Noncommutative tori and the Riemann-Hilbert correspondence

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Outline

Goal: Study interplay noncommutative differential and noncommutative algebraic geometry by means of an example.

**Common principle: trade spaces for algebras**

Examples:

- affine varieties \( \leftrightarrow \) finitely generated domains
  
  \[ X \leftrightarrow A(X) \]

- top. Hausdorff spaces \( \leftrightarrow \) commutative \( C^* \)-algebras

  \[ X \leftrightarrow C(X) \]

- Relax commutativity: \[ \text{noncommutative differential geometry} \]

- Applications of NCG in ordinary geometry: use noncommutative algebra to describe “bad quotients”.
Example 1: Noncommutative torus

- Leaf space of Kronecker foliation of torus: non-Hausdorff space $S^1/\theta \mathbb{Z}$ with $\theta$ irrational.
- Better description by crossed product $A_\theta = C(S^1) \rtimes_\theta \mathbb{Z}$. Explicitly: $A_\theta$ is generated by two unitaries $U_1, U_2$ such that
  \[ U_1 U_2 = e^{2\pi i \theta} U_2 U_1 \]
- Two corresponding derivations: $\delta_1, \delta_2$:
  \[ \delta_j(U_k) = 2\pi i \delta_{jk} U_k. \]
- Vector bundles on $S^1/\theta \mathbb{Z}$ are described by finite projective (right) $A_\theta$-modules.
Holomorphic vector bundles on $\mathbb{T}_\theta$.

Let $\delta_\omega := \omega_1 \delta_1 + \omega_2 \delta_2$ with $\omega_1, \omega_2$ two complex numbers.
If $(\omega_1, \omega_2) = (\tau, 1)$ we set $\delta_\tau := \delta_\omega$.

- A **holomorphic vector bundle** is given by a pair $(E, \nabla_\omega)$ with $E$ finite projective $A_\theta$-module and $\nabla_\omega$ a connection associated to the differential $\delta_\omega$.

- Denote by $\text{Vect}(\mathbb{T}_\theta^\omega)$ the category of holomorphic vector bundles.

**Proposition**

(a) *If $g$ is an element in $SL(2, \mathbb{Z})$, then $\text{Vect}(\mathbb{T}_\theta^{g\omega}) \simeq \text{Vect}(\mathbb{T}_\theta^\omega)$.***

(b) *If $\omega_2 \neq 0$ and $\tau = \frac{\omega_1}{\omega_2}$, then $\text{Vect}(\mathbb{T}_\theta^{\omega}) \simeq \text{Vect}(\mathbb{T}_\theta^{\tau})$.***

The category $\text{Vect}(\mathbb{T}_\theta^{\tau})$ turns out to be an abelian category [PS]

(i.e. kernels and cokernels of morphisms $f : (E, \nabla) \to (E', \nabla')$ are in $\text{Vect}(\mathbb{T}_\theta^{\tau})$)
Example 2: Noncommutative elliptic curves

- $\mathbb{C}^*$ with action of $\mathbb{Z}$: $n \cdot z \mapsto q^n z$ for some $q \in \mathbb{C}$.
- Consider the following category of modules over $\mathcal{O}(\mathbb{C}^*)$:

$$\mathcal{B}_q := \left\{ \text{Finitely presentable } \mathbb{Z}-\text{equivariant } \mathcal{O}(\mathbb{C}^*)\text{-modules} \right\}$$

- If $|q| < 1$, $q = e^{2\pi i \tau}$, then $\mathcal{B}_q \simeq \text{Coh}(X_\tau)$ with $X_\tau$ the elliptic curve (i.e. $\mathbb{C}^*/q\mathbb{Z}$) and $\mathcal{F}$ is a coherent sheaf, i.e.

$$\mathcal{O}_{\mathbb{C}^*}^m \rightarrow \mathcal{O}_{\mathbb{C}^*}^n \rightarrow \mathcal{F} \rightarrow 0$$

- Again, if $q = e^{2\pi i \theta}$ with $\theta$ irrational real then the quotient $\mathbb{C}^*/q\mathbb{Z}$ is not Hausdorff but we can define the category of coherent sheaves on $\mathbb{C}^*/q\mathbb{Z}$ by $\mathcal{B}_q$ [SV].

