Noncommutative spheres

in dimension '3', 4 and 7

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$\theta-$deformed spheres

• $\mathcal{A}(S^4_\theta)$ is complex unital $*$-algebra generated by α, β, x with $x = x^*$ a central element and relations:

$$\alpha\beta = \lambda\beta\alpha; \quad \alpha^*\beta = \overline{\lambda}\beta\alpha^*; \quad \alpha\alpha^* + \beta\beta^* + x^2 = 1; \qquad (\lambda \in S^1 \subset \mathbb{C})$$

• $\mathcal{A}(S^7_\theta)$ is complex unital $*$-algebra generated by $z^1, \ldots, z^4$ with relations

$$z^i z^j = \lambda^{ij} z^j z^i; \quad \overline{z}^i z^j = \lambda^{ji} z^j \overline{z}^i; \quad \sum z^i \overline{z}^i = 1$$

where for $\mu = \sqrt{\lambda}$:

$$\lambda^{ij} = \begin{pmatrix} 1 & 1 & \mu & \mu \\ 1 & 1 & \mu & \mu \\ \overline{\mu} & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \overline{\mu} & 1 & 1 \end{pmatrix}$$

Action of $SU(2)$

$\mathcal{A}(S^7_\theta)$ carries an action of $SU(2)$,

$$\alpha_w : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w^1 & w^2 & 0 & 0 \\ -\overline{w}^2 & \overline{w}^1 & 0 & 0 \\ 0 & 0 & w^1 & w^2 \\ 0 & 0 & -\overline{w}^2 & \overline{w}^1 \end{pmatrix}$$

where $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$. More precisely: the map $w \mapsto \alpha_w$ is a group homomorphism:

$$\boxed{SU(2) \hookrightarrow \mathrm{Aut}(\mathcal{A}(S^7_\theta))}$$

Algebra of polynomials on 'base space' consists of invariants: $x \in \mathcal{A}(S^7_\theta)$ s.t. $\alpha_w(x) = x$. Since $SU(2)$ acts classically, we find that the algebra of invariants is generated by

$$\begin{aligned}
\alpha &= 2(z^1\overline{z}^3 + z^2\overline{z}^4), \\
\beta &= 2(-z^1 z^4 + z^2 z^3), \\
x &= z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4.
\end{aligned}$$

satisfying $\alpha\beta = \lambda\beta\alpha$ etc., i.e. the algebra of invariants is isomorphic to $\mathcal{A}(S^4_\theta)$:

$\mathcal{A}(S^4_\theta) \to \mathcal{A}(S^7_\theta)$ is a principal Hopf-Galois extension (=principal bundle)

Associated modules

Classical $SU(2)$ $\Rightarrow$ irreducible representations given by integer l.

Associated modules consist of equivariant maps

$$\mathrm{Hom}^{SU(2)}(V^{(l)}, \mathcal{A}(S^7_\theta)) := \{\phi : V^{(l)} \to \mathcal{A}(S^7_\theta) : \phi(v \cdot w) = \alpha_w \phi(v)\}$$

In the case $l = 1$, the equivariant maps are given on the basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ by $\phi(e_i) = \langle \psi_i | f \rangle$ for $|f\rangle \in \mathcal{A}(S^4_\theta)^4$, where

$$|\psi_1\rangle = (\overline{z}^1, -z^2, \overline{z}^3, -z^4)^*; \quad |\psi_2\rangle = (\overline{z}^2, z^1, \overline{z}^4, z^3)^*$$

which satisfy $\langle \psi_i | \psi_j \rangle = \delta_{ij}$.

