# On the geometry of noncommutative gauge fields

Walter D. van Suijlekom
(joint work with G. Landi, C. Pagani, C. Reina)

21 Februari, 2008

Radboud University Nijmegen

- Our goal is to find a proper formulation of noncommutative gauge field theories; we do this by studying an example of a noncommutative 4-dimensional sphere $S^4_\theta$; it is the base space of a **noncommutative principal bundle**:

$$SU(2) \longrightarrow S^7_\theta \longrightarrow S^4_\theta$$

- This is a quantization of the Hopf fibration on the *commutative* 4-sphere, that plays a central role in the construction of (charge 1) instantons on $S^4$.
- Recall: moduli space $\mathcal{M}_1$ of charge 1 instantons on $S^4$ is isomorphic to the homogeneous space:

$$\boxed{\mathcal{M}_1 \simeq \mathsf{SL}(2,\mathbb{H}) \;/\; \mathrm{Sp}(2,\mathbb{H})}$$

conformal transformations　　　gauge transformations

- Question: What does the moduli space of (charge 1) instantons on $S^4_\theta$ look like?
- To answer this question, we first need a notion of gauge fields on $S^4_\theta$.

# Toric noncommutative manifolds

Let $M$ a compact Riemannian spin manifold ($\dim M = m$) with a smooth isometrical action of the $n$-torus $\mathbb{T}^n$.

- action $\sigma_s$ of $\mathbb{T}^n$ on $C^\infty(M)$ by automorphisms:

$$\sigma_s(f)(x) = f(s^{-1} \cdot x) \ .$$

- decompose $f = \sum f_r$ in homogeneous elements of degree $r$:

$$\sigma_s(f_r) = e^{2\pi i r \cdot s} f_r \ .$$

- $C^\infty(M)$ is represented on Hilbert space $\mathcal{H} = L^2(M, \mathcal{S})$ of spinors by pointwise multiplication: $\pi : C^\infty(M) \to \mathcal{B}(\mathcal{H})$.
- There is a representation $U$ of $\mathbb{T}^n$ on $\mathcal{H}$ such that

$$U(s) D U(s)^{-1} = D \ ,$$
$$U(s) \pi(f) U(s)^{-1} = \pi(\sigma_s(f)) \ .$$

Given any real $n \times n$ anti-symmetric matrix $\theta = (\theta_{\mu\nu})$ a twisted representation $L_\theta$ of $C^\infty(M)$ is defined by

$$L_\theta(f) = \sum_r f_r U(r_\mu \theta_{\mu 1}, \dots, r_\mu \theta_{\mu n}) \ .$$

and set $C^\infty(M_\theta) := L_\theta(C^\infty(M))$;

- quantization map $L_\theta : C^\infty(M) \to C^\infty(M_\theta)$ satisfying $L_\theta(f \times_\theta g) = L_\theta(f) L_\theta(g)$ with on homogeneous elements:

$$f_r \times_\theta g_{r'} = f_r \sigma_{r\cdot\theta}(g_{r'}) = e^{2\pi i r\cdot\theta\cdot r'} f_r g_{r'} \ .$$

- the triple $(C^\infty(M_\theta), \mathcal{H}, D)$ satisfies all properties of Connes' noncommutative spin geometry of dim $m$

(i.e. an m-summable spectral triple)

Noncommutative integral given by Dixmier trace:

$$\int\!\!\!\!\!\!- \; L_\theta(f) = \mathrm{Tr}_\omega L_\theta(f)|D|^{-m}.$$

### Lemma (GIV05)

*If $f \in C^\infty(M)$ then* $\int\!\!\!\!\!- \; L_\theta(f) = \int_M f \, \mathrm{d}\nu$

Also, Connes-Moscovici local index formula takes a simple form.

### Theorem

*For a projection $p \in M_N(C^\infty(M_\theta))$, the index of the twisted Dirac operator $D_p = pDp$ is given by:*

$$\mathrm{Index}\ D_p = \operatorname*{Res}_{z=0} z^{-1}\,\mathrm{tr}\left(\gamma p |D|^{-2z}\right) + \sum_{k\geq 1} c_k \operatorname*{Res}_{z=0} \mathrm{tr}\left(\gamma\big(p - \frac{1}{2}\big)[D,p]^{2k}|D|^{-2(k+z)}\right)$$

# Differential calculus on $M_\theta$

Let $(\Omega(M), \mathrm{d})$ be the usual differential calculus on $M$.

