Renormalization of quantum gauge theories using Hopf algebras

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CHAPTER 1

Introduction

The goal of these lectures is to understand the structure of perturbative quantum gauge theories in an algebraic manner. We intend to obtain rigorous results using Hopf algebras. These Hopf algebras are generalizations of the Connes–Kreimer Hopf algebras to gauge theories, involving more than one type of vertex and one type of edge. However, the co-structure on Feynman graphs is exactly the same, and the interpretation of renormalization to be captured by the coproduct still holds.

A question that we would like to address is the following:

Q1: What is the coproduct on sums of Feynman graphs, such as the (1PI) Green's functions?

This question is motivated by the existence of gauge symmetries, which would imply certain relations (Ward identities, Slavnov–Taylor identities) between Green's functions. Another interesting question is thus

Q2: What is the coproduct on the Ward identities?

We will see that the answer to the second question is that the Ward (and Slavov–Taylor) identities generate Hopf ideals. This implies that the quotient by this ideal is still a Hopf algebra, and in the meanwhile it provides an answer to the first question: in this quotient Hopf algebra, the Green's functions generate Hopf subalgebras (at each loop order). We have thus obtained a compatibility between renormalization and gauge symmetries on the combinatorial level, which is of completely algebraic nature.

The outline of these lectures is the following:

(1) Local/Lagrangian approach to gauge field theories, BRST, BV-algebra structures
(2) Hopf algebras H of Feynman graphs
(3) and its structure
(4) Applications:
    - Slavnov–Taylor identities in Yang–Mills theories
    - BCFW-recursion.

CHAPTER 2

Lagrangian approach to gauge field theories

1. Local functions and functionals

A *field* ϕ is a section of a vector bundle $E \to M$ on the background manifold M. If the rank of the vector bundle E is r, the field is said to have r components, in which case we can write locally $\phi = \phi^a e_a$ in terms of a basis e_a of E.

EXAMPLE 1. *If $E = M \times \mathbb{C}$, then a section ϕ is a complex scalar field $\phi : M \to \mathbb{C}$; it has one component.*

EXAMPLE 2. *Gauge fields are sections A of $E = \Lambda^1 \otimes (P \times_G \mathfrak{g})$ with P a G-principal bundle and $\mathfrak{g} = \mathrm{Lie}(G)$. In the case that P is trivial, this becomes a $\mathfrak{g}$-valued one-form on M, i.e. A is a section of $\Lambda^1(\mathfrak{g})$. In this case, the rank of the vector bundle is $\dim(M) \cdot \mathrm{rank}(\mathfrak{g})$ which leads to the familiar decomposition*

$$A = A^a_\mu dx^\mu T^a,$$

with $\{T^1, \ldots, T^{\mathrm{rank}(\mathfrak{g})}\}$ a basis for $\mathfrak{g}$ and summation is understood.

DEFINITION 3. *A* **local form** *is the tensor product of a differential form on M with a smooth function (polynomial) in the coordinates x^μ and $\partial_{\mu_1} \cdots \partial_{\mu_{n'}} \phi^a$ $(0 \leq n' \leq n)$ for some finite positive integer n. The* **algebra of local forms** *is denote by* $\mathrm{Loc}(E)$.

EXAMPLE 4. *A scalar field theory is defined by the following Lagrangian $L \in \mathrm{Loc}^{(m,0)}(M \times \mathbb{R})$:*

$$L(x, \phi, \partial_i \phi) = \tfrac{1}{2} d\phi * d\phi - V(\phi)(*1)$$

with $V(\phi)$ a polynomial in the field $\phi \in \Gamma(M \times \mathbb{R})$.

EXAMPLE 5. *Let $A \in \Omega^1(\mathfrak{g})$. The Yang–Mills Lagrangian is given by the local m-form:*

$$L_{\mathrm{ym}}(x, A) = -\mathrm{tr}\ F * F$$

where $F = dA + A^2$ is the curvature of A.

If the vector bundle E carries a grading, $E = \oplus_q E^{(q)}$ the algebra $\mathrm{Loc}(E)$ becomes bigraded, $L \in \mathrm{Loc}(E)$ of bidegree (p, q) if L has degree p as a differential form and degree q as a section. We also use the words *ghost degree* to specify the latter. In this case, we write

$$\mathrm{Loc}(E) = \bigoplus_{p \geq 0, q \in \mathbb{Z}} \mathrm{Loc}^{(p,q)}(E),$$

and we have $\mathrm{Loc}^{(p,q)}(E) \simeq \Omega^p(M) \otimes_{C^\infty(M)} \mathrm{Loc}^{(0,q)}(E)$.

Any Lagrangian L, defined in general as a local m-form of the fields ($m = \dim M$), can be integrated to give the so-called *action*

$$S[\phi] := \int_M L(x, \phi(x)).$$

In general, we make the following definition.

DEFINITION 6. *A* local functional *$F[\phi]$ is the integral of a local m-form, i.e. $F[\phi] = \int_M L(x, \phi(x))$ for $L \in \mathrm{Loc}^{(m,0)}(E)$. The free commutative algebra generated (over $\mathbb{C}$) by local functionals is denoted by $\mathcal{F}([E])$.*

The grading by ghost degree on local m-forms carries over to a grading on local functionals, which we also denote by $\mathrm{gh}(F)$ for $F \in \mathcal{F}([E])$.

2. Fields and BRST-sources

If we consider a set Φ consisting of $2N$ fields, we have specified $2N$ (graded) vector bundles each of which has a corresponding field as its section. As said, we will assume that the fields come in pairs of a field ϕ_i and an BRST-source K_{ϕ_i} $(i = 1, \dots, N)$ and we write E_i and $E_i^\vee$ for the corresponding vector bundles which are of equal rank. In fact, $E_i^\vee$ is the dual vector bundle of E_i, although shifted in degree as we make more precise now. The fields ϕ_i are understood to have a so-called *ghost degree* $\mathrm{gh}(\phi_i) \in \mathbb{Z}$ which is then extended to the BRST-sources by

$$\mathrm{gh}(K_{\phi_i}) := -\mathrm{gh}(\phi_i) - 1.$$

In the physics literature, this is usually called the (total) *ghost number*. Summarizing, the elements of Φ constitute a section of the total vector bundle E_{tot}:

$$(\phi_1, K_{\phi_1}, \dots, \phi_N, K_{\phi_N}) : M \to E_{\mathrm{tot}} = \bigoplus_{i=1}^N E_i \oplus E_i^\vee,$$

The grading on the fields turn E_{tot} into a graded vector bundle. We will write $\mathrm{Loc}(\Phi)$ instead of $\mathrm{Loc}(E_{\mathrm{tot}})$ (and similarly $\mathrm{Loc}^{(p,q)}(\Phi)$), and $\mathcal{F}([\Phi])$ instead of $\mathcal{F}([E_{\mathrm{tot}}])$.

EXAMPLE 7. *In Section 4 below, we will focus on pure Yang-Mills gauge theories. In that case, there is the gauge field A as in Example 2 which (in the trivial bundle case) is a section of* $\Lambda^1 \otimes M \times \mathfrak{g}$*, i.e. an element of* $\Omega^1(\mathfrak{g})$*. The so-called ghost fields* ω *and* $\overline{\omega}$ *are assigned to each generator of* $\mathfrak{g}$*, in components* $\omega = \omega^a T^a$ *and* $\overline{\omega} = \overline{\omega}^a T^a$*. Their ghost degrees are defined to be* 1 *and* -1*, respectively, so that* ω *is a section in* $\Omega^0(\mathfrak{g}[-1])$ *and* $\overline{\omega}$ *in* $\Omega^0(\mathfrak{g}[1])$*. Also, there is the so-called auxiliary – or Nakanishi–Lantrup – field* $h = h^a T^a$*, which is a section in* $\Omega^0(\mathfrak{g})$ *and of degree* 0*.*

Corresponding to these fields, there are the BRST-sources K_A*,* K_ω*,* $K_{\overline{\omega}}$ *and* K_h *which are of respective ghost degree* -1*,* -2*,* 0 *and* -1*. Thus, the field content of pure Yang-Mills gauge theories can be summarized by the following sections*

$$(A, \omega, \overline{\omega}, h) \in \Omega^1(\mathfrak{g}) \oplus \Omega^0(\mathfrak{g}[-1]) \oplus \Omega^0(\mathfrak{g}[1]) \oplus \Omega^0(\mathfrak{g}),$$
$$(K_A, K_\omega, K_{\overline{\omega}}, K_h) \in \mathfrak{X}(\mathfrak{g}^*[1]) \oplus \Omega^0(\mathfrak{g}^*[2]) \oplus \Omega^0(\mathfrak{g}^*) \oplus \Omega^0(\mathfrak{g}^*[1]),$$

where $\mathfrak{X}(\mathfrak{g}^*)$ *denotes* $\mathfrak{g}^*$*-valued vector fields. Taken all together, they form a section of the total bundle.*

3. The anti-bracket

We will now try to elucidate the above 'doubling' of the fields (adding a BRST-source for every field) in terms of the structure of a *Gerstenhaber algebra* on the algebra of local functionals $\mathcal{F}([\Phi])$. Recall that a Gerstenhaber algebra [**12**] is a graded commutative algebra with a Lie bracket of degree 1 satisfying the graded Leibniz property:

$$(x, yz) = (x, y)z + (-1)^{(|x|+1)|y|} y(x, z).$$

Batalin and Vilkovisky encountered this structure in their study of quantum gauge theories [**2, 3, 4**]. In fact, they invented what is now called a *BV-algebra* (see for instance [**16**]): a Gerstenhaber algebra with an additional operator $\widetilde{\Delta}$ that satisfies:

$$(x, y) = \widetilde{\Delta}(xy) - \widetilde{\Delta}(x)y + (-1)^{|x|} x\widetilde{\Delta}(y).$$

We will define such an *anti-bracket* on the algebra of local functionals using the functional derivative.

DEFINITION 8. *The left and right functional derivatives are the distributions defined by*

$$\frac{d}{dt} F[\phi + t\psi_\phi] = \int_M \frac{\delta_L F}{\delta \phi^a(x)} \psi_\phi^a(x) d\mu(x) = \int_M \psi_\phi^a(x) \frac{\delta_R F}{\delta \phi^a(x)} d\mu(x),$$

for test functions ψ_ϕ *of the same ghost degree as* $\phi \in \Phi$*.*

There is the following relation between the two functional derivatives:

$$\frac{\delta_R F}{\delta\phi^a(x)} = (-1)^{\mathrm{gh}(\phi)(\mathrm{gh}(F)-\mathrm{gh}(\phi))}\frac{\delta_L F}{\delta\phi^a(x)}.$$

with gh the ghost degree.

PROPOSITION 9. *The bracket* $(\cdot,\cdot)$ *defined by*

$$(F_1,F_2) = \sum_{i=1}^{N}\sum_{a=1}^{\mathrm{rk}E_i}\int_M \left[\frac{\delta_R F_1}{\delta\phi_i^a(x)}\frac{\delta_L F_2}{\delta K_{\phi_i}^a(x)} - \frac{\delta_R F_1}{\delta K_{\phi_i}^a(x)}\frac{\delta_L F_2}{\delta\phi_i^a(x)}\right] d\mu(x),$$

gives $\mathcal{F}([\Phi])$ *the structure of a Gerstenhaber algebra with respect to the ghost degree. Moreover, with*

$$\widetilde{\Delta}(F) = \sum_{i=1}^{N}\frac{\delta_R}{\delta K_{\phi_i}^a(x)}\frac{\delta_L}{\delta\phi_i^a(x)}(F)$$

it becomes a BV*-algebra.*

In the physics literature, it is common to write this anti-bracket on the fields generators in terms of the Dirac delta distribution as

$$(K_{\phi_i}^a(x),\phi_j^b(y)) = \delta^{ab}\delta_{ij}\delta(x-y), \quad (K_{\phi_i}^a(x),K_{\phi_j}^a(y)) = 0, \quad (\phi_i^a(x),\phi_j^b(y)) = 0$$

which is then extended to $\mathcal{F}([\Phi])$ using the graded Leibniz property.

