb{C}^2),\not{D})$ is a spectral triple.

Proof. We have to check several conditions.
1. It is clear that $\mathcal{A}$ is a $*$-algebra.
2. We have a representation π of $C^\infty(\mathbb{T}^2)$ on $L^2(\mathbb{T}^2,\mathbb{S}^2)=L^2(\mathbb{T}^2,\mathbb{C}^2)$ via multiplication i.e. $(\pi(f)g)(x)=f(x)g(x)$ (strictly speaking this is defined almost everywhere, because $L^2(\mathbb{T}^2,\mathbb{C}^2)$ consists of equivalence classes). Each $\pi(f)$ is bounded because $\|\pi(f)g\|_2\leq\|f\|_\infty\|g\|_2$. The action is faithful, because if $\pi(f_1)=\pi(f_2)$, then in particular for the function

$$g:(x,y)\mapsto(1,1)\quad\text{for all }(x,y)\in\mathbb{T}^2,$$

we have $(f_1(x,y),f_1(x,y))=\pi(f_1)g(x,y)=\pi(f_2)g(x,y)=(f_2(x,y),f_2(x,y))$. Thus $f_1=f_2$ a.e. But f_1 and f_2 are C^∞, hence $f_1=f_2$ everywhere.
3. $\not{D}$ has a compact resolvent and is self-adjoint, this has been shown in Corollary 2.29.
4. Suppose $f\in\mathcal{A}=C^\infty(\mathbb{T}^2)$. Then $f=\sum_{k,l}\alpha_{k,l}\varphi_{k,l}$ for some double sequence of rapid decay $(\alpha_{k,l})_{k,l}$. Since $\not{D}$ is closed we have

$$\begin{aligned}
[\not{D},\pi(f)]d_{m,n,\varepsilon}&=\not{D}\Big(\sum_{k,l}\alpha_{k,l}d_{k+m,l+n,\varepsilon}\Big)-\sum_{k,l}\alpha_{k,l}e_{k,l}\not{D}(d_{m,n,\varepsilon})\\
&=\sum_{k,l}\alpha_{k,l}2\pi((k+m)+i\varepsilon(l+n))d_{k+m,l+n,-\varepsilon}-\alpha_{k,l}2\pi(m+i\varepsilon n)d_{k+m,l+n,-\varepsilon}\\
&=\Big(\sum_{k,l}\alpha_{k,l}2\pi(k+i\varepsilon l)e_{k,l}\Big)d_{m,n,-\varepsilon}
\end{aligned}$$

Observe that $-i\partial_x f=\sum_{k,l}2\pi k\alpha_{k,l}\varphi_{k,l}$ and $\partial_y f=\sum_{k,l}2\pi il\alpha_{k,l}\varphi_{k,l}$. We obtain

$$[\not{D},\pi(f)]d_{m,n,\varepsilon}=\pi(-i\partial_x f+\varepsilon\partial_y f)d_{m,n,-\varepsilon}.$$

Therefore on $\mathrm{span}\{d_{m,n,\varepsilon}\ :\ m,n\in\mathbb{Z},\ \varepsilon\in\{-1,1\}\}$ the operator $[\not{D},\pi(f)]$ can be represented by the matrix

$$\begin{pmatrix}0 & \pi(-i\partial_x f-\partial_y f)\\ \pi(-i\partial_x f+\partial_y f) & 0\end{pmatrix},$$

with respect to the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_{-1}$ (see remark 2.33). Since each entry of this matrix is a bounded operator and $\mathrm{span}\{d_{m,n,\varepsilon}\ :\ m,n\in\mathbb{Z},\ \varepsilon\in\{-1,1\}\}$ is dense in $L^2(\mathbb{T}^2,\mathbb{C}^2)$ the operator $[\not{D},\pi(f)]$ extends to a bounded operator. And thus $(C^\infty(\mathbb{T}^2,\mathbb{C}),L^2(\mathbb{T}^2,\mathbb{C}^2),\not{D})$ is a spectral triple. ⊠

2.3 Summability, regularity and dimension spectrum

The notion of a dimension and the smooth structure of a manifold also translates to an algebraic counterpart for spectral triples. This will be examined in this subsection.

Definition 2.35. Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. The triple is *p-summable* if $\mathrm{Tr}((1+D^2)^{-p/2} < \infty$. The triple is *$p^+$-summable* if for all $\varepsilon > 0$ the triple is $(p+\varepsilon)$-summable. The triple is *finitely summable* if it is p summable for some $p > 0$. The *metric dimension* of a spectral triple is defined to be $m \in \mathbb{N}$ if the triple is m^+-summable, but not $(m-1)^+$-summable. We say the triple is *θ-summable* if for all $t > 0$ it holds that $\mathrm{Tr}(e^{-tD^2}) < \infty$.

Some authors define summability using $|D|$ instead of $(1+D^2)^{1/2}$. We explicitly choose the second possibility because if D is not invertible one has to deal with $\ker(|D|) = \ker(D)$ in some way. This can for example be done as in the Remark 2.36 stated below. Because D is self-adjoint, D^2 is positive and hence $0 \notin \sigma((1+D^2)^{1/2})$, therefore with $(1+D^2)^{1/2}$ one does not have this problem.

Remark 2.36. If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and $\ker(D) \neq \{0\}$, we can define a new spectral triple which is only slightly different. We have $\mathcal{H} = \ker(D)^\perp \oplus \ker(D)$. Define $R : \ker(D) \oplus \ker(D) \to \ker(D) \oplus \ker(D)$ given by the matrix decomposition

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\varepsilon > 0$ and now define a spectral triple $(\mathcal{A}, \tilde{\mathcal{H}}, \tilde{D})$ by

$$\tilde{\mathcal{H}} := \ker(D)^\perp \oplus (\ker(D) \oplus \ker(D)) \qquad \tilde{D} := D|_{\ker(D)^\perp} \oplus \varepsilon R.$$

The action of the algebra $\mathcal{A}$ on $\tilde{\mathcal{H}}$ is given by $a(h_1 \oplus h_2 \oplus h_2) := a(h_1 \oplus h_2) \oplus 0$.
If the spectral triple is even with grading γ we can extend this grading to $\tilde{\gamma}$ by

$$\tilde{\gamma} := \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma|_{\ker(D)} \end{pmatrix},$$

with respect to the decomposition $\tilde{\mathcal{H}} = \mathcal{H} \oplus \ker(D)$. Write $D_1 := D|_{\ker(D)^\perp}$. Since γ anticommutes with D we have

$$\begin{pmatrix} \gamma_{11}D_1 & 0 \\ \gamma_{21}D_1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} -D_1\gamma_{11} & -\gamma_{12}D_1 \\ 0 & 0 \end{pmatrix}.$$

So $\gamma_{12}D_1 = D_1\gamma_{21} = 0$. Since $\ker(D_1) = 0$ this implies $\gamma_{21} = 0$. Since D is self-adjoint, D_1 is self-adjoint and hence $\mathrm{ran}(D_1)^{\perp\perp} = \ker(D)^\perp$, so $\mathrm{ran}(D_1)$ is dense in $\ker(D)^\perp$. Hence $\gamma_{12} = 0$. Thus $\tilde{\gamma}$ looks like

$$\begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & -\gamma_{22} \end{pmatrix},$$

from which it is clear that $\tilde{\gamma}$ anticommutes with $\tilde{D}$ and commutes with $\tilde{\mathcal{A}}$. For this reason we will in some results assume that the Dirac operator D is invertible. If that is not the case one can perform the above construction and then let ε tend to zero.

Locally the torus is homeomorphic to $\mathbb{R}^2$, so we expect that the metric dimension of the torus is also 2. This is indeed the case, see the next example.

Example 2.37. The metric dimension of the torus $\mathbb{T}^2$ is 2. In particular the torus is finitely summable.

Proof. We will show that the triple is 2^+-summable. The Dirac operator D is self-adjoint, thus

$$\operatorname{Tr}\left((1+D^2)^{-p/2}\right) = \sum_{m,n,\varepsilon} \left(1+\lambda^2_{m,n,\varepsilon}\right)^{-p/2}.$$

So establish 2^+-summability it is sufficient to show that the sequence $(\mu_n)_n$ is $\mathcal{O}(n^{-(1/2)})$. Here $(\mu_n)_n$ are the eigenvalues of $(1+D^2)^{-1/2}$ arranged in decreasing order and with multiplicity.
In Lemma 2.28 we computed the eigenvalues of D. These are $\{\varepsilon 2\pi\sqrt{m^2+n^2}\}$. We will ignore the factor 2π, because this will not affect the order of growth of the eigenvalues. Observe that $0 \leq \frac{1}{2}(m-n)^2 = \frac{1}{2}m^2 + \frac{1}{2}n^2 - mn$. So $mn \leq \frac{1}{2}(m^2+n^2)$, hence for $m,n \geq 0$ it holds that $\frac{1}{2}(m+n)^2 = \frac{1}{2}(m^2+n^2) + mn \leq m^2+n^2 \leq (m+n)^2$. So

$$\frac{1}{2}(|m|+|n|)^2 \leq (m^2+n^2)+1 \leq 2(|m|+|n|)^2 \qquad \text{for all } (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$$

and hence

$$\frac{1}{\sqrt{2}(|m|+|n|)} \leq \frac{1}{\sqrt{1+m^2+n^2}} \leq \frac{\sqrt{2}}{|m|+|n|}, \qquad \text{for all } (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

We order the values $\{(|m|+|n|)^{-1} \,:\, (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$. The value $\frac{1}{k}$ appears $4k$ times. To take care of ε we make a new sequence $(a_n)_n$ in which each number $\frac{1}{k}$ appears $8k$ times (again in decreasing order). If we order the set

$$\left\{\left(1+\left(\varepsilon\sqrt{m^2+n^2}\right)^2\right)^{-1/2} \,:\, \varepsilon = \pm 1,\, m,n \in \mathbb{Z}\right\} = \left\{\left(1+m^2+n^2\right)^{-1/2} \,:\, \varepsilon = \pm 1,\, m,n \in \mathbb{Z}\right\}$$

of eigenvalues of $(1+D^2)^{-1/2}$ in decreasing order, say $(\mu_n)_n$. Then $\frac{1}{\sqrt{2}}a_n \leq \mu_n \leq \sqrt{2}a_n$. So it is sufficient to show that the sequence $(a_n)_n$ is $\mathcal{O}(n^{-1/2})$. If $k = 8\frac{n(n+1)}{2}$, then $a_k = \frac{1}{n}$. More generally if $k = 4(x(x+1))$, then

$$x = \frac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot -k}}{2 \cdot 4} = \frac{1}{2}(-1 \pm \sqrt{1+k}).$$

Therefore

$$\left\lfloor \frac{1}{2}(-1+\sqrt{1+k}) \right\rfloor \leq a_k^{-1} \leq \left\lceil \frac{1}{2}(-1+\sqrt{1+k}) \right\rceil.$$

So $a_k = \mathcal{O}(k^{-\frac{1}{2}})$. So we conclude that D is 2^+-summable and not 2-summable. Thus by definition of the metric dimension, the torus has metric dimension 2. ⊠

The summability gives a part of the dimension of a spectral triple. But on can get more information from the dimension spectrum, in some sense this also takes submanifolds into account. We start with the necessary definitions and lemmas.

Definition 2.38. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we define for $a \in \mathcal{A}$ the operator $\delta(a) := [|D|, a]$, it is the unbounded derivation of a. We denote

$$\operatorname{Dom}(\delta) := \{T \in B(\mathcal{H}) \,:\, \delta(T) \text{ is bounded on } \mathcal{H} \text{ and } T\operatorname{Dom}(|D|) \subset \operatorname{Dom}(|D|)\}.$$

If for all $a \in \mathcal{A}$ the operators $a, [D,a] \in \operatorname{Dom}(\delta^k)$, we call the triple a *$QC^k$-triple*, or QC^k for short. If the triple is a QC^k-triple for all $k \geq 1$, we call it QC^∞ or *regular*. An operator $a \in B(\mathcal{H})$ with $a \in \operatorname{Dom}(\delta^k)$ for all $k \in \mathbb{N}$ is called *smooth*.
Also define the following subspace $\mathcal{H}_\infty := \bigcap_n \operatorname{Dom}(D^n)$.

As the name suggests, smooth operators preserve smooth domains, therefore the following results should be no surprise. These results are due to Connes in the paper [7].

Lemma 2.39. *For a self-adjoint operator D on $\mathcal{H}$, the space $\mathcal{H}_\infty \subset \mathcal{H}$ dense.*

Proof. Since D is self-adjoint, by the spectral theorem D has a spectral decomposition $D = \int_{\mathbb{R}} \lambda \, dE$. Suppose $m \in \mathbb{N}$ and $h \in E([-m,m])$, then $D^n h = \int_{[-m,m]} \lambda^n \, dE$ and hence

$$\|D^n h\| = \left\| \left(\int_{[-m,m]} \lambda^n \, dE \right) h \right\| \leq \int_{[-m,m]} \|\lambda^n\| \, dE \, \|h\| \leq 2m^n \|h\| < \infty.$$

So $h \in \mathcal{H}_\infty$. We conclude that $\operatorname{span}(\{E([-n,n]) : n \in \mathbb{N}\}) \subset \mathcal{H}_\infty$. But the linear space $\operatorname{span}(\{E([-n,n]) : n \in \mathbb{N}\}) \subset \mathcal{H}$ dense. So $\mathcal{H}_\infty \subset \mathcal{H}$ dense. ⊠

Lemma 2.40. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple, then for an element $a \in \mathcal{A}$ it holds that $a\mathcal{H}_\infty \subset \mathcal{H}_\infty$.*

Proof. See [7, Lemma 2.1]. We will first prove with induction that

$$|D|^m a = \sum_{k=0}^{m} \binom{m}{k} \delta^k(a) |D|^{m-k}. \tag{2.3}$$

For now we will only prove this as a formal expression, later we will show that this holds as operators on suitable domains.
In the case $m = 0$ there is nothing to prove. Using $|D|\delta^k(a) = [|D|, \delta^k(a)] + \delta^k(a)|D|$ we obtain

$$\begin{aligned}
|D|^{m+1} a &= \sum_{k=0}^{m} \binom{m}{k} |D| \delta^k(a) |D|^{m-k} \\
&= \sum_{k=0}^{m} \binom{m}{k} \delta^{k+1}(a) |D|^{(m+1)-(k+1)} + \delta^k(a) |D|^{m+1-k} \\
&= |D|^{m+1} + \sum_{k=1}^{m} \left(\binom{m}{k-1} + \binom{m}{k} \right) \delta^k(a) |D|^{(m+1)-k} + \delta^{m+1}(a) \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} \delta^k(a) |D|^{(m+1)-k}.
\end{aligned}$$

Now note that for $a \in \operatorname{Dom}(\delta^k)$ it holds that $\delta^{k-1}(a) \in \operatorname{Dom}(\delta)$ and $\operatorname{Dom}(D^{k+1}) \subset \operatorname{Dom}(D)$, hence

$$\delta^k(a) \operatorname{Dom}(D^{k+1}) \subset \delta(\delta^{k-1}(a))(\operatorname{Dom}(D)) \subset \operatorname{Dom}(D). \tag{2.4}$$

To finish the proof of this lemma it is sufficient to prove the following claim: if $a \in \operatorname{Dom}(\delta^m)$, then $a \operatorname{Dom}(D^m) \subset \operatorname{Dom}(D^m)$. For $m = 0$ there is nothing to prove and for $m = 1$ this is included in the definition of δ, so assume $m \geq 2$. Let $h \in \operatorname{Dom}(D^m)$, then $|D|^{m-1-k} h \in \operatorname{Dom}(D^{k+1})$. Hence by (2.4) we obtain $\delta^k(a)|D|^{m-1-k} h \in \operatorname{Dom}(D)$. Combining this with (2.3) gives that

$$|D|^{(m-1)} a h = \sum_{k=0}^{m-1} \binom{m-1}{k} \delta^k(a) |D|^{m-1-k} h \in \operatorname{Dom}(D).$$

Hence $ah \in \operatorname{Dom}(D^m)$ and (2.3) holds as operators on $\operatorname{Dom}(D^m)$. ⊠

In the above proof we have established the following useful result.

Corollary 2.41. *If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and $a \in \operatorname{Dom}(\delta^n)$, then $a(\operatorname{Dom}(D^n)) \subset \operatorname{Dom}(D^n)$.*

Lemma 2.42. *Suppose D is a self-adjoint operator and the operators $a, \delta(a) : \mathcal{H}_\infty \to \mathcal{H}$ are bounded. Then a preserves $\operatorname{Dom}(|D|) = \operatorname{Dom}(D)$ and on $\operatorname{Dom}(D)$ the bounded extension of $\delta(a) = [|D|, a]$ and the commutator $|D|T - T|D|$ coincide.*

Proof. See [7, Lemma 13.1]. By Lemma 2.39 it holds that $\mathcal{H}_\infty \subset \mathcal{H}$ is dense. So let $h \in \mathrm{Dom}(D)$, then there exists a sequence $(h_n)_n \subset \mathcal{H}_\infty$ such that $h = \lim_n h_n$ and $|D|h = \lim_n |D|h_n$. By boundedness of a and $\delta(a)$ it follows that

$$\lim_n ah_n = ah, \qquad \lim_n a|D|h_n = a|D|h, \qquad \lim_n \delta(a)h_n = h.$$

Therefore $(|D|ah_n)_n = (\delta(a)h_n - a|D|h_n)_n$ is convergent. Because $|D|$ is closed and $(ah_n)_n$ is convergent we obtain $\lim_n |D|ah_n = |D|ah$. Hence $ah \in \mathrm{Dom}(|D|) = \mathrm{Dom}(D)$. The other statements of this lemma are trivial. ☒

Most of the times it is way more convenient to work with D^2 instead of with $|D|$. To establish regularity of a spectral triple the following result can be useful. We will not prove this lemma because the proof is very technical and does not give a lot of insight.

Lemma 2.43. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and $a : \mathcal{H}_\infty \to \mathcal{H}$ is a bounded linear operator. Denote $\delta_1(a) := [D^2, a](1 + D^2)^{-1/2}$. Then the following holds:*

(i) If $\delta_1(a)$ and $\delta_1^2(a)$ are bounded, then $\delta(a)$ is bounded;

(ii) The operators $\delta_1^n(a)$ are bounded for all $n \geq 1$ if and only if $\delta^n(a)$ is bounded for all $n \geq 1$.

Proof. See [7, Lemma 13.2]. ☒

There exists another equivalent notion of smoothness of operators which is sometimes used in the literature. Then an operator is called smooth if the map $t \mapsto e^{it|D|}Te^{-it|D|}$ is smooth. We will prove the equivalence.

Lemma 2.44. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. Then for $a \in B(\mathcal{H})$ the following are equivalent*

(i) $a \in \bigcap_{k=1}^n \mathrm{Dom}(\delta^k)$;

(ii) the map $t \mapsto e^{it|D|}ae^{-it|D|}$ is in $C^n(\mathbb{R}, B(\mathcal{H}))$ with respect to the norm topology.

Proof. See also [7, Lemma 13.4]. We start by calculating the derivative of the map $t \mapsto e^{it|D|}ae^{-it|D|}$. For this we will freely use the properties of groups of unitary operators see for example [11, §X.5]. We obtain

$$\begin{aligned}
\frac{d}{dt}|_{t=t_0} e^{it|D|}ae^{-it|D|}h &= \lim_{t\to 0} \frac{1}{t}\big(e^{i(t_0+t)|D|}ae^{-i(t_0+t)|D|}h - e^{it_0|D|}ae^{-it_0|D|}h\big) \\
&= \lim_{t\to 0} \frac{1}{t}\big(e^{i(t_0+t)|D|}ae^{-it_0|D|}\big(e^{-it|D|}h\big) - e^{it_0|D|}ae^{-it_0|D|}\big(e^{it|D|}h\big)\big) \\
&\quad + \lim_{t\to 0} \frac{1}{t}\big(e^{it_0|D|}ae^{-i(t_0+t)|D|}h - e^{it_0|D|}ae^{-it_0|D|}h\big) \\
&= \lim_{t\to 0} e^{it_0|D|}\frac{1}{t}\big(e^{it|D|} - 1\big)ae^{-it_0|D|}\big(e^{-it|D|}h\big) + \lim_{t\to 0} e^{it_0|D|}a\frac{1}{t}\big(e^{it|D|} - 1\big)\big(e^{it_0|D|}h\big) \\
&= e^{it_0|D|}|D|ae^{-it_0|D|}h - e^{it_0|D|}a|D|e^{-it_0|D|}h \\
&= e^{it_0|D|}\delta(a)e^{-it_0|D|}h. \qquad (2.5)
\end{aligned}$$

Because $e^{-it|D|}$ is unitary and commutes with $|D|$, this equality holds for $h \in \mathcal{H}$ with $h \in \mathrm{Dom}(|D|)$ and $ah \in \mathrm{Dom}(|D|)$.
Assume $n = 1$. Suppose (i) holds, then $\delta(a)$ is bounded. Since by assumption (i) $a\,\mathrm{Dom}(D) \subset \mathrm{Dom}(D)$ equality (2.5) holds for $h \in \mathrm{Dom}(|D|)$ which is dense in $\mathcal{H}$. Because the operator $e^{it_0|D|}\delta(a)e^{-it_0|D|}$ is bounded, it is a bounded extension of $\frac{d}{dt}|_{t=t_0} e^{it|D|}ae^{-it|D|}$. So the derivative exists and is continuous because $t \mapsto e^{it|D|}$ is continuous, i.e. $t \mapsto e^{it|D|}ae^{-it|D|}$ is in $C^1(\mathbb{R}, B(\mathcal{H}))$
Conversely if (ii) holds, then (2.5) is true on $\mathrm{Dom}(|D|) \cap a^{-1}(\mathrm{Dom}(|D|))$. We will show, if

$h \in \mathrm{Dom}(|D|)$, then $ah \in \mathrm{Dom}(|D|)$. Since D is self-adjoint, it holds that $h \in \mathrm{Dom}(|D|)$ if $\lim_{t\to 0} \frac{1}{t}\big(e^{it|D|}h - h\big)$ exists (cf. [11, X§5]. By assumption for all $h \in \mathcal{H}$ the limit

$$\lim_{t\to 0} \frac{1}{t}\big(e^{it|D|}ae^{-it|D|}h - h\big)$$

exists. Suppose $h \in \mathrm{Dom}(|D|)$, then

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t}\big(e^{it|D|}ah - ah\big) &= \lim_{t\to 0} \frac{1}{t}\big(e^{it|D|}ah - e^{it|D|}ae^{-it|D|}h\big) + \lim_{t\to 0} \frac{1}{t}\big(e^{it|D|}ae^{-it|D|}h - ah\big) \\ &= \lim_{t\to 0} e^{it|D|}a\frac{1}{t}\big(e^{-it|D|}h - h\big) + \lim_{t\to 0} \frac{1}{t}\big(e^{it|D|}ae^{-it|D|}h - ah\big). \qquad (2.6) \end{aligned}$$

Both limits in (2.6) exist, the first one because $h \in \mathrm{Dom}(|D|)$, the second one by assumption (ii). Hence $ah \in \mathrm{Dom}(|D|)$. Since $\mathrm{Dom}(|D|) \subset \mathcal{H}$ dense, the operator $\delta(a)$ extends to a bounded operator on $\mathcal{H}$. Therefore $a \in \mathrm{Dom}(\delta)$ and (i) holds.
To prove the lemma for higher orders (i.e. $n > 1$) one can proceed by induction. Simply replace the operator a by $\delta^{(n-1)}(a)$ and use Corollary 2.41. ⊠

As we showed in Remark 2.36, invertibility of D is not a big restriction. So we will assume in the following definition and lemmas that D is invertible.

Definition 2.45. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we define $OP^0 := \{T \in B(\mathcal{H}) : T \text{ is smooth}\}$ and $OP^r := \{T \in B(\mathcal{H}) : |D|^{-r}T \in OP^0\}$. Note that it is immediate from this definition that $T \in OP^\alpha$ if and only if $|D|^\beta T \in OP^{\alpha+\beta}$, see also Lemma 2.47.

Notation 2.46. For an operator T denote $\nabla T := D^2T - TD^2$. If there is a possibility of confusion, we will write ∇_D to indicate the dependence on D.

Lemma 2.47. *Let $T \in OP^0$ and $n \in \mathbb{N}$, then*

(i) $\nabla^n(T) \in OP^n$*;*

(ii) $D^{-2}T = \sum_{k=1}^{n} (-1)^{k-1}\nabla^{k-1}(T)D^{-2k} + R_n$.

Here $R_n := (-1)^n D^{-2}\nabla^n(T)D^{-2n} \in OP^{-n-1}$. Furthermore we have $|D|^\alpha T$ is smooth if and only if $T|D|^\alpha$ is smooth.

Proof. [9, Lemma 1.136] ⊠

The importance of this result lies in the property that it is now possible to interchange an element $A \in \mathcal{A}$ with factors D^{-2}. We will use the following result a lot of times.

Lemma 2.48. *If $S \in OP^\alpha$ and $T \in OP^\beta$, then $ST \in OP^{\alpha+\beta}$. And if $\alpha \leq \beta$, then we have an inclusion $OP^\alpha \subset OP^\beta$.*

Proof. By assumption $|D|^{-\alpha}S$ and $|D|^{-\beta}T$ are smooth. Therefore by Lemma 2.47 $T|D|^{-\beta}$ is smooth and thus $|D|^{-\alpha}ST|D|^{-\beta}$ is smooth. Again by this lemma $ST|D|^{-\beta}|D|^\alpha = ST|D|^{-(\alpha+\beta)}$ is smooth. So $ST \in OP^{\alpha+\beta}$.
As before we will assume that D is invertible. Then $|D|^{-r}$ is bounded for all $r \geq 0$ and $|D|^{-r}$ commutes with $|D|$. Thus $\delta^n(|D|^{-r})$ extends to the zero operator, in particular $|D|^{-r} \in \mathrm{Dom}(\delta^n)$ for all $n \in \mathbb{N}$. So $|D|^{-r}$ is smooth. Hence $1 \in OP^r$ for all $r > 0$. Thus by the previous result, if $S \in OP^\alpha$, then $S = S1 \in OP^{\alpha+(\beta-\alpha)} = OP^\beta$. ⊠

Lemma 2.49. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a regular p-summable spectral triple. Then for an element $B \in OP^{-\alpha}$ with $\alpha > p$ the function*

$$\{z \in \mathbb{C} : \mathrm{Re}(z) > m\} \to \mathbb{C}, \qquad z \mapsto \mathrm{Tr}(B|D|^{-z})$$

extends to a function which is holomorphic in $z = 0$.

Proof. Suppose $\alpha > p$ and $B \in OP^{-\alpha}$. By Lemma 2.47 $B|D|^{\alpha} \in OP^0$. In particular $B|D|^{\alpha}$ is bounded. Note

$$\operatorname{Tr}\left(B|D|^{-z}\right) = \operatorname{Tr}\left((B|D|^{\alpha})|D|^{-(z+\alpha)}\right)$$

and $z \mapsto \operatorname{Tr}(|D|^{-z})$ is holomorphic for $s \in \{z \in \mathbb{C} : \operatorname{Re}(s) > p\}$. Thus $z \mapsto \operatorname{Tr}(B|D|^{-z})$ is holomorphic at $z = 0$. ⊠

We will need the following result a couple of times in the upcoming calculations, so we write it down separately.

Lemma 2.50. *Suppose $\mathcal{H}$ is a separable Hilbert Space with orthonormal basis $(e_n)_n$. For $k = 1, \dots, K$ let $T_k : \mathcal{H} \to \mathcal{H}$ be a bounded operator and $D > 0$ a constant (independent of k), such that $\|T_k\| \leq D$. Let $c_k : \mathbb{N} \to \mathbb{C}$, $k = 1, \dots, K$ be complex valued functions and $C > 0$ a constant such that $|c_k(n)| < C$ for all k, n. Suppose $T : \mathcal{H} \to \mathcal{H}$ is defined by*

$$Tx := \sum_{k=1}^{K} T_k\Big(\sum_n c_k(n)\langle x, e_n\rangle e_n\Big).$$

Then T is a bounded operator of norm $\|T\| \leq KCD$.