- Extend the map $L_\theta : C^\infty(M) \to C^\infty(M_\theta)$ to $\Omega(M)$ by imposing it to commute with $\mathrm{d}$. The image $L_\theta(\Omega(M))$ will be denoted by $\Omega(M_\theta)$.
- Similarly, there is a Hodge star operator on $\Omega(M_\theta)$ defined by

$$*_\theta L_\theta(\omega) = L_\theta(*\omega) \ .$$

  with $* : \Omega^p(M) \to \Omega^{m-p}(M)$ the classical Hodge star operator.
- inner product on $\Omega(M_\theta)$:

$$(\alpha, \beta)_2 = \int *_\theta(\alpha^* *_\theta \beta)$$

  since $*_\theta(\alpha^* *_\theta \beta)$ is an element in $C^\infty(M_\theta)$.

# Yang-Mills theory on $M_\theta$

A connection on a right $C^\infty(M_\theta)$-module $E$ is a map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^1(M_\theta)$$

obeying Leibniz rule, curvature $F = \nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^2(M_\theta)$.

- Yang-Mills action for a connection $\nabla$ on a finite projective $C^\infty(M_\theta)$-module $\mathcal{E}$ with curvature $F$ is defined by

$$S(\nabla) = \int\!\!\!\!- \operatorname{tr} *_\theta(F *_\theta F)$$

- Gauge invariance: $S(u^* \nabla u) = S(\nabla)$ for $u \in \mathcal{U}(\mathcal{E})$

  noncommutative vector bundle

  Infinitesimally: $S(\nabla + [\nabla, X]) = S(\nabla)$ with $X \in \mathrm{End}^s_{C^\infty(M_\theta)}(\mathcal{E})$.
- Equations of motion: nc Yang-Mills equations

$$\boxed{[\nabla, *_\theta F] = 0}$$

- Bianchi identiy $[\nabla, F] = 0 \Longrightarrow$ connections with (anti)selfdual curvature $*_\theta F = \pm F$ (instantons) are solutions of the YM equations; absolute minima of YM-action.

# The (Connes-Landi) sphere $S^4_\theta$

With $\theta$ a real parameter, the algebra $\mathcal{A}(S^4_\theta)$ of polynomial functions on the sphere $S^4_\theta$ is generated by elements $z_0 = z_0^*, z_j, z_j^*$, $j = 1, 2$, subject to

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu z_\mu^*, \quad \mu, \nu = 0, 1, 2,$$

with deformation parameters given by

$$(\lambda_{\mu\nu}) = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \bar{\lambda} \\ 1 & \lambda & 1 \end{smallmatrix}\right); \qquad \lambda := e^{2\pi i\theta}$$

and with the spherical relation $\sum_\mu z_\mu^* z_\mu = 1$.

- Isospectral deformation: nc spin geometry $(\mathcal{A}(S^4_\theta), \mathcal{H}, D)$ of dim 4;
- Differential calculus $(\Omega(S^4_\theta), \mathrm{d})$ as before, with $*_\theta : \Omega^p(S^4_\theta) \to \Omega^{4-p}(S^4_\theta)$.

# The sphere $S^7_{\theta'}$

With $\lambda'_{ab} = e^{2\pi i \theta'_{ab}}$ and $(\theta'_{ab})$ a real antisymmetric matrix, the algebra $\mathcal{A}(S^7_{\theta'})$ is generated by elements $\psi_a, \psi_a^*$, $a = 1, \ldots, 4$, subject to

$$\psi_a \psi_b = \lambda'_{ab} \psi_b \psi_a, \quad \psi_a \psi_b^* = \lambda'_{ba} \psi_b^* \psi_a,$$

and the spherical relation:

$$\sum_a \psi_a^* \psi_a = 1.$$

- Differential calculus $(\Omega(S^7_{\theta'}), \mathrm{d})$ as before.

