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The Dirac operator on quantum $SU(2)$

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Introduction

- Noncommutative Geometry vs. Quantum Groups
- Construct q-version of spin geometry on $SU(2)$:
  - Homogeneous space:

$$SU(2) = \frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} = \frac{SU(2) \times SU(2)}{SU(2)} \tag{1}$$

  with $\mathrm{Spin}(3)$ the diagonal $SU(2)$ subgroup of $\mathrm{Spin}(4)$.
  Quotient map: $(p, q) \mapsto pq^{-1}$
  - Action of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ on $SU(2)$:

$$(p, q) \cdot x = pxq^{-1} \tag{2}$$

Noncommutative geometry

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on $\mathcal{H}$ such that

1. The resolvent $(D-\lambda)^{-1}$ is a compact operator on $\mathcal{H}$;

2. $[D, a]$ is a bounded operator for any $a \in \mathcal{A}$.

The spectral triple is called real if there exists an antiunitary operator J on $\mathcal{H}$ satisfying $J^2 = \pm 1, JD = \pm DJ$, and such that $b \rightarrow Jb^* J^{-1}$ determines an action of the opposite algebra $\mathcal{A}^\circ$ on $\mathcal{H}$ that commutes with $\mathcal{A}$:

$$[a, Jb^* J^{-1}] = 0,$$

The first order condition demands that $\mathcal{A}^\circ$ not only commutes with $\mathcal{A}$ but also with $[D, \mathcal{A}]$. This makes D a "differential operator of order one".

Example: canonical spectral triple

The basic example of a spectral triple is constructed by means of the Dirac operator on a compact n-dimensional Riemannian spin manifold M. The canonical spectral triple is defined by

- $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M.
- $\mathcal{H} = L^2(M, S)$ is the Hilbert space of square integrable sections of a spinor bundle on M, on which $\mathcal{A}$ acts by pointswise multiplication.
- D is the Dirac operator associated with the Levi-Civita connection.
- J is defined as the map $\psi \mapsto C\overline{\psi}$ with C the charge conjugation operator.

Algebraic preliminaries

Let q be a positive real number, $q \neq 1$.

Definition. *Define the algebra $\mathcal{A} := \mathcal{A}(SU_q(2))$ of polynomials on $SU_q(2)$ to be the $*$-algebra generated by a and b, subject to the following relations:*

$$ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b$$
$$a^*a + q^2b^*b = 1, \qquad aa^* + bb^* = 1.$$

As a consequence, $a^*b = qba^*$ and $a^*b^* = qb^*a^*$.

Correspondence with [Kl-Schm],[Chakr-Pal],[Con]: $a \leftrightarrow a^*, b \leftrightarrow -b$.

This becomes a Hopf $*$-algebra with

- *the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by*

$$\Delta a := a \otimes a - q\, b \otimes b^*,$$
$$\Delta b := b \otimes a^* + a \otimes b,$$

- *the counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ defined by $\varepsilon(a) = 1$ and $\varepsilon(b) = 0$,*

- *the antipode $S : \mathcal{A} \to \mathcal{A}$ defined as an antilinear map by*

$$Sa = a^*, \qquad Sb = -qb,$$
$$Sb^* = -q^{-1}b^*, \qquad Sa^* = a.$$

Definition. *The* $*$*-algebra* $\mathcal{U} := \mathcal{U}_q(su(2))$ *is generated by elements* e, f, k*, with* k *invertible, satisfying the relations*

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef)$$

Correspondence with [Kl-Schm]: $q \leftrightarrow q^{-1}$*, or, equivalently:* $e \leftrightarrow f$*.*

Hopf $*$-algebra structure given by: coproduct Δ:

$$\Delta k = k \otimes k, \qquad \Delta e = e \otimes k + k^{-1} \otimes e, \qquad \Delta f = f \otimes k + k^{-1} \otimes f,$$

counit $\epsilon(k) = 1, \epsilon(f) = \epsilon(e) = 0$, antipode S,

$$Sk = k^{-1}, \qquad Sf = -qf, \qquad Se = -q^{-1}e,$$

and star structure: $k^* = k, f^* = e$.

