Noncommutative Geometry and Particle Physics

Walter D. van Suijlekom

Institute for Mathematics, Astrophysics and Particle Physics, Faculty of Science, Radboud University Nijmegen, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherlands

E-mail address: waltervs@math.ru.nl

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2.1. Finite spaces

Consider a space X consisting of N points:

$$\underset{1}{\bullet} \qquad \underset{2}{\bullet} \qquad \cdots\cdots \qquad \underset{N}{\bullet}$$

The $*$-algebra $C(X)$ of (continuous) $\mathbb{C}$-valued functions on this discrete space is rather simple, it is just $\mathbb{C}^N$. A fancy way of putting this is to say that C is a functor

$$C : \mathsf{Fin} \to \mathsf{Alg}_f$$

from the category of finite spaces to the category of finite-dimensional $*$-algebras. Indeed, a map $\phi : X_1 \to X_2$ of finite spaces induces a map $\phi^* : C(X_2) \to C(X_1)$ by pullback:

$$\phi^* f = \phi \circ f \in C(X_1)$$

when $f \in C(X_2)$. We arrive at the following
Question: Can we 'invert' the functor C?

In other words, given a finite dimensional algebra A can we obtain a finite space X such that $A \simeq C(X)$? The answer is: not quite, since $C(X)$ is always commutative. This suggest two ways of resolving this issue:

(1) Restrict to the category $\mathsf{CommAlg}_f$ of (fin.dim.) commutative algebras.
(2) Change morphisms in Alg_f.

Before explaining both of these solutions, let us introduce some useful definitions on representations of finite-dimensional algebras, not necessarily commutative, and which moreover extend readily to the infinite-dimensional case.

Definition 2.1. *A representation of $A \in \mathsf{Alg}_f$ is a pair (V, π) where V is an inner product space and π a $*$-algebra map*

$$\pi : A \to L(V)$$

Equivalently, V is a left A-module. A representation (V, π) is irreducible if the only subspaces in V that are invariant under the action of A are $\{0\}$ or V.

Example 2.2. *Consider $A = M_n(\mathbb{C})$. There is a defining representation on $\mathbb{C}^n$, which is irreducile. A reducible representation is given on $\mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in M_n(\mathbb{C})$ acting in block-form:*

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

which thus decomposes in the direct sum of two defining representations. See also Lemma 2.16 below.

Exercise 2.1. *Show that a representation (V, π) of A is irreducible if and only if the commutant $\pi(A)'$ of $\pi(A)$ consists of multiples of the identity. Here*

$$\pi(A)' = \{T \in L(V) : \pi(a)T = T\pi(a) \forall a \in A\}.$$

Definition 2.3. *Two representation (V_1, π_1) and (V_2, π_2) of $A \in \mathsf{Alg}_f$ are unitarily equivalent if there exists a unitary matrix $U : V_1 \to V_2$ such that*

$$\pi_1(a) = U^* \pi_2(a) U.$$

Definition 2.4. *The structure space $\widehat{A}$ of A is the set of all unitary equivalence classes of irreducible representations of A.*

Exercise 2.2. *Show that for any algebra $A \in \mathsf{Alg}_f$ there is a 1-1 correspondence between unitary equivalence classes of representations of A and of $M_n(A)$, the algebra of matrices with entries in A. In other words, $\widehat{M_n(A)} = \widehat{A}$.*

2.1.1. Commutative algebras. We now explain how the first option above resolves the question raised. First, any commutative (fin. dim.) algebra is of the form

$$A \simeq \mathbb{C}^N$$

so it should not be too hard to construct its structure space $\widehat{A}$. Of course, if $A = \mathbb{C}^N$ then $\widehat{A} \simeq \{1, \ldots, N\}$ where each point in $\widehat{A}$ corresponds to the representation of the corresponding copy of $\mathbb{C}$ on $\mathbb{C}$:

$$(f_1, \ldots, f_N) \mapsto \begin{pmatrix} f_1 & & \\ & \ddots & \\ & & f_N \end{pmatrix}$$

acting on $\mathbb{C}^N$.

We conclude that there is an equivalence between the category Fin of finite spaces and the category $\mathsf{CommAlg}_f$ of finite-dimensional commutative $*$-algebras.

2.1.2. Noncommutative algebras. A more interesting perspective is given by the noncommutative alternative above. We adapt the morphisms in Alg_f to obtain a category which is equivalent to Fin.

Definition 2.5. *The category KK_f has as objects finite-dimensional (noncommutative) algebras. The morphisms $\mathrm{Hom}(A, B)$ are given by $A - B$-bimodules E with a B-valued inner product. That is, E is both a left A-module and a right B-module which mutually commute, and there is a hermitian structure $\langle \cdot, \cdot \rangle \to B$ satisfying*

$$\langle e_1, e_2 b \rangle = \langle e_1, e_2 \rangle b; \qquad \langle e_1, e_2 \rangle^* = \langle e_2, e_1 \rangle; \qquad \langle e, e \rangle \geq 0 \text{ with equality iff } e = 0.$$

Composition of morphisms is given by the balanced tensor product:

$$E \circ F := E \otimes_B F, \qquad (E \in \mathrm{Hom}(A, B), F \in \mathrm{Hom}(B, C)),$$

with C-valued inner product given by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes F} = \langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F$$

Exercise 2.3. *Check that this last formula defines a C-valued inner product.*

Example 2.6. *The trivial A-bimodule given by A itself is the identity morphism in* $\mathrm{Hom}(A, A)$.

Example 2.7. *The vector space $E = \mathbb{C}^n$ is a $M_n(\mathbb{C}) - \mathbb{C}$-bimodule; with the standard $\mathbb{C}$-valued inner product it becomes a morphism in KK_f from $M_n(\mathbb{C})$ to $\mathbb{C}$.*

Example 2.8. *Similarly, the vector space $F = \mathbb{C}^n$ is an $\mathbb{C} - M_n(\mathbb{C})$-bimodule by right matrix-multiplications. A $M_n(\mathbb{C})$-valued inner product is given by*

$$\langle v_1, v_2 \rangle = v_1 \overline{v}_2^t \in M_n(\mathbb{C}).$$

As such, it is a morphisms in KK_f from $\mathbb{C}$ to $M_n(\mathbb{C})$.

Observe in the last two examples that we can compose in two ways:

$$E \otimes_{\mathbb{C}} F \simeq M_n(\mathbb{C}); \qquad F \otimes_{M_n(\mathbb{C})} E \simeq \mathbb{C}.$$

This leads us to our first little result, which has a very powerful analogue in the infinite-dimensional case.

Theorem 2.9. *Two finite dimensional algebras are isomorphic in KK_f if and only if they have isomorphic structure spaces (which are finite).*

Since in this case the representation theories of both algebras are equivalent, we call them *Morita equivalent*.

PROOF. Let $A \simeq B$ in KK_f. Then there exists ${}_AE_B$ and ${}_BF_A$ such that

$$E \otimes_B F \simeq A, \qquad F \otimes_A E \simeq B.$$

If $[\pi_B] \in \widehat{B}$ then we can define π_A by setting

$$\pi_A : A \to L(E \otimes_B V), \quad \pi_A(a)(e \otimes v) = ae \otimes v \tag{2.1.1}$$

Vice versa, we construct $\pi_B : B \to L(F \otimes_A W)$ from π_A by setting $\pi_B(b)(f \otimes w) = bf \otimes w$ and these two maps are each other's inverse.

Now let $A, B \in \mathsf{KK}_f$ such that $\widehat{A} \simeq \widehat{B}$. By the Artin-Wedderburn Theorem, any finite-dimensional algebra is a direct sum of matrix algebras so that we may assume

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}); \qquad B = \bigoplus_{i=j}^{M} M_{m_j}(\mathbb{C})$$

and $\widehat{A} \simeq \widehat{B}$ implies that $M = N$. Then, define

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i};$$

with A acting by block-diagonal matrices on the first tensor and B acting in a similar way by right matrix multiplication on the second leg of the tensor product; Also, set

$$F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}$$

with now B acting on the left and A on the right. Then, as above

$$E \otimes_B F \simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i} \simeq A$$

and similarly we obtain that $F \otimes_A E \simeq B$ as required. □

EXERCISE 2.4. *Fill in the gaps in the above proof:*

(a) *Show that the representation π_A defined by Eq.* (2.1.1) *is irreducible if and only if π_B is.*
(b) *Show that the association of the class $[\pi_A]$ from $[\pi_B]$ through Eq.* (2.1.1) *is independent of the choice of representative π_B.*

EXERCISE 2.5. *Show that $\mathsf{CommAlg}_f$ is a full subcategory in KK_f. In other words, show that the inclusion functor $\iota : \mathsf{CommAlg}_f \to \mathsf{KK}_f$ induces an isomorphism* $\mathrm{Hom}_{\mathsf{CommAlg}_f}(A, B) \simeq \mathrm{Hom}_{\mathsf{KK}_f}(A, B)$ *if A, B are finite-dimensional commutative algebras.*

We conclude that there is an equivalence between the category Fin of finite spaces and the category KK_f of finite-dimensional noncommutative algebras with KK-bimodule morphisms. Thus, there are equivalences of categories:

$$\mathsf{CommAlg}_f \sim \mathsf{Fin} \sim \mathsf{KK}_f$$

As should be clear by now, we prefer the category KK_f over the rest.

2.2. Noncommutative geometric finite spaces

Consider again a finite space X described by a noncommutative (finite dimensional) $*$-algebra A. We would like to introduce some geometry on X, in particular, a notion of metric on X.

Question: How can we describe distances between the points in x, say, as embedded in a metric space?

Naively, this could be done by giving an array $\{d_{ij}\}_{i,j\in X}$ of real nonnegative entries, indexed by two elements in X and requiring at least that $d_{ii} = 0$ and $d_{ij} = d_{ji}$.

EXAMPLE 2.10. *The usual discrete metric on the discrete space X is given by such an array:*

$$d_{ij} = \begin{cases} 1 \text{ if } i \neq j \\ 0 \text{ if } i = j. \end{cases}$$

Recall the particular form of A in terms of matrix algebras:

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$$

This $*$-algebra can be represented (irreducibly) on the vector space

$$H = \bigoplus_{i=1}^{N} \mathbb{C}^{n_i}.$$

Moreover, an array $\{d_{ij}\}$ can be implemented on H as a symmetric linear operator on H:

$$D : H \to H$$

In fact, we can define a metric on the structure space $\widehat{A}$

$$d_{ij} = \sup_{f\in A}\{|\operatorname{Tr} f(i) - \operatorname{Tr} f(j)| : \|[D, f]\| \leq 1\}$$

where i labels the defining (irreducible) representation corresponding to the matrix algebra $M_{n_i}(\mathbb{C})$ in the decomposition of A. The norm $\|T\|$ of a matrix T is by definition the square root of the largest eigenvalues of T^*T.

This suggests that the above structure of algebra, representation space and symmetric matrix D is the data that captures a metric structure on the finite space X.

DEFINITION 2.11. *A* finite spectral triple *is a triple (A, H, D) of a $*$-algebra represented faithfully on a finite-dimensional Hilbert space, together with a symmetric matrix $D : H \to H$.*

We will loosely refer to D as a finite Dirac operator. In this vain, a finite spectral triple naturally gives rise to a notion of (discrete) differential forms.

DEFINITION 2.12. *The A-bimodule of Connes' differential one-forms is given by*

$$\Omega^1_D(A) := \left\{ \sum_k a_k[D, b_k] : a_k, b_k \in A \right\}$$

The structure of a finite spectral triple can be further enriched by introducing a $\mathbb{Z}_2$-grading on H, denoted by γ, demanding that A is even and D is odd with respect to this grading:

$$\gamma D = -D\gamma; \qquad \gamma a = a\gamma (a \in A).$$

Also, there is a more symmetric refinement of the notion of finite spectral triple in which H is an A-bimodule.

DEFINITION 2.13. *A* real *finite spectral triple is given by a spectral triple (A, H, D) and a anti-unitary operator $J : H \to H$ such that $a^\circ := Ja^*J^{-1}$ furnishes a right representation of A such that*

$$[a, b^\circ] = 0; \qquad [[D, a], b^\circ] = 0 \tag{2.2.1}$$

for all $a,b \in A$. Moreover, we demand that J, D and (in the even case) γ satisfy commutation relations:

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon''\gamma J \quad \text{(even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n \mod 8$:

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

and determine the KO-dimension of $(A, H, D; J, \gamma)$.

The above two conditions (2.2.1) are called *commutant property* and *first-order condition*, respectively. They imply that left multiplication by an element in A and $\Omega^1_D(A)$ commutes with the right action of A.

REMARK 2.14. *The opposite algebra A° is defined to be A as a vector space but with opposite product $\circ$:*

$$a \circ b := ba.$$

Thus, a° defines a left representation of A°: $(a \circ b)^\circ = a^\circ b^\circ$.

The following exercise is inspired by Tomita-Takesaki theory of Von Neumann algebras.

EXERCISE 2.6. *Suppose that $(A, H, D = 0)$ is a finite spectral triple such that H posesses a cyclic and separating vector ξ for A:*

$$A\xi = H \ \text{(cyclic)};$$
$$a\xi = 0 \implies a = 0 \ \text{(separating)}.$$

Show that the operator $J : H \to H$ defined by

$$J(a\xi) = a^*\xi$$

makes $(A, H, 0)$ a real finite spectral triple.

DEFINITION 2.15. *Two finite spectral triples (A_1, H_1, D_1) and (A_2, H_2, D_2) are called* unitary equivalent *if $A_1 \simeq A_2$ and there exists a unitary intertwining operator $U : H_1 \to H_2$ such that*

$$U\pi_1(a)U^* = \pi_2(a) \qquad (a \in \mathcal{A}_1)$$
$$UD_1U^* = D_2$$

If there exist grading operators γ_1, γ_2 then we also demand that $U\gamma_1 U^* = \gamma_2$. If there exist real structures J_1, J_2 then we also demand that $UJ_1U^* = J_2$.

2.2.1. Classification of finite spectral triples. In this section, we follow Krajewski's work [**4**] on the classification of all real finite spectral triples $(A, H, D; J)$ modulo unitary equivalence. This is accomplished in a nice diagrammatic way, involving what are now called Krajewski diagrams. They play the same role for finite spectral triple as Dynkin diagrams do for simple Lie algebras.

The algebra: Here, there is a shortcut to the matrix-algebra decomposition of A. In fact, since A acts faithfully on a Hilbert space, it is a subalgebra of a matrix algebra $L(H) = M_{\dim(H)}(\mathbb{C})$. It is not hard to see that thus

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}).$$

for some $n_1, \ldots, n_N$.

The Hilbert space: Let us start with a basic result on irreducible representations of $M_n(\mathbb{C})$.

