Noncommutative tori and the Riemann-Hilbert correspondence

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Outline

Goal: Study interplay noncommutative differential and noncommutative algebraic geometry by means of an example.

Common principle: trade spaces for algebras

Examples:

affine varieties	$\leftrightarrow$	finitely generated domains
X		affine coordinate ring $A(X)$
top. Hausdorff spaces	$\leftrightarrow$	commutative C^*-algebras
X		continuous functions $C(X)$

- Relax commutativity: $\begin{cases} \text{noncommutative differential geometry} \\ \text{noncommutative algebraic geometry} \end{cases}$
- Applications of NCG in ordinary geometry: use noncommutative algebra to describe "bad quotients".

Example 1: Noncommutative torus

- Leaf space of Kronecker foliation of torus: non-Hausdorff space $S^1/\theta\mathbb{Z}$ with θ irrational.
- Better description by crossed product $A_\theta = C(S^1) \rtimes_\theta \mathbb{Z}$. Explicitly: A_θ is generated by two unitaries U_1, U_2 such that

$$\boxed{U_1 U_2 = e^{2\pi i\theta} U_2 U_1}$$

- Two corresponding derivations: δ_1, δ_2:

$$\delta_j(U_k) = 2\pi i \delta_{jk} U_k.$$

- **Vector bundles** on $S^1/\theta\mathbb{Z}$ are described by finite projective (right) A_θ-modules.

Holomorphic vector bundles on $\mathbb{T}_\theta$.

Let $\delta_\omega := \omega_1\delta_1 + \omega_2\delta_2$ with ω_1, ω_2 two complex numbers.
If $(\omega_1, \omega_2) = (\tau, 1)$ we set $\delta_\tau := \delta_\omega$.

- A **holomorphic vector bundle** is given by a pair (E, ∇_ω) with E finite projective A_θ-module and ∇_ω a connection associated to the differential δ_ω
- Denote by $\mathrm{Vect}(\mathbb{T}_\theta^\omega)$ the category of holomorphic vector bundles.

Proposition

(a) *If g is an element in $SL(2,\mathbb{Z})$, then* $\mathrm{Vect}(\mathbb{T}_\theta^{g\omega}) \simeq \mathrm{Vect}(\mathbb{T}_\theta^\omega)$.

(b) *If $\omega_2 \neq 0$ and $\tau = \frac{\omega_1}{\omega_2}$, then* $\mathrm{Vect}(\mathbb{T}_\theta^\omega) \simeq \mathrm{Vect}(\mathbb{T}_\theta^\tau)$.

The category $\mathrm{Vect}(\mathbb{T}_\theta^\tau)$ turns out to be an abelian category [PS]

(i.e. kernels and cokernels of morphisms $f : (E, \nabla) \to (E', \nabla')$ are in $\mathrm{Vect}(\mathbb{T}_\theta^\tau)$)

Example 2: Noncommutative elliptic curves

- $\mathbb{C}^*$ with action of $\mathbb{Z}$: $n \cdot z \mapsto q^n z$ for some $q \in \mathbb{C}$.
- Consider the following category of modules over $\mathcal{O}(\mathbb{C}^*)$:

$$\mathcal{B}_q := \left\{ \begin{array}{l} \textbf{Finitely presentable } \mathbb{Z}- \\ \textbf{equivariant } \mathcal{O}(\mathbb{C}^*)\textbf{-modules} \end{array} \right\}$$

- If $|q| < 1$, $q = e^{2\pi i \tau}$, then $\mathcal{B}_q \simeq \mathrm{Coh}(X_\tau)$ with X_τ the elliptic curve (i.e. $\mathbb{C}^*/q^{\mathbb{Z}}$) and $\mathcal{F}$ is a coherent sheaf, i.e.

$$\mathcal{O}^m_{\mathbb{C}^*} \to \mathcal{O}^n_{\mathbb{C}^*} \to \mathcal{F} \to 0$$

- Again, if $q = e^{2\pi i \theta}$ with θ irrational real then the quotient $C^*/q^{\mathbb{Z}}$ is not Hausdorff but we can define the category of coherent sheaves on $\mathbb{C}^*/q^{\mathbb{Z}}$ by $\mathcal{B}_q$ [SV].

