

# Hopf algebras of Feynman graphs for gauge theories

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

## What Feynman graphs are...

Graphs built from a fixed set  $\{v_1, \dots, v_k\}$  of types of vertices (possibly  $k = \infty$  [BK]) and a fixed set  $\{e_1, \dots, e_N\}$  of types of edges.



Examples:

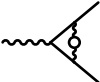

- **Scalar  $\phi^3$ -theory:**

vertex:  , edge: .

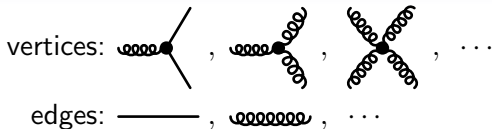
and one constructs graphs such as  , .

- **Electrodynamics:**

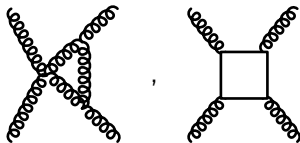
vertex:  , edges:  , .

and one constructs graphs such as  , .

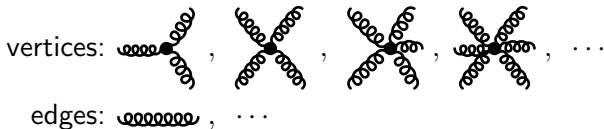
- Yang–Mills theory:



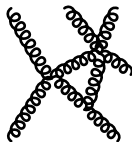
and one constructs graphs such as



- (Gravity):



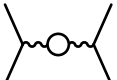

and one constructs graphs such as



## Hopf algebra of Feynman graphs

- Restrict to **one-particle irreducible** graphs (example **not 1PI**: )

- Residue** of a graph:  $\text{res} \left( \text{wavy line} \triangleleft \text{circle} \right) = \text{wavy line} \text{ and } \text{res} \left( \text{wavy line} \text{ --- } \text{blob} \right) = \text{wavy line}$

Eg. a graph not allowed:  since 1PI and residue   $\neq v_i$

### Theorem (Connes–Kreimer, 2000)

The free commutative algebra  $H$  generated by 1PI graphs with residue  $v_i$  and  $e_j$  is a Hopf algebra with counit  $\epsilon(\Gamma) = \delta_{\Gamma, \emptyset}$  and coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over (disjoint unions of) 1PI subgraphs with residue  $v_i$  or  $e_j$ .

Examples of the coproduct with  $v = \begin{array}{c} \diagup \\ \diagdown \end{array}$  and  $e = \text{---}$

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$

$$\Delta \left( \begin{array}{c} \diagup \\ \diagdown \end{array} \right) = \begin{array}{c} \diagup \\ \diagdown \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \diagup \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \otimes \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$\Delta \left( \begin{array}{c} \diagup \\ \diagdown \\ \circ \end{array} \right) = \begin{array}{c} \diagup \\ \diagdown \\ \circ \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \diagup \\ \diagdown \\ \circ \end{array} + \begin{array}{c} \diagup \\ \diagdown \\ \circ \end{array} \otimes \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$\Delta \left( \begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right) = \begin{array}{c} \circ \\ \circ \\ \circ \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \circ \\ \circ \\ \circ \end{array} + 2 \begin{array}{c} \diagup \\ \diagdown \end{array} \otimes \begin{array}{c} \circ \\ \circ \end{array} + 2 \begin{array}{c} \diagup \\ \diagdown \end{array} \otimes \begin{array}{c} \circ \\ \circ \\ \circ \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \otimes \begin{array}{c} \diagup \\ \diagdown \end{array} \otimes \begin{array}{c} \circ \\ \circ \end{array}$$

## Renormalization as decomposition $\text{Hom}_{\mathbb{C}}(H, \mathbb{C})$

- The **Feynman rules** specify an element in the group  $\text{Hom}_{\mathbb{C}}(H, \mathbb{C})$  of characters of  $H$ :

$$U_z : H \rightarrow \mathbb{C}$$

with  $z$  a regularization parameter (cutoff, dim-reg, ...).

