Noncommutative instantons from twisted conformal symmetries

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Outline

- Connections on (right) $\mathcal{A}$-modules: $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ satisfying Leibniz rule; curvature $\mathsf{F} = \nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$

- Gauge transformations: unitaries in $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$;

  Infinitesimal gauge transformations: $\mathsf{X} \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ s.t. $\mathsf{X}^* = -\mathsf{X}$

- Toric noncommutative manifolds: isometrical action $\mathbb{T}^n$ on Riemannian manifold M: this torus is deformed to a noncommutative torus $\mathbb{T}^n_\theta$ $\longrightarrow$ nc manifold M_θ, described by algebra $C^\infty(M_\theta)$

  - quantization map $L_\theta : C^\infty(M) \to C^\infty(M_\theta)$
  - nc spin geometry: $(C^\infty(M_\theta), \mathcal{H}, D)$ isospectral deformation
  - nc vector bundles given by finite projective $C^\infty(M_\theta)$-modules
  - differential calculus $(\Omega(M_\theta), d)$

- Yang-Mills theory on M_θ $(\dim M = 4)$
  - Yang-Mills action: $\int *_\theta F *_\theta F$
  - Instantons: connection with $*_\theta F = \pm F$; minima of YM action

- Example: YM-theory on S^4_θ
  - noncommutative $SU(2)$-Hopf fibration $S^7_{\theta'} \to S^4_\theta$
  - associated modules $S^7_{\theta'} \times_{SU(2)} V$ and index of twisted Dirac operators
  - basic (charge 1) instanton for $V = \mathbb{C}^2$, gauge potential ω

- Action of $\mathcal{U}_\theta(so(5))$ leave basic instanton gauge potential invariant

  Action of $\mathcal{U}_\theta(so(5,1))$ generates 5-dimensional collection of (infinitesimal) instantons

- An index theoretical argument shows that the dimension of the 'tangent' to the moduli space of instantons on S^4_θ (at the basic instanton) equals five, showing that the above set of infinitesimal instantons is complete

Connections on modules

Let $\mathcal{A}$ be an $*$-algebra with a differential calculus $(\Omega\mathcal{A} = \oplus_p \Omega^p \mathcal{A}, \mathrm{d})$.
A connection on a right $\mathcal{A}$-module $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A} \ ,$$

which satisfies Leibniz rule: $\nabla(\eta a) = (\nabla\eta)a + \eta \otimes_{\mathcal{A}} \mathrm{d}a$.
The curvature $F = \nabla^2$ is then an $\mathcal{A}$-linear map $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2 \mathcal{A}$ and it satisfies the Bianchi identity $[\nabla, F] = 0$.

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Finite projective module $\mathcal{E} = p\mathcal{A}^N$ $(p = p^2 = p^* \in M_N(\mathcal{A}))$:
Grassmann connection $\nabla_0 = p\mathrm{d}$ with curvature $F_0 = p\mathrm{d}p\mathrm{d}p$.

Leibniz rule $\Rightarrow$ any connection on $\mathcal{E}$ of the form $\nabla = \nabla_0 + \alpha$ with $\alpha \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A})$.

A Hermitian structure is a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ s.t.

$$\langle \eta a, \xi \rangle = a^* \langle \xi, \eta \rangle \ ,$$
$$\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle \ ,$$
$$\langle \eta, \eta \rangle \geq 0 \ , \langle \eta, \eta \rangle = 0 \iff \eta = 0 \ ,$$

which can be extended to $\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$ by:

$$\langle \eta \otimes_{\mathcal{A}} \omega, \xi \otimes_{\mathcal{A}} \rho \rangle \ = (-1)^{|\eta||\omega|} \omega^* \langle \eta, \xi \rangle \rho \ , \quad \forall \ \eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A} \ , \ \omega, \rho \in \Omega\mathcal{A}.$$

A connection ∇ on $\mathcal{E}$ and a Hermitian structure $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ are said to be compatible if

$$\langle \nabla\eta, \xi \rangle + \langle \eta, \nabla\xi \rangle = \mathrm{d} \langle \eta, \xi \rangle \ , \quad \forall \ \eta, \xi \in \mathcal{E} \ .$$

For $\nabla = p\mathrm{d} + \alpha$ this means that $\alpha^* = -\alpha$.