LEMMA 2.16. *The unique irreducible representation of $M_n(\mathbb{C})$ is given by left matrix multiplication on $\mathbb{C}^n$.*

PROOF. It is clear that $\mathbb{C}^n$ is an irreducible representation of $A \equiv M_n(\mathbb{C})$. Suppose V is irreducible and of dimension K and define a linear map

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{K \text{ copies}} \to V^*; \qquad \phi(a_1, \ldots, a_K) \to e^1 \circ a_1^t + \cdots e^K \circ a_K^t$$

in terms of a basis $\{e^1, \ldots e^K\}$ of the dual vector space V^*. This is a map of $M_n(\mathbb{C})$ representations, provided a matrix a acts on the dual vector space V^* by sending $v \mapsto v \circ a^t$. It is also surjective, so that $\phi^* : V \to (A^K)^*$ is injective. Upon identifying $(A^K)^*$ with A^K as A-representation spaces, it follows that V is a submodule of $A^K \simeq \oplus(\mathbb{C}^n)^{\oplus nK}$. By irreducibility $V \simeq \mathbb{C}^n$. □

We conclude that the irreducible representations of $A = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$ are given by corresponding direct sums:

$$\bigoplus_{i=1}^N \mathbb{C}^{n_i} \equiv \bigoplus_{i=1}^N \mathbf{n}_i$$

on which A acts by left block-diagonal matrix multiplication. We adopt the physics notation to indicate the irreducible representation of $M_{n_i}(\mathbb{C})$ by their dimension.

Now, besides the representation of A, there should also be a representation of A° on H, which is mutually commuting with that of A. In other words, we are looking for the irreducible representations of $A \otimes A^\circ$. If we denote the unique irreducible representation of $M_n(\mathbb{C})^\circ$ by $\mathbb{C}^{n\circ}$, this implies that the irreducible representation of $A \otimes A^\circ$ is given by

$$H = \bigoplus_{i,j=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\circ} \equiv \bigoplus_{i=1}^N \mathbf{n}_i \otimes \mathbf{n}_j^\circ$$

The integers $\mathbf{n}_i$ and $\mathbf{n}_j^\circ$ form the grid of a diagram: Whenever there is a node at the coordinates

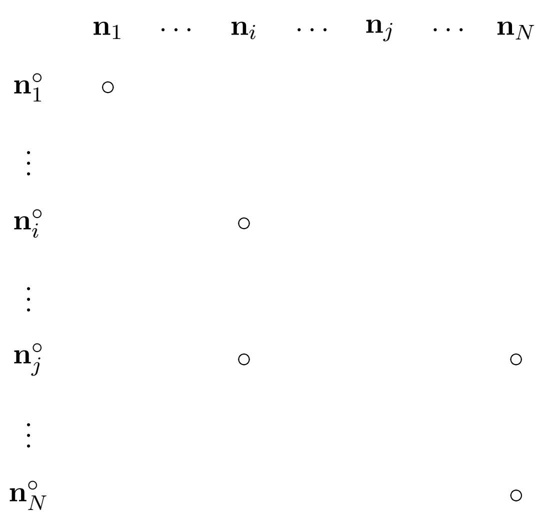


FIGURE 1. The nodes represent the presence of $\mathbf{n}_i \otimes \mathbf{n}_j^\circ$ in H.

$(\mathbf{n}_i, \mathbf{n}_j^\circ)$, the representation $\mathbf{n}_i \otimes \mathbf{n}_j^\circ$ is present in the direct sum decomposition of H.

EXAMPLE 2.17. *Consider the algebra $A = \mathbb{C} \oplus M_2(\mathbb{C})$. The irreducible representations of A are given by* **1** *and* **2**. *The two diagrams*

	1	**2**
1°	○	
2°		○

	1	**2**
1°	○	○
2°		

correspond to $H_1 = \mathbb{C} \oplus M_2(\mathbb{C})$ *and* $H_2 = \mathbb{C} \oplus \mathbb{C}^2$*, respectively. We used that* $\mathbb{C}^2 \otimes \mathbb{C}^{2\circ} \simeq M_2(\mathbb{C})$*. The action of* A *on the left of* H_1 *is given by the matrix*

$$\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix}$$

with $a \in M_2(\mathbb{C})$ *acting on* $M_2(\mathbb{C}) \subset H_1$ *by left matrix multiplication. The right action of* A *on* H_1 *corresponds to the same matrix acting by right matrix multiplication.*

For H_2 *the left action of* A *is given by matrix multiplication by the above matrix on vectors in* $\mathbb{C} \oplus \mathbb{C}^2$*. However, the right action of* $(\lambda, a) \in A$ *is given by scalar multiplication with* λ *on all of* H_2*.*

The real structure: Recall that an anti-unitary operator $J : H \to H$ satisfies

$$\langle J\psi, J\phi \rangle = \langle \phi, \psi \rangle,$$

and is bijective. We can always write $J\psi = K\overline{\psi}$ in terms of a matrix K which now satisfies

$$\langle K\psi, K\phi \rangle = \langle \psi, \phi \rangle$$

and is bijective. In other words, K is a unitary operator on H.

LEMMA 2.18. *The presence of a real structure for the* A*-bimodule* H *requires the corresponding diagram to be symmetric along the diagonal.*

PROOF. The compatibility between the left and right action of A and the operator J implies that for $a = a_1 \oplus \cdots \oplus a_N \in A$ according to the above direct sum decomposition, we have

$$K(a_1^t \oplus \cdots \oplus a_N^t) = (a_1^\circ \oplus \cdots \oplus a_N^\circ)K.$$

This implies that K maps $\mathbf{n}_i \otimes \mathbf{n}_j^\circ$ bijectively to $\mathbf{n}_j \otimes \mathbf{n}_i^\circ$. All the maps $K|_{\mathbf{n}_i \otimes \mathbf{n}_j^\circ \oplus \mathbf{n}_j \otimes \mathbf{n}_i^\circ}$ can be simultaneously diagonalized so that we have a real structure on the A-bimodule H if and only if the corresponding diagram is symmetric along the diagonal. □

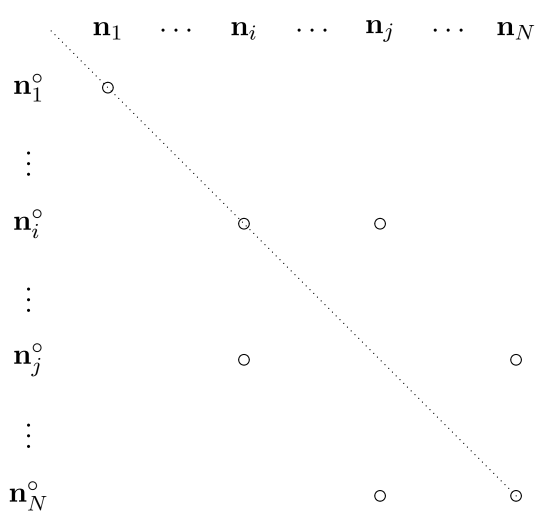


FIGURE 2. The presence of the real structure J implies a symmetry in the diagram along the diagonal.

Remark 2.19. *The condition that* $J^2 = \epsilon$ *implies that* $K^t = \epsilon K$, *on top of unitarity of* K. *This means that there is little choice in the form of* K *(modulo base-change): on the direct summand* $\mathbf{n}_i \otimes \mathbf{n}_j^\circ \oplus \mathbf{n}_j \otimes \mathbf{n}_i^\circ$ *we have*

$$K \sim \begin{pmatrix} 0 & \tau \\ \epsilon\tau & 0 \end{pmatrix}$$

where $\tau : \mathbf{n}_i \otimes \mathbf{n}_j^\circ \to \mathbf{n}_j \otimes \mathbf{n}_i^\circ$ *is the flip* $\tau(v \otimes w^\circ) = w \otimes v^\circ$.

The situation is more subtle on the diagonals of the diagram: $\mathbf{n}_i \otimes \mathbf{n}_i^\circ$ *can not be mapped to itself by a unitary and skew-symmetric matrix. Hence, if* $\epsilon = -1$, *the diagonal* $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i \circ}$ *should appear in pairs in* H *that can be interchanged by a* K *of the above form. We describe this diagrammatically by doubling the node at a given diagonal position. If* $\epsilon = 1$ *then we simply have* $K : \mathbf{n}_i \otimes \mathbf{n}_i^\circ \to \mathbf{n}_i \otimes \mathbf{n}_i^\circ$ *given by transposition (i.e. the flip).*

The finite Dirac operator: Corresponding to the above decomposition of H we can write D as a sum of operators

$$D_{ij,kl} : \mathbf{n}_i \otimes \mathbf{n}_j^\circ \to \mathbf{n}_k \otimes \mathbf{n}_l^\circ,$$

restricted to these subspaces. The condition $D^* = D$ implies that $D_{kl,ij} = D_{ij,kl}^*$. In terms of the above diagrammatic representation of H, we express a non-zero $D_{ij,kl}$ as a single line between the nodes $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ and $(\mathbf{n}_k, \mathbf{n}_l^\circ)$. In such a case, we have a non-zero $D_{kl,ij}$ as well but avoid drawing double lines between all pairs of nodes.

Lemma 2.20. *The condition* $JD = \pm DJ$ *and the order-one condition* $[[D, a], b^\circ] = 0$ *forces the lines in the diagram to run only vertically or horizontally (or between the same node in the case of degeneracies) thereby maintaining the diagonal symmetry in the diagram.*

Proof. The condition $JD = \pm JD$ easily translates into a commuting diagram:

$$\begin{array}{ccc} \mathbf{n}_i \otimes \mathbf{n}_j^\circ & \xrightarrow[D]{} & \mathbf{n}_k \otimes \mathbf{n}_l^\circ \\ {\scriptstyle J}\downarrow & & {\scriptstyle J}\downarrow \\ \mathbf{n}_j \otimes \mathbf{n}_i^\circ & \xrightarrow[D]{} & \mathbf{n}_l \otimes \mathbf{n}_k^\circ \end{array}$$

thus relating $D_{ij,kl}$ to $D_{ji,lk}$: the diagonal symmetry is maintained.

If we write the order-one condition $[[D, a], b^\circ]$ in terms of $a = a_1 \oplus \cdots \oplus a_N$ and $b^\circ = b_1 \oplus \cdots \oplus b_N$ we compute that

$$D_{ij,kl}(a_i - a_k)(b_j^\circ - b_l^\circ) = 0$$

for all $a_i \in M_{n_i}(\mathbb{C})$ and $b_j^\circ \in M_{n_j}(\mathbb{C})^\circ$. As a consequence, $D_{ij,kl} = 0$ whenever $i \neq k$ or $j \neq l$. □

Grading: If, finally, there is a grading $\gamma : H \to H$ then each node in the diagram gets dressed by a plus or minus sign. The rules are that

- D connects nodes with different signs,
- If the node $\mathbf{n}_i \otimes \mathbf{n}_j^\circ$ has sign $\pm$, then the node $\mathbf{n}_j \otimes \mathbf{n}_i^\circ$ has sign $\pm\epsilon''$, according to $J\gamma = \epsilon''\gamma J$.

Remark 2.21. *The fact that the grading is assigned to each node, even if there are degeneracies is a result of the orientation axiom. It says that the grading* γ *can be implemented by elements* $x_i, y_i \in A$ *by* $\gamma \equiv \sum_i x_i y_i^\circ$. *Hence, this is completely dictated by the operator* J *and the representation of* A.

Finally, this leads to Krajewski's diagrammatic classification of real finite spectral triples, extended to any KO-dimension.

For A a finite dimensional algebra, let $\Lambda \times \Lambda^\circ$ be the lattice embedded (symmetrically) in $\mathbb{R}^2$ of all irreducible representations of $A \otimes A^\circ$ (without repetition). A symmetrical embedding means that all points $(\mathbf{n}, \mathbf{n}^\circ)$ $(\mathbf{n} \in \Lambda)$ lie on the diagonal of $\mathbb{R}^2$ and the flip in $\Lambda \times \Lambda^\circ$ corresponds to reflection with respect to the diagonal in $\mathbb{R}^2$.

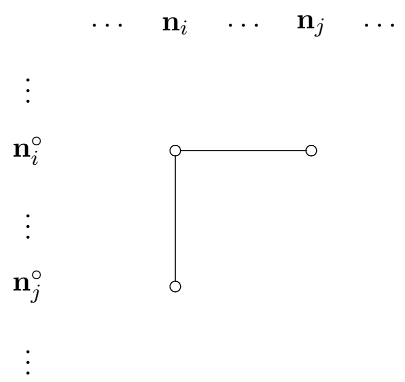


FIGURE 3. The lines between two nodes represent a non-zero $D_{ii,ji} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i^\circ} \to \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ}$, as well as its adjoint $D_{ji,ii} : \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ} \to \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i^\circ}$. The non-zero components $D_{ii,ij}$ and $D_{ij,ii}$ are related to $\pm D_{ii,ji}$ and $\pm D_{ji,ii}$, respectively, according to $JD = \mp DJ$.

DEFINITION 2.22. *A* Krajewski diagram Γ *of KO-dimension n on a finite dimensional algebra A is a directed graph together with an embedding $\iota : \Gamma \to \mathbb{R}^2$ for which the embedded vertices form a subset of the lattice $\Lambda \times \Lambda^\circ \subset \mathbb{R}^2$ such that*

(1) *the embedded graph $\iota(\Gamma)$ is symmetric along the diagonal in $\mathbb{R}^2$, and every row or column in $\Lambda \times \Lambda^\circ$ has non-empty intersection with $\iota(\Gamma)$.*
(2) *the embedded edges connect only horizontally or vertically, or have source and target the same point in $\Lambda \times \Lambda^\circ$.*
(3) *the edges are dressed by arbitrary non-zero complex matrices between the source and target representation spaces of $A \otimes A^\circ$,*
(4) *if n is even, the (embedded) vertices are dressed by ± 1 and the edges only connect opposite signs. Moreover, if n is 0 or 4 then $\pm$-signs at the vertices are symmetric with respect to reflection along the diagonal; if n is 2 or 4 they are anti-symmetric,*
(5) *if n is $2, 3, 4$ or 5 then the inverse image under ι of the diagonal element in $\Lambda \times \Lambda^\circ$ contains an even and positive number of vertices of Γ.*

Note that this definition allows for different vertices of Γ to be mapped to the same point on the lattice $\Lambda \times \Lambda^\circ$; similarly for the edges of Γ.

THEOREM 2.23. *There is a one-to-one correspondence between real finite spectral triples of KO-dimension n and Krajewski diagrams of KO-dimension n. One associates to a Krajewski diagram a spectral triple $(A, H, D; J, \gamma)$ in the following way:*

- *to each vertex v of Γ with $\iota(v) = (\mathbf{n}_i, \mathbf{n}_j^\circ) \in \Lambda \times \Lambda^\circ$ one associates a direct summand $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$ in H, thus furnishing a (faithful) representation of A on H,*
- *to each dressed edge between $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ and $(\mathbf{n}_k, \mathbf{n}_l^\circ)$, one associates a non-zero linear operator $D_{ij,kl} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \to \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l \circ}$ given by the matrix on that edge; the operator D is the direct sum of all these operators and their adjoints,*
- *the grading γ on H is defined by setting γ to be ± 1 on $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \subset H$ according to the dressing of the vertex v for which $\iota(v) = (\mathbf{n}_i, \mathbf{n}_j^\circ) \in \Lambda \times \Lambda^\circ$,*[1]

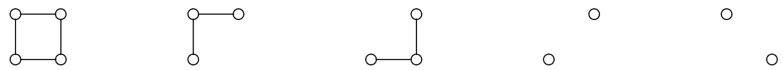
[1]That the grading is the same for all vertices in the inverse image of a given node in $\Lambda \times \Lambda^\circ$ corresponds to orientability, cf. Remark 2.21.