- The **renormalized Feynman rules** are given by the map

$$R_z = C_z^{-1} * U_z$$

where  $C_z$  is the **counterterm map** defined using the coproduct by

$$C_z(X) = \epsilon(X) - T [m(C_z \otimes U_z)(1 \otimes (1 - \epsilon)\Delta(X))]$$

- $R_z$  is **finite** (renormalized) as  $z \rightarrow 0$ .
- All the maps  $U_z, C_z, R_z$  are **characters** in  $\text{Hom}_{\mathbb{C}}(H, \mathbb{C})$ ; such a decomposition works in general for **graded connected Hopf algebras**.

## Gauge theories

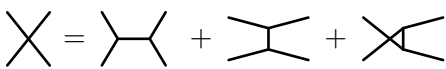
- The **physical (renormalized) Green's functions** are given by

$$\phi_r(p, \mu, \alpha, \pm, \dots) R_{z=0}(G^r)(p, \mu, \alpha, \pm, \dots)$$

with  $r = v_i, e_j$  and  $\phi_r$  the corresponding **formfactors** (depending on momenta, Lorentz and spinor indices, chiralities *et cetera*) and

$$G^{v_i} = 1 + \sum_{\text{res}(\Gamma)=v_i} \frac{\Gamma}{|\text{Aut}(\Gamma)|}, \quad G^{e_j} = 1 - \sum_{\text{res}(\Gamma)=e_j} \frac{\Gamma}{|\text{Aut}(\Gamma)|}$$

- Gauge symmetries** imply certain identities between these **formfactors**, such as in pure Yang–Mills theories:

“  $\phi_{\text{X}} = \phi_{\text{Y}} \frac{1}{\phi_{\text{Z}}} \phi_{\text{W}}$  ”, or 

- For **renormalizability of gauge theories** it is **essential** for these identities to hold at **any loop order**: the **Slavnov–Taylor identities** for the couplings

$$G^{\text{X}} G^{\text{W}} = (G^{\text{Y}})^2, \quad \dots$$

- Thus, we first need an expression for the coproduct on the  $G^r$ 's.



# Structure of $H$

## Gradings

- Grading by **loop number**  $l(\Gamma) = h^1(\Gamma)$ :

$$H = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^l, \quad q_l : H \rightarrow H^l$$

- Multigrading by **number of vertices**:

$$d_i(\Gamma) = \#\text{vertices } v_i \text{ in } \Gamma - \delta_{v_i, \text{res}(\Gamma)}$$

with

$$H = \bigoplus_{n_1, \dots, n_k \in \mathbb{Z}^k} H^{n_1, \dots, n_k}, \quad p_{n_1, \dots, n_k} : H \rightarrow H^{n_1, \dots, n_k}$$

- These are related via  $\sum_{i=1}^k (\text{val}(v_i) - 2)d_i = 2l$ .

**N.B.** **Connected Hopf algebra:**  $H^0 = H^{0, \dots, 0} = \mathbb{C}1$ .

# Structure of $H$

## Hopf subalgebras

- **Question: what is the coproduct on Green's functions?**
- Example: scalar  $\phi^3$ -theory (with one type of vertex  $v = \text{---} \langle$  and one type of edge  $e = \text{---}$ ) :

### Proposition

The elements  $X = G^v(G^e)^{-3/2}$  and  $G^e$  generate a *Hopf subalgebra* in  $H$ :

$$\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X), \quad \Delta(G^e) = \sum_{l=0}^{\infty} G^e X^{2l} \otimes q_l(G^e)$$

- This is recognized as the Hopf algebra dual to (a subgroup of) the group  $\mathbb{C}[[\lambda]]^\times \rtimes \overline{\text{Diff}}(\mathbb{C}, 0)$ . Namely, a character  $\phi$  on this Hopf subalgebra defines
  - 1 An **invertible formal power series** by  $\sum_{l=0}^{\infty} \phi(q_l(G^e))\lambda^l$
  - 2 A **formal diffeomorphism** on  $\mathbb{C}$  by  $\lambda \mapsto \sum_{l=0}^{\infty} \phi(q_l(X))\lambda^{l+1}$ .