Proof. Let $x \in \mathcal{H}$, put $\alpha_n := \langle x, e_n\rangle$. Then

$$\begin{aligned}
\|Tx\| &\leq \sum_{k=1}^{K} \|T_k(\sum_n c_k(n)\alpha_n e_n)\| \\
&\leq \sum_{k=1}^{K} D(\|\sum_n c_k(n)\alpha_n e_n\|^2)^{1/2} \\
&= \sum_{k=1}^{K} D(\sum_n |c_k(n)\alpha_n|^2)^{1/2} \\
&\leq \sum_{k=1}^{K} DC(\sum_n |\alpha_n|^2)^{1/2} \\
&= KDC\|x\|.
\end{aligned}$$

⊠

Example 2.51. We will show that $(C^\infty(\mathbb{T}^2, \mathbb{C}), L^2(\mathbb{T}^2, \mathbb{C}^2), D)$ is a regular triple. We will use Lemma 2.43 because D^2 is a lot easier to work with than $|D|$. Because $d_{m,n,\varepsilon}$ are eigenvectors for D^2 with eigenvalue $4\pi^2(m^2 + n^2)$, the functional calculus for D^2 (Theorem 1.71) gives us

$$(1 + D^2)^{-1/2} d_{m,n,\varepsilon} = (1 + 4\pi^2(m^2 + n^2))^{-1/2}.$$

Then for $f \in C^\infty(\mathbb{T}^2, \mathbb{C})$, $f = \sum_{k,l} \alpha_{k,l}\varphi_{k,l}$ we have

$$\begin{aligned}
\delta_1(\pi(f))d_{m,n,\varepsilon} &= [D^2, \pi(f)](1 + D^2)^{-1/2} d_{m,n,\varepsilon} \\
&= (1 + 4\pi^2(m^2 + n^2))^{-1/2}\Big(\sum_{k,l} \alpha_{k,l}(D^2 d_{k+m,l+n,\varepsilon} - e_{k,l}D^2 d_{m,n,\varepsilon})\Big) \\
&= (1 + 4\pi^2(m^2 + n^2))^{-1/2}\Big(\sum_{k,l} \alpha_{k,l}4\pi^2((k+m)^2 + (l+n)^2 - (m^2 + n^2))d_{k+m,l+n,\varepsilon}\Big) \\
&= (1 + 4\pi^2(m^2 + n^2))^{-1/2}\Big(\sum_{k,l} \alpha_{k,l}4\pi^2(k^2 + 2km + l^2 + 2ln)e_{k,l}\Big)d_{m,n,\varepsilon}. \qquad (2.7)
\end{aligned}$$

Since $\partial_x f = \sum_{k,l} 2\pi i k \varphi_{k,l}$ and $\partial_y f = \sum_{k,l} 2\pi i l \varphi_{k,l}$ Equation (2.7) becomes

$$\Big(\frac{-1}{(1+4\pi^2(m^2+n^2))^{1/2}}(\partial_x^2+\partial_y^2)\,f + \frac{-4\pi m i}{(1+4\pi^2(m^2+n^2))^{1/2}}\,\partial_x f \\ + \frac{-4\pi n i}{(1+4\pi^2(m^2+n^2))^{1/2}}\,\partial_y f\Big)\,d_{m,n,\varepsilon}.$$

Observe that there exists a constant $C>0$ such that for all $m,n\in\mathbb{Z}$

$$\Big|\frac{-1}{(1+4\pi^2(m^2+n^2))^{1/2}}\Big|,\ \Big|\frac{-4\pi m i}{(1+4\pi^2(m^2+n^2))^{1/2}}\Big|,\ \Big|\frac{-4\pi n i}{(1+4\pi^2(m^2+n^2))^{1/2}}\Big| \leq C.$$

Using induction

$$\delta_1^p(\pi(f))d_{m,n,\varepsilon} = \sum_{k,l=1}^{2p} c_{k,l}(m,n)\pi(\partial_x^k\partial_y^l f)d_{m,n,\varepsilon},$$

for some collection of uniformly bounded functions $c_{k,l}:\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$. Since $\operatorname{span}\{d_{m,n,\varepsilon} : m,n\in\mathbb{Z},\ \varepsilon=\pm1\}\subset L^2(\mathbb{T}^2,\mathbb{C}^2)$ dense. It holds that

$$\delta_1^p(\pi(f))\Big(\sum_{m,n,\varepsilon}\alpha_{m,n,\varepsilon}d_{m,n,\varepsilon}\Big) = \sum_{k,l=1}^{2p}\pi(\partial_x^k\partial_y^l f)\Big(\sum_{m,n,\varepsilon}c_{k,l}(m,n)\alpha_{m,n,\varepsilon}d_{m,n,\varepsilon}\Big).$$

Since each of the operators $\pi(\partial_x^k\partial_y^l f)$ is bounded, Lemma 2.50 implies that $\delta_1^p(\pi(f))$ is bounded. Note that $[D,\pi(f)]$ is no longer an element of $C^\infty(\mathbb{T}^2,\mathbb{C})$, but we can show boundedness of $\delta_1^p([D,\pi(f)])$ in a similar way. Recall from example 2.34 that

$$[D,\pi(f)]d_{m,n,\varepsilon} = \pi(-i\partial_x f+\varepsilon\partial_y f)d_{m,n,-\varepsilon} = \sum_{k,l}2\pi(k+i\varepsilon l)\alpha_{k,l}d_{k+m,l+n,-\varepsilon}. \tag{2.8}$$

Then

$$\begin{aligned}
\delta_1([D,\pi(f)])d_{m,n,\varepsilon} &= [D^2,[D,\pi(f)]](1+D^2)^{-1/2}d_{m,n,\varepsilon}\\
&= (1+4\pi^2(m^2+n^2))^{-1/2}(D^2[D,\pi(f)]d_{m,n,\varepsilon} - [D,\pi(f)]D^2 d_{m,n,\varepsilon})\\
&= (1+4\pi^2(m^2+n^2))^{-1/2}\Big(D^2\big(\sum_{k,l}2\pi(k+i\varepsilon l)\alpha_{k,l}d_{k+m,l+n,-\varepsilon}\big)\\
&\qquad - [D,\pi(f)](4\pi^2(m^2+n^2)d_{m,n,\varepsilon})\Big)\\
&= (1+4\pi^2(m^2+n^2))^{-1/2}\Big(\sum_{k,l}2\pi(k+i\varepsilon l)4\pi^2((k+m)^2+(l+n)^2)\alpha_{k,l}d_{m,n,-\varepsilon}\\
&\qquad - 4\pi^2(m^2+n^2)\sum_{k,l}2\pi(k+i\varepsilon l)\alpha_{k,l}d_{k+m,l+n,-\varepsilon}\Big)\\
&= (1+4\pi^2(m^2+n^2))^{-1/2}\Big(\sum_{k,l}8\pi^3(k+i\varepsilon l)(k^2+l^2+2km+2ln)\alpha_{k,l}e_{k,l}\Big)d_{m,n,-\varepsilon}\\
&= (1+4\pi^2(m^2+n^2))^{-1/2}\pi\big((-i\partial_x+\varepsilon\partial_y)(-\partial_x^2-\partial_y^2-4\pi i m\partial_x-4\pi i n\partial_y)f\big)d_{m,n,-\varepsilon}\\
&= (1+4\pi^2(m^2+n^2))^{-1/2}([D,\pi(-\partial_x^2 f)]+[d,\pi(-\partial_y^2 f)]+[D,\pi(-4\pi i m\partial_x f)]\\
&\qquad +[D,\pi(-4\pi i n\partial_y f)])d_{m,n,\varepsilon}.
\end{aligned}$$

Now we can use induction and the same arguments as before to obtain that $\delta_1^p([D,\pi(f)])$ is a bounded operator for each $p\in\mathbb{N}$. From Lemma 2.43 it follows that $\delta^n(a)$ is bounded for all n. Then with Lemma 2.42 it follows that $\delta^n(a)\operatorname{Dom}(D)\subset\operatorname{Dom}(D)$. Hence $\delta^n(a)\in\operatorname{Dom}(\delta)$ for all n and the torus is regular.

Definition 2.52. Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular p^+-summable spectral triple. Let $\mathcal{B}$ be the algebra generated by $\delta^n(a)$ and $\delta^n([D,a])$, $a \in \mathcal{A}$, $n \in \mathbb{N}$. For $b \in \mathcal{B}$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > p$ define $\zeta_b(z) := \mathrm{Tr}(b(1+D^2)^{-z})$. The *dimension spectrum* is defined as the subset $Sd \subset \mathbb{C}$, where $z \in Sd$ if and only if there exits a $b \in \mathcal{B}$ such that (the meromorphic continuation of) ζ_b has a singularity at z.
If Sd is discrete, the dimension spectrum is called *discrete*, if the functions ζ_b have only simple poles, the dimension spectrum is called *simple*.

The definition of the dimension spectrum might seem arbitrary, but in fact it is very useful. In [10] Connes and Moscovici state and prove the local index theorem. With this theorem one can compute the index of an operator by sums of residues of zeta functions ζ_b. One of the assumptions of this theorem is that the dimension spectrum is discrete and simple.
Another motivation for considering the dimension spectrum is that in case of a spin manifold the dimension spectrum consists of $\{0, 1, \ldots, \dim(M)\}$, so the dimension spectrum in some sense also sees the dimensions of the submanifolds of the original one. In some sense it can be seen as a generalisation of the dimension of a manifold. We will use this definition of dimension when we later define the z-dimensional spaces.

In the definition of the dimension spectrum we speak of an analytic continuation, to do so we need a holomorphic function. So we have to prove that $s \mapsto \mathrm{Tr}(b(1 + D^2)^{-s/2})$ gives a holomorphic function on an appropriate domain.

Lemma 2.53. *If a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p^+-summable, then the functions $\{\zeta_b : b \in \mathcal{B}\}$ are holomorphic on $\{z \in \mathbb{C} : \mathrm{Re}(z) > p\}$.*

Proof. Suppose $(\lambda_n)_n$ is the collection of eigenvalues of D. Then $(1 + D^2)^{-1/2}$ has eigenvalues $\mu'_n := (1 + \lambda^2)^{-1/2}$. Order these eigenvalues μ'_n as a decreasing sequence say $(\mu_n)_n$. Then by assumption of the metric dimension the sequence $(\mu_n)_n$ is of order $\mathcal{O}(n^{1/m})$. Thus

$$|\zeta_b(s)| = \left|\mathrm{Tr}(b(1 + D^2)^{-s/2})\right| \leq \|b\| \,\mathrm{Tr}(|(1 + D^2)^{-s/2}|) = \|b\| \sum_{n=0}^{\infty} \mu_n^{\mathrm{Re}(s)},$$

which is convergent for s with $\mathrm{Re}(s) > m$. We also have

$$\frac{d^k}{ds^k}\Big(\sum_{n=0}^{\infty} \mu_n^s\Big) = \sum_{n=0}^{\infty} \log(\mu_n)^k \mu_n^s,$$

which also converges if $\mathrm{Re}(s) > m$. Thus the function $\zeta_b : \{z \in \mathbb{C} : \mathrm{Re}(z) > m\} \to \mathbb{C}$ is holomorphic. ⊠

Using Lemma 2.40 it is easy to prove the following result.

Corollary 2.54. *If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, then for $b \in \mathcal{B}$ the inclusion $b\mathcal{H}_\infty \subset \mathcal{H}_\infty$ holds.*

Proof. By Lemma 2.40 for an element $a \in \mathcal{A}$ the inclusion $a\mathcal{H}_\infty \subset \mathcal{H}_\infty$ holds. Since $D(\mathcal{H}_\infty) \subset \mathcal{H}_\infty$ we have $[D,a] = Da - aD$ where both summands exist. Then $(Da - aD)\mathcal{H}_\infty \subset \mathcal{H}_\infty$. Use induction to n and similar argument to show that $\delta^n(a) = [|D|, \delta^{n-1}(a)]$ preserves $\mathcal{H}_\infty$. ⊠

Connes and Moscovici compute the dimension spectrum of spin manifolds.

Theorem 2.55. *Suppose M is a closed Riemannian spin manifold of dimension p. Then the triple $(C^\infty(M), L^2(M, \mathcal{S}), D)$ has simple dimension spectrum, which is contained in the set $\{n \in \mathbb{N} : n \leq p\}$.*

Proof. See [10, Rem. II.1]. ⊠

We would like to compute the dimension spectrum of the torus ourselves. Because it is very hard to deal with the explicit expressions of the operators in $\mathcal{B}$ we will use a different approach using pseudo-differential operators. We start with a general remark.

Remark 2.56. Suppose A and B are (pseudo-)differential operators of order k respectively l. Then AB and BA are (pseudo-)differential operators of order $k+l$ and the leading symbols are equal; $\sigma_L(AB) = \sigma_L(BA)$. Thus the commutator $[A, B]$ does not have a term of order $k+l$, the highest order terms of AB and BA cancel. Thus $[A, B]$ is a (pseudo-)differential operator of order $k+l-1$.

Lemma 2.57. *The algebra $\mathcal{B}$ of a spin manifold M only contains pseudo-differential operators of order ≤ 0.*

Proof. Note that the operators in $\mathcal{A} = C^\infty(M)$ are just multiplication by smooth functions, so they are differential operators of order 0. Also D is partial differential operator, it is of order 1. The operator $|D|$ is a pseudo-differential operator of order 1. Thus from the previous remark we have if $b \in \mathcal{A} \cup [D, \mathcal{A}]$, then b is of order ≤ 0. Note that $|D| = \left(D^2\right)^{1/2}$, thus $|D|$ is a pseudo-differential operator of order 1. Using induction it immediately follows that $\delta^n(b)$ is also of order ≤ 0 if $b \in \mathcal{A} \cup [D, \mathcal{A}]$. Hence the result follows. ⊠

In Example 2.59 we will compute the dimension spectrum of the torus. In the proof we will use a heat kernel expansion for elements of the form $b(1+D^2)^{-s}$. We will first give the basics about these heat kernel expansions, a detailed treatment can be found in the book [18]. The following remark is based [18, Ch. 1] and [19, Ch. 7].

Remark 2.58. Suppose P is an elliptic partial differential operator. It is possible to show that e^{-tP} has a kernel K. Thus there exists a function K such that

$$e^{-tP} f(x) = \int_M K(t, x, y) f(y)\, dvol(y).$$

Apply this to be^{-tP}, then

$$be^{-tP} f(x) = \int_M bK(t, x, y) f(y)\, dvol(y).$$

Taking the trace gives

$$\mathrm{Tr}(be^{-tP}) = \int_M \mathrm{Tr}_{V_x}(bK(t, x, x))\, dvol(x), \tag{2.9}$$

where Tr_{V_x} is the trace in the fiber V_x. Therefore we would like to have an expansion of $bK(t, x, y)$ near $x = y$. Such an expansion exists: there exists coefficients $e_n(x; b, P)$ such that

$$(bK(t, x, y))|_{x=y} \sim \sum_{n=0}^{\infty} t^{(n-m-a)/d} e_n(x; b, P), \qquad \text{as } t \downarrow 0. \tag{2.10}$$

Here m is the dimension of the manifold M and d and a are the orders of the operators P respectively b. The notation

$$f(t) \sim \sum_{n=0}^{\infty} f_n(t), \qquad \text{as } t \downarrow 0$$

is used if for all N there exists a constant C_N such that $|f(t) - \sum_{n=0}^{N-1} f_n(t)| < C_N |f_N(t)|$ in a neighbourhood of $t = 0$. Combination of (2.9) and (2.10) gives

$$\mathrm{Tr}(be^{-tP}) \sim \sum_{n=0}^{\infty} t^{(n-m-a)/d} a_n(b, P), \qquad \text{as } t \downarrow 0. \tag{2.11}$$

where $a_n(b, P) := \int_M \mathrm{Tr}_{V_x}(e_n(x; b, P))\, dvol(x)$.

Example 2.59. The dimension spectrum of the torus $\mathbb{T}^2$ is contained in the set $2 - \mathbb{N}$.

Proof. For brevity we will write in this proof $P := (1 + D^2)$. Note that the dimension of $\mathbb{T}^2$ is 2 and the operator P is an elliptic partial differential operator of order 2. By lemma 2.57 each $b \in \mathcal{B}$ is of order 0, thus we can apply [19, Eq. (7.56c)] to conclude that the kernel of the operator be^{-tP} has an asymptotic expansion given by

$$\{K_{P,b}(t,x,y)\}|_{x=y} \sim \sum_{k=0}^{\infty} t^{(n-2)/2} e_n(x;b,P), \qquad \text{as } t \downarrow 0, \tag{2.12}$$

for some coefficients $e_n(x;b,P)$ depending on x, b and P. For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$ we have by the functional calculus

$$P^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tP}\, dt.$$

Thus

$$bP^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} bt^{s-1} e^{-tP}\, dt.$$

Interchanging integral and trace gives for $\mathrm{Re}(s) > 2$

$$\zeta_b(s) = \mathrm{Tr}(bP^{-s/2}) = \int_0^{\infty} t^{s/2-1}\, \mathrm{Tr}\left(be^{-tP}\right) dt. \tag{2.13}$$

This switch of integral and trace is allowed. Namely, write the trace as an infinite sum and apply the dominated convergence theorem, because as we show below this integral converges absolutely for $\mathrm{Re}(s) > 2$.

So we have to compute the poles of the analytic continuation of $s \mapsto \int_0^\infty t^{s/2-1}\, \mathrm{Tr}\left(be^{-tP}\right) dt$. We will split this integral in two parts: from 0 to 1 and 1 to ∞. The part $s \mapsto \int_1^\infty t^{s/2-1}\, \mathrm{Tr}\left(be^{-tP}\right) dt$ gives an entire function, because

$$\left| \int_1^{\infty} t^{s/2-1}\, \mathrm{Tr}\left(be^{-tP}\right) dt \right| \leq \int_1^{\infty} \left| t^{s/2-1}\, \mathrm{Tr}\left(be^{-tP}\right) \right| dt \leq \|b\| \int_1^{\infty} t^{\mathrm{Re}(s/2)-1}\, \mathrm{Tr}\left(e^{-tP}\right) dt. \tag{2.14}$$

We know that the eigenvalues λ_n of the operator P increase as $\sim n$ (see the proof of 2.37). Thus the last integral of (2.14) converges for all $s \in \mathbb{C}$. And therefore the first integral of (2.14) gives a holomorphic function.

To compute the dimension spectrum it thus suffices to compute the poles of the meromorphic extension of

$$s \mapsto \int_0^1 t^{s/2-1}\, \mathrm{Tr}\left(be^{-tP}\right) dt. \tag{2.15}$$

Since we are in the situation of remark 2.58 we obtain from (2.11) and (2.12) the following heat kernel expansion

$$\mathrm{Tr}(be^{-tP}) \sim \sum_{n=0}^{\infty} t^{(n-2)/2} a_n(b,P), \qquad \text{as } t \downarrow 0.$$

So for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ we have

$$\begin{aligned}
\int_0^1 t^{s/2-1}\, \mathrm{Tr}(be^{-tP})\, dt &\sim \int_0^1 t^{s/2-1} \Big(\sum_{n=0}^{m} t^{(n-2)/2} a_n(b,P) + r_m(t) \Big)\, dt \\
&= \sum_{n=0}^{m} \int_0^1 t^{(n+s-2)/2-1} a_n(b,P)\, dt + \int_0^1 t^{s/2-1} r_m(t)\, dt \\
&= \sum_{n=0}^{m} \frac{2}{n+s-2} \left[t^{(n+s-2)/2} \right]_{t=0}^{t=1} a_n(b,P) + q_m(s) \\
&= \sum_{n=0}^{m} \frac{2}{n+s-2}\, a_n(b,P) + q_m(s).
\end{aligned}$$

Here the function r_m is the error-term of the asymptotic expansion (2.15). By definition r_m is $\mathcal{O}(t^{(m-2)/2})$. Hence $q_m(s) := \int_0^1 t^{s/2-1} r_m(t)\, dt = \int_0^1 \mathcal{O}(t^{(m+s-2)/2-1})\, dt = \mathcal{O}(t^{(m+s-2)/2})$, which is holomorphic for $s \in \mathbb{C}$, with $\mathrm{Re}(s) > m-2$. So the function (2.13) extends to the meromorphic function

$$\{s \in \mathbb{C} : \mathrm{Re}(s) > m\} \to \mathbb{C}, \qquad s \mapsto \sum_{n=0}^{m} \frac{2}{n+s-2}\, a_n(b,P) + q_m(s),$$

with simple poles of residue $2a_n(b,P)$ at $s = 2-n$. Thus the dimension spectrum is contained in the set $2 - \mathbb{N}$. In particular the dimension spectrum is simple and discrete. ⊠

Note that from the above proof it follows that $m \in 2 - \mathbb{N}$ is in the dimension spectrum of the torus if and only if there exists an $b \in \mathcal{B}$ such that $a_n(b,P) \neq 0$. So if we can calculate these coefficients we know the dimension spectrum precisely.

2.4 Products of spectral triples

Given two spectral triples it is possible to construct the product of those triples. In the case of a spin manifold taking the product of the spectral triples corresponds to taking the cartesian product of the underlying manifolds. For the product of spectral triples we will need some constructions using the tensor product. We will first review those.

This is a very concise overview, more information can be found in textbooks (e.g. [30, Chapter IV]). We will assume knowledge of the (algebraic) tensor product of vector spaces. The tensor product of vector spaces has the universal property that given two vector spaces U, V, W and a bilinear map $T : V \times W \to U$, there exists a unique linear map $\hat{T}$ such that

$$\begin{tikzcd} V \times W \arrow[r, "i"] \arrow[d, "T"'] & V \otimes W \arrow[dl, "\hat{T}"] \\ U & \end{tikzcd}$$

commutes. Here $i : V \times W \to V \otimes W$ denotes the inclusion $i(v,w) := v \otimes w$. In fact one can define the tensor product of vector spaces using this universal property.
For two algebras A and B we can define the algebra $A \otimes B$ as the vector space $A \otimes B$ equipped with the product given by $(a \otimes b) \cdot (a' \otimes b') := (aa') \otimes (bb')$ and extend this linearly and with distributivity. Again we have a property of unique extension of bilinear maps over $A \otimes B$ when considered as vector spaces.
For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ we denote $\mathcal{H} \otimes_{alg} \mathcal{K}$ for the tensor product of $\mathcal{H}$ and $\mathcal{K}$ as vector spaces. This space needs to be equipped with a inner product. Define $\langle h_1 \otimes_{alg} k_1, h_2 \otimes_{alg} k_2 \rangle := \langle h_1, h_2 \rangle_{\mathcal{H}} \langle k_1, k_2 \rangle_{\mathcal{K}}$ for elementary tensors and by linear extension. In general $(\mathcal{H} \otimes_{alg} \mathcal{K}, \langle \cdot, \cdot \rangle)$ is not complete in the norm induced by this inner product. Therefore we let $\mathcal{H} \otimes \mathcal{K}$ be the completion of the pre-Hilbert space $\mathcal{H} \otimes_{alg} \mathcal{K}$ in the norm induced by the inner product $\langle \cdot, \cdot \rangle$.
In the same way it is possible to extend two operators to the tensor product.

Lemma 2.60. *Suppose $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$ are bounded operators, then the map $S \otimes T$ given on the elementary tensors by $S \otimes T(h \otimes k) := (Sh) \otimes (Tk)$ extends linearly and continuously to the full Hilbert space $\mathcal{H} \otimes \mathcal{K}$.*

Proof. It is trivial that we an extends this operator to $\mathcal{H} \otimes_{alg} \mathcal{K}$. To continuously extend this operator to the whole space $\mathcal{H} \otimes \mathcal{K}$ it is sufficient to show that $S \otimes T$ is bounded on the dense subspace $\mathcal{H} \otimes_{alg} \mathcal{K} \subset \mathcal{H} \otimes \mathcal{K}$. Indeed, let $h \in \mathcal{H} \otimes_{alg} \mathcal{K}$ and write $h = \sum_{i=1}^{n} e_i \otimes f_i$, for an orthonormal set $(e_i)_{i=1}^n$, then

$$\begin{aligned}\|S\otimes T(h)\|^2 &= \Big\|\sum_{i=1}^n Se_i\otimes Tf_i\Big\|^2\\ &= \sum_{i,j=1}^n \langle Se_i, Se_j\rangle\langle Tf_i, Tf_j\rangle\\ &\leq \sum_{i,j=1}^n \|S\|^2|\langle e_i, e_j\rangle|\,\|T\|^2|\langle f_i, f_j\rangle|\\ &= \|S\|^2\,\|T\|^2\sum_{i=1}^n |\langle e_i, e_i\rangle|\,|\langle f_i, f_i\rangle|\\ &= \|S\|^2\,\|T\|^2\Big|\sum_{i=1}^n \langle e_i, e_i\rangle\,\langle f_i, f_i\rangle\Big|\\ &= \|S\|^2\,\|T\|^2\Big|\sum_{i,j=1}^n \langle e_i, e_j\rangle\,\langle f_i, f_j\rangle\Big|\\ &= \|S\|^2\,\|T\|^2\|h\|^2.\end{aligned}$$

⊠

Using these bounded operators we can define the tensor product of von Neumann algebras.

Definition 2.61. Suppose for $i = 1, 2$, $\mathcal{M}_i \subset B(\mathcal{H}_i)$ are von Neumann algebras. We define $\mathcal{M}_1 \otimes \mathcal{M}_2$ to be the von Neumann algebra generated by the elements $a_1 \otimes a_2$, for $a_i \in \mathcal{M}_i$. More precise $\mathcal{M}_1 \otimes \mathcal{M}_2 := \{a_1 \otimes a_2 : a_1 \in \mathcal{M}_1, a_2 \in \mathcal{M}_2\}'' \subset B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. This is well-defined, because as we showed before (cf. Lemma 2.60) the operators $a_1 \otimes a_2$ are again bounded operators.

Lemma 2.60 has an analogy for unbounded operators. But this is more subtle because we have to deal with the domains. We will state it as a lemma.

Lemma 2.62. *Suppose $(S, \mathrm{Dom}(S))$ and $(T, \mathrm{Dom}(T))$ are densily defined closable operators on $\mathcal{H}$ respectively $\mathcal{K}$. Then there exists a densily defined closed operator $S\otimes T$ such that $S\otimes T(h\otimes k) = Sh \otimes Tk$ for all $h \in \mathrm{Dom}(S)$ and $h \in \mathrm{Dom}(T)$.*

Proof. Define the operator $S \otimes_{alg} T : \mathrm{Dom}(S) \otimes_{alg} \mathrm{Dom}(T) \to \mathcal{H} \otimes \mathcal{K}$ by $S \otimes_{alg} T(h \otimes k) := (Sh) \otimes (Tk)$. This is operator well-defined. Since S and T are closable $\mathrm{Dom}(S), \mathrm{Dom}(S^*) \subset \mathcal{H}$ dense and $\mathrm{Dom}(T), \mathrm{Dom}(T^*) \subset \mathcal{K}$ dense. Hence $\mathrm{Dom}(S)\otimes\mathrm{Dom}(T), \mathrm{Dom}(S^*)\otimes\mathrm{Dom}(T^*) \subset \mathcal{H}\otimes\mathcal{K}$ dense. Since clearly $S^* \otimes_{alg} T^* \subset (S \otimes_{alg} T)^*$, the operator $(S \otimes_{alg} T)^*$ is densely defined and therefore $S \otimes_{alg} T$ is closable. We define $S \otimes T := \overline{S \otimes_{alg} T}$, which satisfies the requirements. ⊠

These constructions will be applied in the following proposition, which constructs the product of spectral triples.

Proposition 2.63. *Given an even spectral triple $\mathcal{S}_1 = (\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and a spectral triple $\mathcal{S}_2 = (\mathcal{A}_2, \mathcal{H}_2, D_2)$, then the tuple $\mathcal{S}_1 \times \mathcal{S}_2$ given by*

$$\mathcal{S}_1 \times \mathcal{S}_2 := (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2)$$

is a spectral triple. If $\mathcal{S}_2$ is also even with grading γ_2, then the product triple $\mathcal{S}_1 \times \mathcal{S}_2$ is even with grading $\gamma_1 \otimes \gamma_2$.

