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The noncommutative geometry of Yang–Mills fields

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We generalize to topologically non-trivial gauge configurations the description of the Einstein–Yang–Mills system in terms of a noncommutative manifold, as was done previously by Chamseddine and Connes. Starting with an algebra bundle and a connection thereon, we obtain a spectral triple, a construction that can be related to the internal Kasparov product in unbounded KK-theory. In the case that the algebra bundle has typical fiber $M_N(\mathbb{C})$, we construct a $PSU(N)$-principal bundle for which it is an associated bundle. The so-called internal fluctuations of the spectral triple are parametrized by connections on this principal bundle and the spectral action gives the Yang–Mills action for these gauge fields, minimally coupled to gravity. Finally, we formulate a definition for a topological spectral action.

1. Introduction

One of the main applications of noncommutative geometry to theoretical physics is in deriving the Yang–Mills action from purely geometrical data [1]. In fact, the full Lagrangian of the Standard Model of high-energy physics – including the Higgs potential – can be derived by starting with a noncommutative Riemannian spin manifold [2].

It is interesting to confront this with the geometrical approach to Yang–Mills theory (*cf.* [3]), using the language of principal fiber bundles and connections thereon. It turns out that the noncommutative geometrical description of [1] corresponds to topologically trivial $\mathrm{SU}(N)$-principal bundles. It is the goal of this paper to generalize this to topologically non-trivial gauge configurations. As a matter of fact, we derive the Yang–Mills action for gauge fields defined on a non-trivial principal bundle from a noncommutative Riemannian spin manifold, that is, from a spectral triple. Since spectral triples – and more generally, (unbounded) KK-theory – form a natural setting for doing index theory, our construction has potential applications to *e.g.* the study of moduli spaces of instantons in noncommutative geometry.

Our construction will naturally involve algebra bundles and connections thereon, for which – after some preliminaries – we will give a definition in Section 3. There, we will also construct a spectral triple from this data. The above connection plays the same role as it does in the internal Kasparov product in KK-theory and we will explore this relation in some detail in Section 3.3.

In the case that the algebra bundle has typical fiber $M_N(\mathbb{C})$ it is possible to construct a $\mathrm{PSU}(N)$-principal bundle, with the algebra bundle as an associated bundle. We will explore this case in Section 4 and also relate it to the Dixmier–Douady class of continuous trace C^*-algebras. The so-called internal fluctuations of the above spectral triple are parametrized by connections on this principal bundle. Finally, we show that the spectral action principle applied to the spectral triple gives the Yang–Mills action on a topologically non-trivial $\mathrm{PSU}(N)$-principal bundle, minimally coupled to gravity.

In the concluding section, we sketch the definition of a so-called topological spectral action.

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2. Preliminaries

2.1. Spectral triples and the spectral action principle

Spectral triples, as they are introduced in [4] are at the heart of noncommutative geometry. In fact, they generalize spin^c-structures to the noncommutative world.

Definition 2.1 (*[4]*)**.** A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ represented faithfully on the Hilbert space $\mathcal{H}$, together with a densely defined, self-adjoint operator D on $\mathcal{H}$ with the following properties:

- The resolvent operators $(D-\lambda)^{-1}$ are compact on $\mathcal{H}$ for all $\lambda \notin \mathbb{R}$,
- For all $a \in \mathcal{A}$ the operator $[D, a]$ extends to a bounded operator defined on $\mathcal{H}$.

The triple is said to be *even* if there exists an operator γ on $\mathcal{H}$ with the properties

$$\gamma^* = \gamma, \qquad \gamma^2 = 1, \qquad \gamma D + D\gamma = 0, \qquad \gamma a - a\gamma = 0.$$

If such an operator does not exist, then the triple is said to be *odd*.

Example 2.2. The motivating example for the definition of a spectral triple is formed by the *canonical triple*

$$(C^\infty(M), L^2(M,S), \not{D})$$

associated to any compact Riemannian spin manifold M.[1] The Hilbert space $L^2(M,S)$ consists of square-integrable sections of the spinor bundle $S \to M$. The operator $\not{D}$ is the Dirac operator on the spinor bundle. For even dimensional spin manifolds there exists a grading γ on $L^2(M,S)$.

A spectral triple can have additional structure such as reality.

Definition 2.3 (*[5, Definition 1.124]*)**.** A *real structure* on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \quad \text{and} \quad J\gamma = \epsilon''\gamma J, \quad \text{(even case)},$$

where the numbers $\epsilon, \epsilon', \epsilon''$ are ±1. Moreover, there are the following relations between J and elements of $\mathcal{A}$:

$$[a, b^0] = 0, \qquad [[D,a], b^0] = 0 \quad \text{for all } a, b \in \mathcal{A}, \tag{1}$$

where $b^0 = Jb^*J^{-1}$. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ endowed with a real structure J is called a *real spectral triple*.

The signs ϵ, ϵ' and ϵ'' determine the so-called KO-dimension (modulo 8) of the real spectral triple (see [6] for more details).

Example 2.4. For a spin manifold and a given spinor bundle S there exists an operator J_M – called charge conjugation – on $L^2(M,S)$ such that

$$(C^\infty(M), L^2(M,S), \not{D}, J_M)$$

is a real spectral triple. Here the *KO*-dimension is equal to the dimension of the spin manifold M. For more details on the construction of J_M the reader is referred to *e.g.* [7]. When the dimension n is even, the inclusion of the grading operator γ of Example 2.2 to the datum

$$(C^\infty(M), L^2(M,S), \not{D}, J_M, \gamma) \tag{2}$$

yields a real and even spectral triple.

Remark 2.5. Note that the existence of a real structure J turns $\mathcal{H}$ into a bimodule over $\mathcal{A}$. Indeed, condition (1) implies that the right action of $\mathcal{A}$ on $\mathcal{H}$ defined by

$$\xi a := Ja^*J^*\xi, \quad (\xi \in \mathcal{H},\ a \in \mathcal{A})$$

commutes with the left action of $\mathcal{A}$.

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[1] Here and in what follows we work in the category of smooth manifolds.

2.1.1. *Spectral triples and gauge theories*

In this subsection we show how noncommutative spectral triples naturally give rise to gauge theories, following [6]. First of all, note that the most natural notion of equivalence of (unital) noncommutative (C^*-)algebras is Morita equivalence [8]. A unital algebra $\mathcal{A}$ is *Morita equivalent* to a unital algebra $\mathcal{B}$ if and only if there exists a $\mathcal{B}-\mathcal{A}$-module $\mathcal{E}$ which is finitely generated and projective as an $\mathcal{A}$-module such that $\mathcal{B} = \mathrm{End}_{\mathcal{A}}\mathcal{E}$. Commutative algebras are Morita equivalent if and only if they are isomorphic, justifying this notion of equivalence for noncommutative algebras.

If $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is a real and even spectral triple and $\mathcal{B}$ is a unital algebra Morita equivalent to $\mathcal{A}$, then there is natural way to construct a real and even spectral triple for the algebra $\mathcal{B}$. If this is done for the case $\mathcal{B} = \mathcal{A}$ through the module $\mathcal{E} = \mathcal{A}$, the obtained spectral triple is of the form

$$(\mathcal{A}, \mathcal{H}, D_A := D + A + \epsilon' JAJ^{-1}),$$

where $A = \sum_j a_j[D, b_j]$ with $a_j, b_j \in \mathcal{A}$ is a bounded self-adjoint operator on $\mathcal{H}$ (see [6] and also [5, Section 10.8], for more details). It is a straightforward verification that this is again a real and even spectral triple. Thus we get another spectral triple consisting of the same algebra and Hilbert space but with the operator D fluctuated by an element A. The element A will be interpreted as the *gauge potential*.

The *gauge group* of the triple $(\mathcal{A}, \mathcal{H}, D)$ is the subgroup $\mathrm{Inn}(\mathcal{A})$ of $*$-automorphism of $\mathcal{A}$ consisting of all automorphisms of the form $a \mapsto uau^*$ where $u \in \mathcal{A}$ satisfies $uu^* = u^*u = 1$ [5, Section 9.9]. This inner automorphism group acts naturally on the constituents of a spectral triple as an intertwiner. The gauge potential transforms accordingly as $A \mapsto uAu^* + u[D, u^*]$. The action of the gauge group on the Hilbert space is given by $\psi \mapsto uJuJ^*\psi$, where $\psi \in \mathcal{H}$ and $u \in \mathcal{U}(A)$.

