On the geometry of noncommutative gauge fields

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Goal: formulate gauge theories on noncommutative spaces

- **Approach**: study an example of a noncommutative 4-dimensional sphere S^4_θ:
- This forms the base space of a noncommutative principal bundle:

$$SU(2) \to S^7_\theta \to S^4_\theta$$

deforming the **Hopf fibration on S^4**.

Let us start by recalling the classical setup...

Gauge theory on principal bundles

- P is a principal G-bundle on M, G a Lie group
- The gauge group $\mathcal{G}(P)$ consists of automorphisms of P covering the identity on M.
- Sections of the bundle of groups

  $$\boxed{\mathrm{Ad}P = P \times_{\mathrm{Ad}} G}$$

  form a group isomorphic to $\mathcal{G}(P)$.

Gauge fields (=connections)

- Connection one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ (+ conditions), induces splitting $T_pP = H_p \oplus V_p$ via $H_p = \ker \omega_p$.

Equivalent description in terms of associated bundles:

- A connection one-form ω on P induces a covariant derivative (also called connection) on $E = P \times_G V$:

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes_{C^\infty(M)} \Omega^1(M)$$
$$\eta \mapsto d_P\eta + \omega \cdot \eta$$

- $f \in \Gamma(\mathrm{Ad}P)$ (as an equivariant map $f : P \to G$) acts by gauge transformations on a connection :

$$\nabla \mapsto f^{-1}\nabla f \qquad (\iff \omega \mapsto f^{-1}d_P f + f^{-1}\omega f)$$

Yang–Mills action functional

Definition

The **Yang–Mills action functional** of a connection is defined as

$$S(\omega) := \|F(\omega)\|^2 \equiv \int_M \operatorname{tr} F(\omega) \wedge *F(\omega)$$

with $F(\omega) = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(M, \mathrm{ad}P)$ the curvature of ω.

- $S(\omega)$ is invariant under gauge transformations and positive.
- In 4 dimensions, minima of S are obtained by solutions of the (anti-)selfdual equation $*F(\omega) = \pm F(\omega)$;
  such connections are called **instantons**.

Example: Hopf fibration on S^4

$$\begin{array}{ccc} SU(2) & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array} \qquad\qquad \begin{array}{c} S^7 \times_{SU(2)} \mathbb{C}^2 \\ \downarrow \\ S^4 \end{array}$$

- Instantons: connections (charge 1) on $S^7 \times_{SU(2)} \mathbb{C}^2$
- The moduli space of charge 1 instantons (connections on $S^7 \times_{SU(2)} \mathbb{C}^2$) has a geometrical description (Atiyah, 1978):

$$\boxed{\mathcal{M}_{k=1} \simeq \frac{SL(2,\mathbb{H})}{Sp(2,\mathbb{H})}}$$

- Higher charges via the ADHM construction (Atiyah, Drinfel'd, Hitchin and Manin, 1978)

Q: Can we 'quantize' this construction?

Toric noncommutative manifolds

Let M be a compact (Riemannian spin) manifold with a smooth (isometrical) action of the n-torus $\mathbb{T}^n$.

- action σ_s of $\mathbb{T}^n$ on $C^\infty(M)$ via pullback.
- decompose $f = \sum f_r$ in homogeneous elements of degree r:

$$\sigma_s(f_r) = e^{2\pi i r \cdot s} f_r \ .$$

- given a real $n \times n$ anti-symmetric matrix $\theta = (\theta_{\mu\nu})$ define twisted product $\times_\theta$ on homogeneous elements:

$$f_r \times_\theta g_{r'} = f_r \sigma_{r\cdot\theta}(g_{r'}) \equiv e^{2\pi i r\cdot\theta\cdot r'} f_r g_{r'} \ .$$

- Denote by $C^\infty(M_\theta) = (C^\infty(M), \times_\theta)$, "smooth functions on noncommutative manifold M_θ" (Rieffel 1993).
- Similarly, deform $\Omega^p(M)$ to $\Omega^p(M_\theta)$, "noncommutative differential calculus" (with induced d_θ, $*_\theta$, etc.).