Proof. We will give a sketch of the proof which shows the main ideas. Later we will give a full proof when we generalise this result to the case of semifinite spectral triples see theorem 3.10. A very brief construction of the product can be found in [19, §10.5]. In the case $\mathcal{H}_1$ and $\mathcal{H}_2$ are

separable a proof of this proposition can be found in the article [12], this is the proof of which we give a sketch.
We have two faithful representations $\pi_1 : \mathcal{A}_i \to B(\mathcal{H}_i)$. This yields a representation $\pi_1 \otimes \pi_2$ on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ via the familiar construction. $\pi_1 \otimes \pi_2(a_1 \otimes a_2)h_1 \otimes h_2 := \pi_1(a_1)h_1 \otimes \pi_2(a_2)h_2$.
The algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ obeys the requirements of a spectral triple. Since for $i = 1, 2$ the representations π_i of $\mathcal{A}_i$ are faithful and $\mathcal{A}_i$ acts by bounded operators on $\mathcal{H}_i$, we have an embedding of $\mathcal{A}_i$ into $B(\mathcal{H}_i)$. Then also $\mathcal{A}_1 \otimes \mathcal{A}_2 \subset B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) \cong B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, where $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ is the tensor product of the von Neumann algebras $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$. Therefore $\mathcal{A}_1 \otimes \mathcal{A}_2$ is faithfully represented by $\pi_1 \otimes \pi_2$ and acts by bounded operators. The algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ has an involution given on the elementary tensors by $(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*$. And if both $\mathcal{A}_1$ and $\mathcal{A}_2$ have a unit then so does $\mathcal{A}_1 \otimes \mathcal{A}_2$.
Now we will turn our attention to the Dirac operator. We will show that the operator $D := D_1 \otimes 1 + \gamma_1 \otimes D_2$ is densely defined, self-adjoint, $\|[D, \pi_1 \otimes \pi_2(a)]\| < \infty$ for all $a \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and has a compact resolvent.
It is relatively easy to show that the domains of the operators $D_1 \otimes 1$ and $\gamma_1 \otimes D_2$ are given by $\mathrm{Dom}(D_1 \otimes 1) = \mathrm{Dom}(D_1) \otimes \mathcal{H}_2$ and $\mathrm{Dom}(\gamma_1 \otimes D_2) = \mathcal{H}_2 \otimes \mathrm{Dom}(D_2)$, (this is proved in detail in lemma 3.13). Hence $\mathrm{Dom}(D) = \mathrm{Dom}(D_1) \otimes \mathcal{H}_2 \cap \mathcal{H}_2 \otimes \mathrm{Dom}(D_2) = \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2)$ which is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$.
Self-adjointness and compactness of the resolvent requires some more work. In the case $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable there exist orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ consisting of eigenvectors for D_1 respectively D_2. From these bases it is possible (but we will not do that) to construct an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$ consisting of eigenvectors for D. The eigenvalues are of the form $\pm\sqrt{\lambda^2 + \mu^2}$, where λ and μ are eigenvalues of D_1 respectively D_2, counted with multiplicity. Hence the eigenvalues are all real and tend to infinity. So D has a self-adjoint extension and a compact resolvent.
Suppose $a \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is of the form $a = a_1 \otimes a_2$. Then

$$[D, \pi_1 \otimes \pi_2(a_1 \otimes a_2)] = [D_1, \pi_1(a_1)] \otimes \pi_2(a_2) - \pi_1(a_1) \otimes [D_2, \pi_2(a_2)].$$

Because $[D_1, \pi_1(a_1)]$ and $[D_2, \pi_2(a_2)]$ are densely defined and bounded and the operators $\pi_2(a_2)$ and $\pi_1(a_1)$ are bounded, the commutator $[D, \pi_1 \otimes \pi_2(a_1 \otimes a_2)]$ is densely defined and extends to a bounded operator. Now for an arbitrary element $a = \sum_{i=1}^n a_i \otimes b_1$ we can show boundedness by linearity of π_i and D_i.
These results show that $\mathcal{S}_1 \times \mathcal{S}_2$ is a spectral triple.

To prove the last assertion, suppose that $\mathcal{S}_2$ is also even with grading χ_2. A direct computation shows

$$\begin{aligned}(\gamma_1 \otimes \gamma_2)(D_1 \otimes 1 + \gamma_1 \otimes D_2) &= \gamma_1 D_1 \otimes \gamma_2 + \gamma_1^2 \otimes \gamma_2 D_2 \\ &= -D_1\gamma_1 \otimes \gamma_2 + \gamma_1^2 \otimes (-D_2\gamma_2) \\ &= -(D_1 \otimes 1 + \gamma_1 \otimes D_2)(\gamma_1 \otimes \gamma_2).\end{aligned}$$

Similarly for an elementary tensor $a_1 \otimes a_2$ one has

$$\pi_1 \otimes \pi_2(a_1 \otimes a_2)(\gamma_1 \otimes \gamma_2) = \pi_1(a_1)\gamma_1 \otimes \pi_2(a_2)\gamma_2 = \gamma_1\pi_1(a_1) \otimes \gamma_2\pi_2(a_2) = (\gamma_1 \otimes \gamma_2)\pi_1 \otimes \pi_2(a_1 \otimes a_2).$$

Hence $\mathcal{S}_1 \times \mathcal{S}_2$ is an even spectral triple if $\mathcal{S}_1$ and $\mathcal{S}_2$ are. ⊠

We can prove results about the summability and regularity of the product triple. We will do this later in the more general setting of semifinite spectral triples, this can be found in Section 3.2.

3 Semifinite noncommutative geometry

The objective of the next section is to construct spectral triples which satisfy a specific requirement so that they can be considered to be z-dimensional for $z \in (0, \infty)$. After a short examination one easily sees that one cannot find a spectral triple with that property. Therefore we are forced to generalise the spectral triples. This will lead to the semifinite spectral triples. We will take a closer look at these triples in general in this section.

3.1 Semifinite spectral triples and their properties

The difference between an ordinary spectral triple and a semifinite one is that we no longer require that the resolvent of the Dirac operator is compact, but we want it to be compact relative to a trace on a semifinite von Neumann algebra (cf. Subsection 1.91).

Definition 3.1. A *semifinite spectral triple* $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ consists of a Hilbert space $\mathcal{H}$ a semifinite von Neumann algebra $\mathcal{N}$ acting on $\mathcal{H}$ with a faithful normal semifinite trace τ, an involutive algebra $\mathcal{A} \subset \mathcal{N}$ and a self-adjoint operator D affiliated to $\mathcal{N}$. Furthermore we require that for all $a \in \mathcal{A}$ the operator $[D, a]$ is densely defined and extends to a bounded operator on $\mathcal{H}$ and that the operator D is τ-discrete.
If in addition there exists a grading $\gamma \in \mathcal{N}$ such that $\gamma D = -D\gamma$ and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$. Then the tuple $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau, \gamma)$ is called an *even semifinite spectral triple.* If no such grading exists, the tuple $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ is called an *odd semifinite spectral triple.*

The reason for the requirement $\gamma \in \mathcal{N}$ is that we would like to be able to take the trace $\tau(\gamma a)$ for $a \in \mathcal{A}$.
Most of the definitions of the classical case copy to the semifinite setting, in most cases we only have to deal with the substitution of Tr by a trace τ.

Definition 3.2. Suppose $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ is a semifinite spectral triple. We say the triple is *p-τ-summable* if $\tau((1 + D^2)^{-p/2}) < \infty$. The triple is *$\tau$-finitely summable* if it is p-τ-summable for some $p > 0$. The triple is *p^+-τ-summable* if $\tau((1 + D^2)^{-p/2+\varepsilon}) < \infty$ for all $\varepsilon > 0$. The triple is said to be *θ-τ-summable* if $\tau(e^{-tD^2}) < \infty$ for any $t > 0$.

These different notions of τ-summability are related to one another, see the next lemma.

Lemma 3.3. *Suppose $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ is a semifinite spectral triple and $q > p > 0$. If the triple is p-τ-summable, then it is p^+-τ-summable, q-summable and θ-τ-summable.*

Proof. Suppose $1 < p < q$. Because D is affiliated to $\mathcal{N}$ and $(1+D^2)^{-1}$ is bounded, the operator $(1 + D^2)^{-(q-p)/2} \in \mathcal{N}$. Thus by Proposition 1.46 the following inequality holds

$$\tau\big((1+D^2)^{-q/2}\big) = \tau\big((1+D^2)^{-(q-p)/2}(1+D^2)^{-p/2}\big) \le \|(1+D^2)^{-(q-p)/2}\|\tau\big((1+D^2)^{-p/2}\big) < \infty.$$

and thus D is q-τ–summable. If $\varepsilon > 0$, put $q := p + 2\varepsilon$ from which p^+-τ-summability follows.
For the last assertion, we know that for $t > 0$ and $\alpha > 0$ fixed, the function $g_{t,\alpha} : [0, \infty) \to \mathbb{R}$, $g_{t,\alpha}(x) := (1 + x^2)^{\alpha/2} e^{-tx^2}$ is bounded, say by $C_{t,\alpha}$. Then

$$\tau\big(e^{-tD^2}\big) = \tau\big(|g_{t,p}(D)|\,(1 + D^2)^{-p/2}\big) \le C_{t,p}\tau\big((1 + D^2)^{-p/2}\big) < \infty,$$

so the operator is θ-τ-summable. ⊠

Definition 3.4. A regular τ-finitely summable semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ is said to have *dimension spectrum $Sd \subset \mathbb{C}$* if for all $b \in \mathcal{B}$ the zeta function $\zeta_b : z \mapsto \tau(b(1 + D^2)^{-z/2})$ (for $\mathrm{Re}(z)$ large), extends analytically to $\mathbb{C} \setminus Sd$. If Sd is discrete, then $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ is said to have a *discrete dimension spectrum.* If ζ_b has at most simple poles, the dimension spectrum is called *simple.*

3.2 Products of semifinite spectral triples

We can generalise the construction of Proposition 2.63 of the product of two spectral triples to a product of semifinite spectral triples. It is clear that we cannot expect the product to be a classical spectral triple, but we do obtain a semifinite spectral triple. This is the content of Theorem 3.10. Note that such a theorem exceeds Proposition 2.63, because a classical spectral triple is a semifinite spectral triple with type I von Neumann algebra $B(\mathcal{H})$ and trace Tr. We start with the preparations for the proof of this theorem.

Lemma 3.5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Suppose $\mathcal{H}_2$ has an orthonormal basis $(e_i)_{i\in I}$. Denote $\mathcal{K}_i$ for a copy of $\mathcal{H}_1$. Then the map*

$$U : \bigoplus_{i\in I} \mathcal{K}_i \to \mathcal{H}_1 \otimes \mathcal{H}_2, \qquad (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i \otimes e_i$$

is an isometry.

Proof. Note that this sum is well-defined, because if $(x_i)_i \in \bigoplus_{i\in I} \mathcal{K}_i$ at most countably many elements x_i can be non-zero. Now clearly U is bijective and it preserves the norm. Hence it is an isometry. ⊠

Proposition 3.6. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be semifinite von Neumann algebras, then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is a semifinite von Neumann algebra.*

Proof. See [2, Thm. III.2.5.27]. ⊠

Combination of 1.43 and 3.6 shows that given two semifinite von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ with semifinite faithful normal traces τ_1 and τ_2 there exists a semifinite normal trace on $\mathcal{M}_1 \otimes \mathcal{M}_2$. It is possible to construct such a trace from τ_1 and τ_2. The obvious choice is $\tau_1 \otimes \tau_2$, which indeed works.

Proposition 3.7. *Suppose for $i = 1, 2$ $\mathcal{M}_i$ is a semifinite von Neumann algebra with faithful semifinite normal trace τ_i, then $\tau := \tau_1 \otimes \tau_2 : (\mathcal{M}_1 \otimes \mathcal{M}_2)_+ \to [0, \infty]$ is a faithful semifinite normal trace. In particular this trace factors, thus $\tau_1 \otimes \tau_2(a_1 \otimes a_2) = \tau_1(a_1)\tau_2(a_2)$.*

Proof. From [31, VIII.§4] it follows that the map $\tau_1 \otimes \tau_2$ is a faithful semifinite normal weight. (A weight on a C^*-algebra is a function $\varphi : A_+ \to [0, \infty]$ such that $\varphi(0) = 0$, $\varphi(\lambda a) = \lambda\varphi(a)$ for all $a \in A_+$ and $\lambda \geq 0$ and $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in A_+$). So it remains to show that τ has the trace property, i.e. $\tau(aa^*) = \tau(a^*a)$ for all $a \in \mathcal{M}_1 \otimes \mathcal{M}_2$. Let $a = \sum_n x_n \otimes y_n$, then

$$aa^* = \Big(\sum_n x_n \otimes y_n\Big)\Big(\sum_n x_n \otimes y_n\Big)^* = \Big(\sum_n x_n \otimes y_n\Big)\Big(\sum_n x_n^* \otimes y_n^*\Big) = \sum_n\sum_m x_n x_m^* \otimes y_n y_m^*.$$

Since τ_i is a trace we have $\tau_i(ab) = \tau_i(ba)$ for all $a, b \in \mathcal{M}_{i+}$ with $\tau_i(a), \tau_i(b) < \infty$. Therefore τ_i extends to a linear functional on $\operatorname{span}\{a \in \mathcal{M}_+ : \tau_i(a) < \infty\}$. And if $a, b \in \operatorname{span}\{a \in \mathcal{M}_+ : \tau_i(a) < \infty\}$ it holds hat $\tau_i(ab) = \tau_i(ba)$. Observe if $\tau(aa^*) = \infty$ then also $\tau(a^*a) = \infty$. So assume $\tau(aa^*) < \infty$. In that case $a = \sum_n x_n \otimes y_n$ with $|\tau_1(x_n)| < \infty$ and $|\tau_2(y_n)| < \infty$ for all n and hence

$$\begin{aligned}
\tau(aa^*) &= \tau\Big(\sum_n\sum_m x_n x_m^* \otimes y_n y_m^*\Big) \\
&= \sum_n\sum_m \tau_1(x_n x_m^*)\tau_2(y_n y_m^*) \\
&= \sum_n\sum_m \tau_1(x_m^* x_n)\tau_2(y_m^* y_n) \\
&= \tau\Big(\sum_n\sum_m x_m^* x_n \otimes y_m^* y_n\Big) \\
&= \tau(a^*a).
\end{aligned}$$

So indeed $\tau_1 \otimes \tau_2$ is a trace. ⊠

Lemma 3.8. *In the notation of Lemma 3.7 suppose for $i = 1, 2$ the operators $K_i : \mathcal{H}_i \to \mathcal{H}_i$ are τ_i-compact. Then $K_1 \otimes K_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ is a τ-compact operator.*

Proof. Let $\varepsilon > 0$. Select for $i = 1, 2$ operators $R_i \in B(\mathcal{H}_i)$ such that $\|K_i - R_i\| < \varepsilon$, with the property that for the projection P_i on the range of R_i the trace $\tau_i(P_i) < \infty$. Then $P_1 \otimes P_2$ is the projection on the range of $R_1 \otimes R_2$. By the factorisation of τ (Lemma 3.7) we have $\tau(P_1 \otimes P_2) = \tau_1(P_1)\tau_2(P_2) < \infty$. And by the cross-norm property of the norm on the tensor product

$$
\begin{aligned}
\|K_1 \otimes K_2 - R_1 \otimes R_2\| &\leq \|K_1 \otimes K_2 - K_1 \otimes R_2\| + \|K_1 \otimes R_2 - R_1 \otimes R_2\| \\
&\leq \|K_1\| \, \|K_2 - R_2\| + \|K_1 - R_1\| \, \|R_2\| \\
&\leq (\|K_1\| + \|K_2\| + \varepsilon)\varepsilon.
\end{aligned}
$$

⊠

Lemma 3.9. *Suppose $\mathcal{M}$ is a von Neumann algebra acting on $\mathcal{H}_1$. Then $(\mathcal{M} \otimes B(\mathcal{H}_2))' = \mathcal{M}' \otimes \mathbb{C}1$.*

Proof. See [30, Prop. IV.1.6]. ⊠

Theorem 3.10. *Suppose $\mathcal{S}_1 := (\mathcal{A}_1, \mathcal{H}_1, D_1; \mathcal{N}_1, \tau_1, \gamma_1)$ is an even semifinite spectral triple and $\mathcal{S}_2 := (\mathcal{A}_2, \mathcal{H}_2, D_2; \mathcal{N}_2, \tau_2)$ is a semifinite spectral triple. Then*

$$\mathcal{S} := (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2; \mathcal{N}_1 \otimes \mathcal{N}_2, \tau_1 \otimes \tau_2)$$

is a semifinite spectral triple. If in addition also $\mathcal{S}_2$ is even with a grading γ_2, then $\mathcal{S}$ is even with grading $\gamma_1 \otimes \gamma_2$.

Definition 3.11. The triple $\mathcal{S}$ is called the *product* of the triples $\mathcal{S}_1$ and $\mathcal{S}_2$. In accordance with Proposition 2.63 this product triple will be denoted by $\mathcal{S}_1 \times \mathcal{S}_2 := \mathcal{S}$.

Remark 3.12. If we start with two even spectral triples $\mathcal{S}_1$ and $\mathcal{S}_2$, the triples $\mathcal{S}_1 \times \mathcal{S}_2$ and $\mathcal{S}_2 \times \mathcal{S}_1$ are related in the following way. The algebras, Hilbert spaces and von Neumann algebras of these two triples are isomorphic and the Dirac operators $D_1 \otimes 1 + \gamma_1 \otimes D_2$ and $D_1 \otimes \gamma_2 + 1 \otimes D_2$ are unitarily equivalent. Namely for

$$U := \frac{1}{4}(1 \otimes 1 + \gamma_1 \otimes 1 + 1 \otimes \gamma_2 - \gamma_1 \otimes \gamma_2)$$

it holds that [33]

$$U(D_1 \otimes 1 + \gamma_1 \otimes D_2)U^* = D_1 \otimes \gamma_2 + 1 \otimes D_2.$$

We will now prove Theorem 3.10.

Proof. The proof of this theorem is quite lengthy since we have to check several things. It is similar to the proof of Proposition 2.63 but it involves at lot more technical difficulties in particular the compactness of the resolvent is difficult since we no longer have a basis of the Hilbert space consisting of eigenvectors of the Dirac operator.

For the ease of notion we introduce the following objects

$$\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2, \qquad \mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2, \qquad \mathcal{N} := \mathcal{N}_1 \otimes \mathcal{N}_2, \qquad \tau := \tau_1 \otimes \tau_2, \qquad D := D_1 \otimes 1 + \gamma_1 \otimes D_2.$$

Where $\mathrm{Dom}(D) := \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2)$. If we have a second grading γ_2 on $\mathcal{S}_2$ we denote $\gamma := \gamma_1 \otimes \gamma_2$. Note that we are in the special situation that $D_1 \otimes 1$ and $\gamma_1 \otimes D_2$ anti-commute and therefore that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$. Now we will start the actual proof.

It is clear that $\mathcal{N}$ is a von Neumann algebra acting on $\mathcal{H}$. By Lemma 3.6 $\mathcal{N}$ is a semifinite von Neumann algebra and by Lemma 3.7 τ is a semifinite faithful normal trace.
Concerning the algebra, the tensor product of two involutive algebras is again an involutive algebra. And since $\mathcal{A}_i \subset \mathcal{N}_i$, the inclusion $\mathcal{A} \subset \mathcal{N}$ is obvious. If we have a grading γ_2, then obviously $\gamma = \gamma_1 \otimes \gamma_2 \in \mathcal{N}_1 \otimes \mathcal{N}_2$.

Before we can prove self-adjointness of the operator D we will prove the next lemma.

Lemma 3.13. *The operators* $D_1 \otimes 1$ *and* $\gamma_1 \otimes D_2$ *with domains respectively* $\mathrm{Dom}(D_1) \otimes \mathcal{H}_2$ *and* $\mathcal{H}_1 \otimes \mathrm{Dom}(D_2)$ *are self-adjoint.*

Note that $\mathrm{Dom}(D_1) \otimes \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathrm{Dom}(D_1)$ are tensor products of vector spaces and not tensor products of Hilbert spaces because $\mathrm{Dom}(D_i)$ are not Hilbert spaces.
Proof. As in Lemma 3.5 let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}_2$ and let $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \bigoplus_{i\in I} \mathcal{K}_i$ be the isometry $\sum_i x_i \otimes e_i \mapsto (x_i)_i$. Then

$$U(D_1 \otimes 1)U^* : \bigoplus_{i\in I} \mathcal{K}_i \to \bigoplus_{i\in I} \mathcal{K}_i, \qquad (x_i)_i \mapsto (D_1 x_i)_i.$$

Observe $U(\mathrm{Dom}(D_1) \otimes \mathcal{H}_2) = \bigoplus_{i\in I} \mathrm{Dom}(D_1)_i$. Hence

$$\begin{aligned}&\mathrm{Dom}((U(D_1 \otimes 1)U^*)^*)\\ &\quad= \Big\{(y_i)_i \in \bigoplus_{i\in I} \mathcal{K}_i \,:\, \bigoplus_{i\in I} \mathrm{Dom}(D_1)_i \to \mathbb{C};\ (x_i)_i \mapsto \sum_{i\in I} \langle D_1 x_i, y_i\rangle \text{ is bounded}\Big\}.\end{aligned}$$

Suppose $(y_i)_i \in \bigoplus_{i\in I} \mathcal{K}_i$ and there exists an $i_0 \in I$ such that $y_{i_0} \notin \mathrm{Dom}(D_1)$. Then by self-adjointness of D_1 the map

$$\mathrm{Dom}(D_1) \to \mathbb{C};\ x \mapsto \langle D_1 x, y_{i_0}\rangle \tag{3.1}$$

is unbounded. For an element $(x_i)_i$ with $x_i = 0$ if $i \neq i_0$, we have $\sum_{i\in I}\langle D_1 x_i, y_i\rangle = \langle x_{i_0}, y_{i_0}\rangle$. Hence (3.1) shows that for $y \notin \bigoplus_{i\in I} \mathrm{Dom}(D_1)_i$ the map

$$\bigoplus_{i\in I} \mathrm{Dom}(D_1)_i \to \mathbb{C};\ (x_i)_i \mapsto \sum_{i\in I} \langle D_1 x_i, y_i\rangle$$

is unbounded. Thus $\mathrm{Dom}((U(D_1\otimes 1)U^*)^*) \subset \bigoplus_{i\in I} \mathrm{Dom}(D_1)_i$. So $\mathrm{Dom}((D_1\otimes 1)^*) \subset \mathrm{Dom}(D_1)\otimes \mathcal{H}_2$. For the converse inclusion let $y \in \mathrm{Dom}(D_1) \otimes \mathcal{H}_2$, say $y = \sum_{n=1}^N y_1^{(n)} \otimes y_2^{(n)}$. Then for $x = \sum_{m=1}^M x_1^{(m)} \otimes x_2^{(m)}$ we have by self-adjointness of D_1

$$\begin{aligned}\langle (D_1 \otimes 1)x, y\rangle &= \Big\langle (D_1 \otimes 1)\Big(\sum_{m=1}^M x_1^{(m)} \otimes x_2^{(m)}\Big), \sum_{n=1}^N y_1^{(n)} \otimes y_2^{(n)}\Big\rangle\\ &= \sum_{n=1}^N \sum_{m=1}^M \langle D_1 x_1^{(m)}, y_1^{(n)}\rangle \langle x_2^{(m)}, y_2^{(n)}\rangle\\ &= \sum_{n=1}^N \sum_{m=1}^M \langle x_1^{(m)}, D_1 y_1^{(n)}\rangle \langle x_2^{(m)}, y_2^{(n)}\rangle\\ &= \Big\langle \sum_{m=1}^M x_1^{(m)} \otimes x_2^{(m)}, (D_1 \otimes 1)\Big(\sum_{n=1}^N y_1^{(n)} \otimes y_2^{(n)}\Big)\Big\rangle\\ &= \langle x, D_1 \otimes 1 y\rangle.\end{aligned} \tag{3.2}$$

So $\mathrm{Dom}(D_1) \otimes \mathcal{H}_2 \to \mathbb{C}$, $x \mapsto \langle x, (D_1 \otimes 1)y\rangle$, is a bounded map (bounded by $\|(D_1 \otimes 1)y\|$). Hence $\mathrm{Dom}((D_1 \otimes 1)^*) = \mathrm{Dom}(D_1) \otimes \mathcal{H}_2$.

Similarly we have $\mathrm{Dom}((1\otimes D_2)^*)=\mathcal{H}_1\otimes\mathrm{Dom}(D_2)$. The map $(\gamma_1\otimes 1)^*=\gamma_1^*\otimes 1=\gamma_1\otimes 1$ is bounded and maps $\mathcal{H}_1\otimes\mathrm{Dom}(D_2)$ on itself, thus

$$\mathrm{Dom}((\gamma_1\otimes D_2)^*)=\mathrm{Dom}((\gamma_1\otimes 1)^*(1\otimes D_2)^*)=\mathcal{H}_1\otimes\mathrm{Dom}(D_2).$$

The computation in (3.2) shows $D_1\otimes 1$ is symmetric, and we have $\mathrm{Dom}(D_1\otimes 1)=\mathrm{Dom}(D_1)\otimes\mathcal{H}_2=\mathrm{Dom}((D_1\otimes 1)^*)$. So $D_1\otimes 1$ is self-adjoint on the domain $\mathrm{Dom}(D_1)\otimes\mathcal{H}_2$. A similar argument applies to $\gamma_1\otimes D_2=(\gamma_1\otimes 1)(1\otimes D_2)$. ⊠

Now we are able to prove that D is self-adjoint on the domain $\mathrm{Dom}(D_1)\otimes\mathrm{Dom}(D_2)$. The idea is to apply Nelson's theorem 1.56 to the operator D, from which we will obtain that D is essentially self-adjoint and then we will show that D is closed.
From Proposition 1.56 and the fact that D_1 and D_2 are self-adjoint we obtain that $\mathrm{Dom}^b(D_i)\subset\mathcal{H}_i$ dense for $i=1,2$. Let $x\in\mathrm{Dom}^b(D_1)$ and $y\in\mathrm{Dom}^b(D_2)$, we will show that $x\otimes y\in\mathrm{Dom}^b(D_1\otimes 1+\gamma_1\otimes D_2)$. Select $C>0$ such that

$$\|D_1^n x\|\leq C^n\|x\|;\qquad \|D_2^n y\|\leq C^n\|y\|.$$

Observe that $(D_1\otimes 1+\gamma_1\otimes D_2)^2=D_1^2\otimes 1+1\otimes D_2^2$ and that the operators $D_1^2\otimes 1$ and $1\otimes D_2^2$ commute. Hence

$$\begin{aligned}\|D^{2n}(x\otimes y)\| &= \Big\|(D_1^2\otimes 1+1\otimes D_2^2)^n(x\otimes y)\Big\|\\ &= \Big\|\sum_{j=0}^{n}\binom{n}{j}D_1^{2j}x\otimes D_2^{2(n-j)}y\Big\|\\ &\leq \sum_{j=0}^{n}\binom{n}{j}\|D_1^{2j}x\|\|D_2^{2(n-j)}y\|\\ &\leq \sum_{j=0}^{n}\binom{n}{j}C^{2j}C^{2(n-j)}\\ &= (2C^2)^n\leq(2C)^{2n};\end{aligned}$$

and also

$$\begin{aligned}\|D^{2n+1}(x\otimes y)\| &= \Big\|(D_1\otimes 1+\gamma_1\otimes D_2)(D_1^2\otimes 1+1\otimes D_2^2)^n(x\otimes y)\Big\|\\ &= \Big\|(D_1\otimes 1+\gamma_1\otimes D_2)\sum_{j=0}^{n}\binom{n}{j}D_1^{2j}x\otimes D_2^{2(n-j)}y\Big\|\\ &= \Big\|\sum_{j=0}^{n}\binom{n}{j}D_1^{2j+1}x\otimes D_2^{2(n-j)}y+\binom{n}{j}\gamma_1 D_1^{2j}x\otimes D_2^{2(n-j)+1}y\Big\|\\ &\leq \sum_{j=0}^{n}\binom{n}{j}\|D_1^{2j+1}x\|\|D_2^{2(n-j)}y\|+\binom{n}{j}\|\gamma_1\|\|D_1^{2j}x\|\|D_2^{2(n-j)+1}y\|\\ &\leq \sum_{j=0}^{n}\binom{n}{j}C^{2j+1}C^{2(n-j)}+\binom{n}{j}C^{2j}C^{2(n-j)+1}\\ &= \Big(\sum_{j=0}^{n}\binom{n}{j}C^{2n}\Big)2C\\ &= (2C^2)^n 2C\leq(2C)^{2n+1}.\end{aligned}$$

Hence $x\otimes y\in\mathrm{Dom}^b(D)$. Since $\mathrm{Dom}^b(D)$ is a linear subspace and $\mathrm{Dom}^b(D_i)\subset\mathcal{H}_i$ dense, we have $\mathrm{Dom}^b(D_1)\otimes\mathrm{Dom}^b(D_2)\subset\mathrm{Dom}^b(D)\subset\mathcal{H}_1\otimes\mathcal{H}_2$ dense. But since $\mathrm{Dom}^b(D)\subset\mathrm{Dom}^a(D)$ then clearly $\mathrm{Dom}^a(D)\subset\mathcal{H}$ dense. It is also clear that D is symmetric, so D is essentially self-adjoint.