The matrix $p = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$ is a projector with entries in $\mathcal{A}(S^4_\theta)$:

$$p = \frac{1}{2}\begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\mu\beta^* & \overline{\mu}\alpha^* \\ \alpha^* & -\overline{\mu}\beta & 1-x & 0 \\ \beta^* & \mu\alpha & 0 & 1-x \end{pmatrix}$$

If the image of p in $\mathcal{A}(S^4_\theta)^4$ is denoted by $\Gamma(S^4_\theta, E)$ we have

$$\begin{aligned} \mathrm{Hom}^{SU(2)}(\mathbb{C}^2, \mathcal{A}(S^7_\theta)) &\quad\simeq\quad \Gamma(S^4_\theta, E) \\ \phi : e_i \mapsto \langle\psi_i|f\rangle &\quad\leftrightarrow\quad \sigma = p|f\rangle. \end{aligned}$$

For general l, the equivariant maps are are of the form $\phi_{(l)}(e_i) = \langle \phi_i | f \rangle$ on the basis $\{e_1, \ldots, e_{l+1}\}$ of $V^{(l)}$ where now $|f\rangle \in \mathcal{A}(S^4_\theta)^{4^l}$ and $|\phi_i\rangle$ is the completely symmetrized form of the tensor product $|\psi_1\rangle^{\otimes l-i+1} \otimes |\psi_2\rangle^{\otimes i-1}$ for $i = 1, \ldots, l+1$.

We normalize $|\phi_i\rangle$ such that $\langle \phi_i | \phi_j \rangle = \delta_{ij}$, hence

$$p_{(l)} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{l+1}\rangle\langle\phi_{l+1}| \in M_{4^l}(\mathcal{A}(S^4))$$

defines a projector. If we set $\Gamma(S^4_\theta, E^{(l)}) := p_{(l)}(\mathcal{A}(S^4_\theta)^{4^l})$, we have

$$\begin{aligned} \mathrm{Hom}^{SU(2)}(V^{(l)}, \mathcal{A}(S^7_\theta)) \quad &\simeq \quad \Gamma(S^4_\theta, E^{(l)}) \\ \phi_{(l)} : e_i \mapsto \langle \phi_i | f \rangle \quad &\leftrightarrow \quad \sigma_{(l)} = p_{(l)} |f\rangle \end{aligned}$$

Instanton on S^4_θ

A connection on the $\mathcal{A}(S^4_\theta)$–module $\Gamma(S^4_\theta, E)$ is defined to be a map

$$\nabla : \Gamma(S^4_\theta, E) \to \Gamma(S^4_\theta, E) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$$

satisfying Leibniz rule:

$$\nabla(s\omega) = (\nabla s)\omega + s\mathrm{d}\omega; \quad (s \in \Gamma(S^4_\theta, E), \omega \in \Omega^1(S^4_\theta))$$

The basic instanton on S^4_θ is described by the connection $\nabla = p\mathrm{d}$ on $\Gamma(S^4_\theta, E)$. Again, with $F = p(\mathrm{d}p)^2 \in \Omega^2(S^4_\theta)$:

$$*_\theta F = -F$$

Dirac operator on $\mathcal{A}(SU_q(2))$

with L. Dąbrowski, G. Landi, A. Sitarz and J. Varilly

- $\mathcal{A}(SU_q(2))$ is the $*$-algebra generated by a and b, with relations

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a+q^2b^*b = 1, \quad aa^*+bb^* = 1.$$

- The (Hopf) $*$-algebra $\mathcal{U}_q(su(2))$ is generated as an algebra by elements e, f, k, with k invertible, satisfying the relations

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef), \quad (1)$$

- There is a left and right action of $\mathcal{U}_q(su(2))$ on $\mathcal{A}(SU_q(2))$.

- Representation π of $\mathcal{A}(SU_q(2))$ on V in an equivariant manner w.r.t. left and right action of $\mathcal{U}_q(su(2))$ (which are also represented on V).

- Opposite representation $\pi^\circ(a) = J\pi(a)J^{-1}$, with J charge conjugation; π° commutes with π.

- Lift to spinor representation π' on $V \otimes \mathbb{C}^2$. On this, constructed $\mathcal{U}_q(su(2))$ invariant Dirac operator. There is the opposite spinor representation π'° commuting with π'.

Goal: Complete construction of a noncommutative spin geometry on $\mathcal{A}(SU_q(2))$, in the sense of Connes' seven axioms for such a structure.

Problem: Several π'° do not satisfy the first order condition, requiring that $\pi'^\circ(a)$ commutes with derivatives $[D, \pi'(b)]$.