Gauge transformations

Let $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ denote all $\mathcal{A}$-linear (adjointable) maps. The group of gauge transformations is given by:

$$\mathcal{U}(\mathcal{E}) := \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \mid uu^* = u^*u = \mathrm{id}_{\mathcal{E}}\} .$$

It acts on connections by: $(u, \nabla) \mapsto \nabla^u := u^*\nabla u$ and on the corresponding curvature by: $(u, F) \mapsto F^u = u^*Fu$.

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Infinitesimal gauge transformations are given by an element $X \in \mathrm{End}(\mathcal{E})$ s.t. $X^* = -X$. (follows from imposing unitarity of $u = \mathrm{id}_{\mathcal{E}} + tX$ *up to first order* in t). They act on connection and curvature by

$$(X, \nabla) \mapsto \nabla^X = \nabla + [\nabla, X] ; \qquad (X, F) \mapsto F^X = F + [F, X] .$$

Toric noncommutative manifolds

Let M a compact Riemannian spin manifold ($\dim M = m$) with a smooth isometrical action of the n-torus $\mathbb{T}^n$.
$\longrightarrow$ action σ_s of $\mathbb{T}^n$ on $C^\infty(M)$ by automorphisms: $\sigma_s(f)(x) = f(s^{-1} \cdot x)$.

Decomposition $f = \sum f_r$ in homogeneous elements of degree r, i.e.

$$\sigma_s(f_r) = e^{2\pi i r \cdot s} f_r \ .$$

$C^\infty(M)$ is represented on Hilbert space $\mathcal{H} = L^2(M, \mathcal{S})$ of spinors by pointwise multiplication: $\pi : C^\infty(M) \to \mathcal{B}(\mathcal{H})$. There is a double cover $c : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a representation U of $\widetilde{\mathbb{T}}^n$ such that

$$U(s) D U(s)^{-1} = D \ ,$$

$$U(s)\pi(f)U(s)^{-1} = \pi(\sigma_{c(s)}(f)) \ .$$

Given any real $n \times n$ anti-symmetric matrix $\theta = (\theta_{\mu\nu})$ a twisted representation L_θ of $C^\infty(M)$ is defined by

$$L_\theta(f) = \sum_r f_r U(r_\mu \theta_{\mu 1}, \ldots, r_\mu \theta_{\mu n}) \ .$$

Set $C^\infty(M_\theta) := L_\theta(C^\infty(M))$;

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L_θ can be understood as a quantization map $L_\theta : C^\infty(M) \longrightarrow C^\infty(M_\theta)$ satisfying $L_\theta(f \times_\theta g) = L_\theta(f) L_\theta(g)$ with on homogeneous elements:

$$f_r \times_\theta g_{r'} = f_r \sigma_{r \cdot \theta}(g_{r'}) = e^{2\pi i r \cdot \theta \cdot r'} f_r g_{r'} \ .$$

The triple $(C^\infty(M_\theta), \mathcal{H}, D)$ satisfies all properties of a noncommutative spin geometry of dim m: isospectral deformation of spin geometry of M [CL01].

$\longrightarrow$ Noncommutative integral as a Dixmier trace: $⨍ L_\theta(f) = \mathrm{Tr}_\omega f|D|^{-m}$.

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Also, Connes-Moscovici local index formula takes a simple form. For a projection $p \in M_N(C^\infty(M_\theta))$, the index of the twisted Dirac operator $D_p = pDp$ is given by:

$$\mathrm{Index}\ D_p = \operatorname*{Res}_{z=0} z^{-1}\,\mathrm{tr}\left(\gamma p|D|^{-2z}\right) + \sum_{k\geq 1} c_k \operatorname*{Res}_{z=0}\mathrm{tr}\left(\gamma\big(p - \frac{1}{2}\big)[D, p]^{2k}|D|^{-2(k+z)}\right)$$

where $c_k = (k-1)!/(2k)!$.

