Recent advances in noncommutative geometry:
spheres, instantons, sigma models

Florence, 29/3/2004

Instantons on S^4_θ

Walter van Suijlekom
(SISSA, Trieste)

Ref: "Instantons on S^4_θ", G. Landi and W. van Suijlekom

Contents

Hopf fibration $S^7 \to S^4$

- $S^7 := \{(z^1, z^2, z^3, z^4) \in \mathbb{C}^4 : \sum_i |z^i| = 1\}$

- Right $SU(2)-$action

- Bundle projection $S^7 \to S^4$:

$$\begin{aligned} \alpha &= 2(z^1\overline{z}^3 + z^2\overline{z}^4), \\ \beta &= 2(-z^1 z^4 + z^2 z^3), \\ x &= z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4 \end{aligned}$$

with $\alpha\alpha^* + \beta\beta^* + x^2 = (\sum_i z^i\overline{z}^i)^2 = 1$.

- Dual picture: $C_{\mathrm{alg}}(S^4) = \mathrm{Coinv}_{C_{\mathrm{alg}}(SU(2))}(C_{\mathrm{alg}}(S^7))$

Instantons

Instanton on S^4 is a connection on a rank$_\mathbb{C}$ 2 vector bundle, carrying an action of $SU(2)$, with self-dual curvature $*F = F$; instanton number: $k = c_2(\nabla)$. Similarly: anti-instanton with $*F = -F$.

Basic anti-instanton ($k = -1$) on S^4 is described by projector p as $\nabla = p\mathrm{d}$. Defined by $p = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$ with

$$|\psi_1\rangle = \begin{pmatrix} z^1 \\ -\overline{z}^2 \\ z^3 \\ -\overline{z}^4 \end{pmatrix}; \quad |\psi_2\rangle = \begin{pmatrix} z^2 \\ \overline{z}^1 \\ z^4 \\ \overline{z}^3 \end{pmatrix}$$

vector-valued functions on S^7. Explicitly

$$p = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\beta^* & \alpha^* \\ \alpha^* & -\beta & 1-x & 0 \\ \beta^* & \alpha & 0 & 1-x \end{pmatrix}$$

Quantum Hopf Fibration

We call $B \to P$ a Hopf-Galois extension if

- H is a Hopf algebra
- P a right H-comodule algebra, i.e. $\Delta_R : P \to P \otimes H$ is an algebra map.
- $B := \mathrm{Coinv}_{\Delta_R}(P)$
- the canonical map $\chi : P \otimes_B P \to P \otimes H$ defined by

$$\chi := (m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \Delta_R)$$

  is bijective.

$\Rightarrow$ Find $S^7_\theta, SU_\theta(2)$ such that:

- $\mathrm{Coinv}_{\Delta_R}(C_{\mathrm{alg}}(S^7_\theta)) = C_{\mathrm{alg}}(S^4_\theta)$
- $C_{\mathrm{alg}}(S^4_\theta) \to C_{\mathrm{alg}}(S^7_\theta)$ is a Hopf-Galois extension

$\theta-$deformed spheres

- $C_{\mathrm{alg}}(S^7_\theta)$ is complex unital $*$-algebra generated by $z^1, \ldots, z^4$ with relations

$$z^i z^j = \lambda^{ij} z^j z^i; \quad \overline{z}^i z^j = \lambda^{ji} z^j \overline{z}^i$$
$$\sum z^i \overline{z}^i = 1$$

for $\lambda^{ij} = \overline{\lambda^{ji}}$.

- $C_{\mathrm{alg}}(S^4_\theta)$ is complex unital $*$-algebra generated by α, β, x with $x = x^*$ a central element and relations:

$$\alpha\alpha^* = \alpha^*\alpha; \quad \beta\beta^* = \beta^*\beta$$
$$\alpha\beta = \lambda\beta\alpha; \quad \alpha^*\beta = \overline{\lambda}\beta\alpha^*$$
$$\alpha\alpha^* + \beta\beta^* + x^2 = 1$$

