The structure of perturbative quantum gauge theories

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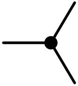
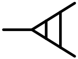
What Feynman graphs are...

Graphs built from a fixed set $\{v_1, \ldots, v_k\}$ of types of vertices and a fixed set $\{e_1, \ldots, e_N\}$ of types of edges.

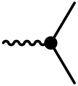
Examples:

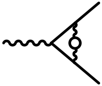
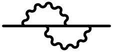
- Scalar ϕ^3-theory:

vertex:  , edge: .

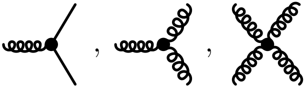
and one constructs graphs such as  , 

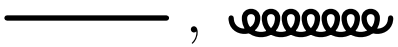
- Quantum electrodynamics:

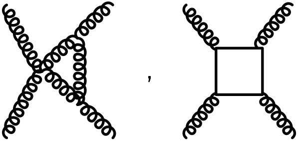
vertex:  , edges: , .

and one constructs graphs such as  , 

- Quantum chromodynamics:

vertices:  , …

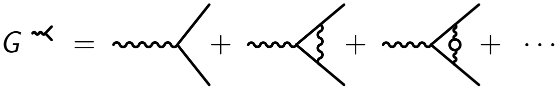
edges:  , …

and one constructs graphs such as 

Perturbative quantum gauge theory

Idea: probability amplitudes for physical processes are given by expansions in Feynman graphs.

- **Example**: interaction of photon with electron (QED)



- A physicist is interested in numbers, and the **Feynman rules** associate a complex number to a Feynman graph Γ

$$\Gamma \mapsto U(\Gamma) \in \mathbb{C}$$

- However, these numbers are typically infinite... $\rightsquigarrow$ need to **renormalize**

Idea of renormalization

1. **Regularization**: introduce a parameter $z \in \mathbb{C}$ and define new Feynman rules U_z:

$$\Gamma \mapsto U_z(\Gamma) \in \mathbb{C}$$

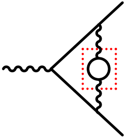
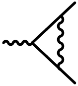
   The previous infinity becomes a **pole at** $z = 0$ of the Laurent series expansion in z.

2. **Subtraction**: get rid of the whole pole part of the Laurent series expansion: this gives the renormalized amplitude

$$\Gamma \mapsto R_z(\Gamma) \in \mathbb{C}$$

   This applies to any Feynman graph, and in particular to subgraphs of Feynman graphs:

For a generic graph Γ: $R_z(\Gamma)$ defined by a recursive procedure

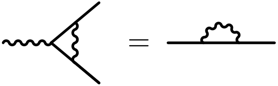
Example: renormalization of  involves  and 

Quantum gauge symmetries

- Gauge field theories possess a gauge symmetry: this forms the infinite dimensional **gauge group**.
- Mathematically, the gauge group can be understood as sections of the bundle of groups $P \times_G G$ associated to a principal G-bundle P (on which the gauge field is a connection).
- After a successful (perturbative) quantization of gauge field theories, the gauge symmetry is lost, but **quantum gauge symmetries** appear:

  There are certain identities between Feynman graphs

  Example: in quantum electrodynamics we have linear Ward identities, eg.



  Instead, in quantum chromodynamics, the (Slavnov–Taylor) identities are *quadratic* in the Feynman graphs.

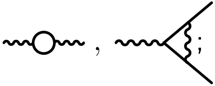
Mathematical structure of renormalization

Group of 'Feynman rules'

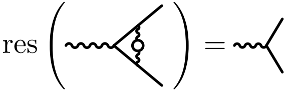
It turns out that the collection of all Feynman rules constitute a group.