2.1.2. *Spectral action principle*

Associated to a spectral triple we have a gauge group, a gauge potential and gauge transformations and in this way a spectral triple forms the setting of a gauge theory. To obtain the dynamics of the theory, the spectral action principle [6,1] is used to calculate an action from the spectral triple. The action consists of two parts: the first part is a fermion part, which is defined by

$$S_f[\psi, A] = \langle \psi, D_A\psi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. Note that the fermionic action depends on the gauge potential A in D_A but that it is invariant under gauge transformations. The other part of the action is the bosonic action which is defined by

$$S_b[A] = \mathrm{Tr}(f(D_A/\Lambda)), \tag{3}$$

where Tr denotes the trace in $\mathcal{H}$, f is a suitable cut-off function with $\Lambda > 0$. Note that, just as for the fermionic action, the expression of S_b is invariant under the transformations $D_A \mapsto uJuJ^*D_AJu^*Ju^*$ for $u \in \mathcal{A}$ unitary.

2.2. *Einstein–Yang–Mills theories and spectral triples*

Chamseddine and Connes showed in [1] that Yang–Mills gauge theory over a compact Riemannian spin manifold M can be obtained from a spectral triple built from the canonical triple associated to this manifold and a matrix algebra. In this subsection we will briefly review their results and relate it to the description of gauge theories in terms of principal fiber bundles. Let us first recall how such fiber bundles enter gauge theories.

Definition 2.6. Let G be a matrix Lie group and let P be a principal G-bundle. A *connection* ω assigns to each local trivialization $\phi_U : \pi^{-1}(U) \to U \times G$ a $\mathfrak{g}$-valued one-form ω_u on U. If ϕ_V is another local trivialization and $g_{uv} : U \cap V \to G$ is the transition function from (U, ϕ_U) to (V, ϕ_V), then we require the following transformation rule for ω:

$$\omega_u = g_{uv}^{-1} dg_{uv} + g_{uv}^{-1}\omega_v g_{uv}. \tag{4}$$

More generally, that is in the case of arbitrary Lie groups, the gauge potential is defined as a global $\mathfrak{g}$-valued connection 1-form on P satisfying some extra conditions. In the case of matrix Lie groups this definition coincides with Definition 2.6 (see for instance [9] for more details). The local one-forms ω_u are the gauge potentials one encounters in physics.

Definition 2.7. A *gauge theory* with Lie group G over a manifold M consists of a principal G-bundle together with a connection 1-form ω on P. The connection 1-form ω on P is also called the *gauge potential*. If $G = (P)\mathrm{SU}(N)$ then the gauge theory is called a $(P)\mathrm{SU}(N)$-Yang–Mills theory.

We now briefly summarize the results of [1] that obtained Yang–Mills theory on a manifold M from a well-chosen spectral triple. For the rest of this section the manifold M is assumed to be a compact 4-dimensional spin manifold.

Consider the following objects:

$$\begin{aligned} &\mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C}), \qquad \mathcal{H} = L^2(M, S) \otimes M_N(\mathbb{C}), \qquad D = \not{D}_M \otimes 1, \\ &J = J_M \otimes (\cdot)^*, \qquad \gamma = \gamma_5 \otimes 1. \end{aligned} \tag{5}$$

The bundle S is the spinor bundle whose fibers are isomorphic to $\mathbb{C}^4$ as in Example 2.2 and the operator $\not{\partial}$ is the Dirac operator on the bundle S. Observe that this triple forms a spectral triple, being the product of the canonical triple $(C^\infty(M), L^2(M,S), \not{\partial})$ and the matrix algebra $M_N(\mathbb{C})$ that acts on itself by left multiplication. We will describe this product structure in more detail in Section 3.3.

The spectral triple in Eq. (5) is real and even since the canonical triple (2) is already real and even. Let us now determine the fluctuated Dirac operator $D_A = D + A + \epsilon' JAJ^*$ for this spectral triple. The fact that $\epsilon' = 1$ in even dimensions implies that

$$A + \epsilon' JAJ^* = \gamma^\mu A_\mu + J\gamma^\mu A_\mu J^*. \tag{6}$$

In even dimensions one has

$$J_M \gamma^\mu J_M^* = -\gamma^\mu,$$

and if we use that left multiplication by $JA_\mu J^*$ is right multiplication by A_μ^*, Eq. (6) turns into $A + JAJ^* = \gamma^\mu \cdot \mathrm{ad}(A_\mu)$, since A is self-adjoint. Thus the fluctuated Dirac operator is of the form:

$$D_A = D + \mathrm{i}\gamma^\mu \mathbb{A}_\mu \tag{7}$$

where $\mathbb{A}_\mu = -\mathrm{i}\,\mathrm{ad}\, A_\mu$. The self-adjointness of A implies that $\mathbb{A}_\mu$ is an anti-hermitian one-form. Since A acts in the adjoint representation the $u(1)$-part drops out and we effectively have a $su(N)$-gauge potential.

It was shown in [1] that the spectral action applied to the above spectral triple (5) describes the Einstein–Yang–Mills system. It contains the Einstein–Hilbert action and higher-order gravitational terms, as well as the Yang–Mills action for a global $su(N)$-valued 1-form A_μ. This is in line with the interpretation of the fluctuation A as a gauge potential. Comparing this with the definition of a PSU(N)-Yang–Mills theory as in Definition 2.7, the fact that the gauge potential A_μ is globally an $su(N)$-valued 1-form means that this corresponds to a gauge theory with a trivial principal PSU(N)-bundle P. The goal of this paper is to generalize the spectral triple (5) in such a way that it determines a topologically non-trivial PSU(N)-gauge theory.

3. Algebra bundles and spectral triples

In this section we will generalize the above spectral triple to obtain a gauge theory on a non-trivial PSU(N)-bundle. The important observation here is that in the trivial case we started with the algebra $C^\infty(M) \otimes M_N(\mathbb{C})$ which is precisely the algebra of smooth sections of a trivial $M_N(\mathbb{C})$-bundle over M. This suggests for the non-trivial case that the algebra in the spectral triple is given by $\Gamma^\infty(M,B)$, where B is an arbitrary locally trivial algebra bundle where the fiber is the $*$-algebra $M_N(\mathbb{C})$. In fact, we will construct such a real and even spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ where the algebra $\mathcal{A}$ is isomorphic to $\Gamma^\infty(M,B)$. This allows for a derivation of Yang–Mills theory for a gauge connection on a non-trivial principal fiber bundle in the next section.

3.1. Definition of algebra bundles

In this paper we take the following definition of an algebra bundle.

Definition 3.1. An *algebra bundle* B is a vector bundle together with a vector bundle homomorphism $\mu : B \otimes B \to B$ such that for all $x \in M$:

$$\mu(p_x \otimes \mu(q_x \otimes r_x)) = \mu(\mu(p_x \otimes q_x) \otimes r_x), \quad \forall p_x, q_x, r_x \in B_x, \tag{8}$$

inducing an associative algebra structure on each of the fibers of B by setting $p_x \cdot q_x = \mu(p_x \otimes q_x)$ for two elements p_x and q_x in the fiber B_x of $x \in M$.

If B_1 and B_2 are two algebra bundles, then a map $\phi : B_1 \to B_2$ is called an *algebra bundle morphism* if it is a vector bundle morphism such that the restriction $f_{|(B_1)_x} : (B_1)_x \to (B_2)_x$ is a homomorphism of algebras.

An algebra bundle B is called an *involutive* or $*$-*algebra bundle*, if there exists in addition an algebra bundle homomorphism $J : B \to \overline{B}^{op}$ such that $J^2 = 1$,[2] giving each fiber the structure of an involutive algebra by setting $p_x^* = J(p_x)$.

If B_1, B_2 are two $*$-algebra bundles, then an $*$-*algebra bundle homomorphism* is a vector bundle homomorphism $f : B_1 \to B_2$ such that the restriction $f_{|(B_1)_x} : (B_1)_x \to (B_2)_x$ is a $*$-algebra homomorphism for every base point $x \in M$.

Let $\mathsf{AlgB(M)}$ ($\mathsf{AlgB^*(M)}$) denote the category whose objects are all (involutive) algebra bundles (over M), and where the morphisms are all (involutive) algebra bundle morphisms.