$\lambda \in S^1 \subset \mathbb{C}$

Quantum group coaction

No $\theta-$deformation of $C_{\rm alg}(SU(2))$
$\Rightarrow$ Classical coaction of $C_{\rm alg}(SU(2))$ on $C_{\rm alg}(S^7_\theta)$:

$$\Delta_R : (z^1, z^2) \quad \mapsto \quad (z^1, z^2) \otimes \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix}$$

$$\Delta_R : (z^3, z^4) \quad \mapsto \quad (z^3, z^4) \otimes \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix}$$

with $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$. $\Delta_R(z^i)\Delta_R(z^j) = \lambda^{ij}\Delta_R(z^j)\Delta_R(z^i) \Rightarrow$

$$\boxed{\lambda^{12} = \lambda^{34} = 1; \qquad \lambda^{14} = \lambda^{23} = \lambda^{24} = \lambda^{13}}$$

$C_{\rm alg}(SU(2))$ coacts classically so

$$\mathrm{Coinv}_{\Delta_R}(C_{\rm alg}(S^7_\theta)) = \mathbb{C}[z^1\overline{z}^3 + z^2\overline{z}^4, -z^1z^4 + z^2z^3, z^1\overline{z}^1 + z^2\overline{z}^2] \text{ (mod. relations)}$$

Again we define

$$\begin{aligned}\alpha &= 2(z^1\overline{z}^3 + z^2\overline{z}^4),\\ \beta &= 2(-z^1z^4 + z^2z^3),\\ x &= z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4,\end{aligned}$$

Imposing $\alpha\beta = \lambda\beta\alpha$ and $\alpha\beta^* = \overline{\lambda}\beta^*\alpha \Rightarrow$

$$\boxed{\lambda^{14} = \lambda^{23} = \lambda^{24} = \lambda^{13} = \lambda^{\frac{1}{2}}}$$

we conclude that for these values of λ^{ij}:

$$\mathrm{Coinv}_{\Delta_R}(C_{\mathrm{alg}}(S^7_\theta)) = C_{\mathrm{alg}}(S^4_\theta)$$

In fact, an inverse to χ can be constructed explicitly so that

$$\boxed{C_{\mathrm{alg}}(S^4_\theta) \to C_{\mathrm{alg}}(S^7_\theta) \text{ is a Hopf-Galois extension}}$$

Proof is very similar to [BCDT].

Basic instanton on S^4_θ

Projector defined as $p = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$ with

$$|\psi_1\rangle = \begin{pmatrix} z^1 \\ -\overline{z}^2 \\ z^3 \\ -\overline{z}^4 \end{pmatrix} ; \quad |\psi_2\rangle = \begin{pmatrix} z^2 \\ \overline{z}^1 \\ z^4 \\ \overline{z}^3 \end{pmatrix}$$

In terms of generators α, β, x of S^4_θ:

$$p = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\lambda^{\frac{1}{2}}\beta^* & \overline{\lambda}^{\frac{1}{2}}\alpha^* \\ \alpha^* & -\overline{\lambda}^{\frac{1}{2}}\beta & 1-x & 0 \\ \beta^* & \lambda^{\frac{1}{2}}\alpha & 0 & 1-x \end{pmatrix}$$

which is (K-)equivalent to projector found in [CL], describing the basic anti-instanton on S^4_θ:

$$p = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\lambda\beta^* & \alpha^* \\ \alpha^* & -\overline{\lambda}\beta & 1-x & 0 \\ \beta^* & \alpha & 0 & 1-x \end{pmatrix}$$

Strong connections

Let $B \to P$ be a Hopf-Galois extension and suppose the antipode of H is invertible. A strong connection one-form [H] is defined by a map $l : H \to P \otimes P$ as

$$\begin{aligned} \omega : H &\to \Omega^1 P \quad (\textit{universal diff. forms}) \\ h &\mapsto l(h) - \epsilon(h) 1 \otimes 1 \end{aligned}$$

with conditions on l:

$$l(1) = 1 \otimes 1; \qquad \chi(l(h)) = 1 \otimes h$$
$$(l \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta_R) \circ l$$
$$(\mathrm{id} \otimes l) \circ \Delta = (\Delta_R \otimes \mathrm{id}) \circ l$$

where $\Delta_L : P \to H \otimes P$ is defined by $p \mapsto S^{-1} p_{(1)} \otimes p_{(0)}$ Then ω satisfies

$$\Delta_{\Omega^1 P} \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{Ad}$$
$$(m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) \circ \omega = 1 \otimes (\mathrm{id} - \epsilon)$$
$$\mathrm{d}p - p_{(0)} \omega(p_{(1)}) \in (\Omega^1 B) P, \forall p \in P$$

where $\mathrm{Ad} : H \to H \otimes H, h \mapsto h_{(2)} \otimes S(h_{(1)}) h_{(3)}$

