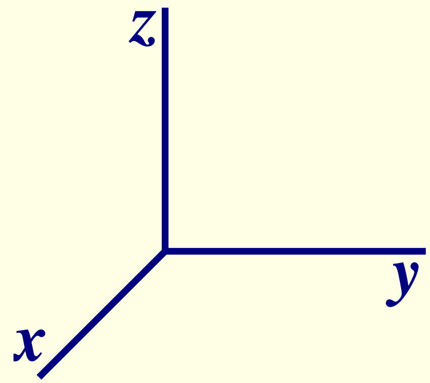
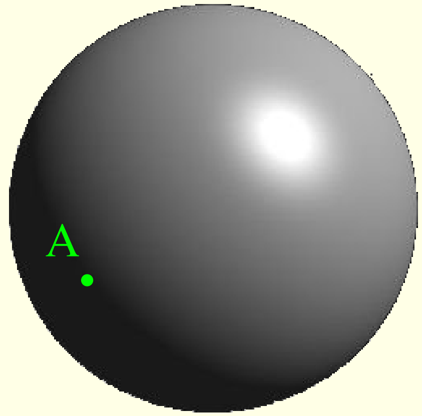
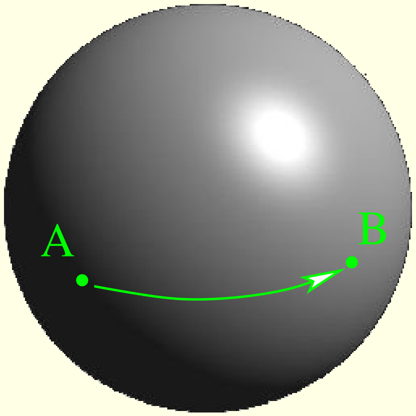
The Geometry of Noncommutative Spheres and their Symmetries

Walter Daniël van Suijlekom

Spheres and symmetries







$$x_A^2 + y_A^2 + z_A^2 = 1 \quad \longrightarrow \quad x_B^2 + y_B^2 + z_B^2 = 1$$

Noncommutative geometry

Extension: x, y do not commute: $xy \neq yx \longrightarrow$ noncommutative geometry

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Hopf algebra:	"symmetries"

Noncommutative Geometry

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on $\mathcal{H}$ s.t.

1. The resolvent $(D - \lambda)^{-1}$ is a compact operator on $\mathcal{H}$;

2. $[D, a] = D \circ a - a \circ D$ is a bounded operator for any $a \in \mathcal{A}$.

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2. $[D, a] = D \circ a - a \circ D$ is a bounded operator for any $a \in \mathcal{A}$.

Its dimension is the nonnegative integer n for which the eigenvalue sums $\sigma_N = \sum_{0 \leq k < N} \mu_k$ of the operator $|D|^{-n}$ satisfy $\sigma_N \sim C \log N$ as $N \to \infty$.

$\longrightarrow$ noncommutative integral on $\mathcal{A}$ defined to be the constant in front of $\log N$ in the eigenvalue sums σ_N (as $N \to \infty$) of the operator $a|D|^{-n}$; it will be denoted by $\int a$.

Example: canonical spectral triple

The basic example of a spectral triple is constructed by means of the Dirac operator on a compact n-dimensional Riemannian spin manifold M.

- $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M.
- $\mathcal{H} = L^2(M, S)$ is the Hilbert space of square integrable sections of a spinor bundle on M, on which $\mathcal{A}$ acts by pointswise multiplication.
- D is the Dirac operator associated with the Levi-Civita connection.

The noncommutative integral reduces to the Riemann integral:

$$⨍ f := \int_M f$$

Connes' reconstruction theorem provides a duality between Riemannian spin manifolds and spectral triples with a commutative algebra.

Quantum Groups

Q: How to encode group structure on a topological group G in terms of the algebra of (smooth, continuous) functions $\mathcal{F}(G)$ on it?

We take $\mathcal{A} := \mathcal{F}(G)$ as the algebra of representative functions, satisfying $\mathcal{F}(G) \otimes \mathcal{F}(G) = \mathcal{F}(G \times G)$.