We start by considering the Feynman rules $\Gamma \mapsto U(\Gamma) \in \mathbb{C}$ as characters on the **free commutative algebra H generated by all 1PI Feynman graphs with residue in** $\{v_1, \ldots, v_k\} \cup \{e_1, \ldots, e_N\}$:

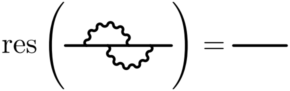
- One-particle irreducible graphs:

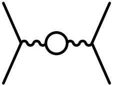
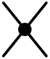
1PI:  ; and **not 1PI (1PR)**: ,

- Residue of a graph:



and



Example of a graph not allowed:  since 1PR and residue  $\neq v_i$

Group structure on characters of H

- Unit $\epsilon \in G := \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ is understood as a **co**unit $\epsilon : H \to \mathbb{C}$.
- Multiplication $* : G \times G \to G$ induced by a **co**product $\Delta : H \to H \otimes H$
- Inverse induced by the antipode $S : H \to H$.

Theorem (Connes–Kreimer, 2000)

There exists a counit, coproduct and antipode on the algebra H of Feynman graphs, turning H into a Hopf algebra (and G a group). The counit is

$$\epsilon(\Gamma) = \begin{cases} 1 & \text{if } \Gamma = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

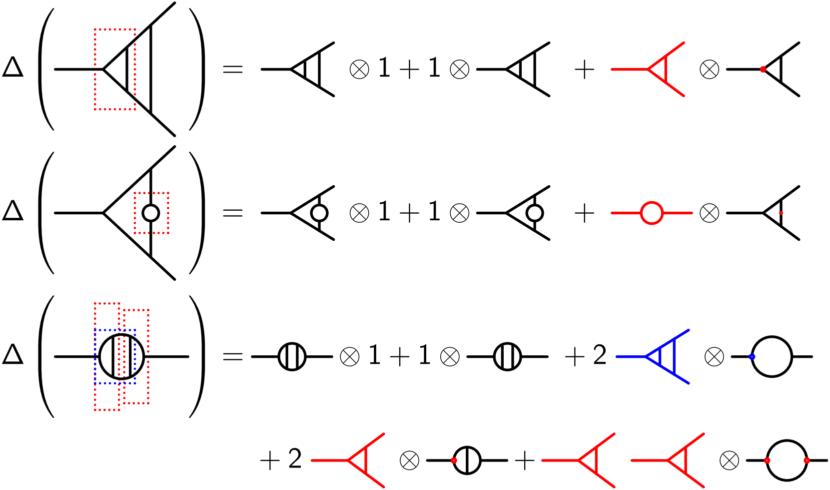
and the coproduct is defined by

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over (disjoint unions of) 1PI subgraphs with residue v_i or e_j.

Examples of the coproduct with $v =$ ⤙ and $e =$ —

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$

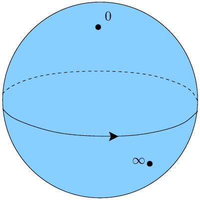


Renormalization as a decomposition in G

- The above Hopf algebra H is the algebraic structure underlying the recursive procedure of renormalization.
- In fact, for a character $U_z : H \to \mathbb{C}$, there exists a character $C_z : H \to \mathbb{C}$ ('counterterm') defined for $z \neq 0$, such that

$$\boxed{R_z = C_z * U_z}$$

  is finite at $z = 0$ [Connes and Kreimer, 2000].
- This decomposition is unique if one requires $C_{z=\infty} = \epsilon$ and can be interpreted as a Birkhoff decomposition in the group $G = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$:



$$U_z = C_z^{-1} * R_z$$

Structure of the Hopf algebra of Feynman graphs

- Grading by loop number $L(\Gamma) = h^1(\Gamma)$:

$$H = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^l, \qquad q_l : H \to H^l$$

- Multigrading by number of vertices:

$$d_i(\Gamma) = \#\text{vertices } v_i \text{ in } \Gamma - \delta_{v_i,\mathrm{res}(\Gamma)}$$

with

$$H = \bigoplus_{n_1,\ldots,n_k \in \mathbb{Z}^k} H^{n_1,\ldots,n_k}, \qquad p_{n_1,\ldots,n_k} : H \to H^{n_1,\ldots,n_k}$$

Physical interest: Green's functions

- For each vertex and edge we define Green's functions

$$G^{v_i} = 1 + \sum_{\mathrm{res}(\Gamma)=v_i} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|}, \qquad G^{e_j} = 1 - \sum_{\mathrm{res}(\Gamma)=e_j} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

  corresponding to the physical probability amplitudes.