It remains to show that D is closed on $\mathrm{Dom}(D_1)\otimes\mathrm{Dom}(D_2)$. To that end, let $(x_n)_n$ be a sequence in $\mathrm{Dom}(D_1)\otimes\mathrm{Dom}(D_2)$ converging to $x\in\mathcal{H}_1\otimes\mathcal{H}_2$ such that $(Dx_n)_n$ is a Cauchy sequence in $\mathcal{H}_1\otimes\mathcal{H}_2$. For the moment fix m,n and write $y=x_n-x_m$. Then $(D_1\otimes 1)y\in\mathcal{H}_1\otimes\mathrm{Dom}(D_2)$ and $(\gamma_1\otimes D_2)y\in\mathrm{Dom}(D_1)\otimes\mathcal{H}_2$. For ease of notation write $S:=\gamma_1\otimes D_2$ and $T:=D_1\otimes 1$. Then $STy=-TSy$ and thus

$$\begin{aligned}\langle Dy,Dy\rangle &= \langle Ty,Ty\rangle+\langle Sy,Sy\rangle+\langle Ty,Sy\rangle+\langle Sy,Ty\rangle\\ &= \langle Ty,Ty\rangle+\langle Sy,Sy\rangle+\langle STy,y\rangle+\langle TSy,y\rangle\\ &= \langle Ty,Ty\rangle+\langle Sy,Sy\rangle+\langle STy-STy,y\rangle\\ &= \langle Ty,Ty\rangle+\langle Sy,Sy\rangle. \end{aligned}\tag{3.3}$$

By Lemma 3.13 $D_1\otimes 1$ and $\gamma_1\otimes D_2$ are self-adjoint and hence closed on the domains $\mathrm{Dom}(D_1)\otimes\mathcal{H}_2$ respectively $\mathcal{H}_1\otimes\mathrm{Dom}(D_2)$. The fact that $(Dx_n)_n$ is Cauchy combined with (3.3) gives that $(Tx_n)_n$ and $(Sx_n)_n$ are Cauchy. By closedness of the operators T and S we have $x\in(\mathrm{Dom}(D_1)\otimes\mathcal{H}_2)\cap(\mathcal{H}_1\otimes\mathrm{Dom}(D_2))=\mathrm{Dom}(D_1)\otimes\mathrm{Dom}(D_2)$. Also the sequences $(Tx_n)_n$ and $(Sx_n)_n$ both have a limit in $\mathcal{H}$, say z_1 respectively z_2 and $Tx=z_1$ and $Sx=z_2$. Thus

$$\lim_{n\to\infty}Dx_n=\lim_{n\to\infty}Tx_n+Sx_n=z_1+z_2=Tx+Sx=Dx$$

and D is closed.

To prove that the operator D is affiliated to the von Neumann algebra $\mathcal{N}$, observe that by self-adjointness of D it is sufficient to show that $\{D\}'\supset\mathcal{N}'$. By assumption $\{D_i\}'\supset\mathcal{N}_i'$. Also $\gamma_1\in\mathcal{N}_1$ hence $\{\gamma_1\}'\supset\mathcal{N}_1'$. Using this inclusion we have for $\sum_n m_k\otimes n_k\in\mathcal{N}_1\otimes\mathcal{N}_2$

$$(\gamma_1\otimes D_2)\Big(\sum_n m_k\otimes n_k\Big)=\sum_n\gamma_1 m_k\otimes D_2 n_k\supset\sum_n m_k\gamma_1\otimes n_k D_2=\Big(\sum_n m_k\otimes n_k\Big)(\gamma_1\otimes D_2).$$

Combination with Lemma 3.9 gives the inclusions

$$\begin{aligned}&\{\gamma_1\otimes D_2\}'\supset(\mathcal{N}_1\otimes\mathcal{N}_2)';\\ &\{D_1\otimes 1\}'\supset\mathcal{N}_1'\otimes B(\mathcal{H}_2)=(\mathcal{N}_1\otimes 1)'\supset(\mathcal{N}_1\otimes\mathcal{N}_2)'.\end{aligned}$$

Thus

$$\{D\}'=\{D_1\otimes 1+\gamma_1\otimes D_2\}'\supset\{D_1\otimes 1\}'\cap\{\gamma_1\otimes D_2\}'\supset(\mathcal{N}_1\otimes\mathcal{N}_2)'=\mathcal{N}'$$

and hence D is affiliated to $\mathcal{N}$.

Suppose $a\in\mathcal{A}_1\otimes\mathcal{A}_2$. Assume $a=a_1\otimes a_2$. Then

$$\begin{aligned}[D,a_1\otimes a_2]&=(D_1\otimes 1-\gamma_1\otimes D_2)(a_1\otimes a_2)-(a_1\otimes a_2)(D_1\otimes 1-\gamma_1\otimes D_2)\\ &=D_1a_1\otimes a_2-\gamma_1a_1\otimes D_2a_2-a_1D_1\otimes a_2+a_1\gamma_1\otimes D_2a_2\\ &=(D_1a_1-a_1D_1)\otimes a_2-a_1\gamma_1\otimes(D_2a_2-a_2D_2)\\ &=[D_1,a_1]\otimes a_2-a_1\gamma_1\otimes[D_2,a_2].\end{aligned}$$

Because $[D_1,a_1]$ and $[D_2,a_2]$ are densely defined and bounded and the operators a_2 and $a_1\gamma_1$ are bounded, the commutator $[D,a_1\otimes a_2]$ is densely defined and extends to a bounded operator. Now for an arbitrary element $a=\sum_{i=1}^n a_i\otimes b_1$ we can show boundedness of the commutant $[D,a]$ using linearity of D_i and bilinearity of $[\cdot,\cdot]$. Hence for all $a\in\mathcal{A}$ the operator $[D,a]$ is densely defined and extends to a bounded operator on $\mathcal{H}$.

Before we will prove τ-discreteness of D we will need the two results proven in the following lemmas 3.14 and 3.15.

Lemma 3.14. *The identity*

$$(D+i)^{-1} = i(\gamma_1\otimes D_2+i)^{-1}(D_1\otimes 1+i)^{-1} - (\gamma_1\otimes D_2+i)^{-1}(D_1\otimes 1+i)^{-1}(\gamma_1\otimes D_2)(D_1\otimes 1)(D+i)^{-1} \tag{3.4}$$

holds as operators on $\mathcal{H}$. Moreover $(\gamma_1 \otimes D_2)(D_1 \otimes 1)(D+i)^{-1}$ is bounded.

Proof. First we make a formal manipulation of the symbols. Then we have to check that the operators involved indeed extend to the whole space $\mathcal{H}$.
Again write $S := \gamma_1 \otimes D_2$ and $T := D_1 \otimes 1$. We have $ST = -TS$ and hence

$$\begin{aligned}
(3.4) &\Leftrightarrow (S+T+i)^{-1} = i(S+i)^{-1}(T+i)^{-1} - (S+i)^{-1}(T+i)^{-1}ST(S+T+i)^{-1}\\
&\Leftrightarrow 1 = i(S+i)^{-1}(T+i)^{-1}(S+T+i) - (S+i)^{-1}(T+i)^{-1}ST\\
&\Leftrightarrow (T+i)(S+i) = i(S+T+i) - ST\\
&\Leftrightarrow TS + iS + iT - 1 = iS + iT - 1 - ST.
\end{aligned}$$

The operators $(D, \mathrm{Dom}(D_1)\otimes\mathrm{Dom}(D_2))$, $(D_1\otimes 1, \mathrm{Dom}(D_1)\otimes\mathcal{H}_2)$ and $(\gamma_1\otimes D_2, \mathcal{H}_2\otimes\mathrm{Dom}(D_2))$ are self-adjoint, thus $-i \notin \sigma(D), \sigma(D_1 \otimes 1), \sigma(\gamma_1 \otimes D_2)$. And hence $(D+i)^{-1}$, $(D_1 \otimes 1 + i)^{-1}$ and $(\gamma \otimes D_2 + i)^{-1}$ are bounded operators on $\mathcal{H}$. Furthermore

$$\mathrm{ran}((D+i)^{-1}) = \mathrm{Dom}(D+i) = \mathrm{Dom}(D) = \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2).$$

Also since γ_1 anticommutes with D_1

$$\begin{aligned}
\mathrm{Dom}((D_1 \otimes 1)(\gamma_1 \otimes D_2)) &= \mathrm{Dom}(\gamma_1 \otimes D_2) \cap (\gamma_1 \otimes D_2)^{-1}(\mathrm{Dom}(D_1 \otimes 1))\\
&= \mathcal{H}_1 \otimes \mathrm{Dom}(D_2) \cap \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2)\\
&= \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2).
\end{aligned}$$

Observe that $\mathrm{Dom}\left(((\gamma_1 \otimes D_2)(D_1 \otimes 1))^*\right) \supset \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2)$. Hence $((\gamma_1 \otimes D_2)(D_1 \otimes 1))^*$ is densely defined and thus $((\gamma_1 \otimes D_2)(D_1 \otimes 1), \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2))$ is closable. We have previously shown that $\mathrm{Dom}(D+i) = \mathrm{Dom}(D) = \mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2)$ and D is self-adjoint on this domain. Hence the operator $(D+i)^{-1}$ maps $\mathcal{H}$ into $\mathrm{Dom}(D_1) \otimes \mathrm{Dom}(D_2)$ and is bounded. Thus $(\gamma_1 \otimes D_2)(D_1 \otimes 1)(D+i)^{-1}$ is a closed operator defined on $\mathcal{H}$ and hence by the closed graph theorem it is a bounded operator. And equality (3.4) holds on $\mathcal{H}$. ⊠

Lemma 3.15. *The operator $(\gamma_1 \otimes D_2 + i)^{-1}(D_1 \otimes 1 + i)^{-1} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ is τ-compact.*

Proof. The grading γ_1 induces a direct sum decomposition of $\mathcal{H}_1$. Write $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$, where $\mathcal{H}_1^\pm$ is the eigenspace of the eigenvalue ± 1 of γ_1. Then also

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cong (\mathcal{H}_1^+ \otimes \mathcal{H}_2) \oplus (\mathcal{H}_1^- \otimes \mathcal{H}_2).$$

Now

$$\begin{aligned}
&\gamma_2 \otimes D_2 + i : \mathcal{H}_1^+ \otimes \mathcal{H}_2 \to \mathcal{H}_1^+ \otimes \mathcal{H}_2, && x \otimes y \mapsto x \otimes (D_2 y + iy)\\
&\gamma_2 \otimes D_2 + i : \mathcal{H}_1^- \otimes \mathcal{H}_2 \to \mathcal{H}_1^- \otimes \mathcal{H}_2, && x \otimes y \mapsto -x \otimes D_2 y + x \otimes iy = -x \otimes (Dy - iy).
\end{aligned}$$

Then the inverse of $\gamma_1 \otimes D_2 + i$ is given in the matrix representation w.r.t. $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$ as

$$(\gamma_1 \otimes D_2 + i)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (D_2 + i)^{-1} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \otimes (D_2 - i)^{-1}. \tag{3.5}$$

Since D_1 anti-commutes with γ_1 and D_1 is self-adjoint, the operator D_1 can be written with respect to the decomposition $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$ as

$$D_1 = \begin{pmatrix} 0 & D_1^+ \\ D_1^- & 0 \end{pmatrix},$$

where $D_1^{+*} = D_1^-$. Since $(D_1 \otimes I + i)^{-1} = ((D_1 + i) \otimes I)^{-1} = (D_1 + i)^{-1} \otimes I$, we have to compute the inverse of

$$D_1 + i = \begin{pmatrix} i & D_1^+ \\ D_1^- & i \end{pmatrix}.$$

Observe that

$$\begin{aligned}
\begin{pmatrix} i & D_1^+ \\ D_1^- & i \end{pmatrix} \begin{pmatrix} (D_1^-)^{-1} & i \\ i & (D_1^+)^{-1} \end{pmatrix} &= \begin{pmatrix} i((D_1^-)^{-1} + D_1^+) & -1 + D_1^+(D_1^+)^{-1} \\ D_1^-(D_1^-)^{-1} - 1 & i(D_1^- + (D_1^+)^{-1}) \end{pmatrix} \\
&= \begin{pmatrix} i((D_1^-)^{-1} + D_1^+) & 0 \\ 0 & i(D_1^- + (D_1^+)^{-1}) \end{pmatrix}.
\end{aligned}$$

Therefore $(D_1 + i)^{-1}$ is given by the matrix

$$\begin{aligned}
&\begin{pmatrix} (D_1^-)^{-1} & i \\ i & (D_1^+)^{-1} \end{pmatrix} \begin{pmatrix} -i((D_1^-)^{-1} + D_1^+)^{-1} & 0 \\ 0 & -i(D_1^- + (D_1^+)^{-1})^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -i(D_1^-)^{-1}((D_1^-)^{-1} + D_1^+)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ ((D_1^-)^{-1} + D_1^+)^{-1} & -i(D_1^+)^{-1}(D_1^- + (D_1^+)^{-1})^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -i(1 + D_1^+D_1^-)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ ((D_1^-)^{-1} + D_1^+)^{-1} & -i(D_1^-D_1^+ + 1)^{-1} \end{pmatrix}. \qquad (3.6)
\end{aligned}$$

Hence $(D_1 \otimes I + i)^{-1} = (D_1 + i)^{-1} \otimes I$ is given by

$$\begin{pmatrix} -i(1 + D_1^+D_1^-)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ ((D_1^-)^{-1} + D_1^+)^{-1} & -i(D_1^-D_1^+ + 1)^{-1} \end{pmatrix} \otimes I. \qquad (3.7)$$

Multiplication of (3.5) and (3.7) gives that $(\gamma_1 \otimes D_2 + i)^{-1}(D_1 \otimes 1 + i)^{-1}$ is represented as

$$\begin{aligned}
&\begin{pmatrix} -i(1 + D_1^+D_1^-)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ 0 & 0 \end{pmatrix} \otimes (D_2 + i)^{-1} \\
&\quad + \begin{pmatrix} 0 & 0 \\ -((D_1^-)^{-1} + D_1^+)^{-1} & i(D_1^-D_1^+ + 1)^{-1} \end{pmatrix} \otimes (D_2 - i)^{-1}. \qquad (3.8)
\end{aligned}$$

Because $(D_1 + i)^{-1}$ is τ_1-compact, there exists a sequence $(F_n)_n$ of τ_1-finite rank operators with $\lim_n \|(D + i)^{-1} - F_n\| = 0$. Decompose F_n as a matrix with respect to $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$ by

$$F_n = \begin{pmatrix} F_n^{11} & F_n^{12} \\ F_n^{21} & F_n^{22} \end{pmatrix}.$$

Thus from the estimate

$$\begin{aligned}
&\left\| \left(\begin{pmatrix} -i(1 + D_1^+D_1^-)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} F_n^{11} & F_n^{12} \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (-i(1 + D_1^+D_1^-)^{-1} - F_n^{11})x + ((D_1^- + (D_1^+)^{-1})^{-1} - F_n^{21})y \\ 0 \end{pmatrix} \right\| \\
&\leq \left\| (F_n - (D_1 - i)^{-1}) \begin{pmatrix} x \\ y \end{pmatrix} \right\|
\end{aligned}$$

and a similar one for the lower entries of the matrices we obtain

$$\lim_{n\to\infty} \left\| \begin{pmatrix} -i(1 + D_1^+D_1^-)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} F_n^{11} & F_n^{12} \\ 0 & 0 \end{pmatrix} \right\| = 0;$$

$$\lim_{n\to\infty} \left\| \begin{pmatrix} 0 & 0 \\ -((D_1^-)^{-1} + D_1^+)^{-1} & i(D_1^-D_1^+ + 1)^{-1} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -F_n^{21} & -F_n^{22} \end{pmatrix} \right\| = 0.$$

Since the range of each of the operators F_n^{ij} is contained in the range of F_n $(i,j=1,2)$ the operators $F_n^{i,j}$ are τ_1 compact. Therefore the operators represented by

$$\begin{pmatrix} -i(1+D_1^+D_1^-)^{-1} & (D_1^- + (D_1^+)^{-1})^{-1} \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ -((D_1^-)^{-1}+D_1^+)^{-1} & i(D_1^-D_1^+ + 1)^{-1} \end{pmatrix}$$

are τ_1 compact. By assumption D_2 is τ_2-discrete, thus $(D_2+i)^{-1}$ and $(D_2-i)^{-1}$ are τ_2-compact. Using Lemma 3.8 the operator given by (3.8) is τ-compact, thus $(D_1\otimes I+i)^{-1}=(D_1+i)^{-1}\otimes I$ is τ-compact. ⊠

Now it has become easy to prove compactness of the resolvent. Since D is self-adjoint, $\sigma(D)\subset\mathbb{R}$. According to Theorem 1.90 it is therefore sufficient to show that $(D+i)^{-1}$ is τ-compact. By Corollary 1.89 the τ-compact operators are an ideal in $B(\mathcal{H}_1\otimes\mathcal{H}_2)$. Combination of this corollary with Lemmas 3.14 and 3.15 imply that $(D+i)^{-1}$ is τ-compact.

This concludes the proof that $\mathcal{S}_1\otimes\mathcal{S}_2$ is a semifinite spectral triple.

In the case that we have a grading γ_2 on the second triple, then for $\sum_k n_1^{(k)}\otimes n_2^{(k)}\in\mathcal{N}$ we have

$$(\gamma_1\otimes\gamma_2)\Big(\sum_k n_1^{(k)}\otimes n_2^{(k)}\Big)=\sum_k \gamma_1 n_1^{(k)}\otimes\gamma_2 n_2^{(k)}=\sum_k n_1^{(k)}\gamma_1\otimes n_2^{(k)}\gamma_2=\Big(\sum_k n_1^{(k)}\otimes n_2^{(k)}\Big)(\gamma_1\otimes\gamma_2),$$

and

$$(\gamma_1\otimes\gamma_2)D=\gamma_1D_1\otimes\gamma_2+\gamma_1^2\otimes\gamma_2D_2=-D_1\gamma_1\otimes\gamma_2-\gamma_1^2\otimes D_2\gamma_2=D(\gamma_1\otimes\gamma_2).$$

Hence γ is a grading on $\mathcal{S}$, thus the product of two even triples is again an even semifinite spectral triple. ⊠

If you take the product of two manifolds say of dimension m and n, the product is of dimension $m+n$. Therefore we might expect that the product of two finitely summable semifinite spectral triples is again finitely summable. This is indeed true.

Lemma 3.16. *Suppose for $i=1,2$ the tuples $(\mathcal{A}_1,\mathcal{H}_i,D_i;\mathcal{N}_i,\tau_i)$ are semifinite spectral triples and the first triple is even with grading γ_1. If the triples are p_i-τ_i-summable, then the semifinite product spectral triple is (p_1+p_2)-τ-summable. If both spectral triples are θ-τ_i-summable, then the product spectral triple is θ-τ-summable.*

Proof. We start with the easiest one, the θ-τ-summability. If both triples are θ-τ_i-summable, then

$$\tau\big(e^{-tD^2}\big)=\tau\big(e^{-t(D_1^2\otimes 1+1\otimes D_2^2)}\big)=\tau\big(e^{-tD_1^2}\otimes e^{-tD_2^2}\big)=\tau_1\big(e^{-tD_1^2}\big)\,\tau_2\big(e^{-tD_2^2}\big)<\infty.$$

Thus the triple is θ-τ-summable. To prove the first statement, suppose that both triples are p_i-τ_i-summable. Note that for $i=1,2$ it holds that $1+D_i^2\le 1+D_1^2+D_2^2$, thus

$$(1+D_1^2+D_2^2)^{-(p_1+p_2)/2}\le(1+D_1^2)^{-p_1/2}(1+D_2^2)^{-p_2/2}.$$

The factorisation of the trace now gives

$$\tau(1+D_1^2+D_2^2)^{-(p_1+p_2)/2}\le\tau_1(1+D_1^2)^{-p_1/2}\tau_2(1+D_2^2)^{-p_2/2}<\infty.$$

⊠

Since the definition of a smooth element $a\in\mathcal{A}$ does not involve the trace Tr, this definition copies from the classical case. So we obtain the following definition.

Definition 3.17. For a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D; \mathcal{N}, \tau)$ we define for $a \in \mathcal{A}$ the operator $\delta(a) := [|D|, a]$, it is the unbounded derivation of a. We denote

$$\mathrm{Dom}(\delta) := \{a \in B(\mathcal{H}) : \delta(a) \text{ is bounded on } \mathcal{H} \text{ and } a\,\mathrm{Dom}(|D|) \subset \mathrm{Dom}(|D|)\}.$$

If for all $a \in \mathcal{A}$ the operators $a, [D, a] \in \mathrm{Dom}(\delta^k)$, we call the triple a *$QC^k$-triple*, or QC^k for short. If the triple is a QC^k-triple for all $k \geq 1$, we call it QC^∞ or *regular*. An operator $a \in B(\mathcal{H})$ with $a \in \mathrm{Dom}(\delta^k)$ for all $k \in \mathbb{N}$ is called *smooth*.

The above definition was just an example of how some results immediately apply to semifinite triples. Other previously introduced terminology or previously proved result of spectral triples, which without any changes apply to the semifinite spectral triples, will not be reformulated.

In the case of spectral triples it has been proved [25] that the product of two regular spectral triples is again regular. We will now prove it in a different way (not involving pseudo-differential operators) for semifinite spectral triples. We will use some previous results: Lemma 2.43 and Lemma 2.44.

Theorem 3.18. *Suppose that for $i = 1, 2$ the tupels $\mathcal{S}_i := (\mathcal{A}_i, \mathcal{H}_i, \mathcal{N}_i, D_i, \tau_i)$ are regular semifinite spectral triples and $\mathcal{S}_1$ is even. Then the product $\mathcal{S}$ of these spectral triples is again regular.*

Proof. We start by showing that $\mathcal{H}_\infty = \mathcal{H}_{1\infty} \otimes \mathcal{H}_{2\infty}$. Recall

$$D^{2m} = (D_1^2 \otimes 1 + 1 \otimes D_2^2)^m = \sum_{k=0}^{m} \binom{m}{k} D_1^{2k} \otimes D_2^{2(m-k)}.$$

Furthermore using Lemma 3.13 one can easily show that $\mathrm{Dom}(D_1^n \otimes 1) = \mathrm{Dom}(D_1^n) \otimes \mathcal{H}_2$, a similar statement holds for D_2. Thus by commutativity of $D_1^k \otimes 1$ and $1 \otimes D_2^j$ we have

$$\begin{aligned}
\mathrm{Dom}\left(D^{2m}\right) &= \bigcap_{k=0}^{m} \mathrm{Dom}\left((D_1^{2k} \otimes 1)(1 \otimes D_2^{2(m-k)})\right) \\
&= \bigcap_{k=0}^{m} \mathrm{Dom}\left(D_1^{2k}\right) \otimes \mathcal{H}_2 \cap \left(D_1^{2k} \otimes 1\right)^{-1} (\mathcal{H}_1 \otimes \mathrm{Dom}\left(D_2^{2(m-k)}\right) \\
&= \bigcap_{k=0}^{m} \mathrm{Dom}\left(D_1^{2k}\right) \otimes \mathcal{H}_2 \cap \mathrm{Dom}\left(D_1^{2k}\right) \otimes \mathrm{Dom}\left(D_2^{2(m-k)}\right) \\
&= \mathrm{Dom}\left(D_1^{2m}\right) \otimes \mathrm{Dom}\left(D_2^{2m}\right).
\end{aligned}$$

If we now use the even case, we obtain the following result for odd powers

$$\begin{aligned}
\mathrm{Dom}\big(D^{2m+1}\big) &= \mathrm{Dom}\left((D_1^2 \otimes 1 + 1 \otimes D_2^2)^m (D_1 \otimes 1 + \gamma_1 \otimes D_2)\right) \\
&= \mathrm{Dom}\left((D_1^2 \otimes 1 + 1 \otimes D_2^2)^m (D_1 \otimes 1)\right) \cap \mathrm{Dom}\left((D_1^2 \otimes 1 + 1 \otimes D_2^2)^m (\gamma_1 \otimes D_2)\right) \\
&= \mathrm{Dom}\left(D_1\right) \otimes \mathcal{H}_2 \cap (D_1 \otimes 1)^{-1} \big(\mathrm{Dom}\left((D_1^2 \otimes 1 + 1 \otimes D_2^2)^m\right)\big) \\
&\quad \cap \mathcal{H}_1 \otimes \mathrm{Dom}\left(D_2\right) \cap (\gamma_1 \otimes D_2)^{-1} \big(\mathrm{Dom}\left((D_1^2 \otimes 1 + 1 \otimes D_2^2)^m\right)\big) \\
&= \mathrm{Dom}\left(D_1\right) \otimes \mathrm{Dom}\left(D_2\right) \cap (D_1^{-1} \otimes 1)\big(\mathrm{Dom}\left(D_1^{2m}\right) \otimes \mathrm{Dom}\left(D_2^{2m}\right)\big) \\
&\quad \cap (\gamma_1 \otimes D_2^{-1})\big(\mathrm{Dom}\left(D_1^{2m}\right) \otimes \mathrm{Dom}\left(D_2^{2m}\right)\big) \\
&= \mathrm{Dom}\left(D_1\right) \otimes \mathrm{Dom}\left(D_2\right) \cap \mathrm{Dom}\left(D_1^{2m+1}\right) \otimes \mathrm{Dom}\left(D_2^{2m}\right) \cap \mathrm{Dom}\left(D_1^{2m}\right) \otimes \mathrm{Dom}\left(D_2^{2m+1}\right) \\
&= \mathrm{Dom}\left(D_1^{2m+1}\right) \otimes \mathrm{Dom}\left(D_2^{2m+1}\right).
\end{aligned}$$

As a result we obtain $\mathcal{H}_\infty = \mathcal{H}_{1\infty} \otimes \mathcal{H}_{2\infty}$.
Recall $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Since $[a_1 \otimes a_2, D] = [a_1, D_1] \otimes a_2 + \gamma_1 a_1 \otimes [a_2, D_2]$ it is sufficient to show that $a_1 \otimes a_2$ is smooth if both a_1 and a_2 are smooth (thus a_i are not necessarily from the algebra $\mathcal{A}_i$). So suppose that a_1 and a_2 are smooth. We have to prove that for all $n \geq 1$ it holds that

$\delta^n(a_1 \otimes a_2)(\mathrm{Dom}(D)) \subset \mathrm{Dom}(D)$ and $\delta^n(a_1 \otimes a_2)$ is bounded.
By regularity of the triple and Lemma 2.40, $a_i(\mathcal{H}_{i\infty}) \subset \mathcal{H}_{i\infty}$. Thus

$$a_1 \otimes a_2(\mathcal{H}_\infty) = a_1 \otimes a_2(\mathcal{H}_{1\infty} \otimes \mathcal{H}_{2\infty}) \subset \mathcal{H}_{1\infty} \otimes \mathcal{H}_{2\infty} = \mathcal{H}_\infty.$$

We will invoke Lemma 2.43 to show that $\delta^n(a_1 \otimes a_2)$ is bounded for all n.

$$\begin{aligned}
\delta_1(a_1 \otimes a_2) &= [D^2, a_1 \otimes a_2](1 + D^2)^{-1/2} \\
&= \big([D_1^2, a_1] \otimes a_2 + a_1 \otimes [D_2^2, a_2]\big)(1 + D^2)^{-1/2} \\
&= \big([D_1, a_1](D_1^2 + 1)^{-1/2} \otimes a_2\big) \left((D_1^2 + 1)^{1/2} \otimes 1\right)(1 + D^2)^{-1/2} \\
&\quad + \left(a_1 \otimes [D_2, a_2](D_2^2 + 1)^{-1/2}\right) \left(1 \otimes (D_2^2 + 1)^{1/2}\right)(1 + D^2)^{-1/2} \\
&= \delta_1(a_1) \otimes a_2\big((D_1^2 + 1)^{1/2} \otimes 1\big)(1 + D^2)^{-1/2} + a_1 \otimes \delta_1(a_2)\big(1 \otimes (D_2^2 + 1)^{1/2}\big)(1 + D^2)^{-1/2}.
\end{aligned} \tag{3.9}$$

We will show that (3.9) is bounded, we will only show that the first summand is bounded, the other one is similar. Note

$$D_1^2 \otimes 1 + 1 \leq D_1^2 \otimes 1 + 1 \otimes D_2^2 + 1 = D^2 + 1,$$

so

$$\begin{aligned}
1 &\leq (D_1^2 \otimes 1 + 1)^{-1/2}(D + 1)(D_1^2 \otimes 1 + 1)^{-1/2}; \\
1 &\geq (D_1^2 \otimes 1 + 1)^{1/2}(D + 1)^{-1}(D_1^2 \otimes 1 + 1)^{1/2}; \\
1 &\geq \left\|(D_1^2 \otimes 1 + 1)^{1/2}(D + 1)^{-1/2}\right\|^2.
\end{aligned}$$

To show that higher powers $\delta_1^n(a_1 \otimes a_2)$ are bounded one can do the same as in (3.9). If one expands $\delta_1^n(a_1 \otimes a_2)$, one gets a sum of products of elements of the form

$$\delta_1^k(a_1) \otimes \delta_1^l(a_2), \qquad \big((D_1^2 + 1)^{1/2} \otimes 1\big)(1 + D^2)^{-1/2}, \qquad \big(1 \otimes (D_2^2 + 1)^{1/2}\big)(1 + D^2)^{-1/2}$$

and they are all bounded. Hence (2.43) $\delta_1^n(a_1 \otimes a_2)$ is bounded for all $n \geq 1$. That the operators $\delta^n(a_1 \otimes a_2)$ preserve the domain of D now has become easy. Since $\delta^n(a)$ is bounded for all n, it is a direct consequence of Lemma 2.42. We conclude that $a \in \mathrm{Dom}(\delta^n)$ for all $n \in \mathbb{N}$. So the triple is regular. ⊠

4 Spaces of real dimension

This section is based on the work of Connes and Marcolli in [9, §1.19.2]. We will provide details in their construction of a collection of semifinite spectral triples which can be considered as noncommutative spaces of dimension $z \in (0, \infty)$ and we will give a slightly different construction. In the second subsection we will compute that the dimension spectrum of such triples consists of a single point z. We will start with the definition and some general properties.

