Noncommutative Geometry and Particle Physics

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This is a short survey on the derivation of the Standard model from a noncommutative manifold.

1. NONCOMMUTATIVE MANIFOLDS AND GAUGE FIELD THEORY

The starting point is a noncommutative spin manifold as described by a *spectral triple* [3] $(\mathcal{A}, \mathcal{H}, D)$ consisting of

- a $*$-algebra $\mathcal{A}$ of bounded operators on
- a Hilbert space $\mathcal{H}$, and
- a self-adjoint operator in $\mathcal{H}$ such that
  - $[D, a]$ is bounded for any $a \in \mathcal{A}$
  - the resolvent of D is compact.

This structure can be further enriched by introducing a *grading* γ on $\mathcal{H}$ and an anti-linear isometry J (*real structure*) in $\mathcal{H}$ such that

$$[a, [D, b]] = [JaJ^{-1}, [D, b]] = 0$$

Moreover, we demand that $\gamma D = D\gamma$, $J\gamma = \pm\gamma J$, $J^2 = \pm$ and $JD = \pm DJ$. The $\pm$-signs determine the *KO-dimension*; they can be found in [4].

The main idea is that the above consists of all structure necessary to define a gauge theory. In fact, the group $U(\mathcal{A})$ of unitary elements in the algebra $\mathcal{A}$ naturally acts on the Hilbert space and as intertwiners on the representation of $\mathcal{A}$ and D. More precisely,

$$\psi \mapsto U\psi; \qquad a \mapsto UaU^*; \qquad D \mapsto UDU^*; \qquad (\psi \in \mathcal{H}, a \in \mathcal{A}),$$

where $U = uJuJ^*$ is the adjoint representation of $u \in U(\mathcal{A})$. It is then only natural to look for invariants under this group action and we work with the following combination

$$S_\Lambda[D, \psi] := \langle \psi, D\psi \rangle + \mathrm{Tr}\, f(D/\lambda)$$

considered as a physical action functional on D and ψ. Here f is an even function, and is such that the trace is well-defined. There are now two ways of introducing gauge fields, the first of physical origin and the second of mathematical.

Observe that the unitaries $u \in U(\mathcal{A})$ act as

$$D \mapsto UDU^* = D + u[D, u^*] \pm Ju[D, u^*]J^{-1}.$$

Thus, as usual in minimal coupling, one replaces D by the operator $D + A \pm JAJ^{-1}$ where $a = \sum a_j[D, b_j]$ with a_j, b_j now arbitrary elements in $\mathcal{A}$. This is our gauge field, which transforms in the usual way:

$$A \mapsto uAu^* + u[D, u^*].$$

From a mathematical point of view there is a nice interpretation of gauge fields as inner fluctuations, generated by Morita equivalence. It is based on the following question: given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an algebra $\mathcal{B}$ that is Morita equivalent to $\mathcal{A}$, is it possible to construct a spectral triple $(\mathcal{B}, \mathcal{H}', D')$?

Not surprisingly, the answer is yes [4]. We will not give the details here, but note that in the case that $\mathcal{B} = \mathcal{A}$ there is still freedom in choosing D' different from D. These are precisely the *inner fluctuations*, and correspond to choosing a connection one-form of the form

$$A = \sum_j a_j [D, b_j]; \qquad (a_j, b_j \in \mathcal{A}).$$

The operator D then becomes $D_A := D + A \pm JAJ^{-1}$ and A transforms as above.

In the rest of this note, we will compute in several examples the leading terms of the spectral action, as an expansion in Λ. The main techniques we will use are the Laplace transform and heat kernel expansions, as we will now briefly sketch. We will write

$$f(x) = \int_{t>0} k(t) e^{-tx^2}.$$

Also define $f_0 = f(0)$, and $f_\alpha = \int_0^\infty f(y) y^{\alpha-1} dy$. Thus, in determing $\mathrm{Tr}\, f(D_A/\Lambda)$ we have to compute the heat kernels $\mathrm{Tr}\, e^{-tD_A^2}$. This is achieved trough a theorem by Gilkey [6]. In fact, in all our examples D_A^2 is of the following form $\nabla^*\nabla + E$. For such an operator with ∇ a connection on a vector bundle, we have

$$\mathrm{Tr}\, e^{-tD_A^2} \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} \int_M a_n(x, D_A^2) \sqrt{g} d^m x$$

where m is the dimension of the manifold M. The *Seeley-de Witt coefficients* $a_n(D_A^2)$ vanish for odd values of n. The first three a_n's for n even are:

$$a_0(x, D_A^2) = (4\pi)^{-m/2}\, \mathrm{Tr}(1) \qquad a_2(x, D_A^2) = (4\pi)^{-m/2}\, \mathrm{Tr}\left(-\frac{R}{6} + E\right)$$

$$a_4(x, D_A^2) = (4\pi)^{-m/2} \frac{1}{360} \mathrm{Tr}\Big(- 60 R\, E + 180 E^2 + 60 E;_\mu{}^\mu + 30 \Omega_{\mu\nu}\, \Omega^{\mu\nu} \\ - 12 R;_\mu{}^\mu + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big)$$

We will now consider several examples of noncommutative manifolds and present the leading terms in the spectral action. For the details, we refer to [2] and [5].