Problem: Write the coproduct on these Green's functions

For example, for scalar ϕ^3-theory (with one type of vertex $v =$  and one type of edge $e =$ —) we have

Proposition

The elements $X = G^v(G^e)^{-3/2}$ *and* G^e *generate a Hopf subalgebra in* H*:*

$$\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X), \qquad \Delta(G^e) = \sum_{l=0}^{\infty} G^e X^{2l} \otimes q_l(G^e)$$

Hopf subalgebras and ideals

- In general (vertices $\{v_1, \ldots, v_k\}$ and edges $\{e_1, \ldots, e_N\}$), we define

$$X_{v_i} := \left(\frac{G^{v_i}}{\prod_j \left(G^{e_j}\right)^{\mathrm{val}_j(v_i)/2}} \right)^{1/\mathrm{val}(v_i)-2}$$

Theorem (vS, 2008)

1. *The elements X_{v_i} and G^{e_j} generate a Hopf subalgebra H' in H with dual group*

$$G := \mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \left(\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]^\times\right)^N \rtimes \overline{\mathbf{Diff}}(\mathbb{C}^k)$$

2. *The ideal $J := \langle X_{v_i} - X_{v_j} \rangle$ in H' is a Hopf ideal, i.e. H'/J is a Hopf algebra with dual group*

$$\mathrm{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset \left(\mathbb{C}[[\lambda]]^\times\right)^N \rtimes \overline{\mathbf{Diff}}(\mathbb{C})$$

Can we explain the existence of these Hopf ideals from the classical gauge symmetry?

Comodule Gerstenhaber algebras

We now establish a connection between the Hopf algebra of renormalization and a Gerstenhaber structure in the context of gauge theories

- In general, we assign to each vertex v_i a parameter λ_i for $i = 1, \ldots, k$.
- To each edge e_j we assign a field ϕ_j and a corresponding antifield $\phi_j^{\ddagger}$ for $j = 1, \ldots, N$ (with certain degrees); the anti-bracket is defined by

$$\left(\phi_i(x), \phi_j^{\ddagger}(y)\right) = \delta_{ij}\delta(x - y).$$

  and zero otherwise.
- This makes the following a **Gerstenhaber algebra**:

$$\boxed{A := \mathcal{F}([\phi_1, \phi_1^{\ddagger}, \ldots, \phi_N, \phi_N^{\ddagger}]) \otimes \mathbb{C}[[\lambda_1, \cdots, \lambda_k]]}$$

  i.e. **a graded algebra with a Lie bracket of degree 1**.

The algebra $A = \mathcal{F}([\phi_1, \phi_1^\ddagger, \ldots, \phi_N, \phi_N^\ddagger]) \otimes \mathbb{C}[[\lambda_1, \cdots, \lambda_k]]$ consists of $\mathbb{C}[[\lambda_1, \cdots, \lambda_k]]$-linear functionals in the fields.

Proposition (vS, 2008)

The algebra A is a Gerstenhaber comodule algebra over H'. In other words, there exists a map $\rho : A \to A \otimes H'$ compatible with the coproduct on H' and respecting the bracket in A.

Consequently, there is an action of $G \subset \left(\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]^\times\right)^N \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^k)$ on A.