4.1 The definition

In this subsection we will follow the lines of [9, §1.19.2]. We want to construct for each $z \in \mathbb{C}$ a spectral triple $(\mathcal{A}, \mathcal{H}, D_z)$ which has the following property

$$\operatorname{Tr}\left(e^{-\lambda D_z^2}\right) = \pi^{z/2}\lambda^{-z/2}, \qquad \text{for all } \lambda > 0. \tag{4.1}$$

This requirement comes from the formula

$$\int_{\mathbb{R}^n} e^{-\lambda p^2}\, d^n p = \left(\frac{\pi}{\lambda}\right)^{n/2}, \text{ for } n \in \{1, 2, 3, \ldots\}.$$

Later (in subsection 5.1) we will use these spectral triples to construct a tool for dimensional regularisation and then this property is essential. For the application to dimensional regularisation we will the need products of spectral triples which we constructed in Section 3.2. The requirement (4.1) may look like an simple one, but it appears to be a great constraint. We will first show that we cannot find such a spectral triple if we demand an ordinary spectral triple. We will need to look in the class of semifinite spectral triples. Later we will show that one can only obey this requirement for $z \in (0, \infty)$.

Following the ideas of Connes and Marcolli, we consider a self-adjoint operator Z affiliated to a type II_∞ factor N with trace Tr_N. The operator Z is given by the spectral measure

$$\tau(1_E(Z)) = \frac{1}{2}\int_E 1\, dx.$$

Existence of such an operator Z is established in [9, Rem. 3.36]. Using this operator they construct the desired operator $\tilde{D}_z := \rho(z)\operatorname{sgn}(Z)|Z|^{1/z}$. Here $\rho(z) := \pi^{-1/2}(\Gamma(z/2+1))^{1/z}$ is a renormalisation constant. Let us denote this triple by

$$(\mathbb{C}, \mathcal{H}, \tilde{D}_z; N, \operatorname{Tr}_N) \tag{4.2}$$

Later in this section we will give an alternative construction of such an operator D_z, without the operator Z. Furthermore we will not need a type II_∞ factor, but a type I von Neumann algebra. Note that Z has spectrum $\sigma(Z) = \mathbb{R}$. Indeed, suppose $x \in \mathbb{R}\setminus\sigma(Z)$. Then, since the spectrum is closed, there must exist an $\varepsilon > 0$ such that $A := (x-\varepsilon, x+\varepsilon) \cap \sigma(Z) = \emptyset$. But then $1_A(Z) = 0$ hence $\tau(1_A(Z)) = 0$. However $\frac{1}{2}\int_A 1\, dy = \varepsilon$ which is a contradiction.

We will give a motivation why Connes and Marcolli consider a semifinite spectral triple

$$(\mathcal{A}, \mathcal{H}, D; \mathcal{M}, \tau),$$

where $\mathcal{M}$ is a type II_∞ factor. We will show why type I or type II_1 factors are not sufficient for existence of such an operator Z.

Lemma 4.1. *If τ is a trace on a type I factor $\mathcal{M}$ (with minimal projection p), there does not exist a Z which satisfies $\tau(1_A(Z)) = \frac{1}{2}\int_A 1\, dy$ for all $A \in \mathcal{B}(\mathbb{R})$.*

Proof. Indeed, suppose such an operator Z exist. Then by Lemma 1.44 there exists a minimal projection p. Therefore for all $A \in \mathcal{B}(\sigma(Z)) = \mathcal{B}(\mathbb{R})$

$$\frac{1}{2}\mu(A) = \frac{1}{2}\int_A 1\, dx = \tau(1_A(Z)) = \tau(E(A)) \in \{n\tau(p) : n \in \mathbb{N}\} \cup \{\infty\},$$

which is a contradiction. ⊠

This lemma implies that an ordinary spectral triple is not sufficient. Because if we have a spectral triple, the corresponding von Neumann algebra is $B(\mathcal{H})$ which is a type I factor.

Remark 4.2. It also does not work for a finite factor. Since if $\mathcal{M}$ is a finite factor, by lemma 1.45 any semifinite normal trace τ is finite. This would imply that $\tau(1) < \infty$ and hence

$$\infty = \frac{1}{2}\int_{\mathbb{R}} 1\,dx = \tau(1_{\mathbb{R}}(Z)) = \tau(1_{\sigma(Z)}(Z)) = \tau(1) < \infty,$$

which is a contradiction.

So if we restrict ourselves to factors we at least need type II_∞. However we considered factors and not von Neumann algebras. So the requirement might be satisfied if we look to von Neumann algebras instead of factors. This is indeed the case.

Notation 4.3. For $z > 0$ consider the following tuple

$$\mathcal{T}_z := (\mathcal{A}_z, \mathcal{H}_z, \not{D}_z, \mathcal{N}_z, \tau_z) := \Big(\mathbb{C}, L^2(\mathbb{R}), \not{D}_z; L^\infty(\mathbb{R}), \frac{1}{2}\int_{\mathbb{R}} \cdot\, dx\Big). \tag{4.3}$$

Here $\lambda \in \mathbb{C}$ is identified with the map $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $h \mapsto \lambda h$. Denote

$$f_z : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \rho(z)\operatorname{sgn}(x)|x|^{1/z},$$

where $\rho(z) := \pi^{-1/2}(\Gamma(z/2+1))^{1/z}$ is a normalisation constant. Now $\not{D}_z$ is given by $\not{D}_z h := f_z h$ on the domain

$$\operatorname{Dom}(\not{D}_z) := \Big\{h \in L^2(\mathbb{R}) \,:\, \int_{\mathbb{R}} |f_z h|^2\,dx < \infty\Big\}.$$

We will use $\mathcal{T}_z$, $\not{D}_z$ etcetera in the rest of this thesis to denote this triple and elements thereof.

Remark 4.4. Note that for $\xi \neq 0$ we have

$$\Big|\frac{d}{d\xi} f_z(\xi)\Big| = \Big|\rho(z)|\xi|^{1/z-1}, \frac{1}{z}\Big|.$$

Using induction for $\xi \neq 0$

$$\Big|\frac{d^n}{d\xi^n} f_z(\xi)\Big| = \Big|\rho(z)|\xi|^{1/z-n}\,\frac{1}{z}\,(\frac{1}{z}-1)\cdots(\frac{1}{z}-n+1)\Big|.$$

Thus

$$\Big|\frac{d^n}{d\xi^n} f_z(\xi)\Big| \leq C_n(1+|\xi|)^{1/z-n}, \qquad \text{allmost everywhere.}$$

So f_z is a symbol of a pseudo-differential operator of order $1/z$. Note that $\operatorname{sgn}(\xi)$ is the symbol of the Hilbert transform [13, Ch. 3] and $\xi^{1/z}$ corresponds to the symbol of "$\frac{1}{z}$ times differentiation".

Proposition 4.5. *For $z > 0$ the tuple $\mathcal{T}_z$ is a semifinite spectral triple, $\not{D}_z$ has spectrum $\sigma(\not{D}_z) = \mathbb{R}$ and the triple satisfies*

$$\tau_z\big(e^{-\lambda \not{D}_z^2}\big) = \pi^{z/2}\lambda^{-z/2}, \qquad \textit{for all } \lambda > 0. \tag{4.4}$$

Proof. It is clear that $L^2(\mathbb{R})$ is a Hilbert space. The type I von Neumann algebra $L^\infty(\mathbb{R})$ acts on $L^2(\mathbb{R})$ by left-multiplication, because $\big(\int_{\mathbb{R}} |fh|^2\,dx\big)^{1/2} \leq \|f\|_\infty \big(\int_{\mathbb{R}} |h|^2\,dx\big)^{1/2}$.
The trace τ_z is faithful because if $f \geq 0$, then $\tau_z(f) = \int_{\mathbb{R}} f\,dx$, which is 0 if and only if $f = 0$ almost everywhere. Thus $\tau_z(f) = 0$ if and only if $f = 0$ in $L^\infty(\mathbb{R})$. From the monotone convergence theorem it is immediate that τ_z is normal. τ_z is semifinite because $\tau_z(1) = \infty$ and for every nonzero $f \in L^\infty(\mathbb{R})$, $f \geq 0$ there exists a $g \in L^\infty(\mathbb{R})$ with $0 \leq g \leq f$ and $\int_{\mathbb{R}} g\,dx < \infty$.

Since $\mathcal{A}_z = \mathbb{C}$ it is obvious that $\mathcal{A}_z \subset \mathcal{N}_z$ and $[\not{D}_z, a] = 0$, which thus extends to a bounded operator on $\mathcal{H}_z$.
We will now show that $(\not{D}_z, \mathrm{Dom}(\not{D}_z))$ is a self-adjoint operator. Since f_z is real-valued $\not{D}_z$ is a symmetric operator, thus $\mathrm{Dom}(\not{D}_z) \subset \mathrm{Dom}(\not{D}_z^*)$. It remains to show the converse inclusion. Suppose $g \in L^2(\mathbb{R})$ but $g \notin \mathrm{Dom}(\not{D}_z)$. Then $\int_{\mathbb{R}} |gf_z|^2\, dx = \infty$. Observe that f_z is a continuous function, so f_z is bounded on compact sets. In particular for each $n \in \mathbb{N}$ there exists a constant C_n such that $|f|_{[-n,n]}| \leq C_n$. Put $g_n := gf_z 1_{[-n,n]}$. Then $|g_n(x)| \leq C_n|g(x)|$ and

$$\int_{\mathbb{R}} |f_z g_n|^2\, dx = \int_{\mathbb{R}} |f_z g f_z 1_{[-n,n]}|^2\, dx \leq C_n^4 \int_{\mathbb{R}} |g|^2\, dx < \infty.$$

So $g_n \in \mathrm{Dom}(\not{D}_z)$. But

$$\langle D_z g_n, g\rangle = \int_{\mathbb{R}} f_z g f_z 1_{[-n,n]} \bar{g}\, dx = \int_{-n}^{n} |f_z g|^2\, dx \to \infty \qquad \text{as } n \to \infty.$$

Thus $g \notin \mathrm{Dom}(\not{D}_z^*)$ and $\not{D}_z$ is self-adjoint.
Since $\not{D}_z$ is self-adjoint, $\sigma(\not{D}_z) \subset \mathbb{R}$. We will show that the converse inclusion also holds. Observe that for $z > 0$ the function $f_z : \mathbb{R} \to \mathbb{R}$ is continuous, bijective and strictly increasing. Since $(\not{D}_z - \lambda)h = (f_z - \lambda)h$, the only possible candidate for $(\not{D}_z - \lambda)^{-1}$ is given by $h \mapsto (f_z - \lambda)^{-1}h$. But if $\lambda \in \mathbb{R}$ the function $f_z - \lambda$ has a zero. By continuity of f_z the function $(f_z - \lambda)^{-1}$ is therefore not essentially bounded. Hence $h \mapsto (f_z - \lambda)^{-1}h$ is an unbounded map and thus $\not{D}_z - \lambda$ is not invertible, which implies that $\sigma(\not{D}_z) = \mathbb{R}$.
Since $\not{D}_z$ is self-adjoint we can give a spectral decomposition of $\not{D}_z$. From this decomposition we can easily show that $\not{D}_z$ is affiliated to $\mathcal{N}$ and that the operator is τ-discrete (i.e. the resolvent is compact relative to $\mathcal{N}$). Observe that the inverse of f_z is given by

$$f_z^{-1}(x) = \mathrm{sgn}(x)\Big(\frac{|x|}{\rho(z)}\Big)^z.$$

Define

$$E : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R})), \qquad E(A)h := 1_{\{f_z^{-1}(A)\}}h.$$

It is clear that E is a spectral measure. For an interval $I = [f_z(a), f_z(b)]$ we have

$$\int_{f_z(a)}^{f_z(b)} f_z(a)\, dE = f_z(a)E([f_z(a), f_z(b)]) = f_z(a)1_{[a,b]}.$$

If we approximate the identity map $id : \mathbb{R} \to \mathbb{R}$ and use the above identity, we see that $\not{D}_z = \int x\, dE$, thus E is the spectral measure for $\not{D}_z$. For each $A \in \mathcal{B}(\mathbb{R})$ it holds that $E(A) \in L^\infty(\mathbb{R}) = \mathcal{N}_z$, thus by Lemma 1.75 $\not{D}_z$ is affiliated with $\mathcal{N}_z$. Also

$$\tau(E([-\lambda, \lambda])) = \frac{1}{2}\int_{\mathbb{R}} 1_{f_z^{-1}([-\lambda,\lambda])}\, dx = \frac{1}{2}\int_{f_z^{-1}(-\lambda)}^{f_z^{-1}(\lambda)} 1\, dx < \infty,$$

hence by Theorem 1.90, $\not{D}_z$ is compact relative to $\mathcal{N}_z$. Thus $\mathcal{S}_z$ is a semifinite spectral triple.
We will now show that $\mathcal{S}_z$ satisfies the property (4.4). For $\lambda > 0$ we have

$$\tau(e^{-\lambda \not{D}_z^2}) = \frac{1}{2}\int_{\mathbb{R}} e^{-\lambda\rho(z)^2|x|^{2/z}}\, dx = \int_0^\infty e^{-\lambda\rho(z)^2 x^{2/z}}\, dx$$

Use the substitution $u = \lambda\rho(z)^2|x|^{2/z}$. Then

$$x = \rho(z)^{-z}\Big(\frac{u}{\lambda}\Big)^{z/2}, \qquad \frac{dx}{du} = \rho(z)^{-z}\frac{z}{2}\lambda^{-z/2}u^{z/2-1}.$$

So we obtain

$$\begin{aligned}\tau(e^{-\lambda \not{D}_z^2}) &= \rho(z)^{-z}\lambda^{-z/2}\,\frac{z}{2}\int_0^\infty e^{-u}u^{z/2-1}\,du\\ &= \rho(z)^{-z}\lambda^{-z/2}\Gamma(z/2+1)\\ &= \pi^{z/2}\lambda^{-z/2}.\end{aligned}$$

In the last line we inserted the defintion of $\rho(z)$. Hence (4.4) holds. ⊠

Of course the triple constructed by Connes and Marcolli satisfies the requirement (4.1).

Lemma 4.6. *The semifinite spectral triple* (4.2) *fulfills equation* (4.1) *when* Tr *is replaced by* $\mathrm{Tr}_{\mathcal{N}}$ *and* $\not{D}_z$ *by* $\tilde{D}_z$.

Proof. The computation of this is basically the same as in the proof of Proposition 4.5. The only difference is that $\mathrm{Tr}_{\mathcal{N}}$ is not explicitly given as an integral. But Theorem 1.83 resolves this difficulty, because the trace of Z is given as an integral.
Let μ denote the Lebesgue measure on $\mathbb{R}$. The self-adjoint operator Z has a spectral measure E given by $\mu_{\tau,E}(A) = \mathrm{Tr}_{\mathcal{N}}(E(A)) = \frac{1}{2}\mu(A)$, for all $A \in \mathcal{B}(\mathbb{R})$. Now suppose $z \in [0,\infty)$ and $\lambda \in (0,\infty)$. Then the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-\lambda\rho(z)|x|^{2/z}}$ is continuous and bounded by 1. So we can apply Theorem 1.83. Then

$$\mathrm{Tr}_{\mathcal{N}}(e^{-\lambda\tilde{D}_z^2}) = \int_{\mathbb{R}} e^{-\lambda\rho(z)^2|x|^{2/z}}\,d\mu_{\tau,E}(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-\lambda\rho(z)^2|x|^{2/z}}\,d\mu(x).$$

Now the calculation is the same as in Proposition 4.5. ⊠

Remark 4.7. For $z \notin \mathbb{R}$ the operator $\not{D}_z$ is not self-adjoint. This is easy to see because for $z \notin \mathbb{R}$ the Lebesgue measure of $\{x : f_z(x) \notin \mathbb{R}\}$ is strictly positive. Hence it does not hold that $f_z = \overline{f_z}$ almost everywhere. But then $\not{D}_z^* \neq \not{D}_z$. So for $z \notin \mathbb{R}$ the tuple $\mathcal{T}_z$ is not a spectral triple. This is the reason why for example $e^{-\lambda\not{D}_z^2}$ is not in the domain of the trace τ_z, see Lemma 4.9. In fact, as the next result shows, we cannot expect that any self-adjoint operator satisfies (4.1) for $z \in \mathbb{C} \setminus (0,\infty)$.

Proposition 4.8. *Suppose $\mathcal{N} \subset B(\mathcal{H})$ is a semifinite von Neumann algebra, with a faithful semifinite normal trace τ. Suppose $z \notin (0,\infty)$. If D is a self-adjoint (unbounded) operator on $\mathcal{H}$ affiliated with $\mathcal{N}$, then there exists a $\lambda > 0$ such that $\tau\big(e^{-\lambda D^2}\big) \neq \pi^{z/2}\lambda^{-z/2}$.*

Proof. Let $\mathcal{N}$, τ, z and D be as in the assumptions of the proposition. Since D is affiliated with $\mathcal{N}$ it holds that

$$e^{-\lambda D^2} \in \{D\}'' \subset \mathcal{N}.$$

Thus $\tau\big(e^{-\lambda D^2}\big)$ is well-defined. Let $\lambda > 0$ and consider the function

$$f : \mathbb{C} \to \mathbb{C},\ f(w) := e^{-\lambda w^2}.$$

Then $f(\mathbb{R}) \subset \mathbb{R}$, in particular $f|_{\sigma(D)} = \bar{f}|_{\sigma(D)}$. By the functional calculus

$$\big(e^{-\lambda D^2}\big)^* = f(D)^* = \bar{f}(D) = f(D) = e^{-\lambda D^2},$$

thus $e^{-\lambda D^2}$ is self-adjoint. Since $\lambda > 0$ and $f(\mathbb{R}) \subset [0,1]$, the spectral radius $r(e^{-\lambda D^2}) \leq 1$, which by Corollary 1.29 and self-adjointness imply that $e^{-\lambda D^2}$ is a bounded operator. Since f is positive, the operator $e^{-\lambda D^2}$ is positive. Therefore it holds that $\tau\big(e^{-\lambda D^2}\big) \in [0,\infty]$ for all $\lambda > 0$. Now suppose $z \notin \mathbb{R}$, then it is impossible that

$$\pi^{z/2}\lambda^{-z/2} = \left(\frac{\pi}{\lambda}\right)^{z/2} \in [0,\infty] \qquad \text{for all } \lambda > 0.$$

This proves the statement for $z \in \mathbb{C} \setminus \mathbb{R}$. If we have $z \in (-\infty, 0]$, let $t > s > 0$. Then clearly

$$t^{-z/2} \geq s^{-z/2}. \tag{4.5}$$

But also $-tD^2 < -sD^2$ and thus $e^{-tD^2} < e^{-sD^2}$. This gives

$$\pi^{z/2} t^{-z/2} = \tau\big(e^{-tD^2}\big) < \tau\big(e^{-sD^2}\big) = \pi^{z/2} s^{-z/2},$$

which is a contradiction with (4.5). ⊠

In the proof of [9, Prop. 1.240], it is stated (but not proved) that for $z \in \mathbb{C}$, $\mathrm{Im}(z) \neq 0$ the operator $e^{-\lambda \tilde{D}_z^2}$ is not in the domain of the trace. We will prove this statement in the next lemma for the operator $\not{D}_z$.

Lemma 4.9. *Let $z \in \{w \in \mathbb{C} : \mathrm{Im}(w) \neq 0,\ \mathrm{Re}(w) > 0\}$, then the operator $e^{-\lambda \not{D}_z^2}$ is not in the domain of the trace τ.*

Proof. Suppose $z \in \{w \in \mathbb{C} : \mathrm{Im}(w) \neq 0,\ \mathrm{Re}(w) > 0\}$. We will show that the map $g : [0,\infty) \to \mathbb{C}$, $g(s) = e^{-\lambda \rho(z)^2 |s|^{2/z}}$ is not essentially bounded with respect to the Lebesgue measure on $[0,\infty)$. This is sufficient, because in that case Lemma 1.82 implies that $\mathrm{Dom}(e^{-\lambda \not{D}_z^2}) = \mathrm{Dom}(\int g\, dE) \subsetneq \mathcal{H}$. So $e^{-\lambda \not{D}_z^2}$ is unbounded and hence not in the domain of the trace.
Observe $\mathrm{Re}(\frac{2}{z}) = \frac{1}{z} + \frac{1}{\bar{z}} = \frac{z+\bar{z}}{z\bar{z}} = \frac{2\,\mathrm{Re}(z)}{|z|^2}$. Put $w := \frac{2}{z}$. Then the computation shows $\mathrm{Re}(w) > 0$. So $e^{t\,\mathrm{Re}(w)} \uparrow \infty$, as $t \to \infty$. Let $c := -\lambda \rho(z)^2$. Since $\mathrm{Im}(z) \neq 0$ for all $t \in [0,\infty)$, there exists $t' > t$ such that $e^{t' i\,\mathrm{Im}(w)} c = |c|$. Now for each $n \in \mathbb{N}$ select a $t_n \in [0,\infty)$, such that $t_n > t_{n-1}$, $t_n > n$ and $e^{t_n i\,\mathrm{Im}(w)} c = |c| > 0$. Then we have

$$\lim_{n\to\infty} \mathrm{Re}(e^{t_n w} c) = \lim_{n\to\infty} \mathrm{Re}(e^{t_n \mathrm{Re}(w)} e^{t_n i\,\mathrm{Im}(w)} c) = \lim_{n\to\infty} \mathrm{Re}(e^{t_n \mathrm{Re}(w)} |c|) = \infty.$$

So the map $h : [0,\infty) \to \mathbb{R}$, $h(s) = \mathrm{Re}(-\lambda \rho(z)^2 s^{2/z}) = \mathrm{Re}(-\lambda \rho(z)^2 e^{\log(s) 2/z})$ is not bounded. But then also the function g is unbounded.
Clearly g is continuous. So if $\mathrm{Re}(g(s_n)) > n$, then there exists a neighborhood U of s_n such that $\mathrm{Re}(g(x)) > n-1$ for all $x \in U$. But this implies that $Re(g)$ is not essentially bounded and therefore g is not essentially bounded. ⊠

A last observation about this spectral triple.

Remark 4.10. The map $\mathbb{R} \to \mathbb{R}$, $x \mapsto -x$ induces an operator on the triple $\mathcal{T}_z$ by

$$\gamma_z : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \gamma_z(f)(x) := f(-x).$$

Since $\mathcal{A}_z = \mathbb{C}$, the operator γ_z clearly commutes with $\mathcal{A}$. Also it holds that

$$\gamma_z(D_z f)(x) = D_z(-x) f(-x) = \rho(z)\,\mathrm{sgn}(-x)|-x|^{1/z} f(-x) = -D_z(\gamma_z f)(x).$$

Clearly γ_z is bounded and that $f \in \mathrm{Dom}(D_z)$ if and only if $\gamma_z f \in \mathrm{Dom}(D_z)$. But since γ_z is not given by a function, it is no element of $\mathcal{N}_z = L^\infty(\mathbb{R})$. So γ_z is not a grading as in Definition 3.1, but it is very similar.

4.2 Dimension spectrum

In the rest of this thesis we will work with the semifinite spectral triple $\mathcal{T}_z$. All the results we obtain also hold for the spectral triple (4.2), because every time we can use the same reduction of the trace Tr_N to the Lebesgue integral using Theorem 1.83 as we did in the proof of Lemma 4.6.
In this section we will establish some facts about the dimension spectrum of the triple $\mathcal{T}_z$. We will calculate in various ways the dimension spectrum because as we will see some problems occur.

Remark 4.11. Connes and Marcolli use $|D|$ in the definition of the dimension spectrum. Since $\mathcal{A} = \mathbb{C}$ they try to compute $\tau_z\big(|\not{D}_z|^{-s}\big) = \tau_z\big((\not{D}_z^2)^{-s/2}\big)$. Because this yields ∞ for all values of s they make an infrared cutoff. The reason behind this problem is that $f_z(0) = 0$ and f_z is continuous, therefore $0 \in \sigma(\not{D}_z)$ and thus $\not{D}_z$ is not invertible. We will start by computing the trace without a cutoff and later we will show how it works with a cutoff as Connes and Marcolli do. Since

$$((f_z(x))^2)^{-s/2} = (\rho(z)^2)^{-s/2}(\mathrm{sgn}(x)^2)^{-s/2}((|x|^{1/z})^2)^{-s/2} = \rho(z)^{-s}|x|^{-s/z},$$

the operator $(\not{D}_z^2)^{-s/2}$ is given by multiplication with the function $x \mapsto \rho(z)^{-s}|x|^{-s/z}$. So

$$\begin{aligned}
\tau_z((\not{D}_z^2)^{-s/2}) &= \frac{1}{2}\int_{\mathbb{R}} \rho(z)^{-s}|y|^{-s/z}\,dy \\
&= \rho(z)^{-s}\int_0^\infty y^{-s/z}\,dy \\
&= \rho(z)^{-s}\left[\frac{1}{-s/z+1}\,y^{-s/z+1}\right]_{y=0}^{y=\infty} \\
&= \rho(z)^{-s}\Big(\lim_{y\to\infty}\big(y^{(-s+z)/z}\frac{z}{-s+z}\big) - \lim_{y\to 0}\big(y^{(-s+z)/z}\frac{z}{-s+z}\big)\Big).
\end{aligned} \tag{4.6}$$

Then for $\mathrm{Re}(\frac{s}{z}) > 1$, we have $\mathrm{Re}(\frac{-s+z}{z}) < 0$. Therefore

$$\lim_{y\to\infty} y^{(-s+z)/z} = 0 \qquad\qquad \lim_{y\to 0} y^{(-s+z)/z} = \infty$$

and (4.6) does not converge. If $\mathrm{Re}(\frac{s}{z}) < 1$, then $\mathrm{Re}(\frac{-s+z}{z}) > 0$. Thus

$$\lim_{y\to\infty} y^{(-s+z)/z} = \infty \qquad\qquad \lim_{y\to 0} y^{(-s+z)/z} = 0$$

and (4.6) does not converge either. In the last case that $\mathrm{Re}(\frac{s}{z}) = 1$, there exists $c \in \mathbb{R}$ such that $\frac{-s+z}{z} = ci$. If c is non-zero, then $\lim_{y\to\infty} y^{(-s+z)/z} = \lim_{y\to\infty} y^{ci}$ which does not exist, and $\lim_{y\to 0} y^{ci} = 0$, thus (4.6) again does not exist. The only possibility remaining is that $\frac{s}{z} = 1$. But then

$$\tau_z((\not{D}_z^2)^{-s/2}) = \rho(z)^{-s}\int_0^\infty y^{-1}\,dy$$

and this integral also does not converge either.