Remark 3.2. We note here that we do not require that the algebra in each fiber is the same. However, the way we introduced the associative algebra structures on the fibers guarantees that the product of two smooth sections is again smooth. This turns $\Gamma^\infty(M,B)$ into an associative algebra.

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[2] Here $\overline{B}^{op}$ is as a vector bundle conjugate to B and has opposite multiplication in the fibers.

In general, the space of smooth sections of a vector bundle on M is a module over $C^\infty(M)$. In the case of algebra bundles, this action is compatible with the multiplication in the fiber. Thus, if B is an (involutive) algebra bundle, then $\Gamma^\infty(M, B)$ is a finitely generated (involutive) module algebra over $C^\infty(M)$. Recall that an R-module algebra is an R-module $\mathcal{A}$ with an associative multiplication $\mathcal{A} \times \mathcal{A} \mapsto \mathcal{A} : (a, b) \mapsto ab$ which is R-bilinear:

$$r(ab) = (ra)b = a(rb) \quad \forall a, b \in \mathcal{A},\ r \in R.$$

An R-module algebra is called involutive if there exists a map $* : \mathcal{A} \to \mathcal{A}$ such that

$$(ab)^* = b^* a^*; \qquad (a + b)^* = a^* + b^*; \qquad (ra)^* = r^* a^*; \quad (r, s \in R,\ a, b \in \mathcal{A}).$$

Recall the well-known Serre–Swan Theorem for (complex) vector bundles over compact manifolds [10].

Theorem 3.3 (*Serre–Swan [10]*)**.** *For every vector bundle E over a compact manifold M the space of sections $\Gamma^\infty(M, E)$ is a finitely generated projective $C^\infty(M)$-module. The association Γ of the space of sections to the vector bundle E establishes an equivalence of categories between the complex vector bundles over M and the category of finitely generated projective $C^\infty(M)$-modules.*

We now extend this result to arrive at an equivalence between $(*)$-algebra bundles and finitely generated projective $C^\infty(M)$-module algebras (with involution). The idea is that the $C^\infty(M)$-linear multiplicative structure on a finitely generated projective $C^\infty(M)$-module algebra $\mathcal{P}$, where $\mathcal{P}$ is identified with the space of sections of some vector bundle B through Theorem 3.3 (here B is unique up to isomorphism), induces a product on the fibers B_x such that $(s \cdot t)(x) = s(x) \cdot t(x)$. The next lemma is crucial for lifting the multiplication structure on $\Gamma^\infty(M, B)$ to the fibers of B.

Lemma 3.4 (*[11, Lemma 11.8b]*)**.** *Let $\pi : B \to M$ be a vector bundle. Suppose s is a section with $s(x) = 0$ for some $x \in M$. Then there exist functions f_i with $f_i(x) = 0$ and sections $s_i \in \Gamma^\infty(M, B)$ so that s can be written as a finite sum $s = \sum_i f_i s_i$.*

We will now step-by-step introduce an (involutive) algebra bundle structure on B.

Proposition 3.5. *For $x \in M$, let $p, q \in B_x$ be given and suppose $s, t \in \mathcal{P}$ are such that $p = s(x)$ and $q = t(x)$. There exists a well-defined fiber multiplication $\mu(p \otimes q) := st(x)$ turning B into an algebra bundle. Consequently, we have $st(y) = s(y)t(y)$ for all $y \in M$ and $s, t \in \Gamma^\infty(M, B)$.*

Proof. We need to show that the definition of the fiber product is independent of the choice of sections s, t with $s(x) = p$ and $t(x) = q$. Therefore, let s', t' be two other sections of the bundle B with $s'(x) = p$ and $t'(x) = q$. Then $s_0 = s' - s$ and $t_0 = t' - t$ are sections for which $s_0(x) = t_0(x) = 0$. According to Lemma 3.4 s_0 and t_0 can be written as $s_0 = \sum_i f_i s_i$, $t_0 = \sum_i g_i t_i$ where $f_i(x) = g_i(x) = 0$ for every i. This gives

$$s't' - st = (s' - s)t' + s(t' - t) = \sum f_i s_i t' + \sum_i g_i s t_i,$$

which evaluated at x gives zero because of the module structure of $\Gamma^\infty(M, B)$. This argument shows that $s't'(x) = st(x)$ and the product is well-defined. Actually, the map $(s, t) \mapsto st$ is $C^\infty(M)$-bilinear so it can be considered as a $C^\infty(M)$-linear map from $\Gamma^\infty(M, B) \otimes_{C^\infty(M)} \Gamma^\infty(M, B)$ to $\Gamma^\infty(M, B)$. This corresponds to a vector bundle homomorphism $\mu : B \otimes B \to B$. If $\mathcal{P} \cong \Gamma^\infty(M, B)$ is unital with unit 1_P, then we can fix a unit in the fiber B_x by setting $1_{B_x} = 1_{\mathcal{P}}(x)$. □

Proposition 3.6. *For given $p \in B_x$, let $s \in \mathcal{P}$ be such that $s(x) = p$. Define $p^* := Jp := s^*(x)$. This is a well-defined involutive structure on the fiber B_x, turning B into an involutive algebra bundle.*

Proof. We will use the same argument as before. Let s' be another such section with $s'(x) = p$. Then with Lemma 3.4 $s_0 = s - s'$ can be written as a sum $\sum_i f_i s_i$ where $s_i \in \mathcal{P}$, $f_i \in C^\infty(M)$ and $f_i(x) = 0$ for all i. This gives

$$s^*(x) - s'^*(x) = (s - s')^*(x) = \sum_i (f_i s_i)^*(x) = \sum_i f_i^*(x) s_i^*(x) = 0,$$

so that the star structure is well-defined. That this is indeed a star structure on the fiber B_x compatible with the algebra structure of the fiber, follows immediately from the definition of a module $*$-algebra. □

The functor $\Gamma : \mathsf{Vect_M} \to \mathsf{FGP_{C^\infty(M)}Mod}$ can be restricted to a functor $\hat{\Gamma}$ from the category $\mathsf{AlgB(M)}$ of algebra bundles to the category of finitely generated projective $C^\infty(M)$-algebras $\mathsf{FGP_{C^\infty(M)}AlgMod}$. A similar statement applies to involutive algebra bundles and involutive module algebra. It follows from Propositions 3.5 and 3.6 that the restricted functor $\hat{\Gamma}$ is still essentially surjective. As a restriction of a faithful functor, $\hat{\Gamma}$ is of course also faithful. To show that Γ is full, let B_1, B_2 be two $(*\text{-})$algebra bundles and $F : \Gamma^\infty(M, B_1) \to \Gamma^\infty(M, B_2)$ be a ($*$-preserving) $C^\infty(M)$-algebra-homomorphism. We will prove that the bundle homomorphism ϕ defined by

$$\phi(p) = F(s)(x), \quad (p \in B_1), \tag{9}$$

where $s \in \Gamma^\infty(M, B_1)$ satisfies $s(x) = p$, is a $*$-algebra bundle homomorphism which is mapped to F by $\hat{\Gamma}$.

Firstly, observe that the map ϕ is well-defined: let s' be another section with $s'(x) = e$. Then $s - s' = \sum_i f_i s_i$, where the s_i are in $\Gamma^\infty(M, B_1)$ and where the f_i are smooth functions on M vanishing at x. This implies that indeed $F(s - s')(x) = 0$.

Secondly, ϕ is a $*$-algebra bundle homomorphism, since

$$\phi(pq) = F(st)(x) = F(s)F(t)(x) = F(s)(x) \cdot F(t)(x) = \phi(p)\phi(q),$$
$$\phi(p^*) = F(s^*)(x) = (F(s))^*(x) = (F(s)(x))^* = \phi(p)^*$$

where $s, t \in \Gamma^\infty(M, B_1)$ are such that $p = s(x)$, $q = t(x)$.

Finally, by construction $(\hat{\Gamma}(\phi)s)(x) = \phi(s(x)) = F(s)(x)$, so that $\hat{\Gamma}(\phi) = F$ as required. Hence, $\hat{\Gamma}$ is a full functor.

Remark 3.7. If B_1, B_2 are unital ($*$-)algebra bundles and $\phi : B_1 \to B_2$ is a unital ($*$-)algebra bundle homomorphism, then $\hat{\Gamma}(\phi)$ is a unital ($*$-preserving) $C^\infty(M)$-algebra-homomorphism. Conversely, if $F : \Gamma^\infty(M, B_1) \to \Gamma^\infty(M, B_2)$ is a unital ($*$-preserving) $C^\infty(M)$-algebra-homomorphism, and $\phi : B_1 \to B_2$ is defined by (9), then

$$\phi(1_x) = F(1)(x) = 1_x.$$

We summarize the results in this subsection in the following theorem.

