The Hopf algebra of Feynman graphs in QED

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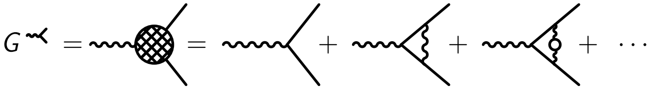
Outline

1. Perturbative quantum field theory; Feynman rules; Renormalization
2. Gauge theories (QED); Ward identities
3. Hopf algebra of Feynman graphs; Birkhoff decomposition
4. Ward identities and the Hopf algebra structure

Perturbative quantum field theory

- Probability amplitudes for physical processes are expressed as sums of Feynman diagrams (Green's functions)
  - **Example**: interaction of photon with electron (QED)

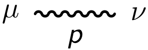


- Feynman rules associate integrals to graphs; dictated by a *Lagrangian*; for QED:

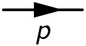
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\psi}\,(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\,\psi,$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

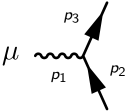
Feynman rules



$$D_{\mu\nu}(p) = -\frac{\delta_{\mu\nu}}{p^2+i\epsilon} + \frac{p_\mu p_\nu}{(p^2+i\epsilon)^2}(1-\xi)$$

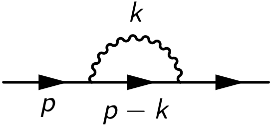


$$S(p) = \frac{1}{\gamma^\mu p_\mu + m}$$



$$e\gamma^\mu \delta(p_1+p_2+p_3)$$

Example: Electron self-energy graph: $\Gamma =$



Feynman amplitude: $U_\Gamma(p) = \int d^4k \; (e\gamma^\mu) S(p-k)(e\gamma^\nu) D_{\mu\nu}(k)$

$$\sim \int d^4k \frac{1}{(k^2+i\epsilon)((p-k)^2+m^2)}$$ *which typically diverges*

Renormalization

1. **Regularization**: dim-reg in $4 - z$ dimensions ($z \in \mathbb{C}$) while assuming that the following holds also for $D \in \mathbb{C}$:

$$\int e^{-\lambda k^2} d^D k = \pi^{D/2} \lambda^{-D/2}$$

   - In the previous example, we obtain the *regularized* Feynman amplitude by integrating in $4 - z$ dimensions:

$$U_\Gamma(p)(z) \sim \Gamma\left(\frac{z}{2}\right) \mathrm{Pol}(p)$$

   with poles at $z = 0, -2, \ldots$

2. **Renormalization**: subtract divergent part,

$$R_\Gamma = U_\Gamma(z) - T\left(U_\Gamma(z)\right)\Big|_{z=0}$$

with T the projection on pole part of the Laurent series in z.

BPHZ subtraction scheme

In general, graphs contain *subgraphs* corresponding to *subdivergences* in the Feynman amplitudes.

The **BPHZ subtraction scheme** gives a recursive procedure to subtract these subdivergences.

- Preparation:

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

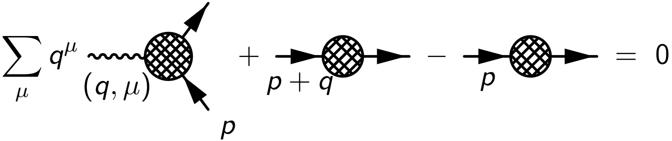
  with $C(\gamma) = -T\left(\overline{R}(\gamma)\right)$ the counterterm for the subgraph γ.
- Renormalized Feynman amplitude:

$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma).$$

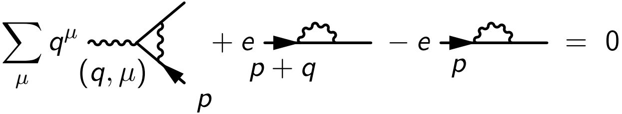
Gauge theories

Physical processes are described by **gauge theories**: symmetry with respect to a gauge group, eg. $U(1)$,SU(2), ...

- Symmetry manifests itself as **Ward identities** between different Green's functions; in QED (gauge group $U(1)$):



- In QED, there are also (Ward-Takahashi) identities between individual amplitudes. Eg.,



The Ward (-Takahashi) identities are compatible with renormalization:

- they are satisfied by the unrenormalized and renormalized amplitudes
- and by the counterterms: $Z_1 = Z_2$ [Ward, 1950] where

$$Z_1 = 1 + \sum_{\Gamma=} \frac{C(\Gamma)}{\mathrm{Sym}(\Gamma)}$$

$$Z_2 = 1 - \sum_{\Gamma=} \frac{C(\Gamma)}{\mathrm{Sym}(\Gamma)}$$

Hopf algebra of Feynman graphs

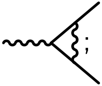
Definition (A. Connes and M. Marcolli)

A Feynman graph *Γ is given by a set $\Gamma^{[0]}$ of vertices and $\Gamma^{[1]}$ of edges, and maps*

$$\partial_j : \Gamma^{[1]} \to \Gamma^{[0]} \cup \{1, 2, \dots, N\}, \qquad j = 0, 1$$

The set $\{1, \dots, N\}$ labels external lines, $\Gamma^{[1]}_{\mathrm{ext}} = \cup_j \partial_j^{-1}\{1, \dots, N\}$.

