Radboud Universiteit Nijmegen

The unbounded Kasparov module of a vertical Dirac operator

Master Thesis

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Abstract

Unbounded Kasparov modules are the cycles for KK-theory. In this thesis, we look at the geometric setting of a vector bundle sitting over a fibre bundle with a Clifford module structure along the fibres. This allows us to define a vertical Dirac operator. This operator together with a Hilbert C^*-module defined by the sections of the vector bundle gives rise to an unbounded Kasparov module.

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1 Introduction

KK-theory is a common generalization of both K-homology and K-theory. It is an additive bivariant functor from the category of separable C^*-algebras to the category of Abelian groups. This notion was first introduced by the Russian mathematician Gennadi Kasparov in 1980.

The cycles in KK-theory are given by bounded Kasparov modules. Any unbounded Kasparov module defines a bounded Kasparov module via the so-called bounded transform. The unbounded Kasparov module consists of a triple $(\mathcal{A}, X, D)$ satisfying some properties. X is a Hilbert C^* module, $\mathcal{A}$ is an unital $*$-sub-algebra of the bounded adjointable operators on X, and D is an unbounded, self-adjoint regular operator on X. In this thesis we will look a specific geometric setting that defines an unbounded Kasparov module.

The geometric data considered consists of a smooth complex vector bundle sitting over a smooth fibre bundle. The base space of the vector bundle is the total space of the fibre bundle. Assuming the vector bundle has a connection and is a Clifford module along the fibres in the base space, one can define a vertical generalized Dirac operator on the smooth sections. This Dirac operator, together with the smooth sections of the vector bundle, give rise to an unbounded Kasparov module.

This thesis will aim to give an almost complete proof. At first we consider the easier case, that of a vector bundle together with a generalized Dirac operator. The general case reduces to this, if we take the base space of the fibre bundle to be a single point. As described in Section 2, this data defines a spectral triple.

In the following Section 3 we briefly cover the definition of Hilbert C^* modules, and define the unbounded Kasparov module. In Section 4 we introduce the more general case of a vector bundle sitting over a fibre bundle. This allows us to define a vertical tangent bundle, and introduce the exact geometric setting we work in to obtain an unbounded Kasparov module. In Section 5 we show how the geometric setting defines an unbounded Kasparov module. This means we define our Hilbert C^* module, we define the vertical Dirac operator, and we show all necessary properties except for D being self-adjoint and regular and the resolvent of D being compact. It turns proving the self-adjointness and regularity of the Dirac operator, and the compactness of the resolvent are both more involved. These assertions rely on localising our Hilbert C^*-module. The definition of localisations and their properties can be found in Section 6. We then use these localisations in Section 7 to prove the Dirac operator is regular and self-adjoint, and that it's resolvent is compact. The compactness of the resolvent will only be proven in the case where the vector bundle is trivial. In Section 8 there is a short discussion on difficulties for proving compactness for a general vector bundle. Finally, Section 9 summarizes the results obtained in this thesis.

2 Clifford modules and spectral triples

This section will outline how the geometric data consisting of a Clifford module and a generalized Dirac operator yields a spectral triple. Let us start of with the definition of a spectral triple.

2.1 Spectral triples

Definition 2.1.1. Let H be a Hilbert space, $D : \mathrm{Dom}(D) \to H$ be an unbounded operator on H. Furthermore, $\mathcal{A} \subset B(H)$ is a unital $*$-subalgebra of the bounded operators on H. We say $(\mathcal{A}, H, D)$ is an odd spectral triple if it satisfies the following properties:

- The resolvent $(i + D)^{-1}$ is a compact operator on H

- Each element $a \in \mathcal{A}$ has $\mathrm{Dom}(D)$ as an invariant subspace.

- The commutator $[D, a] : \mathrm{Dom}(D) \to H$ extends to a bounded operator on H for all $a \in \mathcal{A}$.

We say $(\mathcal{A}, H, D)$ is an even spectral triple if it satisfies all the above properties, and has an extra *grading operator* $\gamma : H \to H$. γ is a self-adjoint unitary operator on H. It commutes with elements of $\mathcal{A}$ and anti-commutes with D.

Throughout this thesis we will only consider odd spectral triples and odd unbounded Kasparov modules. This means we will not see this grading operator.

2.2 Clifford modules

For a given vector space with a quadratic form, we have the associated Clifford algebra. For more information on Clifford algebras, we refer to [LM89] Chapter 1, Section 1, and [BGV92] Section 3.1.

For a vector bundle with a metric, each fibre is a vector space with an inner product, which gives rise to a Clifford algebra. These algebras can be assembled into a fibre bundle. The case of an Riemannian manifold and its co-tangent space is of special interest.

Definition 2.2.1. Suppose we have a smooth Riemannian manifold M. Every co-tangent space $T^*_m M$ of M above m is a real vector space with an inner product, which in particular gives a quadratic form. This means we get a Clifford algebra for each tangent space, denoted by $C(T^*_m M)$. The *bundle of algebras* $C(M)$ for M is the fibre bundle whose fibre above the point $m \in M$ is $C(T^*_m M)$.

Suppose we have a complex vector bundle $\pi : E \to M$. Each fibre above each point in M is a complex vector space. The bundle of algebras gives us a Clifford algebra above each point in M. The bundle of algebras can have an action on the vector bundle.

Definition 2.2.2 (Clifford Module)**.** A vector bundle $\pi : E \to M$ is said to be a *Clifford module* if we have an action of the bundle of algebras $C(M)$ on E. This means the Clifford algebra

$C(T_m^*M)$ has an action on the vector space E_m for each point $m \in M$, and this action varies smoothly for the point m, i.e. it is a bundle map

$$c : C(M) \to \mathrm{End}(E)$$

Because of the properties of the Clifford algebra it is enough to specify the action of $T_m^*M \subset C(T_m^*M)$ on E_m. We have a bundle map

$$c : T^*M \to \mathrm{End}(E)$$

with the property that $c(\omega)^2 = -g(\omega, \omega)$. Here g is the Riemannian metric on M. Because of the relation between smooth vector bundles and $C^\infty(M)$ modules, we can also view c as a map

$$c : \Gamma^\infty(M, T^*M) \to \Gamma^\infty(M, \mathrm{End}(E))$$

which is the description we will use most often.
Definition 2.2.3. A Clifford module $E \to M$ is said to be *self-adjoint* if for

$$c : \Gamma^\infty(M, T^*M) \to \Gamma^\infty(M, \mathrm{End}(E))$$

we know $c(\omega)(m)$ is a skew-symmetric map on E_m.

Suppose the hermitian vector bundle $\pi : E \to M$ (with metric $\langle\ ,\ \rangle_E$) has a hermitian connection ∇. We know ∇_X is a linear map on the sections of E satisfying the Leibniz rule, i.e.

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s.$$

This connection is hermitian when we have to additional property

$$X(\langle s, t\rangle_E) = \langle \nabla_X s, t\rangle_E + \langle s, \nabla_X t\rangle_E\,.$$

Recall that a Clifford module structure also gives a way of acting on sections. For $a \in \Gamma^\infty(M, T^*M)$, let $c(a)$ be the map on $\Gamma^\infty(M, E)$ defined by $(c(a))s(m) = s(m) \cdot a(m)$. These two maps can be related in the following way.

Definition 2.2.4. Let $\pi : E \to M$ be a Clifford module with connection ∇. The connection is said to be a *Clifford connection* if for any $a \in \Gamma^\infty(M, T^*M)$ and X a vectorfield on M, we have:

$$[\nabla_X, c(a)] = c(\nabla_X^{\mathrm{LC}} a) \tag{1}$$

where ∇_X^{LC} is the Levi-Civita connection on M.

Let $\pi : E \to M$ be a complex l-dimensional Clifford module, with metric $\rangle\ ,\ \langle_E$. Suppose that this Clifford module has the following additional properties:

1. The smooth Riemannian manifold M has odd dimension n and is compact,

2. It has a hermitian connection ∇,

3. The connection is a Clifford connection.

This is setting from which we will build a spectral triple. During the rest of this section the manifolds E and M are assumed to have the properties specified above.

2.3 The Dirac operator

We want to define a first order differential operator on $\Gamma^\infty(M,E)$.

Definition 2.3.1. For a smooth complex vector bundle $\pi : E \to M$, a *first-order linear differential operator* on E is a complex-linear map

$$L : \Gamma^\infty(M,E) \to \Gamma^\infty(M,E),$$

which has the following properties:

- If s_1 and s_2 are smooth sections of E which agree on an open set $U \subset M$ then Ls_1 and Ls_2 agree on U

- For each coordinate patch $U \subset M$, choosing coordinates x_j in U and a trivialization of the bundle E over U, L can be represented in the local coordinates by a formula

$$Ls = \sum_j A_j \left(\frac{\partial}{\partial x_j}\right)(s) + Bs,$$

  where A_j and B are smooth, matrix-valued functions on U

The functions A_j and B which appear above depend on the coordinate system. Let $m \in M$ and $\xi = \sum_j \xi_j dx^j$. Form the expression $\sigma_L(m,\xi) = \sum_j A_j \xi_j$. $\sigma_L(m,\xi)$ can be interpreted as an endomorphism of the vector space E_m, independent of the choice of coordinates.

Definition 2.3.2. The *principal symbol* of L is the vector bundle morphism

$$\sigma_L : T^*M \to End(E)$$

defined by the formula above.

Definition 2.3.3. A first order differential operator is said to be *elliptic* if its symbol $\sigma_L(m,\xi)$ is an isomorphism of E_m for all non-zero $\xi \in T^*_m M$.

For general first order differential we have the following relation

Proposition 2.3.4. The principal symbol for a differential operator L has the following property

$$\sigma_L(x, dg) = [L, g] \tag{2}$$

as a map from $\Gamma^\infty(M,E)$ to $\Gamma^\infty(M,E)$. Here g is a smooth function on M, and dg is the exterior derivative of g. For the commutator we mean the multiplication by g on sections.

Proof. We refer to [HR00] Remarks 10.1.2 □

We can define the following first order differential operator $D : \Gamma^\infty(M,E) \to \Gamma^\infty(M,E)$ on the vector bundle, given locally by:

$$D := \sum_{i=1}^{n} c(dx^i)\nabla_{\frac{\partial}{\partial x_i}}.$$

Here $\frac{\partial}{\partial x_i}$ is the local coordinate frame of TM, and dx^i it's associated co-frame. We call the differential operator D the *generalized Dirac operator*.

Proposition 2.3.5. The Dirac operator is a first order linear differential operator

Proof. Pick an open set over which the bundle and M both trivialize. $c(dx^i)$ and ∇_{p_i} are linear, so D is as well. Let $\{s_j\}_{j=1}^l$ be a local frame of the vector bundle. Writing $s = \sum_{j=1}^l f_j s_j$ we see

$$\begin{aligned} D(s) &= \sum_i \sum_j c(dx^i) \nabla_{\frac{\partial}{\partial x_i}} (f_j s_j) \\ &= \sum_i \sum_j c(dx^i) f_j \nabla_{\frac{\partial}{\partial x_i}} (s_j) + c(dx^i) \frac{\partial}{\partial x_i} (f_j) s_j \end{aligned}$$

Both $c(dx^i)$ and $\nabla_{\frac{\partial}{\partial x_i}}(s_j)$ can locally be written as matrices, which gives D the form of a first order differential operator. □

We can directly calculate the commutator of D with a smooth function.
Lemma 2.3.6. For a smooth function $g \in C^\infty(M)$ we have

$$[D, g] = c(dg)$$

Proof. Take any smooth function $g \in C^\infty(M)$. We can write dg in terms of the 1-forms $\{dx^i\}$, which gives us $dg = \sum_i \frac{\partial g}{\partial x_i} dx^i$. Writing out the left of the equation, on any section s supported in a trivializing subset of the bundle in local coordinates gives us

$$\begin{aligned} [D, g] (s) &= (Dg - gD) (s) \\ &= \sum_{i=1}^n c(dx^i) \nabla_{\frac{\partial}{\partial x_i}} (gs) - gc(dx^i) \nabla_{\frac{\partial}{\partial x_i}} (s) \\ &= \sum_{i=1}^n c(dx^i) \frac{\partial g}{\partial x_i} s + c(dx^i) \left(g \nabla_{\frac{\partial}{\partial x_i}} (s) \right) - g \left(c(dx^i) \nabla_{\frac{\partial}{\partial x_i}} (s) \right) \end{aligned}$$

Where we used that for any connection $\nabla_{\frac{\partial}{\partial x_i}}(gs) = \frac{\partial g}{\partial x_i} s + g \nabla_{\frac{\partial}{\partial x_i}}(s)$. Note that the action of smooth functions on sections is defined point-wise, and the $c(dx^i)$ are linear maps on the fibres of E. This means $c(dx^i) \left(g \nabla_{\frac{\partial}{\partial x_i}} (s) \right) = g \left(c(dx^i) \nabla_{\frac{\partial}{\partial x_i}} (s) \right)$, so we get

$$\begin{aligned} [D, g] (s) &= \sum_{i=1}^n c(dx^i) \frac{\partial g}{\partial x_i} s \\ &= c(dg) s \end{aligned}$$

using linearity of c, which proves the lemma. □

The identity from the above lemma allows us to conclude D is elliptic.

