Inner perturbations in noncommutative geometry

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Overview

- Spectral triples as noncommutative manifolds
- Gauge group, semi-group of inner perturbations
- Relation to Morita equivalence
- Examples: Yang–Mills, almost-commutative manifolds, SM

Noncommutative manifolds

Basic device: a spectral triple $(\mathcal{A}, \mathcal{H}, D)$:

- algebra $\mathcal{A}$ of bounded operators on
- a Hilbert space $\mathcal{H}$,
- a self-adjoint operator D in $\mathcal{H}$ with compact resolvent and such that the commutator $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Key example: Riemannian spin geometry

Let M be an compact m-dimensional Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M)$
- $\mathcal{H} = L^2(S)$, square integrable spinors
- $D = \partial\!\!\!/$, Dirac operator

Then D has compact resolvent because $\partial\!\!\!/$ elliptic self-adjoint.
Also $[D, f]$ bounded for $f \in C^\infty(M)$.

Real spectral triples

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extend to a real spectral triple:

$$J : \mathcal{H} \to \mathcal{H} \qquad \text{real structure (anti-unitary)}$$

  such that

$$J^2 = \pm 1; \qquad JD = \pm DJ$$

- Action of $\mathcal{A}^{\text{op}}$ on $\mathcal{H}$: $a^{\text{op}} = Ja^* J^{-1}$ such that

$$a^{\text{op}} b^{\text{op}} = Ja^* J^{-1} Jb^* J^{-1} = J(ba)^* J^{-1} = (ba)^{\text{op}}$$

  We demand

$$[a^{\text{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy the first-order condition if

$$[[D, a], b^{\text{op}}] = 0$$

Spectral invariants

$$\mathrm{Tr}\ f(D/\Lambda) + \frac{1}{2}\langle J\psi, D\psi\rangle$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \qquad U = uJuJ^{-1}$$

- Gauge group: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and blue term vanishes if D satisfies first-order condition

Semi-group of inner perturbations

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j$$

- For real spectral triples we use the map $\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ sending $T \mapsto T \otimes \hat{T}$ so that

$$D \mapsto \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j$$

and this extends the action of the gauge group $\mathcal{G}(\mathcal{A})$.

Morita equivalence

Suppose $\mathcal{A} \sim_M \mathcal{B}$.
Can we construct a spectral triple on $\mathcal{B}$ from $(\mathcal{A}, \mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$.
- The self-adjoint operator $(1 \otimes_{\nabla} D)(\eta \otimes \psi) := \nabla_D(\eta)\psi + \eta \otimes D\psi$ requires a universal connection on $\mathcal{E}$:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

  where ∇_D indicates that $a\delta(b) \in \Omega^1(\mathcal{A})$ is represented as $a[D, b]$.
- Then $(\mathcal{B}, \mathcal{H}', 1 \otimes_{\nabla} D)$ is a spectral triple [Connes, 1996].

Morita equivalence

with real structure

Again, suppose $\mathcal{A} \sim_M \mathcal{B}$.

- If there is a real structure J on $(\mathcal{A}, \mathcal{H}, D)$, then we define

$$\mathcal{H}' := (\mathcal{E} \otimes_\mathcal{A} \mathcal{H}) \otimes_\mathcal{A} \overline{\mathcal{E}}$$

with the conjugate (left $\mathcal{A}$-) module $\overline{\mathcal{E}}$ and define analogously the operator $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1$ on $\mathcal{H}'$, where

$$\overline{\nabla} : \overline{\mathcal{E}} \to \Omega^1(\mathcal{A}) \otimes_\mathcal{A} \overline{\mathcal{E}},$$

and we also define

$$J' : \mathcal{H}' \to \mathcal{H}', \qquad \eta \otimes \psi \otimes \overline{\rho} \mapsto \rho \otimes J\psi \otimes \overline{\eta}$$

Proposition (Chamseddine–Connes–vS, 2013)

We have $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1 = 1 \otimes_\nabla (D \otimes_{\overline{\nabla}} 1)$ *and the tuple* $(\mathcal{B}, \mathcal{H}', (1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1; J')$ *is a real spectral triple.*