# Structure of $H$

## Hopf subalgebras and ideals

- In general (vertices  $\{v_1, \dots, v_k\}$  and edges  $\{e_1, \dots, e_N\}$ ), we define for each vertex  $v$

$$X_v := \left( \frac{G^v}{\prod_i (G^{e_j})^{\text{val}_j(v)/2}} \right)^{1/\text{val}(v)-2}$$

### Proposition (vS, 2008)

The *coproduct on the Green's functions* reads

$$\Delta(G^r) = \sum_{n_1, \dots, n_k} G^r(X_{v_1})^{n_1(\text{val}(v_1)-2)} \dots (X_{v_k})^{n_k(\text{val}(v_k)-2)} \otimes p_{n_1, \dots, n_k}(G^r),$$

- On the elements  $X_{v_i}$  we then have

$$\Delta(X_v) = \sum_{n_1, \dots, n_k} X_v(X_{v_1})^{n_1(\text{val}(v_1)-2)} \dots (X_{v_k})^{n_k(\text{val}(v_k)-2)} \otimes p_{n_1, \dots, n_k}(X_v),$$

It appears that  $X_v$  and  $G^e$  for all vertices  $v$  and edges  $e$  generate a **Hopf subalgebra**, when restricted to each multidegree...

# Structure of $H$

## Hopf subalgebras and ideals

### Theorem (vS, 2008)

- 1 The elements  $X_{v_i}$  and  $G^{e_j}$  generate a Hopf subalgebra  $H'$  in  $H$  with dual group

$$G := \text{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset (\mathbb{C}[[\lambda_1, \dots, \lambda_k]]^\times)^N \rtimes \overline{\text{Diff}}(\mathbb{C}^k)$$

- 2 The ideal  $J := \langle X_{v_i} - X_{v_j} \rangle$  in  $H'$  is a Hopf ideal, i.e.  $H'/J$  is a Hopf algebra with dual group

$$\text{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset (\mathbb{C}[[\lambda]]^\times)^N \rtimes \overline{\text{Diff}}(\mathbb{C})$$

- The relations  $X_{v_i} = X_{v_j}$  in the quotient Hopf algebra  $H'/J$  are called (generalized) **Slavnov–Taylor identities** for the couplings.

# Structure of $H$

The gauge theory theorem

With  $B_+^\gamma$  the (suitably normalized) **grafting operator** on  $H$ , we can define a linear map  $B_+^{k;r} : H \rightarrow H$  by

$$B_+^{k;r} = \sum_{\substack{\gamma \text{ prim} \\ l(\gamma)=k, \text{ res}(\gamma)=r}} \frac{1}{\text{Sym}(\gamma)} B_+^\gamma$$

Denote  $X_{k,r} = G^r(X_v)^{2k} \in H/J$ , independent of the choice of the vertex  $v$ .

Theorem (Kreimer, 2005)

In the **quotient Hopf algebra**  $H/J$ , the following hold

- ①  $G^r = \sum_{k=0}^{\infty} B_+^{k;r}(X_{k,r})$
- ②  $\Delta(B_+^{k;r}(X_{k,r})) = B_+^{k;r}(X_{k,r}) \otimes I + (\text{id} \otimes B_+^{k;r})\Delta(X_{k,r})$ .
- ③  $\Delta(G_k^r) = \sum_{j=0}^k \text{Pol}_j^r(G) \otimes G_{k-j}^r$ .

where  $\text{Pol}_j^r(G)$  is a polynomial in the  $G_m^r$  of degree  $j$ , determined as the order  $j$  term in the loop expansion of  $G^r(X_v)^{2k-2j}$ .

## Application I: ST identities and renormalization

We now apply the above results to (Yang–Mills) gauge theories (eg. QCD).

- Suppose we are given a **gauge invariant regularization scheme**, such that

$$U_z(X_{v_i}) = U_z(X_{v_j}); \quad (\forall i, j = 1, \dots, k).$$

or, more explicitly, in QCD: **Slavnov–Taylor identities** for the couplings,

$$U_z \left( \frac{G^{\times}}{(G^{\leftarrow})^2} \right) = U_z \left( \left( \frac{(G^{\leftarrow})}{(G^{\leftarrow})^{3/2}} \right)^2 \right) \text{ i.e. } U_z(G^{\times}G^{\leftarrow}) = (U_z(G^{\leftarrow}))^2.$$

- Thus, the map  $U_z$  is a character on the quotient  $H'/J$  and there exists a decomposition  $U_z = C_z^{-1} * R_z$  as characters on  $H'/J$ .
- Consequently, both  $C_z$  and  $R_z$  vanish on  $J$ , in other words

$$C_z(X_{v_i}) = C_z(X_{v_j}); \quad R_z(X_{v_i}) = R_z(X_{v_j})$$

In particular, the **Slavnov–Taylor identities for the couplings are satisfied by the renormalized Feynman rules**.