Theorem 3.8 (*Serre–Swan for Algebra Bundles*)**.** *Let M be a compact manifold. The functor $\hat{\Gamma}$ furnishes an equivalence between the category of (unital) (involutive) algebra bundles over M and the category of (unital) (involutive) finitely generated projective $C^\infty(M)$-algebras.*

3.2. Spectral triple obtained from an algebra bundle

In this subsection we construct a real and even spectral triple whose algebra is isomorphic to $\Gamma^\infty(M, B)$. Here B is some locally trivial $*$-algebra bundle whose fibers are copies of a fixed (finite-dimensional) $*$-algebra A. Furthermore, we require that for each x the fiber B_x is endowed with a faithful tracial state τ_x so that for all $s \in \Gamma^\infty(M, B)$ the function $x \mapsto \tau_x s(x)$ is smooth. The corresponding Hilbert–Schmidt inner product in the fiber B_x induced by τ_x is denoted by $\langle \cdot, \cdot \rangle_{B_x}$. Consequently, the $C^\infty(M)$-valued form

$$(\cdot, \cdot)_B : \Gamma^\infty(M, B) \times \Gamma^\infty(M, B) \to C^\infty(M), \qquad (s, t)_B(x) = \langle s(x), t(x) \rangle_{B_x},$$

turns $\Gamma^\infty(M, B)$ into a pre-Hilbert $C(M)$-module.

As in the previous sections, we assume that M is a Riemannian spin manifold, on which $S \to M$ is a spinor bundle and $\not{D} = \mathrm{i}c \circ \nabla^S$ a Dirac operator. Combining the inner product on spinors with the above hermitian structure naturally induces the following inner product on $\Gamma^\infty(M, B \otimes S)$:

$$\langle \xi_1, \xi_2 \rangle_{\Gamma^\infty(M, B\otimes S)} := \int_M \langle \xi_1(x), \xi_2(x) \rangle_{B_x \otimes S_x} \quad (\xi_1, \xi_2 \in \Gamma^\infty(M, B \otimes S)), \tag{10}$$

turning it into a pre-Hilbert space. The completion with respect to the norm induced by this inner product consists of all square-integrable sections of $B \otimes S$, and is denoted by $L^2(M, B \otimes S)$.

Remark 3.9. Note that we can identify $\Gamma^\infty(M, B \otimes S) \cong \Gamma^\infty(M, B) \otimes_{C^\infty(M)} \Gamma^\infty(M, S)$ as $C^\infty(M)$-modules. In what follows, we will use this isomorphism without further notice. The above inner product (10) can be written as

$$\langle s_1 \otimes \psi_1, s_2 \otimes \psi_2 \rangle_{\Gamma^\infty(M,B) \otimes_{C^\infty(M)} \Gamma^\infty(M,S)} = \langle \psi_1, (s_1, s_2)_B \psi_2 \rangle_S,$$

where $(s_1, s_2)_B \in C^\infty(M)$ acts on $\Gamma^\infty(M, S)$ by pointwise multiplication.

Theorem 3.10. *In the above notation, let ∇^B be a hermitian connection (with respect to the Hilbert–Schmidt inner product) on the algebra bundle B and let $D_B = \mathrm{i}c \circ (\nabla^B \otimes 1 + 1 \otimes \nabla^S)$ be the twisted Dirac operator on $B \otimes S$. Then*

$$(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B)$$

is a spectral triple with summability equal to the dimension of M.

Proof. First, it is obvious that fiberwise multiplication of $a \in \Gamma^\infty(M, B)$ on $\Gamma^\infty(M, B \otimes S)$ extends to a bounded operator on $L^2(M, B \otimes S)$ since

$$\|as \otimes \psi\|^2 = \int_M \langle \psi(x), \langle a(x)s(x), a(x)s(x) \rangle_{B_x} \psi(x) \rangle_{S_x} \mathrm{d}x \le \sup_{x \in M} \{\|a(x)\|_x^2\} \|s \otimes \psi\|^2.$$

Here $\|\cdot\|_x$ denotes the fiberwise operator C^*-norm. Compactness of the resolvent and summability follows from ellipticity of the twisted Dirac operator D_B, M being a compact manifold. Moreover, the commutator $[D_B, a]$ is bounded for $a \in \Gamma^\infty(M, B)$ since D_B is a first-order differential operator. More precisely, in local coordinates one computes

$$[D_B, a](s \otimes \psi) = \left(\partial_\mu a + [\omega_\mu^B, a]\right) s \otimes \gamma^\mu \psi$$

where $\nabla^B_\mu = \partial_\mu + \omega^B_\mu$, locally. This operator is bounded on $L^2(M, B \otimes S)$, provided a is differentiable and ω^B_μ is a smooth connection one-form. $\square$

Next, we would like to extend our construction to arrive at a real spectral triple. For this, we introduce an anti-linear operator on $L^2(M, B \otimes S)$ of the form

$$J(s \otimes \psi) = s^* \otimes J_M \psi$$

with J_M charge conjugation on M (*cf.* Example 2.4). For this operator to be a real structure on our spectral triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B)$, we need some extra conditions on the connection ∇_B on B.

Definition 3.11. Let B be a $*$-algebra bundle over a manifold M. A *$*$-algebra connection* ∇ on B is a connection on B that satisfies

$$\nabla(st) = s\nabla t + (\nabla s)t, \qquad (\nabla s)^* = \nabla s^*; \quad (s, t \in \Gamma^\infty(M, B)).$$

If B is a hermitian $*$-algebra bundle and ∇ is also a hermitian connection, then ∇ is called a *hermitian* $*$-algebra connection.

Before we proceed we need to know whether a hermitian $*$-algebra connection exists on any given locally trivial $*$-algebra bundle. A partition of unity argument easily shows how to construct hermitian $*$-algebra connections on arbitrary $*$-algebra bundles.

Lemma 3.12. *Every locally trivial hermitian $*$-algebra bundle B defined over a paracompact space M admits a hermitian $*$-algebra connection.*

Proof. Let $\{U_i\}$ be a locally finite open covering of M such that B is trivialized over U_i for each i. Then on each U_i there exists a hermitian $*$-algebra connection ∇_i, for instance the trivial connection d on U_i. Now, let $\{f_i\}$ be a partition of unity subordinate to the open covering $\{U_i\}$ (all f_i are real-valued). Then the linear map ∇ defined by

$$(\nabla s)(x) = \sum_i f_i(x)(\nabla_i s)(x), \quad (x \in M),$$

is a hermitian $*$-algebra connection on $\Gamma^\infty(M, B)$. $\square$

Remark 3.13. The fact that locally, on some trivializing neighborhood, the exterior derivative d is a hermitian $*$-algebra connection shows that on such a local patch every hermitian $*$-algebra connection is of the form

$$d + \omega,$$

where ω is a real connection 1-form with values in the real Lie algebra of $*$-derivations of the fiber that are anti-hermitian with respect to the inner product on the fiber. For instance, when the fiber is the $*$-algebra $M_N(\mathbb{C})$ endowed with the Hilbert–Schmidt inner product, this Lie algebra is precisely $\mathrm{ad}(\mathfrak{u}(N)) \cong \mathfrak{su}(N)$.

Theorem 3.14. *Suppose in addition to the conditions of Theorem 3.10 that ∇^B is a hermitian $*$-algebra connection and set $\gamma_B = 1 \otimes \gamma_5$ as a self-adjoint operator on $L^2(M, B \otimes S)$. Then $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$ is a real and even spectral triple with KO-dimension equal to the dimension of M.*

Proof. First of all, we check that J is anti-unitary:

$$\begin{aligned}\langle J(s \otimes \psi), J(t \otimes \eta)\rangle &= \langle J_M \psi, (s^*, t^*) J_M \eta\rangle = \langle J_M \psi, J_M \overline{(s^*, t^*)} \eta\rangle \\ &= \langle \overline{(s^*, t^*)} \eta, \psi\rangle = \langle (s, t)\eta, \psi\rangle = \langle t \otimes \eta, s \otimes \psi\rangle,\end{aligned}$$

where we have in the second step that $J_M f = \bar{f} J_M$ for every $f \in C^\infty(M)$, in the third step that J_M is anti-unitary and in the fourth step that $(s, t) = (t^*, s^*)$ (by definition of the hermitian structure as a fiberwise trace). Moreover, since $J_M^2 = \pm 1$ it follows that $J^2 = \pm 1$ (with the same sign, according to the dimension of M).