- Multiplication $G \times G \to G; (g, h) \to gh$ induces a map $\Delta : A \to A \otimes A$ called the coproduct:

$$\Delta(f)(g, h) = f(gh)$$

Associativity of G becomes coassociativity on $\mathcal{A}$:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad \text{or} \quad f\big((gh)k\big) = f\big(g(hk)\big). \qquad \text{(a)}$$

- Unit $e \in G$ induces a map $\epsilon : A \to \mathbb{C}$, called counit, which satisfies:

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta, \quad \text{or} \quad f(ge) = f(eg) = f(g). \qquad \text{(b)}$$

- Inverse map $g \mapsto g^{-1}$ becomes the antipode $S : A \to A$, defined by $S(f)(g) = f(g^{-1})$ satisfying:

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = 1_A \epsilon, \quad \text{or} \quad f(g^{-1}g) = f(gg^{-1}) = f(e). \qquad \text{(c)}$$

where $m : A \otimes A \to A$ denotes pointswise multiplication of functions.

Definition 1. *A Hopf algebra* A *is an algebra* A*, together with two algebra maps* $\Delta : A \otimes A \to A$ *(coproduct),* $\epsilon : A \to \mathbb{C}$ *(counit), and a bijective* $\mathbb{C}$*-linear map* $S : A \to A$ *(antipode), such that equations* (a)–(c) *are satisfied.*

The Noncommutative Spin Geometry of Quantum $SU(2)$

Geometry of (classical) $SU(2)$

Construct q-version of spin geometry on $SU(2)$:

- Homogeneous space:

$$SU(2) = \frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} = \frac{SU(2) \times SU(2)}{SU(2)}$$

  with $\mathrm{Spin}(3)$ the diagonal $SU(2)$ subgroup of $\mathrm{Spin}(4)$.
  Quotient map: $(p, q) \mapsto pq^{-1}$

- Action of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ on $SU(2)$: $(p, q) \cdot x = pxq^{-1}$.

- Spinor bundle $\mathcal{S} := Spin(4) \times_{SU(2)} \mathbb{C}^2$ is parallelizable: $\mathcal{S} = SU(2) \times \mathbb{C}^2$.

Algebraic preliminaries

Definition. *The algebra* $\mathcal{A} := \mathcal{A}(SU_q(2))$ *of polynomials on* $SU_q(2)$ *is the* $*$*-algebra generated by* a *and* b*, subject to:*

$$ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b$$
$$a^*a + q^2b^*b = 1, \qquad aa^* + bb^* = 1.$$

This becomes a Hopf $*$-algebra with

$$\begin{cases} \Delta : \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \otimes \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}, \\ \epsilon : \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ S : \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \mapsto \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix}. \end{cases}$$

Definition. *The $*$-algebra $\mathcal{U} := \mathcal{U}_q(su(2))$ is generated by elements e, f, k, with k invertible, satisfying:*

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef)$$

It is also a Hopf $*$-algebra.

The irreducible finite dimensional representations σ_l of $\mathcal{U}_q(su(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots,$ and given by

$$\begin{aligned} \sigma_l(k)\,|lm\rangle &= q^m\,|lm\rangle, \\ \sigma_l(f)\,|lm\rangle &= \sqrt{[l-m][l+m+1]}\,|l, m+1\rangle, \\ \sigma_l(e)\,|lm\rangle &= \sqrt{[l-m+1][l+m]}\,|l, m-1\rangle, \end{aligned}$$

on the irreducible $\mathcal{U}$-modules $V_l = \mathrm{Span}\{|lm\rangle\}_{m=-l,\dots,l}$.

Action of $\mathcal{U}_q(su(2))$ on $\mathcal{A}(SU_q(2))$

There is a dual pairing $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ inducing left and right action of $h \in \mathcal{U}_q(su(2))$ on $x \in \mathcal{A}(SU_q(2))$:

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle \qquad x \triangleleft h := \langle h, x_{(1)} \rangle \, x_{(2)},$$

where we use Sweedler's notation for the coproduct in $\mathcal{A}(SU_q(2))$:

$$\Delta x = x_{(1)} \otimes x_{(2)}, \qquad (x \in \mathcal{A})$$

Using the antipode, the right action can be transformed into a left action, which we will denote by $h \cdot x$.

