Noncommutative geometry and some of its applications

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Motivation

Mathematics:

- Index theory, foliations
- Quantization

Physics:

- Quantum spacetime, quantum symmetries
- Fine-structure of spacetime: Standard Model of elementary particles

Trading spaces for algebras

- X compact Hausdorff topological space
- Associate to X the algebra $C = C(X)$ of continuous functions
- Vice-versa, consider the space $\widehat{C}$ of its irr. representations.
- Since C is commutative, these are characters $\phi : C \to \mathbb{C}$:

$$\phi(f_1 f_2) = \phi(f_1)\phi(f_2) \qquad (f_1, f_2 \in C(X))$$

- Equip $\widehat{C}$ with the pointwise topology:

$$\phi_n \to \phi \quad \text{iff} \quad \phi_n(f) \to \phi(f) \qquad (\forall f \in C(X))$$

Theorem (Gelfand)

$\widehat{C}$ is a compact Hausdorff topological space and

$$\widehat{C} \simeq X.$$

with $\phi \leftrightarrow p$ via $\phi(f) = f(p)$.

- C : Top $\to$ Comm(C^*)Alg is (anti)equivalence of categories.

Noncommutative topology

- Summarizing, $C(X)$ is an algebra with topology given by the sup-norm, satisfying

$$\|f^* f\|_\infty = \|f\|_\infty^2$$

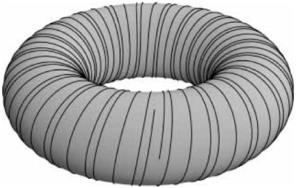
  i.e. a C^*-algebra.
- This motivates to consider a noncommutative C^*-algebra $(A, \|\cdot\|)$ as the 'continuous functions on a noncommutative space'
- Categorically:

$$\mathrm{NCTop} := \mathrm{C}^* - \mathrm{Alg}^{\mathrm{op}}$$

- N.B. For a noncommutative algebra, the space of irreps is typically *not* a good topological space to study (cf. the next example...)

Key example: the noncommutative torus I

- Motivated by Kronecker foliation of $\mathbb{T}^2$



- Consider the corresponding leaf space $S^1/\theta\mathbb{Z}$
- Key result: if $\theta \in \mathbb{Q}$:

$$C(\widehat{S^1/\theta\mathbb{Z}}) \simeq C(\widehat{S^1) \rtimes_\theta}\, \mathbb{Z}$$

- But, when θ is irrational $S^1/\theta\mathbb{Z}$ is not Hausdorff any more, and the algebra $C(S^1) \rtimes_\theta \mathbb{Z}$ is a better description of this quotient.

- Explicitly, $C(S^1) \rtimes_\theta \mathbb{Z}$ is the C^*-algebra generated by U, V, U^*, V^* such that

$$U^* U = 1 = UU^*; \qquad V^* V = 1 = VV^*;$$
$$UV = e^{2\pi i\theta} VU.$$

  This algebra describes the noncommutative torus, denoted $C(\mathbb{T}^2_\theta)$.
- In the limit $\theta \to 0$ it becomes $C(\mathbb{T}^2)$ for the ordinary torus. This is an example of a quantization.

Noncommutative Riemannian geometry

- Given a Riemannian manifold (M, g), there is the Laplace operator

$$\Delta := -g_{\mu\nu}\partial_\mu\partial_\nu.$$

- Δ is a second-order differential operator, which implies that

$$[[\Delta, f_1], f_2] \text{ is a bounded operator on } L^2(M)$$

with f_1, f_2 acting on $L^2(M)$ by pointwise multiplication.

- Δ is elliptic, which implies that

$$(1 + \Delta)^{-1} \text{ is a compact operator}$$

- More convenient, on a Riemannian spin manifold there is a Dirac operator: first-order operator and such that

$$D^2 = \Delta + \text{ lower order}$$

- $[D, f]$ is bounded and $(1 + D^2)^{-1}$ compact.