Vector bundles on M_θ

A σ-equivariant vector bundle E on M is a vector bundle carrying an action V of $\mathbb{T}^n$ covering the action σ:

$$V_s(f\psi) = \sigma_s(f)V_s(\psi); \qquad f \in C^\infty(M),\ \psi \in \Gamma(M, E)\ .$$

The $C^\infty(M_\theta)$-bimodule $\Gamma(M_\theta, E)$ is defined as vector space $\Gamma(M, E)$ but with bimodule structure given by

$$f \triangleright_\theta \psi = \sum_k f_k V_{k\cdot\theta}(\psi); \qquad \psi \triangleleft_\theta f = \sum_k V_{-k\cdot\theta}(\psi) f_k$$

$\Gamma(M_\theta, E)$ is finite projective as right $C^\infty(M_\theta)$-module [CD02]
$\longrightarrow$ 'noncommutative vector bundle on M_θ'.

Differential calculus on M_θ

Let $(\Omega(M), \mathrm{d})$ be the usual differential calculus on M. Extend the map $L_\theta : C^\infty(M) \to C^\infty(M_\theta)$ to $\Omega(M)$ by imposing it to commute with d. The image $L_\theta(\Omega(M))$ will be denoted by $\Omega(M_\theta)$.

Similarly, there is a Hodge star operator on $\Omega(M_\theta)$ defined by

$$*_\theta L_\theta(\omega) = L_\theta(*\omega) \ .$$

with $* : \Omega^p(M) \to \Omega^{m-p}(M)$ the classical Hodge star operator.

$\longrightarrow$ inner product on $\Omega(M_\theta)$:

$$(\alpha, \beta)_2 = \int\!\!\!\!\!- *_\theta(\alpha^* *_\theta \beta)$$

since $*_\theta(\alpha^* *_\theta \beta)$ is an element in $C^\infty(M_\theta)$.

Yang-Mills theory on M_θ

Recall: a connection on a right $\mathcal{A}(M_\theta)$-module $\mathcal{E}$ is a map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}(M_\theta)} \Omega^1(M_\theta)$$

satisfying Leibniz rule, having curvature $F = \nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}(M_\theta)} \Omega^2(M_\theta)$.

Definition. *The Yang-Mills action for a connection* ∇ *on a finite projective* $\mathcal{A}(M_\theta)$*-module* $\mathcal{E}$ *with curvature* F *is defined by* $S = \int \mathrm{tr} *_\theta (F *_\theta F)$.

Equations of motion: noncommutative Yang-Mills equations $[\nabla, *_\theta F] = 0$.

Bianchi identiy $[\nabla, F] = 0 \implies$ connections with (anti)selfdual curvature $*_\theta F = \pm F$ (instantons) are solutions of the YM equations; absolute minima of YM-action.

The sphere S^4_θ

With θ a real parameter, the algebra $\mathcal{A}(S^4_\theta)$ of polynomial functions on the sphere S^4_θ is generated by elements $z_0 = z_0^*, z_j, z_j^*$, $j = 1, 2$, subject to

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu z_\mu^*, \quad z_\mu^* z_\nu^* = \lambda_{\mu\nu} z_\nu^* z_\mu^*, \quad \mu, \nu = 0, 1, 2,$$

with deformation parameters given by

$$\lambda_{12} = \bar{\lambda}_{21} =: \lambda = e^{2\pi i\theta}, \quad \lambda_{j0} = \lambda_{0j} = 1, \quad j = 1, 2$$

and together with the spherical relation $\sum_\mu z_\mu^* z_\mu = 1$.

– Isospectral deformation: nc spin geometry $(\mathcal{A}(S^4_\theta), \mathrm{H}, \mathrm{D})$ of dim 4;

– Differential calculus $(\Omega(S^4_\theta), \mathrm{d})$ as before, with $*_\theta : \Omega^p(S^4_\theta) \to \Omega^{4-p}(S^4_\theta)$.