# Noncommutative Hopf fibration

A minimal requirement for $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ to constitute a noncommutative $SU(2)$–principal bundle is that

there is an action $\alpha$ of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ such that $\mathcal{A}(S^4_\theta)$ can be identified with the subalgebra of invariant elements under this action

- These conditions express $\theta'$ in terms of $\theta$ and we identify:

$$\begin{aligned} z_0 &= \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4 \\ z_1 &= 2(\mu\psi_3^*\psi_1 + \psi_2^*\psi_4) \\ z_2 &= 2(-\mu\psi_4\psi_1^* + \psi_2\psi_3^*) \end{aligned} \qquad \theta'_{ab} = \frac{\theta}{2}\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

  where $\mu = \sqrt{\lambda} = e^{\pi i\theta}$, giving the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$.

- The $*$-action $\alpha$ of SU(2) on $\mathcal{A}(S^7_{\theta'})$ is given by

$$\alpha_w : (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \cdot \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$$

  with $w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \in \mathrm{SU}(2)$.

# Quantization of $S^7 \to S^4$ by an action of $\mathbb{T}^2$

The noncommutative Hopf fibration $S^7_{\theta'} \to S^4_\theta$ is a deformation of $S^7 \to S^4$ by an action of a 2-torus:

- Action of $\mathbb{T}^2 \subset \mathrm{SO}(5)$ on $S^4$ gives deformed algebra $\mathcal{A}(S^4_\theta)$
- Action of double cover $\widetilde{\mathbb{T}}^2 \subset \mathrm{Spin}(5)$ of $\mathbb{T}^2$ on $S^7$ gives deformed algebra $\mathcal{A}(S^7_{\theta'})$
- This lifted action commutes with the action of the structure group SU(2) which guarantees that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a noncommutative SU(2) principal bundle

## Associated vector bundles $S^7_{\theta'} \times_{\mathrm{SU}(2)} V$

We associate $\mathcal{A}(S^4_\theta)$-modules to the noncommutative principal bundle $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ by all finite-dimensional representations of $SU(2)$.

- Let $\rho$ be a representation of $SU(2)$ on $V^{(n)} = \mathbb{C}^{n+1}$:

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)} = \left\{ f \in \mathcal{A}(S^7_{\theta'}) \otimes V^{(n)} \ : (\alpha_w \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho(w)^{-1})(f) \right\}$$

- There are projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ s.t.

finite projective

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)} \ \simeq p_{(n)} \mathcal{A}(S^4_\theta)^{4^n}.$$

- Twisted Dirac operator $D_{(n)} = p_{(n)} D p_{(n)}$ with coefficients in the module $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)}$.

### Proposition

*The index of the Dirac operator $D_{(n)}$ on $S^4_\theta$ is given by*

$$\mathrm{Index}\ D_{(n)} = \frac{1}{6} n(n+1)(n+2)$$

# Basic (charge 1) instanton on $S^4_\theta$

A generic element in the module $\mathcal{E} = \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ can be written as $\Psi^* f$, $f \in \mathcal{A}(S^4_\theta)^4$ with

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \\ \psi_3 & -\psi_4^* \\ \psi_4 & \psi_3^* \end{pmatrix}; \qquad \text{satisfying } \Psi^*\Psi = \mathbb{I}_2.$$

recall that: $\alpha_w : \Psi \mapsto \Psi \cdot \left(\begin{smallmatrix} w & 0 \\ 0 & w \end{smallmatrix}\right)$

Thus, $p = \Psi\Psi^*$ is a projection in $M_4(\mathcal{A}(S^4_\theta))$ and in fact $\mathcal{E} \simeq p\mathcal{A}(S^4_\theta)^4$.
Explicitly:

$$p = \frac{1}{2} \begin{pmatrix} 1+z_0 & 0 & z_1 & -\overline{\mu} z_2^* \\ 0 & 1+z_0 & z_2 & \mu z_1^* \\ z_1^* & z_2^* & 1-z_0 & 0 \\ -\mu z_2 & \overline{\mu} z_1 & 0 & 1-z_0 \end{pmatrix}$$