- **N.B.** This is a completely (Hopf) algebraic proof of this physical fact.

## (Potential) Application II: MHV/BCFW rules [KvS]

Consider the following  $n$ -valence **tree-level gluon amplitude** in QCD:

$$p_k(z) \text{ (diagram)} =: A_n^{\text{tree}}$$

- Shift two external momenta (i.e.  $p_k, p_l$ ) by a **complex amount  $z$** .
- One can show that  $A(z)$  is **meromorphic** in  $z$  with **simple poles**;
- These poles appear in the **propagators that 'split  $A_n(z)$  into two pieces'**:

$$\sum_{m=2}^n : \text{ (diagram) } : \quad \text{implying } A_n = \sum_{m=2}^n A_{m+1}^{\text{tree}} \frac{1}{P_m^2} A_{n-m+1}^{\text{tree}}$$

- At higher loop order, this would become:  $A_n = \sum_{m=2}^n A_{m+1} \frac{1}{A_2} A_{n-m+1}$ ; compare to our Hopf ideal which is generated (in this case) by

$$G^{(m+1)} \frac{1}{G^{(2)}} G^{(n-m+1)} = G^{(n)}$$

# General setup

## Comodule Gerstenhaber algebras

We now establish a connection between the **Hopf algebra of renormalization** and a **Gerstenhaber structure** in the context of gauge theories, with the goal of **explaining the origin of the Hopf ideals  $J$** .

- In general, we assign to each vertex  $v_i$  a **parameter  $\lambda_i$**  for  $i = 1, \dots, k$ .
- To each edge  $e_j$  we assign a **field  $\phi_j$**  and a corresponding **antifield  $\phi_j^\ddagger$**  for  $j = 1, \dots, N$  (with certain degrees); the **anti-bracket** is defined by

$$\left( \phi_i(x), \phi_j^\ddagger(y) \right) = \delta_{ij} \delta(x - y).$$

and zero on other generators.

- This makes the following a **Gerstenhaber algebra**:

$$A := \mathcal{F}([\phi_1, \phi_1^\ddagger, \dots, \phi_N, \phi_N^\ddagger]) \otimes \mathbb{C}[[\lambda_1, \dots, \lambda_k]]$$

i.e. **a graded algebra with a Lie bracket of degree 1.**



The algebra  $A = \mathcal{F}([\phi_1, \phi_1^\dagger, \dots, \phi_N, \phi_N^\dagger]) \otimes \mathbb{C}[[\lambda_1, \dots, \lambda_k]]$  consists of  $\mathbb{C}[[\lambda_1, \dots, \lambda_k]]$ -linear functionals in the fields.

Proposition (vS, 2008)

The algebra  $A$  is a *Gerstenhaber comodule algebra* over  $H'$ . In other words, there exists a map  $\rho : A \rightarrow A \otimes H'$  compatible with the coproduct on  $H'$  and respecting the bracket in  $A$ .

Consequently, there is an *action of  $G \subset (\mathbb{C}[[\lambda_1, \dots, \lambda_k]]^\times)^N \rtimes \overline{\text{Diff}}(\mathbb{C}^k)$*  on  $A$ .

For instance, we have

$$\rho : \phi_j \longmapsto \sum_{n_1 \dots n_k} \phi_j \lambda_1^{n_1} \dots \lambda_k^{n_k} \otimes p_{n_1 \dots n_k} \left( (G^{e_j})^{1/2} \right) \quad (\text{invertible series})$$

$$\rho : \lambda_i \longmapsto \sum_{n_1 \dots n_k} \lambda_i \lambda_1^{n_1} \dots \lambda_k^{n_k} \otimes p_{n_1 \dots n_k} \left( (X_v)^{\text{val}(v_i)-2} \right) \quad (\text{formal diffeos}),$$

where we recall  $X_{v_i} = \left( \frac{G^{v_i}}{\prod_j (G^{e_j})^{\text{val}_j(v_i)/2}} \right)^{1/\text{val}(v_i)-2} \in H'$ .