Spirit: Gabriel-Rosenberg Theorem: reconstruct variety from the category of coherent sheaves.

Goal: link these two examples functorially via a category $\mathcal{B}_q^\tau$ and study its properties.

The category $\mathcal{B}_q^\tau$

Intermediate category between $\mathcal{B}_q$ and $\mathrm{Vect}(\mathbb{T}_\theta^\tau)$;
Objects of $\mathcal{B}_q^\tau$ are triples (M, σ, ∇):

- M: finite presentable $\mathcal{O}(\mathbb{C}^*)$-module, i.e.

$$\mathcal{O}(\mathbb{C}^*)^m \to \mathcal{O}(\mathbb{C}^*)^n \to M \to 0$$

- σ: action of $\theta\mathbb{Z}$ on M that covers the action α on $\mathcal{O}(\mathbb{C}^*)$, i.e.

$$\sigma(m \cdot f) = \sigma(m)\alpha(f)$$

- ∇: $\theta\mathbb{Z}$-equivariant connection on M lifting $\delta = \tau z d/dz$, i.e.

$$\nabla(m \cdot f) = \nabla(m) \cdot f + m \cdot \delta(f); \qquad \nabla(\sigma(m)) = \sigma(\nabla(m)).$$

We impose that ∇ is regular singular so that there exists a module basis e_k for M such that the holomorphic functions $z^{-1}A_{ij}$ defined by $A_{ij}e_j = \nabla(e_i)$ have simple poles at 0.

- The modules M turn out to be **free**: this follows from the equivariance condition or from the fact that they allow a connection.
- $\mathcal{B}_q^\tau$ is an abelian category.

Relation with the noncommutative torus

Consider the following restriction of a holomorphic function on $\mathbb{C}^*$ to the unit circle, embedded in the smooth noncommutative torus:

$$\psi : \mathcal{O}(\mathbb{C}^*) \to \mathcal{A}_\theta$$
$$\sum_{n\in\mathbb{Z}} f_n z^n \mapsto \sum_{n\in\mathbb{Z}} f_n U_1^n$$

- This makes $\mathcal{A}_\theta$ a $\mathcal{O}(\mathbb{C}^*)$-module: $f \cdot a = \psi(f)a$.
- ψ is injective since a holomorphic function that vanishes on S^1, vanishes everywhere on $\mathbb{C}^*$.

Proposition

The association $(M, \sigma, \nabla) \mapsto (\tilde{M}, \tilde{\nabla})$ *defines a functor* ψ_* *from* $\mathcal{B}_q^\tau$ *to* $\mathrm{Vect}(\mathbb{T}_\theta^\tau)$ *where*

$$\tilde{M} = M \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta$$
$$\tilde{\nabla} = 2\pi i \nabla \otimes 1 + 1 \otimes \delta_\tau \qquad (\textit{with } \delta_\tau = \tau\delta_1 + \delta_2).$$

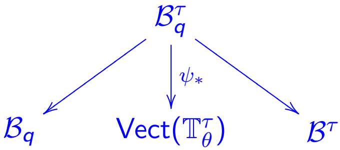
Properties of the functor ψ_*

First of all, it turns out that the module $\mathcal{A}_\theta$ over $\mathcal{O}(\mathbb{C}^*)$ (via ψ) is flat (i.e. tensor product $\_\_ \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta$ preserves exact sequences). As a corollary to this and to injectivity of ψ:

Proposition

The functor $\psi_* : \mathcal{B}_q^\tau \to \mathrm{Vect}(\mathbb{T}_\theta^\tau)$ *is* **exact** *and* **faithful** *(i.e. injective on morphisms).*

Summary:



with all arrows faithful and exact functors.

The Riemann-Hilbert correspondence on $\mathbb{C}^*$

Correspondence between linear differential equations on $\mathbb{C}^*$ and representations of the fundamental group $\pi_1(\mathbb{C}^*, z_0) \simeq \mathbb{Z}$.

