The Higgs boson and noncommutative geometry

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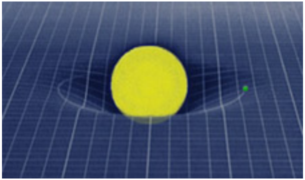
Geometry and physics

Mathematics:

Physics:

Riemannian Geometry

General relativity
gravity



Noncommutative Geometry

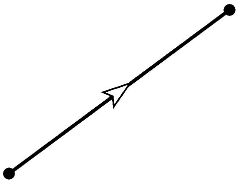
Gauge theories
electromagnetism
weak, strong interactions

Basics of Noncommutative Geometry

Device: spectral triple $(\mathcal{A}, H, D)$

Motivating example is Riemannian (spin) geometry :

- $\mathcal{A}$: algebra $F(M)$ of functions on spacetime M (coordinates)
- H: Hilbert space of spinors $L^2(M, S)$
- D: Dirac operator $\partial\!\!\!/ := \gamma^\mu \partial_\mu$ ('square root' of Laplacian)



Dirac equation describes motion of fermion in spacetime:
'screening geometry'

Essential properties:

1. Commutator $[D, f]$ is a *bounded operator* on H.
2. Eigenvalues of D are $\sim j^{1/\dim M}$.

The full Riemannian (spin) geometry of M can be reconstructed from this spectral triple $(\mathcal{A}, H, D)$

This motivates to define a **noncommutative geometry** as a triple

- $\mathcal{A}$: any algebra, represented on:
- H: a Hilbert space
- D: a hermitian (selfadjoint) operator on H

satisfying similar properties as before: *(1)* $[D, a]$ bounded and *(2)* a condition on the eigenvalues of D.)

There exist many examples of noncommutative geometries, but of interest to us is:

- $\mathcal{A}$: algebra of $n \times n$ matrices with as entries functions on spacetime (i.e. $F(M) \otimes M_n(\mathbb{C})$)
- H: vector space of $n \times n$ matrices with as entries spinors (i.e. $L^2(M, S) \otimes M_n(\mathbb{C})$)
- D: the above $\gamma^\mu \partial_\mu$ acting on each matrix entry

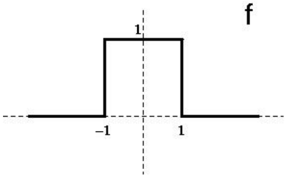
Spectral action

Passage from **metric** to **spectrum**:

- meter first (1791) defined as a part of the circumference of the earth, and made ‘concrete by platinum bar (1799)
- 1960’s: meter defined as a multiple of the wavelength corresponding to specific transitions in Krypton
- second defined as the frequency corresponding to a hyperfine transition in Caesium
- this allowed (1983) for a definition of the meter as the length that light travels in 1/299792458 seconds...

Spectral action:

$$S_\Lambda(D) = \#\text{ eigenvalues of } D \text{ up to cutoff } \Lambda$$
$$= \mathrm{Trace}\,(f(D/\Lambda))\,.$$



Computation of spectral action

Classical case: Riemannian geometry

$\mathcal{A}$ functions on M, H Hilbert space of spinors, D Dirac operator.
Heat kernel expansion:

$$S_\Lambda(D) = \sum_{k\geq 0} f_k \; a_k(D^2/\Lambda^2), \qquad \begin{cases} f_0 = \int_0^\infty t f(t)dt \\ f_2 = \int_0^\infty f(t)dt \\ \vdots \end{cases}$$

and a_k are the Seeley-de Witt coefficients given as certain integrals. Then, the spectral action turns out to be:

$$S_\Lambda(D) = (\Lambda^2)^2 f_0 \frac{1}{4\pi^2} \int_M \sqrt{g}\, dx + \Lambda^2 f_2 \frac{1}{4\pi^2} \frac{1}{12} \int_M \sqrt{g} R dx$$

$$+ f_4 \frac{1}{4\pi^2} \frac{1}{360} \int_M \sqrt{g}[3R^{\mu}_{;\mu} + \frac{5}{4}R^2 - 2R_{\mu\nu}R^{\mu\nu} - \frac{7}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}]dx + \cdots$$

which is (in red) the **Einstein-Hilbert action** of general relativity!
For this, we identify $\frac{\Lambda^2 f_2}{24\pi^2} = \frac{1}{8\pi G}$ (thus, cutoff $\Lambda \sim$ Planck energy).

Now consider: $\mathcal{A}$ algebra of $n \times n$ matrices with entries functions on spacetime, H Hilbert space of $n \times n$ matrices with entries spinors and D acting on each entry.

Spectral action of **perturbed Dirac operator** $D_A = D + A$ by inner fluctuations of the form $A = a[D, b]$ given by elements a, b in the algebra $\mathcal{A}$ can be computed as the spectral action for Riemannian geometry (derived before) plus **Yang-Mills term**:

$$S_\Lambda(D_A) = (\cdots) + \frac{g^2}{48\pi^2} F_{\mu\nu} F^{\mu\nu} + \cdots$$

with $F_{\mu\nu}$ the curvature (field strength) of the $SU(n)$ connection (gauge potential) A_μ defined by $A = g\gamma^\mu A_\mu$.

Inner fluctuations only non-trivial when the algebra is noncommutative

Fermions and their interactions with the gauge fields can be included by adding the term $(\psi, D_A\psi)$ for $\psi \in H$.

The spectral action for the Standard Model

including the Higgs!

With a slightly more advanced triple $(\mathcal{A}, H, D)$ involving not only matrices but a combination of the complex numbers, quaternions and 3×3 matrices, we find that

$S_\Lambda(D_A) =$ full action of the Standard Model including the Higgs boson

Its derivation relies on the following relations:

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4}; \qquad g_3^2 = g_2^2 = \frac{5}{3} g_1^2$$

with g_1 the electromagnetic, g_2 the weak and g_3 the strong coupling (GUT-type relations).
This geometrical form of the theory is supposed to take place at unification scale Λ; the masses, couplings, etc. are bare.

GUT triangle

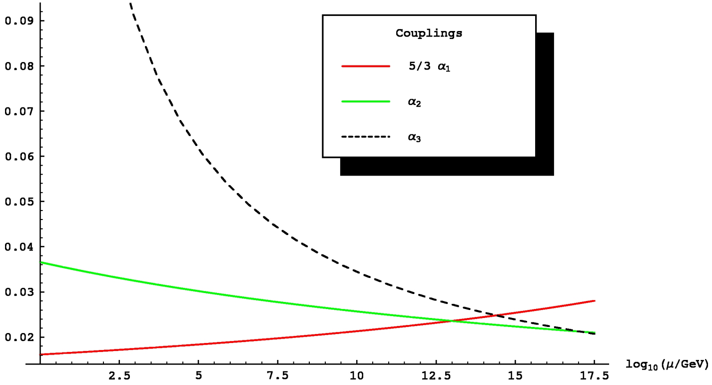


Figure: Running of the three couplings in the SM where $\alpha_i = \frac{g_i^2}{4\pi}$

Predicting the Higgs mass

If we identify the quartic Higgs term in the spectral action, we find a relation for the Higgs scattering parameter $\alpha_h = \frac{m_h^2}{4M_W^2}$:

$$\alpha_h(\Lambda) \sim \frac{8}{3}$$

This holds at the cutoff scale and by using renormalization group techniques, one can run it down and give a prediction of the mass of the Higgs boson:

$$m_h \sim 168 GeV$$

References

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