The typical situation in physics is that there is a distinguished element $S\in\mathcal{F}([\Phi])$ – called the **action** – that satisfies the so-called **master equation**

$$(S,S) = 0. \tag{3.1}$$

Thus, S plays the same role with respect to the Gerstenhaber bracket, as the Poisson bivector does with respect to the Schouten–Nijenhuis bracket. One can easily check that the master equation implies that

$$s(F) = (S,F)$$

is a differential: it satisfies Leibniz rule and $s^2=0$. In the cases we will cover, s will be the BRST-differential.

4. Example: Yang–Mills gauge theory

Let us exemplify this in the case of a pure Yang–Mills theory. Let G be a simple Lie group with Lie algebra $\mathfrak{g}$. The *gauge field* A is a $\mathfrak{g}$-valued one-form, that is, a section of $\Lambda^1\otimes(M\times\mathfrak{g})$. As before, we have in components $A = A_i^a dx^i T^a$ where the $\{T^a\}$ form a basis for $\mathfrak{g}$. The structure constants $\{f_c^{ab}\}$ of $\mathfrak{g}$ are defined by $[T^a,T^b] = f_c^{ab}T^c$ and the normalization is such that $\mathrm{tr}\,(T^aT^b) = -\delta^{ab}$.

The **Yang–Mills action** is given in terms of the curvature $F = dA + gA^2$. as

$$S_{\mathrm{ym}}[A] = -\int_M \mathrm{tr}\,[F*F] = -\int_M \mathrm{tr}\,[dA*dA + gdA*[A,A] + \frac{1}{4}g^2[A,A]*[A,A],$$

where $*$ is the Hodge star operator. Consider the quadratic term in this action; it is locally of the form

$$-\frac{1}{2}\int_M \mathrm{tr}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 = \frac{1}{2}\int_M \mathrm{tr}\,A_\mu(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A_\nu$$

The problem with this free action is that there is no well-defined propagator, since $g_{\mu\nu}\Box - \partial_\mu\partial_\nu$ has zero eigenvectors (namely, $A_\nu = \partial_\nu X$ for a $\mathfrak{g}$-valued function X. This is a consequence of gauge invariance, and we need to fix the gauge.

Following Faddeev and Popov, we introduce *ghost fields* $\omega,\overline{\omega}$ which are sections of $M\times\mathfrak{g}[-1]$ and $M\times\mathfrak{g}[1]$, respectively, and we write $\omega = \omega^aT^a$ and $\overline{\omega} = \overline{\omega}^aT^a$. The auxiliary field – also known as the Nakanishi–Lantrup field – is denoted by $h = h^aT^a$ and is a section of $M\times\mathfrak{g}$.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \\
\mathrm{Loc}^{(0,1)} & \xrightarrow[d]{} & \mathrm{Loc}^{(1,1)} & \xrightarrow[d]{} & \mathrm{Loc}^{(2,1)} & \xrightarrow[d]{} & \cdots \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \\
\mathrm{Loc}^{(0,0)} & \xrightarrow[d]{} & \mathrm{Loc}^{(1,0)} & \xrightarrow[d]{} & \mathrm{Loc}^{(2,0)} & \xrightarrow[d]{} & \cdots \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \\
\mathrm{Loc}^{(0,-1)} & \xrightarrow[d]{} & \mathrm{Loc}^{(1,-1)} & \xrightarrow[d]{} & \mathrm{Loc}^{(2,-1)} & \xrightarrow[d]{} & \cdots \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

The form degree and ghost degree of the fields are combined in the *total degree* and summarized in the following table:

	A	ω	$\overline{\omega}$	h
ghost degree	0	$+1$	-1	0
form degree	$+1$	0	0	0
total degree	$+1$	$+1$	-1	0

The gauge-fixed action S_0 for pure Yang–Mills theory is the local functional

$$
S_0 = \int_M \mathrm{tr}\ \Big[- dA * dA - g dA * [A,A] - \frac{1}{4} g^2 [A,A] * [A,A] \\
- A * dh + d\overline{\omega} * d\omega + \frac{1}{2}\xi h * h + g d\overline{\omega} * [A,\omega] \Big] \tag{4.1}
$$

where ξ is the so-called gauge fixing (real) parameter. Note that

$$
\int_M \tfrac{1}{2}\xi h * h - d^* A * h = \tfrac{1}{2}\xi^{-1}(\xi h - d^* A) * (\xi h - d^* A) - \frac{1}{2\xi} d^* A * d^* A,
$$

where we recognize the last term as the R_ξ gauge fixing term ($\xi = 1$ is Feynman-'t Hooft gauge, $\xi \to 0$ is Landau gauge). This action is invariant under the following symmetry transformation:

$$
sA = -d\omega - g[A,\omega], \qquad s\omega = -\frac{1}{2} g[\omega,\omega], \qquad s\overline{\omega} = -h, \qquad sh = 0
$$

which is extended to all of $\mathrm{Loc}^{(p,q)}(\Phi)$ by the graded Leibniz rule, and imposed to anti-commute with the exterior derivative d. One can easily check that $s^2 = 0$, so s and d form a bicomplex in which $s \circ d + d \circ s = 0$. The fact that $s(S_0) = 0$ is an easy consequence of the fact that the Yang–Mills action S_{ym} is gauge invariant and that

$$
s \int_M \mathrm{tr}\ \left[-\overline{\omega} * \tfrac{1}{2}\xi h - d\overline{\omega} * A) \right] = \int_M \mathrm{tr}\ \left[\tfrac{1}{2}\xi h * h - dh * A + d\overline{\omega} * d\omega + g d\overline{\omega} * [A,\omega] \right]
$$

so that $S_0 = S_{\mathrm{ym}} + s\Psi$ is BRST-closed.

A convenient way to express this compactly is by introducing BRST-sources for each of the above fields, $K_A, K_\omega, K_{\overline{\omega}}$ and K_h. The shift in ghost degree is illustrated by the following table:

	K_A	K_ω	$K_{\overline{\omega}}$	K_h
ghost degree	-1	-2	0	-1
form degree	$+1$	0	0	0
total degree	0	-2	0	-1

With these degrees, we can generate the algebra of local forms $\mathrm{Loc}(\Phi)$, which decomposes before into $\mathrm{Loc}^{(p,q)}(E)$ with p the form degree and q the ghost degree. The total degree is then $p+q$ and $\mathrm{Loc}(E)$ is a graded Lie algebra by setting

$$[X,Y] = XY - (-1)^{\deg(X)\deg(Y)} YX,$$

with the grading given by this total degree. Note the slight abuse of notation, $\mathrm{Loc}(\Phi)$ is generated by $A, \omega, \overline{\omega}$, *et cetera*, and not their components. This bracket should not be confused with the anti-bracket defined on local functionals in Section 3. The present graded Lie bracket is of degree 0 with respect to the total degree, that is, $\deg([X,Y]) = \deg(X) + \deg(Y)$. It satisfies graded skew-symmetry, the graded Leibniz identity and the graded Jacobi identity:

$$\begin{aligned} &[X,Y] = -(-1)^{\deg(X)\deg(Y)}[Y,X], \\ &[XY,Z] = X[Y,Z] + (-1)^{\deg(Y)\deg(Z)}[X,Z]Y. \\ &(-1)^{\deg(X)\deg(Z)}[[X,Y],Z] + (\text{cyclic perm.}) = 0 \end{aligned}$$

$$S = S_0 + \langle sA, K_A\rangle + \langle s\overline{\omega}, K_{\overline{\omega}}\rangle + \langle s\omega, K_\omega\rangle + \langle sh, K_h\rangle \tag{4.2}$$

$$\begin{aligned} &= \int_M \mathrm{tr}\, \Big[- dA * dA - g dA * [A,A] - \frac{1}{4} g^2 [A,A] * [A,A] - A * dh + d\overline{\omega} * d\omega + \frac{1}{2}\xi h * h \\ &\quad + g d\overline{\omega} * [A,\omega] - \Big(\langle d\omega, K_A\rangle + g\langle [A,\omega], K_A\rangle + \langle h, K_{\overline{\omega}}\rangle + \frac{1}{2} g \langle [\omega,\omega], K_\omega\rangle \Big) * 1 \Big] \end{aligned} \tag{4.3}$$

Here $\langle \cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{g}$-valued 1-forms and $\mathfrak{g}^*$-valued vector fields (or $\mathfrak{g}$-valued 0-forms and $\mathfrak{g}^*$-valued 0-forms). Validity of the master equation $(S,S) = 0$ is equivalent to BRST-invariance of the functional S_0 and nilpotence of the derivation defined by $s = (S, \cdot)$.

An interesting functional is the **effective action** S_{eff}, which is defined as

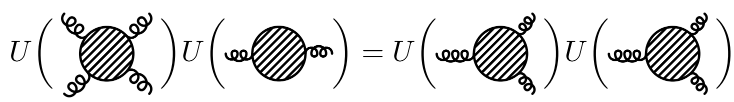
$$S_{\text{eff}} = \sum_{\Gamma \text{ 1PI}} \hbar^{L(\Gamma)} U(\Gamma) m_\Gamma(\Phi)$$

where U is the Feynman amplitude corresponding to the one-particle irreducible graph Γ (see below) and $m_\Gamma(\Phi)$ is the monomial in the fields correspoding to the graph Γ. We project $U(\Gamma)$ onto the relevant form factor, which is then absorbed in $m_\Gamma(\Phi)$. At zeroeth order in $\hbar$, S_{eff} is just the action S. Quite importantly is the **Zinn–Justin equation** that is supposed to be satisfied:

$$(S_{\text{eff}}, S_{\text{eff}}) = 0. \tag{4.4}$$

Note the correspondence of the Zinn–Justin equation for the effective action with the master equation for the action S, the quantum corrections to the action seem to respect the Gerstenhaber bracket. As far as I know, the proof of the Zinn–Justin equation relies heavily on path integral techniques.

Explicitly, in terms of 1PI Green's functions, this would imply for instance that



It is absolutely crucial to obtain these relations for the renormalized Feynman amplitudes as well. Our goal in the next chapters is to understand this compatibility from an algebraic (rigorous) viewpoint, eventually aiming for a proof of this equation.

CHAPTER 3

Hopf algebras and renormalization

1. Commutative Hopf algebras

For convenience, let us briefly recall the definition of a (commutative) Hopf algebra. It is the dual object to a group and, in fact, there is a one-to-one correspondence between groups and commutative Hopf algebras.

Let G be a group with product, inverse and identity element. We consider the algebra of representative functions $H = \mathcal{F}(G)$. This class of functions is such that $\mathcal{F}(G \times G) \simeq \mathcal{F}(G) \otimes \mathcal{F}(G)$. For instance, if G is a (complex) matrix group, then $\mathcal{F}(G)$ could be the algebra generated by the coordinate functions x_{ij} so that $x_{ij}(g) = g_{ij} \in \mathbb{C}$ are just the (i, j)'th entries of the matrix g.

Let us see what happens with the product, inverse and identity of the group on the level of the algebra $H = \mathcal{F}(G)$. The multiplication of the group can be seen as a map $G \times G \to G$, given by $(g, h) \to gh$. Since dualization reverses arrows, this becomes a map $\Delta : H \to H \otimes H$ called the *coproduct* and given for $f \in H$ by

$$\Delta(f)(g, h) = f(gh).$$

The property of associativity on G becomes *coassociativity* on H:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{A1}$$

stating simplify that $f\big((gh)k\big) = f\big(g(hk)\big)$.

The unit $e \in G$ gives rise to a *counit*, as a map $\epsilon : H \to \mathbb{C}$, given by $\epsilon(f) = f(e)$ and the property $eg = ge = g$ becomes on the algebra level

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta, \tag{A2}$$

which reads explicitly $f(ge) = f(eg) = f(g)$.