- Grassmann connection $\nabla_0 = p\mathrm{d} \to$ curvature satisfies

$$\boxed{*_\theta F_0 = F_0}$$

For this reason, $\nabla_0$ is called the basic instanton on $S^4_\theta$, of charge 1, since

$$\left\langle [S^4_\theta], \mathrm{ch}_2(p) \right\rangle = 1$$

- In terms of $f \in \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ we have

$$(\nabla_0 f)_i = \mathrm{d}f_i + \omega_{ij} f_j.$$

where $\omega = \Psi^* \mathrm{d}\Psi$ is the basic instanton gauge potential, a $2 \times 2$-matrix with entries in $\Omega^1(S^7_{\theta'})$ satisfying:

$$\overline{\omega_{ij}} = -\omega_{ji} \ ; \qquad \sum_i \omega_{ii} = 0 \ .$$

noncommutative $su(2)$ gauge field

# Moduli space of (charge 1) instantons

[AHS78] Starting with the basic instanton $\nabla_0$ on $\mathcal{E}$, any other ($su(2)$) connection on $\mathcal{E}$ is given by $\nabla_0 + t\alpha$, with

$$\alpha \in \Omega^1(\mathrm{ad}(S^7_{\theta'})) := \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma(\mathrm{ad}(S^7_{\theta'}))$$

where $\Gamma(\mathrm{ad}(S^7_{\theta'}))$ is the associated module to the adjoint representation of $SU(2)$ on $su(2)$.

- Linearized selfdual equation: $P_-[\nabla_0, \alpha] = 0$; $P_- = \frac{1}{2}(1 - *_\theta)$.
- If $\alpha$ were obtained from an infinitesimal gauge transformation, then $\alpha = [\nabla_0, X]$ with $X \in \Gamma(\mathrm{ad}(S^7_{\theta'}))$.
- Since $P_-[\nabla_0, [\nabla_0, X]] = [P_- F_0, X] = 0$, we have the selfdual complex

$$0 \to \Omega^0(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{[\nabla_0,\cdot]} \Omega^1(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{P_-[\nabla_0,\cdot]} \Omega^2_-(\mathrm{ad}(S^7_{\theta'})) \to 0$$

and look for an element in the first cohomology group $H^1$

We compute the alternating sum $h^0 - h^1 + h^2$ of the dimensions of the cohomology groups as the index of a twisted Dirac operator, with coefficients in the noncommutative vector bundle $\mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})$ on $S^4_\theta$
(N.B. $\mathcal{S}^-$ is isomorphic to the charge $-1$ instanton bundle)

- Local index formula:

$$\begin{aligned}\mathrm{Index}\,(\mathcal{D}) &= \left\langle [S^4_\theta], \mathrm{ch}(\mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})) \right\rangle \\ &= 2 \cdot \left\langle [S^4_\theta], \mathrm{ch}_2(\mathrm{ad}(S^7_{\theta'})) \right\rangle - 3 \cdot \left\langle [S^4_\theta], \mathrm{ch}_2(\mathcal{S}^-) \right\rangle \\ &= 2 \cdot 4 - 3 \cdot 1 = 5\end{aligned}$$

- Using a vanishing argument for $h^0$ and $h^2$, we find that $h^1 = 5$.

**The tangent space of the moduli space at $\nabla_0$ has dimension 5**

$\boxed{\text{Problem: How can we generate a 5-parameter family of instantons?}}$

## Twisted infinitesimal symmetries

Consider the Lie algebra $so(5)$ with generators $H_1, H_2, E_r$ for the eight roots $r = (\pm 1, \pm 1), (0, \pm 1), (\pm 1, 0)$;

- Action on generators $z_0, z_1, z_1^*, z_2, z_2^*$ of $\mathcal{A}(S^4_\theta)$:

$$\begin{aligned} H_1 &= z_1 \partial_1 - z_1^* \partial_1^*, \qquad & H_2 &= z_2 \partial_2 - z_2^* \partial_2^* \\ E_{+1,+1} &= z_2 \partial_1^* - z_1 \partial_2^*, \qquad & & \textit{et cetera} \end{aligned}$$

- Extended to the whole of $\mathcal{A}(S^4_\theta)$ as twisted derivations [Sit01]:

$$\begin{aligned} E_r(ab) &= E_r(a) \lambda^{-r_1 H_2}(b) + \lambda^{-r_2 H_1}(a) E_r(b), \\ H_j(ab) &= H_j(a) b + a H_j(b), \end{aligned}$$

- The action of $so(5)$ by twisted derivations can be lifted to an action on $\mathcal{A}(S^7_{\theta'})$ of the same twisted type and extended to $\Omega^1(S^7_{\theta'})$.