Morita self-equivalence

- If $\mathcal{B} = \mathcal{A}$ (i.e. $\mathcal{E} = \mathcal{A}$) we have $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}} \simeq \mathcal{H}$ and $J' \equiv J$.
- The operator D is perturbed to $D' \equiv D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$ where

$$A_{(1)} := \sum_j a_j[D, b_j], \qquad \widetilde{A}_{(1)} := \sum_j \hat{a}_j[D, \hat{b}_j] = \pm J A_{(1)} J^{-1};$$

$$A_{(2)} := \sum_j \hat{a}_j[A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k[[D, b_k], \hat{b}_j]$$

  and blue terms vanish if D satisfies first-order condition
- Gauge transformations $D' \mapsto U D' U^*$ implemented by

$$A_{(1)} \mapsto u A_{(1)} u^* + u[D, u^*]$$
$$A_{(2)} \mapsto J u J^{-1} A_{(2)} J u^* J^{-1} + J u J^{-1}[u[D, u^*], J u^* J^{-1}]$$

Perturbation semi-group and Morita self-equivalences

Proposition (Chamseddine–Connes–vS, 2013)

- *The linear map* $\eta : \mathrm{Pert}(\mathcal{A}) \to \Omega^1(\mathcal{A})$, $\eta(a \otimes b^{\mathrm{op}}) = a\delta(b)$ *is surjective.*
- *If* $\sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathrm{Pert}(\mathcal{A})$ *then the perturbed operator*

$$\sum_j a_j D b_j = D + \sum_j a_j [D, b_j] \equiv D + A_{(1)}$$

and, for real spectral triples:

$$\sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j = D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

Examples: $\mathrm{Pert}(\mathcal{A})$ for matrix algebras

(joint with Niels Neumann)

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathsf{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- If $\mathcal{A} = \mathbb{C}^N$, then in terms of basis vectors $\{e_i\}$ of $\mathbb{C}^N$:

$$\mathrm{Pert}(\mathbb{C}^N) \simeq \left\{ \sum_{i,j} C_{ij} e_i \otimes e_j \in \mathbb{C}^{N^2} \;\middle|\; \begin{array}{l} C_{ii} = 1 \\ C_{ij} = \overline{C_{ji}} \end{array} \right\} \simeq \mathbb{C}^{N(N-1)/2}$$

Examples: $\mathrm{Pert}(\mathcal{A})$ for matrix algebras

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- If $\mathcal{A} = M_N(\mathbb{C}) \simeq \mathcal{A}^{\mathrm{op}}$, then in terms of basis vectors $\{e_{ij}\}$ of $M_N(\mathbb{C})$:

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq \left\{ \sum_{i,j} C_{ij,kl} e_{ij} \otimes e_{kl} \in M_{N^2}(\mathbb{C}) \,\middle|\, \begin{array}{l} \sum_j C_{ij,jk} = \delta_{ik} \\ C_{ij,kl} = \overline{C_{lk,ji}} \end{array} \right\}$$

$$\simeq \left\{ C \in M_{N^2}(\mathbb{C}) \,\middle|\, \begin{array}{l} Ce_1 = e_1 \\ C\Omega = \Omega \overline{C} \end{array} \right\}$$

where $\Omega = \begin{pmatrix} 1_{N(N+1)/2} & 0 \\ 0 & -1_{N(N-1)/2} \end{pmatrix}$.

Examples: $\mathrm{Pert}(\mathcal{A})$ for function algebras

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- If $\mathcal{A} = C^\infty(M)$ then

$$\mathrm{Pert}(C^\infty(M)) \simeq \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{ll} f(x,x) = 1 & \forall x \in M \\ f(x,y) = \overline{f(y,x)} & \forall x, y \in M \end{array} \right\}$$

- Action of $\mathrm{Pert}(C^\infty(M))$ on $D = \partial\!\!\!/ = i\gamma^\mu \nabla_\mu$ is given by

$$\sum_j a_j \partial\!\!\!/ b_j = \partial\!\!\!/ + i\gamma^\mu \nabla_{\frac{\partial}{\partial y^\mu}} f(x,y)|_{x=y} =: \partial\!\!\!/ + i\gamma^\mu A_\mu$$

with $A_\mu \in C^\infty(M, \mathfrak{u}(1))$

Non-abelian Yang–Mills theory

On a 4-dimensional background:

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = \partial\!\!\!/ \otimes 1$
- $J = C \otimes (.)^*$.

