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Instantons on the noncommutative 4-sphere

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Hopf fibration $S^7 \to S^4$

Basic definitions:

$$\begin{aligned} S^7 &:= \{(z^1, z^2, z^3, z^4) : |z^1|^2 + |z^2|^2 + |z^3|^2 + |z^4|^2 = 1\} \\ S^4 &:= \{(\alpha, \beta, x) : \alpha\overline{\alpha} + \beta\overline{\beta} + x^2 = 1\} \\ \mathrm{SU}(2) &:= \{w \in GL(2, \mathbb{C}) : w^* w = w w^* = 1, \det w = 1\} \end{aligned}$$

S^7 carries right $\mathrm{SU}(2)$–action:

$$\begin{aligned} S^7 \times \mathrm{SU}(2) &\to S^7 \\ (z, w) &\mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \end{aligned}$$

Hopf map $\pi : (z^1, z^2, z^3, z^4) \mapsto (\alpha, \beta, x)$:

$$\begin{aligned} \alpha &= 2(z^1\overline{z}^3 + z^2\overline{z}^4), \\ \beta &= 2(-z^1 z^4 + z^2 z^3), \\ x &= z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4. \end{aligned}$$

such that $\alpha\overline{\alpha} + \beta\overline{\beta} + x^2 = (\sum_i |z^i|^2)^2 = 1$

$\pi : S^7 \to S^4$ is a principal $\mathrm{SU}(2)$–bundle

Dually: $C^\infty(S^4)$ consists of the element in $C^\infty(S^7)$ that are invariant under the action of $\mathrm{SU}(2)$.

Vector bundles and connections

In general, any vector bundle $E \to X$ can be described by the $C^\infty(X)$–module $\Gamma(X,E)$ of its smooth sections.
Theorem. [Serre-Swan] There exists a projection $p \in M_N(C^\infty(X))$ such that $\Gamma(X,E) \simeq p\big(C^\infty(X)\big)^N$.
A connection on E is defined as a map

$$\nabla : \Gamma(X,E) \to \Gamma(X,E) \otimes_{C^\infty(X)} \Omega^1(X)$$

satisfying Leibniz rule:

$$\nabla(sf) = (\nabla s)f + s \otimes \mathrm{d}f; \quad \big(s \in \Gamma^\infty(X,E), f \in C^\infty(S^4)\big)$$

Its curvature F is defined by ∇^2.

Instantons on S^4

An instanton on S^4 is a connection on a rank$_{\mathbb{C}}$ 2 vector bundle, carrying an action of $\mathrm{SU}(2)$, with self-dual curvature $*F = F$.

Basic instanton ($k = 1$) on S^4 is described by projection p as follows. Define isometry (gauge) in terms of polynomials on S^7:

$$u := (u_{ia}) = \begin{pmatrix} z^1 & z^2 \\ -\overline{z}^2 & \overline{z}^1 \\ z^3 & z^4 \\ -\overline{z}^4 & \overline{z}^3 \end{pmatrix}; \qquad (i = 1, \ldots, 4, a = 1, 2)$$

which satisfies $u^* u = \mathbb{I}_2$, or in components $\sum_i \overline{u_{ia}} u_{ib} = \delta_{ab}$.
Then $p = uu^*$ (in components $p_{ij} = u_{ia} \overline{u_{ja}}$) is a projection.

Entries of p are polynomials on S^4, as follows from explicit calculation:

$$p = \frac{1}{2}\begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\beta^* & \alpha^* \\ \alpha^* & -\beta & 1-x & 0 \\ \beta^* & \alpha & 0 & 1-x \end{pmatrix}$$

s.t. $p \in M_4(C^\infty(S^4))$. By Serre-Swan, it defines a vector bundle E on S^4

$$\Gamma(S^4, E) := p\big(C^\infty(S^4)\big)^4$$

of rank 2 ($= \mathrm{Tr}\, p$), and a connection (covariant derivative) by $\nabla = p \circ \mathrm{d}$. Its curvature $F = \nabla^2$ satisfies $*F = F$.

∇ describes the basic (charge 1) instanton on S^4

More intuitive:
Gauge potential corresponding to ∇ is $\omega = u^* \mathrm{d}u$, which is an anti-hermitian matrix with entries in $\Omega^1(S^7)$, i.e. $\omega^* = -\omega$.

Curvature can be expressed as $F = \mathrm{d}\omega + \omega \cdot \omega$.