The inverse map $g \mapsto g^{-1}$, becomes the *antipode* $S : H \to H$, defined by $S(f)(g) = f(g^{-1})$. The property $gg^{-1} = g^{-1}g = e$, becomes on the algebra level:

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = 1_H \epsilon, \tag{A3}$$

where $m : H \otimes H \to H$ denotes pointwise multiplication of functions in H.

From this example, we can now abstract the conditions that define a general Hopf algebra.

DEFINITION 10. *A* Hopf algebra H *is an algebra* H*, together with two algebra maps* $\Delta : H \otimes H \to H$ *(coproduct),* $\epsilon : H \to \mathbb{C}$ *(counit), and a bijective* $\mathbb{C}$*-linear map* $S : H \to H$ *(antipode), such that equations* (A1)–(A3) *are satisfied.*

If the Hopf algebra H is commutative, we can conversely construct a (complex) group from it as follows. Consider the collection $G = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ of multiplicative linear maps from H to $\mathbb{C}$. We will show that G is a group. Indeed, we have the *convolution product* between two such maps ϕ, ψ defined as the dual of the coproduct: $(\phi * \psi)(X) = (\phi \otimes \psi)(\Delta(X))$ for $X \in H$. One can easily check that coassociativity of the coproduct (Eq. (A1)) implies associativity of the convolution product: $(\phi * \psi) * \chi = \phi * (\psi * \chi)$. Naturally, the counit defines the unit e by $e(X) = \epsilon(X)$. Clearly $e * \phi = \phi = \phi * e$ follows at once from Eq. (A2). Finally, the inverse is constructed from the antipode by setting $\phi^{-1}(X) = \phi(S(X))$ for which the relations $\phi^{-1} * \phi = \phi * \phi^{-1} = e$ follow directly from Equation (A3).

With the above explicit correspondence between groups and commutative Hopf algebras, one can translate practically all concepts in group theory to Hopf algebras. For instance, a subgroup $G' \subset G$ corresponds to a *Hopf ideal* $I \subset \mathcal{F}(G)$ in that $\mathcal{F}(G') \simeq \mathcal{F}(G)/I$ and viceversa. The conditions for being a subgroup can then be translated to give the following three conditions defining a Hopf ideal I in a commutative Hopf algebra H

$$\Delta(I) \subset I \otimes H + H \otimes I, \qquad \epsilon(I) = 0, \qquad S(I) \subset I.$$

One can check that then $\mathrm{Hom}_{\mathbb{C}}(H/I, \mathbb{C}) \simeq G'$.

Representations of G correspond one-to-one to *corepresentations* of H. In fact, if V is a G-module, then it is also a *comodule* over H, that is, there exists a map (called *coaction*) $\rho : V \to V \otimes H$ such that $gv = (1 \otimes g)\rho(v)$. If V has additional structure, it is natural to require the coaction to respect this structure.

Finally, we will encounter *connected graded Hopf algebras* for which there is a grading $H = \oplus_{n \in \mathbb{N}} H^n$ that is respected by the product and the coproduct:

$$H^k H^l \subset H^{k+l}; \qquad \Delta(H^n) = \sum_{k=0}^{n} H^k \otimes H^{n-k}.$$

and such that $H^0 = \mathbb{C}1$. Dually, graded Hopf algebras correspond to (pro)-unipotent groups.

2. Hopf algebra of Feynman graphs

We suppose that we have defined a (renormalizable) perturbative quantum field theory and specified the possible interactions between different types of fields. These fields are collected in a set $\Phi = \{\phi_1, \ldots, \phi_{N'}\}$ whereas the different types of interactions – represented by vertices – constitute a set R_V. In the Lagrangian formalism, it is natural to associate to each vertex a local monomial in the fields (present in the Lagrangian); we will denote this map by $\iota : R_V \to \mathrm{Loc}(\Phi)$.

Propagators, on the other hand, are indicated by edges and form a set R_E. Again, one assigns a monomial to each edge via $\iota : R_E \to \mathrm{Loc}(\Phi)$ but now $\iota(e)$ which is now of order 2 in the fields, involving precisely the field (and its conjugate in the case of fermions) that is propagating.

If there are BRST-source terms present in the theory, which means that for each field ϕ_i there is a corresponding source field K_{ϕ_i} in Φ. In other words, the set of fields is of the form

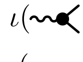
$$\Phi = \{\phi_1, \cdots, \phi_N, K_{\phi_1}, \cdots, K_{\phi_N}\}$$

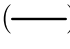
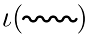
This *even-dimensionality* is a manifestation of the structure on the fields of a Gerstenhaber algebra which we will explore later.

Example 11. *Quantum electrodynamics describes the interaction of charged particles such as electrons with photons, with corresponding fields ψ and A. Their propagation is usually indicated by a straight and a wiggly line (for the electron and photon, respectively). There is only the interaction of an electron emitting a photon: this is indicated by a vertex of valence three; the mass term for the electron is indicated by a vertex of valence two. The dynamical and interactive character of the theory can be summarized by the following sets,*[1]

$R_V = \{$  ,  $\}$; $R_E = \{$, $\}$.

The corresponding monomials in $\mathcal{F}([\Phi])$ are

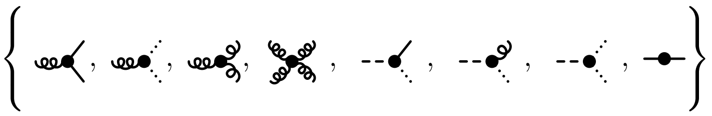
$\iota($$) = -e\overline{\psi}\gamma \circ A\psi$, $\iota($$) = -m\overline{\psi}\psi$,

$\iota($$) = i\overline{\psi}\gamma \circ d\psi$, $\iota($$) = -dA * dA$.

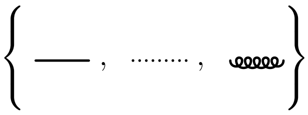
with e and m the electric charge and mass of the electron, respectively.

[1] We specify the type of fields that are involved in the interaction by drawing a small neighborhood around the vertex instead of merely a dot.

Example 12. *Quantum chromodynamics describes the strong interaction between quarks and gluons, described by the fields ψ and A, respectively (see Section 4 below for more details). These are indicated by straight and wiggly lines. In addition, associated to the non-abelian gauge symmetry (with symmetry group $SU(3)$) there is the so-called ghost field ω, indicated by dotted lines, as well as the BRST-sources K_ψ, K_A and K_ω. Between the fields there are four interactions, three BRST-source terms, and a mass term for the quark. This leads to the following sets of vertices and edges,*

$R_V =$ 

with the dashed lines representing the BRST-source terms, and

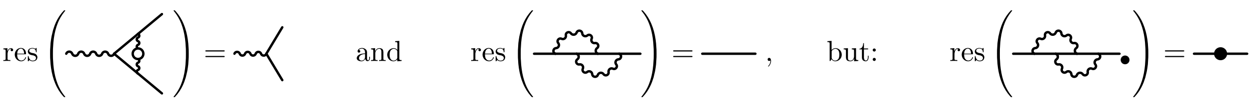
$R_E =$ 

Note that the dashed edges do not appear in R_E, i.e. the source terms do not propagate and in the following will not appear as internal edges of a Feynman graph.

Although these examples motivate our construction, we stress that for what follows it is not necessary to specify the fields nor the vertices and edges in $R = R_V \cup R_E$ explicitly. It is enough to give the sets R_V and R_E of types of vertices and types of edges, respectively, as exemplified above.

A *Feynman graph* is a graph built from the types of vertices present in R_V and the types of edges present in R_E. Naturally, we demand edges to be connected to vertices in a compatible way, respecting the type of vertex and edge. As opposed to the usual definition in graph theory, Feynman graphs have no external vertices. However, they do have *external lines* which come from vertices in Γ for which some of the attached lines remain vacant (i.e. no edge in R_E attached).

If a Feynman graph Γ has two external lines, both corresponding to the same field, we would like to distinguish between propagators and mass terms. In more mathematical terms, since we have vertices of valence two, we would like to indicate whether a graph with two external lines corresponds to such a vertex, or to an edge. A graph Γ with two external lines is dressed by a bullet when it corresponds to a vertex, i.e. we write $\Gamma_\bullet$. The above correspondence between Feynman graphs and vertices/edges is given by the *residue* $\text{res}(\Gamma)$. It is defined as the vertex or edge the graph corresponds to after collapsing all its internal points. For example, we have:



For the definition of the Hopf algebra of Feynman graphs, we restrict to *one-particle irreducible* (1PI) Feynman graphs. These are graphs that are not trees and cannot be disconnected by cutting a single internal edge.

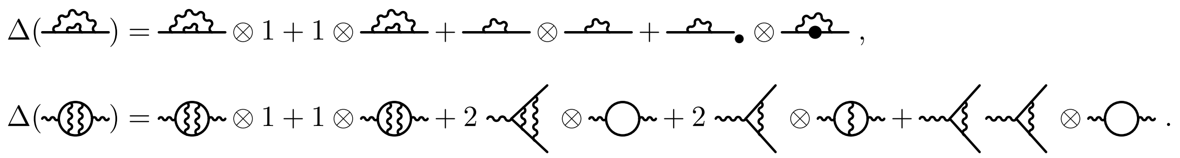
Definition 13 (Connes–Kreimer [**8**]). *The Hopf algebra of Feynman graphs is the free commutative algebra H over $\mathbb{C}$ generated by all 1PI Feynman graphs with residue in $R = R_V \cup R_E$, with counit $\epsilon(\Gamma) = 0$ unless $\Gamma = \emptyset$, in which case $\epsilon(\emptyset) = 1$, coproduct,*

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over disjoint unions of 1PI subgraphs with residue in R. The quotient Γ/γ is defined to be the graph Γ with the connected components of the subgraph contracted to the corresponding vertex/edge. If a connected component γ' of γ has two external lines, then there *are possibly two contributions corresponding to the valence two vertex and the edge; the sum involves the two terms* $\gamma'_\bullet \otimes \Gamma/(\gamma' \to \bullet)$ *and* $\gamma' \otimes \Gamma/\gamma'$. *The antipode is given recursively by,*

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma} S(\gamma)\Gamma/\gamma.$$

Two examples of this coproduct, taken from QED, are:



The above Hopf algebra is an example of a connected graded Hopf algebra: it is graded by the *loop number* $L(\Gamma)$ of a graph Γ. Indeed, one checks that the coproduct (and obviously also the product) satisfy the grading by loop number and H^0 consists of complex multiples of the empty graph, which is the unit in H, so that $H^0 = \mathbb{C}1$. We denote by q_l the projection in H onto H^l.

In addition, there is another grading on this Hopf algebra. It is given by the number of vertices and already appeared in [**8**]. However, since we consider vertices and edges of different types (wiggly, dotted, straight, *et cetera*), we extend to a multigrading as follows. As in [**17**], we denote by $m_{\Gamma,r}$ the number of vertices/internal edges of type r appearing in Γ, for $r \in R$. Moreover, let $n_{\gamma,r}$ be the number of connected components of γ with residue r. For each $v \in R_V$ we define a degree d_v by setting

$$d_v(\Gamma) = m_{\Gamma,v} - n_{\Gamma,v}.$$

The multidegree indexed by R_V is compatible with the Hopf algebra structure as follows easily from the following relation:

$$m_{\Gamma/\gamma,v} = m_{\Gamma,v} - m_{\gamma,v} + n_{\gamma,v},$$

and the fact that $m_{\Gamma\Gamma',v} = m_{\Gamma,v} + m_{\Gamma',v}$, and $n_{\Gamma\Gamma',v} = n_{\Gamma,v} + n_{\Gamma',v}$. This gives a decomposition

$$H = \bigoplus_{(n_1,\dots,n_k)\in\mathbb{Z}^k} H^{n_1,\dots,n_k},$$

where $k = |R_V|$. We denote by $p_{n_1,\dots,n_k}$ the projection onto $H^{n_1,\dots,n_k}$. Note that also $H^{0,\cdots,0} = \mathbb{C}1$.