Proposition (Chamseddine-Connes, 1996)

- $\mathrm{Tr}\ f(D)$: *pure gravity*
- *The perturbations* $A_{(1)} + \widetilde{A}_{(1)}$ *with* $A_{(1)} = \gamma^\mu A_\mu$ *describes an* $\mathfrak{su}(n)$*-gauge field on* M.
- *Gauge group* $\mathcal{G}(\mathcal{A}) \simeq C^\infty(M, SU(n))$
- *The spectral action of the perturbed Dirac operator is given by*

$$\mathrm{Tr}\ f\left(\frac{D + A_{(1)} + \widetilde{A}_{(1)}}{\Lambda}\right) \sim (\cdots) + \frac{f(0)}{24\pi^2}\int_M \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Lambda^{-1})$$

Almost-commutative geometries

A class of examples

$$(C^\infty(M) \otimes A_F, \quad L^2(S) \otimes \mathcal{H}_F, \quad \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F)$$

with real structure $J = J_M \otimes J_F$.

- Gauge group $\mathcal{G}(C^\infty(M) \otimes A_F) = C^\infty(M, \mathcal{G}(A_F))$
- Inner perturbations:

$$D \mapsto D' = \partial\!\!\!/ \otimes 1 + \gamma^\mu \otimes \mathrm{ad} A_\mu + \gamma_5 \otimes \Phi$$

  with $\mathrm{ad} A_\mu$ a $\mathfrak{g}(A_F)$-gauge potential and $\Phi = D_F + \phi + J_F \phi J_F^{-1}$ a map $\mathcal{H}_F \to \mathcal{H}_F$

- Explicitly,

$$A_\mu = -i \sum_j a_j \partial_\mu (b_j); \qquad \phi = \sum_j a_j [D_F, b_j]$$

- As $\mathcal{G}(A_F)$-representations:

$$A_\mu \mapsto u A_\mu u^* - i u \partial_\mu u^*, \qquad \Phi \mapsto U \Phi U^*$$

Proposition (Van den Dungen–vS, 2012)

In the above setting,

$$\mathrm{Tr}\ \left(f\left(\frac{D'}{\Lambda}\right)\right) \sim (\cdots) + \frac{f(0)}{24\pi^2}\mathrm{Tr}\ (F_{\mu\nu}F^{\mu\nu}) - \frac{2f_2\Lambda^2}{4\pi^2}\mathrm{Tr}\ (\Phi^2) + \frac{f(0)}{8\pi^2}\mathrm{Tr}\ (\Phi^4))$$
$$+ \frac{f(0)}{48\pi^2} s\mathrm{Tr}\ (\Phi^2) + \frac{f(0)}{8\pi^2}\mathrm{Tr}\ \big((D_\mu\Phi)(D^\mu\Phi)\big) + \mathcal{O}(\Lambda^{-1}).$$

with f_2 the first moment of f.

The noncommutative Standard Model

$$(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})), L^2(S) \otimes \mathcal{H}_F, \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F)$$

- Fermions are given by:

$$\mathcal{H}_F := \left(\mathcal{H}_l \oplus \mathcal{H}_{\bar{l}} \oplus \mathcal{H}_q \oplus \mathcal{H}_{\bar{q}}\right)^{\oplus 3}.$$

- Algebra acts as:

$$(\lambda, q, m) \xrightarrow{\mathcal{H}_l} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad (\lambda, q, m) \xrightarrow{\mathcal{H}_q} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix} \otimes 1_3,$$

$$(\lambda, q, m) \xrightarrow{\mathcal{H}_{\bar{l}}} \lambda 1_4, \qquad (\lambda, q, m) \xrightarrow{\mathcal{H}_{\bar{q}}} 1_4 \otimes m$$

- Real structure J_F interchanges fermions and anti-fermions.