Categorically,

$$\{\text{vector bundles on } \mathbb{C}^* \text{ with a connection } \nabla\}$$
$$\updownarrow$$
$$\{\text{finite dimensional representations of fundamental group}\}.$$

The correspondence is given by taking germs of local solutions of the differential equation $\nabla U = 0$ at the point z_0; the fundamental group acts on the resulting vector space (monodromy).

The equivariant Riemann-Hilbert correspondence

Theorem

$\mathcal{B}_q^\tau$ is equivalent to the category $\mathrm{Rep}(\mathbb{Z}^2)$ of representations of $\mathbb{Z}^2$.

As a consequence, it is a so-called Tannakian category: a (rigid) tensor category with a fiber functor.

- We define a tensor product on $\mathcal{B}_q^\tau$ as follows:

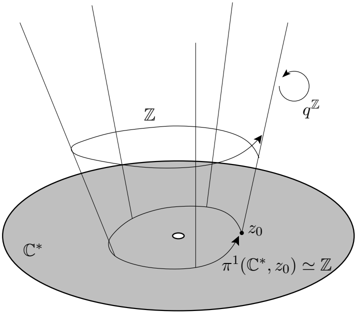
  $(M, \sigma, \nabla) \otimes (N, \sigma', \nabla') = (M \otimes_{\mathcal{O}(\mathbb{C}^*)} N,\ \sigma \otimes \sigma',\ \nabla \otimes 1 + 1 \otimes \nabla')$

- Rigidity: dual object
  - $M^\vee = \mathrm{Hom}_{\mathcal{O}(\mathbb{C}^*)}(M, \mathcal{O}(\mathbb{C}^*))$
  - $\sigma^\vee(f) = \alpha \circ f \circ \sigma^{-1}$
  - $\nabla^\vee(f) = \delta \circ f - f \circ \nabla$.

Fiber functor:

$$\omega : \mathcal{B}_q^\tau \longrightarrow \mathrm{Vect}_\mathbb{C}$$

$$(M, \sigma, \nabla) \mapsto (\ker \nabla)_{z_0}$$



$\mathbb{Z}$
$q^{\mathbb{Z}}$
z_0
$\mathbb{C}^*$
$\pi^1(\mathbb{C}^*, z_0) \simeq \mathbb{Z}$

Proof of Theorem

Proposition

For each object in $\mathcal{B}_q^\tau$ there is an isomorphic object (M, σ, ∇) with

- $M = V \otimes \mathcal{O}(\mathbb{C}^*)$ *with V a vector space*
- $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ *for an invertible constant matrix B.*
- $\nabla = \delta + A$ *with A a constant matrix with all eigenvalues in the same transversal* T *to $\theta\mathbb{Z}$ in $\mathbb{C}$.*

N.B. If (M, σ, ∇) and (M', σ', ∇') are of this form (with same transversal) then $\mathrm{Hom}_{\mathcal{O}(\mathbb{C}^*)}(M, M') \subset \mathrm{Hom}_{\mathbb{C}}(V, V')$.
This implies that $\omega : \mathcal{B}_q^\tau \to \mathrm{Vect}_{\mathbb{C}}$ is faithful.

Equivalence functor

- Fix a transversal T to $\tau\mathbb{Z}$ in $\mathbb{C}^*$ and let (V, ρ_1, ρ_2) be an object in $\mathrm{Rep}(\mathbb{Z}^2)$.
- We can write $\rho_1(1) =: e^{2\pi i A/\tau}$ such that A is the unique matrix with all eigenvalues in T.
- Also, write $\rho_2(1) =: B$ so that A and B are constant matrices.

They define an element in $\mathcal{B}_q^\tau$ denoted $\mathsf{F}_\mathsf{T}(V, \rho_1, \rho_2)$. For a morphism ϕ in $\mathrm{Rep}(\mathbb{Z}^2)$ we set $\mathsf{F}_\mathsf{T}(\phi) = \phi \otimes 1$.

Proposition

The above association defines a functor $\mathsf{F}_\mathsf{T} : \mathrm{Rep}(\mathbb{Z}^2) \to \mathcal{B}_q^\tau$ *which is fully faithful and essentially surjective.*