Lemma 14. *There is the following relation between the grading by loop number and the multigrading by number of vertices:*

$$\sum_{v\in R_V} (N(v) - 2)d_v = 2L$$

where $N(v)$ *is the valence of the vertex* v.

Proof. This can be easily proved by induction on the number of internal edges using invariance of the quantity $\sum_v (N(v) - 2)d_v - 2L$ under the adjoint of an edge. □

The group $\mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ dual to H is called the *group of diffeographism.* This name was coined in [**9**] motivated by its relation with the group of (formal) diffeomorphisms of $\mathbb{C}$, whose definition we recall in the next section. Stated more precisely, they constructed a map from the group of diffeographism to the group of formal diffeomorphisms. We will establish this result in general (i.e. for any quantum field theory) in the next chapter.

3. Formal diffeomorphisms

Another Hopf algebra that will be of interest is that dual to the group $\overline{\mathrm{Diff}}(\mathbb{C},0)$ of formal diffeomorphisms of $\mathbb{C}$ tangent to the identity, it is known in the literature as the Faà di Bruno Hopf algebra (see for instance the short review [**11**]). The elements of this group are given by formal power series:

$$f(x) = x \sum_{n\geq 0} a_n(f) x^n; \qquad a_0(f) = 1 \tag{3.1}$$

with the composition law given by $(f \circ g)(x) = f(g(x))$. The coordinates $\{a_n\}$ generate a Hopf algebra with the coproduct, counit and antipode defined in terms of the pairing $\langle a_n, f\rangle := a_n(f)$ as

$$\langle \Delta(a_n), f\otimes g\rangle = \langle a_n, g\circ f\rangle. \qquad \epsilon(a_n) = \langle a_n, 1\rangle, \qquad \langle S(a_n), f\rangle = \langle a_n, f^{-1}\rangle \tag{3.2}$$

A convenient expression for the coproduct on a_n can be given as follows [**7**]. Consider the generating series

$$A(x) = x\sum_{n\geq 0} a_n x^n; \qquad a_0 = 1$$

where x is considered as a formal parameter. Then the coproduct can be written as

$$\Delta A(x) = \sum_{n\geq 0} A(x)^{n+1} \otimes a_n \tag{3.3}$$

One readily checks that indeed $\langle \Delta A(x), g\otimes f\rangle = f(g(x))$.

Remark 15. *Actually, this Hopf algebra is the dual of the opposite group of* $\overline{\mathrm{Diff}}(\mathbb{C},0)$. *Instead of acting on* $\mathbb{C}$ *as formal diffeomorphisms, the opposite group* $\overline{\mathrm{Diff}}(\mathbb{C},0)^{\mathrm{op}}$ *can be characterized by its action on the algebra* $\mathbb{C}[[x]]$ *of formal power series in* x. *On the generator* x, *the action of* $\overline{\mathrm{Diff}}(\mathbb{C},0)^{\mathrm{op}}$ *is defined by the same formula* (3.1) *but it is extended to all of* $\mathbb{C}[[x]]$ *as an algebra map. We will denote in the following this group by* $\mathrm{Aut}_1(\mathbb{C}[[x]]) := \overline{\mathrm{Diff}}(\mathbb{C},0)^{\mathrm{op}}$.

Clearly, we have an analogous definition of formal diffeomorphisms of $\mathbb{C}^k$ tangent to the identity. The group $\overline{\mathrm{Diff}}(\mathbb{C}^k,0)$ consists of elements:

$$f(x) = \big(f_1(x),\ldots,f_k(x)\big)$$

where each f_i is a formal power series of the following form

$$f_i(x) = x_i\big(\sum a^{(i)}_{n_1\cdots n_k}(f) x_1^{n_1}\cdots x_k^{n_k}\big)$$

with $a^{(i)}_{0,\ldots,0} = 1$ and $x = (x_1,\cdots,x_k)$.

Again, there is a dual Hopf algebra generated by the coordinates $a^{(i)}_{n_1\cdots n_k}$ with the coproduct, counit and antipode defined by the analogous formula to Eq. (3.2).

Lemma 16. *On the generating series* $A_i(x) = x_i(\sum a^{(i)}_{n_1\cdots n_k} x_1^{n_1}\cdots x_k^{n_k})$ *the coproduct equals*

$$\Delta(A_i(x)) = \sum_{n_1,\ldots,n_k} A_i(x)\,(A_1(x))^{n_1}\cdots(A_k(x))^{n_k} \otimes a^{(i)}_{n_1\cdots n_k}.$$

Closely related to these groups of formal diffeomorphisms, is the group of invertible power series in k parameters, denoted $\mathbb{C}[[x_1,\ldots,x_k]]^\times$. As above, it consists of formal series f with non-vanishing first coefficient $a_0(f)\neq 0$, but with product given by the algebra multiplication. The formula for the inverse is given by the Lagrange inversion formula for formal power series.

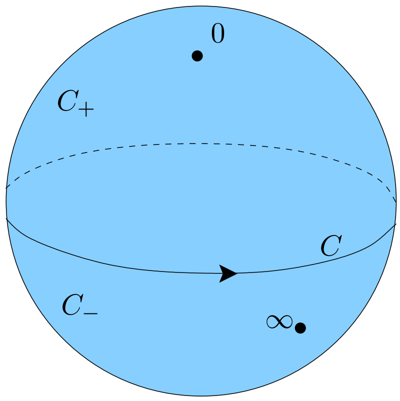
4. Birkhoff decomposition

We now briefly recall how renormalization is an instance of a Birkhoff decomposition in the group of characters of H as established in [**8**]. Let us first recall the definition of a Birkhoff decomposition.

We let $l : C \to G$ be a loop with values in an arbitrary complex Lie group G, defined on a smooth simple curve $C \subset \mathcal{P}_1(\mathbb{C})$. Let $C_\pm$ be the two complements of C in $\mathcal{P}_1(\mathbb{C})$, with $\infty \in C_-$. A **Birkhoff decomposition** of l is a factorization of the form

$$l(z) = l_-(z)^{-1} l_+(z); \qquad (z \in C),$$

where $l_\pm$ are (boundary values of) two holomorphic maps on $C_\pm$, respectively, with values in G. This decomposition gives *a natural way to extract finite values from a divergent expression.* Indeed, although $l(z)$ might not holomorphically extend to C_+, $l_+(z)$ is clearly finite as $z \to 0$.



We now look at the group $G(K) = \mathrm{Hom}_{\mathbb{Q}}(H, K)$ of K-valued characters of a connected graded commutative Hopf algebra H, where K is the field of convergent Laurent series in z.[2] The product, inverse and unit in the group $G(K)$ are defined by the respective equations:

$$\phi * \psi(X) = \langle \phi \otimes \psi, \Delta(X) \rangle,$$
$$\phi^{-1}(X) = \phi(S(X)),$$
$$e(X) = \epsilon(X),$$

for $\phi, \psi \in G(K)$. We claim that a map $\phi \in G(K)$ is in one-to-one correspondence with loops l on an infinitesimal circle around $z = 0$ and values in $G(\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(H, \mathbb{Q})$. Indeed, the correspondence is given by

$$\phi(X)(z) = l(z)(X),$$

and to give a Birkhoff decomposition for l is thus equivalent to giving a factorization $\phi = \phi_-^{-1} * \phi_+$ in $G(K)$. It turns out that for graded connected commutative Hopf algebras such a factorization exists.

THEOREM 17 (Connes–Kreimer [**8**]). *Let H be a graded connected commutative Hopf algebra. The Birkhoff decomposition of $l : C \to G$ (given by an algebra map $\phi : H \to K$) exists and is given dually by*

$$\phi_-(X) = \epsilon(X) - T\left[m(\phi_- \otimes \phi)(1 \otimes (1 - \epsilon)\Delta(X)\right]$$

and $\phi_+ = \phi_- * \phi$.

[2] In the language of algebraic geometry, there is an affine group scheme G represented by H in the category of commutative algebras. In other words, $G = \mathrm{Hom}_{\mathbb{Q}}(H, \ .\ )$ and $G(K)$ are the K-points of the group scheme.

The graded connected property of H assures that the recursive definition of ϕ_- actually makes sense. In the case of the Hopf algebra of Feynman graphs defined above, the factorization takes the following form:

$$\phi_-(\Gamma) = -T\left[\phi(\Gamma) + \sum_{\gamma \subsetneq \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma)\right]$$

$$\phi_+(\Gamma) = \phi(\Gamma) + \phi_-(\Gamma) + \sum_{\gamma \subsetneq \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma)$$

The key point is now that the Feynman rules actually define an algebra map $U : H \to K$ by assigning to each graph Γ the regularized Feynman rules $U(\Gamma)$, which are Laurent series in z. One concludes that the algebra maps U_+ and U_- in the Birkhoff factorization of U are precisely the renormalized amplitude R and the counterterm C, respectively. Summarizing, we can write the BPHZ-renormalization as the Birkhoff decomposition $U = C^{-1} * R$ of the map $U : H \to K$ dictated by the Feynman rules.

CHAPTER 4

The structure of the Hopf algebra of Feynman graphs

1. The Hopf subalgebra of Green's functions

Although the previous construction gives a very nice geometrical description of the process of renormalization, it is a bit unphysical in that it relies on individual graphs. Rather, as mentioned before, in physics the probability amplitudes are computed from the full expansion of Green's functions. Individual graphs do not correspond to physical processes and therefore a natural question to pose is how the Hopf algebra structure behaves at the level of the Green's functions. We will see in the next section that they generate Hopf subalgebras, i.e. the coproduct closes on Green's functions. In proving this, the Slavnov–Taylor identities turn out to play an essential role.

Definition 18. *For a vertex or edge $r \in R$ we define the 1PI Green's function by*

$$G^r = 1 \pm \sum_{\mathrm{res}(\Gamma)=r} \frac{\Gamma}{\mathrm{Sym}(\Gamma)} \tag{1.1}$$

where the sign is $+$ if r is a vertex and $-$ if it is an edge. Finally, we denote the restriction of the sum to graphs Γ at loop order $L(\Gamma) = L$ by G^r_L.

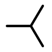
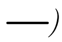
We are particularly interested in the form of the coproduct on 1PI Green's functions, and more generally, the Hopf algebra structure of Green's functions.

1.1. The coproduct on Green's functions.

Lemma 19 ([**19**]).

$$\Delta(G^r) = \sum_{\mathrm{res}(\Gamma)=r} \prod_{v\in R_V} (G^v)^{m_{\Gamma,v}} \prod_{e\in R_E} (G^e)^{-m_{\Gamma,e}} \otimes \frac{\Gamma}{\mathrm{Sym}(\Gamma)}.$$

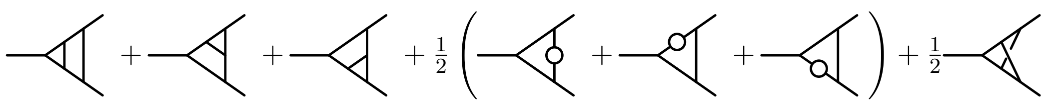
We will not proof this here, but give a low-order example in ϕ^3-theory.