- When considering a particular physical theory, we allow only certain types of vertices and edges.
  - For example, in QED: one vertex and two edges and .
- We restrict to one-particle irreducible (1PI) graphs: Feynman graphs that are not trees and cannot be disconnected by removal of a single internal edge:

1PI:  , ; and **not 1PI (1PR)**: ,

Definition

An automorphism of a Feynman graph Γ *is given by an isomorphism* $g^{[0]}$ *from* $\Gamma^{[0]}$ *to itself, and an isomorphism* $g^{[1]}$ *from* $\Gamma^{[1]}$ *to itself that is the identity on* $\Gamma^{[1]}_{\mathrm{ext}}$ *and s.t.* $\forall e \in \Gamma^{[1]}$,

$$\cup_j g^{[0]}(\partial_j(e)) = \cup_j \partial_j(g^{[1]}(e))$$

The order of the automorphism group is called the symmetry factor of Γ *and is denoted by* $\mathrm{Sym}(\Gamma)$.

Similarly, there is a notion of an isomorphism of two graphs Γ and Γ' as a pair of maps that intertwines the maps ∂_j and ∂'_j.

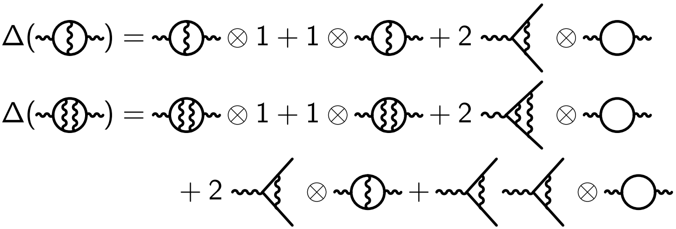
N.B. This definition of isomorphism corrects for the apparent orientation on Feynman graphs by the two maps ∂_0 and ∂_1.

Connes-Kreimer Hopf algebra

The Hopf algebra H of Feynman graphs (for a particular theory) is the free commutative algebra generated by 1PI Feynman graphs:

- product: disjoint union of graphs, identity $1 = \emptyset$
- counit: $\epsilon(\Gamma) = 0, \epsilon(\emptyset) = 1$
- coproduct: $\Delta\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$

  Examples:



Birkhoff decomposition

For any commutative graded connected Hopf algebra H, the Birkhoff decomposition of an algebra map $\phi : H \to K$ to the field of Laurent series in z, is given by the following factorization

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right)$$

$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X')\phi(X'')$$

such that $\phi_+ = \phi_- * \phi$. Here $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

- ϕ_+ is finite at $z = 0$.
- Apply this decomposition to the (regularized) Feynman amplitude understood as a map $U : H \to K$ by $U : \Gamma \mapsto U_\Gamma(z)$ (the Feynman rules) to obtain BPHZ:
  - $U_-(\Gamma) = -T\left(\overline{R}(\Gamma)\right) = C(\Gamma)$; counterterm
  - $U_+(\Gamma) = \overline{R}(\Gamma) + C(\Gamma) = R(\Gamma)$; renormalized amplitude

Ward-Takahashi identities

We define Ward-Takahashi elements $\mathrm{WT}(\Gamma)$ for any electron self-energy graph $\Gamma =$  as follows:

$$\mathrm{WT}(\Gamma) = \sum_{\substack{i \text{ internal} \\ \text{el. lines of } \Gamma}} \Gamma(i) + \Gamma$$

with $\Gamma(i)$ the graph Γ with one external photon line attached to i

Proposition

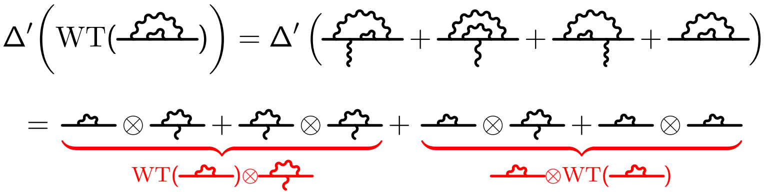
For any electron self-energy graph Γ, we have for the non-primitive part of the coproduct:

$$\Delta' \mathrm{WT}(\Gamma) = \sum_{\gamma \subset \Gamma} \left[\sum_{\gamma_e \subset \gamma} \mathrm{WT}(\gamma_e)(\gamma - \gamma_e) \otimes \Gamma/\gamma(e) + \gamma \otimes \mathrm{WT}(\Gamma/\gamma) \right]$$

Thus, the ideal $\oplus_{\Gamma=}$  I_Γ generated by WT_Γ is a Hopf ideal,

$$\Delta(I) \subset I \otimes H + H \otimes I$$

Example



Sketch of proof.