The map $l(h) := \chi^{-1}(1 \otimes h)$ defines a strong connection $C_{\mathrm{alg}}(S^7_\theta) \to C_{\mathrm{alg}}(S^4_\theta)$

(Here χ^{-1} implicitly denotes the lift to $P \otimes P$ of the inverse of the canonical map $\chi : P \otimes_B P \to P \otimes H$.)

$$
\begin{aligned}
l(w^1) &= \sum_i \langle \psi_1|_i \otimes |\psi_1\rangle_i \\
l(\overline{w}^1) &= \sum_i \langle \psi_2|_i \otimes |\psi_2\rangle_i
\end{aligned}
$$

$\Rightarrow$ Projector associated to $l(w^1), l(\overline{w}^1)$ describes basic instanton

Local expressions

- $C_{\mathrm{alg}}(\mathbb{R}^4_\theta)$ is complex unital $*$-algebra generated by ζ^1, ζ^2 with relations

$$\zeta^1\zeta^2 = \lambda\zeta^2\zeta^1; \quad \zeta^1\overline{\zeta}^2 = \overline{\lambda}\,\overline{\zeta}^2\zeta^1$$

- Enlarge by adjoining self-adjoint central generator $(1+|\zeta|^2)^{-1}$, inverse to $(1+\zeta^1\overline{\zeta}^1+\zeta^2\overline{\zeta}^2)$ and define

$$\tilde{\alpha} = 2\zeta^1(1+|\zeta|^2)^{-1}; \quad \tilde{\beta} = 2\zeta^2(1+|\zeta|^2)^{-1}$$
$$\tilde{x} = \left(1-\zeta^1\overline{\zeta}^1-\zeta^2\overline{\zeta}^2\right)(1+|\zeta|^2)^{-1}$$

  Inverse stereographical projection: Spectrum of $\tilde{\alpha}, \tilde{\beta}, \tilde{x}$ does not contain classical point $x=-1$.

- Local expressions for p:

$$p_{\mathrm{loc}} = (1+|\zeta|^2)^{-1}\begin{pmatrix} 1 & 0 & \zeta^1 & \zeta^2 \\ 0 & 1 & -\lambda\overline{\zeta}^2 & \overline{\zeta}^1 \\ \overline{\zeta}^1 & -\overline{\lambda}\zeta^2 & |\zeta|^2 & 0 \\ \overline{\zeta}^2 & \zeta^1 & 0 & |\zeta|^2 \end{pmatrix}$$

Conformal group

- Classically, the conformal group $SL(2,\mathbb{H})$ of S^4 generates other (anti-)instantons:

$$\boxed{*p(\mathrm{d}p)^2 = \pm p(\mathrm{d}p)^2 \text{ conformal invariant}}$$

- Under a proper isomorphism $\mathbb{H} \simeq \mathbb{C}^2$, $g \in SL(2,\mathbb{H})$ can be written as an element in $GL(4,\mathbb{C})$

$$g = \begin{pmatrix} a^1 & \overline{a}^2 & b^1 & \overline{b}^2 \\ -a^2 & \overline{a}^1 & -b^2 & \overline{b}^1 \\ c^1 & \overline{c}^2 & d^1 & \overline{d}^2 \\ -c^2 & \overline{c}^1 & -d^2 & \overline{d}^1 \end{pmatrix} ; \qquad \det g = 1$$

which acts by left multiplication on $|\psi_i\rangle$:

$$|\psi_i\rangle \mapsto |\psi_i^g\rangle = \frac{1}{\langle\psi_i|g^*g|\psi_i\rangle^{\frac{1}{2}}} g|\psi_i\rangle$$

Note that $\langle\psi_1|g^*g|\psi_1\rangle = \langle\psi_2|g^*g|\psi_2\rangle \equiv \langle\psi|g^*g|\psi\rangle$.

