Noncommutative geometry and particle physics

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Noncommutative manifolds

- Basic device: a spectral triple $(\mathcal{A}, \mathcal{H}, D)$:
  - algebra $\mathcal{A}$ of bounded operators on
  - a Hilbert space $\mathcal{H}$,
  - a self-adjoint operator D with compact resolvent
    such that the commutator $[D, a]$ is bounded for all $a \in \mathcal{A}$.

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- Grading $\gamma : \mathcal{H} \to \mathcal{H}$ such that

$$\gamma^2 = \mathrm{id}, \qquad D\gamma + \gamma D = 0, \qquad \gamma a = a\gamma \quad (a \in A)$$

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$$\gamma^2 = \mathrm{id}, \qquad D\gamma + \gamma D = 0, \qquad \gamma a = a\gamma \quad (a \in A)$$

- Real structure $J : \mathcal{H} \to \mathcal{H}$, anti-unitary operator such that

$$JD = \pm JD, \qquad J\gamma = \pm \gamma J.$$

defining an $\mathcal{A}$-bimodule structure on $\mathcal{H}$ via

$$(a, b) \cdot \psi = aJb^* J^{-1}\psi \quad (\psi \in \mathcal{H})$$

and we require (first order):

$$[[D, a], JbJ^{-1}] = 0$$

Example: Riemannian spin geometry

Let M be an compact m-dimensional Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M)$
- $\mathcal{H} = L^2(S)$, square integrable spinors
- $D = \not{\partial}$, Dirac operator
- $\gamma = \gamma_{m+1}$ if m even (chirality)
- $J = C$ (charge conjugation)

Then D has compact resolvent because $\not{\partial}$ elliptic self-adjoint.
Also $[D, f]$ bounded for $f \in C^\infty(M)$.

Morita equivalence

Suppose $\mathcal{A} \sim_M \mathcal{B}$.

Can we construct a spectral triple on $\mathcal{B}$ from $(\mathcal{A}, \mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$.

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- Definition of operator $D'(\eta, \psi) := \nabla(\eta)\psi + \eta \otimes D\psi$ requires a (compatible) connection on $\mathcal{E}$:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$$

  with respect to the derivation $d := [D, .]$ and the Connes' differential one-forms are

$$\Omega^1_D(\mathcal{A}) = \left\{ \sum_j a_j[D, b_j] : a_j, b_j \in \mathcal{A} \right\}$$

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- Then $(\mathcal{B}, \mathcal{H}', D')$ is a spectral triple [Connes, 1996].

Morita equivalence

with real structure

Again, suppose $\mathcal{A} \sim_M \mathcal{B}$.

- If there is a real structure J on $(\mathcal{A}, \mathcal{H}, D)$, then instead

$$\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}}$$

in terms of the conjugate (left $\mathcal{A}$-) module $\overline{\mathcal{E}}$ and define

$$D'(\eta \otimes \psi \otimes \overline{\rho}) = \nabla(\eta)\psi \otimes \overline{\rho} + \eta \otimes D\psi \otimes \overline{\rho} + \eta \otimes \psi\overline{\nabla}(\overline{\rho})$$

where

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$$
$$\overline{\nabla} : \overline{\mathcal{E}} \to \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \overline{\mathcal{E}},$$

and

$$J' : \mathcal{H}' \to \mathcal{H}', \qquad \eta \otimes \psi \otimes \overline{\rho} \mapsto \rho \otimes J\psi \otimes \overline{\eta}$$

complete the definition of a real spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$.

Morita self-equivalence

- In the case $\mathcal{B} = \mathcal{A}$ (i.e. $\mathcal{E} = \mathcal{A}$) we have of course $\mathcal{H}' \simeq \mathcal{H}$ and $J' \equiv J$.

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- However, the operator D is perturbed to $D' = D_A \equiv D + A \pm JAJ^{-1}$ with $A^* = A \in \Omega^1_D(\mathcal{A})$ the connection one-form (gauge potential) in $\nabla = d + A$. These are the so-called inner fluctuations.

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- The (gauge) group $\mathcal{U}(\mathcal{A})$ of unitary elements in $\mathcal{A}$ acts on $\mathcal{H}$ in the adjoint, i.e. via the unitary $U = uJuJ^{-1}$ for $u \in \mathcal{U}(\mathcal{A})$.