Corollary 2.3.7. The Dirac operator is elliptic, and for the principal symbol we have the following identity

$$\sigma_D(m, \xi) = c(\xi) \tag{3}$$

For ξ a cotangent vector in $T_m M^*$.

Proof. We combine Proposition 2.3.4 and Lemma 2.3.6. We know $c(\xi)$ is invertible away from 0, so the principal symbol is invertible. □

2.4 Constructing the spectral triple

Our goal is to prove this Dirac operator together with $\Gamma^\infty(M,E)$ gives rise to a spectral triple. We first need to define a Hilbert space in which $\Gamma^\infty(M,E)$ lies densely. Recall that, as M is compact we can integrate smooth functions on M using the density defined by the Riemannian metric on M. More information about integration using densities can be found in [Lee02] Chapter 14.

Recall that as we have a metric, for any $s,t \in \Gamma^\infty(M,E)$, we know $\langle s,t\rangle_E$ is a smooth function, which can be integrated. As the integration is linear, we obtain a pre-Hilbert space for $\Gamma^\infty(M,E)$ together with the inner product defined as $\langle s,t\rangle := \int_M \langle s,t\rangle_E \, dV_g$. Here dV_g is the density coming from the Riemannian metric g on M. $\Gamma^\infty(M,E)$ has a completion which we will write as $L^2(M,E)$, which is now a Hilbert space. As the sections of E are dense in $L^2(M,E)$ we see that D is an unbounded operator on this Hilbert space.

We know the smooth sections are a left-module over the smooth (complex) functions on M. These smooth functions $C^\infty(M)$ will give rise to bounded operators on $L^2(M,E)$ by multiplication. For $f \in C^\infty(M)$, we have $\phi(f) : \Gamma^\infty(M,E) \to \Gamma^\infty(M,E)$ defined by $\phi(f)(s) = fs$ for a section s. This gives us an unbounded operator, but these can easily be extended to bounded operators on $L^2(M,E)$.

Proposition 2.4.1. For every $f \in C^\infty(M)$, $\phi(f)$ extends to a bounded operator. These bounded operators form a unital $*$-sub-algebra of the bounded operators on $L^2(M,E)$.

Proof. We see, for any $s \in \Gamma^\infty(M,E)$,

$$\begin{aligned}\langle \phi(f)s, \phi(f)s\rangle &= \int_M \langle fs, fs\rangle_E \, dV_g \\ &\leq \int_M \|f\|_\infty^2 \langle s,s\rangle \, dV_g \\ &\leq \|f\|_\infty^2 \langle s,s\rangle .\end{aligned}$$

so $\phi(f)$ is bounded on $\Gamma^\infty(M,E)$, which means it extends to a bounded map on $L^2(M,E)$. Note that $\phi(f)+\phi(g) = \phi(f+g)$ and $\phi(f)\phi(g) = \phi(fg)$, with the fact that $\phi(\mathbb{1}_M) = \mathrm{Id}_H$ to conclude $C^\infty(M)$ is indeed a unital $*$-sub-algebra of the bounded operators on H. □

This gives us all the ingredients for a spectral triple.

Theorem 2.4.2. The triple $(C^\infty(M), L^2(M,E), \overline{D})$ is an odd spectral triple

Note that we use the closure of D, which has not been proven to exist yet. A first good step towards proving this is to show that D is a symmetric operator. This immediately shows D is closable, and that D^* extends D.

Proposition 2.4.3. The Dirac operator D is a symmetric operator.

Proof. We have to show that for any two sections $s,t \in \Gamma^\infty(M,E)$, we have the following equality:

$$\langle s, Dt\rangle = \langle Ds, t\rangle .$$

Note that this is an equality of scalars, not of smooth functions. Recall that $\langle s, Dt\rangle := \int_M \langle s, D(t)\rangle \, dV_g$. First, let us look at the terms inside the integral. For some $m \in M$, we can find a coordinate frame $\{\partial_i\}$. We know the form of D in local coordinates. Using the compatibility of the connection with the metric, and the fact that we have a Clifford connection we get the following:

$$\begin{aligned}\langle s(x), D(t(x))\rangle &= \sum_{i=1}^{n} \left\langle s(x), c(dx^i)\nabla_{\partial_i}(t)(x)\right\rangle \\ &= -\sum_{i=1}^{n} \left\langle c(dx^i)s(x), \nabla_{\partial_i} t(x)\right\rangle,\end{aligned}$$

As the $c(dx^i)$ are skew symmetric. We now use the compatibility of the connection, by inserting $L_X(\langle s, t\rangle) = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$ into the above. This yields:

$$\begin{aligned}\langle s(x), D(t(x))\rangle &= \sum_i \left\langle \nabla_{\partial_i} c(dx^i)s(x), t(x)\right\rangle - \sum_i L_{\partial_i}\left\langle c(dx^i)s(x), t(x)\right\rangle \\ &= \sum_i \left\langle c(dx^i)\nabla_{\partial_i} s(x), t(x)\right\rangle + \sum_i \left\langle c(\nabla^{\mathrm{LC}}_{\partial_i} dx^i)s(x), t(x)\right\rangle \\ &\quad - \sum_i L_{\partial_i}\left(\left\langle c(dx^i)s(x), t(x)\right\rangle\right) \\ &= \langle Ds, t\rangle(x) + \sum_i \left\langle c(\nabla^{\mathrm{LC}}_{\partial_i} dx^i)s(x), t(x)\right\rangle - \sum_i L_{\partial_i}\left(\left\langle c(dx^i)s, t\right\rangle(x)\right) \qquad (4)\end{aligned}$$

where we used $c(\nabla^{\mathrm{LC}}_{\partial_i} dx^i) = \nabla_{\partial_i} c(dx^i) - c(dx^i)\nabla_{\partial_i}$ in the second line, as the connection is a Clifford connection. We end up with three terms. The term we desire, and two term which have to integrate to zero over M.

Let use rewrite the second term slightly (using linearity of the metric and Clifford multiplication) to $\left\langle c\left(\sum_i \nabla^{\mathrm{LC}}_{\partial_i}(dx^i)\right) s(x), t(x)\right\rangle$. We want to understand what the Clifford action of the 1-form $\sum_i \nabla^{\mathrm{LC}}_{\partial_i}(dx^i)$ is. To do this, we write this 1-form in the basis dx^i by finding smooth maps f_i such that

$$\sum_i \nabla^{\mathrm{LC}}_{\partial_i}(dx^i) = \sum_i f_i dx^i,$$

. Finding these f_i can be done by solving:

$$f_j = \sum_i \nabla^{\mathrm{LC}}_{\partial_i}(dx^i)(\partial_j).$$

Recall that by definition of the dual connection we have: $\nabla^{\mathrm{LC}}_{\partial_i}(dx^i)(\partial_j) = dx^i\left(\nabla^{\mathrm{LC}}_{\partial_i}(\partial_j)\right)$.

The Levi-Civita connection is defined using the Riemannian metric on the tangent bundle of M. $\nabla^{\mathrm{LC}}_{\partial_i}$ is defined by the following expression using the metric on TM:

$$g(\nabla^{\mathrm{LC}}_{\partial_i}\partial_j, \partial_k) = \frac{1}{2}\left(\partial_i(g(\partial_j)(\partial_k)) + \partial_j(g(\partial_k)(\partial_i)) - \partial_k(g(\partial_i)(\partial_j))\right).$$

Writing $(g_{i,j}) = g(\partial_i, \partial_j)$ as a matrix of smooth function, we have

$$g(\nabla^{\mathrm{LC}}_{\partial_i}\partial_j)(\partial_k) = \frac{1}{2}(\partial_i g_{j,k} + \partial_j g_{k,i} - \partial_k g_{i,j}).$$

To compute $dx^i\left(\nabla^{\mathrm{LC}}_{\partial_i}(\partial_j)\right)$ recall we can write dx^i in the following form:

$$dx^i = \sum_p (g^{-1})_{i,p} g(\partial_p, \cdot).$$

We get

$$\begin{aligned} f_j &= \sum_i \sum_p (g^{-1})_{i,p} \left\langle \partial_p, \nabla^{\mathrm{LC}}_{\partial_i} p_j \right\rangle \\ &= \sum_i \sum_p (g^{-1})_{i,p} \left\langle \nabla^{\mathrm{LC}}_{\partial_i} p_j, \partial_p \right\rangle \\ &= \frac{1}{2} \sum_i \sum_p (g^{-1})_{i,p} (\partial_i g_{j,k} + \partial_j g_{k,i} - \partial_k g_{i,j}) \\ &= \frac{1}{2} \sum_i \sum_p (g^{-1})_{i,p} \partial_j g_{p,i} \end{aligned}$$

Let us write $\partial_j(g)$ for the matrix of smooth functions which has entries $\partial_j(g_{p,i})$. We get

$$\begin{pmatrix} g^{-1}_{1,1} & \dots & g^{-1}_{1,n} \\ \vdots & & \vdots \\ g^{-1}_{n,1} & \dots & g^{-1}_{n,n} \end{pmatrix} \cdot \begin{pmatrix} \partial_j g_{1,1} & \dots & \partial_j g_{1,n} \\ \vdots & & \vdots \\ \partial_j g_{n,1} & \dots & \partial_j g_{n,n} \end{pmatrix} = \begin{pmatrix} \sum_p g^{-1}_{1,p} \partial_j g_{p,1} & \sum_p g^{-1}_{1,p} \partial_j g_{p,2} & \dots & \sum_p g^{-1}_{1,p} \partial_j g_{p,n} \\ \vdots & & & \vdots \\ \sum_p g^{-1}_{n,p} \partial_j g_{p,1} & \sum_p g^{-1}_{n,p} \partial_j g_{p,2} & \dots & \sum_p g^{-1}_{n,p} \partial_j g_{p,n} \end{pmatrix},$$

which shows us

$$\begin{aligned} f_j &= \frac{1}{2} \sum_i \sum_p (g^{-1})_{i,p} \partial_j g_{p,i} \\ &= \frac{1}{2} \operatorname{Tr}\left(g^{-1} \partial_j(g)\right) \\ &= \partial_i \left(\frac{1}{2} \operatorname{Tr} \log(g) \right) \end{aligned}$$

In the last equality we use that $\operatorname{Tr}\left(g^{-1}\partial_j(g)\right)$ is equal to $\partial_j\left(\operatorname{Tr}\log(g)\right)$. This equality can be proven by realising that g can be diagonalized. We write $g = QDQ^t$, where Q is a smooth map from U to the orthogonal real matrices and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a fixed diagonal matrix. Note that $g^{-1} = QD^{-1}Q^t$. We see

$$\begin{aligned} \operatorname{Tr}\left(g^{-1}\partial_j(g)\right) &= \operatorname{Tr}\left(QD^{-1}Q^t \partial_j(QDQ^t)\right) \\ &= \operatorname{Tr}\left(QD^{-1}Q^t \left(\partial_j(Q)(DQ^t) + Q\partial_j(DQ^t)\right)\right) \\ &= \operatorname{Tr}\left(Q^t \partial_j(Q)\right) + \operatorname{Tr}\left(QD^{-1}\partial_j(DQ^t)\right) \\ &= \operatorname{Tr}\left(Q^t \partial_j(Q)\right) + \operatorname{Tr}\left(QD^{-1}\partial_j(D)Q^t\right) + \operatorname{Tr}\left(QD^{-1}D\partial_j(Q^t)\right) \end{aligned}$$

As $\partial_j(QQ^t) = 0$, using the Leibniz rule, we see $\partial_j(Q^t) = -Q^t\partial_j(Q)Q$, and get

$$\begin{aligned} \operatorname{Tr}\left(g^{-1}\partial_j(g)\right) &= \operatorname{Tr}\left(Q^t\partial_j(Q)\right) - \operatorname{Tr}\left(QQ^t\partial_j(Q)Q^t\right) + \operatorname{Tr}\left(D^{-1}\partial_j(D)\right) \\ &= \operatorname{Tr}\left(D^{-1}\partial_j(D)\right) \\ &= \sum_i \lambda_i^{-1} \partial_j\left(\lambda_i\right) \end{aligned}$$

For the part with the logarithm we see

$$\begin{aligned}
\partial_j\left(\operatorname{Tr}\log\left(QDQ^t\right)\right) &= \partial_j\left(\operatorname{Tr}\left(Q\log(D)Q^t\right)\right)\\
&= \partial_j\left(\operatorname{Tr}\log(D)\right)\\
&= \partial_j\left(\sum_i \log(\lambda_i)\right)\\
&= \sum_i \lambda_i^{-1}\partial_j\left(\lambda_i\right)
\end{aligned}$$

which proves $\partial_j\left(\operatorname{Tr}\log g\right) = \operatorname{Tr}\left(g^{-1}\partial_j g\right)$. We now have $f_j = \frac{1}{2}\operatorname{Tr}\left(g^{-1}\partial_j(g)\right) = \partial_i\left(\frac{1}{2}\operatorname{Tr}\log(g)\right)$.