1.1. **Einstein's general theory of relativity.** Consider a compact 4-dimensional Riemannian spin manifold (M, g); there is a canonical spectral triple

$$(C^\infty(M), L^2(M, S), \partial\!\!\!/),$$

where $\partial\!\!\!/$ is the ordinary Dirac operator on the spinor bundle $S \to M$. Further, there is a grading given by γ_5 and a real structure by charge conjugation J_M. Since the algebra is commutative the inner fluctuations are trivial. With Lichnerowicz formula one expresses $\partial\!\!\!/^2 = \Delta - \frac{1}{4} R$ in terms of the scalar curvature, resulting in

$$\mathrm{Tr}\, f(\partial\!\!\!/ / \Lambda) = \frac{1}{4\pi^2} \int_M \left(2\Lambda^4 f_4 + \frac{\Lambda^2 f_2}{6} R - \frac{f_0}{80} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) + \mathcal{O}(\Lambda^{-2})$$

in terms of the Weyl curvature tensor $C_{\mu\nu\rho\sigma}$. This action is recognized as the Einstein–Hilbert action, plus additional higher-order gravitational terms.

1.2. **Yang–Mills action.** We make the spectral triple of the previous section 'mildly' noncommutative and consider

$$(C^\infty(M, M_N(\mathbb{C})), L^2(M,S) \otimes M_N(\mathbb{C}), \not{\partial} \otimes 1).$$

In addition, there exist a grading $\gamma = \gamma_5 \otimes 1$ and a real structure $J = J_M \otimes (\cdot)^*$. It turns out that the inner fluctuations are parametrized by a $SU(N)$-gauge field A_μ, the group of unitaries is $C^\infty(M, U(N))$ acting in the adjoint on the Hilbert space. Actually, the fact that the fermions are in the adjoint representation is the origin of supersymmetry in this model as was suggested in [2] and worked out in detail in [1]. One computes that in this case the spectral action contains the Yang–Mills action

$$S_\Lambda[A,\psi] = -\frac{f_0}{24\pi^2} \int_M \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} + \langle \psi, (\not{\partial} + i\gamma^\mu \operatorname{ad} A_\mu)\psi \rangle + \mathcal{O}(\Lambda^{-2}),$$

in addition to the gravitational terms considered in the previous subsection.

1.3. **The Standard Model of high-energy physics.** The previous model generalizes to actually geometrically describe the full Standard Model, including Higgs boson. The spectral triple is now given by

$$(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \ L^2(M,S) \otimes \mathbb{C}^{96}, \ \not{\partial} \otimes 1 + \gamma_5 \otimes D_F).$$

Here 96 is 2 (particles and anti-particles) times 3 (families) times 4 leptons times 4 quarks with 3 colors each. We write the representation of $\mathcal{A}$ in terms of the suggestive basis of $\mathbb{C}^{96}$: $\left(\nu_L \ e_L \ \nu_R \ e_R \ u_L \ d_L \ u_R \ d_R \ \bar\nu_L \ \bar e_L \ \bar\nu_R \ \bar e_R \ \bar u_L \ \bar d_L \ \bar u_R \ \bar d_R \right)^t$. Then, for an element $(\lambda, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

$$\pi(\lambda,q,m) = \begin{pmatrix} q &&&&& \\ & \begin{bmatrix} \lambda & \\ & \bar\lambda \end{bmatrix} &&&& \\ && q\otimes 1_3 &&& \\ &&& \begin{bmatrix} \lambda & \\ & \bar\lambda \end{bmatrix} \otimes 1_3 && \\ &&&& \lambda 1_4 & \\ &&&&& 1_4 \otimes \bar m \end{pmatrix}$$

Here, the quaternion q is considered as a 2×2-matrix. The 96×96-matrix D_F is of the following form: $D_F = \left(\begin{smallmatrix} S & T^* \\ T & \bar S \end{smallmatrix} \right)$ where

$$S = \begin{pmatrix} & \begin{bmatrix} \Upsilon_\nu & \\ & \Upsilon_e \end{bmatrix} && \\ \begin{bmatrix} \Upsilon_\nu^* & \\ & \Upsilon_e^* \end{bmatrix} &&& \\ &&& \begin{bmatrix} \Upsilon_u \otimes 1_3 & \\ & \Upsilon_d \otimes 1_3 \end{bmatrix} \\ && \begin{bmatrix} \Upsilon_u^* \otimes 1_3 & \\ & \Upsilon_d^* \otimes 1_3 \end{bmatrix} & \end{pmatrix}; \quad T = \begin{pmatrix} \begin{bmatrix} 0 &&& \\ & 0 & \Upsilon_R & \\ &&& 0 \end{bmatrix} & \\ & 0_4 \end{pmatrix}$$

in terms of the 3×3 *Yukawa-mixing-matrices* $\Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d$ and a real constant Υ_R responsible for neutrino mass terms.