The noncommutative Standard Model

The finite Dirac operator

$$D_F := \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := S|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \quad S_q \otimes 1_3 := S|_{\mathcal{H}_q} = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix}$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Proposition (Chamseddine–Connes–Marcolli, 2007)

In the above setting,

- *The unimodular gauge group* $S\mathcal{G}(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) = U(1) \times SU(2) \times SU(3)$
- *The inner perturbations of* $\not{\partial} \otimes 1 + \gamma_5 \otimes D_F$ *are parametrized by* $U(1), SU(2)$ *and* $SU(3)$ *gauge fields* $\Lambda_\mu, Q_\mu, V_\mu$ *and a Higgs doublet* H
- *The spectral action is given by*

$$\mathrm{Tr}\ f\left(\frac{D'}{\Lambda}\right) \sim (\cdots) + \frac{f(0)}{\pi^2}\left(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}\ (Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}\ (V_{\mu\nu}V^{\mu\nu})\right)$$
$$+ \frac{bf(0)}{2\pi^2}|H|^4 + \frac{-2af_2\Lambda^2 + ef(0)}{\pi^2}|H|^2 + \frac{af(0)}{2\pi^2}|D_\mu H|^2 + \mathcal{O}(\Lambda^{-1}).$$

Example beyond first-order

[Chamseddine–Connes–vS, 2013]

$$A'_F = \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C}),$$

$$H_F = (\mathbb{C}_R \oplus \mathbb{C}_L) \otimes (\mathbb{C}^2)^\circ \oplus \mathbb{C}^2 \otimes (\mathbb{C}^\circ_R \oplus \mathbb{C}^\circ_L),$$

$$J_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \qquad (C : \text{complex conjugation}),$$

$$D_F = \begin{pmatrix} 0 & k_x \otimes 1_2 & \begin{smallmatrix} \overline{k}_y & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ \overline{k}_x \otimes 1_2 & 0 & 0 & \\ \begin{smallmatrix} k_y & 0 \\ 0 & 0 \end{smallmatrix} & 0 & 0 & 1_2 \otimes \overline{k}_x \\ 0 & 0 & 1_2 \otimes k_x & 0 \end{pmatrix}$$

The algebra action of $(\lambda_R, \lambda_L, m) \in \mathcal{A}$ on $\mathcal{H}$ is given explicitly by

$$\pi(\lambda_R, \lambda_L, m) = \begin{pmatrix} \lambda_R 1_2 & & & \\ & \lambda_L 1_2 & & \\ & & m & \\ & & & m \end{pmatrix}, \pi^{\text{op}}(\lambda_R, \lambda_L, m) = \begin{pmatrix} m^t & & & \\ & m^t & & \\ & & \lambda_R 1_2 & \\ & & & \lambda_L 1_2 \end{pmatrix}.$$

Proposition

The largest (even) subalgebra $A_F \subset A'_F \equiv \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$ for which the first-order condition holds (for the above $\mathcal{H}_F, D_F$ and J_F) is given by

$$A_F = \left\{ \left(\lambda_R, \lambda_L, \begin{pmatrix} \lambda_R & 0 \\ 0 & \mu \end{pmatrix} \right) : (\lambda_R, \lambda_L, \mu) \in \mathbb{C}_R \oplus \mathbb{C}_L \oplus \mathbb{C} \right\}$$

Proposition

The inner perturbed Dirac operator D' is parametrized by three complex scalar fields ϕ, σ_1, σ_2 entering in $A_{(1)}$ and $A_{(2)}$:

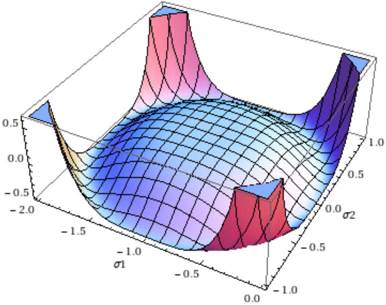
$$D_F + A_{(1)} + \hat{A}_{(1)} + A_{(2)} = \begin{pmatrix} 0 & k_x(1+\phi)\otimes 1_2 & \overline{k}_y \overline{v}\overline{v}^t & 0 \\ \overline{k}_x(1+\overline{\phi})\otimes 1_2 & 0 & 0 & \\ k_y v \ v^t & 0 & 0 & 1_2\otimes \overline{k}_x(1+\overline{\phi}) \\ 0 & 0 & 1_2 \otimes k_x(1+\phi) & 0 \end{pmatrix}$$

with $v = \begin{pmatrix} 1+\sigma_1 \\ \sigma_2 \end{pmatrix}$.