Isospectral deformation

Connes and Landi (2001) defined noncommutative spin geometry on M_θ by constructing a spectral triple $(C^\infty(M_\theta), \mathcal{H}, D)$.

$\rightsquigarrow$ Dixmier trace: $⨍ a \equiv \operatorname*{Res}_{z=0} \operatorname{tr} a|D|^{-2z} \;\; (a \in C^\infty(M_\theta))$

Connes–Moscovici local index formula (1995) takes a simple form.

Theorem (Landi–vS, 2005)

For a projection $p \in M_N(C^\infty(M_\theta))$, the index of the twisted Dirac operator $D_p = pDp$ is given by:

$$\operatorname{Ind} D_p = \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}\left(\gamma p |D|^{-2z}\right) + \sum_{k \geq 1} c_k ⨍ \gamma \big(p - \frac{1}{2}\big) [D, p]^{2k} |D|^{-2k}$$

Yang-Mills theory on M_θ

A connection on a right (finite projective) $C^\infty(M_\theta)$-module $\mathcal{E}$ is

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^1(M_\theta)$$

obeying Leibniz rule, curvature $F := \nabla^2$.

- Yang-Mills action functional is defined by

$$S(\nabla) = ⨍ \mathrm{tr} *_\theta (F *_\theta F)$$

- Gauge invariant under $\nabla \mapsto u^* \nabla u$ for $u \in \mathcal{U}(\mathcal{E})$.
- Connections with (anti)selfdual curvature $*_\theta F = \pm F$ (instantons) are minima of S.

The deformed Hopf fibration

- Consider the following action of $\mathbb{T}^2 \subset \mathrm{SO}(5)$ on S^4:

  $$\sigma_{t_1,t_2}(x_0, x_1+ix_2, x_3+ix_4) \mapsto \left(x_0, e^{2\pi i t_1}(x_1 + ix_2), e^{2\pi i t_2}(x_3 + ix_4)\right)$$

- With $(\theta_{ij}) = \theta\epsilon_{ij}$ $(\theta \in \mathbb{R})$: deformed algebra $C^\infty(S^4_\theta)$ (Connes–Landi, 2001).
- The action of $\mathbb{T}^2$ lifts to S^7 $\rightsquigarrow$ deformed algebra $C^\infty(S^7_\theta)$.
- There is still a (commuting) action α of SU(2) on $C^\infty(S^7_\theta)$ such that $C^\infty(S^4_\theta)$ forms the subalgebra of invariants under this action.

noncommutative SU(2)-principal bundle $S^7_\theta \to S^4_\theta$

Associated vector bundles $S^7_\theta \times_{\mathrm{SU}(2)} V$

- Let V be a (f.d.) representation space of SU(2):

$$\Gamma(S^7_\theta \times_{\mathrm{SU}(2)} V) := \left\{ f \in C^\infty(S^7_\theta) \otimes V : \alpha_w \otimes \mathrm{id}(f) = \mathrm{id} \otimes w^{-1}(f) \right\}$$

- These are finite projective right $C^\infty(S^4_\theta)$-modules:
$\rightsquigarrow$noncommutative vector bundles

Example: defining representation $V = \mathbb{C}^2$

- Grassmann connection $\nabla_0 = p\mathrm{d}$ on $\mathcal{E} := \Gamma(S^7_\theta \times_{\mathrm{SU}(2)} \mathbb{C}^2)$ has selfdual curvature $\boxed{*_\theta F_0 = F_0}$
$\rightsquigarrow$ basic instanton on S^4_θ (of charge 1)
- In terms of $f \in \Gamma(S^7_\theta \times_{\mathrm{SU}(2)} \mathbb{C}^2)$ we have

$$(\nabla_0 f)_i = \mathrm{d}f_i + \omega_{ij} f_j,$$

with ω a traceless, skew-hermitian 2×2-matrix with entries in $\Omega^1(S^7_\theta)$.