### Proposition

*The instanton gauge potential $\omega$ is* **invariant** *under the twisted action of $so(5)$; in other words, $H_j(\omega) = E_r(\omega) = 0$.*

# (Twisted) Conformal Lie algebra

Consider the Lie algebra $so(5,1)$: it consists of the generators of $so(5)$ and generators $H_0, G_r$ with $r = (\pm 1, 0), (0, \pm 1)$.

- Action of $so(5,1)$ on the generators of $\mathcal{A}(S^4_\theta)$:

$$H_0 = \partial_0 - z_0(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*),$$

$$G_{1,0} = 2\partial_1^* - z_1(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + \overline{\lambda} z_2\partial_2 + \lambda z_2^*\partial_2^*),$$

*et cetera*

extended to $\mathcal{A}(S^4_\theta)$ as twisted derivations; lift to $\mathcal{A}(S^7_{\theta'})$ and $\Omega(S^7_{\theta'})$.

## Proposition

1. *The instanton gauge potential* $\omega = \Psi^* \mathrm{d}\Psi$ *transforms to* $\omega + t\delta\omega_i$, $i = 0, \ldots, 4$ *under* $tH_0, tG_r$ $(t \in \mathbb{R})$
2. *The curvature* $F_0$ *of basic instanton transforms to* $F_0 + t\delta F_i + \mathcal{O}(t^2)$ *with* $\delta F_0 = -2z_0 F_0$,

$$\delta F_1 = -2z_1 \lambda^{H2} F_0; \qquad \delta F_2 = -2z_2 \lambda^{H1} F_0;$$

*and* $\delta F_3 = \delta F_1^*, \delta F_4 = \delta F_2^*$, *which are still* **selfdual**.

# Drinfel'd twists

This twisting of $so(5)$ and $so(5,1)$ can be understood as a **Drinfel'd twist**:

- Universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra with coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X$, counit $\epsilon(X) = 0$ and $S(X) = -X$ for $X \in \mathfrak{g}$.
- This structure is twisted by an invertible element $\mathcal{F} \in H \otimes H$:

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}.$$

- The twist of $\mathcal{U}(so(5))$ and $\mathcal{U}(so(5,1))$ is given by the element

$$\boxed{\mathcal{F} = \lambda^{\frac{1}{2}(-H_1 \otimes H_2 + H_2 \otimes H_1)}; \qquad \lambda = e^{2\pi i\theta}}$$

  yielding the Hopf algebras $\mathcal{U}_\theta(so(5))$ and $\mathcal{U}_\theta(so(5,1))$.

- We have a 5-parameter family of *infinitesimal* instantons.
- Since the tangent space at $\nabla_0$ of the moduli space of charge 1 instantons on $S^4_\theta$ is 5-dimensional, this is a complete family.

How to integrate this infinitesimal family? Two possibilities:

1. Imitate [AHS78] to integrate tangent spaces to obtain local charts on moduli space $\mathcal{M}_1$, then patch these together.
2. Dualize Drinfel'd twist of *so*$(5,1)$ and *so*$(5)$ and quantize [Rie93] the corresponding Lie groups SL$(2,\mathbb{H})$ and $\mathrm{Sp}(2,\mathbb{H})$; then generate instantons via a coaction of these quantum groups on the noncommutative Hopf fibration.

Problem with the first option is that we need a global version of (noncommutative) SU(2) gauge transformations, corresponding to the infinitesimal gauge transformations given by $\mathrm{ad}(S^7_{\theta'})$. This is an known problem in noncommutative geometry: determinant not central.