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- The element A is the gauge field and it acts as $A \pm JAJ^{-1}$ on $\mathcal{H}$, that is, in the adjoint representation.

Spectral action principle

Given a (real) spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we define an action functional on $A \in \Omega^1_D(\mathcal{A})$ and $\psi \in \mathcal{H}$ as

$$S_\Lambda[A, \psi] := \mathrm{Tr}\,(f(D_A/\Lambda)) - \mathrm{Tr}\,(f(D/\Lambda)) + \langle \psi, D_A \psi \rangle$$

with f a function on $\mathbb{R}$ (...) and $\Lambda \in \mathbb{R}$ a cut-off.

- Gauge invariance: $S_\Lambda[u^* A u + u^*[D, u], u\psi] = S_\Lambda[A, \psi]$.

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with f a function on $\mathbb{R}$ (...) and $\Lambda \in \mathbb{R}$ a cut-off.

- Gauge invariance: $S_\Lambda[u^* A u + u^*[D, u], u\psi] = S_\Lambda[A, \psi]$.
- The part $\mathrm{Tr}\,(f(D/\Lambda))$ is purely 'gravitational' (this terminology is justified by applying it to the commutative spectral triple associated to M).

Heat kernel expansion

One obtains an explicit expression for

$$\mathrm{Tr}\ (f(D_A/\Lambda))$$

in terms of the heat expansion for the operator $e^{-t(D_A/\Lambda)^2}$.

- Assume simple dimension spectrum for D and a heat expansion

$$\mathrm{Tr}\ e^{-tD_A^2} \sim \sum_\alpha t^\alpha a_\alpha(D_A) \qquad (t \to 0)$$

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- Assume simple dimension spectrum for D and a heat expansion

$$\mathrm{Tr}\; e^{-tD_A^2} \sim \sum_\alpha t^\alpha a_\alpha(D_A) \qquad (t \to 0)$$

- Then write f as a Laplace transform of ϕ

$$\mathrm{Tr}\;(f(D_A/\Lambda)) = \int_{t>0} \phi(t) e^{-t(D_A/\Lambda)^2} dt = \sum_\alpha f_{-\alpha} \Lambda^{-\alpha} a_\alpha(D_A)$$

Example: Yang–Mills theory

Given a compact 4-dimensional Riemannian spin manifold M, consider

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = \partial\!\!\!/ \otimes 1$
- $\gamma = \gamma_5 \otimes 1, \qquad J = C \otimes (.)^*.$

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Proposition (Chamseddine-Connes)

- *The self-adjoint operator $A + JAJ^{-1}$ with $A = A^* \in \Omega^1_D(\mathcal{A})$ describes an $\mathfrak{su}(n)$-valued one-form on M.*
- *The gauge group $\mathcal{U}(\mathcal{A}) \simeq C^\infty(M, U(n))$ acts on $\mathcal{H}$ in the (usual) adjoint representation.*
- *The spectral action is given by*

$$S_\Lambda[A,\psi] = \frac{f(0)}{24\pi^2}\int_M \mathrm{Tr}\; F_A \wedge * F_A + \langle \psi, (\partial\!\!\!/ + \mathrm{ad} A)\psi\rangle + \mathcal{O}(\Lambda^{-1})$$

with F_A the curvature of the connection one-form corresponding to A.

We make two observations.

1. The $\mathfrak{su}(n)$-valued one-form defines a connection one-form on the trivial principal bundle $M \times SU(n)$.

Is there a spectral triple that describes Yang–Mills theory on topologically non-trivial principal bundles?

We make two observations.

1. The $\mathfrak{su}(n)$-valued one-form defines a connection one-form on the trivial principal bundle $M \times SU(n)$.

   Is there a spectral triple that describes Yang–Mills theory on topologically non-trivial principal bundles?

2. With the fermions in the adjoint representation of $\mathcal{U}(\mathcal{A})$, the above action is a candidate for defining a supersymmetric theory.

   How does supersymmetry appear, and can we extend it to physically realistic models? (eg. MSSM)

Geometry of Yang–Mills fields

Let $P \to M$ be a G-principal bundle. A convenient way to define connections on P is through covariant derivatives on the associated bundle(s).