Moduli space of (charge 1) instantons

Starting with the basic instanton ∇_0 on $\mathcal{E}$, any other $(su(2))$ connection on $\mathcal{E}$ is given by $\nabla_0 + t\alpha$, with

$$\alpha \in \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma(\mathrm{ad}(S^7_\theta))$$

- Linearized selfdual equation: $P_-[\nabla_0, \alpha] = 0$; $P_- = \frac{1}{2}(1 - *_\theta)$.
- If α were obtained from an infinitesimal gauge transformation, then $\alpha = [\nabla_0, X]$ with $X \in \Gamma(\mathrm{ad}(S^7_\theta))$.
- This gives the selfdual complex

$$0 \to \Omega^0(\mathrm{ad}(S^7_\theta)) \xrightarrow{[\nabla_0,\cdot]} \Omega^1(\mathrm{ad}(S^7_\theta)) \xrightarrow{P_-[\nabla_0,\cdot]} \Omega^2_-(\mathrm{ad}(S^7_\theta)) \to 0$$

  and look for an element in the first cohomology group H^1
- Compute: $h^1 = \mathrm{Ind} D_p = 5$ for some projection p.

- Thus (Landi-vS, 2007):

$$\boxed{\dim T_{\nabla_0}\mathcal{M}_{\theta,k=1} = 5}$$

- A 5-dimensional (infinitesimal) family of instantons was obtained by means of Drinfel'd twists $\mathcal{U}_\theta(so(5,1))$ and $\mathcal{U}_\theta(so(5))$ acting on $C^\infty(S^7_\theta)$ and $C^\infty(S^4_\theta)$ by twisted derivations

Global structure of the moduli space

How to integrate this infinitesimal family? Two possibilities:

1. Integrate tangent spaces to obtain local charts on moduli space $\mathcal{M}_{\theta,k=1}$, then patch these together [AHS78].
2. Dualize Drinfel'd twist of $so(5,1)$ and $so(5)$ and quantize (Rieffel, 1993) the Lie groups $\mathrm{SL}(2,\mathbb{H})$ and $\mathrm{Sp}(2,\mathbb{H})$; then generate instantons via a coaction of these quantum groups on the noncommutative Hopf fibration.

Problem with the first option is that we need a definition of noncommutative gauge group;

- How to define $\Gamma(S^7_\theta \times_{\mathrm{SU}(2)} \mathrm{SU}(2))$?

We take the second approach and quantize these Lie groups

The quantum group $\mathrm{SL}_\theta(2,\mathbb{H})$

- We have $\mathbb{T}^2 \subset \mathrm{SL}(2,\mathbb{H})$ by setting:

$$(t_1, t_2) \mapsto \begin{pmatrix} e^{2\pi i t_1} & 0 \\ 0 & e^{2\pi i t_2} \end{pmatrix}.$$

- Adjoint action of $\mathbb{T}^2$ on $\mathrm{SL}(2,\mathbb{H})$ gives deformed product $\times_\theta$:

$$C^\infty(\mathrm{SL}_\theta(2,\mathbb{H})) := (C^\infty(\mathrm{SL}(2,\mathbb{H})), \times_\theta)$$

- In fact,$(C^\infty(\mathrm{SL}_\theta(2,\mathbb{H})), \Delta, \epsilon, S)$ with undeformed coproduct, counit and antipode form a Hopf algebra.

Noncommutative family of instantons

- There is a coaction:

$$\Delta_L : C^\infty(S^7_\theta) \to C^\infty(\mathrm{SL}_\theta(2,\mathbb{H})) \otimes C^\infty(S^7_\theta),$$

  which descends to $C^\infty(S^4_\theta)$.
- The basic instanton projection $p \in M_4(C^\infty(S^4_\theta))$ is mapped to $P := \Delta_L(p) \in M_4(C^\infty(\mathrm{SL}_\theta(2,\mathbb{H})) \otimes C^\infty(S^4_\theta))$.