We next establish $DJ = JD$ by a local calculation:

$$\begin{aligned}(JD - DJ)(s \otimes \psi) &= J(\nabla^B_\mu s \otimes \mathrm{i}\gamma^\mu \psi + s \otimes \not{D}\psi) - D_B(s^* \otimes J_M \psi) \\ &= (\nabla^B_\mu s)^* \otimes (-\mathrm{i}) J_M \gamma^\mu \psi + s^* \otimes J_M \not{D}\psi - \nabla^B_\mu s^* \otimes \mathrm{i}\gamma^\mu J_M \psi - s^* \otimes \not{D} J_M \psi \\ &= -\mathrm{i}\left((\nabla^B_\mu s)^* - \nabla^B_\mu s^*\right) \otimes J_M \gamma^\mu \psi = 0,\end{aligned}$$

since in even dimensions $\{J_M, \gamma^\mu\} = [\not{D}, J_M] = 0$, and the last step is established by the condition of a $*$-algebra connection, *i.e.* $(\nabla s)^* = \nabla s^*$ for all $s \in \Gamma^\infty(M, B)$.

The commutant property follows easily:

$$\begin{aligned}[a, b^0](s \otimes \psi) &= aJb^*J^*(s \otimes \psi) - Jb^*J^*a(s \otimes \psi) = aJ(b^*s^* \otimes J_M^*\psi) - Jb^*(s^*a^* \otimes J_M^*\psi) \\ &= asb \otimes \psi - asb \otimes \psi = 0,\end{aligned}$$

where $a, b \in \Gamma^\infty(M, B)$ and $s \otimes \psi \in \Gamma^\infty(M, B) \otimes_{C^\infty(M)} \Gamma^\infty(M, S)$. Since $[a, b^0] = 0$ on $\Gamma^\infty(M, B) \otimes_{C^\infty(M)} \Gamma^\infty(M, S) \cong \Gamma^\infty(M, B \otimes S)$, it is zero on the entire Hilbert space $L^2(M, B \otimes S)$. It remains to check the order one condition for the Dirac operator. First note that

$$[[D, a], b^0](s \otimes \psi) = \mathrm{i}c([[\nabla, a], b^0](s \otimes \psi)) \quad (a, b, s \in \Gamma^\infty(M, B)).$$

This is zero because $[[\nabla, a], b^0](s \otimes \psi)$ is zero:

$$\begin{aligned}([\nabla, a]sb) \otimes \psi - Jb^*J^*([\nabla, a]s \otimes \psi) &= \nabla(asb) \otimes \psi - a\nabla(sb) \otimes \psi - Jb^*J^*\nabla(as) \otimes \psi + Jb^*J^*a(\nabla s) \otimes \psi \\ &= \nabla(asb) \otimes \psi - a\nabla(sb) \otimes \psi - \nabla(as)b \otimes \psi + a(\nabla s)b \otimes \psi \\ &= ((\nabla a)sb + a(\nabla s)b + as(\nabla b) - a(\nabla s)b \\ &\quad - as(\nabla b) - (\nabla a)sb - a(\nabla s)b + a(\nabla s)b) \otimes \psi, \\ &= 0\end{aligned}$$

using the defining property for ∇^B to be a $*$-algebra connection. Thus, J fulfils all of the necessary conditions of a real structure on $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B)$. The conditions on γ_B to be a grading operator for this spectral triple are easily checked. □

In the next section we show that the triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$ gives a non-trivial Yang–Mills theory over the manifold M. The Serre–Swan Theorem 3.8 plays an essential role in the proof. First, we explore the form of this spectral triple in the context of Kasparov's KK-theory.

3.3. Relation with the unbounded Kasparov internal product

In this section we establish that the spectral triple of Theorem 3.10 is an unbounded Kasparov product of two unbounded KK-cycles [12,13]. Since the KK-groups are naturally formulated for C^*-algebras we will work with C^*-algebras in this section. Therefore, we consider $\Gamma^\infty(M, B)$ as a dense subalgebra of $\Gamma(M, B)$, the C^*-algebra of continuous sections from M to B. The norm on $\Gamma(M, B)$ is given by $\|s\| = \sup_x \|s(x)\|_x$, where $\| \cdot \|_x$ denotes the operator norm on the fiber B_x. In particular, we consider $C^\infty(M)$ as a dense subalgebra of $C(M)$, the C^*-algebra of continuous complex-valued functions on M.

Let us briefly recall some elementary notions from (unbounded) KK-theory. Denote by $\mathcal{B}(E)$ the bounded endomorphisms of a right Hilbert B-module E and by $\mathcal{K}(E)$ the compact endomorphisms.

Definition 3.15. Let A and B be $\mathbb{Z}_2$-graded C^*-algebras. A Kasparov A–B-module consists of a triple (E, ϕ, F) where E is a countably generated $\mathbb{Z}_2$-graded Hilbert-B-module, ϕ is a graded $*$-homomorphism $A \to \mathcal{B}(E)$ and F is a bounded operator of degree 1, such that $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, and $(F - F^*)\phi(a)$ are in $\mathcal{K}(E)$.

There are the natural notions of unitary and homotopy equivalence and under the direct sum the set of equivalence classes of Kasparov $A-B$-modules forms an abelian group which is denoted by $KK(A, B)$ [14]. One of the key properties of KK-theory is the existence of the internal Kasparov product.

Definition 3.16. Let E_1 be an A–B-module and E_2 a B–C-module, and define an A–C-module by $E := E_1 \otimes_B E_2$. A Kasparov module (E, ϕ, F) is called a *Kasparov product* for (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) if

- $(E, \phi_1 \otimes \mathrm{Id}, F) \in KK(A, C)$;
- for every $x \in E_1$ of homogeneous degree $\#x$, the operator $T_x : E_2 \to E$ defined by $T_x(e) = x \otimes e$ satisfies

$$\begin{aligned}&T_x \circ F_2 - (-1)^{\#x} F \circ T_x \in \mathcal{K}(E_2, E), \\ &F_2 \circ T_x^* - (-1)^{\#x} T_x^* \circ F \in \mathcal{K}(E, E_2);\end{aligned}$$

- for all $a \in A$ the graded commutator $\phi(a)[F_1 \otimes \mathrm{Id}, F]\phi(a^*) \geq 0 \bmod \mathcal{K}(E)$.

If A is separable and B is σ-unital, then there is a Kasparov-product for (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) and any of these products are homotopic.[3] Therefore, the internal Kasparov product defines a bilinear map $\otimes_B : KK(A, B) \times KK(B, C) \to KK(A, C)$.

The Kasparov internal product of Definition 3.16 can be captured in terms of unbounded Kasparov-modules [13].

Definition 3.17 (*[13,16]*). Let A and B be graded C^*-algebras. An *unbounded Kasparov module* is a triple (E, ϕ, D) where E is a graded Hilbert-B-module, $\phi : A \to \mathcal{B}(E)$ a graded $*$-homomorphism, and D a self-adjoint regular operator in E, homogeneous of degree 1 such that $(1 + D^2)^{-1}\phi(a)$ extends to an element of $\mathcal{K}(E)$ for all $a \in A$, and the set of all $a \in A$ such that $[D, \phi(a)]$ extends to an element in $\mathcal{B}(E)$ is dense in A.

The set of all unbounded Kasparov modules is denoted by $\Psi(A, B)$.

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[3] Actually, they are even operator homotopic (*cf.* [14] or [15]).

Example 3.18. The canonical spectral triple $(C^\infty(M), L^2(M, S), \not{D})$ is an element in $\Psi(C(M), \mathbb{C})$. Another example is given as follows: let B be a locally trivial $*$-algebra bundle with a smoothly-varying faithful tracial state on the fibers. Then $(\Gamma(M, B), \lambda, 0)$ is an element of $\Psi(\Gamma(M, B), C(M))$, and even in $KK(\Gamma(M, B), C(M))$. The $C(M)$-Hilbert module $\Gamma(M, B)$ is equipped with the fiberwise Hilbert–Schmidt norm, and λ is the continuous action of $\Gamma(M, B)$ (as a C^*-algebra) on this Hilbert module by pointwise left matrix multiplication. Note that the algebras $\Gamma(M, B)$ and $C(M)$ are trivially graded.