To fix this Connes and Marcolli impose a infrared cutoff, that is they compute the integral on the subset $(-\infty, 1] \cup [1, \infty)$. Then for $\mathrm{Re}(\frac{s}{z}) > 1$ we have

$$\begin{aligned}
\rho(z)^{-s}\int_1^\infty y^{-s/z}\,dy &= \rho(z)^{-s}\Big(\lim_{y\to\infty}\big(y^{(-s+z)/z}\frac{z}{-s+z}\big) - 1^{(-s+z)/z}\frac{z}{-s+z}\Big) \\
&= 0 - \rho(z)^{-s}\frac{z}{-s+z} \\
&= \rho(z)^{-s}\frac{z}{s-z},
\end{aligned} \tag{4.7}$$

The right hand side indeed only has a pole for $s = z$. Hence the meromorphic continuation of the left hand side has only a simple pole at $s = z$.
The function $h : \mathbb{R} \to [0, 1]$ defined by

$$h(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

is a smooth and strictly increasing function. Therefore the function $g : [0, \infty) \to [0, 1]$ given by

$$g(x) := \begin{cases} 0 & \text{if } x \geq \frac{1}{2} \\ e^{-1/(x-\frac{1}{2})} & \text{if } x < \frac{1}{2} \end{cases},$$

is smooth and strictly decreasing. The interval $[0, \frac{1}{2}]$ is compact, g' is continuous, so g' is bounded on $[0, \frac{1}{2}]$. Let $c > 0$ be such that $cg'(x) \leq -\frac{1}{2}$ for all $x \in [0, \frac{1}{2}]$. Now we define $f : [0, \infty) \to [0, \infty)$, $f(x) := x + cg(x)$. Then

- $f(x) = x$ if $x \geq \frac{1}{2}$
- $f'(x) \geq \frac{1}{2}$ for all $x \in [0, \infty)$, thus f is strictly increasing
- $f(0) = 0 + cg(0) > 0$.

Define an operator E_z by $E_z h(x) := \rho(z) \operatorname{sgn}(x) f(|x|)^{1/z} h(x)$, it is a smooth modification of $\not{D}_z$ near $x = 0$. We have

$$\begin{aligned} \left|\tau\left(e^{-\lambda \not{D}_z^2}\right) - \tau\left(e^{-\lambda E_z^2}\right)\right| &= \frac{1}{2}\left| \int_{\mathbb{R}} e^{-\lambda\rho(z)^2|y|^{2/z}} - e^{-\lambda\rho(z)^2 f(|y|)^{2/z}} \, dy\right| \\ &= \left| \int_0^{\frac{1}{2}} e^{-\lambda\rho(z)^2|y|^{2/z}} - e^{-\lambda\rho(z)^2 f(|y|)^{2/z}} \, dy\right| \\ &\leq \int_0^{\frac{1}{2}} \left|e^{-\lambda\rho(z)^2|y|^{2/z}} - 1\right| + \left|1 - e^{-\lambda\rho(z)^2 f(|y|)^{2/z}}\right| dy. \end{aligned} \tag{4.8}$$

The function $s \mapsto e^s$ is smooth, in particular differentiable. And $e^0 = 1$. Thus $\left|\frac{e^s-1}{s}\right|$ is bounded for bounded $|s|$. This implies that there exists a constant C such that $|e^u - 1| < C \max\{a, b\}$ for all $u \in [a, b]$. Since f is strictly increasing, λ and $\rho(z)$ are some constants, $|-\lambda\rho(z)^2|y|^{2/z}|$ and $|-\lambda\rho(z)^2|f(y)|^{2/z}|$ are bounded for $y \in [0, \frac{1}{2}]$, there exists a constant C such that by (4.8) we have for

$$R(\lambda, z) := \tau(e^{-\lambda \not{D}_z^2}) - \tau(e^{-\lambda E_z^2})$$

that

$$|R(\lambda, z)| \leq C\left| -\lambda\rho(z)^2\left|\frac{1}{2}\right|^{2/z}\right| + C\left| -\lambda\rho(z)^2 f(\left|\frac{1}{2}\right|)^{2/z}\right| \leq 2C|\lambda\rho(z)^2|\left|\frac{1}{2}^{2/z}\right|.$$

Since $z \mapsto e^z$ is holomorphic, $\lambda \mapsto R(\lambda, z)$ is holomorphic on $z \in \{z : \operatorname{Re}(z) > 0, |z| < 1\}$ $|\lambda| < 2^{\operatorname{Re}(z)}$. This implies that $\lambda \mapsto \tau(e^{-\lambda \not{D}_z^2})$ has a pole at λ_0 if and only if $\lambda \mapsto \tau(e^{-\lambda E_z^2})$ has a pole at λ_0. One can check that therefore the substitution $\not{D}_z \leftrightarrow E_z$ does not influence the Poles in proposition 5.11. Also as Proposition 4.12 shows that the dimension spectrum is unaltered by this substitution, so for this use infrared cut-off is allowed. We will not go in further details because we will choose a different approach.

The operator E_z is very closely related to $\not{D}_z$, therefore we expect that the meromorphic continuation of $s \mapsto \tau_z(|E_z|^{-s})$ has the same poles as the meromorphic continuation of (4.7), the cutoff of $\tau_z(|\not{D}_z|^{-s})$.

Proposition 4.12. *The meromorphic continuation of $s \mapsto \tau_z(|E_z|^{-s})$ is holomorphic on $\mathbb{C} \setminus \{z\}$ and has a simple pole at $s = z$ with residue*

$$\operatorname{res}_{s=z} \tau_z(|E_z|^{-s}) = 2\,\frac{\pi^{z/2}}{\Gamma(\frac{z}{2})}.$$

Proof. From the construction in the above remark 4.11, we have that

$$\begin{aligned}
\tau\big(|E_z|^{-s}\big) &= \int_0^\infty \rho(z)^{-s} f(x)^{-s/z}\,dx \\
&= \rho(z)^{-s}\int_0^{\frac{1}{2}} f(x)^{-s/z}\,dx + \rho(z)^{-s}\int_{\frac{1}{2}}^\infty x^{-s/z}\,dx \\
&= \rho(z)^{-s}\int_0^{\frac{1}{2}} f(x)^{-s/z}\,dx + \rho(z)^{-s}(-\frac{s}{z}+1)^{-1}\big[x^{-s/z+1}\big]_{x=\frac{1}{2}}^{x=\infty} \\
&= \rho(z)^{-s}\int_0^{\frac{1}{2}} f(x)^{-s/z}\,dx + \rho(z)^{-s}(-\frac{s}{z}+1)^{-1}\Big(\frac{1}{2}\Big)^{-s/z+1}
\end{aligned}$$

Observe that f is continuous and non-zero, so the integral $\int_0^{\frac{1}{2}} f(x)^{-s/z}\,dx$ exists for all s and therefore does not create any singularities. The second term $\rho(z)^{-s}(-\frac{s}{z}+1)^{-1}\big(\frac{1}{2}\big)^{-s/z+1}$ has preceisely a simple pole at $s=z$.
Since $(-\frac{s}{z}+1)^{-1} = \frac{z}{z-s}$,

$$\operatorname{res}_{s=z}\tau_z\big(|E_z|^{-s}\big) = \rho(z)^z z\Big(\frac{1}{2}\Big)^{-z/z+1} = \pi^{z/2} z\Big(\Gamma\big(\frac{z}{2}+1\big)\Big)^{-1} = \pi^{z/2} z\,\frac{2}{z}\,\Big(\Gamma\big(\frac{z}{2}\big)\Big)^{-1} = 2\,\frac{\pi^{z/2}}{\Gamma(\frac{z}{2})}.$$

⊠

In Theorem 4.14 we will compute the dimension spectrum of $\not{D}_z$. This requires the machinery of hypergeometric functions. We state the properties that will be used.

Notation 4.13. We will use the shorthand notation $F(a,b;c;z) = {}_2F_1(a,b;c;z)$ to denote the hypergeometric function. The definition and properties of this function can be found in several books, for example in [24, Ch. 15] or [32, Ch. 5]. We mention the properties that we are going to use

$$F(0,b;c;z) = F(a,0;c;z) = 1 \qquad \text{for all } a,b,c,z; \tag{4.9}$$

$$\begin{aligned}
F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a,1-c+a;1-b+a;1/z) \\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(b,1-c+b;1-a+b;1/z).
\end{aligned} \tag{4.10}$$

Equality (4.10) holds if the following requirements are satisfied $|\arg(-z)|<\pi$ and $1-b+a,\ 1-a+b\notin\{0,-1,-2,\ldots\}$.
The last property we will use is that the indefinite integral of $f(x)=(1+x^p)^q$ is given by

$$x \mapsto xF\Big(\frac{1}{p},-q;1+\frac{1}{p};-x^p\Big). \tag{4.11}$$

This property will need a proof.

Proof. By the binomial theorem we have

$$(1+x^p)^q = \sum_{k=0}^\infty \binom{q}{k} x^{pk}.$$

Here $\binom{q}{k}$ for $q\in\mathbb{C}$ is defined by $\binom{q}{k} := \frac{(q)_k}{k!}$, with $(q)_k := q(q-1)\cdots(q-k+1)$. Taking primitives on both sides yields

$$
\begin{aligned}
\int (1+x^p)^q &= \sum_{k=0}^{\infty} \binom{q}{k} \frac{1}{pk+1}\, x^{pk+1} \\
&= x \sum_{k=0}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!(pk+1)}\, (x^p)^k \\
&= x \sum_{k=0}^{\infty} \frac{\frac{1}{p}(-q)(-q+1)\cdots(-q+k-1)}{(k+\frac{1}{p})k!}\, (-x^p)^k \\
&= x \sum_{k=0}^{\infty} \frac{\frac{1}{p}(\frac{1}{p}+1)\cdots(\frac{1}{p}+k-1)(-q)(-q+1)\cdots(-q+k-1)}{(\frac{1}{p}+1)\cdots(\frac{1}{p}+k-1)(\frac{1}{p}+k)k!}\, (-x^p)^k \\
&= xF\Big(\frac{1}{p}, -q; 1+\frac{1}{p}; -x^p\Big),
\end{aligned}
$$

as desired. ⊠

Theorem 4.14. *Assume $z \in [0,\infty)$. The triple $\mathcal{T}_z$ is z^+-summable, regular and the dimension spectrum is simple and consists of $\{z\}$. The residue of $s \mapsto \tau_z((1+\not{D}_z^2)^{s/2})$ at $s = z$ is $\frac{\pi^{z/2}}{\Gamma(z/2)}$.*

Note that we have a factor 2 difference between the residue at $s = z$ of $s \mapsto \tau_z((1+\not{D}_z^2)^{-s/2})$ and $s \mapsto \tau_z(|E_z|^{-s})$.

Proof. Since $\mathcal{A}_z = \mathbb{C}$, the commutators $[\not{D}_z, a] = [|\not{D}_z|, a] = 0$ for all $a \in \mathcal{A}_z$, thus it is obvious that the triple is regular. For the dimension spectrum we have to compute the poles of $s \mapsto \tau_z(b(1+\not{D}_z^2)^{-s/2})$. We can take $b = 1$, since $\mathcal{A}_z = \mathbb{C}$. Thus we consider the meromorphic function

$$
\begin{aligned}
s \mapsto \tau_z(1+\not{D}_z^2)^{-s/2} &= \frac{1}{2}\int_{\mathbb{R}} \big(1+(\rho(z)\operatorname{sgn}(x)|x|^{1/z})^2\big)^{-s/2} \\
&= \int_0^\infty (1+\rho(z)^2 x^{2/z})^{-s/2}\, dx \\
&= \rho(z)^{-z} \int_0^\infty (1+y^{2/z})^{-s/2}\, dy.
\end{aligned}
$$

The last equality follows from the substition $y = \rho(z)^z x$. Note that the constant $\rho(z)^z$ does not affect the location of the poles. By Equation (4.11) we have

$$
\int_0^\infty (1+x^{2/z})^{-s/2}\, dx = \lim_{x\to\infty} xF\Big(\frac{z}{2}, \frac{s}{2}; 1+\frac{z}{2}; -x^{2/z}\Big).
$$

Since $2/z \in [0,\infty)$, for every $x > 0$ we have $-x^{2/z} < 0$. So $|\arg(--x^{2/z})| = |\arg(x^{2/z})| = 0 < \pi$. Therefore if $1 - z/2 + s/2 \notin \{0, -1, -2, \ldots\}$, i.e. if $s \notin \{-2+z, -4+z, \ldots\}$ we can apply (4.10) and (4.9) to obtain

$$
\begin{aligned}
&\lim_{x\to\infty} xF\Big(\frac{z}{2}, \frac{s}{2}; 1+\frac{z}{2}; -x^{2/z}\Big) \qquad (4.12) \\
&\quad = \lim_{x\to\infty} \Big(x\, \frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{s}{2}-\frac{z}{2})}{\Gamma(\frac{s}{2})\Gamma(1)}\, (x^{2/z})^{-z/2} F\big(\frac{z}{2}, 0; 1-\frac{s}{2}+\frac{z}{2}; (-x^{2/z})^{-1}\big). \\
&\qquad + x\, \frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{z}{2}-\frac{s}{2})}{\Gamma(\frac{z}{2})\Gamma(1+\frac{z}{2}-\frac{s}{2})}\, (x^{2/z})^{-s/2} F\big(\frac{s}{2}, -\frac{z}{2}+\frac{s}{2}; 1-\frac{z}{2}+\frac{s}{2}; (-x^{2/z})^{-1}\big)\Big) \\
&\quad = \frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{s}{2}-\frac{z}{2})}{\Gamma(\frac{s}{2})} \\
&\qquad + \lim_{x\to\infty} \frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{z}{2}-\frac{s}{2})}{\Gamma(\frac{z}{2})\Gamma(1+\frac{z}{2}-\frac{s}{2})}\, x^{-s/z+1}\, F\big(\frac{s}{2}, -\frac{z}{2}+\frac{s}{2}; 1-\frac{z}{2}+\frac{s}{2}; -x^{-2/z}\big). \qquad (4.13)
\end{aligned}
$$

Now suppose $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > z$, then $\mathrm{Re}(-s/z+1) < 0$. On the disk $\{z \in \mathbb{C} : |z| < 1\}$ the function $z \mapsto F(a,b;c;z)$ is holomorphic. Since $F(a,b;c;0) = 1$ we have

$$\lim_{x\to\infty} x^{-s/z+1}\, F\big(\frac{s}{2}, -\frac{z}{2}+\frac{s}{2}; 1-\frac{z}{2}+\frac{s}{2}; -x^{-2/z}\big) = 0. \tag{4.14}$$

Inserting (4.14) in (4.13) gives for $\mathrm{Re}(s) > z$

$$\lim_{x\to\infty} xF\big(\frac{z}{2}, \frac{s}{2}; 1+\frac{z}{2}; -x^{2/z}\big) = \frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{s}{2}-\frac{z}{2})}{\Gamma(\frac{s}{2})}.$$

By analytic continuation this holds for any $s \in \mathbb{C} \setminus \{-2+z, -4+z, \ldots\}$. Recall that $\rho(z) = \pi^{-1/2}(\Gamma(1+\frac{z}{2}))^{1/z}$, so

$$\tau_z\big((1+\not{D}_z^2)^{-s/2}\big) = \frac{\Gamma(\frac{s}{2}-\frac{z}{2})}{\Gamma(\frac{s}{2})}\,\pi^{z/2}, \qquad \text{for } s \in \mathbb{C} \setminus \{-2+z, -4+z, \ldots\}.$$

Observe that the function $s \mapsto \Gamma(\frac{s}{2}-\frac{z}{2})$ has simple poles for $\frac{s}{2}-\frac{z}{2} \in \{\ldots,-2,-1,0\}$. Note that $w \mapsto \big(\Gamma(\frac{w}{2})\big)^{-1}$ is holomorphic at $w = s$. Thus $s \mapsto \tau_z(1+\not{D}_z^2)^{-s/2}$ has a simple pole at $s = z$. For $m \in \mathbb{N}$ the residue of the gamma function are given by $\mathrm{res}_{w=-m}\,\Gamma(w) = \frac{(-1)^m}{m!}$. Thus it follows that

$$\mathrm{res}_{s=z}\,\tau_z\big((1+\not{D}_z^2)^{-s/2}\big) = \mathrm{res}_{s=z}\,\frac{\Gamma(\frac{s}{2}-\frac{z}{2})}{\Gamma(\frac{s}{2})}\,\pi^{z/2} = \frac{\pi^{z/2}}{\Gamma(\frac{z}{2})}.$$

For every pole s we have $s \leq z$. Therefore $\tau_z(1+\not{D}_z^2)^{-s/2} < \infty$ for all $s > z$, hence the triple is z^+-summable.

It remains to show that z is the only pole. Recall that the transformation (4.13) was only valid for $1 - z/2 + s/2 \notin \{0,-1,-2,\ldots\}$. We have to deal with these points in a different way. Fix $x_0 \in (1,\infty)$ and $s_0 \in \{\ldots,-4+z,-2+z\}$. Say $-\frac{z}{2}+\frac{s_0}{2} = -m$. Let s be close to s_0, then

$$\begin{aligned}
F\big(&\frac{s}{2}, -\frac{z}{2}+\frac{s}{2}; 1-\frac{z}{2}+\frac{s}{2}; -x_0^{2/z}\big)\\
&= \sum_{j=0}^{\infty} \frac{(\frac{s}{2})_j(-\frac{z}{2}+\frac{s}{2})(-\frac{z}{2}+\frac{s}{2}+1)\cdots(-\frac{z}{2}+\frac{s}{2}+j-1)}{(-\frac{z}{2}+\frac{s}{2}+1)\cdots(-\frac{z}{2}+\frac{s}{2}+j-1)(-\frac{z}{2}+\frac{s}{2}+j)j!}\,(-x_0^{-2/z})^j\\
&= \sum_{j\in\mathbb{N}\setminus\{m\}} \frac{(\frac{s}{2})_j}{j!}\,\frac{-\frac{z}{2}+\frac{s}{2}}{-\frac{z}{2}+\frac{s}{2}-j}\,(-x_0^{z/2})^j + \frac{(\frac{s}{2})_m(-\frac{z}{2}+\frac{s}{2})}{(-\frac{z}{2}+\frac{s}{2}+m)m!}\,(-x_0^{-2/z})^m.
\end{aligned} \tag{4.15}$$

We will now consider the limit $s \to s_0$. To prove that (4.12) does not have a pole at $s = s_0$ we compute (still for x_0 fixed) the residue of (4.12) and show that it equals 0. Since we singled out m in the infinite sum, near s_0 the function

$$s \mapsto \sum_{j\in\mathbb{N}\setminus\{m\}} \frac{(\frac{s}{2})_j}{j!}\,\frac{-\frac{z}{2}+\frac{s}{2}}{-\frac{z}{2}+\frac{s}{2}-j}\,(-x_0^{-2/z})^j$$

is holomorphic. To conclude that (4.13) has a removable singularity at s_0 we compute the following residue

$$\begin{aligned}
\mathrm{res}_{s=s_0}&\frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{s}{2}-\frac{z}{2})}{\Gamma(\frac{s}{2})} + \frac{\Gamma(1+\frac{z}{2})\Gamma(\frac{z}{2}-\frac{s}{2})}{\Gamma(\frac{z}{2})\Gamma(1+\frac{z}{2}-\frac{s}{2})}\,x_0^{-s/z+1}\,\frac{(\frac{s}{2})_m(-\frac{z}{2}+\frac{s}{2})}{(-\frac{z}{2}+\frac{s}{2}+m)m!}(-x_0^{-2/z})^m\\
&= \frac{\Gamma(1+\frac{z}{2})}{\Gamma(\frac{s_0}{2})}\,\frac{(-1)^m}{m!} + \frac{\Gamma(1+\frac{z}{2})\Gamma(m)}{\Gamma(\frac{z}{2})\Gamma(m+1)}\,x_0^{-s_0/z+1}\,\frac{(\frac{s_0}{2})_m(-m)}{m!}(-1)^m(x_0^{-2/z})^m\\
&= \frac{\Gamma(1+\frac{z}{2})}{\Gamma(\frac{s_0}{2})}\,\frac{(-1)^m}{m!} - \frac{\Gamma(1+\frac{z}{2})}{\Gamma(\frac{z}{2})}\,\frac{(m-1)!m}{m!}\,\frac{1}{m!}\,\frac{\Gamma(\frac{s_0}{2}+m)}{\Gamma(\frac{s_0}{2})}(-1)^m\\
&= \frac{\Gamma(1+\frac{z}{2})}{\Gamma(\frac{s_0}{2})}\,\frac{(-1)^m}{m!} - \frac{\Gamma(1+\frac{z}{2})}{\Gamma(\frac{z}{2})}\,\frac{(-1)^m}{m!}\,\frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{s_0}{2})}\\
&= 0.
\end{aligned}$$

Indeed, this residue is independent of x_0. So the dimension spectrum of $\not{D}_z$ consists of $z = s$. ⊠

It is interesting to compare the dimension spectrum of $\not{D}_z$ and E_z, the latter operator is the operator introduced in Remark 4.11. With the previous theorem it has become easy to compute the poles of $s \mapsto \tau_z((1+E_z^2)^{-s/2})$.

Proposition 4.15. *Suppose $z \in (0,\infty)$, then the dimension spectrum of E_z equals the dimension spectrum of $\not{D}_z$.*

Proof. From the definition of the trace τ_z and the operator E_z it immediately follows that

$$\begin{aligned}\tau_z\big((1+E_z^2)^{-s/2}\big) &= \int_0^\infty \big(1+(\rho(z)\operatorname{sgn}(x) f(|x|)^{1/z})^2\big)^{-s/2}\,dx\\ &= \int_0^{1/2} \big(1+\rho(z)^2 f(|x|)^{2/z}\big)^{-s/2}\,dx + \int_{1/2}^\infty \big(1+\rho(z)^2 x^{2/z}\big)^{-s/2}\,dx.\end{aligned}$$

It follows from the proof of Theorem 4.14 that it suffices to show that the functions

$$s \mapsto \int_0^{1/2} \big(1+\rho(z)^2 x^{2/z}\big)^{-s/2}\,dx, \qquad s \mapsto \int_0^{1/2} \big(1+\rho(z)^2 f(|x|)^{2/z}\big)^{-s/2}\,dx$$

are holomorphic on $\mathbb{C}$. We start with the first one.

$$\int_0^{1/2} \big(1+\rho(z)^2 x^{2/z}\big)^{-s/2}\,dx = \rho(z)^{-z}\int_0^{1/2} (1+x^{2/z})^{-s/2}\,dx = \rho(z)^{-z}\frac{1}{2}\,F\Big(\frac{z}{2},\frac{s}{2};1+\frac{z}{2};-\big(\frac{1}{2}\big)^{2/z}\Big).$$

Note $-(1/2)^{2/z} \in (-\infty,0)$, thus [24, §15.2] implies that

$$s \mapsto \rho(z)^{-z}\frac{1}{2}F\Big(\frac{z}{2},\frac{s}{2};1+\frac{z}{2};-\big(\frac{1}{2}\big)^{2/z}\Big)$$

is holomorphic on $\mathbb{C}$. Now the second one. The function f is smooth, strictly increasing, $f(0) > 0$ and $f(1/2) = 1/2$. Thus there exist $\delta, D > 0$ such that $\delta < 1+\rho(z)^2 f(x)^{2/z} < D$ for all $x \in [0,1/2]$. But from these bounds it is clear that for each $s \in \mathbb{C}$ the function

$$[0,1/2] \to \mathbb{C}, \qquad x \mapsto \big(1+\rho(z)^2 f(|x|)^{2/z}\big)^{-s/2}$$

is bounded. Thus

$$s \mapsto \int_0^{1/2} \big(1+\rho(z)^2 f(|x|)^{2/z}\big)^{-s/2}\,dx$$

does not have any poles in $\mathbb{C}$. ⊠

In Theorem 3.18 it was proved that the product of two regular semifinite spectral triples is again regular. Since $\mathcal{A}_z = \mathbb{C}$ and thus $\mathcal{T}_z$ is regular, the following corollary is immediate.

Corollary 4.16. *Suppose $\mathcal{S}$ is an even regular semifinite spectral triple. Then the product triple $\mathcal{S} \times \mathcal{T}_z$ is also regular.*

Since the tensor product of a spectral triple with $\mathcal{S}_z$ is regular, one can compute its dimension spectrum. We expect that the whole spectrum shifts over the vector z, but we cannot prove this fact in its full generality. However we can prove the result for the largest pole. We will use Fubini's theorem for traces cf. Proposition 1.79.

Proposition 4.17. *Suppose $\mathcal{S} := (\mathcal{A},\mathcal{H},D;\mathcal{N},\tau,\gamma)$ is a finitely summable regular semifinite spectral triple with $1 \in \mathcal{A}$. Denote $\mathcal{S}_z := (\mathcal{A},\tilde{\mathcal{H}},D_z;\tilde{\mathcal{N}},\tau') := \mathcal{S}\times\mathcal{T}_z$. Suppose $w \in \mathbb{C}$ is an element of the dimension spectrum of $\mathcal{S}$ such that $\operatorname{Re}(w) > 0$ and for all w' in the dimension spectrum of $\mathcal{S}$ we have $\operatorname{Re}(w') \le \operatorname{Re}(w)$. Then if $0 < z < \operatorname{Re}(w)$, the function $s \mapsto \tau'((D_z^2+1)^{-s/2})$ has a pole for $s = w+z$ and all other poles w'' of the zeta functions ζ_b satisfy $\operatorname{Re}(w'') \le \operatorname{Re}(w)+z$.*

Proof. The main idea of the proof of this lemma is to write the operator $(D_z+1)^{-s/2}$ as an elementary tensor and then use the factorisation of the trace τ'. This can be done by writing this operator as an integral of exponential functions. Then we will need the previous Lemma 1.79 to interchange the integral and the trace.
We start with the identity

$$\int_0^\infty e^{-tx}t^{a-1}\,dt = x^{-a}\Gamma(a) \qquad \mathrm{Re}(a)>0.$$

In the following calculation we will interchange two times an integral with a trace. We will justify those manipulations later.

$$\begin{aligned}
\zeta_1(s) &= \tau'((D_z^2+1)^{-s/2})\\
&= \tau'\Big(\frac{1}{\Gamma(s/2)}\int_0^\infty e^{-t(1+D^2\otimes 1+1\otimes \not{D}_z^2)}t^{s/2-1}\,dt\Big)\\
&= \frac{1}{\Gamma(s/2)}\int_0^\infty \tau'\big(e^{-t(1+D^2)}\otimes e^{-t\not{D}_z^2}\big)t^{s/2-1}\,dt \qquad (4.16)\\
&= \frac{1}{\Gamma(s/2)}\int_0^\infty \tau\big(e^{-t(1+D^2)}\big)\,\tau_z\big(e^{-t\not{D}_z^2}\big)t^{s/2-1}\,dt\\
&= \frac{1}{\Gamma(s/2)}\int_0^\infty \tau\big(e^{-t(1+D^2)}\big)\,\pi^{z/2}t^{-z/2}t^{s/2-1}\,dt\\
&= \pi^{z/2}\,\frac{1}{\Gamma(s/2)}\,\tau\Big(\int_0^\infty e^{-t(1+D^2)}\,t^{(s-z)/2-1}\,dt\Big) \qquad (4.17)\\
&= \pi^{z/2}\,\frac{\Gamma((s-z)/2)}{\Gamma(s/2)}\,\tau\big((1+D^2)^{-(s-z)/2}\big).
\end{aligned}$$

By assumption the function $s\mapsto \tau((1+D^2)^{-s/2})$ has a pole at w, thus $s\mapsto\tau'(D_z^2+1)^{-s/2})$ has a pole at $s-z=w$ i.e. at $s=w+z$. Since $z<\mathrm{Re}(w)$, this is the largest pole of ζ_1, because Γ only has poles at the non-positive integers.
Suppose $b\in\mathcal{B}$, then the following estimate shows that one does not obtain any poles in the half-plane $\{s\in\mathbb{C} : \mathrm{Re}(s)>\mathrm{Re}(w)=z\}$.

$$|\zeta_b(s)| = |\tau_z\big(b(D_z^2+1)^{-s/2}\big)| \le \tau'(|b(D_z^2+1)^{-s/2}|) \le \|b\|\tau'(|(D_z^2+1)^{-s/2}|),$$

which converges for s with $\mathrm{Re}(s)>\mathrm{Re}(w)+z$.
It remains to show why one can interchange the integral and trace in (4.16) and (4.17) we use Proposition 1.79. Suppose E is an unbounded self-adjoint operator on $\mathcal{K}$. By analytic continuation it is sufficient to prove the switch for $s\in\mathbb{R}$ with $s>2$. Fix such an s. Consider

$$f:[0,\infty)\to B(\mathcal{K}), \qquad t\mapsto \frac{1}{\Gamma(s)}\,e^{-t(1+E^2)}t^{s/2-1}.$$

We check the conditions of Proposition 1.79. We know that if a,b are positive, then the map $[0,\infty)\to\mathbb{R}$, $t\mapsto e^{-ta}t^b$ is uniformly bounded. Using the functional calculus shows that $f(E)$ is norm-bounded.
To prove the second requirement note that $f(\cdot)h \leftrightsquigarrow f(\cdot)^*h$ corresponds to $s\leftrightsquigarrow\overline{s}$. Thus it is sufficient to show that $f(\cdot)h$ is measurable. We will prove continuity of $f(\cdot)h$.

$$\begin{aligned}
&\|f(t_0+t)h-f(t_0)h\| \le \frac{1}{\Gamma(s)}\,\big\|e^{(-t_0+t)(1+E^2)}(t_0+t)^{s/2-1}-e^{-t_0(1+E^2)}(t_0+t)^{s/2-1}\big\|\,\|h\|\\
&\le \frac{1}{\Gamma(s)}\,\Big(\big\|e^{-t_0(1+E^2)}\big\|\,\big\|e^{-t(1+E^2)}-1\big\|\,|t_0+t|^{s/2-1}+\big\|e^{-t_0(1+E^2)}\big\|\,\big|(t_0+t)^{s/2-1}-t_0^{s/2-1}\big|\Big)\|h\|
\end{aligned}$$

which tends to 0 as $t\to 0$.
And the last requirement

$$\rho(|f(t)|) = \frac{1}{\Gamma(s)}\rho\big(e^{-t(1+E^2)}\big)\big|t^{s/2-1}\big|. \qquad (4.18)$$

If we now let $E = D_z$ and $\rho = \tau'$ then it is obvious that as a function of t, (4.18) is uniformly bounded on $[0,\infty)$, in particular on the interval $[0,n]$. Thus using the fact that $[D^2 \otimes 1, 1 \otimes \not{D}_z^2] = 0$ and Proposition 1.79 we obtain that for all $n > 0$:

$$\tau'\Big(\frac{1}{\Gamma(s)}\int_0^n e^{-t(1+D^2\otimes 1+1\otimes \not{D}_z^2)}t^{s/2-1}\,dt\Big) = \frac{1}{\Gamma(s)}\int_0^n \tau'\big(e^{-t(1+D^2)}\otimes e^{-t\not{D}_z^2}\big)t^{s/2-1}\,dt.$$

Taking the limit $n \to \infty$ gives the desired Equality (4.16). For (4.17), we can do the same trick, but we have to replace E by D and ρ by τ. ⊠

4.3 Minkowski dimension of the spectrum

At the end of the Paragraph 1.19.2, Connes and Marcolli pose the question to give an upperbound on the Minkowski dimension of the spectrum of the operator $\not{D}_z$. The answer appears to be 1. Before we will give the proof of this result we will start with the definitions.