We take the second approach and quantize these Lie groups

# Quantum groups $\mathrm{SL}_\theta(2,\mathbb{H})$ and $\mathrm{Sp}_\theta(2,\mathbb{H})$

- We have $\mathbb{T}^2 \subset \mathrm{Sp}(2,\mathbb{H}) \subset \mathrm{SL}(2,\mathbb{H})$ by setting:

$$(t_1, t_2) \mapsto \begin{pmatrix} e^{2\pi i t_1} & 0 \\ 0 & e^{2\pi i t_2} \end{pmatrix}.$$

- Adjoint action of $\mathbb{T}^2$ on $\mathrm{Sp}(2,\mathbb{H}) \subset \mathrm{SL}(2,\mathbb{H})$ give deformed products $\times_\theta$ on the algebras of (polynomial) functions $\mathcal{A}(\mathrm{Sp}(2,\mathbb{H}))$ and $\mathcal{A}(\mathrm{SL}(2,\mathbb{H}))$.

$$\mathcal{A}(\mathrm{Sp}_\theta(2,\mathbb{H})) = (\mathcal{A}(\mathrm{Sp}(2,\mathbb{H})), \times_\theta)$$
$$\mathcal{A}(\mathrm{SL}_\theta(2,\mathbb{H})) = (\mathcal{A}(\mathrm{SL}(2,\mathbb{H})), \times_\theta)$$

- Since $\mathrm{Sp}(2,\mathbb{H})$ and $\mathrm{SL}(2,\mathbb{H})$ are Lie groups, the algebras $\mathcal{A}(\mathrm{Sp}(2,\mathbb{H}))$ and $\mathcal{A}(\mathrm{SL}(2,\mathbb{H}))$ are Hopf algebras, with coproduct $\Delta$, counit $\epsilon$ and antipode $S$.
- It turns out that $(\mathcal{A}(\mathrm{Sp}_\theta(2,\mathbb{H})), \Delta, \epsilon, S)$ and $(\mathcal{A}(\mathrm{SL}_\theta(2,\mathbb{H})), \Delta, \epsilon, S)$ are still Hopf algebras.

undeformed coproduct, counit and antipode

# Noncommutative family of instantons

- We can 'embed' $S^7_{\theta'}$ in $\mathbb{H}^2_\theta$ to obtain a coaction:

$$\Delta_L : \mathcal{A}(S^7_{\theta'}) \to \mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^7_{\theta'}),$$

  and similarly for $\mathcal{A}(\mathrm{Sp}_\theta(2,\mathbb{H}))$ which is a quotient of $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))$.
- This induces a coaction of $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))^{\mathbb{Z}_2} =: \mathcal{A}(\mathsf{SO}_\theta(5,1))$ on $\mathcal{A}(S^4_\theta)$
- The basic instanton projection $p \in M_4(\mathcal{A}(S^4_\theta))$ is mapped to $P := \Delta_L(p)$ which is an element in $M_4(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$.
- The projection $P$ is understood as a noncommutative family of projections parametrized by the noncommutative space $\mathsf{SL}_\theta(2,\mathbb{H})$.

## Proposition

1. *The family of connection* $\widetilde{\nabla}_0 = P \circ (\mathrm{id} \otimes \mathrm{d})$ *is self-dual:*

$$(\mathrm{id} \otimes *_\theta) P((\mathrm{id} \otimes \mathrm{d})P)^2 = P((\mathrm{id} \otimes \mathrm{d})P)^2.$$

2. *The projection $P$ is (Murray-von Neumann) equivalent to the projection $1 \otimes p$ in the algebra $M_4\left(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta)\right)$.*

# Index pairing

Since $P \sim 1 \otimes p$, it follows that $\mathrm{ch}_n(P) \in \mathsf{HC}_{2n}(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$ coincides with the pushforward $\phi_* \mathrm{ch}_n(p)$ of $\mathrm{ch}_n(p) \in \mathsf{HC}_{2n}(\mathcal{A}(S^4_\theta))$ under the algebra map

$$\phi : \mathcal{A}(S^4_\theta) \to \mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta), \quad a \mapsto 1 \otimes a.$$