Proposition 3.19 *([13]). If $(E, \phi, D) \in \Psi(A, B)$ then $(E, \phi, F) \in \mathbb{E}(A, B)$ where $F = D(1 + D^2)^{-\frac{1}{2}}$. If A is separable, the map $(E, \phi, D) \mapsto \left[(E, \phi, D(1 + D^2)^{-\frac{1}{2}})\right]$ is a surjective map $\Psi(A, B) \to KK(A, B)$.*

Thus, classes in $KK(A, B)$ can be represented by unbounded cycles in $\Psi(A, B)$. The following theorem is due to Kucerovsky [16] and introduces a Kasparov product for unbounded KK-modules. This was further worked out by Mesland [17].

Theorem 3.20 *(Kucerovsky). Suppose that $(E_1 \otimes_B E_2, \phi_1 \otimes_B \mathrm{Id}, D) \in \Psi(A, C)$, $(E_1, \phi_1, D_1) \in \Psi(A, B)$ and $(E_2, \phi_2, D_2) \in \Psi(B, C)$ are such that*

(1) *for all x in some dense subset of $\phi_1(A)E_1$, the operator*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T_x^* & 0 \end{pmatrix}\right]$$

is bounded on $\mathrm{Dom}\, D \oplus \mathrm{Dom}\, D_2$;

(2) *the resolvent of D is compatible with D_1: that is, there is a dense submodule W such that $D_1(i\mu + D)^{-1}(i\mu_1 + D_1)^{-1}$ is defined on W for all $\mu, \mu_1 \in \mathbb{R} - \{0\}$;*

(3) *there exists a $c \geq 0$ such that $\langle D_1 x, Dx\rangle + \langle Dx, D_1 x\rangle \geq c\langle x, x\rangle$, for all x in the domain;*

where $x \in E_1$ is homogeneous and $T_x : E_2 \to E$ maps $e \mapsto x \otimes_B e$. Then $(E_1 \otimes_B E_2, \phi_1 \otimes_B \mathrm{Id}, D)$ represents the Kasparov-product of $(E_1, \phi_1, D_1) \in \Psi(A, B)$ and $(E_2, \phi_2, D_2) \in \Psi(B, C)$.

Using Theorem 3.20 we show that the spectral triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B)$ of Theorem 3.10 comes from an unbounded Kasparov product.

Proposition 3.21. *Let B be a locally trivial hermitian unital $*$-algebra bundle on a compact Riemannian spin manifold M with fibers isomorphic to some complex $*$-algebra A. Let ∇^B be a hermitian connection on B and D_B the corresponding twisted Dirac operator. Then the spectral triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B)$ is an unbounded Kasparov product of $(\Gamma(M, B), \lambda, 0) \in \Psi(\Gamma(M, B), C(M))$ and $(L^2(M, S), m, \not{D}) \in \Psi(C(M), \mathbb{C})$, where λ is action of $\Gamma(M, B)$ on itself by left multiplication and where m denotes the representation of $C(M)$ on $L^2(M, S)$ induced by pointwise multiplication with elements in $\Gamma(M, S)$.*

Proof. Most of the assertions are straightforward to prove. To prove the last statement note that $L^2(M, B \otimes S) := \mathrm{compl}\left(\Gamma^\infty(M, B) \otimes_{C^\infty(M)} \Gamma^\infty(M, S)\right) \cong \Gamma(M, B) \otimes_{C(M)} L^2(M, S)$. We will now check the first condition of Theorem 3.20 since the other two are trivial (because $D_1 = 0$). It suffices to check that

$$D \circ T_a - T_a \not{D} \in \mathcal{B}(L^2(M, S), L^2(M, B \otimes S)),$$
$$\not{D} T_a^* - T_a^* D \in \mathcal{B}(L^2(M, B \otimes S), L^2(M, S)),$$

for all $a \in \Gamma^\infty(M, B)$. For the first condition, we have for $\psi \in L^2(M, S)$ that

$$(D \circ T_a - T_a D_2)(\psi) = D(a \otimes \psi) - a \otimes \not{D}\psi = c(\nabla^B a) \otimes \psi$$

so that $D \circ T_a - T_a \not{D}$ extends to a bounded operator. Now the second one:

$$\begin{aligned}(\not{D} T_a^* - T_a^* D)(s \otimes \psi) &= \not{D}(\langle a, s\rangle \psi) - \langle a, s\rangle \not{D}\psi - \langle a, c(\nabla^B s)\rangle \psi \\ &= [\not{D}, \langle a, s\rangle]\psi - ([\not{D}, \langle a, s\rangle] - \langle c(\nabla^B a), s\rangle)\psi \\ &= \langle c(\nabla^B a), s\rangle \psi,\end{aligned}$$

which is again uniformly bounded. This completes the proof. □

Another proof of this fact follows by adopting the direct construction of the unbounded Kasparov products by Mesland [17]. Indeed, the spectral triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B) \in \Psi(C(M), \mathbb{C})$ is by construction the internal product of $(\Gamma(M, B), \lambda, 0) \in \Psi(\Gamma(M, B), C(M))$ and the spectral triple $(C^\infty(M), L^2(M, S), \not{D}) \in \Psi(C(M), \mathbb{C})$.

4. Yang–Mills theory as a noncommutative manifold

The real spectral triple $(\Gamma^\infty(M,B), L^2(M,B\otimes S), D_B, J, \gamma_B)$ that we obtained in Theorem 3.14 will turn out to be the correct triple to describe a non-trivial PSU(N)-gauge theory on the spin manifold M if the fibers of B are taken to be isomorphic to the $*$-algebra $M_N(\mathbb{C})$. Not only does it describe a non-trivial PSU(N)-gauge theory, every PSU(N)-gauge theory on M is described by such a triple. In this section we will prove these claims by first showing how a principal PSU(N)-bundle can be constructed from this spectral triple (in fact, the algebra $\Gamma^\infty(M,B)$ is already sufficient for this). As in the topologically trivial case [1], the spectral action applied to this triple will give the Einstein–Yang–Mills action, but now the gauge potential can be interpreted as a connection 1-form on the PSU(N)-bundle P. In fact, the original algebra bundle B will turn out to be an associated bundle of the principal bundle P.

4.1. From the spectral triple to principal bundles

According to Theorem 3.8 we are able to reconstruct the unital $*$-algebra bundle B from $\Gamma^\infty(M,B)$. Note that in this theorem the ($*$-)algebra bundles are not required to be locally trivial as a ($*$-)algebra bundle (they *are* locally trivial as a *vector* bundle). In Section 3.2 we were able to construct a spectral triple out of $\Gamma^\infty(M,B)$ if we assumed that B was locally trivial and had a faithful smoothly-varying tracial state. In this Subsection we will need the additional condition that the fibers of B are $*$-isomorphic to $M_N(\mathbb{C})$ and we endow such B with a smoothly-varying tracial state by taking the usual trace on $M_N(\mathbb{C})$ in the fibers.

In order to construct a principal PSU(N)-bundle P out of B, first of all note that since all $*$-automorphisms of $M_N(\mathbb{C})$ are obtained by conjugation with a unitary element $u \in M_N(\mathbb{C})$ the transition functions of the bundle $\Gamma^\infty(M,B)$ have their values in $\mathrm{Ad}U(N) \cong U(N)/Z(U(N)) \cong \mathrm{PSU}(N)$. Thus the bundle B provides us with a open covering of $\{U_i\}$ and transition functions $\{g_{ij}\}$ with values in PSU(N). Using the reconstruction theorem for principal bundles we can construct a principal PSU(N)-bundle. By construction, the bundle B is an associated bundle to P.

Furthermore, for the real and even spectral triple $(\Gamma^\infty(M,B), L^2(M,B\otimes S), D_B, J, \gamma_B)$ of Theorem 3.14 the hermitian connection ∇^B on the bundle B can locally be written as $\nabla^B = d+\omega$, where ω is a $\mathfrak{su}(N)$-valued 1-form, (*cf.* Remark 3.13). Moreover, the transformation rule for ω is $\omega_i = g_{ij}^{-1}dg_{ij} + g_{ij}^{-1}\omega_j g_{ij}$ with g_{ij} the PSU(N)-valued transition function of B. Comparing this expression with the transformation property of a connection 1-form in Definition 2.6 one concludes that the hermitian $*$-algebra connection ∇^B on B induces a connection 1-form on the principal bundle P constructed in the previous paragraph (and vice versa).