Equivariant representation of $\mathcal{A}(SU_q(2))$

We establish the left regular representation of $\mathcal{A}$ as an equivariant representation with respect to two copies of $\mathcal{U}$ acting via $\cdot$ and $\triangleright$ on the left.

Definition. *Let λ and ρ be mutually commuting representations of the Hopf algebra $\mathcal{U}$ on a vector space V. A representation π of the algebra $\mathcal{A}$ on V is (λ, ρ)-equivariant if the following compatibility relations hold:*

$$\lambda(h)\,\pi(x)\xi = \pi(h_{(1)} \cdot x)\,\lambda(h_{(2)})\xi,$$

$$\rho(h)\,\pi(x)\xi = \pi(h_{(1)} \triangleright x)\,\rho(h_{(2)})\xi,$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$.

Representation space: $V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l$;

The two copies of $\mathcal{U}_q(su(2))$ act via the irreducible representations σ like

$$\lambda(h) = \sigma_l(h) \otimes \mathrm{id}, \qquad \rho(h) = \mathrm{id} \otimes \sigma_l(h) \qquad \text{on } V_l \otimes V_l.$$

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \ldots, l$.

Proposition. *A (λ, ρ)-equivariant $*$-representation π of $\mathcal{A}(SU_q(2))$ on V is necessarily given by the left regular representation. Explicitly:*

$$\pi(a)\,|lmn\rangle = A^+_{lmn}|l + \tfrac{1}{2}, m + \tfrac{1}{2}, n + \tfrac{1}{2}\rangle + A^-_{lmn}|l - \tfrac{1}{2}, m + \tfrac{1}{2}, n + \tfrac{1}{2}\rangle,$$
$$\pi(b)\,|lmn\rangle = B^+_{lmn}|l + \tfrac{1}{2}, m + \tfrac{1}{2}, n - \tfrac{1}{2}\rangle + B^-_{lmn}|l - \tfrac{1}{2}, m + \tfrac{1}{2}, n - \tfrac{1}{2}\rangle,$$

where for example the constants A^+_{lmn} are given by

$$A^+_{lmn} = q^{(-2l+m+n-1)/2}\left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]}\right)^{\frac{1}{2}},$$

Spinor representation

We amplify the representation π of $\mathcal{A}$ to spinor representation $\pi' = \pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$, and set $\rho' = \rho \otimes \mathrm{id}$, but λ' as the tensor product of the representations λ on V and $\sigma_{\frac{1}{2}}$ on $V_{\frac{1}{2}} = \mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}).$$

Proposition. *The representation π' of $\mathcal{A}$ is (λ', ρ')-equivariant.*

Clebsch-Gordan decomposition:

$$V \otimes \mathbb{C}^2 = \left(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l \right) \otimes V_{\frac{1}{2}} \simeq \underbrace{\bigoplus_{2j=0}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j)}_{|j\mu n\uparrow\rangle} \oplus \underbrace{\bigoplus_{2j=1}^{\infty} (V_{j-\frac{1}{2}} \otimes V_j)}_{|j\mu n\downarrow\rangle}.$$

$\Longrightarrow$ expressions for π' in basis $\{|j\mu n\uparrow\rangle, |j\mu n\downarrow\rangle\}$ contain off-diagonal terms.

Invariant Dirac operator

Proposition. *Any self-adjoint operator on* $\mathcal{H} = (V \otimes \mathbb{C}^2)^{\mathrm{cl}}$*, that commutes with both actions* ρ', λ' *of* $\mathcal{U}_q(su(2))$ *is of the form*

$$D|j\mu n{\uparrow}\rangle = d_j^{\uparrow}\,|j\mu n{\uparrow}\rangle, \qquad D|j\mu n{\downarrow}\rangle = d_j^{\downarrow}\,|j\mu n{\downarrow}\rangle.$$

Moreover, if $d_j^{\uparrow}, d_j^{\downarrow}$ *are linear in* j*, then* $[D, \pi'(x)] \in \mathcal{B}(\mathcal{H})$*,* $(x \in \mathcal{A})$*.*

Spectrum (with multiplicities) of D coincides with that of the classical Dirac operator on the round sphere $S^3 \simeq SU(2)$.