Now we have expressed the f_j in terms of the Riemannian metric and the coordinate frame, we can rewrite the last two terms in equation 4 to

$$\begin{aligned}
\sum_i \left\langle c(\nabla^{\mathrm{LC}}_{\partial_i}dx^i)s(x), t(x)\right\rangle - \sum_i L_{\partial_i}\left(\left\langle c(dx^i)s, t\right\rangle(x)\right) &= \left\langle c(\sum_i \nabla^{\mathrm{LC}}_{\partial_i}dx^i)s(x), t(x)\right\rangle - \sum_i L_{\partial_i}\left(\left\langle c(dx^i)s, t\right\rangle(x)\right)\\
&= \sum_i\left(\left\langle f_i c(dx^i)s, t\right\rangle(x) + L_{\partial_i}\left\langle c(dx^i)s, t\right\rangle(x)\right)\\
&= \sum_i f_i h_i + \partial_i(h_i),
\end{aligned}$$

where we wrote $h_i := \left\langle c(dx^i)s, t\right\rangle$.

Our goal is to use the divergence theorem. It tells us that for a vectorfield $X = \sum_i X_i\partial_i$ we know

$$\int_M \operatorname{div}(X)dV_g = 0$$

where $\operatorname{div}(X)dV_g = \sum_i \partial_i\left(\sqrt{\det(g)}X_i\right)$. By setting $X = \sum_i h_i\partial_i$, we can show that $\operatorname{div}(X)dV_g = \left(\sum_i f_i h_i + \partial_i(h_i)\right)\sqrt{\det(g)}$, which completes the proof.

We can see, as the determinant of g is the sum of its eigenvalues that

$$\det(g) = \exp(\operatorname{Tr}(\log(g))).$$

And note $\sqrt{\det(g)} = \exp(\frac{1}{2}\operatorname{Tr}(\log(g)))$. We see, using the chain rule

$$\begin{aligned}
\partial_i\left(\sqrt{\det(g)}\right) &= \partial_i\left(\exp(\frac{1}{2}\operatorname{Tr}(\log(g)))\right)\\
&= \frac{\partial}{\partial x_i}\left(\exp\circ\left(\frac{1}{2}\operatorname{Tr}(\log(g))\right)\circ\phi^{-1}\right)\\
&= \exp\left(\frac{1}{2}\operatorname{Tr}\log(g)\right)\partial_i\left(\frac{1}{2}\operatorname{Tr}\log(g)\right)\\
&= \sqrt{\det(g)}f_i
\end{aligned}$$

For every coordinate patch (U, ϕ), we have the smooth function $\left(\sum_i f_i h_i + \partial_i(h_i)\right)\sqrt{\det(g)}$, which using the above is in fact the divergence of X.

$$\begin{aligned}
\left(\sum_i f_i h_i + \partial_i(h_i)\right)\sqrt{\det(g)} &= \sum_i \sqrt{\det(g)} f_i h_i + \sqrt{\det(g)}\partial_i(h_i) \\
&= \sum_i \partial_i\left(\sqrt{\det(g)} h_i\right) \\
&= \operatorname{div}(X) dV_g
\end{aligned}$$

for $X = \sum_i h_i \partial_i$. Combining all the above we can prove the theorem.

Let $\mathcal{U} = \{U_j\}$ be a cover of M by coordinate patches (U_j, ϕ_j) and let ψ_j be a partition of unity subordinate to $\mathcal{U}$. We have

$$\begin{aligned}
\langle s, Dt\rangle &= \int_M \langle s(x), D(t(x))\rangle \, dV_g \\
&= \sum_j \int_{U_j} \psi_j \langle s(x), D(t(x))\rangle \, dV_g \\
&= \sum_j \int_{U_j} \psi_j \left(\langle Ds, t\rangle (x)\right) dV_g \\
&\quad + \sum_j \int_{U_j} \psi_j \left(\sum_i \left\langle c(\nabla^{\mathrm{LC}}_{\partial_i} dx^i) s(x), t(x)\right\rangle - \sum_i L_{\partial_i}\left(\left\langle c(dx^i) s, t\right\rangle (x)\right)\right) dV_g \\
&= \langle Ds, t\rangle + \sum_j \int_{U_j} \psi_j \left(\sum_i f_i h_i + \partial_i(h_i)\right) dV_g \\
&= \langle Ds, t\rangle + \sum_j \int_{U_j} \psi_j \left(\sum_i f_i h_i + \partial_i(h_i)\right) \sqrt{\det(g)} dx^1 \wedge \ldots \wedge dx^n \\
&= \langle Ds, t\rangle + \sum_j \int_{U_j} \psi_j \operatorname{div}(X) dV_g \\
&= \langle Ds, t\rangle + \int_M \operatorname{div}(X) dV_g \\
&= \langle Ds, t\rangle,
\end{aligned}$$

Where we used the divergence theorem in the last line. This proves D is symmetric. □

We now know D is symmetric, and therefore closable and extended by its adjoint. The closure of D is the smallest closed extension so we get:

$$D \subset \overline{D} \subset D^*.$$

Applying the adjoint to the above, and using the fact $(D^*)^* = \overline{D}$, we get: $\overline{D} \subset \overline{D}^* \subset D^*$. This proves $\overline{D}$ is symmetric as well. We only need to prove $\overline{D}$ is self-adjoint. To do this, we use a general result about symmetric first order differential operators.

Proposition 2.4.4. The closure $\overline{D}$ is self-adjoint.

Proof. For the proof of this proposition we refer to [HR00] lemma 10.2.5, and the corollary 10.2.6. It tells us that any symmetric, closed operator on a compact manifold, has a self-adjoint closure. □

We now look at the resolvent. As the operator $\overline{D}$ is self adjoint, we know the operators $i + \overline{D}$ and $i - \overline{D}$ are bijective maps from $\mathrm{Dom}(\overline{D})$ to $L^2(M, E)$. This means the operator $\left(i + \overline{D}\right)^{-1}$ exists, and we know it is bounded. Our next goal is to prove the operator $(i + \overline{D})^{-1}$ is compact. We use the Rellich Lemma and Gårding's inequality (See [HR00] 10.4.3 and 10.4.4). We would like to stress that Gårding's inequality uses the ellipticity of D.

Proposition 2.4.5. The operator $(i + \overline{D})^{-1}$ is compact.

Proof. Suppose we have a sequence $\{\xi_n\}$ in the unit ball of $L^2(M, E)$. Our goal is to prove the sequence $\{(i + \overline{D})^{-1}\xi_n\}$ has a Cauchy subsequence. We know, using Gårding's inequality that

$$\|(i + \overline{D})^{-1}\xi_n\|_1 \leq \frac{1}{c}\|\xi_n\|$$

So by rescaling the ξ_n, we know $(i\overline{D})\xi_n$ lie in the unit ball of the Sobolev space $L^2_1(M, E)$. Now using the Rellich Lemma (which says $i : L^2_1(M, E) \to L^2(M, E)$ is compact), we see the sequence $\{(i + \overline{D})^{-1}\xi_n\}$ has a Cauchy subsequence (in $L^2(M, E)$). □

For the last part of proving we have a spectral triple, we need to show multiplication by $C^\infty(M)$ preserves $\mathrm{Dom}(\overline{D})$ and that the operators $[\overline{D}, g] : \mathrm{Dom}(\overline{D}) \to L^2(M, E)$ extend to bounded operators.

Proposition 2.4.6. Multiplication by g maps $\mathrm{Dom}(\overline{D})$ into itself, and the commutator $[\overline{D}, g] : \mathrm{Dom}(\overline{D}) \to L^2(M, E)$ extends to a bounded operator on $L^2(M, E)$.

Proof. Recall $[D, g] = c(dg)$ as a map on $\Gamma^\infty(M, E)$. Locally the Clifford action $c(dg)$ is given by a smooth function taking values in the n by n matrices, so the map $c(dg)$ is certainly bounded as a map on $L^2(M, E)$.

Pick any $x \in \mathrm{Dom}(\overline{D})$ and $g \in C^\infty(M)$. We know the graph of D, $\mathcal{G}(D)$ is dense in $\mathcal{G}(\overline{D})$, so we have a sequence $(x_n, D(x_n)) \to (x, \overline{D}(x))$. As multiplication with g is a bounded map, we see $gx_n \to gx$.

If we prove that $D(gx_n)$ is convergent in X, we know $gx \in \mathrm{Dom}(\overline{D})$, as we have found a sequence $(gx_n, D(gx_n))$ that converges, and $\overline{D}$ is closed.

We know $c(dg)$ is bounded, and $[D, g](x_n) = c(dg)(x_n)$ is convergent. Note that $gD(x_n)$ converges to $g\overline{D}(x)$, as $D(x_n)$ converges. We conclude $D(gx_n)$ converges to $\overline{D}(gx)$, which means g maps $\mathrm{Dom}(\overline{D})$ into itself.

We see that $[D, g] = c(dg)$ is continuous on $\Gamma^\infty(M, E)$. $\Gamma^\infty(M, E)$ is dense in X and for $x \in \Gamma^\infty(M, E)$, $[\overline{D}, g](x) = [D, g](x)$. This means $[\overline{D}, g]$ is bounded. □

Let us repeat the main theorem of this first part again to conclude the spectral triple case.

Theorem 2.4.7. The triple $(C^\infty(M), L^2(M, E), \overline{D})$ is an odd spectral triple

Proof. Proposition 2.4.4 proves $\overline{D}$ is self-adjoint, and Proposition 2.4.5 proves the resolvent is compact. The above argumentation shows the commutator of $\overline{D}$ and smooth functions on M extends to a bounded map. These are all properties stated in Definition 2.1.1, the definition of a spectral triple. □

3 Unbounded Kasparov Modules

3.1 Hilbert C^* modules

In order to understand the generalisation of a spectral triple, we will need to introduce a so-called Hilbert C^*-module. A good source for information on these spaces and the theory concerning them is [Lan95]. In this section we will quickly give an outline of the definitions essential to Hilbert C^*-modules.

A Hilbert space is a Banach-space where the norm comes from an inner product. The inner product takes values in the complex numbers, which in particular form a C^*-algebra. The generalisation of a Hilbert space will be a Banach-space with an inner product that takes values in some C^*-algebra. Let us first introduce this idea without the demand for the space to be complete.

Definition 3.1.1. Let $\mathcal{A}$ be a C^*-algebra. An *inner-product $\mathcal{A}$-module* is a linear space X which is a right $\mathcal{A}$-module (with compatible scalar multiplication) together with a map $\langle\ ,\ \rangle : X \times X \to \mathcal{A}$ with the following four properties

$$\begin{aligned} \langle x, \alpha y + \beta z\rangle &= \alpha \langle x, y\rangle + \beta \langle x, z\rangle && (x, y, z \in E, \alpha, \beta \in \mathbb{C}), \\ \langle x, ya\rangle &= \langle x, y\rangle a && (x, y \in E, a \in \mathcal{A}), \\ \langle y, x\rangle &= \langle x, y\rangle^* && (x, y \in E), \\ \langle x, x\rangle &\geq 0; \quad \text{if } \langle x, x\rangle = 0 \text{ then } x = 0 && (x \in E). \end{aligned}$$

In the last inequality, the ≥ 0 indicates positivity in the C^* sense.

Suppose E is an inner-product $\mathcal{A}$ module. Generally the norm on a Hilbert space is given the square root of the inner product, i.e. $\|v\| = \sqrt{\langle v, v\rangle}$. This time, in order to define a norm we use both the $\mathcal{A}$-valued inner product, together with the norm on $\mathcal{A}$. We set for $x \in E$:

$$\|x\| = \sqrt{\| \langle x, x\rangle \|_{\mathcal{A}}}.$$

Proposition [Lan95] shows this is a norm. Using the norm we can define

Definition 3.1.2. An inner-product $\mathcal{A}$-module which is complete with respect to its norm is called a *Hilbert C^*-module* over $\mathcal{A}$.

3.2 Definition of the unbounded Kasparov module

In this section we define the Hilbert C^*-module equivalent of an unbounded spectral triple. This generalisation replaces the Hilbert space with a Hilbert C^*-algebra.

Definition 3.2.1. Let $\mathcal{B}$ be a C^*-algebra. An *odd unbounded Kasparov module* is a triple $(\mathcal{A}, X, D)$ where

1. X is a Hilbert C^*-module for C^*-algebra $\mathcal{B}$.
2. $\mathcal{A}$ is an unital $*$-sub algebra of the bounded adjointable operators on X.

3. D is a self-adjoint regular operator on X,

with the following additional properties for D

1. $(i + D)^{-1}$ is an element of $\mathcal{K}(X)$.
2. $a(Dom(D)) \subset Dom(D) \quad \forall a \in \mathcal{A}$.
3. $[D, \phi(a)]$ has a bounded extension for all $a \in \mathcal{A}$.

Note that other than replacing the Hilbert space by a Hilbert C^*-module very little changed. The most important different is that D is self adjoint and *regular*. The regularity is needed, because for a self-adjoint map on a Hilbert C^* module, the resolvent may not exist.