- *the real structure J is given by $J\psi = K\overline{\psi}$ where K is defined on the direct summand $\mathbf{n}_i \otimes \mathbf{n}_j^\circ \oplus \mathbf{n}_j \otimes \mathbf{n}_i^\circ$ ($\mathbf{n}_i \neq \mathbf{n}_j$) by*

$$K \sim \begin{pmatrix} 0 & \tau \\ \epsilon\tau & 0 \end{pmatrix} \tag{2.2.2}$$

where $\tau : \mathbf{n}_i \otimes \mathbf{n}_j^\circ \to \mathbf{n}_j \otimes \mathbf{n}_i^\circ$ is the flip $\tau(\psi \otimes \phi^\circ) = \phi \otimes \psi^\circ$. If $\mathbf{n}_i = \mathbf{n}_j$ then in KO-dimensions $0, 1, 6$ and 7 we set $K \equiv \tau$ on $\mathbf{n}_i \otimes \mathbf{n}_i^\circ$; in KO-dimension $2, 3, 4$ and 5 we set K to be of the above form (2.2.2) on the pair $\mathbf{n}_i \oplus \mathbf{n}_i^\circ \oplus \mathbf{n}_i \oplus \mathbf{n}_i^\circ$.

EXAMPLE 2.24. *Consider the case that $A = \mathbb{C} \oplus \mathbb{C}$. There are five possible Krajewski diagrams in KO-dimension 0; in terms of $\Lambda = \{\mathbf{1_1}, \mathbf{1}_2\}$:*



understanding a dressing of the diagonal vertices with a plus-sign, and the off-diagonal vertices with a minus-sign.

EXERCISE 2.7. *Use the five diagrams of the previous example to show that on $A = \mathbb{C} \oplus \mathbb{C}$ a finite spectral triple of KO-dimension 6 must have vanishing finite Dirac operator.*

2.3. Spectral triples

DEFINITION 2.25. *A* **spectral triple** *$(\mathcal{A}, \mathcal{H}, D)$ is given by an unital $*$- algebra $\mathcal{A}$ represented as operators in a Hilbert space $\mathcal{H}$ and a self-adjoint operator D such that $(1+D^2)^{-1}$ is a compact operator and $[D, a]$ bounded for $a \in \mathcal{A}$.*

A spectral triple is **even** *if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading γ such that $[\gamma, a] = 0$ and $\{\gamma, D\} = 0$.*

A **real structure** *of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$ on a spectral triple is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that*

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon'' \gamma J \quad \text{(even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n \mod 8$:

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

Moreover, with $b^0 = Jb^*J^{-1}$ we impose that

$$[a, b^0] = 0, \qquad [[D, a], b^0] = 0, \qquad \forall\, a, b \in \mathcal{A},$$

A spectral triple with a real structure is called a **real spectral triple**.

The basic example is the commutative spin geometry of a Riemannian spin manifold given by the triple

- $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions on M.
- $\mathcal{H} = L^2(M, S)$, the Hilbert space of square integrable sections of a spinor bundle $S \to M$.
- D, the Dirac operator associated with the Levi-Civita connection.

If the manifold is even dimensional, there is a grading defined by $\Gamma := -\gamma^1\gamma^2 \cdots \gamma^{\dim M}$, where γ^μ are the Dirac gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The real structure is given by the charge conjugation.

A spectral triple is called **regular** (or **smooth**) if the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ lies within the smooth domain $\bigcap_{n=0}^\infty \operatorname{Dom} \delta^n$ of the operator derivation $\delta(T) := |D|T - T|D|$.

This condition permits to introduce the analogue of Sobolev spaces $\mathcal{H}^s := \mathrm{Dom}(1+D^2)^{s/2}$ for $s \in \mathbb{R}$. One can develop this theory to an abstract differential calculus, cf. the notes by Higson.

DEFINITION 2.26. *The* **dimension spectrum** *of a regular spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is the subset* $\Sigma \subset \mathbb{C}$ *of singularities of the meromorphic functions*

$$\zeta_b(z) = \mathrm{Tr}(b|D|^{-z})$$

where b *is an element in the algebra generated by* $\delta^k(\mathcal{A})$ *and* $\delta^k([D, \mathcal{A}])$ *for all* $k \geq 0$.

Corresponding to the direct product of manifolds, one can take the product of spectral triples as follows. Suppose $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ are even real spectral triples, then we define the (exterior) **product spectral triple** by

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 \\
\mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \\
D &= D_1 \otimes 1 + \gamma_1 \otimes D_2 \\
\gamma &= \gamma_1 \otimes \gamma_2 \\
J &= J_1 \otimes J_2
\end{aligned}$$

Note that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$ since the cross-terms vanish due to $\gamma_1 D_1 = -D_1 \gamma_1$. The dimension spectrum Σ of the product is the sum (in $\mathbb{C}$) of $\Sigma_1 + \Sigma_2$.

2.3.1. Noncommutative differential forms. Let $\mathcal{A}$ be an algebra with unit over $\mathbb{C}$. The universal differential algebra $\Omega_{\mathrm{un}}(\mathcal{A})$ is the graded algebra generated by $a \in \mathcal{A}$ of degree 0 and symbols δa, $a \in \mathcal{A}$ of degree 1, such that

$$\delta(ab) = (\delta a)b + a\delta b \quad \delta(\alpha a + \beta b) = \alpha\delta a + \beta\delta b; \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}).$$

We can write $\Omega_{\mathrm{un}}(\mathcal{A})$ as a direct sum of subspaces $\Omega_{\mathrm{un}}^p(\mathcal{A})$ generated by linear combinations of $a_0\delta a_1 \cdots \delta a_p$. Furthermore, there is the isomorphism of vector spaces

$$\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes p} \simeq \Omega_{\mathrm{un}}^p(\mathcal{A}), \tag{2.3.1}$$

where $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{C}\mathrm{I}$. The operator δ is defined on $\Omega_{\mathrm{un}}(\mathcal{A})$ by

$$\begin{aligned}
\delta(a_0\delta a_1 \cdots \delta a_p) &= \delta a_0 \delta a_1 \cdots \delta a_p, \\
\delta(\delta a_1 \cdots \delta a_p) &= 0.
\end{aligned}$$

By construction, the algebra $\Omega_{\mathrm{un}}(\mathcal{A})$ is also a $\mathcal{A}$-bimodule. As the name suggests, the universal differential algebra satisfies the following universal property.

PROPOSITION 2.27. *Let* (Ω, d) *be a graded differential algebra and let* ρ *be a morphism of unital algebras. Then, there exists a unique extension of* ρ *to a morphism of graded differential algebras* $\widetilde{\rho} : \Omega_{\mathrm{un}}(\mathcal{A}) \to \Omega$ *such that* $\widetilde{\rho} \circ \delta = d \circ \widetilde{\rho}$.

An example of a frequently used differential calculus in the text and more generally, in noncommutative geometry, is Connes' differential calculus [**2**]. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The $\mathcal{A}$-bimodule $\Omega_D^p(\mathcal{A})$ of Connes' differential p-forms is made of classes of operators of the form

$$\omega = \sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j], \quad a_i^j \in \mathcal{A},$$

modulo the sub-bimodule of operators

$$\Big\{ \sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{p-1}^j] : \ b_i^j \in \mathcal{A}, \ b_0^j [D, b_1^j] \cdots [D, b_{p-1}^j] = 0 \Big\}.$$

The exterior differential d_D is given by

$$d_D \Big[\sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j] \Big] = \sum_j [D, a_0^j][D, a_1^j] \cdots [D, a_p^j].$$

In the case of the canonical triple $(C^\infty(M), \mathcal{H}, D)$ of a Riemannian spin manifold M, this differential calculus is isomorphic to the de Rham differential calculus.

2.3.2. Modules and connections. We recall some basic definitions on modules and connections thereon. We derive a general Bianchi identity for the curvature of such connections and link with gauge theory.

2.3.2.1. *Modules.* Let $\mathcal{A}$ be an algebra over the complex numbers $\mathbb{C}$.

Definition 2.28. *A* right module $\mathcal{E}$ *is a vector space over* $\mathbb{C}$ *that carries a right representation of* $\mathcal{A}$*, i.e. there is a map* $\mathcal{E} \times \mathcal{A} \ni (\eta, a) \to \eta a$ *such that*

$$\begin{aligned} \eta(ab) &= (\eta a)b, \\ \eta(a+b) &= \eta a + \eta b, \\ (\eta + \xi)a &= \eta a + \xi a, \end{aligned}$$

for any $\eta, \xi \in \mathcal{E}$ *and* $a, b \in \mathcal{A}$*.*

There is the natural notion of a morphism of (right) $\mathcal{A}$-modules as linear maps that respect this structure. Thus, a morphism between two (right) $\mathcal{A}$-modules $\mathcal{E}$ and F is a linear map $\rho : \mathcal{E} \to \mathcal{F}$ that is also right $\mathcal{A}$-linear:

$$\rho(\eta a) = \rho(\eta)a; \qquad \forall \eta \in \mathcal{E}, a \in \mathcal{A}.$$

Left modules and morphisms of left modules are defined similarly. A *bimodule* over an algebra $\mathcal{A}$ is both a left and a right $\mathcal{A}$-module such that the left and right action of $\mathcal{A}$ commute:

$$(a\eta)b = a(\eta b); \qquad \forall \eta \in \mathcal{E}, a, b \in \mathcal{A}.$$

Given a right $\mathcal{A}$-module, we define its *dual module* $\mathcal{E}'$ as the collection of all morphisms from $\mathcal{E}$ into $\mathcal{A}$, where $\mathcal{A}$ is seen as the trivial right $\mathcal{A}$-module; in other words:

$$\mathcal{E}' := \left\{ \phi : \mathcal{E} \to \mathcal{A} \mid \phi(\eta a) = \phi(\eta)a, \ \eta \in \mathcal{E}, a \in \mathcal{A} \right\} .$$

Definition 2.29. *A right* $\mathcal{A}$*-module* $\mathcal{E}$ *is said to be* finite projective *if there exists an idempotent* $p = p^2 \in M_N(\mathcal{A})$ *such that* $\mathcal{E} \simeq p\mathcal{A}^N$ *as right* $\mathcal{A}$*-modules.*

Here $M_N(\mathcal{A}) \simeq M_N(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}$ denotes the algebra of $N \times N$ matrices with entries in $\mathcal{A}$ whereas $\mathcal{A}^N := \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}$ which can be thought of as the set of N-dimensional vectors with entries in $\mathcal{A}$, and is clearly a right $\mathcal{A}$-module.

2.3.2.2. *Connections.* Let us suppose we have an algebra $\mathcal{A}$ with a differential calculus $(\Omega(\mathcal{A}) = \oplus_p \Omega^p(\mathcal{A}), \mathrm{d})$. We now review the notion of a (gauge) connection on a (finite projective) module $\mathcal{E}$ over $\mathcal{A}$ with respect to the given calculus; we take a right module structure.

A *connection* on the right $\mathcal{A}$-module $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) \mapsto \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}(\mathcal{A}) \ ,$$

defined for any $p \geq 0$, and satisfying the Leibniz rule

$$\nabla(\omega\rho) = (\nabla\omega)\rho + (-1)^p \omega \mathrm{d}\rho \ , \ \forall \ \omega \in \Omega^p(\mathcal{A}) \ , \ \rho \in \Omega(\mathcal{A}) \ .$$

A connection is completely determined by its restriction

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \ ,$$

which satisfies

$$\nabla(\eta a) = (\nabla\eta)a + \eta \otimes_{\mathcal{A}} \mathrm{d}a \ , \ \forall \ \eta \in \mathcal{E} \ , \ a \in \mathcal{A} \ ,$$

and which is extended by the Leibniz rule. It is the latter property that implies that the composition,

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^2(\mathcal{A}) \ ,$$

is $\Omega(\mathcal{A})$-linear. Indeed, for any $\omega \in \Omega^p(\mathcal{A}),\ \rho \in \Omega(\mathcal{A})$,

$$\begin{aligned}\nabla^2(\omega\rho) &= \nabla\left((\nabla\omega)\rho + (-1)^p\omega \mathrm{d}\rho\right)\\ &= (\nabla^2\omega)\rho + (-1)^{p+1}(\nabla\omega)\mathrm{d}\rho + (-1)^p(\nabla\omega)\mathrm{d}\rho + \omega\mathrm{d}^2\rho\\ &= (\nabla^2\omega)\rho\ .\end{aligned}$$

The restriction of ∇^2 to $\mathcal{E}$ is the *curvature*

$$F : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})\ , \tag{2.3.2}$$

of the connection. It is $\mathcal{A}$-linear, $F(\eta a) = F(\eta)a$ for any $\eta \in \mathcal{E}, a \in \mathcal{A}$, and satisfies

$$\nabla^2(\eta \otimes_{\mathcal{A}} \rho) = F(\eta)\rho\ ,\ \forall\ \eta \in \mathcal{E}\ ,\ \rho \in \Omega(\mathcal{A})\ .$$

Thus, $F \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}))$, the collection of (right) $\mathcal{A}$-linear endomorphisms of $\mathcal{E}$, taking values in the two-forms $\Omega^2\mathcal{A}$.

Connections always exist on a projective module. On the free module $\mathcal{E} = \mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A} \simeq \mathcal{A}^N$, a connection is given by the operator

$$\nabla_0 = 1 \otimes \mathrm{d} : \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p(\mathcal{A}) \to \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1}(\mathcal{A})\ .$$

With the canonical identification $\mathbb{C}^N \otimes_{\mathbb{C}} \otimes_{\mathcal{A}} = (\mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A}) \otimes_{\mathcal{A}} \Omega(\mathcal{A})\ \simeq\ (\Omega(\mathcal{A}))^N$, one thinks of ∇_0 as acting on $(\Omega(\mathcal{A}))^N$ as the operator $\nabla_0 = (\mathrm{d}, \mathrm{d}, \cdots, \mathrm{d})$ (N-times).
For a generic projective module $\mathcal{E}$ one has a canonical inclusion map, $\lambda : \mathcal{E} \to \mathcal{A}^N$, which identifies $\mathcal{E}$ as a direct summand of the free module $\mathcal{A}^N$ and a canonical idempotent $p : \mathcal{A}^N \to \mathcal{E}$ which allows to identify $\mathcal{E} = p\mathcal{A}^N$. Using these maps as well as their natural extension to $\mathcal{E}$-valued forms, on $\mathcal{E}$ a connection ∇_0 is given by the composition

$$\mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) \ \xrightarrow{\lambda}\ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p(\mathcal{A}) \ \xrightarrow{1\otimes\mathrm{d}}\ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1}(\mathcal{A}) \ \xrightarrow{p}\ \Omega^{p+1}(\mathcal{A})$$

(we have also used canonical identifications for the free module). This connection is called the *Levi-Civita* or *Grassmann connection* and is explicitly given by

$$\nabla_0 = p \circ (1 \otimes \mathrm{d}) \circ \lambda$$

although one simply indicates it by

$$\nabla_0 = p\mathrm{d}. \tag{2.3.3}$$

2.3.3. Unitary and Morita equivalence of spectral triples. In the previous chapter we have seen the prominent role that is played by symmetries in physics. We now consider symmetries of $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$, an even real spectral triple. The first candidate is unitary equivalence.