Spectral action

Spectral action gives rise to a scalar potential

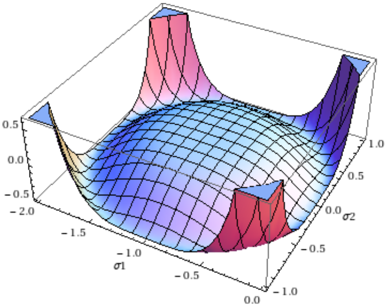
$$V(\phi,\sigma_1,\sigma_2) = -\frac{f_2}{\pi^2}\Lambda^2\left(4|k_x|^2|\phi|^2 + |k_y|^2(|1+\sigma_1|^2+|\sigma_2|^2)^2\right)$$
$$+\frac{f_0}{4\pi^2}\Bigg(4|k_x|^4|\phi|^4 + 4|k_x|^2|k_y|^2|\phi|^2(|1+\sigma_1|^2+|\sigma_2|^2)^2$$
$$+|k_y|^4(|1+\sigma_1|^2+|\sigma_2|^2)^4\Bigg)$$



Spontaneous symmetry breaking to first-order

Proposition

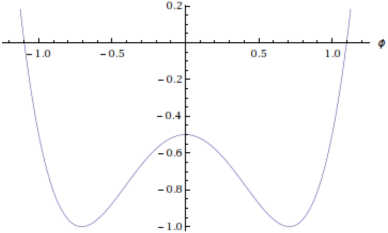
The potential $V(\phi = 0, \sigma_1, \sigma_2)$ *has a local minimum at* $(\sigma_1, \sigma_2) = (-1 + \sqrt{w}, 0)$ *with* $w = \sqrt{2f_2\Lambda^2/(f_0|k_y|^2)}$ *and this point spontaneously breaks the symmetry group* $\mathcal{U}(A'_F)$ *to* $\mathcal{U}(A_F)$*.*



"Usual" SSB

After the fields (σ_1, σ_2) have reached their vevs $(-1+\sqrt{w}, 0)$, there is a remaining potential for the ϕ-field:

$$V(\phi) = -\frac{2f_2}{\pi^2}\Lambda^2|k_x|^2|\phi|^2 + \frac{f_0}{\pi^2}|k_x|^4|\phi|^4.$$



Selecting one of the minima of $V(\phi)$ spontaneously breaks the symmetry further from $\mathcal{U}(A_F) = U(1)_R \times U(1)_L \times U(1)$ to $U(1)_L \times U(1)$, and generates mass terms for the $L - R$ abelian gauge field.

The Standard Model revisited

[Chamseddine–Connes–vS, 2013]

A similar, but more elaborate treatment can be given for the Standard Model algebra, through the inclusion

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

- This subalgebra is the maximal subalgebra that satisfies first-order for a so-called irreducible spectral triple on $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$, and can be selected via spontaneous symmetry breaking.
- Physically, this corresponds to the Pati–Salam model with $SU(2)_R \times SU(2)_L \times SU(4)$ symmetry, spontaneously breaking to the $U(1) \times SU(2)_L \times SU(3)$ Standard Model symmetry.

Spectral action: pure gravity

Proposition

For the canonical triple $(C^\infty(M), L^2(M,S), \not{\partial})$, *the spectral action is*

$$\mathrm{Tr}\ \left(f\left(\frac{\not{\partial}}{\Lambda}\right)\right) \sim \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s + \frac{f(0)}{16\pi^2}\left(\frac{1}{30}\Delta s - \frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\right).$$

Coefficients NCSM

$$
\begin{aligned}
a &= \mathrm{Tr}\,\left(Y_\nu^* Y_\nu + Y_e^* Y_e + 3 Y_u^* Y_u + 3 Y_d^* Y_d\right),\\
b &= \mathrm{Tr}\,\left((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + 3(Y_d^* Y_d)^2\right),\\
c &= \mathrm{Tr}\,\left(Y_R^* Y_R\right),\\
d &= \mathrm{Tr}\,\left((Y_R^* Y_R)^2\right),\\
e &= \mathrm{Tr}\,\left(Y_R^* Y_R Y_\nu^* Y_\nu\right).
\end{aligned}
\tag{1}
$$