Spirit: Gabriel-Rosenberg Theorem: reconstruct variety from the category of coherent sheaves.

Goal: link these two examples functorially via a category $\mathcal{B}_q^{\tau}$ and study its properties.
The category $\mathcal{B}_q^\tau$

Intermediate category between $\mathcal{B}_q$ and $\text{Vect}(\mathbb{T}_\theta^\tau)$;

Objects of $\mathcal{B}_q^\tau$ are triples $(M, \sigma, \nabla)$:

- $M$: finite presentable $\mathcal{O}(\mathbb{C}^*)$-module, i.e.
  \[
  \mathcal{O}(\mathbb{C}^*)^m \rightarrow \mathcal{O}(\mathbb{C}^*)^n \rightarrow M \rightarrow 0
  \]

- $\sigma$: action of $\theta\mathbb{Z}$ on $M$ that covers the action $\alpha$ on $\mathcal{O}(\mathbb{C}^*)$, i.e.
  \[
  \sigma(m \cdot f) = \sigma(m)\alpha(f)
  \]

- $\nabla$: $\theta\mathbb{Z}$-equivariant connection on $M$ lifting $\delta = \tau zd/dz$, i.e.
  \[
  \nabla(m \cdot f) = \nabla(m) \cdot f + m \cdot \delta(f); \quad \nabla(\sigma(m)) = \sigma(\nabla(m)).
  \]

We impose that $\nabla$ is regular singular so that there exists a module basis $e_k$ for $M$ such that the holomorphic functions $z^{-1}A_{ij}$ defined by $A_{ij}e_j = \nabla(e_i)$ have simple poles at 0.

- The modules $M$ turn out to be free: this follows from the equivariance condition or from the fact that they allow a connection.
- $\mathcal{B}_q^\tau$ is an abelian category.
Relation with the noncommutative torus

Consider the following restriction of a holomorphic function on $\mathbb{C}^*$ to the unit circle, embedded in the smooth noncommutative torus:

$$\psi : \mathcal{O}(\mathbb{C}^*) \rightarrow A_\theta$$

$$\sum_{n \in \mathbb{Z}} f_n z^n \longmapsto \sum_{n \in \mathbb{Z}} f_n U_1^n$$

- This makes $A_\theta$ a $\mathcal{O}(\mathbb{C}^*)$-module: $f \cdot a = \psi(f)a$.
- $\psi$ is injective since a holomorphic function that vanishes on $S^1$, vanishes everywhere on $\mathbb{C}^*$.

Proposition

The association $(M, \sigma, \nabla) \mapsto (\tilde{M}, \tilde{\nabla})$ defines a functor $\psi_*$ from $\mathcal{B}_q^\tau$ to $\text{Vect}(\mathbb{T}_\theta^\tau)$ where

$$\tilde{M} = M \otimes_{\mathcal{O}(\mathbb{C}^*)} A_\theta$$

$$\tilde{\nabla} = 2\pi i \nabla \otimes 1 + 1 \otimes \delta_\tau \quad (\text{with} \ \delta_\tau = \tau \delta_1 + \delta_2).$$
Properties of the functor $\psi_*$

First of all, it turns out that the module $A_\theta$ over $O(C^*)$ (via $\psi$) is flat (i.e. tensor product $\bigotimes_{O(C^*)} A_\theta$ preserves exact sequences). As a corollary to this and to injectivity of $\psi$:

**Proposition**

The functor $\psi_* : B^\tau_q \to Vect(T^\tau_\theta)$ is exact and faithful (i.e. injective on morphisms).