Definition 2.30. *Two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ are called* **unitary equivalent** *if $\mathcal{A}_1 \simeq \mathcal{A}_2$ and there exists a unitary intertwining operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a) \qquad (a \in \mathcal{A}_1)\\ UD_1U^* &= D_2\end{aligned}$$

If there exist grading operators γ_1, γ_2 then we also demand that $U\gamma_1U^* = \gamma_2$. If there exist real structures J_1, J_2 then we also demand that $UJ_1U^* = J_2$.

As a special case, we consider the **gauge group** $\mathcal{U}(\mathcal{A})$, defined as the unitary elements in the algebra $\mathcal{A}$ of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. It implements unitary transformations from the spectral triple to itself, transforming

$$D \mapsto D + u[D, u^*].$$

If the spectral triple is real, the unitary intertwiner is given by $U = uJu^*J^{-1}$ for $u \in \mathcal{U}(\mathcal{A})$, thus transforming

$$D \mapsto D + u[D, u^*] + \epsilon' Ju[D, u^*]J^{-1}.$$

Effectively, the unitary group acts as automorphisms on $\mathcal{A}$ by conjugation, $a \mapsto uau^*$. Such automorphisms are called **inner**, in contrast to the outer automorphisms which are defined as the quotient $\mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$. This is nicely summarized by

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1.$$

Note that if $\mathcal{A} = C^\infty(M)$ is commutative, there are no non-trivial inner automorphisms and $\mathrm{Out}(\mathcal{A}) = \mathrm{Diff}(M)$.

We have seen that a non-abelian gauge group appears naturally when $\mathcal{A}$ is noncommutative. Even more, noncommutative algebras allow for a more general – and more natural – notion of equivalence than automorphisms, namely, Morita equivalence. Let us see if we can lift Morita equivalence to the level of spectral triples.

Recall that given an algebra $\mathcal{A}$, a Morita equivalent algebra $\mathcal{B}$ is the algebra of endomorphisms of a finite projective (right) module $\mathcal{E}$ over $\mathcal{A}$,

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}).$$

Let $(\mathcal{A}, \mathcal{H}, D)$ be a given spectral triple and try to construct a spectral triple $(\mathcal{B}, \mathcal{H}', D')$. Naturally, $\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ carries an action of $\phi \in \mathcal{B}$:

$$\phi(\eta \otimes \psi) = \phi(\eta) \otimes \psi \qquad (\eta \in \mathcal{E}, \psi \in \mathcal{H}).$$

The naive choice of an operator D' by $D'(\eta \otimes \psi) = \eta \otimes D\psi$ will not do, because it does not respect the ideal defining the tensor product over $\mathcal{A}$, being generated by elements of the form

$$\eta a \otimes \psi - \eta \otimes a\psi; \qquad (\eta \in \mathcal{E}, a \in \mathcal{A}, \psi \in \mathcal{H}).$$

A better definition is

$$D'(\eta \otimes \psi) = \eta \otimes D\psi + \nabla(\eta)\psi.$$

where $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_D(\mathcal{A})$ is a connection associated to the differential $d : a \mapsto [D, a]$ $(a \in \mathcal{A})$.

Theorem 2.31. *If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and ∇ is a connection on a finite projective right $\mathcal{A}$-module $\mathcal{E}$, then $(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, \nabla \otimes 1 + 1 \otimes D)$ is a spectral triple.*

Analogously, we define for a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ a real spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$ by setting $\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ$. Here $\mathcal{E}^\circ$ is the **conjugate module** to $\mathcal{E}$:

$$\mathcal{E}^\circ = \{\overline{\xi} : \xi \in \mathcal{E}\}$$

with a left $\mathcal{A}$ action defined by $a\overline{\xi} = \overline{\xi a^*}$ for any $a \in \mathcal{A}$. Still $\phi \in \mathcal{B}$ acts on $\mathcal{H}'$ by

$$\phi(\eta \otimes \psi \otimes \overline{\xi}) = \phi(\eta) \otimes \psi \otimes \overline{\xi}$$

and

$$D'(\eta \otimes \psi \otimes \overline{\xi}) = (\nabla\eta)\psi \otimes \overline{\xi} + \eta \otimes D\psi \otimes \overline{\xi} + \eta \otimes \psi(\overline{\nabla\xi})$$
$$J'(\eta \otimes \psi \otimes \overline{\xi}) = \xi \otimes J\psi \otimes \overline{\eta}$$

Theorem 2.32. *If $(\mathcal{A}, \mathcal{H}, D, J)$ is a real spectral triple, then $(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ, \nabla \otimes 1 \otimes 1 + 1 \otimes D \otimes 1 + 1 \otimes 1 \otimes \overline{\nabla})$ is a real spectral triple.*

Finally, for even spectral triples one defines a grading $\gamma' = 1 \otimes \gamma \otimes 1$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ$.

We now focus on **Morita self-equivalences**, for which $\mathcal{B} = \mathcal{A}$ and consequently $\mathcal{E} = \mathcal{A}$. Let us look at connections

$$\nabla : \mathcal{A} \to \Omega^1_D(\mathcal{A}).$$

Clearly, by the Leibniz rule $\nabla = d + A$ where $A \equiv \nabla(1) = \sum_j a_j[D, b_j]$ is a generic element in $\Omega^1_D(\mathcal{A})$. Similarly, $\psi\overline{\nabla}\overline{a} = (\epsilon' J da J^{-1} + \epsilon' J A a J^{-1})\psi$. Since $\mathcal{H}' \simeq \mathcal{H}$ we have

$$D'(\psi) \equiv D'(1 \otimes \psi \otimes \overline{1}) = \nabla(1)\psi + \psi\overline{\nabla}(\overline{1}) + D\psi = D\psi + A\psi + \epsilon' J A J^{-1}\psi.$$

In other words, D is 'innerly perturbed' to $D_A := D + A + \epsilon' J A J^{-1}$ where $A^* = A \in \Omega^1_D(\mathcal{A})$ is called the **gauge field**. Another name used for these fields is **inner fluctuations** of the Dirac operator, since it is the algebra $\mathcal{A}$ that generates – through Morita self-equivalences – the fields A.

On the new spectral triple $(\mathcal{A}, \mathcal{H}, D_A)$ there is an action of the unitary group $\mathcal{U}(\mathcal{A})$ by unitary equivalences. Recall that $U = uJu^*J^{-1}$ so that

$$D_A \mapsto U D_A U^*$$

or, equivalently,

$$A \mapsto uAu^* + u[D, u^*]$$

which is the usual rule for a gauge transformation on a gauge field.

2.3.3.1. *Spectral action functional.* Having identified the gauge group canonically assoaciated to a spectral triple, and derived the gauge fields, we are ready to find action functionals of these fields that are invariant under the gauge group.

Definition 2.33. *Let f be a positive and even function from $\mathbb{R}$ to $\mathbb{R}$. The* **spectral action** *is defined by*

$$S_b[A] := \operatorname{Tr} f(D_A/\Lambda)$$

where Λ is a real cutoff parameter. The fermionic action is defined as the inner product

$$S_f[A, \psi] := (\psi, D_A\psi).$$

We will assume that f is given by a Laplace–Stieltjes transform:

$$f(x) = \int_{t>0} e^{-tx^2} d\mu(t).$$

with μ a measure on $\mathbb{R}$, and further that there exists the following **heat kernel expansion**:

$$\operatorname{Tr} e^{-tD^2} = \sum_\alpha t^\alpha c_\alpha$$

as $t \to 0$. Note that this is defined for the unperturbed operator D, but similar expression hold for any bounded perturbation such as D_A of D.

Lemma 2.34. *For $\alpha < 0$ we have*

$$\operatorname{res}_{z=-2\alpha} \zeta_1(z) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

with $\zeta_b(z) = \operatorname{Tr} b|D|^{-z}$ as before.

Proof. This follows from the Mellin transform:

$$|D|^{-z} = \frac{1}{\Gamma(z/2)} \int_0^\infty e^{-tD^2} t^{z/2-1}\ dt$$

or, after inserting the heat expansion:

$$\begin{aligned} \operatorname{Tr}|D|^{-z} &= \frac{1}{\Gamma(z/2)} \sum_\alpha \int_0^\infty c_\alpha t^{\alpha+z/2-1}\ dt \\ &= \frac{1}{\Gamma(z/2)} \sum_\alpha \int_0^1 c_\alpha t^{\alpha+z/2-1}\ dt + \text{ holomorphic} \\ &= \sum_\alpha \frac{c_\alpha}{\Gamma(z/2)(\alpha + z/2)}. \end{aligned}$$

Taking resisues at $z = -2\alpha$ on both sides gives the desired result. □

Using the Laplace–Stieltjes transform, we now derive an asymptotic expansion of the spectral in terms of the heat coefficients c_α.

Proposition 2.35. *Let Σ be the dimension spectrum of $(\mathcal{A},\mathcal{H},D)$. Then*

$$\operatorname{Tr} f(D/\Lambda) = \sum_{\beta\in\Sigma} f_\beta \Lambda^\beta \frac{\Gamma(\beta/2)}{2} c_{-\frac{1}{2}\beta} + f_0 c_0 + \mathcal{O}(\Lambda^{-1})$$

where $f_\beta := \int_0^\infty f(v)v^{\beta-1}dv$ and $f_0 := f(0)$.

Proof. Inserting the heat expansion in the Laplace–Stieltjes transform gives

$$\operatorname{Tr} f(D/\Lambda) = \int_{t>0} t^\alpha \Lambda^{-2\alpha} c_\alpha \; d\mu(t).$$

The terms with $\alpha > 0$ are of order Λ^{-1}; if $\alpha < 0$, then

$$t^\alpha = \frac{1}{\Gamma(\alpha)} \int_{v>0} e^{-tv} v^{-\alpha-1} \; dv.$$

Applying this to the above intergral gives

$$\begin{aligned}
\Lambda^{-2\alpha} c_\alpha \int_{t>0} t^\alpha \; d\mu(t) &= \Lambda^{-2\alpha} c_\alpha \int_{t>0}\int_{v>0} e^{-tv} v^{-\alpha-1} \; dv d\mu(t) \\
&= \Lambda^{-2\alpha} c_\alpha \int_{t>0}\int_{v>0} e^{-tv^2} v^{-2\alpha-1} \; dv d\mu(t) \\
&= \Lambda^{-2\alpha} c_\alpha \int_{v>0} f(v) v^{-2\alpha-1} \; dv \equiv \Lambda^{-2\alpha} c_\alpha f_{-2\alpha}
\end{aligned}$$

substituting $v \mapsto v^2$ in the going to the second line. Since $c_\alpha = 0$ unless $-2\alpha \in \Sigma$ we substitute $\beta = -2\alpha$ to obtain the claimed formula. □

Corollary 2.36. *For the perturbed operator D_A we have*

$$S_b[A] = \sum_{\beta\in\Sigma} f_\beta \Lambda^\beta \operatorname{res}_{z=\beta} \operatorname{Tr} |D_A|^{-z} + f_0 \operatorname{Tr} |D|^{-z}\big|_{z=0} \mathcal{O}(\Lambda^{-1})$$

CHAPTER 3

Almost commutative (AC) manifolds and gauge theories

3.1. Background: gauge theories in physics

In this section we will give a crash course to gauge theories, put into a historical context.

3.1.1. Dirac and the dawn of quantum electrodynamics. In 1928, Dirac asked the question whether there exists a differential operator D such that its square is equal to the Laplace (d'Alembert) operator:

$$D^2 = \sum_\mu \eta^{\mu\nu} \partial_\mu \partial_\nu$$

where $\eta = \mathrm{diag}(+1, -1, -1, -1)$ The motivation for this was to find a relativistic version of the Schrödinger equation:

$$\sum_{j=1}^{3} \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x_j} \right)^2 \psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t) \tag{3.1.1}$$

Indeed, the fact that the left-hand-side involves a second-order differential operator and the right-hand-side a first-order, breaks the special relastivistic symmetry between space and time.

Two solutions can be found to this question, based on replacing the non-relativistic relation $E = p^2/2m$ by Einstein's relations:

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{3.1.2}$$

keeping in mind the quantum mechanical identification $p_j = -i\hbar\partial/\partial x_j$.

3.1.1.1. *The Klein–Gordon equation.* The first solution is to square the right-hand-side involving the time-derivative, leading to the **Klein–Gordon equation**:

$$\left(\sum_{j=1}^{3} \left(-i\hbar \frac{\partial}{\partial x_j} \right)^2 + m^2c^4 \right) \psi(x,t) = \left(i\hbar \frac{\partial}{\partial t} \right)^2 \psi(x,t)$$

or, equivalently, $(\Box + m^2c^2/\hbar^2)\psi(x,t) = 0$ with $\Box$ the d'Alembert operator:

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \left(\frac{\partial}{\partial x_j} \right)^2.$$

More compactly, $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ after writing $x_0 = ct$. The Klein–Gordon equation describes the relativistic motion of a free scalar particle with mass m.

3.1.1.2. *The Dirac equation.* Another solution to the above problem is to try to take the square root at the left-hand-side of (3.1.1). Dirac postulated a first-order differential operator H by setting

$$H = \alpha_0 mc^2 + c \sum_{i=1}^{3} \alpha_i \left(-i\hbar \frac{\partial}{\partial x_j} \right)$$

and then demanding that $H^2 = \sum_i c^2(-i\hbar\partial/\partial x_i)^2 + m^2c^4$, according to Einstein's relation (3.1.2) for the energy. One computes:

$$H^2 = \alpha_0^2 m^2 c^4 + \sum_i (\alpha_0\alpha_i + \alpha_i\alpha_0) mc^3 (-i\hbar\partial_i) + \sum_i \alpha_i^2 c^2 (-i\hbar\partial_i)^2 + \sum_{i>j} (\alpha_i\alpha_j + \alpha_j\alpha_i) c^2 (-i\hbar\partial_i)(-i\hbar\partial_j)$$

which implies that the α_μ satisfy

$$\alpha_0^2 = 1 \qquad \alpha_0\alpha_i + \alpha_i\alpha_0 = 0$$
$$\alpha_i^2 = 1 \qquad \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad (i \neq j)$$

Clearly, these relations cannot be satisfied by ordinary numbers, and the smallest representation of this (Clifford) algebra such that H is a symmetric operator is four dimensional.