Conversely, given a PSU(N)-gauge theory (P,ω) on some compact Riemannian spin manifold, then we can construct the locally trivial hermitian $*$-algebra bundle $B := P\times_{\mathrm{PSU}(N)} M_N(\mathbb{C})$, where PSU$(N)$ acts on $M_N(\mathbb{C})$ in the usual way. Moreover, the connection ω on P induces a hermitian $*$-algebra connection on B. By following the steps in the previous section it is not difficult to see that the gauge theory (P_B,ω_B) obtained from this spectral triple $(\Gamma^\infty(M,B), L^2(M,B\otimes S), ic(\nabla^B\otimes 1 + 1\otimes\nabla^S), J, \gamma)$ is isomorphic to (P,ω). This is in accordance with the approach to almost commutative manifolds taken in [18].

Proposition 4.1. *Let $(\Gamma^\infty(M,B), L^2(M,B\otimes S), D_B, J, \gamma_B)$ be as before, with B a locally trivial $M_N(\mathbb{C})$-$*$-algebra bundle. Then there exists a principal* PSU(N)*-bundle P such that B is an associated bundle of P, and a connection 1-form ω on P. Moreover, every* PSU(N)*-gauge theory on M is determined by such a spectral triple.*

Remark 4.2. A special case occurs when B is an endomorphism bundle. A consequence of a result by Dixmier and Douady in [19] (*cf.* also [20]) is that a bundle B with continuously varying trace is an endomorphism bundle if and only if the Dixmier–Douady class $\delta(\Gamma(M,B)) \in H^3(M,\mathbb{Z})$ of the C^*-algebra of continuous sections $\Gamma(M,B)$ is equal to zero. Because the Dixmier–Douady class of the bundle B vanishes one can lift the PSU(N)-valued transition functions g_{ij} to $U(N)$-valued functions μ_{ij} such that $g_{ij} = \mathrm{Ad}\,\mu_{ij}$, and $\mu_{ij}\mu_{jk} = \mu_{ik}$ (see for instance [20, Theorem 4.85]). One could therefore equally well construct a principal $U(N)$-bundle instead of a PSU(N)-bundle if and only if B is an endomorphism bundle.

4.2. Spectral action

In this section, we will calculate the spectral action for the real spectral triple of Theorem 3.14. We will show that the spectral action applied to the spectral triple $(\Gamma^\infty(M,B), L^2(M,B\otimes S), D_B, J, \gamma_B)$ produces the Einstein–Yang–Mills action for a 1-form A that defines a connection 1-form on the PSU(N)-bundle P. If B is a trivial algebra bundle, this reduces to the result of [1]. In fact, much of their local computations can be adopted in this case as well, since locally, the bundle B is trivial. Nevertheless, for completeness we include the computation in the case at hand.

First of all, already in Remark 3.13 we noticed that locally, on some local trivialization U_i, ∇^B is expressed as $d+\omega_i$ where ω_i is an $su(N)$-valued 1-form that acts in the adjoint representation on $\Gamma^\infty(M,B)$. Therefore, according to Definition 2.6 ω already induces a connection 1-form on P. To get the full gauge potential we need to take the fluctuation of the Dirac operator into account as well.

Inner fluctuations of the Dirac operator are given by a perturbation term of the form

$$A = \sum_j a_j[D,b_j], \quad (a_j, b_j \in \Gamma(M,B)),$$

with the additional condition that $\sum_j a_j[D, b_j]$ is a self-adjoint operator. Explicitly, we have

$$A = \sum_j c \circ (a_j[\nabla, b_j] \otimes 1), \tag{11}$$

where $c : \Omega^1(M) \otimes_{C^\infty(M)} \Gamma^\infty(M, B \otimes S) \to \Gamma^\infty(M, B \otimes S)$ is given by

$$c(\omega \otimes s \otimes \psi) = s \otimes c(\omega)\psi, \quad (\omega \in \Omega^1(M),\ s \otimes \psi \in \Gamma^\infty(M, B \otimes S)).$$

Also, $\sum_j a_j[\nabla, b_j]$ is an element of $\Gamma^\infty(T^*M \otimes B)$.

Locally, on some trivializing neighborhood U, the expression in Eq. (11) can be written as

$$A = \gamma^\mu A_\mu,$$

where A_μ are the components of the 1-form $\sum_j a_j[\nabla, b_j]$ with values in $\Gamma^\infty(M, B)$. Since A is self-adjoint the 1-form A_μ can be considered as a real 1-form taking values in the hermitian elements $\Gamma^\infty(M, B)$.

Similarly, the expression $A + JAJ^*$ is locally written as

$$\gamma^\mu A_\mu - \gamma^\mu J A_\mu J^*,$$

since γ^μ anti-commutes with J in even dimensions. Writing out the second term gives:

$$(\gamma^\mu J A_\mu J^*)(s \otimes \psi) = s A_\mu \otimes \gamma^\mu \psi, \quad \forall s \otimes \psi \in \Gamma^\infty(M, B \otimes S)$$

so that on this local patch $A + JAJ^*$ can be written as

$$\gamma^\mu \operatorname{ad} A_\mu.$$

Consequently, $A + JAJ^*$ eliminates the $\mathrm{i}u(1)$-part of A, making it natural to impose the uni-modularity condition

$$\operatorname{Tr} A = 0.$$

Thus, $-\mathrm{i} \operatorname{ad} A_\mu$ is a one-form on M with values in $\Gamma^\infty(M, \operatorname{ad} P)$. We denote this 1-form by $\mathbb{A}^{\mathrm{pert}}$; it is defined on the whole of M.

The local and global expression for $D + A + JAJ^*$ are given respectively by

$$D_A = \mathrm{i}\gamma^\mu (\nabla^B_\mu \otimes 1 + 1 \otimes \nabla^S_\mu - \mathrm{i} \operatorname{ad} A_\mu \otimes 1)$$

and

$$D_A = \mathrm{i}c \circ (1 \otimes \nabla^S + \nabla_B \otimes 1 + \mathbb{A}^{\mathrm{pert}}).$$

On some trivializing neighborhood U_i $(i \in I)$ the connection ∇^B can be expressed as $d + \mathbb{A}^0_i$ for a unique $su(N)$-valued 1-form $\mathbb{A}^0_i$ on U_i. Thus, on U_i the fluctuated Dirac operator can be rewritten as

$$D_A = \mathrm{i}c \circ (d + 1 \otimes \omega^s + (\mathbb{A}^0_i + \mathbb{A}^{\mathrm{pert}}) \otimes 1).$$

We interpret $(\mathbb{A}^0_i + \mathbb{A}^{\mathrm{pert}})$ as the full gauge potential on U_i; it acts in the adjoint representation on the spinors. The natural action of $g \in \operatorname{Inn}(\Gamma^\infty(M, B)) \simeq \Gamma^\infty(M, \operatorname{Ad} P)$ by conjugation on D_A induces the familiar gauge transformation:

$$\mathbb{A}^0 + \mathbb{A}^{\mathrm{pert}} \mapsto (g^{-1}\mathbb{A}^0 g + g^{-1}(dg)) + g^{-1}\mathbb{A}^{\mathrm{pert}} g = g^{-1}(dg) + g(\mathbb{A}^0 + \mathbb{A}^{\mathrm{pert}})g^{-1},$$

where the first two terms are the transformation of A^0 under a change of local trivialization, and the last term is the transformation of $\mathbb{A}^{\mathrm{pert}}$. Since P is an associated bundle of B it follows from Definition 2.6 that $\mathbb{A}^0 + \mathbb{A}^{\mathrm{pert}}$ induces a $su(N)$-valued connection 1-form on the principal $\mathrm{PSU}(N)$-bundle P that acts on $\Gamma^\infty(M, B)$ in the adjoint representation. Let us summarize what we have obtained so far.

Proposition 4.3. *Let* $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$ *and* P *be as before, so that* $P \times_{\mathrm{PSU}(N)} M_N(\mathbb{C}) \simeq B$. *Then*

(1) *The group of inner automorphisms* $\operatorname{Inn}(\Gamma^\infty(M, B))$ *is isomorphic to the gauge group* $\Gamma^\infty(M, \operatorname{Ad} P)$ *where* $\operatorname{Ad} P = P \times_{\mathrm{PSU}(N)} \mathrm{PSU}(N)$.