Definition 4.18. Suppose $E \subset \mathbb{R}^n$ is a bounded subset. We denote by $N_E(\varepsilon)$ the number of boxes with sides of length ε needed to cover the set E. If no confusion will arise, we will omit the subscript $_E$. We define

$$\dim_M(E) := \lim_{\varepsilon\downarrow 0}\frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}.$$

If this limit exists, the quantity $\dim_M(E)$ is called the *Minkowski dimension* of E. We can also define the *upper Minkowski dimension* and *lower Minkowski dimension*. These are given by respectively

$$\dim_{M\,up}(E) := \limsup_{\varepsilon\downarrow 0}\frac{\log(N(\varepsilon))}{\log(1/\varepsilon)} \qquad \text{and} \qquad \dim_{M\,low}(E) := \liminf_{\varepsilon\downarrow 0}\frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}.$$

It is immediate from this definition that $\dim_{M\,up}(E) \leq \dim_{M\,low}(E)$. Furthermore we have equality if and only if $\dim_M(E)$ exists, in which case $\dim_{M\,up}(E) = \dim_{M\,low}(E) = \dim_M(E)$. Also if $F \subset E$, and for both sets the Minkowski dimensions exist, then $\dim_M(F) \leq \dim_M(E)$. A similar statement holds for $\dim_{M\,up}$ and $\dim_{M\,low}$.

Example 4.19. The Minkowski dimension of $E := [a,b]$ is 1. Namely $N(\varepsilon) = \lceil \frac{b-a}{\varepsilon}\rceil$. Thus $\frac{b-a}{\varepsilon} \leq N(\varepsilon) < \frac{b-a}{\varepsilon} + 1$. Note

$$\lim_{\varepsilon\downarrow 0}\frac{\log(\frac{b-a}{\varepsilon})}{\log(\frac{1}{\varepsilon})} = \lim_{\varepsilon\downarrow 0}\frac{\log(b-a)+\log(\frac{1}{\varepsilon})}{\log(\frac{1}{\varepsilon})} = 1.$$

And also

$$\lim_{\varepsilon\downarrow 0}\frac{\log(\frac{b-a}{\varepsilon}+1)}{\log(\frac{1}{\varepsilon})} = \lim_{\varepsilon\downarrow 0}\frac{\log(b-a+\varepsilon)+\log(\frac{1}{\varepsilon})}{\log(\frac{1}{\varepsilon})} = 1.$$

Thus

$$1 \leq \lim_{\varepsilon\downarrow 0}\frac{\log(N(\varepsilon))}{\log(\frac{1}{\varepsilon})} \leq 1.$$

Note that this result is independent of a and b.

If a set $E \subset \mathbb{R}^n$ is unbounded, the number of boxes $N_E(\varepsilon)$ is infinite and the limit $\lim_{\varepsilon\downarrow 0}\frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}$ will also be infinite. But this is not what we want, because we expect that $\mathbb{R}^n$ has Minkowski dimension n. This can be fixed in the following way.

Definition 4.20. Let $E \subset \mathbb{R}^n$ be a subset (not necessarily bounded). We define

$$\dim_{M\,up}(E) := \sup\{\dim_{M\,up}(F) \,:\, F \subset E,\, F \text{ is bounded}\};$$
$$\dim_{M\,low}(E) := \sup\{\dim_{M\,low}(F) \,:\, F \subset E,\, F \text{ is bounded}\}.$$

If $\dim_{M\,up}(E) = \dim_{M\,low}(E)$ we define $\dim_{M\,up}(E) := \dim_{M\,up}(E)$.

We cannot define $\dim_{M\,up}(E) = \sup\{\dim_M(F) : F \subset E,\ F \text{ is bounded}\}$, because $\dim_M(F)$ does not to exist for all F.

Example 4.21. We have $\dim_M(\mathbb{R}) = 1$. Indeed, if $F \subset \mathbb{R}$ is a bounded set, then there exists $a, b \in \mathbb{R}$ such that $F \subset [a, b]$. Which implies $N_F(\varepsilon) \leq N_{[a,b]}(\varepsilon)$, so $\dim_{M\,up}(F) \leq 1$ and $\dim_{M\,low}(F) \leq 1$ for any bounded F. Note that for some sets F we have equality, for example if F is an interval. Thus $\dim_{M\,up}(\mathbb{R}) = 1$ and $\dim_{M\,low}(\mathbb{R}) = 1$, hence by definition $\dim_M(\mathbb{R}) = 1$.

Remark 4.22. In [9, §1.19.2] Connes and Marcolli guess that an upperbound for the Minkowski dimension of the spectrum of $\not{D}_z$ is given by $\frac{1}{1/\operatorname{Re}(z)}$. However this cannot be true. Because we know that $\sigma(\not{D}_z) = \mathbb{R}$ if $z \in (0, \infty)$, thus by Example 4.21 it follows that $\dim_M(\sigma(\not{D}_z)) = 1$. Now select $z \in (0, 1)$, then it does not hold that $\dim_M(\sigma(\not{D}_z)) \leq \frac{1}{1/\operatorname{Re}(z)}$. It is possible to explicitly calculate the Minkowski dimension of the spectrum of $\not{D}_z$.

Theorem 4.23. *If $z \in \mathbb{C}$ and $\operatorname{Re}(z) > 0$, then the Minkowski dimension $\dim_M(\sigma(\not{D}_z)) = 1$. Here we consider $\sigma(\not{D}_z) \subset \mathbb{C} \cong \mathbb{R}^2$.*

To obtain an idea of what the spectrum of D_z looks like, take a look at figure 1. Since $\sigma(D_z) = \{\rho(z)\operatorname{sgn}(x)|x|^{1/z} : z \in \mathbb{R}\}$ we plotted the function $f_z : \mathbb{R} \to \mathbb{C}$ (cf. Notation 4.3) for $z = \frac{1}{2} + 2i$. The red curve indicates (a part) of the positive real axis, the blue curve (a part of) the negative real axis. The spiral continues towards zero and infinity, but of course this can not be captured in the plot.

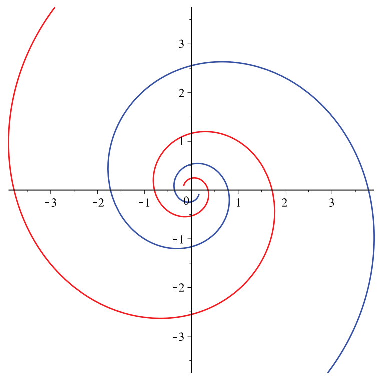


Figure 1: Plot of the spectrum of D_z for $z = \frac{1}{2} + 2i$

Proof. We have to compute the dimension of the set $\sigma(\not{D}_z) = \{\rho(z)\operatorname{sgn}(x)|x|^{1/z}\}$. If $\operatorname{Im}(z) \neq 0$, then this is a double spiral as given in figure 1. Fix $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$. We start by computing the Minkowski dimension of $E := \{x^w : x \in [0, \infty)\}$, where $w := \frac{1}{z}$. Write $w = a + bi$.
The idea of the proof is to calculate the length of the spiral E in the region $\{z \in \mathbb{C} : |z| \leq R\}$ for some R. Once we know the length, we can use this to estimate the number of boxes needed to cover the spiral. And then we can calculate its Minkowski dimension. Using the Minkowski dimension of E we can calculate the Minkowski dimension of $\sigma(\not{D}_z)$.
Notice that $x \mapsto |x^w|$ is a strictly increasing function. It holds that $|x^w| = x^a$, so we have $r \leq |x^w| \leq R$ if and only if $r^{1/a} \leq x \leq R^{1/a}$. We impose this lower bound, because near the origin the spiral crops up. Let γ be the path

$$\gamma : \left[r^{1/a}, R^{1/a}\right] \to \mathbb{C}, \qquad x \mapsto x^w.$$

The length of γ is given by

$$\begin{aligned} l(\gamma) &= \int_{r^{1/a}}^{R^{1/a}} \left|\frac{d}{dx}\gamma(x)\right| dx = \int_{r^{1/a}}^{R^{1/a}} \left|wx^{w-1}\right| dx = |w| \int_{r^{1/a}}^{R^{1/a}} x^{a-1}\, dx = \frac{|w|}{a}\left[x^a\right]_{x=r^{1/a}}^{x=R^{1/a}} \\ &= \frac{|w|}{a}(R-r). \end{aligned}$$

We let $F_R := \{z \in E \,:\, |z| \leq R\}$. The number of boxes of size $\varepsilon \times \varepsilon$ needed to cover F_R is at least the diameter of $\{z \in \mathbb{C} \,:\, |z| \leq R\}$ divided by ε and it is at most $1 + \frac{l(\gamma)}{\varepsilon}$, where γ is the curve $\gamma : [\varepsilon/2, R] \to \mathbb{C}$. Here the 1 comes from the box centered at 0 (because the spiral crops up around 0) and now we can cover the rest of the spiral outside this centered box by $\frac{l(\gamma)}{\varepsilon}$ boxes. So we have

$$\frac{2R}{\varepsilon} \leq N_{F_R}(\varepsilon) \leq 1 + \frac{l(\gamma)}{\varepsilon} = 1 + \frac{\frac{|w|}{a}\left(R - \frac{\varepsilon}{2}\right)}{\varepsilon}.$$

We compute the Minkowski dimension of F_R. It follows from $\log(ab) = \log(a) + \log(b)$ and continuity of the numerators that

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} \frac{\log\left(\frac{2R}{\varepsilon}\right)}{\log\left(\frac{1}{\varepsilon}\right)} &= \lim_{\varepsilon\downarrow 0} \frac{\log(2R) + \log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{\varepsilon}\right)} = 1; \\ \lim_{\varepsilon\downarrow 0} \frac{\log\left(1 + \frac{\frac{|w|}{a}(R-\frac{\varepsilon}{2})}{\varepsilon}\right)}{\log\left(\frac{1}{\varepsilon}\right)} &= \lim_{\varepsilon\downarrow 0} \frac{\log\left(\frac{1}{\varepsilon}\right) + \log\left(\varepsilon + \frac{|w|}{a}(R - \frac{\varepsilon}{2})\right)}{\log\left(\frac{1}{\varepsilon}\right)} = 1. \end{aligned}$$

So the Minkowski dimension $\dim_M(F_R) = 1$.
Suppose $F' \subset E$ is a bounded set. Then there exists an $R > 0$ such that $F' \subset F_R$. Therefore

$$\dim_{M\,low}(F') \leq \dim_{M\,up}(F') \leq \dim_{M\,up}(F) = \dim_M(F_r) = 1.$$

Taking the supremum over all F' yields

$$\dim_{M\,low}(E) \leq \dim_{M\,up}(E) \leq 1.$$

Since for F_R it holds that $\dim_M(F_r) = 1$, we obtain $1 \leq \dim_{M\,low}(E)$. And so $\dim_M(E) = 1$.
Now it remains to show that $\dim_M(\sigma(\not{D}_z) = \dim_M(E)$. Note that $\sigma(\not{D}_z) = (\rho(z)E) \cup (-\rho(z)E)$, where we denote $cE = \{cz \,:\, z \in E\}$. Now the proof immediately follows from the next lemma. ☒

Lemma 4.24. *If $E \subset \mathbb{R}^n$ and E has Minkowski dimension λ and $c \neq 0$, then cE and $E \cup -E$ have Minkowski dimension λ.*

Proof. It is sufficient to prove this statement for bounded sets, because the general statement follows by taking suprema. If E is bounded, then $N_{cE}(|c|\varepsilon) = N_E(\varepsilon)$. So

$$\begin{aligned} \dim_M(cE) &= \lim_{\varepsilon\downarrow 0} \frac{\log(N_{cE}(\varepsilon))}{\log(\frac{1}{\varepsilon})} = \lim_{\delta\downarrow 0} \frac{\log(N_{cE}(|c|\delta))}{\log(\frac{1}{|c|\delta})} = \lim_{\delta\downarrow 0} \frac{\log(N_E(\delta))}{\log(\frac{1}{|c|}) + \log(\frac{1}{\delta})} \\ &= \lim_{\delta\downarrow 0} \frac{\log(N_E(\delta))}{\log(\frac{1}{\delta})} \cdot \lim_{\delta\downarrow 0} \frac{\log(\frac{1}{\delta})}{\log(\frac{1}{|c|}) + \log(\frac{1}{\delta})} = \dim_M(E). \end{aligned}$$

For the other assertion, we have

$$N_E(\varepsilon) \leq N_{E\cup -E}(\varepsilon) \leq 2N_E(\varepsilon),$$

now apply a similar argument. ☒

5 Application to physics

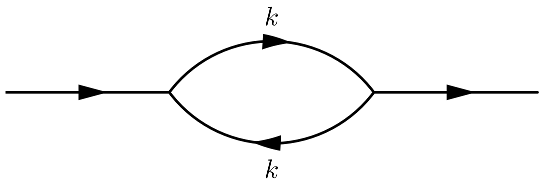
In section 4 we described a set of spectral triples which can be considered as being z-dimensional. In this section we will apply these spectral triples to describe phenomena which occur in physics. We start with dimensional regularisation, continue with computations on so-called anomalous graphs which we will apply to zeta function regularisation in the last subsection.

5.1 Dimensional regularisation

In quantum field theory, the integrals considered are typically of the form

$$\int_{\mathbb{R}^4} \frac{1}{(k^2+m^2)^2} d^4k. \tag{5.1}$$

This specific integral corresponds to the following Feynman diagram



It represents a particle with mass m which propagates and self-interacts. Since these integrals are divergent, 't Hooft and Veltman [21] developed dimensional regularisation to deal with these integrals. Their method is based on the following formula:

$$\int_{\mathbb{R}^D} e^{-\lambda p^2}\, d^D p = \left(\frac{\pi}{\lambda}\right)^{D/2}. \tag{5.2}$$

This equality is valid for $D \in \{1, 2, \ldots\}$. For non-integer values of D this cannot be proved, because their is no such thing as a (Lebesgue) integral in D-dimensions. But instead of proving it, the right hand side is used as a definition for the left hand side if $D \notin \{1, 2, \ldots\}$. In this way we obtain a method to integrate in D dimensions. We will work out an example.

Example 5.1. We will compute (5.1) in dimension $4 - w$ for $\mathrm{Re}(w) > 0$. To start, note that

$$\int_0^\infty e^{-t(p^2+m^2)}\, dt = \frac{1}{p^2+m^2}.$$

So

$$\frac{1}{(p^2+m^2)^2} = \int_0^\infty \int_0^\infty e^{-s(p^2+m^2)} e^{-t(p^2+m^2)}\, ds\, dt.$$

Substitute $s = (1-x)\lambda$ and $t = x\lambda$, then

$$\det \begin{pmatrix} \frac{ds}{d\lambda} & \frac{dt}{d\lambda} \\ \frac{ds}{dx} & \frac{dt}{dx} \end{pmatrix} = \det \begin{pmatrix} 1-x & x \\ -\lambda & \lambda \end{pmatrix} = \lambda(1-x) + \lambda x = \lambda$$

and

$$\begin{aligned} \frac{1}{(p^2+m^2)^2} &= \int_0^1 \int_0^\infty e^{-(1-x)\lambda(p^2+m^2)} e^{-x\lambda(p^2+m^2)} \lambda\, d\lambda\, dx \\ &= \int_0^1 \int_0^\infty e^{-\lambda(p^2+m^2)} \lambda\, d\lambda\, dx \\ &= \int_0^\infty e^{-\lambda(p^2+m^2)} \lambda\, d\lambda. \end{aligned} \tag{5.3}$$

Then we obtain

$$\int \frac{1}{(p^2+m^2)^2}\, d^{4-w} p = \int \int_0^\infty e^{-\lambda(p^2+m^2)} \lambda\, d\lambda\, d^{4-w} p.$$

Now we interchange the order of integration, although we do not have a theorem of Fubini at our disposal, we just change the order. We do this, because this is a way to give a meaning to the integral in $4-w$-dimensions. After this we insert the essential identity (5.2) to obtain

$$\begin{aligned}\int \frac{1}{(p^2+m^2)^2}\, d^{4-w}p &= \int_0^\infty \int e^{-\lambda(p^2+m^2)} \lambda\, d^{4-w}p\, d\lambda \\ &= \int_0^\infty \Big(\int e^{-\lambda p^2}\, d^{4-w}p \Big) \lambda e^{-\lambda m^2}\, d\lambda \\ &= \int_0^\infty \Big(\frac{\pi}{\lambda}\Big)^{(4-w)/2} \lambda e^{-\lambda m^2}\, d\lambda \\ &= \pi^{(4-w)/2} \int_0^\infty \Big(\frac{\mu}{m^2}\Big)^{-1+w/2} e^{-\mu}\, \frac{1}{m^2}\, d\mu \\ &= \pi^{(4-w)/2} m^{-w} \int_0^\infty e^{-\mu} \mu^{-1+w/2}\, d\mu \\ &= \pi^{(4-w)/2} m^{-w} \Gamma\Big(\frac{w}{2}\Big).\end{aligned}$$

The general theory can for example be found in [14, Chapter 7]. What one usually does is introduce new variables (in our case x and λ), rewrite the integrand as an exponential function and use (5.2).

Remark 5.2. We however do not need to use (5.2) as a definition, but using our previous developed machinery we can explicitly compute

$$\tau\big(e^{-\lambda \not{D}_z}\big) = \Big(\frac{\pi}{\lambda}\Big)^{z/2}$$

and use this as a definition of an integral in z dimensions instead. So if we want to calculate an integral in z dimensions, we have to replace the variable over which we integrate by the operator $\not{D}_z$ and the integral by the trace τ_z. Via this method we have a genuine calculation and not just a formal manipulation. We will illustrate this with an example, we compute again (5.1) but now with $\not{D}_z$ and τ_z.

Example 5.3. We want to compute $\int \frac{1}{(k^2+m^2)^2}\, d^z k$ for $z \in (0,\infty)$. To do this, replace k by $\not{D}_z$ and $\int \cdot\, d^z k$ by the trace τ_z. We have

$$\int \frac{1}{(k^2+m^2)^2}\, d^z k := \tau_z\Big(\frac{1}{(\not{D}_z^2+m^2)^2}\Big)$$

As before use (5.3), we interchange integral and trace and finally we use (4.4) to obtain

$$\begin{aligned}\tau_z\Big(\frac{1}{(\not{D}_z^2+m^2)^2}\Big) &= \tau_z\Big(\int_0^\infty e^{-\lambda \not{D}_z^2} e^{-\lambda m^2} \lambda\, d\lambda\Big) \\ &= \int_0^\infty \tau_z\big(e^{-\lambda \not{D}_z^2}\big) e^{-\lambda m^2} \lambda\, d\lambda \\ &= \int_0^\infty \Big(\frac{\pi}{\lambda}\Big)^{z/2} e^{-\lambda m^2} \lambda\, d\lambda.\end{aligned}$$

Note that it is valid to interchange trace and integral, because by the definition of τ_z we have

$$\tau_z\Big(\int_0^\infty e^{-\lambda \not{D}_z^2} e^{-\lambda m^2} \lambda\, d\lambda\Big) = \int_0^\infty \int_0^\infty e^{-\lambda \rho(z)^2 x^{2/z}} e^{-\lambda m^2} \lambda\, d\lambda\, dx.$$

Since $z > 0$ it holds that $e^{-\lambda\rho(z)^2 x^{2/z}} e^{-\lambda m^2} \lambda \geq 0$ for all $x, \lambda \in [0,\infty)$. Thus by Fubini interchanging the integrals is allowed and therefore we can interchange trace and integral.
To finish the calculation, we can copy the end of Example 5.1. So

$$\tau_z\Big(\frac{1}{(\not{D}_z^2+m^2)^2}\Big) = \pi^{z/2} m^{-z} \Gamma(2-z/2),$$

which has precisely a simple pole at $z = 4$. Note that this expression is well defined for all $z > 0$ and has a meromorphic extension to $z \in \mathbb{C}$.

5.2 Anomalies

In this section we will give another application of the the semifinite triple $\mathcal{T}_z$. It can be used to compute anomalies which are of interest in quantum field theory. This computation has been performed in [9, §1.19.4], we will provide details. For physicists these anomalies are interesting on their own, but we will use these anomalies in the next section on zeta function regularisation. More information about quantum field theory can be found in a lot of books, a good book for mathematicians is [14].

Assume we have a space given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. As described in Remark 2.36 we can assume that the Dirac operator of a spectral triple is invertible. For use in this subsection fix a regular, finitely summable, even spectral triple $\mathcal{S} := (\mathcal{A}, \mathcal{H}, D; \gamma)$ where D is an *invertible* Dirac operator. We will consider the tensor product of the triple $\mathcal{S}$ with the semifinite triple $\mathcal{T}_z$. Denote as before

$$\mathcal{S}_z := \mathcal{S} \times \mathcal{T}_z = \Big(\mathcal{A} \otimes \mathbb{C}, \mathcal{H} \otimes L^2(\mathbb{R}), D \otimes 1 + \gamma \otimes \not{D}_z; B(\mathcal{H}) \otimes L^\infty(\mathbb{R}), \operatorname{Tr} \otimes \tau_z\Big)$$

for the product of the spectral triple $\mathcal{S}$ with the semifinite spectral triple $\mathcal{T}_z$ as described in Theorem 3.10. To avoid an abundance of tensor products we introduce the following notation for the product of the spectral triples

$$\mathcal{S}_z =: (\mathcal{A}, \tilde{\mathcal{H}}, D_z; \tilde{\mathcal{N}}, \tau')$$

The use of $\mathcal{A}$ instead of $\tilde{\mathcal{A}}$ is no typo, because $\mathcal{A} \otimes \mathbb{C} \cong \mathcal{A}$.

Notation 5.4. For an even spectral triple $(\mathcal{A}, \mathcal{H}, D; \gamma)$, denote by $OP(\mathcal{A}, \mathcal{H}, D; \gamma)$ the algebra generated by $\mathcal{A}$, D and γ. If it is clear which spectral triple is considered we will write OP.

We have the following important result, it is one of the results of the local index theorem.

Theorem 5.5 (Connes & Moscovici [10])**.** *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a regular, finitely summable spectral triple with a simple dimension spectrum. Consider the functional*

$$⨍ P := \operatorname{res}_{s=0} \operatorname{Tr}(P|D|^{-s}),$$

this defines a trace on the algebra generated by $\mathcal{A}, [D, \mathcal{A}]$ and $|D|^s$ with $s \in \mathbb{C}$,

Note that the assumption of discrete dimension spectrum is necessary for existence of the residues. The aim of this subsection is to relate for $A \in \Omega^1_D$ the trace $⨍(AD^{-1})^n$ to the behaviour of $\tau'((A \otimes 1)D_z^{-1})^n$ as $z \to 0$. This relation will be described in Proposition 5.11. The proof of this result needs some preparations, we follow the line of [9, §1.19.4]. We start with the observation that D_z is invertible and continue with a technical result.

Lemma 5.6. *If D is an invertible Dirac operator, then D_z is invertible as well.*

Proof. Note that the spectrum of an operator is closed. Since D is invertible, D^2 is invertible. So there exists an $\varepsilon > 0$ such that $\varepsilon \leq D^2$. Clearly $0 \leq \not{D}_z$. Hence $\varepsilon \leq D^2 \otimes 1 + 1 \otimes \not{D}_z^2 = D_z^2$. So $0 \notin \sigma(D_z)^2$ and hence $0 \notin \sigma(D_z)$. So D_z is invertible. ⊠

Lemma 5.7. *Let $P \in OP(\mathcal{A}, \mathcal{H}, D; \gamma)$. Let $k, n \in \mathbb{N}$ with $0 < k < n$ and suppose there exists an $z_0 > 0$ such that for all $0 < z < z_0$ the operator $(\gamma_1 \otimes \not{D}_z)^{2k}(P \otimes 1)D_z^{-2n})$ is bounded, then*

$$\lim_{z\to 0} \tau'\big((\gamma \otimes \not{D}_z)^{2k}(P \otimes 1)D_z^{-2n}\big) = -\frac{1}{2}B(k, n-k) ⨍ PD^{-2(n-k)}.$$

Here B denotes the beta-function, given by $B(p, q) := \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

The requirement "$(\gamma_1 \otimes \not{D}_z)^{2k}(P \otimes 1)D_z^{-2n})$ is bounded" is necessary because otherwise the trace of this operator is not defined.

Proof. Note that by Lemma 5.6 the operator D_z is invertible. Again as in Proposition 4.17 we will interchange trace and integral, because then we can write the operator as an elementary tensor. For $a > 0$ consider the map

$$g : [0, \infty) \to [0, \infty), \qquad x \mapsto \int_0^\infty e^{-tx^2} t^{a-1}\, dt.$$

For $a > 0$ we have the following identity

$$\begin{aligned} g(x) &= \int_0^\infty e^{-tx^2} t^{a-1}\, dt \\ &= \int_0^\infty e^{-s} \Big(\frac{s}{x^2}\Big)^{a-1} \frac{1}{x^2}\, ds \\ &= x^{-2a} \int_0^\infty e^{-s} s^{a-1}\, ds \\ &= x^{-2a}\Gamma(a), \end{aligned}$$

From this we obtain that $g|[\varepsilon, \infty)$ is continuous and bounded if $a > 0$. Note that since D is invertible $0 \notin \sigma(D)$. The spectrum is closed, so there exists $\varepsilon > 0$ such that $|\lambda| > \varepsilon$ for all $\lambda \in \sigma(D)$. Hence $\varepsilon^2 < D^2$ and thus $D_z^2 = D^2 \otimes 1 + 1 \otimes \not{D}_z^2 > \varepsilon^2$. Using the functional calculus we obtain

$$\Gamma(m - l - u/2)\, |D_z|^{u+2(l-m)} = \int_0^\infty e^{-tD_z^2} t^{m-1-u/2-l}\, dt. \tag{5.4}$$

So we have

$$\begin{aligned} \tau'\big((\gamma_1 \otimes \not{D}_z)^{2k}(P \otimes 1)D_z^{-2n}\big) &= \tau'\big((P \otimes \not{D}_z^{2k})(D^2 \otimes 1 + 1 \otimes \not{D}_z^2)^{-n}\big) \\ &= \tau'\Big((P \otimes \not{D}_z^{2k})\frac{1}{\Gamma(n)} \int_0^\infty e^{-t(D^2 \otimes 1)} e^{-t(1 \otimes \not{D}_z^2)} t^{n-1}\, dt\Big) \\ &= \frac{1}{\Gamma(n)}\, \tau'\Big(\int_0^\infty \big(Pe^{-tD^2}\big) \otimes \big(\not{D}_z^{2k} e^{-t\not{D}_z^2}\big)\, t^{n-1}\, dt\Big). \end{aligned} \tag{5.5}$$

We compute $\tau\big(\not{D}_z^{2k} e^{-t\not{D}_z^2}\big)$. Successively we do the following: apply the transformation $y := t\rho(z)^2 x^{2/z}$, partial integration and use the definitions of the gamma-function and the constant $\rho(z) = \pi^{-1/2}(\Gamma(z/2 + 1))^{1/z}$.

$$\begin{aligned} \tau\big(\not{D}_z^{2k} e^{-t\not{D}_z^2}\big) &= \frac{1}{2} \int_{\mathbb{R}} (\rho(z) \operatorname{sgn}(x)|x|^{1/z})^{2k} e^{-t(\rho(z)\operatorname{sgn}(x)|x|^{1/z})^2}\, dx \\ &= \int_0^\infty (\rho(z)^2 x^{2/z})^k e^{-t\rho(z)^2 x^{2/z}}\, dx \\ &= \frac{z}{2}\rho(z)^{-z} t^{-k-z/2} \int_0^\infty y^{k+z/2-1} e^{-y}\, dy \\ &= \frac{z}{2}\, \pi^{z/2}\Gamma(z/2+1)^{-1} t^{-k-z/2}\Gamma(k + z/2) \\ &= \frac{z}{2}\, \pi^{z/2}\Gamma(z/2+1)^{-1} t^{-k-z/2}(k + z/2 - 1)(k + z/2 - 2)\cdots(z/2 + 1)\Gamma(z/2 + 1) \\ &= \pi^{z/2} t^{-k-z/2}\, \frac{z(z + 2)\cdots(z + 2k - 2)}{2^k}. \end{aligned} \tag{5.6}$$

We will now show that we can interchange the trace and the integral in Equation (5.5). The idea is clear, but it is some work since we have to check many things. Later we will refer to this result, so we will isolate it as a lemma.