The projection P is understood as a noncommutative family of projections parametrized by the noncommutative space $\mathrm{SL}_\theta(2,\mathbb{H})$.

Proposition

The family of connections $\widetilde{\nabla}_0 = P \circ (\mathrm{id} \otimes \mathrm{d})$ *has self-dual curvature:* $(\mathrm{id} \otimes *_\theta)(\widetilde{\nabla}_0)^2 = (\widetilde{\nabla}_0)^2$

Index pairing

- Denote $A = C^\infty(S^4_\theta)$ and $B = C^\infty(\mathrm{SL}_\theta(2,\mathbb{H}))$
- Pairing between $K_0(B \otimes A) \equiv KK(\mathbb{C}, B \otimes A)$ and $K^0(A) \equiv KK(A, \mathbb{C})$ given by Kasparov product:

$$\begin{array}{ccc} KK(\mathbb{C}, B\otimes A) & & KK(A,\mathbb{C}) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow B\otimes\_ \\ KK(\mathbb{C}, B\otimes A) & \times & KK(B\otimes A, B\otimes \mathbb{C}) \longrightarrow KK(\mathbb{C}, B) \end{array}$$

- P is (Murray-von Neumann) equivalent to $1 \otimes p$, and one computes that the **instanton charge of the family $\widetilde{\nabla}_0$ is 1** ($\in K_0(B)$).

The family of connections

We can express $\widetilde{\nabla}_0$ in terms of a family of connection one-forms:

$$\widetilde{\omega}_{ij} = \Delta_L(\omega_{ij}) \in C^\infty(\mathsf{SL}_\theta(2,\mathbb{H}) \otimes \Omega^1(S^7_\theta)$$

Proposition

The connection ω is invariant under the coaction of a quantum subgroup $\mathrm{Sp}_\theta(2,\mathbb{H}) \subset \mathsf{SL}_\theta(2,\mathbb{H})$ (a toric deformation of $\mathrm{Sp}(2,\mathbb{H})$)

$\rightsquigarrow$ the family of connections is parametrized by the **quantum quotient $\mathbf{SL}_\theta(2,\mathbb{H})/\mathrm{Sp}_\theta(2,\mathbb{H})$**, defined by the algebra:

$$C^\infty(\mathsf{SL}_\theta(2,\mathbb{H})/\mathrm{Sp}_\theta(2,\mathbb{H})) := \{a \in C^\infty(\mathsf{SL}_\theta(2,\mathbb{H})) \mid \Delta_R(a) = 1 \otimes a\}.$$

where $\Delta_R = (\pi \otimes \mathrm{id})\Delta$ and $\pi : C^\infty(\mathsf{SL}_\theta(2,\mathbb{H})) \to C^\infty(\mathrm{Sp}_\theta(2,\mathbb{H}))$.

Conclusions

- Noncommutative Hopf fibration $\mathrm{SU}(2) \to S^7_\theta \to S^4_\theta$.
- Instanton connection ∇_0 on $S^7_\theta \times_{\mathrm{SU}(2)} \mathbb{C}^2$.
- Tangent to moduli space (at ∇_0, $k = 1$) is 5-dimensional, $\mathcal{U}_\theta(so(5,1)) - \mathcal{U}_\theta(so(5))$.
- Noncommutative family of instantons: $\dfrac{\mathrm{SL}_\theta(2,\mathbb{H})}{\mathrm{Sp}_\theta(2,\mathbb{H})}$

Open questions:

- What about higher charge?
- What is the right notion of a (quantum) gauge group?
  - Possibly $\mathrm{Aut}_{C^\infty(S^4_\theta)}(C^\infty(S^7_\theta))$?
  - Relate to $\Gamma(S^7_\theta \times_{\mathrm{SU}(2)} su(2))$

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