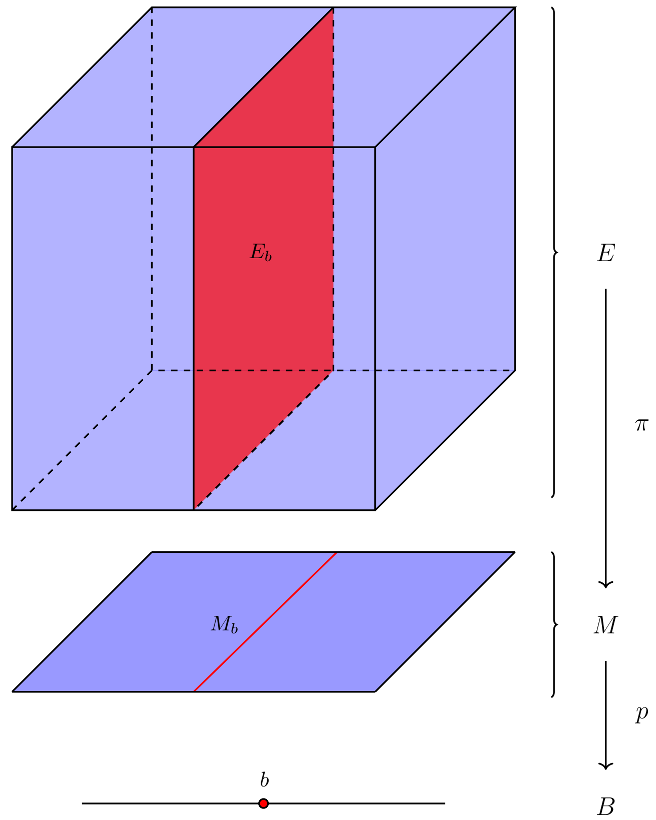
In the previous section we looked at spectral triples, and a geometric example that gives such a spectral triple. Now that we have a generalisation of the spectral triple, we are going to look at a more general case of our geometric setting.

4 Vector bundle above a fibre bundle, the geometric setting

In section 2.4 we proved that a smooth complex vector bundle over a compact base space gives an unbounded spectral triple. The goal of this thesis is to give a slightly more general setting, and proving that this gives an unbounded Kasparov module.

Essentially we will add a fibre-bundle structure to the base space of the smooth vector bundle. We get $\pi : E \to M$ a smooth complex vector bundle, and $p : M \to B$ a smooth fibre bundle with fibre Z. The base space of the vector bundle is the total space of the fibre bundle. This extra fibre bundle structure on M adds extra structure in two important ways.

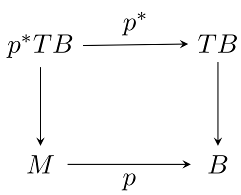
First, it allows us to define a smooth vector bundle for each $b \in B$, by restricting the bundle $E \to M$ to a slice that lies above $b \in B$. We write $M_b := p^{-1}\{b\}$ for $b \in B$ and $E_b := \pi^{-1}(M_b)$ for $b \in B$. Because both p and π are smooth submersions, we know M_b and E_b are smooth manifolds (of dimension $dim(M) - dim(B)$ and $dim(E) - dim(B)$ respectively). Note that each M_b is isomorphic to the fibre Z. Restricting π to these subspaces gives us a complex vector sub-bundle $\pi_{|M_b} : E_b \to M_b$. As we use these sub-manifolds extensively, let us underline the idea with a conceptual drawing.



Secondly, as M is the total space of a fibre bundle, it allows us to look at tangent-vectors which run along the fibres defined by $M \to B$. The sub-bundle of $T(M)$ consisting of all tangent-vectors which run along the fibre will be called the vertical bundle. In the next sections we will quickly explore it's definition and properties.

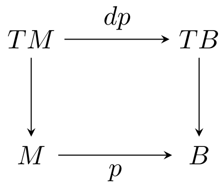
4.1 The vertical bundle

We are going to define a sub-bundle of the tangent bundle of M, which consists of tangent-vectors which lie along the fibres in M. Let us first look at the pull-back bundle defined by p, which is the bundle p^*TB over M.



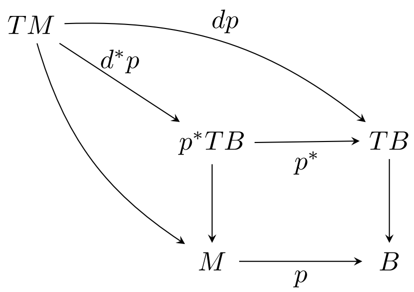
where $p^*TB = \{(m, (b, X)) \in M \times TB \mid p(m) = b\}$ is the pull-back bundle of p with the obvious projections to M and TB. It essentially is given by pasting the tangent space T_bB above all points $m \in M$, for which $p(m) = b$.

We can also look at the differential of p, which is a bundle map $dp : TM \to TB$, given by pushing forward all the tangent vectors.



For a $X \in T_m M$, $dp(X)$ is defined by $dp(X)(f) = X(f \circ p)$ for all $f \in C^\infty(M, \mathbb{R})$.

We use the two maps above (p^* and dp) together with the universal property of the pull-back to obtain a map from TM to p^*TB.



Because p is a smooth submersion, we get the following exact sequence

$$TM \overset{d^*p}{\twoheadrightarrow} p^*TB \to 0$$

The kernel of d^*p will be a smooth vector sub-bundle of TM called the *vectical bundle* above M, denoted by $T(M/B)$. For clarity let us write out the specific vector space above a point $m \in M$:

$$T_m(M/B) = \{X \in T_m M \mid X(f \circ p) = 0 \quad \forall f \in C^\infty(B)\}$$

The bundle $T(M/B)$ contains all tangent-vectors of M which run along the fibres defined by the bundle $M \to B$.

4.2 Splitting and the vertical connection

The vertical bundle, in combination with the Riemannian metric allows the definition of the horizontal bundle

Definition 4.2.1. The *horizontal bundle* is a sub-bundle of TM defined by

$$T_H M = \coprod_m (\ker(d^* p)m)^\perp.$$

It is isomorphic to $p^* TB$.

We can write $TM = T_H M \oplus T(M/B)$ and get an orthogonal projection $P : TM \to T(M/B)$ onto $T(M/B)$. We can use this projection to define a connection $\nabla^{M/B}$ on $T(M/B)$. Recall that, as M is Riemannian we have the Levi-Civita connection on TM, which will be denoted by ∇^{LC}. Using this define

$$\begin{aligned} \nabla_X^{M/B} : \Gamma^\infty (M, TM/B) &\to \Gamma^\infty (M, TM/B) \\ s &\mapsto P\nabla_X^{\mathrm{LC}}(s), \end{aligned}$$

for $X \in \Gamma^\infty (M, TM)$.

Proposition 4.2.2. $\nabla_X^{M/B}$ is a Riemannian connection on $T(M/B)$.

Proof. Let $f \in C^\infty(M)$.

$$\begin{aligned} \nabla_{fX}^{M/B}(s) &= P\left(\nabla_{fX}^{\mathrm{LC}}(s)\right) \\ &= P\left(f\nabla_X^{\mathrm{LC}}(s)\right) \\ &= f\nabla_X^{M/B}(s) \end{aligned}$$

and

$$\begin{aligned} \nabla_X^{M/B}(fs) &= P\left(\nabla_X^{\mathrm{LC}}(fs)\right) \\ &= P\left(f\nabla_X^{\mathrm{LC}}(s) + LX(f)s\right) \\ &= f\nabla_X^{M/B}(s) + X(f)s, \end{aligned}$$

because $(X(f)s)(m) = X(f) \cdot s(m) \in T_m(M/B)$ if $s \in \Gamma^\infty (M, TM/B))$ which means $X(f)s = P(X(f)s)$. ☐

Because of orthogonality, this connection is still Riemannian, i.e. compatible with g (restricted to $T(M/B)$.

Proof.

$$\begin{aligned} \left\langle \nabla_X^{M/B} Y, Z \right\rangle + \left\langle Y, \nabla_X^{M/B} Z \right\rangle &= \left\langle P\nabla_X^E Y, Z \right\rangle + \left\langle Y, P\nabla_X^E Z \right\rangle \\ &= \left\langle \nabla_X^E Y, PZ \right\rangle + \left\langle PY, \nabla_X^E Z \right\rangle \\ &= \left\langle \nabla_X^E Y, Z \right\rangle + \left\langle Y, \nabla_X^E Z \right\rangle \\ &= X(\langle Y, Z \rangle). \end{aligned}$$

☐

This connection behaves like a 'vertical' Levi-Cevita connection. When we restrict to slices M_b in the upcoming sections, we will see $\nabla^{M/B}$ restricts to such a slice, and is equal to the Levi-Cevita connection on that sub-bundle.

4.3 Clifford module along fibres

Let $\pi : E \to M$ be a smooth hermitian vector bundle. Let $p : M \to B$ will be a smooth fibre bundle, with the associated bundle of vertical vectors denoted by M/B. Let ∇^E be a hermitian connection on the vector bundle E.

Definition 4.3.1. We say E is a self-adjoint *Clifford module along the fibres of* M/B if we have a skew-adjoint action

$$c : C(T^*(M/B)) \to End(E)$$

of the bundle of algebra's of the co-tangent bundle of the vertical vectors

As before, this action and the connection on E can be compatible.

Definition 4.3.2. Let E be a Clifford module along the fibres of M/B. The connection ∇^E is a Clifford connection if

$$\left[\nabla^E_X, c(\alpha)\right] = c\left(\nabla^{M/B}_X \alpha\right).$$

for $X \in \Gamma^\infty(M, TM)$ and $\alpha \in \Gamma^\infty(M, T^*(M/B))$.

Note that on the right, we actually use the dual connection of $\nabla^{M/B}$. There is a 1 to 1 correspondence between connections and dual connections.

4.4 The geometric setting of the unbounded Kasparov module

We consider the following situation

1. M is a compact Riemannian smooth manifold, B is a compact manifold,
2. $\pi : E \to M$ is a smooth hermitian vector bundle,
3. $p : M \to B$ is a smooth fibre bundle with fibre Z,
4. ∇^E is a hermitian connection on E,
5. E is a self-adjoint Clifford module along the fibres of M/B.

In all the sections to follow we will assume the above data.

5 Proving geometric data gives an unbounded Kasparov module

We would like to prove that the geometric data gives us an unbounded Kasparov module. We start with defining the Hilbert C^*-module. After this is established, we introduce the vertical generalized Dirac operator.

5.1 Hilbert $C(B)$-module

Analogously to the spectral triple case, we want to build a Hilbert C^*-module from $\Gamma^\infty(M, E)$. We do this by integrating fibre-wise. This means that for every $b \in B$, we integrate over M_b. On $\Gamma^\infty(M, E)$ this defines a map taking values in $C^\infty(B)$. This map has a lot of the properties of an $C(B)$ valued inner product. By looking at the completion of $\Gamma^\infty(M, E)$ in the norm induced by this inner-product, we end up with a Hilbert C^*-module over $C(B)$. We start with the fibre-wise integration

In the Section 2 we introduced an inner product by integration. Using the idea of applying this fibre-wise over the slices $E_b \to M_b$ we define

$$\langle s, t\rangle (b) = \int_{M_b} \left\langle s_{|M_b}, t_{|M_b}\right\rangle_E dV_{g_b}$$

as a map from B to $\mathbb{C}$ which is defined point-wise. Here dV_{g_b} is the density defined by the Riemannian metric g restricted to M_b. Note that we write $\langle\ ,\ \rangle_E$ for the result of applying the metric on E to two sections in $\Gamma^\infty(M, E)$ to avoid confusion with the map just defined above. This map $\langle\ ,\ \rangle$ defines a continuous function on B.

Proposition 5.1.1. $\langle\ ,\ \rangle$ is a map from $\Gamma^\infty(M, E) \times \Gamma^\infty(M, E)$ to $C^\infty(B)$.

Proof. Take any $b \in B$, and take a chart (U, α) of B which contains b, i.e.

$$\alpha : U \to \alpha(U)$$

with $\alpha(U)$ an open subset of $\mathbb{R}^n$.

As M is a fibre bundle over B, we can choose U small enough such that we also have a local trivialisation

$$\psi_M : p^{-1}(U) \to U \times Z.$$

Z is a smooth manifold of dimension k. Suppose we have a chart of Z, so (V, β) with

$$\beta : V \to \beta(V) \subset \mathbb{R}^k.$$

Note that ψ_M is an isomorphism, so $\psi_M^{-1}(U \times V) \subset p^{-1}(U)$ is an open set in M. ψ_M together with α and β give us the following chart for M:

$$\psi := \psi_M \circ (\alpha, \beta) : \psi_M^{-1}(U \times V) \overset{\psi_M}{\to} U \times V \overset{(\alpha,\beta)}{\to} \alpha(U) \times \beta(V) \subset \mathbb{R}^n \times \mathbb{R}^k.$$

For every $b \in U$, we also have charts for M_b by restricting ψ, namely

$$\psi_b : \phi^{-1}(U \times V) \cap M_b \overset{\psi}{\to} \{b\} \times V \overset{(\alpha,\beta)}{\to} \{\alpha(b)\} \times \beta(V) \cong \beta(V) \subset \mathbb{R}^k.$$

For convenience in using the Riemann volume form, we can write all the charts in their coordinate functions. Writing $\alpha = (x^1, \ldots, x^n)$ for the coordinate functions, and $\beta = (y^1, \ldots, y^k)$, we automatically get the coordinate functions $\psi = (\tilde{x}^1, \dots \tilde{x}^n, \tilde{y}^1 \dots \tilde{y}^k)$ where $\tilde{x}^i = x^i \circ \psi_M$ and $\tilde{y}^i = y^i \circ \psi_M$. $\psi_b = (\tilde{y}^1, \dots, \tilde{y}^k)$.

