

Noncommutative tori and the Riemann-Hilbert correspondence

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Outline

Goal: Study interplay noncommutative differential and noncommutative algebraic geometry by means of an example.

Common principle: trade spaces for algebras

Examples:

affine varieties X \leftrightarrow finitely generated domains
affine coordinate ring $A(X)$

top. Hausdorff spaces X \leftrightarrow commutative C^* -algebras
continuous functions $C(X)$

- Relax commutativity: $\left\{ \begin{array}{l} \text{noncommutative differential geometry} \\ \text{noncommutative algebraic geometry} \end{array} \right.$
- Applications of NCG in ordinary geometry: use noncommutative algebra to describe “bad quotients”.

Example 1: Noncommutative torus

- Leaf space of Kronecker foliation of torus: non-Hausdorff space $S^1/\theta\mathbb{Z}$ with θ irrational.
- Better description by crossed product $A_\theta = C(S^1) \rtimes_\theta \mathbb{Z}$.
Explicitly: A_θ is generated by **two unitaries** U_1, U_2 such that

$$U_1 U_2 = e^{2\pi i \theta} U_2 U_1$$

- Two corresponding **derivations**: δ_1, δ_2 .
Let $\delta_\omega := \omega_1 \delta_1 + \omega_2 \delta_2$ and if $(\omega_1, \omega_2) = (\tau, 1)$ we set $\delta_\tau := \delta_\omega$.
- A **holomorphic vector bundle** is a pair (E, ∇_ω) with E finite projective A_θ -module and ∇_ω a connection associated to δ_ω
- Denote by $\text{Vect}(\mathbb{T}_\theta^\omega)$ the category of holomorphic vector bundles.

Proposition

- (a) If g is an element in $SL(2, \mathbb{Z})$, then $\text{Vect}(\mathbb{T}_\theta^{g\omega}) \simeq \text{Vect}(\mathbb{T}_\theta^\omega)$.
- (b) If $\omega_2 \neq 0$ and $\tau = \frac{\omega_1}{\omega_2}$, then $\text{Vect}(\mathbb{T}_\theta^\omega) \simeq \text{Vect}(\mathbb{T}_\theta^\tau)$.

The category $\text{Vect}(\mathbb{T}_\theta^\tau)$ turns out to be an abelian category [PS]

Example 2: Noncommutative elliptic curves

- \mathbb{C}^* with action of \mathbb{Z} : $n \cdot z \mapsto q^n z$ for some $q \in \mathbb{C}$.
- Consider the following category of modules over $\mathcal{O}(\mathbb{C}^*)$:

$$\mathcal{B}_q := \left\{ \begin{array}{l} \text{Finitely presentable } \mathbb{Z}\text{-} \\ \text{equivariant } \mathcal{O}(\mathbb{C}^*)\text{-modules} \end{array} \right\}$$

- If $|q| < 1$, $q = e^{2\pi i\tau}$, then \mathcal{B}_q is the category of coherent sheaves \mathcal{F} on the elliptic curve X_τ (i.e. $\mathbb{C}^*/q^{\mathbb{Z}}$), i.e.

$$\mathcal{O}_{\mathbb{C}^*}^m \rightarrow \mathcal{O}_{\mathbb{C}^*}^n \rightarrow \mathcal{F} \rightarrow 0$$

- Again, if $q = e^{2\pi i\theta}$ with θ irrational real then the quotient $\mathbb{C}^*/q^{\mathbb{Z}}$ is not Hausdorff but we can define **the category of coherent sheaves on it by \mathcal{B}_q [SV]**.

Goal: link these two examples functorially via a category \mathcal{B}_q^T and study its properties.

The category \mathcal{B}_q^τ

Intermediate category between \mathcal{B}_q and $\text{Vect}(\mathbb{T}_\theta^\tau)$;

Objects of \mathcal{B}_q^τ are triples (M, σ, ∇) :

- M : finite presentable $\mathcal{O}(\mathbb{C}^*)$ -module
- σ : action of $\theta\mathbb{Z}$ on M that covers the action α on $\mathcal{O}(\mathbb{C}^*)$
- ∇ : $\theta\mathbb{Z}$ -equivariant connection on M lifting $\delta = \tau zd/dz$

We impose that ∇ is regular singular so that there exists a module basis e_k for M such that the holomorphic functions $z^{-1}A_{ij}$ defined by $A_{ij}e_j = \nabla(e_i)$ have simple poles at 0.

- The modules M turn out to be **free**: this follows from the equivariance condition or from the fact that they allow a connection.
- \mathcal{B}_q^τ is an abelian category.

Relation with the noncommutative torus

Consider the following **restriction of a holomorphic function on \mathbb{C}^* to the unit circle, embedded in the smooth noncommutative torus:**

$$\begin{aligned}\psi : \mathcal{O}(\mathbb{C}^*) &\rightarrow \mathcal{A}_\theta \\ \sum_{n \in \mathbb{Z}} f_n z^n &\mapsto \sum_{n \in \mathbb{Z}} f_n U_1^n\end{aligned}$$

- This makes \mathcal{A}_θ a $\mathcal{O}(\mathbb{C}^*)$ -module: $f \cdot a = \psi(f)a$.
- ψ is **injective** since a holomorphic function that vanishes on S^1 , vanishes everywhere on \mathbb{C}^* .