The sphere $S^7_{\theta'}$

With $\lambda'_{ab} = e^{2\pi i \theta'_{ab}}$ and (θ'_{ab}) a real antisymmetric matrix, the algebra $\mathcal{A}(S^7_{\theta'})$ is generated by elements ψ_a, ψ_a^*, $a = 1, \ldots, 4$, subject to

$$\psi_a \psi_b = \lambda'_{ab} \psi_b \psi_a, \quad \psi_a \psi_b^* = \lambda'_{ba} \psi_b^* \psi_a,$$

and the spherical relation:

$$\sum_a \psi_a^* \psi_a = 1.$$

- Differential calculus $(\Omega(S^7_{\theta'}), \mathrm{d})$ as before.

Noncommutative Hopf fibration

A minimal requirement for $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ to constitute a noncommutative $SU(2)$–prinicipal bundle is that

there is an action α of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ such that $\mathcal{A}(S^4_\theta)$ can be identified with the subalgebra of invariant elements under this action

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These conditions express θ' in terms of θ and we identify:

$$
\begin{aligned}
z_0 &= \psi_1^* \psi_1 + \psi_2^* \psi_2 - \psi_3^* \psi_3 - \psi_4^* \psi_4 \\
z_1 &= 2(\mu \psi_3^* \psi_1 + \psi_2^* \psi_4) \\
z_2 &= 2(-\mu \psi_4 \psi_1^* + \psi_2 \psi_3^*)
\end{aligned}
\qquad
\theta'_{ab} = \frac{\theta}{2} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}
$$

where $\mu = \sqrt{\lambda} = e^{\pi i \theta}$.

Associated modules

We associate $\mathcal{A}(S^4_\theta)$-modules to the noncommutative principal bundle $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ by all (f.d) representations of $SU(2)$. Let ρ be a representation of $SU(2)$ on $V^{(n)} = \mathbb{C}^{n+1}$:

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)} := \Big\{ f \in \mathcal{A}(S^7_{\theta'}) \otimes V^{(n)} \;:\; (\alpha_w \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho(w)^{-1})(f) \Big\}$$

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There are projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ s.t. $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)} \simeq p_{(n)} \mathcal{A}(S^4_\theta)^{4^n}$.

$\rightarrow$ twisted Dirac operator $D_{(n)} = p_{(n)} D p_{(n)}$ with coefficients in the module $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)}$. Its index ($\dim\ker\ - \dim\mathrm{coker}$) equals: $\frac{1}{6} n(n+1)(n+2)$

Basic (charge 1) instanton on S^4_θ

A generic element in the module $\mathcal{E} = \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ can be written as $\Psi^* f$, $f \in \mathcal{A}(S^4_\theta)^4$ with

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \\ \psi_3 & -\psi_4^* \\ \psi_4 & \psi_3^* \end{pmatrix}; \qquad \text{satisfying } \Psi^*\Psi = \mathbb{I}_2.$$

Thus, $p = \Psi\Psi^*$ is a projection in $M_4(\mathcal{A}(S^4_\theta))$ and in fact $\mathcal{E} \simeq p\mathcal{A}(S^4_\theta)^4$. Explicitly:

$$p = \frac{1}{2} \begin{pmatrix} 1+z_0 & 0 & z_1 & -\bar{\mu} z_2^* \\ 0 & 1+z_0 & z_2 & \mu z_1^* \\ z_1^* & z_2^* & 1-z_0 & 0 \\ -\mu z_2 & \bar{\mu} z_1 & 0 & 1-z_0 \end{pmatrix}$$

Grassmann connection $\nabla_0 = p\mathrm{d} \to$ curvature satisfies $*_\theta F_0 = F_0$: basic instanton on S^4_θ.