Using all these charts we can rephrase the problem to a local one, which can be solved using some elementary fact about integration. Suppose for now that $\langle s,t \rangle_E$ is compactly supported in $\psi_M^{-1}(U \times V)$. Note

$$\langle s,t \rangle_E \circ \psi^{-1} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{C}$$

is smooth and compactly supported by definition, and

$$\left\langle s_{|M_b}, t_{|M_b} \right\rangle_E \circ \psi_b^{-1} = \langle s,t \rangle_E \circ \psi^{-1}(\alpha(b), \cdot) : \mathbb{R}^k \to \mathbb{C}$$

. We compute

$$\begin{aligned}\int_{M_b} \left\langle s_{|M_b}, t_{|M_b} \right\rangle_E dV_{g_b} &= \int_{\psi_(M_b)} (\psi_b^{-1})^* \left(\left\langle s_{|M_b}, t_{|M_b} \right\rangle_E dV_{g_b} \right) \\ &= \int_{\psi_b} \left(\left\langle s_{|M_b}, t_{|M_b} \right\rangle_E \circ \psi_b^{-1} \right) \sqrt{\det(g_b)} d\tilde{y}^1 \dots d\tilde{y}^k\end{aligned}$$

This proves the map

$$\begin{aligned}\alpha(U) &\to \mathbb{C} \\ \alpha(b) &\mapsto \int_{M_b} \left\langle s_{|M_b}, t_{|M_b} \right\rangle_E dV_{g_b}\end{aligned}$$

is smooth, because we are integrating a smooth map over $n+k$ variables over the last k variables, which still yields a smooth map in the first n variables. The above map is coordinate representative of $\langle s,t \rangle$ in the coordinate patch (U, α), which means $\langle s,t \rangle$ is a smooth map from U to $\mathbb{C}$.

To complete the proof, note that the sets $\psi_M^{-1}(U_i \times V_j)$ cover M if the U_i cover B and the V_j cover Z. Let $\rho_{i,j}$ be a smooth partition of unity subordinate to this cover. In every patch we will integrate $\rho_{i,j} \langle s,t \rangle_E$, which is a smooth function compactly supported in $\psi_M^{-1}(U \times V)$.

$$\begin{aligned}\langle s,t \rangle \circ \alpha_i^{-1} : \alpha_i(U_i) &\to \mathbb{C} \\ \alpha_i(b) &\mapsto \sum_j \int_{M_b} \rho_{i,j} \left\langle s_{|M_b}, t_{|M_b} \right\rangle_E dV_{g_b}\end{aligned}$$

is continuous for all the patches (U_i, α_i), which means by definition that $\langle s,t \rangle$ is a smooth map. □

We know $C^\infty(B)$ is a pre-C^*-algebra under the sup-norm (everything is defined point-wise), and $C(B)$ is a C^*-algebra with the sup-norm. The closure of $C^\infty(B)$ is $C(B)$, because $C^\infty(B)$ is dense in $C(B)$.

Lemma 5.1.2. *$C^\infty(B)$ is dense in $C(B)$ (for the sup-norm)*

Proof. Proving this is an application of the complex version of the Stone-Weierstrass theorem. It states that the smooth functions are dense C^*-algebra of the continuous functions on B if this set separates points.

Take charts (U_i, α_i) for B and a partition of unity ρ_i subordinate to this cover. Now take different points $b_1, b_2 \in B$. Either they lie in different charts and some ρ_i separates them. Or they lie in the same chart, say (U_j, α_j). We can modify ρ_j such that $\rho_j(b_2) \neq 0$ (by adding some constant), write $\tilde{\rho}_j$ for this map. Define $f : \mathbb{R}^n \to \mathbb{C}$ by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) - \alpha_j^{-1}(b_2)$. The smooth map $\tilde{\rho}_j \cdot (f \circ \alpha_j)$ separates b_1 and b_2. □

We show that $\Gamma^\infty(M, E)$ is an inner-product module for the pre-C^*-algebra $C(B)$ under $\langle\ ,\ \rangle$ in the sense that is has all the properties of Definition 3.1.1. The only difference is that $C^\infty(B)$ is not a C^* algebra. As outlined in the first chapter of [Lan95], we then know that the closure of $\Gamma^\infty(M, E)$ under the norm induces by $\langle\ ,\ \rangle$ is a Hilbert C^* module for $C(B)$ by completion. $\langle\ ,\ \rangle$ extends to a $C(B)$ valued inner product on this closure. We will denote the closure by X.

Lemma 5.1.3. $\Gamma^\infty(M, E)$ is an inner-product module for $C^\infty(B)$ and $\langle\ ,\ \rangle$.

Proof. We know $\Gamma^\infty(M, E)$ is a complex-linear space, and a module over $C^\infty(M)$ by point-wise multiplication. By setting $(s \cdot f)(m) = s(m)f(p(m))$ for $f \in C^\infty(B)$, we make $\Gamma^\infty(M, E)$ into a $C^\infty(B)$ module. This means we only need to check the four properties outlined in Definition 3.1.1.

Pick $s, t, r \in \Gamma^\infty(M, E)$, $\alpha, \beta \in \mathbb{C}$ and $f, g \in C^\infty(B)$. First, note that

$$\begin{aligned}\langle s, \alpha t + \beta r \rangle (b) &= \int_{M_b} \langle s, \alpha t + \beta r \rangle_E \, dV_{g_b} \\ &= \int_{M_b} \alpha \langle s, t \rangle_E \, dV_{g_b} + \int_{M_b} \beta \langle s, r \rangle_E \, dV_{g_b} \\ &= \alpha \langle s, t \rangle (b) + \beta \langle s, r \rangle (b)\end{aligned}$$

which proves the first property.

$$\langle s, t \cdot f \rangle (b) = int_{M_b} \left\langle s_{|M_b}, (t \cdot f)_{|M_b} \right\rangle_E dV_{g_b}$$

We know $(t \cdot f)_{|M_b} = f(b) t_{|M_b}$. This means we can just move $f(b)$ out of the inner-product and out of the integral (both are complex-linear) to get

$$\langle s, t \cdot f \rangle (b) = \langle s, t \rangle (b) f(b)$$

which proves $\langle s, t \cdot f \rangle = \langle s, t \rangle f$, the second property.

The third property is straightforward, as both the integral and the inner product on the vector bundle have this property. Similarly, the last property holds because both the inner product and the integral are positive definite. □

This means we have found our Hilbert $C(B)$ module, denoted by X. As a vector-space it is the closure of $\Gamma^\infty(M, E)$ under the norm given by $\| \cdot \| = \| \langle \cdot, \cdot \rangle \|_\infty^{\frac{1}{2}}$. The module action by $C(B)$ is just point-wise multiplication.

5.2 The action of $C^\infty(M)$.

This action can be defined by using the action of $C^\infty(M)$ on $\Gamma^\infty(M, E)$ by multiplication and extending to X. Write

$$\begin{aligned}\phi(f) : \Gamma^\infty(M, E) &\to \Gamma^\infty(M, E)\\ s &\mapsto fs\end{aligned}$$

for the multiplication by a smooth function f. We need to prove the $\phi(f)$ for $f \in C^\infty(M)$ form a unital $*$-sub-algebra of the bounded adjointable operators on X.

Proposition 5.2.1. For every $f \in C^\infty(M)$, $\phi(f)$ extends to a bounded adjointable operator on X. The set $\{\phi(f) \mid f \in C^\infty(M)\}$ is a unital $*$-sub-algebra of $\mathcal{L}(X)$.

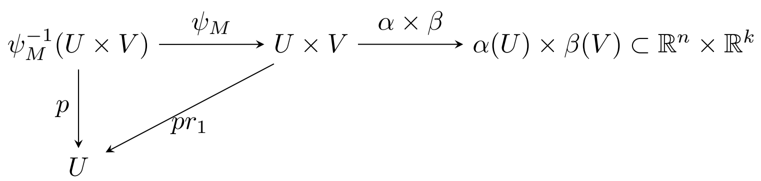
Proof. Take any $f \in C^\infty(M)$.

$$\begin{aligned}\|\phi(f)s\| &= \| \langle f \cdot s, f \cdot s\rangle \|^{\frac{1}{2}}\\ &= \left(\sup_{b\in B} \left| \int_{M_b} \langle f \cdot s, f \cdot s\rangle \, dV_{g_b} \right|\right)^{\frac{1}{2}}\\ &= \left(\sup_{b\in B} \left| \int_{M_b} |f(m)|^2 \langle s, s\rangle \, dV_{g_b} \right|\right)^{\frac{1}{2}}\\ &\leq \|f\|_\infty \langle s, s\rangle \|^{\frac{1}{2}},\end{aligned}$$

This means $\phi(f)$ can be uniquely extended to a bounded operator on X. The adjoint is obviously given by $\phi(\overline{f})$, as we just multiply point-wise. We have $\phi(f)+\phi(g) = \phi(f+g)$, $\phi(fg) = \phi(f)\phi(g)$, $\phi(\overline{f}) = \phi(f)^*$ and $\phi(\mathbb{1}_M) = \mathrm{Id}_X$. This proves $\{\phi(f) \mid f \in C^\infty(M)\}$ is a unital $*$ sub-algebra in $\mathcal{L}(X)$. ☐

5.3 The vertical Dirac operator

We define a Dirac operator on X. It behaves like the generalised Dirac operator defined for the spectral triple case. The key difference is that we only have a Clifford action along the fibres. We therefore define an operator that only derives in the fibre-direction. To be completely clear, recall we have the following charts for M, for some U covering B and V covering Z.



We write $\{\frac{\partial}{\partial x_i}\}_{i=1}^{i=n+k}$ for the associated coordinate tangent frame. Here n is the dimension of B, and k is the dimension of Z. For M_b we have the charts, which we will call ψ_b

$$\begin{array}{ccccc} \psi_M^{-1}(U \times V) \cap M_b & \xrightarrow{\psi_M} & \{b\} \times V & \xrightarrow{\alpha \times \beta} & \alpha(\{b\}) \times \beta(V) \cong \beta(V) \subset \mathbb{R}^k \\ \Big\downarrow p & \swarrow_{pr_1} & & & \\ U & & & & \end{array}$$

We will write $\{\frac{\partial}{\partial y_i}\}_{i=1}^{i=k}$ for the associated coordinate tangent frame.

Recall $T_m(M/B)$ are all tangent vectors in T_mM such that $X(f \circ p) = 0$ for all $f \in C^\infty(B)$. For the chart ψ note that

$$\begin{aligned} \left.\frac{\partial}{\partial x_{n+i}}\right|_m (f \circ p) :&= \left.\frac{\partial}{\partial x_{n+i}}\right|_{\psi(m)} \left(f \circ p \circ \psi^{-1}\right) \\ &= \left.\frac{\partial}{\partial x_{n+i}}\right|_{\psi(m)} \left(f \circ p \circ \psi^{-1} \circ (\alpha, \beta)^{-1}\right), \end{aligned}$$

for all $i \in \{1, \ldots, k\}$. From the above diagram we see $p \circ \psi^{-1} = pr_1$. So

$$\begin{aligned} \left.\frac{\partial}{\partial x_{n+i}}\right|_m (f \circ p) &= \left.\frac{\partial}{\partial x_{n+i}}\right|_{\psi(m)} \left(f \circ pr_1 \circ (\alpha, \beta)^{-1}\right) \\ &= 0 \end{aligned}$$

So we conclude $\frac{\partial}{\partial x_{n+i}}|_m$ lie in $T_m(M/B))$ and in fact form a basis (as they are k independent tangent vectors). In other words, in the charts for M defined above, the last k coordinate functions define a local basis for the vertical bundle $T(M/B)$. The $\frac{\partial}{\partial x_{n+i}}|_m$ also define an associated cotangent basis on $T^*_m(M/B))$. Let us write dx^{i+n} for the associated cotangent frame of $T^*_m(M/B))$.