Representation theory of $\mathcal{U}_q(su(2))$

The irreducible finite dimensional representations σ_l of $\mathcal{U}_q(su(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and given by

$$\sigma_l(k)\,|lm\rangle = q^m\,|lm\rangle,$$
$$\sigma_l(f)\,|lm\rangle = \sqrt{[l-m][l+m+1]}\,|l, m+1\rangle,$$
$$\sigma_l(e)\,|lm\rangle = \sqrt{[l-m+1][l+m]}\,|l, m-1\rangle,$$

on the irreducible $\mathcal{U}$-modules $V_l = \mathrm{Span}\{|lm\rangle\}_{m=-l,\ldots,l}$.

The brackets denote q-integers: $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, provided $q \neq 1$.

Action of $\mathcal{U}_q(su(2))$ on $\mathcal{A}(SU_q(2))$

Dual pairing $\langle .,. \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ induces left and right action of $h \in \mathcal{U}_q(su(2))$ on $x \in \mathcal{A}(SU_q(2))$:

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle \qquad x \triangleleft h := \langle h, x_{(1)} \rangle \, x_{(2)},$$

where we use Sweedler's notation for the coproduct in $\mathcal{A}(SU_q(2))$:

$$\Delta x = x_{(1)} \otimes x_{(2)}, \qquad \big(x \in \mathcal{A}\big)$$

Using the antipode, the right action can be transformed into a left action, which we will denote by $h \cdot x$.

Left regular representation of $\mathcal{A}(SU_q(2))$

We establish the left regular representation of $\mathcal{A}$ as an equivariant representation with respect to two copies of $\mathcal{U}$ acting via $\cdot$ and $\triangleright$ on the left.

Definition. *Let λ and ρ be mutually commuting representations of the Hopf algebra $\mathcal{U}$ on a vector space V. A representation π of the algebra $\mathcal{A}$ on V is (λ, ρ)-equivariant if the following compatibility relations hold:*

$$\lambda(h)\,\pi(x)\xi = \pi(h_{(1)} \cdot x)\,\lambda(h_{(2)})\xi,$$
$$\rho(h)\,\pi(x)\xi = \pi(h_{(1)} \triangleright x)\,\rho(h_{(2)})\xi,$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$.

Equivariant representation of $\mathcal{A}(SU_q(2))$

Representation space:

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l$$

The two copies of $\mathcal{U}_q(su(2))$ act via the irreducible representations σ on the first and the second leg of the tensor product, respectively:

$$\lambda(h) = \sigma_l(h) \otimes \mathrm{id}, \qquad \rho(h) = \mathrm{id} \otimes \sigma_l(h) \qquad \text{on } V_l \otimes V_l. \tag{3}$$

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \dots, l$.

Proposition. *A* (λ, ρ)*-equivariant* $*$*-representation* π *of* $\mathcal{A}(SU_q(2))$ *on* V *is necessarily given by the left regular representation. Explicitly:*

$$\pi(a)\,|lmn\rangle = A^+_{lmn}|l+\tfrac{1}{2}, m+\tfrac{1}{2}, n+\tfrac{1}{2}\rangle + A^-_{lmn}|l-\tfrac{1}{2}, m+\tfrac{1}{2}, n+\tfrac{1}{2}\rangle,$$
$$\pi(b)\,|lmn\rangle = B^+_{lmn}|l+\tfrac{1}{2}, m+\tfrac{1}{2}, n-\tfrac{1}{2}\rangle + B^-_{lmn}|l-\tfrac{1}{2}, m+\tfrac{1}{2}, n-\tfrac{1}{2}\rangle,$$

where for example the constants $A^\pm_{lmn}$ *are given by*

$$A^+_{lmn} = q^{(-2l+m+n-1)/2}\left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]}\right)^{\frac{1}{2}},$$
$$A^-_{lmn} = q^{(2l+m+n+1)/2}\left(\frac{[l-m][l-n]}{[2l][2l+1]}\right)^{\frac{1}{2}}.$$

Spinor representation

We amplify representation π of $\mathcal{A}$ to the spinor representation defined by $\pi' = \pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$, and set $\rho' = \rho \otimes \mathrm{id}$, but λ' as the tensor product of the representations λ on V and $\sigma_{\frac{1}{2}}$ on $V_{\frac{1}{2}} = \mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}). \tag{4}$$

Proposition. *The representation π' of $\mathcal{A}$ is (λ', ρ')-equivariant.*

Clebsch-Gordan decomposition:

$$V \otimes \mathbb{C}^2 = \left(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l \right) \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j). \tag{5}$$