The **Dirac equation** is given by

$$H\psi = i\hbar\frac{\partial}{\partial t}\psi$$

Introducing the so-called **Dirac gamma matrices** $\gamma^\mu = (\alpha^0, \alpha^0\alpha^i)$ this is equivalent to

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi(t,x) = 0. \tag{3.1.3}$$

It describes the relativistic motion of a free electron, or, more generally, of a free fermion. The Dirac matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta^{\mu\nu}1_4$$

Of course, on a (pseudo)-Riemannian spin manifold one has the analogous Dirac equation written concisely as $(D-m)\psi = 0$ in terms of the **Dirac operator** $D = i\gamma \circ \nabla^S$ with ∇^S a spin connection, and a section ψ of a spinor bundle S. In this case, the square of the Dirac operator is not precisely the d'Alembertian, but we have

THEOREM 3.1 (Weitzenböck). *If $\Delta = g_{\mu\nu}\nabla^S_\mu\nabla^S_\nu$ is the Laplace–Beltrami operator, then*

$$D^2 = \Delta - \frac{1}{4}R$$

3.1.1.3. *Principal of extremal action and electrodynamics.* In physics, it is convenient to work with **action functionals** and obtain equation of motions – such as the Dirac equation – as extremas of this functional. Let us illustrate this in the equation of interest, that is, the Dirac equation. A fermionic action functional is given as the inner product on $\Gamma(M,S)$:

$$S_f[\psi] = (\psi, (D-m)\psi); \qquad (\psi \in \Gamma(M,S)).$$

In terms of the hermitian structure, we have $S_f[\psi] = \int_M \langle\psi, (D-m)\psi\rangle_x d\mu(x)$. Now, ψ extremizes the action S_f means that the directional derivative

$$S'_f[\psi][\chi] = \lim_{t\to 0}(S_f[\psi + t\chi] - S_f[\psi])/t,$$

vanishes for all $\chi \in \Gamma(M,S)$. One computes that this happens if and only if $(D-m)\psi = 0$. We conclude that the fermionic action S_f describes the physical system of a relativistic particle moving in spacetime M. More generally, the vanishing of the directional derivative of an action functional gives the equation of motion for the corresponding physical system.

REMARK 3.2. *Note that the Klein–Gordon equation can be obtained as the equation of motion of the action functional*

$$S_{kg}[\phi] = \frac{1}{2}\int_M \eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{mc^2}{\hbar}\phi^2.$$

An interesting observation is that the above action S_f is invariant under the following **global** $U(1)$ symmetry:

$$\psi \to e^{i\theta}\psi; \qquad (\theta \in [0, 2\pi]).$$

Indeed, with $\langle\cdot,\cdot\rangle_x$ being anti-linear and linear in the first and second entry, respectively, we find that $S_f[e^{i\theta}\psi] = S_f[\psi]$.

Next, suppose that this symmetry transformation on ψ is position dependent, $\theta = \theta(x)$. Clearly, S_f is not invariant under this local $U(1)$-symetry unless we make the following **minimal replacement**:

$$\nabla^S \rightsquigarrow \nabla^S + ieA$$

Here we introduce a new field $A \in \Omega^1(M)$ that transforms under a $U(1)$-transformation as

$$A \mapsto A - e^{-1} d\theta$$

The fermionic action now depends on two fields A and ψ:

$$S_f[A, \psi] = \int_M \langle \psi, (i\gamma^\mu \nabla^S_\mu - e\gamma^\mu A_\mu - m)\psi \rangle_x d\mu(x)$$

Invariance of this functional under the $U(1)$-action follows:

$$S_f[A + ie^{-1}d\theta, e^{i\theta}\psi] = S_f[A, \psi] - \int_M \langle \psi, \gamma^\mu(\partial_\mu \theta)\psi\rangle_x d\mu(x) + \int_M \langle \psi, \gamma^\mu(\partial_\mu \theta)\psi\rangle_x d\mu(x)$$

The second term on the right-hand-side comes from the Leibniz rule for ∇^S on $e^{i\theta}\psi$, and the last term comes from the transformation of the A-field.

If we extremize the new action S'_f with respect to ψ we obtain the equation of motion

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\,\psi = 0$$

This describes the relativistic motion of an electron in the presence of an electromagnetic field A_μ. As such, A_μ satisfies Maxwell's equations; let us derive them here from the principle of extremal action.

The **curvature** of A is defined to be $F = dA \in \Omega^2(M)$. In local coordinates, we write:

$$F = \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

where $F_{\mu\nu} = \partial_\mu A_v - \partial_\nu A_\mu$. Maxwell's theory is described by the following action

$$S_{em}[A] := \int_M F \wedge (*F) \equiv \frac{1}{4}\int_M F_{\mu\nu}F^{\mu\nu}.$$

The so-called **Hodge star operator** $* : \Omega^r(M) \to \Omega^{n-r}(M)$ is given locally by

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(n-r)!} \epsilon^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_{n-r}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-r}}$$

with $\epsilon_{\mu_1 \cdots \mu_n} = \pm 1$ depending on whether $\mu_1 \cdots \mu_n$ is an even or odd permutation of $12 \cdots n$. The equation of motion for $S[A]$ is

$$d(*F) = 0$$

which together with the **Bianchi identity** $dF = 0$, which is always satisfied, forms **Maxwell's equation** for electromagnetism. More explicitly, we identify the components of $F_{\mu\nu}$ with the electric and magnetic field as

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

so that $d(*F) = 0 = dF$ become

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

3.1.2. Non-abelian gauge theories. The above $U(1)$-symmetry principle led Yang and Mills to consider theories with a non-abelian symmetry group. At first, there seemed to be no direct physical application of such theories. However, at the beginning of the 1960s, Glashow, later joined by Weinberg and Salam, used a $U(1) \times SU(2)$-symmetry as underlying their electroweak theory. The **Standard Model of elementary particles** was finally completed by adding a $SU(3)$ quark color symmetry. This model has been tested up to previously unencountered precision, the only missing piece being the Higgs particle.

Remark 3.3. *As said, the weak interactions correspond to a $SU(2)$ gauge group. Matter is supposed to be in a representation of this group. For example, the neutron and proton are supposed to be organized in the definining representation:*

$$g \cdot \begin{pmatrix} p \\ n \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix}$$

Historically, this occurence of $SU(2)$-symmetry was first motivated by the similarity in mass for the neutron and proton (940 and 938 $\mathrm{MeV}c^{-2}$, respectively). The slight difference in mass was later explained by interpreting the proton as a combination of two up and one down quark, and the neutron as one up and two down quarks. Then, the up and down quark are combined in a defining representation of $SU(2)$, which in addition both constitute a representation of $SU(3)$: the three colors of the quarks.

More generally, one considers matter as representations of a Lie group G, typically a matrix group such as $SU(N)$. One might consider N-vectors in the defining representation of $SU(N)$:

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$$

with each $\psi_i \in \Gamma(S)$ a spinor. The action functional that describes the dynamics of these N free massless particles is given by

$$S_f[\psi_1, \dots, \psi_N] = \int_M \langle \Psi, i\gamma^\mu \nabla^S_\mu \Psi \rangle_x d\mu(x) = \sum_{j=1}^N \int_M \langle \psi_j, i\gamma^\mu \nabla^S_\mu \psi_j \rangle_x d\mu(x)$$

Indeed, the extremal points of this action are precisely sections ψ_i that satisfy the Dirac equation. This action has a global $SU(N)$-invariance, since $U \in SU(N)$ acting as

$$\Psi \mapsto U\Psi.$$

leaves S_f invariant. Again, by promoting this to a **local** $SU(N)$-symmetry – *i.e.* $U = U(x)$, requires replacing

$$\nabla^S \rightsquigarrow \nabla^S + A$$

The **gauge field** A is an element in $\Omega^1(M) \otimes su(N)$ that transforms under a $SU(N)$-transformation as

$$U : A \mapsto UAU^* + UdU^* \tag{3.1.4}$$

The **curvature** of A is defined to be

$$F_A = dA + A \wedge A$$

and is an element in $\Omega^2(M) \otimes su(N)$. It transform as $F \mapsto UFU^*$ under a $SU(N)$-transformation.

Yang and Mills then introduce an action functional for such a field, now carrying their name. For $A \in \Omega^1(M) \otimes su(N)$ the **Yang–Mills action functional** is given by

$$S_{ym}[A] = -\int_M \mathrm{Tr}\, F_A \wedge *F_A$$

One checks that it is invariant under the action of $U(x) \in G$, as in (3.1.4)

3.1.3. Yang–Mills gauge theory: mathematical setup. As the previous example should indicate, the proper mathematical setting for gauge theories is vector bundles and connections thereon.

Definition 3.4. *Let $E \to M$ be a vector bundle. A* **connection** ∇^E *on E is a map*

$$\nabla^E : \Gamma E \to \Omega^1(M) \otimes_{C^\infty(M)} \Gamma E$$

such that the Leibniz rule is satisfied, i.e.

$$\nabla^E(f\eta) = f\nabla^E(\eta) + df \otimes_{C^\infty(M)} \eta; \qquad (f \in C^\infty(M), \eta \in \Gamma E).$$

The **curvature** F *of* ∇^E *is given by*

$$F = (\nabla^E)^2 : \Gamma E \to \Omega^2(M) \otimes_{C^\infty(M)} \Gamma E$$

In other words, $F \in \Omega^2(M) \otimes_{C^\infty(M)} \Gamma \operatorname{End}(E)$.

Locally, we can always write $\nabla^E = d + A$ with $A \in \Omega^1(M) \otimes_{C^\infty(M)} \Gamma \operatorname{End}(E)$. Similarly, $F \in \Omega^2(M) \otimes_{C^\infty(M)} \Gamma \operatorname{End}(E)$.

Suppose that there is a (smooth) action of a Lie group G on the fibers of $E \to M$. For instance, if E is an associated vector bundle to a G-principal bundle $P \to M$ it naturally comes with such an action. Indeed, one considers the associated fiber bundle

$$\operatorname{Ad} P = P \times_G G$$

so that the **gauge group** $\mathcal{G} := \Gamma \operatorname{Ad} P$ acts fiberwise on ΓE. The Lie algebra of $\mathcal{G}$ is the vector space of section $\Gamma \operatorname{ad} P$ where

$$\operatorname{ad} P = P \times_G \mathfrak{g}$$

In this case, it is also natural to assume that ∇^E comes from a connection on the principal bundle G, so that it is given by a $\mathfrak{g}$-valued one-form ω on P, satisfying:

$$\omega(X^*) = X; \qquad R_g^* \omega = \operatorname{Ad}_{g^{-1}} \omega \qquad (X \in \mathfrak{g}).$$

Of course, we can also write locally $\nabla^E = d + A$ with $A \in \Omega^1(M) \otimes \mathfrak{g}$. There is an action of the gauge group $\mathcal{G}$ on A:

$$A \mapsto uAu^* + udu^*; \qquad (u \in \mathcal{G})$$

identifying $\mathcal{G}$ locally with maps from M to G.

Next, we consider the **tensor product** of the bundle E with the spinor bundle $S \to M$. Essentially, one takes the fiberwise tensor product, and on $S \otimes E$ one can define the **tensor product connection**:

$$\nabla^{S\otimes E} = \nabla^S \otimes 1 + 1 \otimes \nabla^E$$

In local coordinates, one writes

$$\nabla^{S\otimes E} = d + \omega + A.$$

DEFINITION 3.5. *The* **Dirac operator with coefficients in** E *is given (locally) as*

$$D_E = i\gamma^\mu \nabla_\mu^{S\otimes E} = i\gamma^\mu \nabla_\mu^S + i\gamma^\mu A_\mu$$

The corresponding fermionic action is

$$S_f[A, \psi] = (\psi, (D_E - m)\psi)$$

with ψ *a section of* $S \otimes E$.

The dynamics of the field A is described by the Yang–Mills action functional, now in its full form.

DEFINITION 3.6. *The* **Yang–Mills action functional** *is defined for a connection* ∇^E *locally of the form* $d + A$ *with* $A \in \Omega^1(M) \otimes \mathfrak{g}$:

$$S_{ym}[A] = -\int_M \operatorname{Tr} F \wedge *F.$$

where $F := (\nabla^E)^2$ *is the curvature of* ∇^E *and* Tr *the Killing form on* $\mathfrak{g}$ *(and minus the identity on the abelian part of* $\mathfrak{g}$*).*

The Yang–Mills action functional transforms under the action of $\mathcal{G}$ as

$$F \mapsto uFu^*$$

Thus, $S_{ym}[A]$ is invariant under the $\Gamma \operatorname{Ad} P$-action.

The equations of motion of the above action reads

$$[\nabla^E, *F] = 0$$

This is called the **Yang–Mills equation**. Note its similarity with the Bianchi identity, which is simply $[\nabla^E, *F] = 0$ and is always satisfied. This is the starting point of **instantons**, *i.e.* connections with a selfdual curvature $F = *F$. For these connections, the Bianchi identity implies the Yang–Mills equation so that instantons are minima of the Yang–Mills action. It was realized later, through the work of Donaldson, that the moduli space of instantons plays a key role in the classification of smooth structures on four-dimensional manifolds.

3.1.3.1. *Higgs mechanism.* Although the above is intriguing from a mathematical viewpoint, nature is (as usual) slightly more complicated. In fact, the $U(1) \times SU(2)$-symmetry discussed above is not observed in nature, only a residual $U(1)$-symmery (namely, electrodynamics). Let us describe the mathematical structure underlying this symmetry breaking.

Suppose that $H \subset G$, and that Φ is a scalar vector, that is, a section of $P \times_G V$. Then, one considers the action functional:

$$S_h[\Phi, A] = \frac{1}{2} \int_M g^{\mu\nu} \nabla^E_\mu \Phi \cdot \nabla^E_\nu \Phi - V(\Phi).$$

Here $V(\Phi)$ is a potential: a polynomial in the components Φ_i. The minima of this potential are supposed to be only invariant under a subgroup H, rather than under the full group G. Physically, this means that when the field Φ 'rolls down' to such a minimum, the symmetry group G is **spontaneously broken** to H.

Let us illustrate this in an example of great physical interest, namely the Weinberg–Salam electroweak theory. In this case, $G = U(1) \times SU(2)$ which will be broken to $H = U(1)$ as follows. The field Φ has two components, Φ_1 and Φ_2 on which $(e^{i\theta}, U) \in U(1) \times SU(2)$ acts by matrix multiplication:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \mapsto e^{i\theta} U \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

The potential in the above action S_h is taken of the form

$$V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$$

where, conventionally, $\mu^2 < 0$. This potential has the form of a mexican hat, with minima at $|\Phi|^2 = -\mu^2/2\lambda =: v$. After choosing a basis of V, we can assume that a minimum is of the form

$$\Phi_0 = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

Clearly, this is not invariant any more under the $U(1) \times SU(2)$ action, however Φ_0 is invariant under the subgroup

$$H = \left\{ (e^{i\theta}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \right\} \simeq U(1)$$

This is the celebrated **Higgs spontaneous symmetry breaking mechanism**, with the Higgs field h appearing as quantum fluctuations $v \rightsquigarrow v + h$. Note that when we put $\Phi = \Phi_0$ in the action S_h, one generates in this way terms of the form $v^2 g^{\mu\nu} A_\mu A_\nu$, which are interpreted as mass terms for the gauge fields. More precisely, the physical gauge fields are the photon A, the Z-boson and the $W^\pm$-bosons. They are a linear combination (a rotation) of the gauge fields $(B, W) \in \Omega^1(M) \otimes u(1) \oplus su(2)$ that would arise from the previous discussion:

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B \\ W^3 \end{pmatrix}; \qquad W = \begin{pmatrix} W^3 & W^+ \\ W^- & -W^3 \end{pmatrix}$$

where θ_w is the so-called Weinberg angle.