Later on, we will will be restricting to slices of the form $E_b \to M_b$. It is important to note that the tangent space of M_b only consists of tangent-vectors which lie in the fibre direction. We can in fact identify $T_m(M/B)$ and $T_m(M_b)$ using the following lemma

Lemma 5.3.1. Take any $b \in B$. For all $m \in M_b$ the tangent spaces T_mM_b and $T_m(M/B)$ are isomorphic under the tangent map induced by the inclusion $i_b : M_b \hookrightarrow M$. The local frame $\frac{\partial}{\partial y_i}$ of TM_b is mapped to the local frame $\frac{\partial}{\partial x_{i+n}}$ of $T(M/B)$ under this isomorphism.

$$\begin{array}{ccccc} TM_b & \xrightarrow{di_b} & TM & \xrightarrow{d^*p} & p^*TB \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ M_b & \xrightarrow[i_b]{} & M & \longrightarrow & M \end{array}$$

Proof.

The kernel of d^*p is $T(M/B)$. We know for every $m \in M_b$ that T_mM_b and $T_m(M/B)$ are of dimension k. di_b is injective. We prove that the image of di_b lies inside the kernel of d^*p. Take any $f \in C^{\infty}(B, \mathbb{R})$ and $X \in T_m M_b$.

$$\begin{aligned} d^*p(di_b(X))(f) &= X(f \circ p \circ i) \\ &= 0, \end{aligned}$$

as $(f \circ p \circ i)(b') = f(b) \quad \forall b' \in B$. Furthermore, $\frac{\partial}{\partial y_i}$ is mapped to $\frac{\partial}{\partial x_{i+n}}$. □

The vertical Dirac operator can now be defined using only the vertical co-tangent frame $\{dx^{i+n}\}_{i=1}^{i=k}$.
Definition 5.3.2. The *vertical generalized Dirac operator* D is locally defined by

$$D(s) = \sum_{i=1}^{k} c(dx^{i+n}) \nabla^E_{\frac{\partial}{\partial x_{i+n}}} s.$$

D is a densely defined operator. It is also again a first order differential operator. An important difference is that D is no longer elliptic (it only acts on the vertical part).

5.4 The commutator

In order to show X together with D yields an unbounded Kasparov module, we need to check that the commutator of D and the action of a smooth functions exists, and extends to a bounded operator. The proof of this fact is straightforward and will be covered before the more difficult parts about the self-adjointness and compactness of the resolvent.
Lemma 5.4.1. Let f be a smooth map on M supported in. The commutator $[D, f]$ on $\Gamma^{\infty}(M, E)$ is given by

$$[D, f] = c(\sum_{i=1}^{k} \frac{\partial f}{\partial x_{i+n}} dx^{i+n}),$$

where $\{dx^j\}_{j=1}^{n+k}$ is a local frame for the co-tangent bundle of M, and the last k are a local frame of the vertical tangent bundle of M. Here df is locally written as $df = \sum_{j=1}^{n+k} \frac{\partial f}{\partial x_j} dx^j$. In other words, the commutator of D and f is given by the Clifford action of the vertical derivative of f.

Proof. We write it out. We know that locally we can write $df = \sum_{j=1}^{n+k} \frac{\partial f}{\partial x_j} dx^j$. Now

$$\begin{aligned} [D, f](s) &= Df(s) - fD(s) \\ &= \sum_{i=1}^{k} c(dx^{i+n}) \nabla_{\frac{\partial}{\partial x_{i+n}}} (fs) + f \left(c(dx^{i+n}) \nabla_{\frac{\partial}{\partial x_{i+n}}} (s) \right) \\ &= \sum_{i=1}^{k} c(dx^{i+n}) f \nabla_{\frac{\partial}{\partial x_{i+n}}} (s) + c(dx^{i+n}) \frac{\partial}{\partial x_{i+n}} (f) s + f \left(c(dx^{i+n}) \nabla_{\frac{\partial}{\partial x_{i+n}}} (s) \right) \end{aligned}$$

We know the action by f is point-wise so $fc(dx^{i+n}) = c(dx^{i+n})f$, which means

$$\begin{aligned}[D, f](s) &= \sum_{i=1}^{k} c(dx^{i+n}) \frac{\partial}{\partial x_{i+n}}(f)s \\ &= \sum_{i=1}^{k} c(\frac{\partial}{\partial x_{i+n}}(f) dx^{i+n})s\end{aligned}$$

□

We can again use this to prove that multiplication by $C^\infty(M)$ maps $\mathrm{Dom}(\overline{D})$ into itself, and that the commutator $[\overline{D}, f]$ extends to a bounded operator for all $f \in C^\infty(M)$. The proof is the same as proposition 2.4.6 in section 2.4.

6 States and Hilbert C^*-modules

Proving that the vertical generalized Dirac operator is regular and self-adjoint requires extra machinery. One could probably still prove self-adjointness directly, but the regularity is hard to assert directly. Therefore we look at so-called localisations of our Hilbert $C(B)$-module.

In general a localisation of a Hilbert C^* module gives, for every state on the C^*-algebra an associated Hilbert space. All these localisations together can provide information about the Hilbert C^*-module.

We first outline the localisation process, and give additional properties in case of a Hilbert $C(B)$-module. In the last subsection, a pivotal theorem will be introduced which allows us to extract global regularity and self-adjointness from local self-adjointness.

6.1 Localisations of unbounded operators

Let us recall the definition of a state.
Definition 6.1.1. Let $\mathcal{A}$ be a C^* algebra. A *state* is a positive linear functional $\rho : \mathcal{A} \to \mathbb{C}$ with norm one.

Let ρ be a state on C^*-algebra $\mathcal{A}$. Let X be a Hilbert C^*-module for $\mathcal{A}$ with inner product $\langle\ ,\ \rangle$. Using ρ we can define the following subspace of X:

$$N_\rho := \{x \in X \mid \rho(\langle x, x\rangle) = 0\}.$$

We construct the localisation from the quotient of X by the sub-space N_ρ. This quotient sits densely inside a Hilbert space by defining the following inner product.

$$\begin{aligned}\langle\ ,\ \rangle_\rho : E\big/N_\rho \times E\big/N_\rho &\to \mathbb{C}\\ ([x],[y]) &\mapsto \rho(\langle x, y\rangle)\end{aligned}$$

The quotient is not necessarily closed under the norm induced by the above map, so we take the localisation of X for the state ρ to be the closure of the quotient under the inner-product.
Proposition 6.1.2. For every state ρ, $X\big/N_\rho$ together with $\langle\ ,\ \rangle_\rho$ is a pre-Hilbert space

Proof. The linearity and conjugate-symmetry follow immediately from the fact that X is a Hilbert C^*-module for $\mathcal{A}$ with inner product $\langle\ ,\ \rangle$. For positive definiteness we see

$$\begin{aligned}\langle [x],[x]\rangle &= \rho(\langle x, x\rangle)\\ &\geq 0,\end{aligned}$$

as ρ is a positive functional, and $\langle x, x\rangle \geq 0$ (as an element of $\mathcal{A}$) because $\langle\ ,\ \rangle$ is positive definite. Suppose $\langle [x],[x]\rangle = \rho(\langle x, x\rangle) = 0$, then we find $x \in N_\rho$ so $[x] = 0$. ☐

We define X_ρ to be the Hilbert space obtained by completing $X\big/N_\rho$ for the inner product $\langle\ ,\ \rangle_\rho$. We have the following sequence

$$X \xrightarrow{q} X\big/N_\rho \xrightarrow{i} X_\rho.$$

Given an operator on a Hilbert C^* module, we can also localise the operator for every state. Suppose we have an unbounded densely defined operator $D : \mathrm{Dom}(D) \to X$. Note q is continuous and surjective, so $q(\mathrm{Dom}(D))$ is dense in $X \big/ N_\rho$. As $X \big/ N_\rho$ is dense in X_ρ we conclude $i(q(\mathrm{Dom}(D)))$ is dense in X_ρ. The following map is an unbounded operator on $\mathrm{Dom}((D_\rho)_0) = i(q(\mathrm{Dom}(D)))$ under certain circumstances.

$$\begin{aligned}(D_\rho)_0 : \mathrm{Dom}((D_\rho)_0) &\to X_\rho \\ [x] &\mapsto [D(x)]\end{aligned}$$

The only property we need for $(D_\rho)_0$ to be a densely defined unbounded operator is that D maps N_ρ into N_ρ. There are different ways to go about this. In Section 7 we directly prove it. In most literature we cite, we assume D is symmetric. This is enough to prove $(D_\rho)_0$ is well-defined. Pick any $x \in N_\rho$, which means $[x] = 0$. We see for all $\eta \in \mathrm{Dom}(D^*)$ (which is also dense)

$$\begin{aligned}|\langle [D(x)], [\eta] \rangle_\rho| &= |\rho(\langle D(x), \eta \rangle)| \\ &= |\rho(\langle x, D^*(\eta) \rangle)| \\ &= |\langle [x], [D^*(\eta)] \rangle_\rho| \\ &\leq \|[x]\| \|[D^*(\eta)]\| \\ &= 0\end{aligned}$$

As $i(q(\mathrm{Dom}\, D^*))$ is dense we conclude $[D(x)] = 0$. This means $(D_\rho)_0$ is a densely defined operator.

Suppose D is symmetric. We then see $(D_\rho)_0$ is symmetric for every ρ by writing

$$\begin{aligned}\langle D_\rho [x], [y] \rangle_\rho &= \rho(\langle Dx, y \rangle) \\ &= \rho(\langle x, Dy \rangle) \\ &= \langle [x], D_b[x] \rangle\end{aligned}$$

In the next section we discuss the case where $\mathcal{A} = C(B)$ and will see that the converse also holds.

If D is closed and symmetric, $(D_\rho)_0$ is not necessarily closed. We do know though that $(D_\rho)_0$ has a densely defined adjoint and is closable. We set $D_\rho = \overline{(D_\rho)_0}$. This operator will be called the *localisation* of D for the state ρ.

The pivotal theorem below, due to [KL12] and [Pie06], gives a strong connection between self-adjointness and regularity of an operator, and the self-adjointness of it's localisations.

Theorem 6.1.3. Let D be a closed and symmetric operator. D is a self-adjoint and regular operator if and only if D_ρ is self-adjoint for all pure states ρ.

The proof of this theorem is an article by itself. Both the articles [KL12] (Theorem 4.2 together with Theorem 5.8) and [Pie06] prove the above.

6.2 Hilbert $C(B)$-modules

An important example (and the reason we are considering this construction) is the case where $\mathcal{A}$ is $C(B)$, the continuous functions for our base space B (which is compact and Hausdorff). From now on X will be a Hilbert $C(B)$ module as outlined in section 5.1. Every $b \in B$ gives us a state (evaluation in b).

$$\begin{aligned} \mathrm{ev}_b : C(B) &\to \mathbb{C} \\ f &\mapsto f(b) \end{aligned}$$

To see this is a state, recall positive elements of $C(B)$ are positive functions, so ev_b is a positive linear map. The unit in $C(B)$ is $\mathbb{1}_B$, and $\mathrm{ev}_b(\mathbb{1}_B) = 1$ so $\| \mathrm{ev}_b \| = \mathrm{ev}_b(\mathbb{1}_B) = 1$. For an abelian C^*-algebra, a state is pure if it is a character, which is the case for ev_b. See [Mur90], Theorem 5.1.6. For $C(B)$, all pure states are an evaluation for some $b \in B$.

Let us write out the localisation explicitly for a state b, and Hilbert $C(B)$-module X. We get

$$X \to X / N_b \to X_b$$

Note $N_b = \{x \in X \mid \langle x, x\rangle(b) = 0\}$ and $\langle [x], [x]\rangle_b = \langle x, x\rangle(b)$. In the case of sections we get $\langle [s], [s]\rangle_b = \langle s, s\rangle(b) = \int_{M_b} \langle s_{|M_b}, s_{|M_b}\rangle\, dV_{g_b}$. Because the integral and inner product are positive definite, we see a sections s lies in N_b if and only if $s_{|M_b} = 0$. So essentially the space X_b only retains the information on a single slice $E_b \to M_b$. This can be made more precise by showing that X_b and $L^2(M_b, E_b)$ are equivalent as Hilbert-spaces.

Recall $\pi_{E_b} : E_b \to M_b$ is a complex vector bundle, where M_b is a compact Riemannian manifold. We conclude that, as in the section 2.4, $\Gamma^\infty(M_b, E_b)$ is a pre-Hilbert space under the inner-product $\langle s, t\rangle = \int_{M_b} \langle s, t\rangle\, dV_{g_b}$. The closure of this space is $L^2(M_b, E_b)$.

Proposition 6.2.1. X_b is isomorphic to $L^2(M_b, E_b)$ as Hilbert-spaces. The pre-Hilbert space $q(\Gamma^\infty(M, E))$ is isomorphic to $\Gamma^\infty(M_b, E_b)$.