Conformal group

Other (charge 1) instantons are generated from the basic instanton by the group $\mathrm{SL}(2, \mathbb{H})$ of conformal transformations on S^4.
For each $g \in \mathrm{SL}(2, \mathbb{H})$, the isometry u transform as

$$u \mapsto u^g := \rho \; g \cdot u$$

where $\rho^{-2} = (g \cdot u)^*(g \cdot u)$ is a normalization such that $(u^g)^* u^g = \mathbb{I}$.

We obtain a projection $p_g = u^g (u^g)^*$ and connection $\nabla_g = p_g \circ \mathrm{d}$, with selfdual curvature: $*F_g = F_g$.

The gauge potential becomes

$$\omega^g = u^g \mathrm{d}u^g = \rho^2 \big[(g \cdot u)^* (g \cdot \mathrm{d}u) - (g \cdot \mathrm{d}u)^* (g \cdot u) \big]$$

For $g \in \mathrm{Sp}(2, \mathbb{H}) = \{g \in \mathrm{SL}(2, \mathbb{H}) : g^* g = 1\}$, we have $\omega^g = \omega$. These are the gauge transformations, so we obtain the moduli space of gauge equivalence classes of (charge 1) instantons as the five-dimensional quotient space

$$\mathcal{M} := \frac{\mathrm{SL}(2, \mathbb{H})}{\mathrm{Sp}(2, \mathbb{H})}$$

$\theta-$deformed spheres

• $\mathcal{A}(S^4_\theta)$ is complex unital $*$-algebra generated by α, β, x with $x = x^*$ a central element and relations:

$$\alpha\beta = \lambda\beta\alpha; \quad \alpha^*\beta = \overline{\lambda}\beta\alpha^*; \quad \alpha\alpha^* + \beta\beta^* + x^2 = 1; \qquad (\lambda \in S^1 \subset \mathbb{C})$$

• $\mathcal{A}(S^7_\theta)$ is complex unital $*$-algebra generated by $z^1, \ldots, z^4$ with relations

$$z^i z^j = \lambda^{ij} z^j z^i; \quad \overline{z}^i z^j = \lambda^{ji} z^j \overline{z}^i; \quad \sum z^i \overline{z}^i = 1$$

where $\lambda^{ij} = \begin{pmatrix} 1 & 1 & \mu & \mu \\ 1 & 1 & \mu & \mu \\ \overline{\mu} & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \overline{\mu} & 1 & 1 \end{pmatrix}$ with $\mu = \sqrt{\lambda}$

Differential forms on S^4_θ

$\Omega(S^4_\theta)$ is the complex unital graded $*$-algebra generated by the elements α, β, x of degree 0 and $d\alpha, d\beta, dx$ of degree 1 with relations:

$$\alpha d\beta - \lambda d\beta\alpha = 0, \qquad d\alpha\beta - \lambda\beta d\alpha = 0,$$

$$\alpha d\beta^* - \overline{\lambda} d\beta^*\alpha = 0, \qquad d\alpha^*\beta - \overline{\lambda}\beta d\alpha^* = 0,$$

$$d\alpha d\beta + \lambda d\beta\alpha = 0, \qquad d\alpha d\beta^* + \lambda d\beta^*\alpha = 0,$$

$$adx - dxa = 0, \quad dadx + dxda = 0, \qquad (a = \alpha, \beta)$$

There is a unique differential d such that $\mathrm{d} : a \to da$ on a generator a of $\mathcal{A}(S^4_\theta)$, and the involution is such that $(\mathrm{d}\omega)^* = \mathrm{d}\omega^*$.

Hodge star operator can be defined as a map: $*_\theta : \Omega^k(S^4_\theta) \to \Omega^{4-k}(S^4_\theta)$.

Similarly, we define $\Omega(S^7_\theta)$, being the $*$-algebra generated by the elements $z^1, \ldots, z^4$ of degree 0 and $dz^1, \ldots, dz^4$ of degree 1 with relations:

$$dz^\mu dz^\nu + \lambda^{\mu\nu} dz^\nu dz^\mu = 0; \quad d\overline{z}^\mu dz^\nu + \lambda^{\nu\mu} dz^\nu d\overline{z}^\mu = 0$$

$$z^\mu dz^\nu = \lambda^{\mu\nu} dz^\nu z^\mu; \quad \overline{z}^\mu dz^\nu = \lambda^{\nu\mu} dz^\nu \overline{z}^\mu$$

with $\lambda^{\mu\nu}$ of the above form.