3.2. Almost commutative manifolds

3.2.1. Gauge symmetries and inner automorphisms.

3.2.2. Gauge fields and Morita equivalence.

3.3. Spectral action on AC manifolds

3.3.1. Yang–Mills theories.

3.3.2. The Standard Model of elementary particles.

CHAPTER 4

In progress: Perturbative quantization of AC manifolds

4.1. Dimension 0: Matrix models

4.2. Dimension 4: Higher-derivative Yang–Mills theory

APPENDIX A

Basics of gauge theory

A.1. Preliminaries

We start by recalling some concepts from differential geometry that will be essential in the mathematical setup for gauge theories. For further reading, we refer to the standard text-books on differential geometry, for instance, [**5, 3**]

Let M be a smooth manifold of dimension m; by this we mean a Hausdorff topological space, such that each point has a neighborhood homeomorphic to $\mathbb{R}^m$. A *curve* through a point $x \in M$ is a smooth map $\gamma : (a,b) \to M$ $(a < 0 < b)$ such that $\gamma(0) = x$. Two curves γ_1 and γ_2 through x are called *equivalent* if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for some homeomorphism $\varphi : U \to \mathbb{R}^m$ on a neighborhood U of x. A *tangent vector* is an equivalence class of curves and will be denoted by $\gamma'(0)$ or $d/dt|_{t=0}\gamma(t)$. The set of all tangent vectors at a point $x \in M$ is denoted by T_xM. The *derivative along* X_x of a smooth function f on M is given by $X_x[f] = (f \circ \gamma)'(0)$ for $\gamma'(0) = X_x \in T_xM$. A *vector field* X is an assignment of an element $X_x \in T_xM$ to each point x of M such that the function $x \to X_x[f]$ is a smooth function on M; the latter function on M will be denoted by $X[f]$. Also, the set of vector fields on M will be denoted by $\Gamma(TM)$.

If $f : M \to N$ is a smooth map, we can push a vector field on M forward to a vector field on N as follows. For every $x \in M$, we define $f_{*x} : T_xM \to T_{f(x)}M$ as $f_{*x}(\gamma'(0)) = (f \circ \gamma)'(0)$. If $Y \in \Gamma(TM)$ and $\widetilde{Y} \in \Gamma(TN)$ are such that $f_{*x}(Y_x) = \widetilde{Y}_{f(x)}$, then we write $\widetilde{Y} = f_*Y$ and call $\widetilde{Y}$ the *push-forward* of Y by f.

The *Lie bracket* of two vector fields X, Y is given in terms of their action on smooth functions,

$$[X,Y][f] = X[Y[f]] - Y[X[f]].$$

A more geometric definition of the Lie bracket can be given as follows. Let φ_t be the one-parameter group of diffeomorphisms generated by the vector field X; in other words, $\varphi_t : M \to M$ is given by $\varphi_t(x) = \gamma_x(t)$, with $\gamma'_x(0) = X_x$. We then set in terms of the push-forward φ_{t*}^{-1},

$$[X,Y]_x = \frac{d}{dt}\varphi_{t*}^{-1}\left(Y_{\varphi_t(x)}\right)\big|_{t=0}.$$

The Lie bracket thus has the following interpretation: it measures the change of Y when transported along the vector field X.

Dual to tangent vectors in T_xM, there are *cotangent vectors* in T_x^*M. A *differential k-form* is an assignment of an element ω_x in the exterior algebra $\Lambda^k(T_x^*M)$ of T_x^*M to each point of x in M such that the following pairing with vector fields depends smoothly on $x \in X$:

$$\omega(X_1, \ldots, X_k)(x) = \omega_x(X_{1x}, \ldots, X_{kx}).$$

The set of all differential k-forms is denoted by $\Omega^k(M)$; it becomes a graded differential algebra when equipped with the following *wedge product* $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$,

$$(\alpha \wedge \beta)(X_1, \cdot, X_{k+l}) = \frac{1}{k!l!}\sum_\sigma (-1)^\sigma \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}),$$

and *exterior derivation* $d : \Omega^k(M) \to \Omega^{k+1}(M)$,

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \left[\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})\right] + \sum_{1 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}.$$

Dual to the push-forward of vector fields, there is the *pull-back* of k-forms; let $f : M \to N$ be a smooth map, then the pull-back $f^*\omega$ of $\omega \in \Omega^k(N)$ is the element in $\Omega^k(M)$ defined by,

$$(f^*\omega)_x(X_1, \ldots, X_k) = \omega_{f(x)}(f_{*x}X_1, \ldots, f_{*x}X_k)$$

The pull-back has the following properties:

$$df^*\omega = f^*d\omega; \qquad f^*(\omega \wedge \omega') = f^*\omega \wedge f^*\omega'; \qquad (f \circ g)^*\omega = g^*f^*\omega.$$

Suppose now that (M, g) is an oriented Riemannian manifold. Then there is an inner product on $\Lambda^k(T_x^*M)$ for each x, given by $g_x(\omega, \omega')$ which is such that $x \to g_x(\omega, \omega')$ is smooth. We introduce the *Hodge star operator* $* : \Omega^k(M) \to \Omega^{m-k}(M)$ as follows; for every $\omega' \in \Omega^k(M)$ we define $*\omega' \in \Omega^{m-k}(M)$ to be the unique element for which $g(\omega, \omega') = \omega \wedge *\omega'$. If the support of either ω or ω' is compact, we introduce the *inner product on k-forms* as

(A.1.1) $$\big(\omega, \omega'\big) = \int_M \omega \wedge *\omega'.$$

The *codifferential* is defined by $\delta = (-1)^k *^{-1} d* : \Omega^k(M) \to \Omega^{k-1}(M)$ and satisfies $(\omega, \delta\omega') = (d\omega, \omega')$ so that $\delta = d^*$.

Remark A.1. *We have the following local expressions for the Hodge star operator. If x^μ are local coordinates on a neighborhood U of x, we can write a k-form ω as*

$$\omega = \omega_{\mu_1 \cdots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k},$$

with summation over repeated indices understood and $\omega_{\mu_1 \cdots \mu_k}$ a smooth function on U. The above inner product g_x between two k-forms α and β is then given explicitly by

$$g_x(\alpha, \beta) = \frac{1}{k!} g^{\mu_1\nu_1}(x) \cdots g^{\mu_k\nu_k}(x) \alpha_{\mu_1 \cdots \mu_k}(x) \beta_{\nu_1 \cdots \nu_k}(x),$$

with $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$. The Hodge star operator is then given explicitly by

(A.1.2) $$*\omega = \sqrt{\det(g)} \frac{1}{k!} g^{\mu_1\nu_1} \cdots g^{\mu_k\nu_k} \omega_{\mu_1 \cdots \mu_k} \epsilon_{\nu_1 \cdots \nu_k \nu_{k+1} \nu_m} dx^{\nu_{k+1}} \wedge dx^{\nu_m}.$$

with ϵ the total anti-symmetric tensor with $\epsilon_{12 \cdots m} = 1$.

A.2. Principal bundles and connections

We start by introducing the central mathematical object in gauge theories, encoding the *gauge symmetries* in physical theories in terms of a topological datum. For more details, see for instance [**3**] and [**1**].

Definition A.2. *Let M be a manifold and G a Lie group. A* **principal fibre bundle** *over M with group G consists of a manifold P and an action R of G on P satisfying the following conditions*

1. *G acts freely on P on the right: $(p, g) \in P \times G \mapsto pg = R_g(p) \in P$.*
2. *M is the quotient space of P by the equivalence relation induced by G, $M = P/G$ and the canonical projection $\pi : P \to M$ is differentiable.*
3. *P is locally trivial: every point $x \in M$ has a neighborhood U such that there is a diffeomorphism $\psi : \pi^{-1}(U) \widetilde{\to} U \times G$ such that $\psi(p) = (\pi(p), \phi(p))$ where ϕ is a map from $\pi^{-1}(U)$ to G satisfying $\phi(pg) = \phi(p)g$ for all $p \in \pi^{-1}(U)$ and $g \in G$.*

We call P the *total space*, M the *base space* and G the *structure group*. Moreover, $\pi^{-1}(x)$ is called the *fiber* over x; it is a closed submanifold of P that is diffeomorphic to G.

EXAMPLE A.3 (Hopf fibration on S^4). *Given the description of S^4 as quaternionic projective space $P_1(\mathcal{H})$, we can define a map from $S^7 \subset \mathcal{H}^2$ to S^4 by $(p,q) \mapsto [p:q]$, with the quaternions p, q satisfying $p\overline{p}+q\overline{q}=1$. There is an action of quaternions of modulus 1 (i.e. of the symplectic group $\mathrm{Sp}_1 \simeq \mathcal{SU}(2)$) on S^7, such that $\pi : S^7 \to S^4$ is a principal bundle with group $\mathcal{SU}(2)$. It is also known as the Hopf fibration on S^4 and plays a central role in the description of instantons of charge 1 on the four-sphere.*

Next, we introduce the concept of connection on a principal bundle. We give two equivalent definitions, the first one being of geometrical nature, the second more analytical.

DEFINITION A.4. *A* **connection** *assigns to each $p \in P$ a subspace $H_p \subset T_pP$ such that for $V_p := \{X \in T_pP : \pi_*(X) = 0\}$ we have*

$$T_pP = H_p \oplus V_p$$

We require that $R_{g*}(H_p) = H_{pg}$ and that H_p depends smoothly on p (in the sense that there are $m = \dim M$ vector fields on a neighborhood U of p that span H_q for each $q \in U$.

We call V_p the *vertical subspace* of T_pP and H_p the *horizontal subspace.*

DEFINITION A.5. *A* **connection** *is a $\mathfrak{g}$-valued 1-form ω on P, with $\mathfrak{g} = \mathrm{Lie}(G)$ such that the following conditions hold.*

(a) *Let $A \in \mathfrak{g}$ and let A^* be the vector field on P defined by*

$$A^*_p = \frac{d}{dt} p \exp(tA)\big|_{t=0}; \qquad \textit{(fundamental vector field)}$$

Then $\omega(A^*_p) = A$.

(b) *$R^*_g\omega = \mathrm{ad}_{g^{-1}}(\omega)$ for all $g \in G$.*

The 1-form ω is called a connection one-form.

THEOREM A.6. *Definitions A.4 and A.5 are equivalent.*

PROOF. Suppose that ω is a connection one-form. We set $H_p := \{X \in T_pP : \omega_p(X) = 0\}$ and show that $p \to H_p$ is a connection in the sense of Definition A.4. Since $A \mapsto A^*$ defines a vector space isomorphism between $\mathfrak{g}$ and V_p (due to the freeness of the action of G on P). We see from *(a)* in Definition A.5 that $H_p \oplus V_p = T_pP$. Also $R_{g*}(H_p) = H_{pg}$, since $\omega(R_{g*}X) = (R^*_g\omega)(X) = \mathrm{ad}_{g^{-1}}(\omega(X)) = 0$ for $X \in H_p$.

Conversely, suppose that $p \mapsto H_p$ is a connection in the sense of Definition A.4. For A^*_p as in (a) and $X_p \in H_p$ we define

$$\begin{aligned} \omega_p &: T_pP \to \mathfrak{g} \\ A^*_p &+ X_p \mapsto A \end{aligned}$$

Then condition (a) of Definition A.5 holds, and for (b) we need to prove that $\omega_{pg}(R_{g*}X) = \mathrm{ad}_{g^{-1}}(\omega_p(X))$ for any $X \in T_pP$ and $g \in G$. If $X \in H_p$, then $R_{g*}X \in H_{pg}$ and both sides vanish. If $X = A^*_p$ for some $A \in \mathfrak{g}$, then

$$\begin{aligned}
\omega_{pg}(R_{g*}A^*_p) &= \omega_{pg}\left(\frac{d}{dt} p\,(\exp tA)\, g\big|_{t=0}\right) \\
&= \omega_{pg}\left(\frac{d}{dt} pgg^{-1}\,(\exp tA)\, g\big|_{t=0}\right) \\
&= \omega_{pg}\left(\frac{d}{dt} pg\left(\exp t\ \mathrm{ad}_{g^{-1}} A\right)\big|_{t=0}\right) \\
&= \omega_{pg}\left((\mathrm{ad}_{g^{-1}} A)^*_{pg}\right) = \mathrm{ad}_{g^{-1}}(A) = \mathrm{ad}_{g^{-1}}\left(\omega_p(A^*_p)\right).
\end{aligned}$$

□

Remark A.7. *Besides these two definitions, there is a third definition that is common especially in the physics literature. It gives a connections in terms of local expressions as follows. Let $\sigma_U : U \to P$ be a local section (such that $\pi \circ \sigma_U = \mathrm{id_U}$) we can obtain a local expression for ω by pulling it back to U: $A_U = \sigma_U^*\omega \in \Omega^1(U) \otimes \mathfrak{g}$. This is called the* **gauge potential** *in physics. In can be shown that this notion of a connection is equivalent to the above two in that the $A_U{}'s$ for a covering of M such that $\pi^{-1}(U) \simeq U \times G$ can be glued together to obtain a connection one-form ω, such that $\sigma_U^*\omega = A_U$.*

A connection on a principal G-bundle P allows to lift a vector field on the base space M to a horizontal vector field on the total space.

Proposition A.8. *Given a vector field X on M, there is a unique vector field $\widetilde{X}$ on P such that $\omega(\widetilde{X}) = 0$ and $\pi_*(\widetilde{X}_p) = X_{\pi(p)}$ for all $p \in P$. This* **horizontal lift** *is invariant by the action of G, i.e. $R_{g*}\widetilde{X} = \widetilde{X}$.*
Conversely, every horizontal vector field $\widetilde{X}$ on P that is invariant by G is the lift of a vector field X on M.

Proof. Existence and uniqueness of $\widetilde{X}$ follow from the fact that $\pi_* : H_p \to T_{\pi(p)}M$ isomorphically for each $p \in P$. Smoothness of $\widetilde{X}$ is proved as follows. Let U be a neighborhood of x such that $\pi^{-1}(U) \simeq U \times G$. Under this isomorphism, we obtain from X a vector field Y on $\pi^{-1}(U)$ such that $\pi_*Y = X$. Then $\widetilde{X}$ is the horizontal component of Y; this is a smooth vector field. Observe that $\pi_*(R_{g*}\widetilde{X}_p) = (\pi \circ R_g)_*(\widetilde{X}_p) = \pi_*(\widetilde{X}_p) = X_{\pi(p)}$ so that $R_{g*}\widetilde{X}_p = \widetilde{X}_{pg}$.
Finally, let $\widetilde{X}$ be a horizontal vector field on P invariant by G. For every $x \in M$, take a point $p \in P$ such that $\pi(p) = x$ and define $X_x = \pi(\widetilde{X}_p)$. The vector X_x is independent of the choice of p since $\pi_*(\widetilde{X}_{pg}) = \pi_*(R_{g*}\widetilde{X}_p) = \pi(\widetilde{X}_p)$. Clearly, $\widetilde{X}$ is the horizontal lift of X. □

Before we introduce the concept of curvature, we prove the following two useful lemmas. Recall the definition of fundamental vector fields in (a) of Definition A.5.