One can further enrich this spectral triple by a grading γ_F which is $+1$ on all L-particles, and -1 on all R-particles; the total grading is then $\gamma_5 \otimes \gamma_F$. The anti-linear operator J is a combination of charge conjugation on S and the (anti-linear) matrix $J_F = \begin{pmatrix} & 1_{48} \\ 1_{48} & \end{pmatrix}$.

The rest then follows from a long calculation; the inner fluctuations are $D_A = \not{\partial} + i\gamma_\mu A_\mu + \gamma_5(D_F + \mathbb{M}(\Phi))$ with

$$A_\mu = \begin{pmatrix} \frac{g_1}{2}B_\mu - \frac{g_2}{2}W_\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g_1 B_\mu \end{pmatrix} \oplus \begin{pmatrix} -\frac{g_2}{2}W_\mu \otimes 1_3 - \frac{g_1}{6}B_\mu \otimes 3 & 0 & 0 \\ 0 & -\frac{2g_1}{3}B_\mu \otimes 1_3 & 0 \\ 0 & 0 & \frac{g_1}{3}B_\mu \otimes 1_3 \end{pmatrix} - 1_4 \otimes \frac{g_3}{2}V_\mu$$

$$\mathbb{M}(\Phi) = \begin{pmatrix} & \begin{matrix} \Upsilon_\nu \phi_1 & \Upsilon_\nu \phi_2 \\ -\Upsilon_e \bar{\phi}_2 & \Upsilon_e \bar{\phi}_1 \end{matrix} \\ \text{(h.c.)} & \end{pmatrix} \oplus \begin{pmatrix} & \begin{matrix} \Upsilon_u \phi_1 & \Upsilon_u \phi_2 \\ -\Upsilon_d \bar{\phi}_2 & \Upsilon_d \bar{\phi}_1 \end{matrix} \\ \text{(h.c.)} & \end{pmatrix}$$

Here B_μ, W_μ, V_μ are $U(1), SU(2)$ and $SU(3)$-gauge fields, resp. and $\Phi = (\phi_1\ \phi_2)^t$ two scalar (Higgs) fields. The spectral action is modulo gravitational terms:

$$S_\Lambda = \frac{-2af_2\Lambda^2 + ef_0}{\pi^2} \int |\phi|^2 + \frac{f_0}{2\pi^2} \int a|D_\mu \phi|^2 - \frac{f_0}{12\pi^2} \int aR|\phi|^2$$
$$- \frac{f_0}{2\pi^2} \int \left(g_3^2 G_\mu^i G^{\mu i} + g_2^2 F_\mu^a F^{\mu\nu a} + \frac{5}{3} g_1^2 B_\mu B^\mu \right) + \frac{f_0}{2\pi^2} \int b|\phi|^4 + \mathcal{O}(\Lambda^{-2})$$

with a, b, c, d, e constants depending on the Yukawa parameters. For example,

$$a = \mathrm{Tr}\left(\Upsilon_\nu^* \Upsilon_\nu + \Upsilon_e^* \Upsilon_e + 3\left(\Upsilon_u^* \Upsilon_u + \Upsilon_d^* \Upsilon_d\right)\right)$$
$$b = \mathrm{Tr}\left((\Upsilon_\nu^* \Upsilon_\nu)^2 + (\Upsilon_e^* \Upsilon_e)^2 + 3\left((\Upsilon_u^* \Upsilon_u)^2 + (\Upsilon_d^* \Upsilon_d)^2\right)\right)$$

When we add the fermionic term $\langle J\psi, D_A\psi\rangle$ to S_Λ, we obtain the Standard Model Lagrangian, including the Higgs boson, provided we have

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \qquad g_3^2 = g_2^2 = \frac{5}{3} g_1^2.$$

These GUT-type relations between the coupling constants allows for predictions. For example, one identifies the mass of the W as $2M_W = \sqrt{a/2}$ so that the Higgs vacuum reads $2M/g_2$. The above relation for a then gives a postdiction for the mass of the top quark as $m_t \leq 180$ GeV. Moreover, the mass of the Higgs is $m_H = 8\lambda M^2/g_2^2$ with $\lambda = g_3^2 b/a^2$ resulting in a prediction of $m_H \sim 168$ GeV.

References

[1] T. van den Broek and W. D. van Suijlekom. Supersymmetric QCD and noncommutative geometry. In preparation.

[2] A. H. Chamseddine and A. Connes. The spectral action principle. *Commun. Math. Phys.* 186 (1997) 731–750.

[3] A. Connes. *Noncommutative Geometry.* Academic Press, San Diego, 1994.

[4] A. Connes. Gravity coupled with matter and the foundation of non-commutative geometry. *Commun. Math. Phys.* 182 (1996) 155–176.

[5] A. Connes and M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives.* AMS, Providence, 2008.

[6] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, volume 11 of *Mathematics Lecture Series.* Publish or Perish Inc., Wilmington, DE, 1984.