Proof. Take the space $\Gamma^\infty(M, E)$, which is dense in X. Write $\Gamma^\infty(M, E) / N_b := q(\Gamma^\infty(M, E)$, which is still a dense subspace of X / N_b. Consider the following map:

$$\begin{aligned} \Gamma^\infty(M, E) / N_b &\to \Gamma^\infty(M_b, E_b) \\ [s] &\mapsto s_{|M_b} \end{aligned}$$

Lets proof this map is well defined. Suppose $[s] = [t]$, which means

$$\begin{aligned} [s - t] \in N_b & \qquad \Leftrightarrow \\ \langle s - t, s - t\rangle(b) = 0 & \qquad \Leftrightarrow \\ \int_{M_b} \left\langle (s - t)_{|M_b}, (s - t)_{|M_b} \right\rangle_E dV_{g_b} = 0 & \qquad \Leftrightarrow \\ \left\langle (s - t)_{|M_b}(m), (s - t)_{|M_b}(m) \right\rangle_E = 0 \quad \forall m \in M_b & \qquad \Leftrightarrow \\ s_{|M_b} = t_{|M_b} & \end{aligned}$$

This map is also linear (operations are defined point-wise). It is also injective as the image of [0] is the restriction of the zero section, which is 0 again. For sujectivity, it is enough to show that a section of E_b can be extended to a section of E. As M_b is a closed subset of M, this can be done. See lemma 5.6 in [Lee02].

Recall $\langle [s],[t]\rangle_b = \langle s,t\rangle(b) = \int_{M_b} \langle s_{|M_b}, t_{|M_b}\rangle \, dV_{g_b} = \langle s_{|M_b}, t_{|M_b}\rangle$. This means $\Gamma^\infty(M,E)/N_b$ and $\Gamma^\infty(M_b,E_b)$ are isomorphic as pre-Hilbert spaces. Their closures are therefore equal, resulting in the proof of the proposition. □

Let us write down the isomorphism between $\Gamma^\infty(M,E)/N_b$ and $\Gamma^\infty(M_b,E_b)$ down explicitly, as we will use it a lot in the next section.

$$\begin{aligned}\Gamma^\infty(M,E)/N_b &\to \Gamma^\infty(M_b,E_b)\\ [s] &\mapsto s_{|M_b}\end{aligned}$$

and

$$\begin{aligned}\Gamma^\infty(M_b,E_b) &\to \Gamma^\infty(M,E)/N_b\\ s &\mapsto [\tilde{s}]\end{aligned}$$

where $\tilde{s}$ is any extension of s such that $\tilde{s}_{|M_b} = s$.

This means we can extract information about D by looking at the localisations D_b, which are now unbounded maps on $L^2(M_b,E_b)$.

Rewriting theorem 6.1.3 for a Hilbert $C(B)$-module we get
Theorem 6.2.2. A symmetric closed operator D on a Hilbert $C(B)$-module is regular and self adjoint if all localisation D_b are self-adjoint.

All the data in our geometric setting is essentially defined fibre-wise. We will see that all this data can be restricted to the slices $E_b \to M_b$. Every slice is a complex vector bundle, which together with a generalized Dirac operator gives a spectral triple. It will turn out that the localisation of our Hilbert $C(B)$ module is exactly the Hilbert space associated to the slice $E_b \to M_b$, and that the localisation of the vertical Dirac operator is exactly the generalized Dirac operator in every slice. The idea is then to extract local data, from the spectral triple, to prove properties of the global operator and Hilbert $C(B)$-module.

7 Self-adjointness and the compact resolvent

The ultimate goal is to construct an unbounded Kasparov module from the Hilbert C^*-module X with the vertical Dirac operator. We will prove that D has a self-adjoint and regular closure. The last difficult part will then prove the closure of D has a compact resolvent. Due to time-constraints we can only cover the case where E is a trivial vector bundle.

7.1 Self-adjointess and regularity

The main idea behind proving that $\overline{D}$ is self-adjoint and regular is by looking at the localisations and using Theorem 6.2.2. Once we show the localisations of D are well-defined and self-adjoint, we are essentially done.

In the first step, we show the maps $(D_b)_0$ are well-defined unbounded operators. This can be done directly by using some facts about connections. These operators are defined on $\Gamma^\infty(M,E)/N_b \cong \Gamma^\infty(M_b, E_b)$.

In the second part, we prove that the data on our vector-bundle and fibre-bundle (the metric, connection and Clifford connection) can be restricted to every vector-bundle slice $E_b \to M_b$. This allows us to define a generalized Dirac operator on $\Gamma^\infty(M_b, E_b)$ for every b. Thanks to Section 2.4, we know these Dirac operators are essentially self-adjoint. A short calculation shows that the maps $(D_b)_0$ is in fact equal to this generalized Dirac operator for the restricted data. We now know the operators $(D_b)_0$ are essentially self-adjoint.

In the final part, we show that D is symmetric. Once we know D is symmetric, we know it's closure $\overline{D}$ exists. This is a closed and symmetric unbounded operator. We also know the localisations D_b exist, and are self-adjoint. Finally we prove that $\overline{D}_b = D_b$ and show $\overline{D}$ is regular and self-adjoint using Theorem 6.2.2.

7.1.1 Localisations exist

For existence of the non-closed localisations $(D_b)_0$, we use the following fact about connections:
Lemma 7.1.1. Let ∇^E be a connection on a vector bundle E and look at $\nabla^E_X(s)(x_0)$ with $X \in \Gamma^\infty(M, TM)$, $s \in \Gamma^\infty(M, E)$,$x_0 \in M$. This expression vanishes in each of the following cases:

- s arbitrary but $X(x_0) = 0$
- X arbitrary but there exists

  $$\gamma : (-\epsilon, \epsilon) \to M$$

  a path with $\gamma(0) = x_0$ and $\dot{\gamma}(0) = X_{x_0}$, such that $s(\gamma(t)) = 0$ for all t near 0.

Let us use the second part of this lemma to prove that the connection of a vertical vectorfield on a section that is zero on M_b actually is itself a section that is zero on M_b.

Proposition 7.1.2. Suppose $s_{|M_b} = 0$. We have, for $1 \leq i \leq k$,

$$\nabla^E_{\frac{\partial}{\partial x_{i+n}}}(s)(m) = 0 \quad \forall m \in M_b.$$

Proof. Fix the i for $\frac{\partial}{\partial x_{i+n}}$. Picking a path that goes along the fibre is enough. Pick a coordinate chart $x_0 \in \phi^{-1}(U \times V)$. We can now work over a small open box in $\mathbb{R}^n \times \mathbb{R}^k$. There exists some $\epsilon > 0$ such that $\beta(x_0) = (z_1, \ldots, z_i \pm \delta, z_k) \in \beta(V) \subset \mathbb{R}^k$. Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n \times \mathbb{R}^k$ defined by $t \mapsto \alpha(x_0) \times (z_1, \ldots, z_i + t, \ldots, z_n)$. The $(i+n) - th$ partial derivative is 1, all others are 0. Setting $\tilde{\gamma} = \phi^{-1} \circ (\alpha \times \beta)^{-1} \circ \gamma$, gives a path from $(-\epsilon, \epsilon)$ to M with $\tilde{\gamma}(0) = x_0$ and $\tilde{\gamma}(t) \in M_b$. As $[s] = 0$, we see $s_{|M_b} = 0$. We conclude $\nabla^E_{\frac{\partial}{\partial x_{i+n}}}(s)(x_0) = 0$ for all $1 \leq i \leq k$ and $x_0 \in M_b$. □

We can use the above to state the following useful corollary.

Corollary 7.1.3. For s such that $s_{|M_b} = 0$ and $X \in \Gamma^\infty(M, T(M/B))$ a vertical vectorfield, we see

$$\nabla^E_X(s)(m) = 0 \quad \forall m \in M_b$$

.

Proof. We know $\{\frac{\partial}{\partial x_{i+n}}\}$ for $1 \leq i \leq k$ form a local frame for $T(M/B)$. The connection is linear for vector fields, so using the above proposition 7.1.2 the corollary is proven. □

A priori the maps $(D_b)_0$ do not have to be well-defined. They are well-defined when D maps $N_b \cap \mathrm{Dom}(D)$ into itself, which is easily proven using Proposition 7.1.2.

Proposition 7.1.4. The localisations $(D_b)_0$ are well defined. If all $(D_b)_0$ are symmetric, then D is symmetric.

Proof. Recall that locally $D = \sum_i c(dx^{i+n})\nabla^E_{\frac{\partial}{\partial x_{i+n}}}$. Suppose $s \in N_b \cap \Gamma^\infty(M, E)$. Using proposition 7.1.2, we see $D(s)(m) = 0 \quad \forall m \in M_b$. So $D(s) \in N_b$. This means the localisations $(D_b)_0$ are well defined for all $b \in B$.

Using $\langle Ds, t\rangle(b) = \langle [Ds], [t]\rangle_b = \langle (D_b)_0[s], [t]\rangle$, we see that D is symmetric if and only if all localisations are symmetric □

7.1.2 Restricting data to slices

In this Section, we use extensions and restrictions of functions, sections and vectorfields to translate the data on our vector bundle to data on the slices $E_b \to M_b$. We have five things to work with

- A Riemann metric g on M.
- A metric on the vector bundle $E \to M$.
- The connection ∇^E on $E \to M$.
- The connection $\nabla^{M/B}$ on $T(M/B) \to M$.
- The Clifford module structure of E along the fibres of B

An important tool to restrict these maps is lemma 5.6 in [Lee02]. It tells us that

- Any section $s \in \Gamma^\infty(M_b, E_b)$ can be extended to a section $\tilde{s} \in \Gamma^\infty(M, E)$ such that $\tilde{s}(m) = s(m) \quad \forall m \in M_b$.
- Any vectorfield $X \in \Gamma^\infty(M_b, TM_b)$ can be extended to a vertical vectorfield $\tilde{X} \in \Gamma^\infty(M, T(M/B))$, such that $\tilde{X}(m) = X(m) \quad \forall m \in M_b$. Given X we first post-compose with di_b^{-1} in order to obtain $di_b^{-1} \circ X \in \Gamma^\infty(M_b, T(M/B)_b)$, a section of the sub-bundle of the vertical tangent bundle. We then use the same extension principal to obtain $\tilde{X}$. We will suppress the isomorphism di_b.
- Any smooth (real) function $f \in C^\infty(M_b)$ can be extended to a smooth (real) function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}(m) = f(m) \quad \forall m \in M_b$.

More on the identification of $T_m M_b$ and $T_m(M/B)$ can be found in lemma 5.3.1. We can use these extensions (and restrictions) to, when given either a map on $\Gamma^\infty(M, E)$ or on $\Gamma^\infty(M, T(M/B))$, walk around the following diagrams

$$\begin{array}{ccc}
\Gamma^\infty(M, E) & \longrightarrow & \Gamma^\infty(M, E) \\
{\scriptstyle s \mapsto \tilde{s}} \uparrow & & \downarrow {\scriptstyle s \mapsto s_{|M_b}} \\
\Gamma^\infty(M_b, E_b) & \longrightarrow & \Gamma^\infty(M_b, E_b)
\end{array}$$

$$\begin{array}{ccc}
\Gamma^\infty(M, T(M/B)) & \longrightarrow & \Gamma^\infty(M, T(M/B)) \\
{\scriptstyle X \mapsto \tilde{X}} \uparrow & & \downarrow {\scriptstyle X \mapsto X_{|M_b}} \\
\Gamma^\infty(M_b, T(M/B)_b) & & \Gamma^\infty(M_b, T(M/B)_b) \\
{\scriptstyle di_b} \uparrow & & \downarrow {\scriptstyle di_b^{-1}} \\
\Gamma^\infty(M_b, TM_b) & \longrightarrow & \Gamma^\infty(M_b, TM_b)
\end{array}$$

When the top map only acts locally along fibres (i.e. it maps N_b into itself), we can prove the bottom maps are well defined. We use this to define a restricted metric, connection, Levi-Cevita connection and Clifford action.

Proposition 7.1.5. The Riemann metric $g = \{g_m\}_{m\in M}$ on M restricts to a metric $g_b = \{g_{b,m}\}_{m\in M_b}$ on M_b. Here $g_{m,b} = g_m \circ (di_b^{-1} \times di_b^{-1})$

Proof. $g_{m,b}$ as defined above is a real inner product on $T_m M_b$ for every $m \in M_b$. The $\{g_{m,b}\}_{m\in M_b}$ vary smoothly because the $\{g_m\}$ do, and di_b^{-1} is a smooth bundle map. □

We now look at the hermitian metric.

Proposition 7.1.6. The metric on the vector-bundle $E \to M$ restricts to a metric on the vector sub-bundle $E_b \to M_b$.

Proof. We know $\langle\ ,\ \rangle$ on $E \to M$ is a family of complex inner products on every fibre that varies smoothly. We just restrict this family to a family of complex inner products on the fibres of $E_m \to M_b$ to obtain an inner product. □

Let us look at the connection.

Proposition 7.1.7. ∇^{E_b} defined by

$$\begin{aligned} \nabla^{E_b} : \Gamma^\infty(M_b, TM_b) \times \Gamma^\infty(M_b, E_b) &\to \Gamma^\infty(M_b, E_b) \\ (X, s) &\mapsto \left(\nabla^E_{\tilde{X}}(\tilde{s})\right)_{|M_b} \end{aligned}$$

is a connection on the vector bundle $\Gamma^\infty(M_b, E_b)$.

Proof. We walk around the diagram as given above. The first goal is to prove the particular extension of s and X do not matter, making ∇^{E_b} into a well defined map.