Noncommutative Riemannian geometry: a definition

- With $C(M)$ encoding the topology of M, we capture a Riemannian spin manifold by the triple:

$$(C^\infty(M) \subset C(M), L^2(M,S), D)$$

- The geodesic distance $d(p,q)$ between points can be obtained as:

$$d(\phi,\chi) = \sup_f \{|\phi(f) - \chi(f)| : \|[D,f]\| \leq 1\}$$

since $\|[D,f]\| = \|f\|_{\text{Lip}}$ and $\phi \leftrightarrow p$, $\chi \leftrightarrow q$.

- Drop commutativity: a noncommutative Riemannian spin manifold:

$$(\mathcal{A} \subset A, \mathcal{H}, D)$$

such that $[D,a]$ $(a \in \mathcal{A})$ is bounded and $(1+D^2)^{-1}$ compact.

- Connes' reconstruction theorem

Key example: the noncommutative torus II

- Recall: $C(\mathbb{T}^2_\theta)$ generated by 2 unitaries s.t. $UV = e^{2\pi i\theta}VU$.
- Two derivations on $C(\mathbb{T}^2_\theta)$:

$$\delta_1(U) = U; \qquad \delta_1(V) = 0$$
$$\delta_2(V) = 0; \qquad \delta_2(V) = V.$$

and extended by Leibniz rule: $\delta_i(ab) = \delta_i(a)b + a\delta_i(b)$.

- Self-adjoint operator on $L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2$:

$$D = \begin{pmatrix} 0 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & 0 \end{pmatrix}$$

Theorem (Connes)

$(C(\mathbb{T}^2_\theta), L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2, D)$ *is a noncommutative Riemannian spin manifold.*

- Complex structure $\partial = \frac{1}{\tau - \overline{\tau}}(-\overline{\tau}\delta_1 - \delta_2)$

Noncommutative toric manifolds

- Consider a Riemannian spin manifold M, carrying an isometric action of a torus $\mathbb{T}^2$.
- Define $C(M_\theta) \subset C(M) \otimes C(\mathbb{T}^2_\theta)$ as the fixed points under the diagonal action of the torus $\mathbb{T}^2$.

Theorem (Connes–Landi)

$(C(M_\theta), L^2(M, S), D_M)$ *is a noncommutative Riemannian spin manifold*

Quantum SU(2)

- $C(\mathrm{SU}_q(2))$: algebra generated by a, b, a^*, b^* such that

$$U = \begin{pmatrix} a & -qb^* \\ b & a^* \end{pmatrix},$$

is unitary: $UU^* = U^*U = 1$. We take $0 < q < 1$.

- $\mathcal{U}_q(su(2))$: algebra generated by K, K^{-1}, E, F such that

$$KE = qEK; \qquad KF = q^{-1}FK; \qquad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

- $\mathcal{U}_q(su(2))$ acts on $C(\mathrm{SU}_q(2))$: K-degree decomposition

$$C(\mathrm{SU}_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

and E, F act as laddering operators.

Complex projective space $\mathbb{C}P^1_q$

(joint with Khalkhali and Landi)

- The degree 0 elements $\mathcal{L}_0 =: C(\mathbb{C}P^1_q)$ define the 'functions' on the quantum projective line.
- Principal $U(1)$ Hopf fibration: $C(\mathbb{C}P^1_q) \hookrightarrow C(\mathrm{SU}_q(2))$
- $\mathcal{L}_n$ are sections of associated 'vector bundles $\mathbb{C}P^1_q \times_{U(1)} \mathbb{C}$'; of which we denote

$$\Omega^{(1,0)}(\mathbb{C}P^1_q) := \mathcal{L}_2; \qquad \Omega^{(0,1)}(\mathbb{C}P^1_q) := \mathcal{L}_{-2}$$

- Complex structure:

$$\partial := E : C(\mathbb{C}P^1_q) \to \Omega^{(1,0)}(\mathbb{C}P^1_q);$$
$$\overline{\partial} := F : C(\mathbb{C}P^1_q) \to \Omega^{(0,1)}(\mathbb{C}P^1_q);$$

Theorem

The kernel of $\overline{\partial}$ in $C(\mathbb{C}P^1_q)$ is $\mathbb{C}$, i.e. there are no non-trivial holomorphic functions on $\mathbb{C}P^1_q$.