For instance, we have

$$\rho : \phi_j \longmapsto \sum_{n_1 \cdots n_k} \phi_j \lambda_1^{n_1} \cdots \lambda_k^{n_k} \otimes p_{n_1 \cdots n_k}\left((G^{e_j})^{1/2}\right) \qquad \text{(invertible series)}$$

$$\rho : \lambda_i \longmapsto \sum_{n_1 \cdots n_k} \lambda_i \lambda_1^{n_1} \cdots \lambda_k^{n_k} \otimes p_{n_1 \cdots n_k}\left((X_v)^{\mathrm{val}(v_i)-2}\right) \qquad \text{(formal diffeos)},$$

where we recall $X_{v_i} = \left(\frac{G^{v_i}}{\prod_j (G^{e_j})^{\mathrm{val}_j(v_i)/2}}\right)^{1/\mathrm{val}(v_i)-2} \in H'$.

Master equation

- Next, one considers an element $S \in A$ (the *action*) of the following form

$$\boxed{S = \sum_{j=1}^{N} \int dx \;\; m(e_j)(x) + \sum_{i=1}^{k} \lambda_i \int dx \;\; m(v_i)(x)}$$

  with $m(e_j), m(v_i)$ monomials in the fields that interact/propagate at e_j, v_i, resp..
- The ideal $I = \langle (S, S) \rangle$ implements the 'master equation' $(S, S) = 0$ and

$$I = \langle p_\alpha(\lambda_1, \ldots, \lambda_k) \rangle_\alpha, \qquad p_\alpha \text{ polynomals}$$

- A theory (defined by S) is called simple if $I = \langle \lambda_i - \lambda^{\mathrm{val}(v_i)-2} \rangle_i$ with $\lambda := \lambda_j$ corresponding to some fixed valence 3 vertex v_j.

Proposition (vS, 2008)

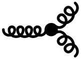
If S defines a simple theory, then the subgroup $G^I \subset G$ leaving I invariant is dual to H'/J (with $J = \langle X_{v_i} - X_{v_j} \rangle_{i,j}$), i.e.

$$G^I \simeq \mathrm{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset (\mathbb{C}[[\lambda]]^\times)^N \rtimes \overline{\mathrm{Diff}}(\mathbb{C}).$$

Application to Yang–Mills theory

- The action is (essentially) the Yang–Mills action for a connection one-form ω (with simple gauge group):

$$S = \|F(\omega)\|^2 = \int F^a_{\mu\nu} F^{\mu\nu}_a, \qquad \text{with } F = d\omega + \tfrac{1}{2}\omega \wedge \omega.$$

- Feynman graphs are built from $v_3 =$  and $v_4 =$ 
- The corresponding Hopf algebra H coacts on the Gerstenhaber algebra A of $\mathbb{C}[[\lambda_3, \lambda_4]]$-linear functionals in $\omega, \omega^\ddagger, \ldots$, and

$$\boxed{\mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^2)}$$

- Gauge symmetry $\rightsquigarrow$ master equation $(S, S) = 0 \iff \boxed{\lambda_4 - \lambda_3^2 = 0}$
- The subgroup G^I of $\mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^2)$ that leaves this equation invariant is dual to the Hopf algebra H'/J with

$$J = \langle X_{\text{3g}} - X_{\text{4g}} \rangle$$

so that $G^I \subset \mathbb{C}[[\lambda]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C})$ identifying $\lambda_4 = \lambda_3^2 \equiv \lambda^2$.

- Thus, in H'/J the identities $X_{\text{3g}} = X_{\text{4g}}$ hold, or, explicitly

$$\boxed{G^{\text{4g}} = \frac{(G^{\text{3g}})^2}{G^{\text{gg}}}}$$

These "quantum gauge symmetries" are known in physics as the Slavnov–Taylor identities for the coupling constants.

- Their appearance as generators of a Hopf ideal proves that the ST-identities are compatible with renormalization.
- In fact, if U_z satisfies ST-identities, it follows that R_z, C_z do so as well:

the renormalized and counterterm maps satisfy ST-identities

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