$\longrightarrow$ $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ is a spectral triple of dimension 3.

Isospectral deformation: H and D unchanged, only algebra deformed.

Analysis of the spin geometry on $SU_q(2)$

- In general, a real structure on a spectral triple $(\mathcal{A}, H, D)$ gives H the structure of an A-bimodule, with the conditions that the right action π'° of $\mathcal{A}$ on H commutes with the left action of $[D, \mathcal{A}]$.

  In our case, those conditions turn out to be *almost* satisfied. The ideal $\mathcal{K}_q$ in $\mathcal{B}(H)$ generated by $L_q|j\mu n\rangle := q^j|j\mu n\rangle$ contains all defects:

$$[\pi'(x), \pi'^\circ(y)] \in \mathcal{K}_q; \qquad \big[[D, \pi'(x)], \pi'^\circ(y)\big] \in \mathcal{K}_q.$$

- Connes-Moscovici local index theorem: noncommutative generalization of the Atiyah-Singer index theorem.

The Geometry of Gauge Fields
on Toric Noncommutative Manifolds

Classical Yang-Mills theory

A central role is played by the theory of principal bundles.

Definition. *A* G*–principal bundle on* X *is a fiber bundle* $P \to X$ *with typical fiber a Lie group* G *such that* G *acts freely and transitively on* P*.*
The associated bundle to P *by a representation* V *of* G *is given by the vector bundle:* $E := P \times_G V$*.*

The space of sections $\Gamma(X, E)$ of E has the structure of a $C(X)$-module, defined in terms of pointwise multiplication: $(sf)(x) = s(x)f(x)$.

Definition. *A connection (or covariant derivative) on* $\Gamma(E)$ *is a map*

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes_{C(X)} \Omega^1(X)$$

satisfying $\nabla(sf) = \nabla(s)f + s \otimes df$*.*
Its curvature F *is defined to be the map* $F = \nabla^2 : \Gamma(E) \to \Gamma(E) \otimes_{C(X)} \Omega^2(X)$

Hodge star operator on X is a map $* : \Omega^p(X) \to \Omega^{\dim X - p}(X)$; if $\dim X = 4$, it maps $\Omega^2(X)$ onto itself.

Definition. *The Yang-Mills action for a connection* ∇ *with curvature* F *is defined by* $S = \int_X \mathrm{tr}(F \wedge *F)$.

Its equations of motion are called the Yang-Mills equations: $\nabla * F = 0$.

Connections with (anti)selfdual curvature $*F = \pm F$ satisfy YM equations since $\nabla F = 0$.
$\longrightarrow$ Instantons: absolute minima of the YM action.

Goal: Construct Yang-Mills theory and noncommutative instantons on toric noncommutative manifolds.

Example: noncommutative $SU(2)$-principal Hopf fibration $S^7_{\theta'} \to S^4_\theta$

The sphere S^4_θ

With θ a real parameter, the algebra $\mathcal{A}(S^4_\theta)$ of polynomial functions on the sphere S^4_θ is generated by elements $z_0 = z_0^*, z_j, z_j^*$, $j = 1, 2$, subject to

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu z_\mu^*, \quad z_\mu^* z_\nu^* = \lambda_{\mu\nu} z_\nu^* z_\mu^*, \quad \mu, \nu = 0, 1, 2,$$

with deformation parameters given by

$$\lambda_{12} = \bar{\lambda}_{21} =: \lambda = e^{2\pi i \theta}, \quad \lambda_{j0} = \lambda_{0j} = 1, \quad j = 1, 2$$

and together with the spherical relation $\sum_\mu z_\mu^* z_\mu = 1$.