Lemma 5.8. *We have the following equalities*

$$\tau'\Big(\int_0^\infty \big(Pe^{-tD^2}\big)\otimes\big(\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)\,t^{n-1}\,dt\Big) = \int_0^\infty \tau'\big(Pe^{-tD^2}\otimes\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)\,t^{n-1}\,dt \tag{5.7}$$

$$= \int_0^\infty \operatorname{Tr}\big(Pe^{-tD^2}\big)\tau\big(\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)\,t^{n-1}\,dt. \tag{5.8}$$

Proof. Equality (5.8) follows directly from the factorisation of the trace τ' (cf. Proposition 3.7). For the first equality we would like to apply Proposition 1.79, but we cannot do this directly since our measure space $([0,\infty),\mathcal{B}([0,\infty)),\mu)$ is not finite. Luckily it is σ-finite, we will exploit that fact later. First we check that the requirements of Proposition 1.79 are satisfied.
By construction $\tilde{\mathcal{N}}$ is a semifinite von Neumann algebra and τ' is a semifinite faithful normal trace. Define

$$f:[0,\infty)\to B(\tilde{\mathcal{H}}),\qquad f(t):=\big(Pe^{-tD^2}\big)\otimes\big(\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)t^{n-1}.$$

Since $P\in OP(\mathcal{A}_1,\mathcal{H}_1,D_1)$ the operator P is finite linear combination of products of elements $a\in\mathcal{A}$, D and γ. Therefore $f(t)$ is affiliated with $\tilde{\mathcal{N}}$. Observe that the commutators $[a,D]$ are bounded and D anti-commutes with γ. Therefore in these products we can pull all the terms D to the right and we can write $P=\sum_{i=0}^n b_iD^i$ for some bounded operators b_i and $l\in\mathbb{N}$. Since D is invertible

$$P=\sum_{i=0}^l b_iD^i=\sum_{i=0}^l b_iD^{i-l}D^l=TD^l$$

for some bounded operator T. Then

$$\begin{aligned}\|f(t)\| &= \|\big(TD^le^{-tD^2}\big)\otimes\big(\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)t^{n-1}\|\\ &\le \|T\|\,\|t^{(n-1)/2}D^le^{-tD^2}\|\,\|t^{(n-1)/2}\not{D}_z^{2k}e^{-t\not{D}_z^2}\|\le C<\infty,\end{aligned}$$

for some constant $C>0$ independent of t. Hence f maps into $\mathcal{N}$.
We will show $f(\cdot)h:[0,\infty)\to\tilde{\mathcal{N}}_z$ is a continuous function. Almost the same argument can be used for $f(\cdot)^*h$. We establish continuity if h is an elementary tensor $h=h_1\otimes h_2$. In general approximate an element $h\in\tilde{\mathcal{H}}$ by linear combinations of elementary tensors. Then

$$\begin{aligned}&\|\big(TD^le^{-(t_0+t)D^2}h_1\big)\otimes\big(\not{D}_z^{2k}e^{-(t_0+t)\not{D}_z^2}h_2\big)-\big(TD^le^{-(t_0)D^2}h_1\big)\otimes\big(\not{D}_z^{2k}e^{-(t_0)\not{D}_z^2}h_2\big)\|\\ &\le\|TD^le^{-t_0D^2}(1-e^{-tD^2})h_1\|\,\|\not{D}_z^{2k}e^{-(t_0+t)\not{D}_z^2}h_2\|\\ &\quad+\|TD^le^{-t_0D^2}h_1\|\,\|(\not{D}_z^{2k}e^{-t_0\not{D}_z^2}(1-e^{-t\not{D}_z^2})h_2\|,\end{aligned}$$

this converges to 0 as $t\to0$. Since $t\mapsto t^{n-1}$ is continuous, the function $f(\cdot)h$ is continuous. Note that a continuous function is measurable. Hence f is $*$-measurable. So

$$f\in\mathcal{L}_\infty^{so^*}([0,\infty),\mu;\mathcal{L}^1(\tilde{\mathcal{N}}_z,\tau')).$$

The trace τ' factorises so by (5.8)

$$\begin{aligned}\tau'(|f(t)|)&=\operatorname{Tr}\big(\big|Pe^{-tD^2}\big|\big)\tau\big(\big|\not{D}_z^{2k}e^{-t\not{D}_z^2}\big|\big)\,|t^{n-1}|\\ &=\pi^{z/2}\frac{z}{2}\,\frac{z+2}{2}\cdots\frac{z+2k-2}{2}\operatorname{Tr}\big(\big|t^{-k-z/2+n-1}TD^le^{-tD^2}\big|\big).\end{aligned}$$

This is clearly uniformly bounded for t large. For $t=0$ we have $\tau'(|f(0)|)=0$. By continuity it follows that $\tau'(|f(t)|)$ is uniformly bounded on $[0,\infty)$. Hence f is uniformly $\mathcal{L}^1(\tilde{\mathcal{N}},\tau')$-bounded. So we can apply Proposition 5.8 to the intervals $[0,n]$ which are finite measure spaces. We obtain

$$\tau'\Big(\int_0^n \big(Pe^{-tD^2}\big)\otimes\big(\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)\,t^{n-1}\,dt\Big)=\int_0^n\tau'\big(Pe^{-tD^2}\otimes\not{D}_z^{2k}e^{-t\not{D}_z^2}\big)\,t^{n-1}\,dt. \tag{5.9}$$

Suppose P is a positive operator then the operators involved in (5.9) are positive. By normality of the trace and the monotone convergence theorem we obtain (5.8) as we let $n \to \infty$. If P is not positive we can write P as a linear combination of four positive operators and then apply the previous. ⊠

We will continue our proof of Lemma 5.7. We plug in equation (5.6) in (5.8), as before we swap the integral and trace and we use again the identity (5.4) to obtain

$$\begin{aligned}
&\tau'\big((\gamma\otimes \not{D}_z)^{2k}(P\otimes 1)(D_z)^{-2n}\big) && (5.10)\\
&\quad= \frac{1}{\Gamma(n)}\int_0^\infty \operatorname{Tr}\big(Pe^{-tD^2}\big)\pi^{z/2}t^{n-k-z/2-1}\,\frac{z(z+2)\dots(z+2k-2)}{2^k}\,dt\\
&\quad= \frac{1}{\Gamma(n)}\,\pi^{z/2}\,\frac{z(z+2)\dots(z+2k-2)}{2^k}\int_0^\infty \operatorname{Tr}\big(Pe^{-tD^2}\big)t^{n-k-z/2-1}\,dt\\
&\quad= \frac{1}{\Gamma(n)}\,\pi^{z/2}\,\frac{z(z+2)\dots(z+2k-2)}{2^k}\,\operatorname{Tr}\Big(P\int_0^\infty e^{-tD^2}t^{n-k-z/2-1}\,dt\Big)\\
&\quad= \frac{1}{\Gamma(n)}\,\pi^{z/2}\,\frac{z}{2}\big(\frac{z}{2}+1\big)\dots\big(\frac{z}{2}+k-1\big)\Gamma\big(n-\frac{z}{2}-k\big)\operatorname{Tr}\big(P|D|^{z-2(n-k)}\big). && (5.11)
\end{aligned}$$

Observe

$$\lim_{z\to 0}\frac{1}{\Gamma(n)}\big(\frac{z}{2}+1\big)\dots\big(\frac{z}{2}+k-1\big)\,\Gamma\big(n-\frac{z}{2}-k\big)=\frac{\Gamma(k)\Gamma(n-k)}{\Gamma(n)}=B(k,n-k).$$

Thus by (5.11) we have

$$\begin{aligned}
\lim_{z\to 0}\tau'\big((\gamma\otimes \not{D}_z)^{2k}(P\otimes 1)(D_z)^{-2n}\big) &= \lim_{z\to 0}\frac{z}{2}\,B(k,n-k)\operatorname{Tr}\big(P|D|^{z-2(n-k)}\big)\\
&= \operatorname{res}_{z=0} -\frac{1}{2}\,B(k,n-k)\operatorname{Tr}\big(P|D|^{-z-2(n-k)}\big)\\
&= -\frac{1}{2}\,B(k,n-k)⨍ P|D|^{-2(n-k)},
\end{aligned}$$

as desired. ⊠

Using this lemma and the theory of the generalised pseudo-differential operators (the classes OP^α) we described in Subsection 2.3, we are able to prove the result which we are after.

Definition 5.9. Define the *gauge potentials* of a spectral triple $(\mathcal{A},\mathcal{H},D)$ by

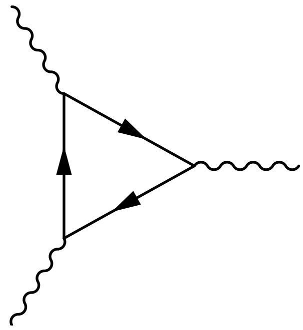
$$\Omega^1_D(\mathcal{A}) := \Big\{\sum_j a_j[D,b_j]\ :\ a_j,b_j\in\mathcal{A}\Big\}.$$

Remark 5.10. Since $\mathcal{A}\otimes\mathbb{C}\cong\mathcal{A}$ as algebras it is no surprise that $\Omega^1_{D_z}(\mathcal{A}\otimes\mathbb{C})=\Omega^1_D(\mathcal{A})\otimes\mathbb{C}$. Indeed, since γ commutes with $\mathcal{A}$ we obtain

$$(a\otimes 1)[D'_z,b\otimes 1]=(a\otimes 1)[D\otimes 1+\gamma\otimes D_z,b\otimes 1]=(a\otimes 1)([D,b]\otimes 1+\gamma b[D_z,1])=a[D,b]\otimes 1,$$

which gives the result. So from now on we identify $\Omega^1_D(\mathcal{A})\cong\Omega^1_{D_z}(\mathcal{A}\otimes\mathbb{C})$.

Now we can prove the key result of this subsection. We can compute one-loop graphs of the following form:



The result for $n = 3$ of the proposition stated below corresponds to this graph.

Proposition 5.11. *Let $A \in \Omega^1_D(\mathcal{A})$ and $n \in \mathbb{N}$, $n > 0$. Then the function*

$$z \mapsto \tau'\big(((A \otimes 1)D_z^{-1})^n\big)$$

has at most a simple pole at $z = 0$, with residue given by

$$\mathrm{res}_{z=0}\, \tau'\big(((A \otimes 1)D_z^{-1})^n\big) = -\!\!\!\!\!\!\int (AD^{-1})^n.$$

Proof. We start by computing $((A \otimes 1)D_z^{-1})^n)$. For this we will use the identity $D_z^{-1} = D_z D_z^{-2}$ and Lemma 2.47. Will we proceed by induction to show the following result. If $k > 0$, then $((A \otimes 1)D_z^{-1})^n$ can be written as

$$((A \otimes 1)D_z^{-1})^n = \sum_{\alpha} c_\alpha (A \otimes 1)D_z(\nabla_D^{\alpha_1}(A) \otimes 1)D_z \dots (\nabla_D^{\alpha_{n-1}}(A) \otimes 1)D_z D_z^{-2\alpha_n} + R, \quad (5.12)$$

with $R \in OP^{-k}$. Here α is some multiindex and $c_\alpha \in \mathbb{Z}$ are some combinatorial constants to count the multiplicities.
The case $n = 1$ is trivial. Now fix k. Observe that since $\not{D}_z$ commutes with 1 and $D_z^2 = D^2 \otimes 1 + 1 \otimes \not{D}_z^2$ it holds that

$$\nabla_{D_z}(A \otimes 1) = \nabla_D(A) \otimes 1.$$

Repeated use of Lemma 2.47 gives

$$\begin{aligned} D_z^{-2n}(A \otimes 1) &= D_z^{-2n+2} \sum_{k_1=1}^{m_1} (-1)^{k_1-1}(\nabla_D^{k_1-1}(A) \otimes 1)D_z^{-2k_1} + D_z^{-2n+2}R_{m_1} \\ &= \sum_{k_n=1}^{m_n} \cdots \sum_{k_1=1}^{m_1} (-1)^{k_1+\ldots+k_n-n}(\nabla_D^{k_1+\ldots+k_n-n}(A) \otimes 1)D_z^{-2(k_1+\ldots+k_n)} + D_z^{-2n+2}R_{m_1} \\ &\quad + \sum_{k_1=1}^{m_1} D_z^{-2n+4}R_{m_2}D_z^{-2k_1} + \ldots + \sum_{k_{n-1}=1}^{m_{n-1}} \cdots \sum_{k_1=1}^{m_1} R_{m_{n-1}}D_z^{-2(k_1+\ldots+k_{n-1})} \end{aligned}$$

Observe that the error terms $R_{m_i} \in OP^{-m_i-1}$, thus

$$D_z^{-2n+2}R_{m_1} \in OP^{-m_1-2n+1}$$

$$\sum_{k_1=1}^{m_1} D_z^{-2n+4}R_{m_2}D_z^{-2k_1} \in OP^{-m_2-2n+1}$$

$$\sum_{k_{n-1}=1}^{m_{n-1}} \cdots \sum_{k_1=1}^{m_1} R_m D_z^{-2(k_1+\ldots+k_{n-1})} \in OP^{-m_{n-1}-2n+1}.$$

Hence

$$D_z^{-2n}(A\otimes 1)=\sum_\beta c_\beta(\nabla_D^\beta(A)\otimes 1)D_z^{-2\beta}+R, \qquad \text{with } R\in OP^{-2n+1-\min\{m_1,\dots,m_{n-1}\}}. \tag{5.13}$$

Select m_j such that $-n-\min\{m_1,\dots,m_{n-1}\}<-k$. We will use the Expression (5.13) to prove the induction step

$$\begin{aligned}
\big((A\otimes 1)D_z^{-1}\big)^{n+1} &= \big((A\otimes 1)D_z^{-1}\big)^n(A\otimes 1)D_z^{-1}\\
&=\Big(\sum_\alpha c_\alpha(A\otimes 1)D_z(\nabla_D^{\alpha_1}(A)\otimes 1)D_z\dots(\nabla_D^{\alpha_{n-1}}(A)\otimes 1)D_zD_z^{-2\alpha_n}+R_1\Big)(A\otimes 1)D_zD_z^{-2}\\
&=\sum_\xi d_\xi(A\otimes 1)D_z(\nabla_D^{\xi_1}(A)\otimes 1)D_z\dots(\nabla_D^{\xi_{n-1}}(A)\otimes 1)D_z(\nabla_D^{\xi_n}(A)\otimes 1)D_zD_z^{-2\xi_{n+1}}\\
&\qquad+\Big(\sum_\alpha c_\alpha(A\otimes 1)D_z(\nabla_D^{\alpha_1}(A)\otimes 1)D_z\dots(\nabla_D^{\alpha_{n-1}}(A)\otimes 1)D_z\Big)R_2D_z^{-2}+R_1(A\otimes 1)D_zD_z^{-2}\\
&=\sum_\xi d_\xi(A\otimes 1)D_z(\nabla_D^{\xi_1}(A)\otimes 1)D_z\dots(\nabla_D^{\xi_{n-1}}(A)\otimes 1)D_z(\nabla_D^{\xi_n}(A)\otimes 1)D_zD_z^{-2\xi_{n+1}}+R_3
\end{aligned}$$

Since $R_1\in OP^{-(n+1)}$ and $R_2\in OP^{-2n+1-\min\{m_1,\dots,m_{n-1}\}}$, the operator $R_3\in OP^{-k}$. This closes the induction.
The next step is to take care of each of the summands of (5.12). We split up $D_z=D_1\otimes 1+\gamma\otimes \not{D}_z$. Since $\gamma\otimes\not{D}_z$ anticommutes with the terms $A\otimes 1$ and $\nabla_D^{\alpha_i}(A)\otimes 1$, in a summand of (5.12) we can pull all the terms $\gamma\otimes\not{D}_z$ to the left and write such a summand as

$$c_\alpha(A\otimes 1)D_z(\nabla_D^{\alpha_1}(A)\otimes 1)\dots(\nabla_D^{\alpha_{n-1}}(A)\otimes 1)D_zD_z^{-2\alpha_n}=\sum_{j=0}^n\sum_{\beta_j}d_{\beta_j}(\gamma\otimes\not{D}_z)^j(A\otimes 1)P_{\beta_j}D_z^{-2\alpha_n}.$$

Here β_j are again a multiindices, the constants $d_{\beta_j}\in\mathbb{Z}$ count the multiplicity and P_{β_j} are certain products of the elements $\nabla_D^{\alpha_i}(A)\otimes 1$ and $D\otimes 1$, in particular $(A\otimes 1)P_{\beta_j}=Q_{\beta_j}\otimes 1$ for an element $Q_{\beta_j}\in OP(\mathcal{A},\mathcal{H},D;\gamma)$.

Consider an operator

$$(\gamma\otimes\not{D}_z)^j(Q_{\beta_j}\otimes 1)D_z^{-2\alpha_n}=\gamma^jQ_{\beta_j}\otimes\not{D}_z^j(D_1^2\otimes 1+1\otimes\not{D}_z^2)^{-\alpha_n}.$$

Note that $\not{D}_z$ was given by multiplication with the function $f_z:\mathbb{R}\to\mathbb{R}$, which is odd i.e. $f_z(-x)=-f_z(x)$. So if j is odd, then f_z^j is an odd function. In particular $\tau_z(\not{D}_z^j)=\int_\mathbb{R}f_z^j(x)\,dx=0$. Note that 1 and $\not{D}_z^2$ are even, so $(D_1^2\otimes 1+1\otimes\not{D}_z^2)^{-\alpha_n}$ is even. Since $\tau_z=\mathrm{Tr}\otimes(\int_\mathbb{R}\cdot\,dx)$, it holds that

$$\tau'(\gamma^jQ_{\beta_j}\otimes\not{D}_z^j(D_1^2\otimes 1+1\otimes\not{D}_z^2)^{-\alpha_n}=0 \qquad \text{for } j \text{ odd}.$$

Because $(\gamma\otimes\not{D}_z)^j(Q_{\beta_j}\otimes 1)D_z^{-2\alpha_n}\in OP^{-k}$ is bounded, Lemma 5.7 shows that the function

$$z\mapsto\tau'\big((\gamma\otimes\not{D}_z)^j(Q_{\beta_j}\otimes 1)D_z^{-2\alpha_n}\big) \tag{5.14}$$

does not have a pole at $z=0$ if $j>0$ is even. Since the functions given by (5.14) does not have a poles at $z=0$ for $j>0$ we only have to compute the residues at $z=0$ of the function

$$z\mapsto\tau'\big((Q\otimes 1)D_z^{-2a}\big), \tag{5.15}$$

for $Q\in OP(\mathcal{A},\mathcal{H},D;\gamma)$. For this we will use the same method as in the proof of Lemma 5.8. In a similar fashion as before one can check that the conditions of Proposition 1.79 are satisfied.

$$\begin{aligned}
\tau'\big((Q\otimes 1)D_z^{-2a}\big) &= \tau'\Big(\frac{1}{\Gamma(a)}\int_0^\infty (Q\otimes 1)e^{-t(D^2\otimes 1)}e^{-t(1\otimes \not{D}_z^2)}t^{a-1}\,dt\Big) \\
&= \frac{1}{\Gamma(a)}\int_0^\infty \tau'\big(Qe^{-tD^2}\otimes e^{-t\not{D}_z^2}t^{a-1}\big)\,dt \\
&= \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{Tr}\,\big(Qe^{-tD^2}\big)\tau\big(e^{-t\not{D}_z^2}\big)t^{a-1}\,dt \\
&= \frac{\pi^{z/2}}{\Gamma(a)}\,\mathrm{Tr}\,\Big(Q\int_0^\infty e^{-tD^2}t^{a-z/2-1}\,dt\Big) \\
&= \pi^{z/2}\,\frac{\Gamma(a-z/2)}{\Gamma(a)}\,\mathrm{Tr}\,\big(Q(D^2)^{z/2-a}\big) \\
&= \pi^{z/2}\,\frac{\Gamma(a-z/2)}{\Gamma(a)}\,\mathrm{Tr}\,\big(QD^{-2a}|D|^z\big)
\end{aligned}$$

Therefore we obtain

$$\mathrm{res}_{z=0}\,\big(z\mapsto \tau'\big((Q\otimes 1)D_z^{-2a}\big) = \mathrm{res}_{z=0} - \mathrm{Tr}\,\big(QD^{-2a}|D|^{-z}\big) = -\!\!\!\!\!\!\int QD^{-2a}. \tag{5.16}$$

If we use exactly the same powers and coefficients in the expansion of $(AD^{-1})^n$ as we did for $((A\otimes 1)D_z^{-1})^n$ (cf. Equation (5.12)), then

$$(AD^{-1})^n = \sum_\alpha c_\alpha AD\nabla_D^{\alpha_1}(A)D\ldots\nabla_D^{\alpha_{n-1}}(A)DD^{-2\alpha_n} + R', \tag{5.17}$$

with $R'\in OP^{-k}(\mathcal{A},\mathcal{H},D)$. These terms correspond 1-1 with the terms $(Q_{\beta_j}\otimes 1)D_z^{-2\alpha_n}$ of (5.15) which might have a pole at $z=0$ (i.e. where $j=0$). By (5.16) for a summand of the expansion (5.17) we have

$$\mathrm{res}_{z=0}\,\tau'\big(((c_\alpha AD\ldots\nabla_D^{\alpha_{n-1}}(A)D)\otimes 1)D_z^{-2\alpha_n} = -\!\!\!\!\!\!\int c_\alpha AD\ldots\nabla_D^{\alpha_{n-1}}(A)DD^{-2\alpha_n}.$$

Summing over α gives the result we are after. ⊠

5.3 Zeta function regularisation

In this subsection we will show that it is also possible to use semifinite spectral triples for zeta function regularisation [20].
In zeta function regularisation on typically has to deal with integrals of the form

$$\int_0^\infty \frac{1}{t}e^{-t}\,dt.$$

For zeta function regularisation one replaces $\frac{1}{t}$ by t^{-1+s} for some $s\in\mathbb{C}$, $\mathrm{Re}(s)>0$ and one investigates what happens if $s\to 0$. For example the integral $\int_0^\infty t^{-1}e^{-t}\,dt$ is divergent, but by definition of the gamma function it holds that

$$\int_0^\infty t^{-1+s}e^{-t}\,dt = \Gamma(s),$$

which has a simple pole at $s=0$ with residue 1. Note that this result is very similar to dimensional regularisation. In both cases we introduce a new parameter in the integral and then examine

what happens if that parameter is sent to 0. In both cases a pole of the gamma function appears which describes the divergence of the integral.
The aim of this subsection is to give a reasonable definition of $\det((D+A)D^{-1})$ for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a gauge potential $A \in \Omega^1_D(\mathcal{A})$. For trace-class operators it is possible to define a so called Fredholm determinant, but we choose a different way to give meaning to a determinant. We will start with an informal motivation. Recall from linear algebra the identity $\det(\exp(T)) = \exp(\mathrm{Tr}(T))$ for any square matrix T. Thus using functional calculus we have $\log(\det(T)) = \mathrm{Tr}(\log(T))$. Then

$$\log\big(\det((D+A)D^{-1})\big) = \mathrm{Tr}\big(\log((D+A)D^{-1})\big) = \mathrm{Tr}\big(\log(1+AD^{-1})\big).$$

Therefore we are interested in computing $\mathrm{Tr}\big(\log(1+AD^{-1})\big)$. However, this is not finite, because the operators considered are not trace class. So we would like to give a different meaning to it. We want to introduce a new parameter and examine in what way the result diverges as a function of this parameter. One way is to consider

$$s \mapsto \mathrm{Tr}\big(\log(1+AD^{-1})|D|^{-s}\big)$$

and compute the residue at $s = 0$. This is what Connes and Chamseddine have done, see Theorem 5.13 below. But we can also use the triple $\mathcal{T}_z$, to compute

$$z \mapsto \tau'\big(\log(1+AD_z^{-1})\big).$$

The result appears to be the same, see Theorem 5.15.

Notation 5.12. If D is a self-adjoint operator, denote $\zeta_D(s) := \mathrm{Tr}(|D|^{-s})$.

To prove our main theorem of this section, we will need the following result.

Theorem 5.13 (Connes & Chamseddine [8])**.** *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a finitely summable, regular spectral triple and $A \in \Omega^1_D(\mathcal{A})$ is a self-adjoint gauge potential, then*

(i) the function ζ_{D+A} extends to a meromorphic function with a most simple poles;

(ii) the function ζ_{D+A} is regular at $s = 0$;

(iii) the following equality holds

$$\zeta_{D+A}(0) - \zeta_D(0) = -⨍ \log\big(1+AD^{-1}\big) = \sum_n \frac{(-1)^n}{n} ⨍ \big(AD^{-1}\big)^n.$$

Proof. See [8, Thm. 2.4] ⊠

Remark 5.14. The above use of $\log(1+AD^{-1})$ needs some comments. Namely, $1+AD^{-1}$ need not to be positive, hence $\log(1+AD^{-1})$ is undefined. What we have is

$$\log(1+x) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n, \text{ for } |x| < 1.$$

Note that for n large due to summability of the spectral triple the residue $⨍(AD^{-1})^n = 0$. So the series $-\sum_n \frac{(-1)^n}{n} ⨍(AD^{-1})^n$ is in fact a finite sum. And this sum is well defined for all gauge potentials A and Dirac operators D. So we use $-\sum_n \frac{(-1)^n}{n} ⨍(AD^{-1})^n$ as a definition for $⨍\log(1+AD^{-1})$. Therefore $-\sum_n \frac{(-1)^n}{n} ⨍(AD^{-1})^n$ gives a meaning to the determinant of $(D+A)D^{-1}$.
We know that if D is invertible and if $A \in \Omega^1_D(\mathcal{A})$ is a gauge potential, then A is bounded. So there exists a constant $a > 0$ such that $\|AD^{-1}\| < a$. For such an a,

$$\log\big(1+\frac{1}{a}AD^{-1}\big)$$

is a well-defined operator and of course both interpretations in 5.13 are equal.

Theorem 5.15. *Suppose* $(\mathcal{A}, \mathcal{H}, D; \gamma)$ *is an even* p^+*-summable regular spectral triple,* $A \in \Omega^1_D(\mathcal{A})$ *is a self-adjoint gauge potential and* D *is invertible. Then*

$$\zeta_{D+A}(0) - \zeta_D(0) = \sum_n \frac{(-1)^n}{n} ⨍ \big(AD^{-1}\big)^n = \sum_n \frac{(-1)^n}{n} \operatorname{res}_{z=0} \tau'\big((AD_z^{-1})^n\big). \tag{5.18}$$

Proof. The proof is now a simple combination of the results we obtained in this section. The operator D_z is invertible because of Lemma 5.6. The first equality of (5.18) is given by Theorem 5.13 and the second equality follows from Proposition 5.11. ⊠

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