1. We split the sum in $\Delta'(\sum_i \Gamma(i))$ over subgraphs γ that contain i and those that do not,

$$\Delta'\left[\sum_i \Gamma(i)\right] = \sum_{\gamma\subseteq\Gamma}\left[\sum_{\gamma_e\subset\gamma}(\gamma-\gamma_e)\sum_{i\in\gamma_e}\gamma_e(i)\otimes\Gamma/\gamma(e) + \sum_{i\notin\gamma}\gamma\otimes\Gamma(i)/\gamma\right],$$

   with γ_e an electron self-energy graph and e its image under the quotient map $\Gamma\to\Gamma/\gamma_e$.

2. The second term can be split into: one term for which i is an external edge for a $\gamma_e\subset\gamma$, and one for which it is not,

$$\sum_{\gamma\subseteq\Gamma}\sum_{i\notin\gamma}\gamma\otimes\Gamma(i)/\gamma = \sum_{\gamma\subseteq\Gamma}\left[\sum_{i\in\partial\gamma_E}\gamma\otimes\Gamma(i)/\gamma + \sum_{i\notin\gamma\cup\partial\gamma_E}\gamma\otimes\Gamma/\gamma(i)\right]$$
$$= \sum_{\gamma\subseteq\Gamma}\sum_{\gamma_e\subset\gamma}(\gamma-\gamma_e)\gamma_e\otimes\Gamma/\gamma(e) + \gamma\otimes\sum_{j\in\Gamma/\gamma}\Gamma/\gamma(j).$$

3. The last term in red combines with $\Delta'(\Gamma)=\sum_\gamma \gamma\otimes\Gamma/\gamma$.

Ward identities in QED

Green's functions:

$$G^{\rightsquigarrow\!<} = 1 + \sum_{\Gamma^{\rightsquigarrow\!<}} \frac{\Gamma}{\mathrm{Sym}(\Gamma)} = \sum_L G_L^{\rightsquigarrow\!<}$$

$$G^{\boldsymbol{-}} = 1 - \sum_{\Gamma^{\boldsymbol{-}}} \frac{\Gamma}{\mathrm{Sym}(\Gamma)} = \sum_L G_L^{\boldsymbol{-}}$$

We define Ward elements $W_L = G_L^{\rightsquigarrow\!<} - G_L^{\boldsymbol{-}}$ for each L

Proposition

The ideal I generated by W_L for all L is a Hopf ideal,

$$\Delta(I) \subset I \otimes H + H \otimes I$$

Birkhoff decomposition and Ward identities

- The quotient Hopf algebra $\widetilde{H} = H/I$ is again a (graded connected commutative) Hopf algebra
- Feynman rules induce an algebra map $U : \widetilde{H} \to K$
- Birkhoff decomposition applies: algebra maps $U_{\pm} : \widetilde{H} \to K$

Renormalized amplitudes and counterterms U_+, U_- automatically satisfy the Ward identities

$Z_1 - Z_2 = U_-(G$  $- G$ $) = 0$

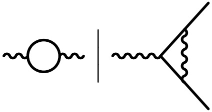
Sketch of proof:

- For a vertex/edge r, the coproduct on the corresponding Green's function can be written as

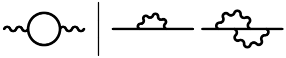
$$\Delta(G^r_L) = \sum_{K=0}^{L} \sum_{\gamma_K, \Gamma^r_{L-K}} \frac{\Gamma \mid \gamma}{\mathrm{Sym}(\gamma)\mathrm{Sym}(\Gamma)} \gamma \otimes \Gamma$$

with $\Gamma \mid \gamma$ the number of insertion places for (the connected components of) γ in Γ *allowing insertion of multiple edge graphs in* γ *on the same edge in* Γ [Kreimer 2005].

Examples



$= 2;$



$= 6.$

Remark

In QED, $\Gamma^r_L \mid \gamma$ only depends on loop number L and external structure $r = \rightsquigarrow\!\!<\, , \boldsymbol{-}$ of Γ; we write $(L, r) \mid \gamma = \Gamma^r_L \mid \gamma$.

- For any $L \geq 0$, we have

$$\Delta(W_L) = \sum_{K+K'=0}^{N} W_K \sum_{\gamma_{K'}} \frac{(L-K-K', \boldsymbol{-}) \mid \gamma}{\mathrm{Sym}(\gamma)} \gamma \otimes G^{\rightsquigarrow\!<}_{L-K-K'} + \sum_{K=0}^{L} \sum_{\gamma_K} \frac{(L-K, \boldsymbol{-}) \mid \gamma}{\mathrm{Sym}(\gamma)} \gamma \otimes W_{L-K}$$

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