Lemma A.9. *If $A, B \in \mathfrak{g}$, then $[A, B]^* = [A^*, B^*]$ as fundamental vector fields.*

Proof. Set $\varphi_t : P \to P$ to be $\varphi_t(p) = p\exp(tA)$. Then, by the definition of the Lie bracket of vector fields:

$$\begin{aligned}
[A^*, B^*]_p &= \frac{d}{dt}\varphi_{t*}^{-1}\left(B^*_{\varphi_t(p)}\right) = \frac{d}{dt}\frac{d}{ds}\left(\varphi_t(p)\exp(sB)\exp(-tA)\right) \\
&= \frac{d}{dt}\frac{d}{ds}\left(p\exp(tA)\exp(sB)\exp(-tA)\right) \\
&= \frac{d}{dt}\frac{d}{ds}\left(p\exp(s\,\mathrm{ad}_{\exp(tA)}B)\right) \\
&= \frac{d}{ds}\left(p\exp(s\frac{d}{dt}\,\mathrm{ad}_{\exp(tA)}B)\right) \\
&= \frac{d}{ds}\left(p\exp(s[A, B])\right) = [A, B]^*_p,
\end{aligned}$$

with all derivatives evaluated at 0. □

Lemma A.10. *If $A \in \mathfrak{g}$ and X is a vector field on M, then $[A^*, \widetilde{X}] = 0$, where $\widetilde{X}$ is the horizontal lift of X.*

Proof. With φ_t as in the proof of the previous Lemma, we have $\varphi_{t*}(\widetilde{X}) = \widetilde{X}$ so that $[A^*, \widetilde{X}] = d/dt\ \varphi_{t*}^{-1}(\widetilde{X}) = 0$. □

Consider the set of all $\mathfrak{g}$-valued k-forms on P, denoted by $\Omega^k(P, \mathfrak{g}) = \Omega^k(P) \otimes \mathfrak{g}$. It is a *differential graded Lie algebra*, with Lie bracket given by

$$[\omega_1 \otimes A_1, \omega_2 \otimes A_2] = \omega_1 \wedge \omega_2 \otimes [A_1, A_2]$$

and extended linearly to the whole of $\Omega^k(P,\mathfrak{g})$. Indeed, with this definition it is not difficult to check the following three properties of a differential graded Lie algebra; for $\phi \in \Omega^k(P,\mathfrak{g})$, $\psi \in \Omega^l(P,\mathfrak{g})$ and $\rho \in \Omega^m(P,\mathfrak{g})$

(1) $[\psi,\phi] = (-1)^{kl+1}[\phi,\psi]$,
(2) $(-1)^{mk}[[\phi,\psi],\rho] + (-1)^{lm}[[\rho,\phi],\psi] + (-1)^{lk}[[\phi,\rho],\phi] = 0$,
(3) $d[\phi,\psi] = [d\phi,\psi] + (-1)^k[\phi,d\psi]$.

Let ω be a connection one-form on a principal bundle $\pi : P \to M$ with structure group G and write $X \in T_pP$ in terms of its vertical and horizontal part as $X = X^V + X^H$. If $\phi \in \Omega^k(P,\mathfrak{g})$, then we set $\phi^H \in \Omega^k(P,\mathfrak{g})$ by $\phi^H(X_1,\dots,X_k) = \phi(X_1^H,\dots,X_k^H)$ for $X_i \in TP$. The *exterior covariant derivative* of $\phi \in \Omega^k(P,\mathfrak{g})$ associated to ω is defined as $D_\omega\phi = (d\phi)^H$.

Definition A.11. *The* **curvature** *of the connection ω is $\Omega_\omega = D_\omega\omega \in \Omega^2(P,\mathfrak{g})$.*

A central result in the theory of connections on principal bundles is the following convenient expression for the curvature.

Theorem A.12 (Structure equation of Cartan). *Let ω be a connection and Ω_ω its curvature. Then $\Omega_\omega = d\omega + \frac{1}{2}[\omega,\omega]$.*

Proof. Note that $\frac{1}{2}[\omega,\omega](Y,Z) = [\omega(Y),\omega(Z)]$. Thus, we need to prove for all $Y,Z \in T_pP$, the equation

$$d\omega(Y^H,Z^H) = d\omega(Y,Z) + [\omega(Y),\omega(Z)] \tag{A.2.1}$$

By linearity, we need to consider only 3 cases:

(1) Both Y and Z are horizontal; in this case $\omega(Y) = \omega(Z) = 0$ and $Y^H = Y, Z^H = Z$.
(2) Both Y and Z are vertical; let $Y = A_p^*$ and $Z = B_p^*$ for some $A,B \in \mathfrak{g}$. Then $d\omega(Y,Z) = A^*[\omega(B^*)] - B^*[\omega(A^*)] - \omega([A^*,B^*])$ which, because $\omega(B^*) = B, \omega(A^*) = A$ are constant, reduces to

$$d\omega(Y,Z) = -\omega([A^*,B^*]) = -\omega([A,B]^*) = -[A,B] = -[\omega(A^*),\omega(B^*)] = -[\omega(Y),\omega(Z)]$$

where we applied Lemma A.9 in the second equality. We conclude that both sides of Eq. (A.2.1) vanish.
(3) Y vertical and Z horizontal; let $Z_p = \widetilde{X}_p$ for some $X \in T_{\pi(p)}M$ and $Y = A_p^*$ for some $A \in \mathfrak{g}$. Now,

$$d\omega(Y,Z) = A^*[\omega(\widetilde{X})] - \widetilde{X}[\omega(A^*)] - \omega([A^*,\widetilde{X}]) = 0$$

since $\widetilde{X}$ is horizontal, $\omega(A^*) = A$ is constant and $[A^*,\widetilde{X}] = 0$ by Lemma A.10.

□

Theorem A.13 (Bianchi identity). *If ω is a connection P with curvature Ω_ω, then $D_\omega\Omega_\omega = 0$.*

Proof. Using the structure equation, we can write $d\Omega_\omega = d^2\omega + \frac{1}{2}[d\omega,\omega] - \frac{1}{2}[\omega,d\omega]$. Since $d^2 = 0$ and ω vanishes on horizontal vector fields, $(d\Omega_\omega)^H$ vanishes. □

Proposition A.14. *For all $g \in G$, $R_g^*\Omega_\omega = \mathrm{ad}_{g^{-1}}\Omega_\omega$.*

Proof.

$$\begin{aligned} R_g^*\Omega_\omega &= R_g^*(d\omega + \frac{1}{2}[\omega,\omega] = d(R_g^*\omega) + \frac{1}{2}[R_g^*\omega, R_g^*\omega] \\ &= d(\mathrm{ad}_{g^{-1}}\omega) + \frac{1}{2}[\mathrm{ad}_{g^{-1}}(\omega),\mathrm{ad}_{g^{-1}}(\omega)] = \mathrm{ad}_{g^{-1}}\left(d\omega + \frac{1}{2}[\omega,\omega]\right) = \mathrm{ad}_{g^{-1}}(\Omega_\omega). \end{aligned}$$

□

A.3. Associated vector bundles and connections

Let $\pi : P \to M$ be a principal bundle with group G. Let V be a finite dimensional representation of G.

Definition A.15. *The* **associated vector bundle** $P \times_G V \to M$ *is defined as the space of equivalence classes in* $P \times V$ *under the relation*

$$(p, v) \sim (pg, g^{-1} \cdot v), \qquad (p \in P, v \in V, g \in G).$$

One indeed checks that this is a vector bundle over M with quotient map $\pi' : P \times_G V \to M$ induced by π as $\pi'(p, v) = \pi(p)$. Moreover, the fiber $\pi^{-1}(x)$ of this vector bundle is isomorphic to the vector space V. It is not difficult to see that the space of (smooth) section $\Gamma(P \times_G V)$ of the associated vector bundle can be described as the space of G-equivariant (smooth) maps from P to V, i.e. maps $\phi : P \to V$ such that,

$$\phi(pg) = g^{-1} \cdot \phi(g).$$

As a particular case of this construction, we have the adjoint bundle $\operatorname{ad} P = P \times_G \mathfrak{g}$ when we consider $\mathfrak{g}$ in its adjoint representation over G.

Definition A.16. *Let* $\overline{\Omega}^k(P, V)$ *be the space of* V*-valued differential* k*-forms* ϕ *on* P *such that the following conditions hold*

(a) $R_g^* \phi = g^{-1} \cdot \phi$ *for all* $g \in G$.
(b) *If one of the vector fields* $X_1, \ldots, X_k$ *is vertical, then* $\phi(X_1, \ldots, X_k) = 0$.

In particular $\overline{\Omega}^0(P, V) \simeq \Gamma(P \times_G V)$, as defined above. The above verticality condition (b) means that we are considering k-forms on the base space M, tensored with sections of the associated bundle (condition (a)).

Proposition A.17. *Two connection one-forms* ω, ω' *on* P *differ by an element in* $\overline{\Omega}^1(P, \mathfrak{g})$.

Proof. By (a) of Definition A.5, $(\omega - \omega')(X) = 0$ if $X = A^*$ is vertical. On the other hand, $R_g^*(\omega - \omega') = \operatorname{ad}_{g^{-1}}(\omega - \omega')$ by linearity of R_g^* and $\operatorname{ad}_{g^{-1}}$. □

In addition, it follows from Proposition A.14 and the fact that Ω_ω vanishes on vertical vector fields by definition, that also $\Omega_\omega \in \overline{\Omega}^2(P, \mathfrak{g})$.

Definition A.18. *For a connection* ω *on* P *we define the* **covariant derivative** *on* $P \times_G V$ *as the map*

$$\nabla_\omega : \overline{\Omega}^k(P, V) \to \overline{\Omega}^{k+1}(P, V),$$
$$\nabla_\omega(\phi) = (d\phi)^H.$$

Note that $\nabla_\omega(\phi)$ is indeed an element in $\overline{\Omega}^{k+1}(P, V)$, since

$$R_g^*(\nabla_\omega(\phi)) = R_g^*(d\phi)^H = (dR_g^*\phi)^H = \big(d(g^{-1} \cdot \phi)\big)^H = g^{-1} \cdot (d\phi)^H = g^{-1} \cdot \nabla_\omega(\phi).$$

The adjoint action of $\mathfrak{g}$ on V can be combined with the wedge product to give an action of $\overline{\Omega}^k(P, \mathfrak{g})$ on $\overline{\Omega}^k(P, V)$; this action will be denoted by $\dot\wedge$. Explicitly, if $\phi \in \overline{\Omega}^k(P, \mathfrak{g})$ and $\psi \in \overline{\Omega}^l(P, V)$ it is given as the following element $\phi \dot\wedge \psi \in \Omega^{k+l}(P, V)$,

$$\phi \dot\wedge \psi(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_\sigma (-1)^\sigma \phi\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right) \cdot \psi\left(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}\right).$$

Theorem A.19. *For* $\tau \in \overline{\Omega}^k(P, V)$, *we have* $\nabla_\omega \tau = d\tau + \omega \dot\wedge \tau$.

Proof. We must verify for $X_1, \ldots, X_{k+1} \in T_pP$,

$$d\tau\left(X_1^H, \ldots, X_{k+1}^H\right) = d\tau\left(X_1, \ldots, X_{k+1}\right) + \frac{1}{k!} \sum_\sigma (-1)^\sigma \omega(X_{\sigma(1)}) \cdot \tau\left(X_{\sigma(2)}, \ldots, X_{\sigma(k+1)}\right)$$

We will distinguish three cases:

(1) If $X_1, \ldots, X_{k+1}$ are all horizontal, then $\omega(X_i) = 0$ and $X_i^H = X_i$ and both sides coincide.
(2) If two or more of $X_1, \ldots X_{k+1}$ are vertical, then since τ vanishes on vertical vectors and $X_i^H = 0$ for some i, the above equation becomes $0 = d\tau(X_1, \ldots, X_{k+1})$. Now,

$$\begin{aligned} d\tau(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i \left[\tau\left(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}\right)\right] \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \tau\left([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}\right). \end{aligned}$$

Since $[X_i, X_j]$ is vertical if X_i and X_j are vertical (Lemma A.9 and since τ vanishes on vertical vectors, $d\tau(X_1, \ldots, X_{k+1}$ indeed vanishes.
(3) If precisely one of $X_1, \ldots, X_{k+1}$ is vertical, say X_1, then $[X_1, X_i] = 0$ by Lemma A.10. We thus need to check

$$X_1 \left[\tau(X_2, \ldots, X_{k+1})\right] + \omega(X_1) \cdot \tau(X_2, \ldots, X_{k+1}) = 0.$$

Let $X_1 = A^*$ and write $g_t = \exp tA$. Then, with all derivatives evaluated at 0,

$$\begin{aligned} X_{1p} \left[\tau(X_2, \ldots, X_{k+1})\right] &= \frac{d}{dt} \left[\tau(R_{g_t *} X_2, \ldots, R_{g_t *} X_{k+1})\right] \\ &= \frac{d}{dt} \left[g_t^{-1} \cdot \tau(X_2, \ldots, X_{k+1})\right] \\ &= -A \cdot \tau(X_2, \ldots, X_{k+1}) = -\omega(X_1) \cdot \tau(X_2, \ldots, X_{k+1}). \end{aligned}$$

□

COROLLARY A.20. *If* $\tau \in \overline{\Omega}^k(P, \mathfrak{g})$ *then* $\nabla_\omega \tau = d\tau + [\omega, \tau]$.

PROPOSITION A.21. *For* $\tau \in \overline{\Omega}^k(P, \mathfrak{g})$ *we have* $\nabla_\omega^2 \tau = F_\omega \dot{\wedge} \tau$*, where* $F_\omega = d\omega + \omega \wedge \omega \in \overline{\Omega}^2(P, \mathfrak{g})$ *is the* **curvature** *of* ∇_ω.

PROOF. Indeed,

$$\begin{aligned} \nabla_\omega(\nabla_\omega \tau) &= d(\nabla_\omega \tau) + \omega \dot{\wedge} (\nabla_\omega \tau) \\ &= d^2\tau + d(\omega \dot{\wedge} \tau) + \omega \dot{\wedge} d\tau + \omega \dot{\wedge} \omega \dot{\wedge} \tau \\ &= (d\omega + \omega \wedge \omega) \dot{\wedge} \tau. \end{aligned}$$

Also, $F_\omega \in \overline{\Omega}^2(P, \mathfrak{g})$, since

$$\begin{aligned} F_\omega(X_1, X_2) &= d\omega(X_1, X_2) + \omega(X_1)\omega(X_2) - \omega(X_2)\omega(X_1) \\ &= X_1[\omega(X_2)] - X_2[\omega(X_1)] + \omega([X_1, X_2]) = 0, \end{aligned}$$

if one of X_1, X_2 – and hence also $[X_1, X_2]$ by a combination of the Lemma's A.9 and A.10 – is vertical. □

PROPOSITION A.22 (Bianchi identity). $\nabla_\omega(F_\omega) = 0$.

PROOF.

$$\nabla_\omega(F_\omega) = dF_\omega + [\omega, F_\omega] = d^2\omega + d(\omega \wedge \omega) + [\omega, d\omega] + [\omega, \omega \wedge \omega] = 0$$

by the defining properties of a differential graded Lie algebra (see previous Section). □

REMARK A.23. *If* $\sigma : U \to P$ *is a local section of the principal bundle* P*, we have already obtained in Remark A.7 the gauge potential* A_U *as an element in* $\Omega^1(U) \otimes \mathfrak{g}$. *The* **field strength** F_U *is defined similarly as the pull-back* $\sigma^* F_\omega$ *of the curvature of* ω*; it is an element in* $\Omega^2(U) \otimes \mathfrak{g}$.