Isospectral deformation: canonical triple on sphere $(\mathcal{A}(S^4), \mathrm{H}, \mathrm{D})$ is deformed to a spectral triple $(\mathcal{A}(S^4_\theta), \mathrm{H}, \mathrm{D})$ of dimension 4;

H and D unchanged, only difference in algebra and its representation on H.

Differential forms on S^4_θ

Differential calculus $\Omega(S^4_\theta)$ is graded differential $*$-algebra with generators z_μ, z_μ^* of degree 0 and elements dz_μ, dz_μ^* of degree 1 satisfying:

$$dz^\mu dz^\nu + \lambda^{\mu\nu} dz^\nu dz^\mu = 0; \qquad d\bar{z}^\mu dz^\nu + \lambda^{\nu\mu} dz^\nu d\bar{z}^\mu = 0;$$
$$z^\mu dz^\nu = \lambda^{\mu\nu} dz^\nu z^\mu; \qquad \bar{z}^\mu dz^\nu = \lambda^{\nu\mu} dz^\nu \bar{z}^\mu.$$

There is a Hodge star operator $*_\theta$ that maps $\Omega^p(S^4_\theta)$ to $\Omega^{4-p}(S^4_\theta)$. Together with the noncommutative integral on $\mathcal{A}(S^4_\theta)$ coming from the spectral triple, this gives an inner product on $\Omega^p(S^4_\theta)$:

$$(\alpha, \beta)_2 := \int\!\!\!\!\!- *_\theta(\alpha^* *_\theta \beta)$$

The sphere $S^7_{\theta'}$

With $\lambda'_{ab} = e^{2\pi i\theta'_{ab}}$ and (θ'_{ab}) a real antisymmetric matrix, the algebra $\mathcal{A}(S^7_{\theta'})$ is generated by elements ψ_a, ψ_a^*, $a = 1, \ldots, 4$, subject to

$$\psi_a\psi_b = \lambda'_{ab}\psi_b\psi_a, \quad \psi_a\psi_b^* = \lambda'_{ba}\psi_b^*\psi_a, \quad \sum_a \psi_a^*\psi_a = 1.$$

Differential calculus $\Omega(S^7_{\theta'})$ is graded differential $*$-algebra with generators ψ_a, ψ_a^* of degree 0 and the elements $d\psi_a, dz_a^*$ of degree 1 with relations:

$$d\psi_a d\psi_b + \lambda'_{ab} d\psi_b d\psi_a = 0; \qquad d\psi_a^* d\psi_b + \lambda'_{ba} d\psi_b d\psi_a^* = 0;$$
$$\psi_a d\psi_b = \lambda'_{ab} d\psi_b \psi_a; \qquad \psi_a^* d\psi_b = \lambda'_{ba} d\psi_b \psi_a^*.$$

Noncommutative Hopf fibration

A minimal requirement for $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ to constitute a noncommutative $SU(2)$–prinicipal bundle is that there is

- an inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$
- an action α of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ such that $\mathcal{A}(S^4_\theta)$ can be identified with the subalgebra of invariant elements under this action.

It turns out that these two conditions express θ' in terms of θ and we identify:

$$
\begin{aligned}
z_0 &= \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4 \\
z_1 &= 2(\mu\psi_3^*\psi_1 + \psi_2^*\psi_4) \\
z_2 &= 2(-\mu\psi_4\psi_1^* + \psi_2\psi_3^*)
\end{aligned}
\qquad
\theta'_{ab} = \frac{\theta}{2}\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}
$$

Noncommutative vector bundles

A vector space is called a right $\mathcal{A}$-module if it carries a right representation of the algebra $\mathcal{A}$.

Definition. *A right $\mathcal{A}$-module $\mathcal{E}$ is said to be finite projective if there exists an projector $p = p^* = p^2 \in M_N(\mathcal{A})$ such that $\mathcal{E} \simeq p\mathcal{A}^N$ as right $\mathcal{A}$-modules.*

In the classical case, any vector bundle E on a topological space X can be described by the $C(X)$–module $\Gamma(X, E)$ of its sections.
Theorem. [Serre-Swan] There exists a projector $p \in M_N(C(X))$ such that $\Gamma(X, E) \simeq p\big(C(X)\big)^N$.
Here $\big(C(X)\big)^N := \mathbb{C}^N \otimes C(X)$.