- A covariant derivative (or, connection) on $E = P \times_G V$ is a map

$$\nabla : \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(M)} \Omega^1(M))$$

satisfying the Leibniz rule $\nabla(sf) = \nabla(s)f + s \otimes df$. This implies that $\nabla = \nabla_0 + A$ with $A \in \Gamma^\infty(\mathrm{ad}P) \otimes_{C^\infty(M)} \Omega^1(M)$ for any two ∇, ∇_0.

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- The curvature of ∇ is $F_\nabla := \nabla^2 \in \Gamma^\infty(\mathrm{ad}P) \otimes_{C^\infty(M)} \Omega^2(M)$.
- The gauge group $\mathrm{Aut}_1(P) \simeq \Gamma^\infty(\mathrm{Ad}P)$ acts on ∇

$$\nabla \mapsto g\nabla g^{-1}$$

  and, consequently, $F_\nabla \mapsto gF_\nabla g^{-1}$.

Yang–Mills action

- Given the above, we may define the Yang–Mills action functional (for simplicity, assume $G = U(n)$ or $SU(n)$)

$$S_{YM}[A] = \int_M \mathrm{Tr}\, F_\nabla \wedge *F_\nabla$$

writing $\nabla = \nabla_0 + A$ for some fixed connection ∇_0

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- This describes the dynamics and self-interactions of a single gauge boson (eg. photon, W-boson, gluon, ...)
- Physical matter (fermions) can be included (on a spin manifold) as sections of the tensor product of the spinor bundle S the associated bundles E to P:

$$S_M[A, \psi] = \langle \psi, \gamma \circ \nabla \psi \rangle$$

  (eg. electrons, quarks, ...)

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  (eg. electrons, quarks, ...)
- Example: QCD has $G = SU(3)$. Gluons are $\mathfrak{su}(3)$-valued one-forms on M; quarks are sections of $E = P \times_{SU(3)} \mathbb{C}^3$. Their dynamics and interaction are described by $S_{YM} + S_M$.

Yang–Mills theory (non-trivial)

Let $\mathcal{A}$ be a finitely generated, projective $C^\infty(M)$-module $*$-algebra. Thus, the module structure is compatible with the $*$-algebra structure:

$$f(ab) = (fa)b = a(fb), \qquad (fa)^* = \bar{f} a^*, \qquad \textit{et cetera}.$$

Recall that an algebra bundle $B \to M$ is a vector bundle with an algebra structure on the fibers; also, the local trivializations are algebra maps.

Yang–Mills theory (non-trivial)

Algebra

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Proposition (Serre-Swan for algebra bundles)

There is a one-to-one correspondence between finite rank (involutive) algebra bundles on M and finitely generated projective $C^\infty(M)$-module $(*)$-algebras.

The correspondence being $\mathcal{A} \simeq \Gamma^\infty(M, B)$ for an algebra bundle $B \to M$.

Yang–Mills theory (non-trivial)

Hilbert space and Dirac operator

We define a Hilbert space $\mathcal{H} := \mathcal{A} \otimes_{C^\infty(M)} L^2(M, S)$. Let $\nabla\!\!\!/_0$ be a (compatible) connection on the finitely generated projective module $\mathcal{A}$:

$$\nabla\!\!\!/_0 : \mathcal{A} \to \mathcal{A} \otimes_{C^\infty(M)} \Omega^1_{\partial\!\!\!/}(C^\infty(M))$$

A self-adjoint operator D on $\mathcal{H}$ is defined as $D = \nabla\!\!\!/_0 \otimes 1 + 1 \otimes \partial\!\!\!/$.

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Theorem

- *The set of data $(\mathcal{A}_{C^\infty(M)}, \mathcal{H}, D)$ defines a spectral triple.*

Also, there exists a grading $\gamma = 1 \otimes \gamma_5$ (assuming M even dimensional) and a real structure $J = (.)^* \otimes C$.
Next, we study the inner fluctuations of this spectral triple.

Yang–Mills theory (non-trivial)

Principal bundles

From the transition functions $t_{\alpha\beta}$ of the algebra bundle B (for which $\mathcal{A} \simeq \Gamma^\infty(M, B)$) we build a $SU(n)$-principal bundle:

- Assume for simplicity that the fiber of B is isomorphic to $M_n(\mathbb{C})$.