In the physics literature, one usually works with the components A_μ and $F_{\mu\nu}$ of A_U and F_U respectively. They are related via,

$$A_U = A_\mu dx^\mu,$$
$$F_U = F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

Another common notation is A^a_μ and $F^a_{\mu\nu}$, which can be obtained from A_μ and $F_{\mu\nu}$ after choosing a set of generators t_a for the Lie algebra $\mathfrak{g}$:

$$A_\mu = A^a_\mu t_a,$$
$$F_{\mu\nu} = F^a_{\mu\nu} t_a.$$

A.4. Gauge transformations

Let $\pi : P \to M$ be a principal bundle with group G.

Definition A.24. *A* **gauge transformation** *of P is a G-equivariant automorphism $f : P \to P$, i.e. such that, $\pi(f(p)) = \pi(p)$ and $f(pg) = f(p)g$.*

The *gauge group* is the group of all gauge transformations with the product given by composition, and will be denoted by $\mathcal{G}$.

Proposition A.25. *Let $\Gamma(P \times_G G)$ be the space of equivariant maps $f : P \to G$, with G acting on itself via the adjoint action $g' \mapsto gg'g^{-1}$, i.e. $f(pg) = g^{-1} f(p) g$. Then $\Gamma(P \times_G G)$ becomes a group with pointwise product and inverse which is isomorphic to the gauge group $\mathcal{G}$.*

Proof. If $f \in \mathcal{G}$, then $\widetilde{f} : P \to G$ defined by $f(p) = p\widetilde{f}(p)$ satisfies $\widetilde{f}(pg) = g^{-1}\widetilde{f}(p)g$. Conversely, if $\widetilde{f} : P \to G$, then $f(p) = p\widetilde{f}(p)$ is G-equivariant. □

In the following we will take $\Gamma(P \times_G P)$ as a working definition of the gauge group and also denote $P \times_G P = \mathrm{Ad}\, P$.

Remark A.26. *In physics, the gauge group consists of G-valued functions on spacetime. Again, this is related to the above notion of gauge group via the pull-back $\sigma^*_U(\mathrm{Ad}\, P)$ by a local section $\sigma : U \to P$ of the principal bundle ($U \subset M$ such that $\pi^{-1}(U) \simeq U \times G$). Indeed, one can identify $\sigma^*_U(\mathrm{Ad}\, P) \simeq U \times G$.*

A gauge transformation $f \in \mathcal{G}$ acts on $\Gamma(P \times_G V)$ by pointwise product on P in combination with the representation of G on V:

$$(f \cdot \phi)(p) = f(p) \cdot \phi(p), \tag{A.4.1}$$

Similarly, there is an action of $\mathcal{G}$ on $\overline{\Omega}^k(P, V)$ given by

$$f \cdot \phi(X_1, \ldots, X_k) = f(p) \cdot \phi(X_1, \ldots, X_k), \qquad (X_i \in T_pP).$$

The covariant derivative transforms under a gauge transformation $f \in \mathcal{G}$ in the following way

$$\nabla_\omega \mapsto \nabla^f_\omega := f \nabla_\omega f^{-1} \tag{A.4.2}$$

so that $f \cdot (\nabla_\omega \phi) = \nabla^f_\omega (f \cdot \phi)$. This implies for the curvature $F_\omega = \nabla_\omega$ that it transforms as,

$$F_\omega \mapsto F^f_\omega := f F_\omega f^{-1} \tag{A.4.3}$$

This implies for the connection one-form that it transforms as follows.

Proposition A.27. *We have $\nabla^f_\omega = \nabla_{f(\omega)}$ where $f(\omega) = f(df^{-1}) + f\omega f^{-1}$.*

Proof. This follows from the definition of ∇^f_ω together with an application of Leibniz' rule:

$$\nabla^f_\omega \phi = \left(f \nabla_\omega f^{-1}\right) \phi = (f df^{-1} + f\omega f^{-1})\phi = (f(df^{-1}) + d + f\omega f^{-1})\phi = (d + \omega^f)\phi.$$

□

We remark that the action of a gauge transformation on a connection one-form ω can be made even more intrinsic when taking a G-equivariant automorphism of P; the transformed connection one-form is the pull-back of ω by such an automorphism. We choose to work with $\Gamma(\mathrm{Ad}\, P)$ as a gauge group since it is closer to the physics definition of a gauge transformation and will prove more convenient in the description of the moduli space of instantons below.

We next construct the Lie algebra of infinitesimal gauge transformations. Geometrically, it is given by pulling back the tangent space along the fibers $T_F(\mathrm{Ad}\, P)$ by the canonical section of $\mathrm{Ad}\, P$ defined by the identity element. Concretely, it is given by the sections of the adjoint bundle $\mathrm{ad}\, P = P \times_G \mathfrak{g}$, with G acting on $\mathfrak{g}$ in the adjoint representation. There is a map $\mathrm{Exp} : \Gamma(\mathrm{ad}\, P) \to \Gamma(\mathrm{Ad}\, P)$ given by

$$\mathrm{Exp}(H)(p) = \exp(H(p)), \quad (p \in P).$$

Indeed, one has

$$\mathrm{Exp}(H)(pg) = \exp(H(pg)) = \exp\left(\mathrm{ad}_{g^{-1}} H(p)\right) = \mathrm{Ad}_{g^{-1}}\left(\exp(H(p))\right) = \mathrm{Ad}_{g^{-1}}(\mathrm{Exp}(H))(p),$$

as required. Moreover, $t \mapsto \mathrm{Exp}(tH)$ defines a one-parameter subgroup of $\mathcal{G}$. The Lie algebra structure on $\mathfrak{g}$ induces a Lie algebra structure on $\Gamma(\mathrm{ad}\, P)$ as follows,

$$[H, H'](p) = [H(p), H'(p)],$$

from which it is direct that $[H, H'] \in \Gamma(\mathrm{ad}\, P)$. Putting all this together, we arrive at the following definition.

DEFINITION A.28. *The Lie algebra* $\Gamma(\mathrm{ad}\, P)$ *is the* **gauge algebra of infinitesimal gauge transformations***.*

The gauge algebra acts on $\overline{\Omega}^k(P, V)$ in the following way,

$$(H \cdot \phi)(X_1, \dots, X_k) = H(p) \cdot \phi(X_1, \dots, X_k),$$

where $X_i \in T_p$, $H \in \Gamma(\mathrm{ad}\, P)$ and $\phi \in \overline{\Omega}^k(P, V)$. Equivalently, it can be obtained from the action of $\mathcal{G}$ on $\overline{\Omega}^k(P, V)$ via the one-parameter subgroup defined by H:

$$H \cdot \phi = \frac{d}{dt}\left(\mathrm{Exp}(tH) \cdot \phi\right)\big|_{t=0}.$$

The induced action of $\Gamma(\mathrm{ad}\, P)$ on the covariant derivative and its curvature can be obtained in like manner. If $f_t = \mathrm{Exp}(tH)$, then we set

$$\nabla_\omega^H := \frac{d}{dt} \nabla_\omega^{f_t}\big|_{t=0}, \qquad F_\omega^H := \frac{d}{dt} F_\omega^{f_t}\big|_{t=0}.$$

One easily computes that

$$\nabla_\omega^H = -[\nabla_\omega, H] \equiv -\nabla_\omega(H), \qquad F_\omega^H = -[F_\omega, H] \equiv -F_\omega \dot{\wedge} H. \tag{A.4.4}$$

A.5. Yang-Mills action

We have now introduced all concepts needed for the definition of the Yang-Mills action functional. Let ω be a connection on a principal bundle $P \to M$ with group G and let ∇_ω be the covariant derivative on an associated vector bundle $P \times_G V$. We have derived in Proposition A.21 that its curvature F_ω is an element in $\overline{\Omega}^2(P, \mathfrak{g})$. Moreover, we can introduce an inner product on $\overline{\Omega}^k(P, \mathfrak{g})$ by combining the inner product of forms defined by a metric on M in Eq. (A.1.1) with the trace on the Lie algebra (in the case of matrix groups, more generally one takes the Killing form). In other words, for $\phi, \psi \in \overline{\Omega}^k(P, \mathfrak{g})$ we set

$$\langle \phi, \psi \rangle = \int_M \mathrm{Tr}(\phi \wedge *\psi).$$

Positive definiteness of this inner product is a consequence of that property for $(\cdot, \cdot)$ and the trace on $\mathfrak{g}$.

Definition A.29. *The* **Yang-Mills action functional** *is given by*

$$\mathrm{YM}(\omega) = \langle F_\omega, F_\omega \rangle = \int_M \mathrm{Tr}\,(F_\omega \wedge *F_\omega)$$

Proposition A.30. *The Yang-Mills action is positive and invariant under the gauge group* $\mathcal{G}$*,*

$$\mathrm{YM}(f(\omega)) = \mathrm{YM}(\omega), \qquad (f \in \mathcal{G}).$$

Proof. Positivity follows from the positive-definiteness of the inner product $\langle \cdot, \cdot \rangle$. For gauge invariance recall that $F_\omega^f = f F_\omega f^{-1}$ so that $\mathrm{Tr}(F_\omega^f \wedge *F_\omega^f) = \mathrm{Tr}(F_\omega \wedge *F_\omega)$. □

The Yang-Mills equations are derived as follows. First, recall from Proposition A.17 that two connection one-forms differ by an element $\eta \in \overline{\Omega}^1(P, \mathfrak{g})$. Let us then consider the family of connections defined by $\omega_t = \omega + t\eta$. The curvature F_{ω_t} of ω_t is computed to be

$$F_{\omega_t} = F_\omega + \nabla_\omega(\eta) + t^2\eta^2. \tag{A.5.1}$$

Indeed, when acting on an element $\phi \in \overline{\Omega}^k(P, V)$,

$$\begin{aligned} F_{\omega_t} \dot{\wedge} \phi &= (d + \omega + t\eta)^2 \phi \\ &= ((d\omega) + \omega \wedge \omega)\, \dot{\wedge}\phi + t\,((d\eta) + \omega \wedge \eta + \eta \wedge \omega)\, \dot{\wedge}\phi + t^2(\eta \wedge \eta)\dot{\wedge}\phi \\ &= \left(F_\omega + t\nabla_\omega(\eta) + t^2 \eta \wedge \eta\right) \dot{\wedge}\phi. \end{aligned}$$

Let us now assume that $\mathrm{YM}(\omega)$ is at a local extremum, so that

$$\frac{d}{dt}\mathrm{YM}(\omega_t)\big|_{t=0} = 0$$

This implies that

$$\frac{d}{dt}\langle F_{\omega_t}, F_{\omega_t}\rangle\big|_{t=0} = \frac{d}{dt}\langle F_\omega + \nabla_\omega(\eta), F_\omega + \nabla_\omega(\eta)\rangle\big|_{t=0} = 2\langle \nabla_\omega(\eta), F_\omega\rangle$$

vanishes. In other words, $\langle \eta, \nabla_\omega^*(F_\omega))\rangle = 0$ for all $\eta \in \overline{\Omega}^1(P, \mathfrak{g})$. Since $\nabla_\omega^* = *\nabla_\omega *$, we have derived the *Yang-Mills equations*,

$$\nabla_\omega(*F_\omega) = 0. \tag{A.5.2}$$

which is satisfied by an extremum ω of YM.

Let us now restrict to a four-dimensional manifold M. The Hodge star operator then maps 2-forms to itself, and the following definition makes sense.

Definition A.31. *An* **instanton** *is a connection* ω *with a selfdual curvature* $*F_\omega = F_\omega$*.*

Similarly, an anti-instanton is defined as a connection with an anti-selfdual curvature $*F_\omega = -F_\omega$. It follows directly from the Bianchi identity $\nabla_\omega(F_\omega) = 0$ which is satisfied by all connections (Proposition A.22) that (anti-) instantons are solutions of the Yang-Mills equations. Moreover, they are absolute minima of the Yang-Mills action. In order to see this, let us introduce the following topological action.

Definition A.32. *The* **topological charge** *is given for a connection one-form* ω *by,*

$$\mathrm{Top}(\omega) = \int_M \mathrm{Tr}(F_\omega \wedge F_\omega).$$

The topological charge is topological in the sense that is independent of the choice of a connection one-form on P. Indeed, if we consider the above family ω_t of connection one-forms, then on inserting the above expression (A.5.1) for F_{ω_t},

$$\frac{d}{dt}\mathrm{Top}(\omega_t)\big|_{t=0} = \frac{d}{dt}\langle F_{\omega_t}, *F_{\omega_t}\rangle\big|_{t=0} = \langle \eta, *\nabla(F_\omega)\rangle$$

which vanishes by the Bianchi identity $\nabla_\omega(F_\omega) = 0$. Moreover, it is (proportional to) an integer,

$$\mathrm{Top}(\omega) = 8\pi^2 k, \qquad (k \in \mathbb{Z}).$$

Other names for the topological charge are *topological action*, the *Chern number*, *Pontryagin number*, depending on the context one is working with.

The Hodge star operator splits $\overline{\Omega}^2(P, \mathfrak{g})$ into two a selfdual and anti-selfdual space,

$$\Omega^2(P, \mathfrak{g}) = \Omega^2_+(P, \mathfrak{g}) \oplus \Omega^2_-(P, \mathfrak{g}),$$

and this decomposition is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$. We can thus write the Yang-Mills action functional and the topological action as follows:

$$\mathrm{YM}(\omega) = \langle F_+, F_+ \rangle + \langle F_-, F_- \rangle$$
$$\mathrm{Top}(\omega) = \langle F_+, F_+ \rangle - \langle F_-, F_- \rangle$$

with $F = F_\omega$. It is then immediate that the topological charge forms a lower bound for the Yang-Mills action, $\mathrm{YM}(\omega) \geq |\mathrm{Top}(\omega)|$. Equality is obtained precisely when $F = F_+$ or $F = F_-$, in other words, when ω is an instanton or an anti-instanton.

Remark A.33. *In the case that the structure group is abelian, $G = \mathcal{U}(1)$, the Yang-Mills equations become simply $d(*F_\omega) = 0$ where now $F_\omega = d\omega$. Together with the Bianchi identity $dF_\omega = 0$, these form Maxwell's equations of electromagnetism, which are linear equations in the curvature (field strenth). On the other hand, if the gauge group is non-abelian, the Yang-Mills equations (as well as the Bianchi identity) are non-linear and hence difficult to solve. However, the particular solutions given by the instantons are the solutions of a linear equation in the curvature, namely $*F = \pm F$. It is for this reason that instantons play such an important role in non-abelian gauge theories.*

Remark A.34. *For later convenience, we give the selfdual equations in terms of the field strength (cf. Remark A.23). If we write $F_U = F_{\mu\nu} dx^\mu \wedge dx^\nu$, the components of $*F$ are given by Equation* (A.1.2) *as*

$$*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\mu'\nu'} F^{\mu'\nu'},$$

assuming an Euclidean metric $g^{\mu\nu} = \delta^{\mu\nu}$ on this local chart.

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