In terms of $f \in \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ we have

$$(\nabla_0 f)_i = \mathrm{d}f_i + \omega_{ij} f_j.$$

where $\omega = \Psi^* \mathrm{d}\Psi$ is the basic instanton gauge potential, a 2×2-matrix with entries in $\Omega^1(S^7_{\theta'})$ satisfying:

$$\overline{\omega_{ij}} = -\omega_{ji}; \qquad \sum_i \omega_{ii} = 0.$$

Symmetry of the basic instanton

Consider a deformation $\mathcal{U}_\theta(so(5))$ of the universal enveloping algebra of the Lie algebra $so(5)$, defined as the Hopf algebra:

1. $\mathcal{U}_\theta(so(5))$ is $\mathcal{U}(so(5)$ as an algebra:
   - Generators H_1, H_2, E_r for the eight roots $r = (\pm1, \pm1), (0, \pm1), (\pm1, 0)$;
   - Relations given by Lie brackets of $so(5)$.

2. Counit $\epsilon(H_j) = \epsilon(E_r) = 0$, but twisted coproduct and antipode:

$$\Delta_\theta(E_r) = E_r \otimes \lambda^{-r_1 H_2} + \lambda^{-r_2 H_1} \otimes E_r,$$

$$\Delta_\theta(H_j) = H_j \otimes \mathbb{I} + \mathbb{I} \otimes H_j.$$

$$S(E_r) = -\lambda^{r_2 H_1} E_r \lambda^{r_1 H_2}, \quad S(H_j) = -H_j$$

$\mathcal{U}_\theta(so(5))$ acts on the generators of the algebra $\mathcal{A}(S^4_\theta)$ by the following operators:

$$H_1 = z_1\partial_1 - z_1^*\partial_1^*, \qquad H_2 = z_2\partial_2 - z_2^*\partial_2^*$$

$$E_{+1,+1} = z_2\partial_1^* - z_1\partial_2^*, \qquad E_{+1,-1} = z_2^*\partial_1^* - z_1\partial_2\,,$$

$$E_{+1,0} = \frac{1}{\sqrt{2}}(2z_0\partial_1^* - z_1\partial_0), \qquad E_{0,+1} = \frac{1}{\sqrt{2}}(2z_0\partial_2^* - z_2\partial_0)\,,$$

and extended to the whole of $\mathcal{A}(S^4_\theta)$ as twisted derivations [Sit01]:

$$E_r(ab) = m\Delta_\theta(E_r)(a\otimes b) = E_r(a)\lambda^{-r_1H_2}(b) + \lambda^{-r_2H_1}(a)E_r(b),$$

$$H_j(ab) = m\Delta_\theta(H_j)(a\otimes b) = H_j(a)b + aH_j(b),$$

The action of $\mathcal{U}_\theta(so(5))$ can be lifted to an action on $\mathcal{A}(S^7_{\theta'})$;

When extended to $\Omega^1(S^7_{\theta'})$, one finds that $\boxed{H_j(\omega) = E_r(\omega) = 0}$

Action of $\mathcal{U}_\theta(so(5,1))$ on basic instanton

- $so(5,1)$ consists of the generators of $so(5)$ and generators H_0, F_r with $r = (\pm 1, 0), (0, \pm 1)$.

- $\mathcal{U}_\theta(so(5,1))$ is $\mathcal{U}(so(5,1))$ as an algebra; but coproduct and antipode become twisted:

$$\Delta_\theta(F_r) = F_r \otimes \lambda^{-r_1 H_2} + \lambda^{-r_2 H_1} \otimes F_r, \qquad S(F_r) = -\lambda^{r_2 H_1} F_r \lambda^{r_1 H_2},$$
$$\Delta_\theta(H_0) = H_0 \otimes 1 + 1 \otimes H_0, \qquad S(H_0) = -H_0,$$

and the counit: $\varepsilon(F_r) = \varepsilon(H_0) = 0$.