Proposition

The association $(M, \sigma, \nabla) \mapsto (\tilde{M}, \tilde{\nabla})$ defines a functor ψ_* from \mathcal{B}_q^τ to $\text{Vect}(\mathbb{T}_\theta^\tau)$ where

$$\begin{aligned}\tilde{M} &= M \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta \\ \tilde{\nabla} &= \nabla \otimes 1 + 1 \otimes \delta_\tau \quad (\text{with } \delta_\tau = \tau\delta_1 + \delta_2).\end{aligned}$$

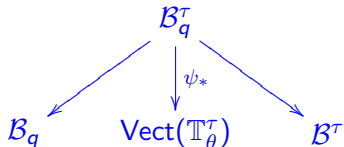
Properties of the functor ψ_*

First of all, it turns out that the module \mathcal{A}_θ over $\mathcal{O}(\mathbb{C}^*)$ (via ψ) is flat (i.e. tensor product $- \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta$ preserves exact sequences). As a corollary to this and to injectivity of ψ :

Proposition

The functor $\psi_* : \mathcal{B}_q^\tau \rightarrow \text{Vect}(\mathbb{T}_\theta^\tau)$ is **exact** and **faithful** (i.e. injective on morphisms).

Summary:



with all arrows faithful and exact functors.

The Riemann-Hilbert correspondence on \mathbb{C}^*

Correspondence between linear differential equations on \mathbb{C}^* and representations of the fundamental group $\pi_1(\mathbb{C}^*, z_0) \simeq \mathbb{Z}$.

Categorically,

{vector bundles on \mathbb{C}^* with a connection ∇ }



{finite dimensional representations of fundamental group}.

The correspondence is given by taking **germs of local solutions** of the differential equation $\nabla U = 0$ at the point z_0 ; the fundamental group acts on the resulting vector space (**monodromy**).

The equivariant Riemann-Hilbert correspondence

Theorem

\mathcal{B}_q^τ is equivalent to the category $\text{Rep}(\mathbb{Z}^2)$ of representations of \mathbb{Z}^2 .

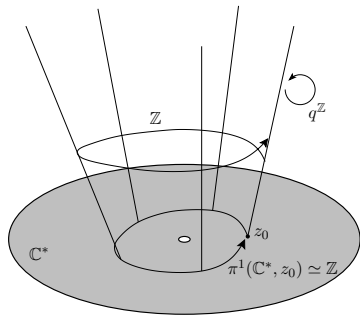
As a consequence, it is a so-called Tannakian category: a (rigid) tensor category with a fiber functor.

- We define a **tensor product** on \mathcal{B}_q^τ as follows:

$$(M, \sigma, \nabla) \otimes (N, \sigma', \nabla') = (M \otimes_{\mathcal{O}(\mathbb{C}^*)} N, \sigma \otimes \sigma', \nabla \otimes 1 + 1 \otimes \nabla')$$

Fiber functor:

$$\begin{aligned} \omega : \mathcal{B}_q^\tau &\rightarrow \text{Vect}_{\mathbb{C}} \\ (M, \sigma, \nabla) &\mapsto (\ker \nabla)_{z_0} \end{aligned}$$



Proof of Theorem

Proposition

For each object in \mathcal{B}_q^T there is an isomorphic object (M, σ, ∇) with

- $M = V \otimes \mathcal{O}(\mathbb{C}^*)$ with V a vector space
- $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ for an invertible constant matrix B .
- $\nabla = \delta + A$ with A a constant matrix with all eigenvalues in the same transversal T to $\theta\mathbb{Z}$ in \mathbb{C} .

N.B. If (M, σ, ∇) and (M', σ', ∇') are of this form (with same transversal) then $\text{Hom}_{\mathcal{O}(\mathbb{C}^*)}(M, M') \subset \text{Hom}_{\mathbb{C}}(V, V')$.

This implies that $\omega : \mathcal{B}_q^T \rightarrow \text{Vect}_{\mathbb{C}}$ is faithful.

Equivalence functor

- Fix a transversal T to $\tau\mathbb{Z}$ in \mathbb{C}^* and let (V, ρ_1, ρ_2) be an object in $\text{Rep}(\mathbb{Z}^2)$.
- We can write $\rho_1(1) =: e^{2\pi i A/\tau}$ such that A is the unique matrix with all eigenvalues in T .
- Also, write $\rho_2(1) =: B$ so that A and B are constant matrices.

They define an element in \mathcal{B}_q^τ denoted $F_T(V, \rho_1, \rho_2)$. For a morphism ϕ in $\text{Rep}(\mathbb{Z}^2)$ we set $F_T(\phi) = \phi \otimes 1$.

Proposition

The above association defines a functor $F_T : \text{Rep}(\mathbb{Z}^2) \rightarrow \mathcal{B}_q^\tau$ which is fully faithful and essentially surjective.