Thus, a noncommutative vector bundle over $\mathcal{A}(S^4_\theta)$ is described by a finite projective $\mathcal{A}(S^4_\theta)$-module, given by a projector p.

Associated modules

We associate $\mathcal{A}(S^4_\theta)$-modules to the noncommutative principal bundle $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ by all (f.d) representations of $SU(2)$. Let ρ be a representation of $SU(2)$ on $V^{(n)} = \mathbb{C}^{n+1}$:

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)} := \left\{ f \in \mathcal{A}(S^7_{\theta'}) \otimes V^{(n)} \ : \ (\alpha_w \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho(w)^{-1})(f) \right\}$$

There are projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ such that the right $\mathcal{A}(S^4_\theta)$-modules $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)} \simeq p_{(n)} \mathcal{A}(S^4_\theta)^{4^n}$.

$\longrightarrow$ twisted Dirac operator $D_{(n)} = p_{(n)} D p_{(n)}$ with coefficients in the module $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho V^{(n)}$. Its index $(\dim \ker \ - \dim \mathrm{coker}\ )$ equals:

$$\mathrm{Index}\ D_{(n)} = \frac{1}{6} n(n+1)(n+2).$$

Yang-Mills theory on S^4_θ

Definition. *A connection on a module $\mathcal{E}$ over $\mathcal{A}(S^4_\theta)$ is a map*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$$

satisfying Leibniz rule: $\nabla(\eta a) = \nabla(\eta)a + \eta da$ *for* $\eta \in \mathcal{E}, a \in \mathcal{A}(S^4_\theta)$*.*

Its curvature is given by $F = \nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}(S^4_\theta)} \Omega^2(S^4_\theta)$*.*

Definition. *The Yang-Mills action for a connection* ∇ *on a finite projective* $\mathcal{A}(S^4_\theta)$*-module* $\mathcal{E}$ *with curvature* F *is defined by* $S = \int \mathrm{tr} *_\theta (F *_\theta F)$*.*

Equations of motion: noncommutative Yang-Mills equations $[\nabla, *_\theta F] = 0$.

Bianchi identiy $[\nabla, F] = 0 \implies$ connections with (anti)selfdual curvature $*_\theta F = \pm F$ (instantons) are solutions of the YM equations; absolute minima of YM-action.

Basic (charge 1) instanton on S^4_θ

A generic element in the module $\mathcal{E} = \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ can be written as $\Psi^* f$, $f \in \mathcal{A}(S^4_\theta)^4$ with

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \\ \psi_3 & -\psi_4^* \\ \psi_4 & \psi_3^* \end{pmatrix}; \qquad \text{satisfying } \Psi^*\Psi = \mathbb{I}_2.$$

Thus, $p = \Psi\Psi^*$ is a projection in $M_4(\mathcal{A}(S^4_\theta))$ and in fact $\mathcal{E} \simeq p\mathcal{A}(S^4_\theta)^4$.

Connection $\nabla_0 = pd \to$ curvature satisfies $*_\theta F = F$: basic instanton on S^4_θ.

On $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ we have

$$(\nabla_0 f)_i = df_i + \omega_{ij} f_j; \qquad \text{with } \omega = \Psi^* d\Psi.$$

Construction of instantons on S^4_θ

- Basic instanton gauge potential ω is invariant under the twisted action of the Hopf algebra $\mathcal{U}_\theta(so(5))$.

- New (charge 1) instantons are constructed by the action of twisted conformal transformations forming the Hopf algebra $\mathcal{U}_\theta(so(5,1))$, resulting in a five-dimensional set of (infinitesimal) instantons.

- Using an index theoretical argument, we compute that the dimension of the 'tangent' to the moduli space of instantons on S^4_θ equals five, showing that the above set of infinitesimal instantons is complete.