Action of $SU(2)$

$\mathcal{A}(S^7_\theta)$ carries an action of $SU(2)$,

$$\alpha_w : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w^1 & w^2 & 0 & 0 \\ -\overline{w}^2 & \overline{w}^1 & 0 & 0 \\ 0 & 0 & w^1 & w^2 \\ 0 & 0 & -\overline{w}^2 & \overline{w}^1 \end{pmatrix}$$

where $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$.
More precisely: the map $w \mapsto \alpha_w$ is a group homomorphism:

$$\boxed{SU(2) \hookrightarrow \mathrm{Aut}(\mathcal{A}(S^7_\theta)}$$

Algebra of polynomials on 'base space' consists of invariants: $x \in \mathcal{A}(S^7_\theta)$ s.t. $\alpha_w(x) = x$. Since $SU(2)$ acts classically, we find that the algebra of invariants is generated by

$$\begin{aligned} \alpha &= 2(z^1\overline{z}^3 + z^2\overline{z}^4), \\ \beta &= 2(-z^1 z^4 + z^2 z^3), \\ x &= z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4. \end{aligned}$$

satisfying $\alpha\beta = \lambda\beta\alpha$ etc., i.e. the algebra of invariants is isomorphic to $\mathcal{A}(S^4_\theta)$:

$\mathcal{A}(S^4_\theta) \to \mathcal{A}(S^7_\theta)$ is a principal extension (=principal bundle)

Vector bundles and connections on S^4_θ

In the spirit of the Serre-Swan theorem, a vector bundle on S^4_θ, is a (finitely generated projective) right $\mathcal{A}(S^4_\theta)$-module, given by a projection $p \in M_N(\mathcal{A}(S^4_\theta))$ as $\Gamma(S^4_\theta, E) = p\big(\mathcal{A}(S^4_\theta)\big)^N$

A connection on $\Gamma(S^4_\theta, E)$ is defined to be a map

$$\nabla : \Gamma(S^4_\theta, E) \to \Gamma(S^4_\theta, E) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$$

satisfying Leibniz rule:

$$\nabla(sa) = (\nabla s)a + s \otimes \mathrm{d}a; \quad \big(s \in \Gamma(S^4_\theta, E), a \in \mathcal{A}(S^4_\theta)\big)$$

The curvature $F := \nabla^2$ is a map $\Gamma(S^4_\theta, E) \to \Gamma(S^4_\theta, E) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^2(S^4_\theta))$.

Instantons on S^4_θ

An instanton on S^4_θ is a connection on a module $\Gamma(S^4_\theta, E) = p\big(\mathcal{A}(S^4_\theta)\big)^N$, with $\mathrm{Tr}\ (p) = 2$ and selfdual curvature $*F = F$.

Again, define isometry (gauge) in terms of polynomials in $\mathcal{A}(S^7_\theta)$:

$$u := \begin{pmatrix} z^1 & z^2 \\ -\overline{z}^2 & \overline{z}^1 \\ z^3 & z^4 \\ -\overline{z}^4 & \overline{z}^3 \end{pmatrix}$$

which satisfies $u^* u = \mathbb{I}_2$ so that $p = uu^*$ is a projection.

Its entries are elements in $\mathcal{A}(S^4_\theta)$:

$$p = \frac{1}{2}\begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\lambda^{\frac{1}{2}}\beta^* & \bar{\lambda}^{\frac{1}{2}}\alpha^* \\ \alpha^* & -\bar{\lambda}^{\frac{1}{2}}\beta & 1-x & 0 \\ \beta^* & \lambda^{\frac{1}{2}}\alpha & 0 & 1-x \end{pmatrix}$$

which is (K-)equivalent to projection found in [CL].

A connection can be defined by $\nabla = p \circ \mathrm{d}$, which turns out to have selfdual curvature:

$$*_\theta F = F$$

∇ describes the basic instanton on S^4_θ

Again, we can associate a gauge potential to ∇: $\omega = u^* \mathrm{d}u$, which is again anti-hermitian, $\omega^* = -\omega$.

In progress...

- Definition of a quantum group $\mathcal{A}(\mathrm{SL}_\theta(2,\mathbb{H}))$ which coacts on the Hopf fibration $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_\theta)$.

- Induced coaction $u_{ia} \mapsto A_{ij} \otimes u_{ja}$ defines transformed projection p^A.

- Transformed gauge potential $\omega^A_{ab} = \overline{A_{ik}} A_{il} \otimes \overline{u_{ka}} \mathrm{d} u_{lb}$ where $\overline{A_{ik}} A_{il}$ are the generators of the quantum quotient $\mathcal{M}_\theta$ of $\mathrm{SL}_\theta(2,\mathbb{H})$ with $\mathrm{Sp}_\theta(2)$.

- $\mathcal{M}_\theta$ is the noncommutative moduli space of instantons.