**Summary:**

\[
\begin{array}{ccc}
B^\tau_q & \xrightarrow{\psi_*} & B^\tau \\
\downarrow & & \downarrow \\
B_q & \xleftarrow{\psi_*} & Vect(T^\tau_\theta)
\end{array}
\]

with all arrows faithful and exact functors.
The Riemann-Hilbert correspondence on $\mathbb{C}^*$

Correspondence between linear differential equations on $\mathbb{C}^*$ and representations of the fundamental group $\pi_1(\mathbb{C}^*, z_0) \simeq \mathbb{Z}$.

Categorically,

$$\{\text{vector bundles on } \mathbb{C}^* \text{ with a connection } \nabla\} \xleftrightarrow{\text{}} \{\text{finite dimensional representations of fundamental group}\}.$$ 

The correspondence is given by taking germs of local solutions of the differential equation $\nabla U = 0$ at the point $z_0$; the fundamental group acts on the resulting vector space (monodromy).
The equivariant Riemann-Hilbert correspondence

Theorem

$\mathcal{B}_q^\tau$ is equivalent to the category $\text{Rep}(\mathbb{Z}^2)$ of representations of $\mathbb{Z}^2$.

As a consequence, it is a so-called Tannakian category: a (rigid) tensor category with a fiber functor.

- We define a tensor product on $\mathcal{B}_q^\tau$ as follows:

  $$ (M, \sigma, \nabla) \otimes (N, \sigma', \nabla') = (M \otimes \mathcal{O}(\mathbb{C}^*) N, \sigma \otimes \sigma', \nabla \otimes 1 + 1 \otimes \nabla') $$

- **Rigidity:** dual object
  
  - $M^\vee = \text{Hom}_{\mathcal{O}(\mathbb{C}^*)}(M, \mathcal{O}(\mathbb{C}^*))$
  
  - $\sigma^\vee(f) = \alpha \circ f \circ \sigma^{-1}$
  
  - $\nabla^\vee(f) = \delta \circ f - f \circ \nabla$

Fiber functor:

$$ \omega : \mathcal{B}_q^\tau \rightarrow \text{Vect}_\mathbb{C} $$

$$ (M, \sigma, \nabla) \mapsto (\ker \nabla)_{z_0} $$
\[ \pi^1(C^*, z_0) \cong \mathbb{Z} \]
Proof of Theorem

Proposition

For each object in $\mathcal{B}_q^T$ there is an isomorphic object $(M, \sigma, \nabla)$ with

- $M = V \otimes \mathcal{O}(\mathbb{C}^*)$ with $V$ a vector space
- $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ for an invertible constant matrix $B$.
- $\nabla = \delta + A$ with $A$ a constant matrix with all eigenvalues in the same transversal $T$ to $\theta \mathbb{Z}$ in $\mathbb{C}$.

N.B. If $(M, \sigma, \nabla)$ and $(M', \sigma', \nabla')$ are of this form (with same transversal) then $\text{Hom}_{\mathcal{O}(\mathbb{C}^*)}(M, M') \subset \text{Hom}_{\mathbb{C}}(V, V')$.

This implies that $\omega : \mathcal{B}_q^T \rightarrow \text{Vect}_\mathbb{C}$ is faithful.
Fix a transversal $T$ to $\tau \mathbb{Z}$ in $\mathbb{C}^*$ and let $(V, \rho_1, \rho_2)$ be an object in $\text{Rep}(\mathbb{Z}^2)$.

We can write $\rho_1(1) =: e^{2\pi iA/\tau}$ such that $A$ is the unique matrix with all eigenvalues in $T$.

Also, write $\rho_2(1) =: B$ so that $A$ and $B$ are constant matrices.

They define an element in $\mathcal{B}_q^T$ denoted $F_T(V, \rho_1, \rho_2)$. For a morphism $\phi$ in $\text{Rep}(\mathbb{Z}^2)$ we set $F_T(\phi) = \phi \otimes 1$.

**Proposition**

The above association defines a functor $F_T : \text{Rep}(\mathbb{Z}^2) \rightarrow \mathcal{B}_q^T$ which is fully faithful and essentially surjective.