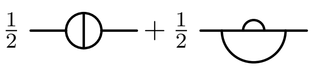
Example 20. *In ϕ^3-theory, Green functions (with $v =$*  *and $e =$* *)*

$$G^v = 1 + \sum_{\Gamma=} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|}$$

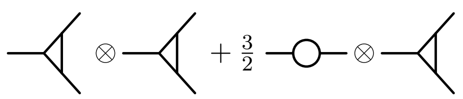
$$G^e = 1 - \sum_{\Gamma=} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|}$$

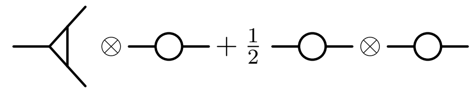
At lowest loop order:

$q_2(G^v) =$ 

$q_2(G^e) =$ 

and

$\Delta\left(q_2(G^v)\right) = q_2(G^v) \otimes 1 + 1 \otimes q_2(G^v) + 3$ 

$\Delta\left(q_2(G^e)\right) = q_2(G^e) \otimes 1 + 1 \otimes q_2(G^e) + \frac{2}{2}$ 

THEOREM 21 ([**19**]). *If we label our vertices as $R_V = \{v_1, \cdots, v_k\}$, then*

$$\Delta(G^r) = \sum_{n_1,\ldots,n_k \in \mathbb{Z}} G^r \prod_{i=1}^{k} \left(\frac{G^{v_i}}{\prod_e (G^e)^{N_e(v_i)/2}} \right)^{n_i} \otimes p_{n_1,\ldots,n_k}(G^r).$$

PROOF. A counting of the number of edges and numbers of vertices in Γ gives the following relations:

$$2m_{\Gamma,e} + N_e(\mathrm{res}(\Gamma)) = \sum_{v \in R_V} N_e(v) m_{\Gamma,v}$$

where $N_e(r)$ is the number of lines (of type e) attached to $r \in R$. For instance $N_e($  $)$ equals 2 if e is an electron line and 1 if e is a photon line. One checks the above equality by noting that the left-hand-side counts the number of internal half lines plus the external lines which are connected to the vertices that appear at the right-hand-side, taken into account their valence.

With this formula, we can write the expression of Lemma 19 as

$$\Delta(G^r) = \prod_e (G^e)^{N_e(r)/2} \sum_{\mathrm{res}(\Gamma)=r} \prod_v \left(\frac{G^v}{\prod_e (G^e)^{N_e(v)/2}} \right)^{m_{\Gamma,v}} \otimes \frac{\Gamma}{\mathrm{Sym}(\Gamma)}. \tag{1.2}$$

It now remains to observe that $m_{\Gamma,v} = d_v(\Gamma)$ unless $v = r$ (the residue of Γ) in which case $m_{\Gamma,v} = d_v(\Gamma) + 1$. This yields the extra factor G^r. □

PROPOSITION 22. *Define elements $Y_v \in H$ for $v \in R_V$ as formal expansions:*

$$Y_v := \frac{G^v}{\prod_e (G^e)^{N_e(v)/2}}$$

The coproduct on $(Y_v)^\alpha$ with $\alpha \in \mathbb{R}$ is given by

$$\Delta(Y_v^\alpha) = \sum_{n_1 \cdots n_k} Y_v^\alpha Y_{v_1}^{n_1} \cdots Y_{v_k}^{n_k} \otimes p_{n_1 \cdots n_k}(Y_v^\alpha),$$

where $p_{n_1 \cdots n_k}$ is the projection onto graphs containing n_i vertices of the type v_i ($i = 1, \ldots, k = |R_V|$).

PROOF. First, one can obtain from Theorem 21 the coproduct on G^r as

$$\Delta(G^r) = \sum_{n_1,\ldots,n_k} G^r Y_{v_1}^{n_1} \cdots Y_{v_k}^{n_k} \otimes p_{n_1,\ldots,n_k}(G^r)$$

which holds for any $r \in R$. A long but straightforward computation involving formal power series expansions yields the following expression for real powers (in the above sense) of the Green's functions:

$$\Delta((G^r)^\alpha) = \sum_{n_1,\ldots,n_k} (G^r)^\alpha Y_{v_1}^{n_1} \cdots Y_{v_k}^{n_k} \otimes p_{n_1,\ldots,n_k}((G^r)^\alpha), \tag{1.3}$$

for $r \in R$ and $\alpha \in \mathbb{R}$. Together with the fact that Δ is an algebra map, this yields the desired cancellations so as to obtain the stated formula. □

There is a striking similarity between the above formula for $\Delta(Y_v)$ and the coproduct in the Hopf algebra dual to $\overline{\mathrm{Diff}}(\mathbb{C}^k, 0)$, as in Lemma 16. In fact, we have the following

COROLLARY 23. *There is a surjective map from the Hopf algebra dual to the group $\overline{\mathrm{Diff}}(\mathbb{C}^k, 0)^{\mathrm{op}}$ to the Hopf subalgebra in H generated by $p_{n_1 \cdots n_k}(Y_v)$.*

PROOF. Whenever $(n_1, \ldots, n_k) \neq (0, \ldots, 0)$, we map the coordinates $a^{(i)}_{n_1 \ldots, n_k}$ of $\overline{\mathrm{Diff}}(\mathbb{C}^k, 0)$ to the elements $p_{n_1,\ldots,n_i-1,\ldots,n_k}(Y_{v_i}) \in H$, with $k = |R_V|$. Indeed, $p_{n_1 \cdots n_k}(Y_{v_i})$ vanishes for all $n_j < 0$ ($j \neq i$) and $n_i < -1$, explaining the shift in the i-th index. Moreover, both $a^{(i)}_{0,\ldots,0}$ and $p_{0,\ldots,0}(Y_{v_i})$ are equal to the identity. □

Actually, with Equation (1.3) above it is easy to see that the algebra generated by $p_{n_1\cdots n_k}(G^v)$ and $p_{n_1\cdots n_k}(G^e)$ for $v \in R_V$ and $e \in R_E$ is a Hopf subalgebra, which we denote by H_R. In Proposition 38 below we will show that the corresponding dual group is in fact a subgroup of the semi-direct product $(\mathbb{C}[[x_1,\ldots,x_k]]^\times)^{|R_E|} \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^k,0)$.

1.2. Example: Quantum electrodynamics. Let us now apply the above formula to the case of quantum electrodynamics (QED). In (massless) quantum electrodynamics, there is only the vertex of valence three, describing the interaction of the photon with a pair of electrons. There are two types of edges corresponding to the photon (wiggly edge) and the electron (straight edge). Summarizing, we have in the notation of the previous section: $R = R_V \cup R_E$ with

$$R_V = \{ \sim\!\!< \};$$
$$R_E = \{ \text{—} , \sim\sim \}.$$

In particular, this means that in the process of renormalization, only three types of graphs are of importance: the vertex graph, the electron self-energy graph and the vacuum polarization. Correspondingly, we have the three 1PI Green's functions,

$$G^{\sim<} = 1 + \sum_{\Gamma^{\sim<}} \frac{\Gamma}{\mathrm{Sym}(\Gamma)};$$
$$G^{e} = 1 - \sum_{\Gamma^{e}} \frac{\Gamma}{\mathrm{Sym}(\Gamma)},$$

with $e = \text{—}, \sim$.

Since there is only one vertex in QED, we can use Lemma 14 to simplify Theorem 21 above.

Proposition 24 ([**18**]). *For* $r = \sim\!<$, — *or* $\sim$ *the following holds*

$$\Delta(G^r) = \sum_{l=0}^{\infty} G^r \left(\frac{G^{\sim<}}{G^e \sqrt{G^{\sim}}} \right)^{2l} \otimes q_l(G^r)$$

with q_l *the projection onto graphs of loop order* l. □

Corollary 25. *The elements* $q_l(G^{\sim<}) - q_l(G^{\text{—}}) \in H$ *for* $l = 1, 2, \ldots$ *generate a Hopf ideal* I, *i.e.*

$$\Delta(I) \subseteq I \otimes H + H \otimes I, \qquad \epsilon(I) = 0, \qquad S(I) \subseteq I.$$

Proof. This follows easily by applying Proposition 24 to the coproduct evaluated on the difference $G^{\sim<} - G^{\text{—}}$, in combination with the recursive definition of the antipode. □

The identities $G^{\sim<} = G^{\text{—}}$ which hold in the corresponding quotient Hopf algebra H/I have a physical meaning: they are the famous *Ward identities of quantum electrodynamics* [**21**]. The above claim that they can be implemented on the Hopf algebra of Feynman graphs corresponds to the physical statement that the Ward identities are compatible with renormalization. In fact, we have the following.

Proposition 26. *Suppose the regularized (but unrenormalized) Feynman rules* $U : H \to K$ *satisfy the Ward identities. Then the counterterms* C *and the renormalized Feynman rules* R *satisfy the Ward identities:*

$$C(G^{\sim<}) = C(G^{\text{—}}); \qquad R(G^{\sim<}) = R(G^{\text{—}})$$

Note that the first equation is usually written as $Z_1 = Z_2$ [**21**].

Proof. This follows directly from the Birkhoff decomposition (cf. Theorem 17 above) applied to the character group of the graded connected commutative Hopf algebra H/I. □

2. Hopf ideals, Ward, and Slavnov–Taylor identities

We will next establish that a quotient of the Hopf algebra generated by $p_{n_1,\dots,n_k}(Y_v)$ by a certain Hopf ideal is isomorphic to the Hopf algebra dual to (a subgroup of) $\mathrm{Aut}_1(\mathbb{C}[[x]]) \equiv \overline{\mathrm{Diff}}(\mathbb{C},0)^{\mathrm{op}}$. The latter is indeed a subgroup of $\overline{\mathrm{Diff}}(\mathbb{C}^k,0)^{\mathrm{op}}$ under the diagonal embedding.

THEOREM 27. [**19**] *The ideal J' in H_R generated by $q_l\left(Y_{v'}^{N(v)-2} - Y_v^{N(v')-2}\right)$ for $v', v \in R_V$ of valence greater than 2 ($l \geq 0$), and Y_v for all v of valence 2 is a Hopf ideal, i.e.*

$$\Delta(J') \subset J' \otimes H_R + H_R \otimes J'.$$

PROOF. First of all, with Proposition 22, the coproduct on Y_v for $\mathrm{val}(v) = 2$ is readily found to be an element in $J' \otimes H_R + H_R \otimes J'$. With Proposition 22, we can write the coproduct on the other generators of J' as

$$\Delta\left(Y_{v'}^{N(v)-2} - Y_v^{N(v')-2}\right) = \sum_n Y_v^{N(v')-2} Y_{v_1}^{n_1} \cdots Y_{v_k}^{n_k} \otimes p_n\left(Y_{v'}^{N(v)-2} - Y_v^{N(v')-2}\right)$$
$$+ \sum_n \left[Y_{v'}^{N(v)-2} - Y_v^{N(v')-2}\right] Y_{v_1}^{n_1} \cdots Y_{v_k}^{n_k} \otimes p_n\left(Y_{v'}^{N(v)-2}\right)$$

with n the multi-index $(n_1, \dots, n_k)$. The second term is clearly an element in $J' \otimes H_R$. For the first term, note that each n_i'th power of Y_{v_i} can be written as

$$Y_{v_i}^{n_i} = Y_{v_i}^{n_i \frac{N(v)-2}{N(v)-2}} = Y_v^{n_i \frac{N(v_i)-2}{N(v)-2}} + J'.$$

Hence, the first term becomes *modulo* $J' \otimes H_R$

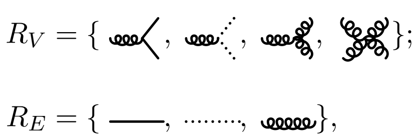
$$\sum_{n_1 \cdots n_k} \left(Y_v^{1/N(v)-2}\right)^{n_1(N(v_1)-2)+\cdots+n_k(N(v_k)-2)} \otimes p_{n_1\cdots n_k}\left(Y_{v'}^{N(v)-2} - Y_v^{N(v')-2}\right).$$

Appealing to Lemma 14 now allows us to write this in terms of the loop number l to finally obtain for the first term

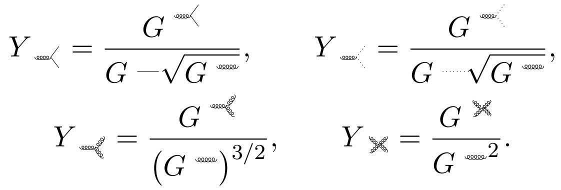
$$\sum_{l=0}^{\infty} Y_v^{\frac{2l+1}{N(v)-2)}} \otimes q_l\left(Y_{v'}^{N(v)-2} - Y_v^{N(v')-2}\right).$$

which is indeed an element in $H_R \otimes J'$. □

As a consequence, the quotient Hopf algebra $\widetilde{H}_R = H_R/J'$ is well-defined. In $\widetilde{H}_R$ the relations $Y_v^{N(v')-2} = Y_{v'}^{N(v)-2}$ are satisfied. In physics these identities are called *Slavnov–Taylor identities* for the couplings; we will see later how they appear naturally from the relations between coupling constants. Moreover, the fact that we put $Y_v = 0$ for vertices of valence 2 means that we consider a massless theory. Let us make this more explicit in the case of quantum chromodynamics (without BRST-sources). Recall that the Feynman graphs are now built from the sets:



where the plain, dotted and curly lines represent the quark, ghost and gluon, respectively. The four 'couplings' are



The Hopf ideal implements the Slavnov–Taylor relations for the couplings:

(2.1)
$$\begin{aligned} G^{ggg} G^{\bar q q g} &= G^{gggg} G^{\bar q q}; \\ G^{ggg} G^{\bar c c g} &= G^{gggg} G^{\bar c c}; \\ G^{ggg} G^{ggg} &= G^{gggg} G^{gg}. \end{aligned}$$

The compatibility of these relations is now best captured by the following result.