- Transformed projector

$$p \mapsto p^g = |\psi_1^g\rangle\langle\psi_1^g| + |\psi_2^g\rangle\langle\psi_2^g| = \frac{1}{\langle\psi|g^* g|\psi\rangle} g p g^*.$$

- Gauge transformations: $g \in SL(2,\mathbb{H})$ with $g^* g = 1$ give gauge equivalent instantons: Sp(2)

$\Rightarrow$ $\boxed{SL(2,\mathbb{H})/Sp(2)}$: 5-parameter family of instantons (locally: 'scaling' (1d) and 'translation' (4d))

Conformal transformations on S^4_θ

Does $\mathbb{C}^4_\theta$ (for special λ^{ij}) allow action of $SL(2,\mathbb{H})$?

$$\rho : z^1 \mapsto a^1 z^1 - \overline{a}^2 \overline{z}^2 + b^1 z^3 - \overline{b}^2 \overline{z}^4$$
$$\rho : \overline{z}^2 \mapsto a^2 z^1 + \overline{a}^1 \overline{z}^2 + b^2 z^3 + \overline{b}^1 \overline{z}^4$$
$$\rho : z^3 \mapsto c^1 z^1 - \overline{c}^2 \overline{z}^2 + d^1 z^3 - \overline{d}^2 \overline{z}^4$$
$$\rho : \overline{z}^4 \mapsto c^2 z^1 + \overline{c}^1 \overline{z}^2 + d^2 z^3 + \overline{d}^1 \overline{z}^4$$

$\rho(z^i)\rho(z^j) = \lambda^{ij}\rho(z^j)\rho(z^i)$ allows only 4 possibilities which all reduce modulo gauge transformations to:

$$(z^1, z^2) \mapsto (r z^1, r z^2)$$
$$(z^3, z^4) \mapsto (\tilde{r} z^3, \tilde{r} z^4)$$

with $\tilde{r} = r^{-1} \in \mathbb{R}_{>0}$

Transformed projector:

$$p^r = (r^2(1+x)+\tilde{r}^2(1-x))^{-1} \begin{pmatrix} r^2(1+x) & 0 & \alpha & \beta \\ 0 & r^2(1+x) & -\lambda\beta^* & \alpha^* \\ \alpha^* & -\overline{\lambda}\beta & \tilde{r}^2(1-x) & 0 \\ \beta^* & \alpha & 0 & \tilde{r}^2(1-x) \end{pmatrix}$$

Local expressions

This projector reads locally:

$$p^r_{\rm loc}(\zeta) = (r^2 + \tilde{r}^2|\zeta|^2)^{-1} \begin{pmatrix} r^2 & 0 & \zeta^1 & \zeta^2 \\ 0 & r^2 & -\lambda\bar{\zeta}^2 & \bar{\zeta}^1 \\ \bar{\zeta}^1 & -\bar{\lambda}\zeta^2 & \tilde{r}^2|\zeta|^2 & 0 \\ \bar{\zeta}^2 & \zeta^1 & 0 & \tilde{r}^2|\zeta|^2 \end{pmatrix}$$

$\Rightarrow p^r_{\rm loc}(\zeta) = p_{\rm loc}(\tilde{r}^2\zeta)$

All non-gauge-equivalent instantons generated from the basic instanton by conformal transformations arise by 'scaling'

Abscence of 'translated instantons' can be seen from the commutation relations on $\mathbb{R}^4_\theta$:

$$\zeta^1\zeta^2 = \lambda\zeta^2\zeta^1$$

since it does not allow automorphism $\zeta^i \mapsto \zeta^i + a^i$ with $a^i \in \mathbb{C}$.

Outlook

- Calculate dimension of moduli space of $(k = 1)-$instantons on S^4_θ:
  - If $\dim = 1$: all instantons on S^4_θ are obtained from the basic instanton by scaling
  - If $\dim > 1$: classical methods to generate instantons not sufficient

- Understand structure of Hopf-Galois extension and relation with projectors; Find principal bundles describing instanton with $|k| > 1$ (even classical!)

References

- M. F. Atiyah. *The Geometry of Yang-Mills Fields*. Fermi Lectures. Scuola Normale, Pisa, 1979.
- F. Bonechi, N. Ciccoli, L. Dąbrowski, and M. Tarlini. Bijectivity of the canonical map for the noncommutative instanton bundle. `arXiv:math.QA/0306114`.
- T. Brzeziński, S. Majid. Quantum group theory on quantum space *Commun. Math. Phys.*, 157:591–638, 1993.
- A. Connes and G. Landi. Noncommutative manifolds: The instanton algebra and isospectral deformations. *Commun. Math. Phys.*, 221:141–159, 2001.
- M. Dubois-Violette and A. Connes. Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. *Commun. Math. Phys.*, 230:539–579, 2002.
- P. M. Hajac. Strong connections on quantum principal bundles *Commun. Math. Phys.*, 182:579–617, 1996
- G. Landi. Deconstructing monopoles and instantons. *Rev. Math. Phys.*, 12(10):1367–1390, 2000.