As a consequence, both $\mathrm{ch}_0(P)$ and $\mathrm{ch}_1(P)$ are zero since $\mathrm{ch}_0(p)$ and $\mathrm{ch}_1(p)$ vanish [CL01].
Then, one uses the map $\phi$ to pull back the fundamental class $[S^4_\theta] \in \mathsf{HC}^4(\mathcal{A}(S^4_\theta))$ to a class $\phi^*[S^4_\theta]$ in $\mathsf{HC}^4(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$. When paired with $\mathrm{ch}_2(P)$ it gives

$$\left\langle \phi^*[S^4_\theta], \mathrm{ch}_2(P) \right\rangle = \left\langle \phi^*[S^4_\theta], \phi_*\mathrm{ch}_2(p) \right\rangle = \left\langle [S^4_\theta], \mathrm{ch}_2(p) \right\rangle = 1,$$

In other words, the **instanton charge of the family $P$ is 1**.

# The family of connections

We can express $\widetilde{\nabla}_0$ in terms of a family of connection one-forms:

$$\widetilde{\omega}_{ij} = \Delta_L(\omega_{ij}) \in \mathcal{A}(\mathsf{SL}_\theta(2, \mathbb{H}) \otimes \mathcal{A}(S^7_{\theta'})$$

## Proposition

*The instanton connection 1-form $\omega$ is invariant under the coaction of the quantum group $\mathrm{Sp}_\theta(2, \mathbb{H})$, that is for this quantum group one has*

$$\Delta_L(\omega_{ab}) = 1 \otimes \omega_{ab}.$$

In other words, the family of connections is parametrized by the **quantum quotient** $\mathcal{M}_\theta := \mathsf{SL}_\theta(2, \mathbb{H})/\mathrm{Sp}_\theta(2, \mathbb{H})$, defined in terms of its function algebra:

$$\mathcal{A}(\mathcal{M}_\theta) := \{a \in \mathcal{A}(\mathsf{SL}_\theta(2, \mathbb{H})) \mid \Delta_R(a) = 1 \otimes a\}.$$

where $\Delta_R = (\pi \otimes \mathrm{id}) \circ \Delta$ is the (right) coaction of $\mathcal{A}(\mathrm{Sp}_\theta(2, \mathbb{H}))$ on $\mathcal{A}(\mathsf{SL}_\theta(2, \mathbb{H}))$ via the quotient map $\pi$.

## Structure of the noncommutative parameter space $\mathcal{M}_\theta$

The algebra $\mathcal{A}(\mathcal{M}_\theta)$ is generated by $m, n, g_1, g_2$ with relations:

$$g_1 g_2 = \lambda g_2 g_1, \qquad g_1 g_2^* = \overline{\lambda} g_2^* g_1,$$
$$mn - (g_1^* g_1 + g_2^* g_2) = 1,$$

Thus, $\mathcal{M}_\theta$ is a deformation of a hyperboloid in 6 dimensions.

In fact, if we write $w = \frac{1}{2}(m+n)$, $Y = \frac{1}{2}w^{-1}(m-n)$ and $G_1 = w^{-1}g_1, G_2 = w^{-1}g_2$:

$$Y^2 + G_1^* G_1 + G_2^* G_2 = 1 - w^{-2}.$$

and in the 'limit $w \to \infty$' (the 'boundary' of $\mathcal{M}_\theta$), we find noncommutative 4-spheres $S^4_\theta$.

# References


- G. Landi, C. Pagani, C. Reina and W. D. van Suijlekom. Noncommutative families of instantons. [arXiv:0710.0721]
- G. Landi and W. D. van Suijlekom. Noncommutative Bundles and Instantons in Tehran. In *An invitation to noncommutative geometry*, Eds. M. Khalkhali and M. Marcolli, [hep-th/0603053]
- G. Landi and W. D. van Suijlekom. Noncommutative instantons from twisted conformal symmetries. *Commun. Math. Phys.* 271 (2007) 591-634 [math.qa/0601554]
- G. Landi and W. D. van Suijlekom. Principal fibrations from noncommutative spheres. *Commun. Math. Phys.* 260 (2005), 203-225 [math.qa/0410077]