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- Then $t_{\alpha\beta}(x) \in \mathrm{Aut}(M_n(\mathbb{C})) = SU(n)$

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$$B = P \times_{SU(n)} M_n(\mathbb{C})$$

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- The connection $\not\nabla_0$ defines a covariant derivative ∇_0 on the associated bundle B.
- The inner fluctuations $D \mapsto D' = D + A + JAJ^{-1}$ give rise to connections ∇ on B, such that $D' = \gamma \circ \nabla$. They are parametrized by elements in $\Omega^1(\mathrm{ad}P)$.

Yang–Mills theory (non-trivial)

We collect this in a

Theorem

- $(\mathcal{A}_{C^\infty(M)}, \mathcal{A}\otimes_{C^\infty(M)} L^2(S), D = \nabla\!\!\!/_0 \otimes 1 + 1 \otimes \partial\!\!\!/, \gamma = 1 \otimes \gamma_5, J = (.)^* \otimes C)$ *is an even real spectral triple.*
- *The self-adjoint operator* $A + JAJ^{-1}$ *with* $A = A^* \in \Omega^1_D(\mathcal{A})$ *describes a section of* $\mathrm{ad}P \times \Lambda^1(M)$.
- *The gauge group* $\mathcal{U}(\mathcal{A}) \simeq \Gamma^\infty(\mathrm{Ad}P)$, *and acts on* $\mathcal{H}$ *in the adjoint representation.*
- *The spectral action is given by*

$$S_\Lambda[A,\psi] = \frac{f(0)}{24\pi^2}\int_M \mathrm{Tr}\; F_A \wedge *F_A + \langle \psi, (\partial\!\!\!/ + \mathrm{ad}A)\psi\rangle + \mathcal{O}(\Lambda^{-1})$$

with F_A *the curvature of the connection* ∇ *corresponding to* $D + A + JAJ^{-1}$.

Outlook (Part 1)

- The noncommutative torus for rational θ is of the above type.
- More generally, one can construct from a spectral triple $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ and a (fin.gen.proj.) $\mathcal{A}_0$-module algebra $\mathcal{A}$, equipped with a D_0-connection ∇ another spectral triple

$$(\mathcal{A}, \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{H}, \nabla \otimes 1 + 1 \otimes D_0)$$

  (similar to Morita equivalence)
  Relation to the work of Ćaćić (MPIM,Caltech)?
- Include topological terms through addition of $\mathrm{Tr}\,(\gamma f(D_A/\Lambda))$.

Reference: J. Boeijink. *Noncommutative geometry of Yang–Mills fields*, Master's thesis, Radboud University Nijmegen. (http://www.math.ru.nl/∼waltervs)

Supersymmetric Yang–Mills theory

Again, consider the spectral triple $(C^\infty(M) \otimes M_n(\mathbb{C}), L^2(S) \otimes M_n(\mathbb{C}), \partial\!\!\!/ \otimes 1)$ and the spectral action

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- With the fermions $\psi \in \mathcal{H}$ in the adjoint representation of the gauge group $\mathcal{U}(\mathcal{A})$, it might be possible to exchange $\psi \leftrightarrow A$ (in some way), while leaving the spectral action invariant.

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- First, we need to obtain the correct degrees of freedom:

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  - Instead of $\langle \psi, D_A \psi \rangle$ we consider

$$\langle \widetilde{\chi}, D_A \widetilde{\psi} \rangle$$

    in terms of a anti-chiral $\widetilde{\chi}$ and chiral $\widetilde{\psi}$ (this is in accordance with the usual independent variables $\overline{\psi}$ and ψ in the Lorentzian case [vNW]).

Supersymmetric Yang–Mills theory

Again, consider the spectral triple $(C^\infty(M) \otimes M_n(\mathbb{C}), L^2(S) \otimes M_n(\mathbb{C}), \partial\!\!\!/ \otimes 1)$ and the spectral action

$$S_\Lambda[A, \psi] = \frac{f(0)}{24\pi^2} \int_M \mathrm{Tr}\ F \wedge *F + \langle \psi, D_A \psi \rangle + \mathcal{O}(\Lambda^{-1})$$