Basis vectors ($j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$):

$$|j\mu n{\uparrow}\rangle := C_{j+1,\mu}\,|j+\tfrac{1}{2},\mu-\tfrac{1}{2},n\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle - S_{j+1,\mu}\,|j+\tfrac{1}{2},\mu+\tfrac{1}{2},n\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle,$$

where $\mu = -j, \dots, j$ and $n = -(j+\frac{1}{2}), \dots, j+\frac{1}{2}$

$$|j\mu n{\downarrow}\rangle := S_{j\mu}\,|j-\tfrac{1}{2},\mu-\tfrac{1}{2},n\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle + C_{j\mu}\,|j-\tfrac{1}{2}\mu+\tfrac{1}{2}n\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle,$$

where $\mu = -j, \dots, j$ and $n = -(j-\frac{1}{2}), \dots, j-\frac{1}{2}$, and the q-Clebsch-Gordan coefficients come from the well-known representation theory of $\mathcal{U}_q(su(2))$:

$$C_{j\mu} := q^{-(j+\mu)/2}\,\frac{[j-\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}, \qquad S_{j\mu} := q^{(j-\mu)/2}\,\frac{[j+\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}.$$

$\Longrightarrow$ expressions for π' in basis $\{|j\mu n{\uparrow}\rangle, |j\mu n{\downarrow}\rangle\}$ contain off-diagonal terms.

Invariant Dirac operator

Proposition. *Any self-adjoint operator on* $\mathcal{H} = (V \otimes \mathbb{C}^2)^{\mathrm{cl}}$*, that commutes with both actions* ρ', λ' *of* $\mathcal{U}_q(su(2))$ *is of the form*

$$D|j\mu n{\uparrow}\rangle = d_j^{\uparrow}\,|j\mu n{\uparrow}\rangle, \qquad D|j\mu n{\downarrow}\rangle = d_j^{\downarrow}\,|j\mu n{\downarrow}\rangle.$$

Restrict form of eigenvalues by imposing bounded commutator condition:

$$[D, \pi'(x)] \in \mathcal{B}(\mathcal{H}), \qquad (x \in \mathcal{A}).$$

- D with as eigenvalues q-analogues of the classical Dirac operator (like $[j]$) gives unbounded commutators (cf. [Bib-Kul]).

- ‘Classical’ Dirac operator with D eigenvalues linear in j with opposite signs on the $\uparrow$ and $\downarrow$-eigenspaces, respectively.

Proposition. *If D has eigenvalues linear in j, the commutators $[D, \pi'(x)]$ ($x \in \mathcal{A}$) are bounded operators.*

The spectrum of D coincides with that of the classical Dirac operator on the round sphere $S^3 \simeq SU(2)$. We make the following choice:

$$D|j\mu n\rangle\rangle = \begin{pmatrix} 2j + \frac{3}{2} & 0 \\ 0 & -2j - \frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle,$$

and conclude that $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ is a (3^+-summable) spectral triple.

Relation with [Gos], with unbounded commutators?
Difference in definition of spinor space: $\mathbb{C}^2 \otimes V$ (instead of $V \otimes \mathbb{C}^2$).
Define on $\mathbb{C}^2 \otimes V$:

$$\pi'(x) = \mathrm{id} \otimes \pi(x);$$
$$\rho'(h) = \mathrm{id} \otimes \rho(h);$$
$$\lambda'(h) = \sigma_{\frac{1}{2}}(h_{(1)}) \otimes \lambda(h_{(2)}).$$

Let us (naïvely) define the Dirac operator to be diagonal in the $\uparrow - \downarrow$ basis obtained from the Clebsch-Gordan decomposition, with j-linear eigenvalues. This is exactly [Gos]. A computation shows that $[D, \pi'(x)]$ is unbounded.

However, this π' is not (λ', ρ')-equivariant, so that the choice of $\mathbb{C}^2 \otimes V$ is not allowed, because $\mathcal{U}_q(su(2))$ is not cocommutative.

