The Higgs boson and noncommutative geometry

Walter D. van Suijlekom

April 19, 2007
Geometry and physics

Mathematics:

- Riemannian Geometry
- Noncommutative Geometry

Physics:

- General relativity
  - gravity
- Gauge theories
  - electromagnetism
  - weak, strong interactions
Motivating example is Riemannian (spin) geometry:

- $\mathcal{A}$: algebra $F(M)$ of functions on spacetime $M$ (coordinates)
- $H$: Hilbert space of spinors $L^2(M, S)$
- $D$: Dirac operator $\mathcal{D} := \gamma^\mu \partial_\mu$ (‘square root’ of Laplacian)

Dirac equation describes motion of fermion in spacetime: ‘screening geometry’

Essential properties:

1. Commutator $[D, f]$ is a bounded operator on $H$.
2. Eigenvalues of $D$ are $\sim j^{1/\dim M}$.

The full Riemannian (spin) geometry of $M$ can be reconstructed from this spectral triple $(\mathcal{A}, H, D)$.
This motivates to define a **noncommutative geometry** as a triple

- $\mathcal{A}$: any algebra, represented on:
  - $H$: a Hilbert space
  - $D$: a hermitian (selfadjoint) operator on $H$

satisfying similar properties as before: (1) $[D, a]$ bounded and (2) a condition on the eigenvalues of $D$.

There exist many examples of noncommutative geometries, but of interest to us is:

- $\mathcal{A}$: algebra of $n \times n$ matrices with as entries functions on spacetime (i.e. $F(M) \otimes M_n(\mathbb{C})$)
- $H$: vector space of $n \times n$ matrices with as entries spinors (i.e. $L^2(M, S) \otimes M_n(\mathbb{C})$)
- $D$: the above $\gamma^\mu \partial_\mu$ acting on each matrix entry
Spectral action

Passage from **metric** to **spectrum**:

- meter first (1791) defined as a part of the circumference of the earth, and made ‘concrete by platinum bar (1799)
- 1960’s: meter defined as a multiple of the wavelength corresponding to specific transitions in Krypton
- second defined as the frequency corresponding to a hyperfine transition in Caesium
- this allowed (1983) for a definition of the meter as the length that light travels in $1/299792458$ seconds...

**Spectral action:**

\[ S_\Lambda(D) = \# \text{ eigenvalues of } D \text{ up to cutoff } \Lambda \]
\[ = \text{Trace} \left( f \left( D/\Lambda \right) \right) . \]
Computation of spectral action

Classical case: Riemannian geometry

A functions on $M$, $H$ Hilbert space of spinors, $D$ Dirac operator. Heat kernel expansion:

$$S_{\Lambda}(D) = \sum_{k \geq 0} f_k \ a_k(D^2/\Lambda^2),$$

\[
\begin{align*}
    f_0 &= \int_0^\infty tf(t)dt \\
    f_2 &= \int_0^\infty f(t)dt \\
    \vdots
\end{align*}
\]

and $a_k$ are the Seeley-de Witt coefficients given as certain integrals. Then, the spectral action turns out to be:

$$S_{\Lambda}(D) = (\Lambda^2)^2 f_0 \frac{1}{4\pi^2} \int_M \sqrt{g} \, dx + \Lambda^2 f_2 \frac{1}{4\pi^2} \frac{1}{12} \int_M \sqrt{g} R \, dx$$

$$+ f_4 \frac{1}{4\pi^2} \frac{1}{360} \int_M \sqrt{g} [3 R_{\mu\nu} R^{\mu\nu} + \frac{5}{4} R^2 - 2 R_{\mu\nu} R^{\mu\nu} - \frac{7}{4} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] \, dx + \cdots$$

which is (in red) the Einstein-Hilbert action of general relativity! For this, we identify $\frac{\Lambda^2 f_2}{24\pi^2} = \frac{1}{8\pi G}$ (thus, cutoff $\Lambda \sim$ Planck energy).
Now consider: Algebra of $n \times n$ matrices with entries functions on spacetime, Hilbert space of $n \times n$ matrices with entries spinors and $D$ acting on each entry.

Spectral action of perturbed Dirac operator $D_A = D + A$ by inner fluctuations of the form $A = a[D, b]$ given by elements $a, b$ in the algebra $A$ can be computed as the spectral action for Riemannian geometry (derived before) plus Yang-Mills term:

$$S_A(D_A) = (\cdots) + \frac{g^2}{48\pi^2} F_{\mu\nu} F^{\mu\nu} + \cdots$$

with $F_{\mu\nu}$ the curvature (field strength) of the $SU(n)$ connection (gauge potential) $A_\mu$ defined by $A = g \gamma^\mu A_\mu$.

Inner fluctuations only non-trivial when the algebra is noncommutative

Fermions and their interactions with the gauge fields can be included by adding the term $(\psi, D_A \psi)$ for $\psi \in H$. 
The spectral action for the Standard Model including the Higgs!

With a slightly more advanced triple \((\mathcal{A}, H, D)\) involving not only matrices but a combination of the complex numbers, quaternions and \(3 \times 3\) matrices, we find that

\[
S_\Lambda(D_A) = \text{full action of the Standard Model including the Higgs boson}
\]

Its derivation relies on the following relations:

\[
\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4}, \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2
\]

with \(g_1\) the electromagnetic, \(g_2\) the weak and \(g_3\) the strong coupling (GUT-type relations).

This geometrical form of the theory is supposed to take place at unification scale \(\Lambda\); the masses, couplings, etc. are bare.
GUT triangle

Figure: Running of the three couplings in the SM where $\alpha_i = \frac{g_i^2}{4\pi}$
Predicting the Higgs mass

If we identify the quartic Higgs term in the spectral action, we find a relation for the Higgs scattering parameter $\alpha_h = \frac{m_h^2}{4M_W^2}$:

$$\alpha_h(\Lambda) \sim \frac{8}{3}$$

This holds at the cutoff scale and by using renormalization group techniques, one can run it down and give a prediction of the mass of the Higgs boson:

$$m_h \sim 168\text{GeV}$$
References