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    in terms of a anti-chiral $\widetilde{\chi}$ and chiral $\widetilde{\psi}$ (this is in accordance with the usual independent variables $\overline{\psi}$ and ψ in the Lorentzian case [vNW]).
  - Write $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C}) \simeq L^2(S) \otimes_{\mathbb{R}} \mathfrak{u}(n)$ and

    $$\langle \widetilde{\chi}, D_A \widetilde{\psi} \rangle = \langle \mathrm{Tr}\ \widetilde{\chi}, D \mathrm{Tr}\ \widetilde{\psi} \rangle + \langle \chi, D_A \psi \rangle$$

    where $\widetilde{\psi} = \mathrm{Tr}\ \widetilde{\psi} + \psi$, $\widetilde{\chi} = idem$ is the decomposition according to $\mathfrak{u}(n) = \mathbb{R} \oplus \mathfrak{su}(n)$. Thus, the spinors $\mathrm{Tr}\ \widetilde{\chi}$ and $\mathrm{Tr}\ \widetilde{\psi}$ decouple.

- We restrict the inner product to χ and ψ in $L^2(S) \otimes_{\mathbb{R}} \mathfrak{su}(n)$ and consider

$$S_{SYM}[A, \chi, \psi] = \frac{f(0)}{24\pi^2} \int_M \mathrm{Tr}\ F_A \wedge *F_A + \langle \chi, D_A \psi \rangle$$

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- Consider the following supersymmetry transformations

$$\delta A := c_1 \gamma^\mu \otimes (\epsilon_-, \gamma_\mu \psi) + c_2 \gamma^\mu \otimes (\chi, \gamma_\mu \epsilon_+)$$
$$\delta \psi := c_3 F \epsilon_+ \qquad \qquad \delta \chi := c_4 F \epsilon_- .$$

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Proposition

For certain constants c_i the action functional S_{SYM} is invariant under the supersymmetry transformations:

$$\frac{d}{dt} S_{SYM}[A + t\delta A, \chi + t\delta\chi, \psi + t\delta\psi]\Big|_{t=0} = 0$$

Guided by physics: super-QCD

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- Quarks are fermions in the defining representation of $SU(3)$ rather then in the adjoint representation. We therefore extend our finite-dimensional Hilbert space $M_3(\mathbb{C})$ to

$$V := \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}^3}$$

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  (eventually combined with the real structure on M).
- Thus, the algebra $\mathcal{A} = C^\infty(M) \otimes M_3(\mathbb{C})$ acts on $\mathcal{H} = L^2(S) \otimes V$ and $J = C \otimes J_V$ defines an anti-unitary operator on $\mathcal{H}$.

Deriving the squarks

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- This motivates to replace the operator $\partial\!\!\!/ \otimes 1$ on $\mathcal{H}$ by

$$D = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_V$$

with $D_V : V \to V$ given by

$$D_V := \begin{pmatrix} 0 & d & 0 \\ d^* & 0 & e^* \\ 0 & e & 0 \end{pmatrix}$$

with $d : M_3(\mathbb{C}) \to \mathbb{C}^3, g \mapsto g \cdot v$ and $e : M_3(\mathbb{C}) \to \overline{\mathbb{C}^3}, g \mapsto g^t \cdot \overline{v}$ for some vector $v \in \mathbb{C}^3$.

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Proposition

$(C^\infty(M) \otimes M_3(\mathbb{C}), L^2(S) \otimes V, D, \gamma_5 \otimes 1, J)$ *defines a real, even spectral triple*

Deriving the squarks

Again, consider $(C^\infty(M) \otimes M_3(\mathbb{C}), L^2(S) \otimes V, D, \gamma_5 \otimes 1, J)$.

- The inner fluctuations $D_A = D + A + JAJ^{-1}$ of D can be written as

$$D + \mathbb{A} + \mathbb{A}_{\widetilde{q}}$$

where $\mathbb{A}$ is parametrized by an $u(3)$-valued one-form and $\mathbb{A}_{\widetilde{q}}$ by an element $\widetilde{q} \in C^\infty(M) \otimes \mathbb{C}^3$. In fact, we can write

$$\mathbb{A}_{\widetilde{q}}(q_1, g, \overline{q}_2) = (g\widetilde{q}, \overline{q}_1\overline{\widetilde{q}}^t + \widetilde{q}\overline{q_2}^t, g^t\overline{\widetilde{q}})$$

Deriving the squarks

Inner fluctuations

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Proposition

- *The gauge group $\mathcal{U}(\mathcal{A}) \simeq C^\infty(M, U(3))$ acts on the Hilbert space as:*

$$(q_1, g, \overline{q_2}) \mapsto (uq_1, ugu^*, \overline{uq}_2)$$

- *The gauge transformation $D_A \to UD_AU^*$ transforms the gauge fields as*

$$\mathbb{A} \mapsto u\mathbb{A}u^* + u[D, u^*]; \qquad \mathbb{A}_{\widetilde{q}} \mapsto \mathbb{A}_{u\widetilde{q}}$$

The spectral action

Interestingly, $[\partial\!\!\!/ + \mathbb{A}, \mathbb{A}_{\widetilde{q}}] = \gamma^\mu \mathbb{A}_{(\partial_\mu + \mathbb{A}_\mu)\widetilde{q}}$.