Real structure

A real structure J on the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ defines a representation of the opposite algebra $\mathcal{A}(SU_q(2))^\circ$:

$$\pi'^\circ(x) = J\pi'(x^*)J^{-1} \text{ satisying } \pi'^\circ(xy) = \pi'^\circ(y)\pi'^\circ(x)$$

Definition. *The real structure J is the antilinear operator on $\mathcal{H}$ which is defined on the orthonormal spinor basis by*

$$J\,|j\mu n{\uparrow}\rangle := i^{2(2j+\mu+n)}\,|j,-\mu,-n,\uparrow\rangle;$$
$$J\,|j\mu n{\downarrow}\rangle := i^{2(2j-\mu-n)}\,|j,-\mu,-n,\downarrow\rangle.$$

$\Longrightarrow$ The Dirac operator D commutes with J.

Conditions such as the commutant property and first-order condition entail that J maps both $\mathcal{A}$ and $[D, \mathcal{A}]$ to the commutant of $\mathcal{A}$.

In the case of $\mathcal{A}(SU_q(2))$, they are almost satisfied.

Definition. *The ideal $\mathcal{K}_q$ is defined as the two-sided ideal in $\mathcal{B}(\mathcal{H})$ generated by the positive traceclass operator: $L_q|j\mu n\rangle := q^j|j\mu n\rangle$.*

$\mathcal{K}_q$ is contained in the ideal of infinitesimals of order α, that is, compact operators whose n-th singular value μ_n satisfies $\mu_n = O(n^{-\alpha})$, for all $\alpha > 0$.

Proposition. *With D and J as above, the commutant property and the first-order condition are satisfied up to infinitesimals of arbitrary order:*

$$[\pi'(x), J\pi'(y)J^{-1}] \in \mathcal{K}_q;$$

$$[\pi'(x), [D, J\pi'(y)J^{-1}]] \in \mathcal{K}_q; \qquad \big(\forall x, y \in \mathcal{A}(SU_q(2))\big)$$

Connes-Moscovici local index formula

The local index formula of Connes and Moscovici provides a powerful method to compute the index of a twisted Dirac operator in terms of (easier) local expressions.

In general, this index map is defined as a map from the K-theory of $\mathcal{A}$. Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and denote by F the sign of D.

(even) $\mathrm{Index}(eFe)$ for a projection $e \in M_N(\mathcal{A})$

(odd) $\mathrm{Index}(PUP)$ for a unitary $U \in M_N(\mathcal{A})$ and $P = \frac{1}{2}(1 + F)$.

Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is regular: i.e. both $\mathcal{A}$ and $[D, \mathcal{A}]$ belong to the smooth domain of $\delta := [|D|, \ \cdot \ ]$ where $|D| = FD$.

The local index formula of Connes and Moscovici expresses this index as a sum of residues of zeta-functions like:

$$\textit{(even)}\ \operatorname*{Res}_{z=0} \mathrm{Tr}(p[D,p]^{[k_1]} \cdots [D,p]^{[k_n]} |D|^{-2(|k|+n)-z})$$

$$\textit{(odd)}\ \operatorname*{Res}_{z=0} \mathrm{Tr}(U^*[D,U]^{[k_1]} \cdots [D,U^*]^{[k_n]} |D|^{-2(|k|+n)-z})$$

where $T^{[k]}$ denotes the j'th commutator with D^2.

In the case of the canonical spectral triple, it reduces to the Atiyah-Singer index theorem.

Local index formula for $SU_q(2)$

Important here is that we can work modulo the infinitesimals of arbitrary high order. The quotient map is understood geometrically as a symbol map:

$$\rho : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-})$$

which maps onto the "cosphere bundle" $\mathbb{S}^*_q$. Here $\mathcal{B}$ is the algebra $\delta^n(\mathcal{A}) \cup \delta^n([D, \mathcal{A}])$, and $D^2_{q\pm}$ are two noncommutative disks.

If F denotes the sign of D ($+$ on $\uparrow$ and - on $\downarrow$), we can express the Chern character $\mathrm{ch}(a_0, a_1) := \mathrm{tr}(a_0[F, a_1])$ of $\mathcal{A}(SU_q(2))$ in terms of three (relatively simple) linear functionals $\tau_0^\uparrow, \tau_0^\downarrow, \tau_1$ on the disks as

$$2(\tau_0^\uparrow \otimes \tau_0^\downarrow)\big(\rho(a_0 \delta a_1)\big) - \big(\tau_1 \otimes \tau_0^\downarrow + \tau_0^\uparrow \otimes \tau_1\big)\big(\rho(a_0 \delta^2 a_1)\big) + \frac{2}{3}(\tau_1 \otimes \tau_1)\big(\rho(a_0 \delta^3 a_1)\big)$$

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