Line bundles on $\mathbb{C}P^1_q$

- The 'line bundles' of degree n on $\mathbb{C}P^1_q$ are described by the $C(\mathbb{C}P^1_q)$-bimodules $\mathcal{L}_n$.
- Holomorphic structure:

$$\nabla := E : \mathcal{L}_n \to \Omega^{(1,0)}(\mathbb{C}P^1_q) \otimes_{C(\mathbb{C}P^1_q)} \mathcal{L}_n;$$
$$\overline{\nabla} := F : \mathcal{L}_n \to \Omega^{(0,1)}(\mathbb{C}P^1_q) \otimes_{C(\mathbb{C}P^1_q)} \mathcal{L}_n;$$

Theorem

Let n be a positive integer. Then

1. $H^0(\mathcal{L}_n, \overline{\nabla}) = 0$,
2. $H^0(\mathcal{L}_{-n}, \overline{\nabla}) \simeq \mathbb{C}^{n+1}$.

Homology ring of line bundles

- Tensor product $\mathcal{L}_m \otimes_{C(\mathbb{C}P^1_q)} \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{m+n}$.

Theorem

The space $R = \bigoplus_{n \geq 0} H^0(\mathcal{L}_{-n}, \overline{\nabla})$ *carries a ring structure and is isomorphic to the quantum plane:* $R \simeq \mathbb{C}\langle a, b\rangle/(ab - qba)$

- A similar result was derived by Artin, Tate and Van den Bergh in the context of noncommutative algebraic geometry:
  - Consider line bundle on $\mathbb{C}P^1$ and an automorphism σ of $\mathbb{C}P^1$.
  - With the pullback $\sigma^* \mathcal{L}$ denoted by $\mathcal{L}^\sigma$, ATV defined the twisted homogeneous coordinate ring by

$$R' = \bigoplus_{n \geq 0} H^0(\mathbb{C}P^1, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}})$$

  - (Keeler) If $q \in [0, 1]$ define an automorphism $\sigma : (z_1 : z_2) \to (qz_1 : z_2)$; one computes that $R' \simeq \mathbb{C}\langle a, b\rangle/(ab - qba)$

Noncommutative spin geometry of $\mathbb{C}P^1_q$

- Define a Hilbert space, with a representation of $C(\mathbb{C}P^1_q)$:

$$\mathcal{H} = (\mathcal{L}_1 \oplus \mathcal{L}_{-1})^{\mathrm{cl}}$$

- Since $\partial : \mathcal{L}_{-1} \to \mathcal{L}_1$, we define $D = \begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix}$
- One then computes for all $x \in C(\mathbb{C}P^1_q)$:

$$[D, x] = \begin{pmatrix} 0 & q^{-\frac{1}{2}}\partial(x) \\ q^{\frac{1}{2}}\overline{\partial}(x) & 0 \end{pmatrix}$$

and eigenvalues of D are
$\pm[l+1]_q = (q^{-l-1} - q^{l+1})/(q^{-1} - q)$ with $l \geq 0$.

Theorem (Dabrowski–Sitarz)

$(C(\mathbb{C}P^1_q), \mathcal{H}, D)$ *is a noncommutative Riemannian spin manifold.*

Noncommutative spin geometry of $\mathrm{SU}_q(2)$

Surprisingly, we also have

Theorem (Dabrowski–Landi–Sitarz-vS–Várilly)

$(C(\mathrm{SU}_q(2)), L^2(\mathrm{SU}(2), S), D_{\mathrm{SU}(2)})$ *is a noncommutative Riemannian spin manifold.*

incorporating for the first time an example from quantum group theory in noncommutative geometry.