## Master equation

- Next, one considers an element  $S \in A$  (the *action*) of the following form

$$S = \sum_r \int \phi_r m(r)$$

with  $m(r)$  monomials in the fields that interact/propagate at  $r$  and  $\phi_r$  are the formfactors. For a vertex:  $\phi_{v_i} = \lambda_i$  (modulo momenta, ...).

- The ideal  $I = \langle (S, S) \rangle$  implements the 'master equation'  $(S, S) = 0$  and

$$I = \langle p_\alpha(\lambda_1, \dots, \lambda_k) \rangle_\alpha, \quad p_\alpha \text{ polynomials}$$

- A theory (defined by  $S$ ) is called **simple** if  $I = \langle \lambda_i - \lambda^{\text{val}(v_i)-2} \rangle_i$  with  $\lambda := \lambda_j$  corresponding to some fixed valence 3 vertex  $v_j$ .

### Proposition (vS, 2008)

If  $S$  defines a simple theory, then the subgroup  $G' \subset G$  leaving  $I$  invariant is dual to  $H'/J$  (with  $J = \langle X_{v_i} - X_{v_j} \rangle_{i,j}$ ), i.e.

$$G' \simeq \text{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset (\mathbb{C}[[\lambda]]^\times)^N \rtimes \overline{\text{Diff}}(\mathbb{C}).$$

## Renormalization group

- The renormalization group appears as a subgroup of  $\text{Hom}_{\mathbb{C}}(H, \mathbb{C})$  when one realizes that the **regularized Feynman rules actually depend on a mass scale  $\mu$** . In fact, there is a decomposition

$$U_{\mu,z} = C_z^{-1} * R_{\mu,z}$$

- The **effective coupling constants** depend on  $\mu$  and are defined via the action of  $R_{\mu,z}$ :

$$\lambda_i(\mu) = R_{\mu,z=0} \lambda_i$$

The **beta-function** is then defined as

$$\beta(\lambda_i(\mu)) = \mu \frac{d}{d\mu} \lambda_i(\mu).$$

### Proposition (vS,2008)

*For a simple theory, all  $\beta$ -functions are expressed in terms of  $\beta(\lambda)$  for the fundamental coupling constant  $\lambda$ :*

$$\beta(\lambda_i) = \beta(\lambda^{\text{val}(v_i)-2}).$$

## Application to Yang–Mills theory

- The action is (essentially) the **Yang–Mills action** for a connection one-form  $\omega$  (with simple gauge group):

$$S = \|F(\omega)\|^2 + \dots = \int F_{\mu\nu}^a F_a^{\mu\nu} + \dots, \quad \text{with } F = d\omega + \frac{1}{2}\omega \wedge \omega.$$

- Feynman graphs are built from  $v_3 = \text{triple vertex}$  and  $v_4 = \text{quadruple vertex}$
- The corresponding Hopf algebra  $H$  coacts on the Gerstenhaber algebra  $A$  of  $\mathbb{C}[[\lambda_3, \lambda_4]]$ -linear functionals in  $\omega, \omega^\dagger, \dots$ , and

$$\text{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \overline{\text{Diff}}(\mathbb{C}^2)$$

- Gauge symmetry  $\rightsquigarrow$  master equation  $(S, S) = 0 \iff \lambda_4 - \lambda_3^2 = 0$
- The subgroup  $G'$  of  $\text{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \overline{\text{Diff}}(\mathbb{C}^2)$  that leaves this equation invariant is dual to the Hopf algebra  $H'/J$  with

$$J = \langle X_{\leftarrow} - X_{\times} \rangle$$

so that  $G' \subset \mathbb{C}[[\lambda]]^\times \rtimes \overline{\text{Diff}}(\mathbb{C})$  identifying  $\lambda_4 = \lambda_3^2 \equiv \lambda^2$ .