(2) *The inner fluctuations of* D_B *are parametrized by a section* $\mathbb{A}^{\mathrm{pert}}$ *of* $\Gamma^\infty(M, \operatorname{ad} P)$ *where* $\operatorname{ad} P = P \times_{\mathrm{PSU}(N)} su(N)$.

Moreover, the action of $\operatorname{Inn}(\Gamma^\infty(M, B))$ *on the inner fluctuations* $D_B + A + JAJ^{-1}$ *by conjugation coincides with the adjoint action of* $\Gamma^\infty(M, \operatorname{Ad} P)$ *on* $\Gamma^\infty(M, \operatorname{ad} P)$.

Let us now proceed to compute the spectral action for these inner fluctuations. First, we recall some results on heat kernel expansions and Seeley–DeWitt coefficients, which will be useful later on; for more details we refer to [21].

If V is a vector bundle on a compact Riemannian manifold (M, g) and if $Q : C^\infty(V) \to C^\infty(V)$ is a second-order elliptic differential operator of the form

$$Q = -(g^{\mu\nu}\partial_\mu\partial_\nu + K^\mu\partial_\mu + L) \tag{12}$$

with $K^\mu, L \in \Gamma^\infty(\mathrm{End}(V))$, then there exist a unique connection ∇ and an endomorphism E on V such that $Q = \nabla\nabla^* - E$. In this situation we can make an asymptotic expansion (as $t \to 0$) of the trace of the operator e^{-tQ} in powers of t:

$$\mathrm{Tr}\ \mathrm{e}^{-tQ} \sim \sum_{n\geq 0} t^{(n-m)/2}a_n(Q), \quad a_n(Q) := \int_M a_n(x, Q)\sqrt{g}\mathrm{d}^m x,$$

where m is the dimension of M and the coefficients $a_n(x, Q)$ are called the *Seeley–DeWitt coefficients*. It turns out [21, Theorem 4.8.16] that $a_n(x, Q) = 0$ for n odd and that the first three even coefficients are given (modulo boundary terms) by

$$\begin{aligned}
a_0(x, Q) &= (4\pi)^{-m/2}\mathrm{Tr}(\mathrm{Id})\\
a_2(x, Q) &= (4\pi)^{-m/2}\mathrm{Tr}\left(-\frac{R}{6}\mathrm{Id} + E\right)\\
a_4(x, Q) &= (4\pi)^{-m/2}\frac{1}{360}\mathrm{Tr}\left(5R^2 - 2R^{\mu\nu}R_{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RE + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\right),
\end{aligned}$$

where $\Omega_{\mu\nu}$ is the curvature of the connection ∇.

This can be used in the computation of the spectral action as follows. Assume that the inner fluctuations give rise to an operator D_A for which D_A^2 is of the form (12) on some vector bundle V on a compact Riemannian manifold M. Then, on writing f as a Laplace transform,[4] we obtain

$$f(D_A/\Lambda) = \int_{t>0} \tilde{g}(t)\mathrm{e}^{-tD_A^2/\Lambda^2}\,\mathrm{d}t. \tag{13}$$

We now restrict to the case where the dimension of the manifold is equal to 4. One calculates that in 4 dimensions the heat expansion (up to order $n = 4$) of the spectral action (3) is given by

$$\begin{aligned}
\mathrm{Tr}\,(f(D/\Lambda)) &\sim f(0)\Lambda^0 a_4(D^2) + \sum_{n=0,2}\Lambda^{4-n}a_n(D^2)\frac{2}{\Gamma\left(\frac{4-n}{2}\right)}\int_0^\infty f(w)w^{(4-n)-1}\mathrm{d}w\\
&= f(0)\Lambda^0 a_4(D^2) + 2f_2\Lambda^2 a_2(D^2) + 2\Lambda^4 f_4 a_0(D^2)
\end{aligned}$$

where the f_k are moments of the function f:

$$f_k := \int_0^\infty f(w)w^{k-1}\mathrm{d}w; \quad (k > 0).$$

Lemma 4.4. *For the spectral triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$, the square of the fluctuated Dirac operator D_A^2 is locally of the form $-g_{\mu\nu}\partial_\mu\partial_\nu + K_\mu\partial_\mu + L$ and we have the following expressions for $\Omega_{\mu\nu}$ and E:*

$$\begin{aligned}
E &= -\frac{1}{4}R \otimes 1_{N^2} - \sum_{\mu<\nu}\gamma^\mu\gamma^\nu \otimes F_{\mu\nu}\\
\Omega_{\mu\nu} &= \frac{1}{4}R^{ab}_{\mu\nu}\gamma_{ab} \otimes 1_{n^2} + \mathrm{id}_4 \otimes F_{\mu\nu},
\end{aligned}$$

where $F_{\mu\nu}$ is the curvature of the connection $\nabla^B + \mathbb{A}^{\mathrm{pert}}$.

This result allows us to compute the bosonic spectral action for the fluctuated Dirac operator D_A, essentially reducing the computation in terms of a local trivialization to the trivial case of [1] with the following result.

Theorem 4.5. *For the spectral triple $(\Gamma^\infty(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$, the spectral action equals the Yang–Mills action for $\nabla^B + \mathbb{A}^{\mathrm{pert}}$ minimally coupled to gravity:*

$$\mathrm{Tr}\,(f(D_A/\Lambda)) \sim \frac{f(0)}{24\pi^2}\int_M \mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}\sqrt{g}\mathrm{d}^4x + \frac{1}{(4\pi)^2}\int_M \mathcal{L}(g_{\mu\nu})\sqrt{g}\mathrm{d}^4x + \mathcal{O}(\Lambda^{-2}),$$

---

[4] We thank the anonymous referee for pointing out that the Laplace transform can be avoided by working with spectral densities, cf. [22] and [23, Sect. 8.4] for more details.

where $\mathcal{L}(g^{\mu\nu})$ *is given by*

$$\mathcal{L}(g^{\mu\nu}) = 2N^2\Lambda^4 f_4 + \frac{N^2}{6}\Lambda^2 f_2 R - \frac{N^2 f_0}{80} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma},$$

ignoring topological and boundary terms. Here C *denotes the Weyl-tensor and* f_i *are the i'th moments of the function* f.

5. Conclusions and outlook

We have generalized the noncommutative description of the Einstein–Yang–Mills system by Chamseddine and Connes [1] to the case where the principal bundle describing the gauge field is non-trivial. We have obtained a spectral triple from an algebra bundle and related its construction to the internal Kasparov product in unbounded KK-theory. If the typical fiber of the algebra bundle is $M_N(\mathbb{C})$, we have shown that its internal fluctuations are parametrized by a PSU(N)-gauge field. In fact, we reconstructed a PSU(N)-principal bundle for which the algebra bundle is an associated bundle, and on which the gauge field defines a connection one-form. Finally, we have applied the spectral action principle to these inner fluctuations of the spectral triple and derived the Yang–Mills action for a PSU(N)-gauge field, minimally coupled to gravity.

A natural question that arises in this topologically non-trivial context is how to incorporate, besides the Yang–Mills action, a topological action functional. Given an (even) spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$, we introduce – besides the spectral action (3) – an invariant by

$$S_{\text{top}}[A] = \operatorname{Tr}\left(\gamma f(D_A/\Lambda)\right). \tag{14}$$

We will call this the *topological spectral action* (cf. also Section 7 in [24]). It is clearly invariant under the action of the group of unitaries in the algebra $\mathcal{A}$, acting on γ by conjugation.

If we again write f as a Laplace transform (13) and use the McKean–Singer formula,

$$\operatorname{Tr} e^{-tD_A^2} = \operatorname{Index} D_A,$$

then we can prove that asymptotically

$$S_{\text{top}}[A] \sim f(0)\operatorname{Index} D_A.$$

In our case of interest, *i.e.* the setting of Theorem 4.5, we thus find with the Atiyah–Singer index theorem an extra contribution of the form

$$S_{\text{top}}[A] \sim \frac{f(0)}{(2\pi i)^{n/2}}\int_M \hat{A}(M)\operatorname{ch}(B)$$

in terms of the $\hat{A}$ genus of M and the Chern character of the algebra bundle B.

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