$\longrightarrow$ Hopf algebra $\mathcal{U}_\theta(so(5,1))$

Action of $\mathcal{U}_\theta(so(5,1))$ on the generators of $\mathcal{A}(S^4_\theta)$:

$$H_0 = \partial_0 - z_0(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*),$$
$$F_{1,0} = 2\partial_1^* - z_1(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + \bar{\lambda} z_2\partial_2 + \lambda z_2^*\partial_2^*),$$
$$F_{0,1} = 2\partial_2^* - z_2(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*),$$

which are extended to the whole of $\mathcal{A}(S^4_\theta)$ as twisted derivations:

$$F_r(ab) = F_r(a)\lambda^{-r_1 H_2}(b) + \lambda^{-r_2 H_1}(a)F_r(b),$$
$$H_0(ab) = H_0(a)b + aH_0(b).$$

Equivalently [Sit01], $T \in \mathcal{U}_\theta(so(5,1))$ with classical counterpart $t \in \mathcal{U}(so(5,1))$ is defined to act on $\mathcal{A}(S^4_\theta) = L_\theta(\mathcal{A}(S^4))$ by

$$T \cdot L_\theta(a) = L_\theta(t \cdot a)$$

- Action extended to $\Omega(S^4_\theta)$ by defining it to commute with d

- $\mathcal{U}_\theta(so(5,1))$ leaves Hodge $*_\theta$-structure of $\Omega(S^4_\theta)$ invariant: $T(*_\theta\omega) = *_\theta(T\omega)$

- Action lifted to $\mathcal{A}(S^7_{\theta'})$ and $\Omega(S^7_{\theta'})$;

  - Instanton gauge potential $\omega = \Psi^* \mathrm{d}\Psi$ transforms (infinitesimally) to $\omega + t\delta\omega_i$, $i = 0, \ldots, 4$ under tH_0, tF_r $(t \in \mathbb{R})$.

- Curvature F_0 of basic instanton transforms to $F_0 + t\delta F_i + \mathcal{O}(t^2)$ with $\delta F_0 = -2z_0 F_0$,

$$\delta F_1 = -2z_1 \lambda^{H2} F_0; \qquad \delta F_3 = -2z_1^* \lambda^{-H2} F_0;$$
$$\delta F_2 = -2z_2 \lambda^{H1} F_0; \qquad \delta F_4 = -2z_2^* \lambda^{-H1} F_0.$$

$\Longrightarrow$ Connections $\nabla_{t,i} = \nabla_0 + t\delta\omega_i$ are infinitesimal instantons

Completeness

[AHS78] Starting with the basic instanton ∇_0 on $\mathcal{E}$, any other $(su(2))$ connection on $\mathcal{E}$ is given by $\nabla_0 + t\alpha$, with

$$\alpha \in \Omega^1(\mathrm{ad}(S^7_{\theta'})) := \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma(\mathrm{ad}(S^7_{\theta'}))$$

where $\Gamma(\mathrm{ad}(S^7_{\theta'}))$ is the associated module to the adjoint representation of $SU(2)$ on $su(2)$.

- Linearized selfdual equation: $P_-[\nabla_0, \alpha] = 0$ with $P_- = \frac{1}{2}(1 - *_\theta)$.

- If α were obtained from an infinitesimal gauge transformation, then $\alpha = [\nabla_0, X]$ with $X \in \Gamma(\mathrm{ad}(S^7_{\theta'}))$.

- Since $P_-[\nabla_0, [\nabla_0, X]] = [P_-F_0, X] = 0$, we have the selfdual complex

$$0 \to \Omega^0(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{[\nabla_0,\cdot]} \Omega^1(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{P_-[\nabla_0,\cdot]} \Omega^2_-(\mathrm{ad}(S^7_{\theta'})) \to 0$$

and look for an element in the first cohomology group H^1

- We compute the alternating sum $h^0 - h^1 + h^2$ of the dimensions of the cohomology groups as the index of a twisted Dirac operator. Using a vanishing argument for h^0 and h^2, we find that $h^1 = 5$.

$\Longrightarrow$ **The collection $\nabla_{t,i}$ of infinitesimal instantons is complete**