Proposition 28. *Suppose the regularized (but unrenormalized) Feynman rules $U : H \to K$ satisfy the Slavnov–Taylor identities for the couplings. Then the counterterms C and the renormalized Feynman rules R satisfy the Slavnov–Taylor identities:*

$$C(Y_{\bar q q g}) = C(Y_{\bar c c g}) = C(Y_{ggg}) = C(Y_{\sqrt{gggg}})$$
$$R(Y_{\bar q q g}) = R(Y_{\bar c c g}) = R(Y_{ggg}) = R(Y_{\sqrt{gggg}})$$

Again, the first equation is typically written in terms of Z-factors. This would lead to the key relation:

(2.2)
$$\frac{Z^{\bar q q g}}{Z^{\bar q q}\sqrt{Z^{gg}}} = \frac{Z^{\bar c c g}}{Z^{\bar c c}\sqrt{Z^{gg}}} = \frac{Z^{ggg}}{\left(Z^{gg}\right)^{3/2}} = \frac{\sqrt{Z^{gggg}}}{Z^{gg}},$$

where the notation is as above: $Z^r := C(G^r)$.

Proof. As in the case of QED, this follows from the Birkhoff decomposition in the character group of the quotient Hopf algebra H/I. □

Returning to the general case, in $\widetilde{H}_R$ we can drop the subscript v and use the notation $X := Y_v^{1/N(v)-2}$ independent of $v \in R_V$ as long as $\mathrm{val}(v) > 2$.

Theorem 29. *The coproduct in $\widetilde{H}_R$ takes the following form on the element X:*

$$\Delta(X) = \sum_{l=0}^{\infty} (X)^{2l+1} \otimes q_l(X).$$

where q_l is the projection in $\widetilde{H}_R$ onto graphs of loop number l.

Proof. This follows directly by substituting X for X_v in the expression for $\Delta(X_v)$ in Proposition 22 and using the relation from Lemma 14 between the number of vertices and the loop number. □

Thus, the Hopf algebra $\widetilde{H}_R$ contains a Hopf subalgebra that is generated by $q_l(X)$ and a comparison with Eq. (3.3) yields – after identifying $q_l(X)$ with a_{2l} – the following result.

Theorem 30. *The graded Hopf subalgebra in $\widetilde{H}_R$ generated by $q_l(X)$ for $l = 0, 1, \ldots$ is isomorphic to the Hopf algebra of the group of odd formal diffeomorphisms of $\mathbb{C}$ tangent to the identity. In other words, there is a homomorphism from the group of diffeographisms to $\overline{\mathrm{Diff}}(\mathbb{C}, 0)^{\mathrm{op}} \equiv \mathrm{Aut}_1(\mathbb{C}[[x]])$.*

This generalizes the result of [**9**] where such a map was constructed explicitly in the case of (massless) ϕ^3-theory; for other theories a map has been constructed by Cartier and Krajewski. In the next section, we will explore its relation with the group of formal diffeomorphisms acting on the space of coupling constants.

3. Dyson–Schwinger equations and Hochschild cohomology

In this section, we will review how Hochschild cohomology fits nicely in the context of renormalization Hopf algebras, following [**10, 5**] and [**15**]. In particular, we will relate it to the Dyson–Schwinger equations and prove the so-called gauge theory theorem that was announced in [**15**].

Let us first recall the definition of Hochschild cohomology for Hopf algebras, – or, more generally, for bialgebras – with values in a bicomodule. This dualizes the definition of Hochschild cohomology for algebras to bialgebras. Let H be a bialgebra and M an H-bicomodule, i.e. there are two cocommuting left and right coactions $\rho_L : M \to H \otimes M$ and $\rho_R : M \to M \otimes H$. We denote by $C^n(H, M)$ the space of linear maps $\phi : M \to H^{\otimes n}$ and define the *Hochschild coboundary map* $b : C^n(H, M) \to C^{n+1}(H, M)$ by

$$b\phi = (\mathrm{id} \otimes \phi)\rho_L + \sum_{i=1}^{n} (-1)^n \Delta_i \phi + (-1)^{n+1}(\phi \otimes \mathrm{id})\rho_R.$$

where Δ_i denotes the application of the coproduct on the i'th factor in $H^{\otimes n}$. Coassociativity implies that b is a differential, i.e. that $b^2 = 0$.

Definition 31. *The Hochschild cohomology $HH^\bullet(H, M)$ of the bialgebra H with values in the H-comodule M is defined as the cohomology of the complex $(C^\bullet(H, M), b)$ defined above.*

We are interested in the particular case that $M = H$ is a comodule over itself, with $\rho_L = \Delta$ but with $\rho_R = (\mathrm{id} \otimes \epsilon)\Delta$. We denote the Hochschild cohomology groups in this case by $HH^\bullet(H, H_\epsilon)$ or simply $HH_\epsilon^\bullet(H)$ as in [**10**]. Let us consider the case $n = 1$, then $\phi \in HH_\epsilon^1(H)$ means simply that

$$\Delta\phi = (\mathrm{id} \otimes \phi)\Delta + (\phi \otimes \mathrm{I}).$$

where $(\phi \otimes \mathrm{I})(h) \equiv \phi(h) \otimes 1$ for $h \in H$. As was observed in [**10**] the grafting operator on rooted trees is an example of such a 1-cocycle. We will give an example in the case of the Hopf algebra of Feynman graphs (cf. (2) of Theorem 32 below), following [**15**].

This starts with the observation that the Green's functions can be dissected as follows [**15**, Theorem 4]:

$$G^r = \sum_{\gamma \text{ prim}} B_+^\gamma \left(\frac{\prod_{v \in R_V} (G^v)^{m_{\gamma,v}}}{\prod_{e \in R_E} (G^e)^{m_{\gamma,e}}} \right) = \sum_{\gamma \text{ prim}} B_+^\gamma (G^{\mathrm{res}(\gamma)} X^{2l(\gamma)}) \tag{3.1}$$

where B_+^γ is the (normalized) grafting operator that inserts in γ the graphs given as its argument on the appropriate insertion places. The sum is over all primitive graphs γ, i.e. satisfying $\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$. It is clear that any graph in G^r is of the form $B_+^\gamma(\Gamma_1 \cdots \Gamma_N)$ for some 1PI graphs $\Gamma_1, \ldots, \Gamma_N$ but this decomposition is highly non-unique. In order to correct for the overcounting, the grafting operators have to be normalized appropriately as was done in [**15**]. We will instead take Eq. (3.1) as a *definition* of the normalized maps B_+^γ, without explicitly describing this normalization. The sum of the B_+^γ over all primitive 1PI Feynman graphs at a given loop order and with given residue will be denoted by $B_+^{k;r}$, as in *loc. cit.*. More precisely,

$$B_+^{k;r} = \sum_{\substack{\gamma \text{ prim} \\ l(\gamma)=k \\ \mathrm{res}(\gamma)=r}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^\gamma$$

and, of course, $G^r = \sum_{k,r} B_+^{k;r}(X_{k,r})$, where we have denoted $X_{k,r} = G^r X^{2k}$. With this and the formulas of the previous section on QCD, we can prove the *gauge theory theorem* as formulated in [**15**, Theorem 5]:

Theorem 32. *Let $\widetilde{H} = H/I$ be the Hopf algebra of QCD Feynman graphs (cf. Sect. 2) with the Slavnov–Taylor identities for the couplings imposed.*

(1) $G^r = \sum_{k=0}^{\infty} B_+^{k;r}(X_{k,r})$
(2) $\Delta(B_+^{k;r}(X_{k,r})) = B_+^{k;r}(X_{k,r}) \otimes \mathrm{I} + (\mathrm{id} \otimes B_+^{k;r})\Delta(X_{k,r})$.
(3) $\Delta(G_k^r) = \sum_{j=0}^{k} \mathrm{Pol}_j^r(G) \otimes G_{k-j}^r$.

where $\mathrm{Pol}_j^r(G)$ *is a polynomial in the* G_m^r *of degree* j*, determined as the order* j *term in the loop expansion of* $G^r X^{2k-2j}$.

Proof. The first claim is just the definition of the $B_+^{k;r}$. For (2), we first enhance the result of Eq. (1.3) to partial sums in G^r over graphs that have 'primitive residue' isomorphic to a fixed primitive graph γ. In other words, if $G^{r,\gamma}$ is the part of G^r that sums only over graphs that are obtained by inserting graphs into the primitive graph γ, then

$$\Delta(G^{r,\gamma}) = G^{r,\gamma} \otimes 1 + \sum_{l=1}^{\infty} G^r X^{2l} \otimes q_l(G^{r,\gamma}).$$

Here we have imposed the Slavnov–Taylor identities for the couplings to write this in terms of a single coupling, X. Combing Eq. (1.3) and Theorem 29 we obtain for the coproduct of $X_{k,r} = G^r X^{2k}$:

$$\Delta(X_{k,r}) = \sum_{l=0}^{\infty} G^r X^{2l+2k} \otimes q_l X_{k,r}.$$

Since $G^{r,\gamma} = B_+^{\gamma}(X_{k,r})$, it follows by a combination of the above two formula that

$$\Delta(B_+^{\gamma}(X_{k,r})) = B_+^{\gamma}(X_{k,r}) \otimes \mathrm{I} + (\mathrm{id} \otimes B_+^{\gamma})\Delta(X_{k,r}).$$

and summing over all primitive graphs with residue r at loop order k gives the desired result.

Finally, (3) follows by combining Theorem 21 with Proposition 22, thereby taking into account the Slavnov–Taylor identities. □

Remark 33. *We have corrected for the apparent misprint in* [**15**, Eq. (83)].

In fact, this proves the slightly stronger result that every B_+^{γ} defines a Hochschild 1-cocycle:

Proposition 34. *For* γ *a primitive graph at loop order* k *and residue* r*, we have*

$$\Delta(B_+^{\gamma}(X_{k,r})) = B_+^{\gamma}(X_{k,r}) \otimes \mathrm{I} + (\mathrm{id} \otimes B_+^{\gamma})\Delta(X_{k,r}).$$

Actually, the above results apply in full generality for any Hopf algebra as defined in Definition 13. However, the meaning of the Hopf ideals as imposing Slavnov–Taylor identities can only be given in the context of a non-abelian gauge theory. Moreover, the above Hochschild cocycles $B_+^{k;r}$ play an important role in that they give quantum equation of motions. These *Dyson–Schwinger equations* are the recursive construction of the 1PI Green's functions G^r from the lower order Green's functions in $X_{k,r}$ forming the argument of the B_+-operations. In fact, Equation (3.1) are precisely the Dyson–Schwinger equations for quantum chromodynamics.

CHAPTER 5

Coaction of Feynman graphs on fields and couplings

We now connect the above approach to renormalization in the BV-formalism, which differs slightly from the usual wave function and coupling constant renormalization (see for instance [**1**, Section 6]). We will not give proofs of the results, but refer instead to [**20**] for all details.