- Thus, in  $H'/J$  the identities  $X_{\leftarrow} = X_{\times}$  hold, or, explicitly

$$G_{\times} = \frac{(G_{\leftarrow})^2}{G_{\leftarrow}}$$

- As said, their appearance as generators of a Hopf ideal proves that the **ST-identities are compatible with renormalization**, so that if  $U_z$  satisfies them, it follows that  $R_z, C_z$  do so as well.
- Also **the beta functions for  $\lambda_3$  and  $\lambda_4$  are the same** so that for the effective couplings:  $\lambda_4(\mu) = \lambda_3(\mu)^2$  for all  $\mu$  and similarly for the quark and ghost couplings.

## Application to Yang–Mills theory

- Fields:  $A, \omega, \bar{\omega}, h$ .
- The gauge-fixed **Yang–Mills action**  $S_0$  plus BRST-sources (anti-fields) is

$$S = \int_M \text{tr} \left[ -dA * dA - \lambda_3 dA * [A, A] - \frac{1}{4} \lambda_4 [A, A] * [A, A] \right. \\ \left. - A * dh + d\bar{\omega} * d\omega + \frac{1}{2} \xi h * h + \lambda_2 d\bar{\omega} * [A, \omega] \right. \\ \left. + (sA) * A^\dagger + (s\omega) * \omega^\dagger + (s\bar{\omega}) * \bar{\omega}^\dagger + (sh) * h^\dagger. \right]$$

with

$$sA = -d\omega - \lambda_5 [A, \omega], \quad s\omega = -\frac{1}{2} \lambda_6 [\omega, \omega], \quad s\bar{\omega} = -h, \quad sh = 0$$

- The **master equation**  $(S, S)$  implies
  - $s^2 = 0$  which happens iff  $\lambda_5 = \lambda_6$
  - $s(S_0) = 0$  which happens iff  $\lambda_5 = \lambda_2 = \lambda_3$  and  $\lambda_3^2 = \lambda_4$ .

- In the quotient by  $I = \langle (S, S) \rangle$  we then have in terms of  $\lambda$ :

$$S = \int_M \text{tr} \left[ -dA * dA - \lambda dA * [A, A] - \frac{1}{4} \lambda [A, A] * [A, A] \right. \\ \left. - A * dh + d\bar{\omega} * d\omega + \frac{1}{2} \xi h * h + \lambda d\bar{\omega} * [A, \omega] \right. \\ \left. + (sA) * A^\dagger + (s\omega) * \omega^\dagger + (s\bar{\omega}) * \bar{\omega}^\dagger + (sh) * h^\dagger \right]$$

with  $sA = -d\omega - \lambda[A, \omega]$ ,  $s\omega = -\frac{1}{2}\lambda[\omega, \omega]$ ,  $s\bar{\omega} = -h$ ,  $sh = 0$

- The subgroup  $G'$  of  $\text{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_2, \dots, \lambda_6]]^\times \rtimes \overline{\text{Diff}}(\mathbb{C}^2)$  that leaves this equation invariant is **dual to the Hopf algebra  $H'/J$**  with  $J = \langle X_{v_i} - X_{v_j} \rangle$ ; so that  $G' \subset \mathbb{C}[[\lambda]]^\times \rtimes \overline{\text{Diff}}(\mathbb{C})$  identifying  $\lambda_2^2 = \lambda_3^2 = \lambda_4 = \lambda_5^2 = \lambda_6^2 \equiv \lambda^2$ .

- In  $H'/J$  we have:  $\frac{G^{\times}}{(G^{\text{---}})^2} = \left( \frac{(G^{\text{---}})}{(G^{\text{---}})^{3/2}} \right)^2, \dots$

- As said, their appearance as generators of a Hopf ideal proves that the **ST-identities are compatible with renormalization**
- Also **the beta functions for  $\lambda_2, \dots, \lambda_6$  are the same** so that  $\lambda_2^2(\mu) = \lambda_3^2(\mu) = \lambda_4(\mu) = \lambda_5^2(\mu) = \lambda_6^2(\mu) \equiv \lambda^2(\mu)$ .

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