Proposition

In addition to the Yang–Mills action, we have in the (bosonic) spectral action:

$$\int_M \left[-\left(\frac{6f_2}{\pi^2\Lambda^2} + 3R \right) \Lambda^2 |\widetilde{q}(x)|^2 + \frac{f(0)}{4\pi^2} (8|\widetilde{q}(x)|^4 + 6|(\partial_\mu + A_\mu)\widetilde{q}(x)|^2 \right] d\mu_g(x$$

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Proposition

The fermionic action $\langle \psi, D_A \psi \rangle$ contains in addition

$$\langle \psi_q, (\partial\!\!\!/ + A)\psi_q \rangle + \langle \chi_g, (\partial\!\!\!/ + \mathrm{ad} A)\psi_g \rangle + \langle \overline{\psi}_q, (\partial\!\!\!/ + \overline{A})\overline{\psi}_q \rangle + \langle \psi_q, \psi_g \widetilde{q} \rangle + \langle \chi_g \widetilde{q}, \chi_q \rangle + \langle \chi_g^t \overline{\widetilde{q}}, \overline{\psi}_q \rangle + \langle \overline{\psi}_q, \psi_g^t \overline{\widetilde{q}} \rangle$$

where $\psi = \psi_q \oplus (\psi_g \oplus \chi_g) \oplus \overline{\psi}_q$

Interpretation/comparison with the MSSM

So, in addition to the previous SYM terms, we have

$$\int_M \left[-\left(\frac{6f_2}{\pi^2\Lambda^2} + 3R \right) \Lambda^2 |\widetilde{q}(x)|^2 + \frac{f(0)}{4\pi^2}(8|\widetilde{q}(x)|^4 + 6|(\partial_\mu + A_\mu)\widetilde{q}(x)|^2 \right] d\mu_g(x$$

$$\langle \psi_q, (\partial\!\!\!/ + A)\psi_q \rangle + \langle \chi_g, (\partial\!\!\!/ + \mathrm{ad}A)\psi_g \rangle + \langle \overline{\psi}_q, (\partial\!\!\!/ + \overline{A})\overline{\psi}_q \rangle +$$

$$\langle \psi_q, \psi_g \widetilde{q} \rangle + \langle \chi_g \widetilde{q}, \chi_q \rangle + \langle \chi_g^t \overline{\widetilde{q}}, \overline{\psi}_q \rangle + \langle \overline{\psi}_q, \psi_g^t \overline{\widetilde{q}} \rangle$$

We recognize from the MSSM [Kramml]:

- squark kinetic term $\propto |\partial_\mu \widetilde{q}|^2$.
- squark mass term $\propto |\widetilde{q}|^2$.
- squark quartic self-interaction $\propto |\widetilde{q}|^4$.
- squark-gluon interactions $\propto |(\partial_\mu + A_\mu)\widetilde{q}|^2$.
- squark-quark-gluino interaction $\propto \langle \chi_g \widetilde{q}, \psi_q \rangle$.

Outlook (Part 2)

- Unimodularity to reduce $U(n)$ to $SU(n)$. Fermion doubling. [CCM].
- An essential further step is to identify the coefficients of the terms just considered. However, the literature is on the MSSM, whereas we considered only part of that, namely super-QCD.
- Future plan is to include the electro-weak sector as well, exploiting the same ideas. This could lead to a noncommutative geometrical description of the MSSM, whose Lagrangian is highly non-trivial to write down. We hope to derive it as the spectral action of some noncommutative manifold.

Reference: T. van den Broek. *Supersymmetric gauge theories in noncommutative geometry. First steps towards the MSSM*, Master's thesis, Radboud University Nijmegen. (http://www.math.ru.nl/∼waltervs)