We make the following natural assumptions on the map $\iota : R \to \mathrm{Loc}(\Phi)$

(1) Whenever a fermionic field, say ψ, interacts at a vertex $v \in R_V$ which does not involve a BRST-source, then $\iota(v)$ involves *both* ψ and $\overline{\psi}$.
(2) There is only one vertex for every BRST-source.
(3) There are no valence two vertices involving two different fields (thus, still allowing mass terms).

Physically, the last condition means that we require order two polynomials other than mass terms in the Lagrangian not to be radiatively corrected.

We further assume that we are given elements $C^\phi \in H$ for each $\phi \in \Phi$ such that the following hold:

(1) If ϕ only appears linearly in the Lagrangian then $C^\phi C^{\phi_{i_1}} \cdots C^{\phi_{i_1}} = 1$ for $\iota(v) \propto \phi\phi_{i_1} \cdots \phi_{i_m}$.
(2) If $\iota(e) \propto \phi\overline{\phi}$ then $C^\phi C^{\overline{\phi}} = G^e$.
(3) For any field ϕ_i we have $C^{K_{\phi_i}} C^{\phi_i} = 1$.

Note that in general the C^ϕ's are not uniquely determined from these conditions. However, in theories of interest such as Yang–Mills gauge theories, they actually are as illustrated by the next example.

Example 35. *For pure Yang–Mills gauge theories (see for notation Example 2 below) we have*

$$C^A = \sqrt{G^{\sim\sim}}; \quad C^\omega = (G^{\cdots\cdots})\sqrt{G^{\sim\sim}}; \quad C^{\overline{\omega}} = (G^{\sim\sim})^{-\frac{1}{2}}; \quad C^h = (G^{\sim\sim})^{-\frac{1}{2}},$$

and $C^{K_\phi} = (C^\phi)^{-1}$ for $\phi = A, \omega, \overline{\omega}, h$. Note that $C^\omega C^{\overline{\omega}} = G^{\cdots\cdots}$ which – as we shall see in Section 4 below – will be the usual wave function renormalization for the ghost propagator.

Returning to the general setup, we assume that we have defined such elements C^ϕ for all $\phi \in \Phi$. One can easily check that if v does not involve a BRST-source term then

$$\frac{G^v}{\prod_\phi (C^\phi)^{N_\phi(v)}} = \frac{G^v}{\prod_{e \in R_E} (G^e)^{N_e(v)/2}}$$

since a fermionic field ϕ will always be accompanied by the field $\overline{\phi}$ on a vertex that does not involve a BRST-source (cf. above), thus reducing the above formula to Eq. (11) in *loc. cit.*. It is sufficient to consider only the case of no BRST-sources since in either case (for r with or without BRST-source) the v's appearing in the above formula will never involve a BRST-source.

One can also show that

$$\Delta(C^\phi) = \sum_{n_1,\ldots,n_k} C^\phi Y_{v_1}^{n_1} \cdots Y_{v_k}^{n_k} \otimes p_{n_1,\ldots,n_k}(C^\phi), \tag{0.2}$$

1. The comodule BV-algebra of coupling constants and fields

Since the coupling constants measure the strength of the interactions, we label them by the elements $v \in R_V$ and write accordingly λ_v. We consider the algebra A_R generated by local functionals in the fields and formal power series (over $\mathbb{C}$) in the coupling constants λ_v. In other words, we define $A_R := \mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]] \otimes_{\mathbb{C}} \mathcal{F}([\Phi])$ where $k = |R_V|$. The BV-algebra structure on $\mathcal{F}([\Phi])$ defined in the previous section induces a natural BV-algebra structure on A_R; we denote the bracket on it by $(\cdot, \cdot)$ as well.

Recall the notation H_R for the Hopf subalgebra generated by the elements $p_{n_1,\ldots,n_k}(Y_v)$ ($v \in R_V$) and $p_{n_1,\ldots,n_k}(C^\phi)$ ($e \in R_E$) in the Hopf algebra of Feynman graphs.

THEOREM 36. *The algebra A_R is a comodule BV-algebra for the Hopf algebra H_R. The coaction $\rho : A_R \to A_R \otimes H_R$ is given on the generators by*

$$\rho : \lambda_v \longmapsto \sum_{n_1 \cdots n_k} \lambda_v \lambda_{v_1}^{n_1} \cdots \lambda_{v_k}^{n_k} \otimes p_{n_1 \cdots n_k}(Y_v),$$

$$\rho : \phi \longmapsto \sum_{n_1 \cdots n_k} \phi \; \lambda_{v_1}^{n_1} \cdots \lambda_{v_k}^{n_k} \otimes p_{n_1 \cdots n_k}(C^\phi),$$

for $\phi \in \Phi$, while it commutes with partial derivatives on ϕ.

COROLLARY 37 ([**20**]). *The Green's functions $G^v \in H_R$ can be obtained when coacting on the monomial $\int_M \lambda_v \iota(v)(x) d\mu(x) = \int_M \lambda_v \partial_{\vec{\mu}_1} \phi_{i_1}(x) \cdots \partial_{\vec{\mu}_M} \phi_{i_M}(x) d\mu(x)$ for some index set $\{i_1, \ldots, i_M\}$. Explicitly,*

$$\rho\left(\int_M \lambda_v \partial_{\vec{\mu}_1} \phi_{i_1}(x) \cdots \partial_{\vec{\mu}_M} \phi_{i_M}(x)\right) = \sum_{n_1 \cdots n_k} \lambda_v \lambda_{v_1}^{n_1} \cdots \lambda_{v_k}^{n_k} \int_M \partial_{\vec{\mu}_1} \phi_{i_1}(x) \cdots \partial_{\vec{\mu}_M} \phi_{i_M}(x) \otimes p_{n_1 \cdots n_k}(G^v).$$

Combining Theorem 36 with Corollary 23 yields an induced coaction on $\mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]]$ of the Hopf algebra dual to the group of diffeomorphisms on $\mathbb{C}^k$ tangent to the identity. The formula for this coaction can be obtained by substituting $a^{(i)}_{n_1 \cdots n_k}$ for $p_{n_1 \cdots n_k}(Y_{v_i})$ in the above formula for $\rho(\lambda_v)$. It induces a group action of $\overline{\mathrm{Diff}}(\mathbb{C}^k, 0)$ on $\mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]]$ by $f(a) := (1 \otimes f)\rho(a)$ for $f \in \overline{\mathrm{Diff}}(\mathbb{C}^k, 0)$ and $a \in \mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]]$. In fact, we have the following

PROPOSITION 38 ([**20**]). *Let G be the group consisting of BV-algebra maps $f : A_R \to A_R$ given on the generators by*

$$f(\lambda_v) = \sum_{n_1 \cdots n_k} f^v_{n_1 \cdots n_k} \lambda_v \lambda_{v_1}^{n_1} \cdots \lambda_{v_k}^{n_k}; \qquad (v \in R_V),$$

$$f(\phi_i) = \sum_{n_1 \cdots n_k} f^i_{n_1 \cdots n_k} \phi_i \lambda_{v_1}^{n_1} \cdots \lambda_{v_k}^{n_k}; \qquad (i = 1, \ldots, N),$$

where $f^v_{n_1 \cdots n_k}, f^i_{n_1 \cdots n_k} \in \mathbb{C}$ are such that $f^v_{0 \cdots 0} = f^i_{0 \cdots 0} = 1$. Then the following hold:

1. *The character group G_R of the Hopf algebra H_R generated by $p_{n_1 \cdots n_k}(Y_v)$ and $p_{n_1 \cdots n_k}(C^\phi)$ with coproduct given in Proposition 22, is a subgroup of G.*
2. *The subgroup $N := \{f : f(\lambda_v) = \lambda_v\}$ of G is normal and isomorphic to $(\mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]]^\times)^{|R_E|}$.*
3. *$G \simeq (\mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]]^\times)^{|R_E|} \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^k, 0)$.*

The action of (the subgroup of) $(\mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]]^\times)^{|R_E|} \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^k, 0)$ on A_R has a natural physical interpretation: the invertible formal power series act on every propagating field as wave function renormalization whereas the diffeomorphisms act on the coupling constants $\lambda_1, \ldots, \lambda_k$. The similarity with the semi-direct product structures obtained (via different approaches) in [**13**] for a scalar field theory and in [**6, 7**] for quantum electrodynamics is striking.

EXAMPLE 39. *Consider again pure Yang–Mills theory with fields $A, \omega, \overline{\omega}$ and h. Then, under the counterterm map $\gamma_-(z) \in G_R$ (cf. Section 4) we can identify $(C^A)^2 = G^{\text{~~~~}}$ with wave* *function renormalization for the gluon propagator, and the combination* $C^\omega C^{\overline{\omega}} = G$ *with wave function renormalization for the ghost propagator. The above action of* $\gamma_-(z)$ *on the fields* $A, \omega, \overline{\omega}$ *is thus equivalent to wave function renormalization. We will come back to Yang–Mills theories in more detail in Section 4 below.*

1.1. The master equation. The dynamics and interactions in the physical system is described by means of a so-called *action* S. In our formalism, S will be an element in A_R of polynomial degree ≥ 2 of the form,

$$S[\phi] = \sum_{e \in R_E} \int d\mu(x)\ \iota(e)(x) + \sum_{v \in R_V} \int d\mu(x)\ \lambda_v\ \iota(v)(x) \tag{1.1}$$

The first sum in S describes the free field theory containing the propagators of the (massless) fields. The second term describes the interactions *including* the mass terms. Note that due to the restrictions in the sums, the action has finitely many terms, that is, it is a (local) polynomial functional in the fields rather than a formal power series.

The action S is supposed to be invariant under some group of gauge transformations.[1] We accomplish this in our setting by imposing the (classical) *master equation*,

$$(S, S) = 0, \tag{1.2}$$

as relations in the BV-algebra A_R.

PROPOSITION 40 ([**20**]). *The BV-ideal* $I = \langle (S,S) \rangle$ *is generated by polynomials in* λ_v $(v \in R_V)$*, independent of the fields* $\phi \in \Phi$*.*

We still denote the image of the action S in A_R/I under the quotient map by S; it satisfies the master equation (1.2) with the brackets as defined before. If we make the natural assumption that S is at most of order one in the BRST-sources, we can write

$$S = S_0[\lambda_v, \phi_i] + \sum_{i=1}^{N} \sum_{a=1}^{\mathrm{rk} E_i} \int d\mu(x) (s\phi_i)^a(x) K^a_{\phi_i}(x). \tag{1.3}$$

with $s\phi_i$ dictated by the previous form of S. Of course, this is the familiar BRST-differential acting on the field ϕ_i as a graded derivation and obviously satisfies $s\phi_i(x) = (S, \phi_i(x))$. As usual, validity of the master equation $(S,S) = 0$ implies that s is nilpotent:

$$s^2(\phi_i) = (S, (S, \phi_i)) = \pm((S,S), \phi_i) = 0$$

using the graded Jacobi identity. Moreover, the action S_0 depending on the fields is BRST-closed, i.e. $sS_0 = 0$, which follows by considering the part of the master equation $(S,S) = 0$ that is independent of the BRST-sources.

The following result establishes an action and coaction on the quotient BV-algebra A_R/I.

THEOREM 41 ([**20**]). *Let* G^I_R *be the (closed) subgroup of* G_R *defined in Proposition 38 consisting of diffeomorphisms* f *that leave* I *invariant, i.e. such that* $f(I) \subset I$*.*

(1) *The group* G^I_R *acts on the quotient BV-algebra* A_R/I*.*
(2) *The ideal in* H_R *defined by*

$$J := \left\{ X \in H_R : f(X) = 0 \text{ for all } f \in G^I_R \right\} \tag{1.4}$$

is a Hopf ideal.

Consequently, $G^I_R \simeq \mathrm{Hom}_{\mathbb{C}}(H_R/J, \mathbb{C})$ *and the quotient Hopf algebra* $\widetilde{H}_R = H_R/J$ *coacts on* A_R/I*.*

[1] In addition, it is supposed to be invariant under the symmetry group of the underlying spacetime one works on, typically the Lorentz group. However, these transformations are linear in the fields and will consequently not give rise to non-linear equations such as the master equation discussed here. See for instance [**14**] for more details.

We denote the coaction of $\widetilde{H}_R := H_R/J$ on A_R by $\widetilde{\rho}$; it is given explicitly by

$$\widetilde{\rho}(a+I) = (\pi_I \otimes \pi_J)\,\rho(a), \tag{1.5}$$

for $a \in A_R$; also, π_I and π_J are the projections onto the quotient algebra and Hopf algebra by I and J respectively.

Let us now justify the origin of the explicit Hopf ideals that we have encountered in the previous section in the case that all coupling constants coincide. This happens for instance in the case of Yang–Mills theory with a simple gauge group, which is discussed in Section 4. In general, we make the following definition.

Definition 42. *A theory defined by S is called* simple *when the following holds modulo the ideal $\langle \lambda_v \rangle_{\mathrm{val}(v)=2}$:*

$$I = \langle \lambda_{v'}^{N(v)-2} - \lambda_v^{N(v')-2} \rangle_{\mathrm{val}(v),\mathrm{val}(v')>2} \tag{1.6}$$

In other words, if we put the mass terms in S to zero, then the ideal I should be generated by the differences $\lambda_{v'}^{N(v)-2} - \lambda_v^{N(v')-2}$ for vertices with valence greater than 2. We denote by I' the ideal in Eq. (1.6) modulo $\langle \lambda_v \rangle_{\mathrm{val}(v)=2}$. A convenient choice of generators for I' is the following. Fix a vertex $v \in R_V$ of valence three,[2] and define $g := \lambda_v$ as the 'fundamental' coupling constant. Then I' is generated by λ_v with $\mathrm{val}(v) = 2$ and $\lambda_{v'} - g^{N(v')-2}$ with $\mathrm{val}(v') > 2$. Recall the ideal J' from the previous section.

Theorem 43 ([**20**]). *Let S define a simple theory in the sense described above.*

1. *The subgroup $G^{I'}$ of diffeomorphisms that leave I' invariant is isomorphic to* $\mathrm{Hom}_{\mathbb{C}}(H_R/J', K)$.
2. *The Hopf algebra H_R/J' coacts on $\mathbb{C}[[g,\phi]] := A_R/I'$ via the map*

$$\widetilde{\rho}' : g \longmapsto \sum_{l=0}^{\infty} g^{2l+1} \otimes q_l(X),$$

$$\widetilde{\rho}' : \phi \longmapsto \sum_{l=0}^{\infty} g^{2l}\phi \otimes q_l(C^{\phi}).$$

Corollary 44. *The group $G_R^{I'}$ acts on A_R/I' as a subgroup of* $(\mathbb{C}[[g]]^{\times})^{|R_E|} \rtimes \overline{\mathrm{Diff}}(\mathbb{C},0)$.

This last result has a very nice physical interpretation: the invertible formal power series act on the $|R_E|$ propagating fields as wave function renormalization whereas the diffeomorphisms act on one fundamental coupling constant g. We will appreciate this even more in the next section where we discuss the renormalization group flow.

2. Example: pure Yang–Mills theory

In the setting of Section 1.1, the action S for pure Yang–Mills theory is the local functional

$$\begin{aligned} S = \int_M \mathrm{tr}\, \Bigg[&- dA * dA - \lambda_{A^3} dA * [A,A] - \frac{1}{4}\lambda_{A^4}[A,A]*[A,A] - A * dh + d\overline{\omega} * d\omega + \frac{1}{2}\xi h * h \\ &+ \lambda_{\overline{\omega}A\omega} d\overline{\omega} * [A,\omega] \Bigg] - \langle d\omega, K_A\rangle + \lambda_{A\omega K_A}\langle [A,\omega], K_A\rangle + \langle h, K_{\overline{\omega}}\rangle + \frac{1}{2}\lambda_{\omega^2 K_\omega}\langle [\omega,\omega], K_\omega\rangle \end{aligned} \tag{2.1}$$

where $*$ denotes the Hodge star operator and ξ is the so-called gauge fixing (real) parameter. Also $\langle \cdot,\cdot \rangle$ denotes the pairing between 1-forms and vector fields (or 0-forms and 0-forms). In contrast with the usual formula for the action in the literature, we have inserted the different coupling constants λ_v for each of the interaction monomials in the action. We will now show that validity of the master equation $(S,S) = 0$ implies that all these coupling constants are expressed in terms of one single coupling.

[2] We suppose that there exists such a vertex; if not, the construction works equally well by choosing the vertex of lowest valence that is present in the set R_V.

First, using Eq. (1.3) we derive from the above expression the BRST-differential on the generators

$$sA = -d\omega - \lambda_{A\omega K_A}[A,\omega], \qquad s\omega = -\frac{1}{2}\lambda_{\omega^2 K_\omega}[\omega,\omega], \qquad s\overline{\omega} = -h, \qquad sh = 0$$

The BRST-differential is extended to all of $\mathrm{Loc}^{(p,q)}(\Phi)$ by the graded Leibniz rule, and imposing it to anti-commute with the exterior derivative d. Actually, rather than on $\mathrm{Loc}(\Phi)$, the BRST-differential is defined on the algebra $\mathbb{C}[[\lambda_{A^3}, \lambda_{A^4}, \lambda_{\overline{\omega}A\omega}, \lambda_{A\omega K_A}, \lambda_{\omega^2 K_\omega}]] \otimes \mathrm{Loc}(\Phi)$. However, in order not to loose ourselves in notational complexities, we denote this tensor product by $\mathrm{Loc}(\Phi)$ as well.

Now, validity of the master equation implies that $s^2 = 0$. One computes using the graded Jacobi identity that

$$s^2(A) = \left(\lambda_{A\omega K_A} - \lambda_{\omega^2 K_\omega}\right)[d\omega,\omega] + \frac{1}{2}\left(\lambda^2_{A\omega K_A} - \lambda_{A\omega K_A}\lambda_{\omega^2 K_\omega}\right)[A,[\omega,\omega]].$$

from which it follows that $\lambda_{A\omega K_A} = \lambda_{\omega^2 K_\omega}$. Thus, with this relation the s becomes a differential, and forms – together with the exterior derivative – the previously described bicomplex in which $s \circ d + d \circ s = 0$.

Next, the master equation implies that $sS_0 = 0$ and a lengthy computation yields for the first three terms in S_0 that

$$\begin{aligned} s\left(-dA * dA - \lambda_{A^3} dA * [A,A] - \frac{1}{4}\lambda_{A^4}[A,A]*[A,A]\right) = \\ 2\left(\lambda_{A\omega K_A} - \lambda_{A^3}\right) dA * [A, d\omega] + (\lambda_{A^4} - \lambda_{A^3}\lambda_{A\omega K_A})[d\omega, A] * [A,A] \\ + \lambda_{A\omega K_A}\left(-dA * dA - \lambda_{A^3} dA * [A,A] - \frac{1}{4}\lambda_{A^4}[A,A]*[A,A]\ ,\ \omega\right). \end{aligned}$$

The last term is a commutator on which the trace vanishes and one is thus left with the equalities $\lambda_{A\omega K_A} = \lambda_{A^3}$ and $\lambda_{A^4} = \lambda_{A^3}\lambda_{A\omega K_A}$. The remaining terms in S_0 yield under the action of s

$$\begin{aligned} s\left(-A * dh + d\overline{\omega} * d\omega + \frac{1}{2}\xi h * h + \lambda_{\overline{\omega}A\omega} d\overline{\omega} * [A,\omega]\right) = \\ (\lambda_{A\omega K_A} - \lambda_{\overline{\omega}A\omega})[A,\omega] * dh + (\lambda_{\omega^2 K_\omega} - \lambda_{\overline{\omega}A\omega}) d\overline{\omega} * [d\omega,\omega]. \end{aligned}$$

Thus, the master equation implies $\lambda_{A\omega K_A} = \lambda_{\overline{\omega}A\omega}$ and $\lambda_{\omega^2 K_\omega} = \lambda_{\overline{\omega}A\omega}$.

Finally, if we write $g = \lambda_{A^3}$, the master equation implies that

$$\lambda_{A^4} = g^2 \text{ and } \lambda_{\overline{\omega}A\omega} = \lambda_{A\omega K_A} = \lambda_{\omega^2 K_\omega} = g. \tag{2.2}$$

This motivates our definition of a simple theory in Section 1.1 above. Imposing these relations reduces the action S to the usual

$$S = \int_M \mathrm{tr}\left[- F * F - A * dh + d\overline{\omega} * d\omega + g d\overline{\omega} * [A,\omega] + \frac{1}{2}\xi h * h + sA * K_A + s\omega * K_\omega + s\overline{\omega} * K_{\overline{\omega}}\right]$$

with the field strength F given by $F = dA + \frac{g}{2}[A,A]$ and the BRST-differential now given by

$$sA = -d\omega - g[A,\omega], \qquad s\omega = -\frac{1}{2}g[\omega,\omega], \qquad s\overline{\omega} = -h, \qquad sh = 0.$$

The extension to include fermions is straightforward, leading to similar expressions of the corresponding coupling constants in terms of g.

2.1. The action of G_R. As alluded to before, when the counterterm map – seen as an element in G_R – acts on the action S, it coincides with wave function renormalization. Let us make this precise in the present case. Clearly, wave function renormalization is given by the following factors:

$$Z_A = \gamma_-(z)(G); \quad Z_\omega = Z_{\overline{\omega}} = \gamma_-(z)(G).$$

With this definition and Theorem 36 we find that $\gamma_-(z)$ acts as

$$\gamma_-(z)\cdot(dA*dA) = \gamma_-(z)\left((C^A)^2\right) dA*dA = Z_A\ dA*dA$$
$$\gamma_-(z)\cdot(d\overline{\omega}*d\omega) = \gamma_-(z)(C^\omega C^{\overline{\omega}})d\overline{\omega}*d\omega = Z_\omega\ d\overline{\omega}*d\omega$$

by definition of the C^ϕ's. This is precisely wave function renormalization for the gluon and ghost fields. Thus, renormalizing through the coefficients $\gamma_-(z)(C^\phi)$ – although more appropriate for the BV-formalism – is completely equivalent to the usual wave function renormalization (see also [**1**, Section 6]).

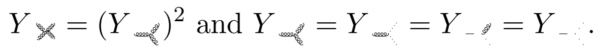
By construction, the terms $-A*dh$ and $\langle h, K_{\overline{\omega}}\rangle$ do not receive radiative corrections. Indeed, this follows from the relations:

$$C^bC^A = 1; \qquad C^{K_{\overline{\omega}}}C^b = 1,$$

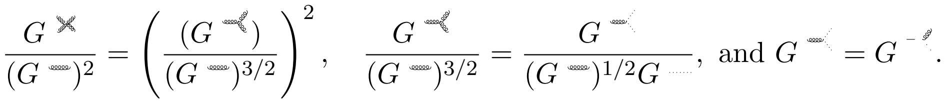
in H_R. Consequently, G_R – and in particular the counterterm map $\gamma_-(z)$ – acts as the identity on these monomials.

In fact, one realizes that $S_0 = \gamma_-(z)\cdot S$ is the *renormalized action*, and since $\gamma_-(z)\in G_R$ acts as a BV-algebra map, also S_0 satisfies the master equation $(S_0, S_0) = 0$.

2.2. The Slavnov–Taylor identities. We now use Theorem 43 to obtain the relations between the Green's function in Yang–Mills equations that are induced by the above master equation $(S,S) = 0$. In fact, the action S defines a simple theory in the sense defined before and Equation (2.2) implies that the following relations hold in the quotient Hopf algebra H_R/J':



In terms of the Green's functions the most relevant read



These are precisely the Slavnov–Taylor identities for the coupling constants for pure Yang–Mills theory with a simple Lie group.

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