Suppose $\tilde{s}_1$ and $\tilde{s}_2$ are two extensions of s. Then $\tilde{s}_1 - \tilde{s}_2 \in N_b$. Using corollary 7.1.3, we see that $\left(\nabla^E_{\tilde{X}}(\tilde{s}_1 - \tilde{s}_2)\right)_{|M_b} = 0$. Using linearity of the connection ∇^E, we see $\left(\nabla^E_{\tilde{X}}(\tilde{s}_1)\right)_{|M_b} = \left(\nabla^E_{\tilde{X}}(\tilde{s}_2)\right)_{|M_b}$. So the choice of extension does not matter, essentially because the connection for vertical vector fields maps N_b into itself.

Suppose $\tilde{X}_1$ and $\tilde{X}_2$ are two vertical vector fields that extend X. Note that $\left(\tilde{X}_1 - \tilde{X}_2\right)(m) = 0 \quad \forall m \in M_b$. Using the first part of lemma 7.1.1, we conclude $\left(\nabla^E_{\tilde{X}_1 - \tilde{X}_2}(\tilde{s})\right)_{M_b} = 0$. Using linearity of the connection the independence on the choice of $\tilde{X}$ follows.

Because ∇^E is a linear map, ∇^{E_b} is as well. The only thing left to prove is the linearity for smooth real functions in the vectorfield slot, and the Leibniz rule in the section slot.

Let $f \in C^\infty(M, \mathbb{R})$ be a smooth real function. Note that $\tilde{f}\tilde{X}$ is an extension of fX.

$$\begin{aligned}\nabla^{E_b}_{fX}(s) &= \left(\nabla^{E}_{\tilde{f}\tilde{X}}(\tilde{s})\right)_{|M_b} \\ &= \left(\tilde{f}\nabla^{E}_{\tilde{X}}(\tilde{s})\right)_{|M_b} \\ &= f\nabla^{E_b}_{X}(s)\end{aligned}$$

Let $f \in C^\infty(M, \mathbb{C})$ be a smooth complex function. Note that $\tilde{f}\tilde{s}$ is an extension of fs.

$$\begin{aligned}\nabla^{E_b}_{X}(fs) &= \left(\nabla^{E}_{\tilde{X}}(\tilde{f}\tilde{s})\right)_{|M_b} \\ &= \left(\tilde{f}\nabla^{E}_{\tilde{X}}(\tilde{s})\right)_{|M_b} + \left(\tilde{X}(\tilde{f})(\tilde{s})\right)_{|M_b} \\ &= f\nabla^{E_b}_{X}(s) + X(f)(s),\end{aligned}$$

because $\tilde{X}(\tilde{f})(\tilde{s})(m) := \tilde{X}_m(\tilde{f}) \cdot \tilde{s}(m)$. We have $\tilde{s}(m) = s(m)$ and

$$\begin{aligned}\tilde{X}_m(\tilde{f}) &= di_b(X_m)(\tilde{f}) \\ &= X_m(\tilde{f} \circ i) \\ &= X_m(f).\end{aligned}$$

So ∇^{E_b} is a connection. □

Proposition 7.1.8. The connection ∇^{E_b} is compatible with the metric on the vector sub-bundle $E_b \to M_b$.

Proof. Pick a vectorfield $X \in \Gamma^\infty(M_b, TM_b)$ and two sections $s, t \in \Gamma^\infty(M_b, E_b)$. Let us denote extensions to the whole space by over-setting it with a tilde. We see, for all $m \in M$

$$\left\langle \nabla_{\tilde{X}}(\tilde{s}), \tilde{t} \right\rangle(m) + \left\langle \tilde{s}, \nabla_{\tilde{X}}(\tilde{t}) \right\rangle(m) = \left\langle \nabla_X s, t \right\rangle_b(m) + \left\langle s, \nabla_X t \right\rangle_b(m)$$

and note that as the isomorphism between $T_m M_b$ and $T_m M/B$ is given by the differential of the inclusion that $\tilde{X}(m)(\tilde{f}) = X(m)(f)$. We see $\widetilde{\langle s, t \rangle} = \langle \tilde{s}, \tilde{t} \rangle$ so $\tilde{X}(\langle \tilde{s}, \tilde{t} \rangle) = X(\langle s, t \rangle)$. □

We use the same procedure for the vertical Levi-Civita connection.

Proposition 7.1.9. $\nabla^{M/B} : \Gamma^\infty(M, TM) \times \Gamma^\infty(M, T(M/B)) \to \Gamma^\infty(M, T(M/B))$ restricts to a connection

$$\begin{aligned}\nabla^{M_b} : \Gamma^\infty(M_b, TM_b) \times \Gamma^\infty(M_b, TM_b) &\to \Gamma^\infty(M_b, TM_b) \\ (X, Y) &\mapsto \left(\nabla^{M/B}_{\tilde{X}}(\tilde{Y})\right)_{|M_b}\end{aligned}$$

This connection is torsion-free and compatible with the metric on M_b, i.e. it is the Levi-Civita connection.

Proof. The proof that this is a connection is the same as proposition 7.1.7.

To see the connection is torsion-free, take $X, Y \in \Gamma^\infty(M_b, TM_b)$. For all $m \in M_b$ we have

$$\begin{aligned}\nabla^{M_b}_X(Y)(m) - \nabla^{M_b}_Y(X)(m) &= \nabla^{M/B}_{\tilde{X}}(\tilde{Y})(m) - \nabla^{M/B}_{\tilde{Y}}(\tilde{X})(m) \\ &= P\left(\nabla^g_{\tilde{Y}}(\tilde{X}) - \nabla^g_{\tilde{Y}}(\tilde{X})\right)(m) \\ &= P\left(\left[\tilde{X}, \tilde{Y}\right]\right)(m)\end{aligned}$$

The Lie bracket of two vertical vector fields is again vertical. To see this, recall that $T(M/B)$ is the kernel of dp for the projections $p : M \to B$. We have for $f \in C^\infty(B, \mathbb{R})$,

$$\begin{aligned}dp\left(\left[\tilde{X}, \tilde{Y}\right]\right)(f) &= \left[\tilde{X}, \tilde{Y}\right](f \circ p) \\ &= \tilde{X}(\tilde{Y}(f \circ p)) - \tilde{Y}(\tilde{X}(f \circ p)) \\ &= \tilde{X}(dp(\tilde{Y}(f))) - \tilde{Y}(dp(\tilde{X}(f))) &= 0\end{aligned}$$

as $\tilde{X}$ and $\tilde{Y}$ are vertical, and lie in the kernel of dp. So $\left[\tilde{X}, \tilde{Y}\right]$ lies in the kernel of dp and is therefore vertical. This means $P\left(\left[\tilde{X}, \tilde{Y}\right]\right) = \left[\tilde{X}, \tilde{Y}\right]$. But for points $m \in M_b$ we see $\left[\tilde{X}, \tilde{Y}\right](m) = [X, Y]$ (where $T_m(M/B)$ and TM_b are identified using the isomorphism di_b coming from the inclusion $i_b : M_b \hookrightarrow M$).

For every

$$\begin{aligned}g(\nabla^{M_b}_X(Y), Z) + g(Y, \nabla^{M_b}_X(Z)) &= g(P\nabla^g_{\tilde{X}}, Z) + g(Y, P\nabla^g_{\tilde{Z}}) \\ &= g(\nabla^g_{\tilde{X}}, Z) + g(Y, \nabla^g_{\tilde{X}} Z) \\ &= X(g(Y, Z))\end{aligned}$$

□

The Levi-Civita connection is the unique metric compatible torsion-free connection. Therefore the above defined connection is just the Levi-Civita on M_b.

Proposition 7.1.10. As E is a Clifford module along the fibres of B, we have a bundle morphism

$$c : C(T^*(M/B)) \to End(E)$$

over M. This morphism restricts to a bundle morphism

$$c_b : C(T^*M_b) \to End(E_b),$$

making E_b into a Clifford module. This Clifford module is self-adjoint if the Clifford module along the fibres is self-adjoint.

Proof. Recall that the fibre of the bundle of algebra's above a point m is the Clifford algebra of the co-tangent space. So $C(T^*(M/B))_m = C(T^*_m(M/B))$. The isomorphism di_b between $T_m M_b$ and $T_m(M/B)$ induces an isomorphism on the co-tangent spaces by pre-composition with di_b. We have

$$\begin{aligned}di_b^* : T^*_m(M/B) &\overset{\cong}{\to} T^*_m M_b \\ dX &\mapsto dX \circ di_b\end{aligned}$$

Note that this isomorphism sends the local co-tangent basis $\{dx^i\}$ of $T^*(M/B)$ to the local co-tangent basis $\{dy^i\}$ of T^*M_b. Such an isomorphism of the vector spaces induces an isomorphism of the associated Clifford algebras (because of the universal property). Let us call this map $\phi : C(T^*_m(M/B)) \overset{\cong}{\to} C(T^*_m M_b)$. We now define

$$\begin{aligned} c_b : C(T^*M_b) &\to End(E_b) \\ (m, \alpha \in C(T^*_m M_b)) &\mapsto \big(c(\phi^{-1}(\alpha)) : E_m \to E_m\big) \end{aligned}$$

□

Proposition 7.1.11. For every $b \in B$, the connections ∇^{E_b} is a Clifford connection for the Clifford action c_b.

Proof. For $a \in \Gamma^\infty(M_b, T^*M_b)$, $X \in \Gamma^\infty(M_b, E_b)$ and $m \in M_b$ we have

$$\begin{aligned} \left[\nabla^{E_b}_X, c_b(a)\right] s &= \nabla^{E_b}_X(c_b(a)(s)) - c_b(a)\left(\nabla^{E_b}_X(s)\right) \\ &= \nabla^E_{\tilde{X}}(\widetilde{c_b(a)(s)})_{|M_b} - c_b(a)\left(\nabla^E_{\tilde{X}}(\tilde{s})_{|M_b}\right), \end{aligned}$$

Note that $\phi^{-1} \circ a$ lies in $\Gamma^\infty(M_b, T^*(M/B)_b)$ which is a closed sub bundle of $T^*(M/B) \to M$. This means an extension $\widetilde{\phi^{-1} \circ a}$ exists. We see that $c(\widetilde{\phi^{-1} \circ a})\tilde{s}$ is an extension of $c_b(a)s$. We get that

$$\begin{aligned} \left[\nabla^{E_b}_X, c_b(a)\right] s &= \left(\nabla^E_{\tilde{X}}(c(\widetilde{\phi^{-1} \circ a})(\tilde{s})) - c(\widetilde{\phi^{-1} \circ a})\left(\nabla^E_{\tilde{x}}(\tilde{s})\right)\right)_{|M_b} \\ &= \left(c(\nabla^{M/B}_{\tilde{X}}(\widetilde{\phi^{-1} \circ a}))\tilde{s}\right)_{|M_b} \end{aligned}$$

where we used that c is a Clifford action along the fibres. Writing out the definition of the dual connection we get

$$\nabla^{M/B}_{\tilde{X}}(\widetilde{\phi^{-1} \circ a}) = \widetilde{\phi^{-1} \circ a}\nabla^{M/B}_{\tilde{X}}(\cdot)$$

For points $m \in M_b$ this expression is equal to $\phi^{-1}(a(\nabla^{M_b}_X(\cdot)))$, which is $\phi^{-1}(\nabla^{M_b}_X a)$. We conclude

$$c(\phi^{-1}(\nabla^{M_b}_X a)) = c_b(\nabla^{M_b}_X a)$$

So ∇^{E_b} indeed is a Clifford connection for each $b \in B$. □

We now know all the geometric data restricts to slices $E_b \to M_b$. This data is exactly what is assumed for a vector-bundle in Section 2.4. This means we have a generalized Dirac operator D'_b, which has a self-adjoint closure on $L^2(M_b, E_b)$. Quickly recall the following facts about the local tangent frames

- $\{\frac{\partial}{\partial x_{i+n}}\}_{i=1}^k$ is a local basis for the vertical tangent bundle, with associated co-tangent basis dx^i.
- $\{\frac{\partial}{\partial y_i}\}_{i=1}^k$ is a local basis for the tangent bundle of TM_b, with it's associated co-tangent basis dy^i.

- The isomorphism $di_b : T_m M_b \to T_m(M/B)$ maps $\frac{\partial}{\partial y_i}$ to $\frac{\partial}{\partial x_{i+n}}$, the induced map di_b^* on the co-tangent spaces maps dx^i to dy^i.

A localisation $(D_b)_0 : \Gamma^\infty(M, E) \big/ N_b \to X_b$ induces a linear map on $\Gamma^\infty(M_b, E_b)$ which has the following form for any $m \in M_b$.

$$\begin{aligned}(D_b)_0 s(m) &= (D\tilde{s})(m)\\ &= \sum_{i=1}^{k} c(dx^i)\nabla^E_{\frac{\partial}{\partial x_{i+n}}}\tilde{s}(m)\\ &= \sum_{i=1}^{k} c_b(dy^i)\nabla^{E_b}_{\frac{\partial}{\partial y_i}}(s)(m)\end{aligned}$$

This means the localisations $(D_b)_0$ are the generalised Dirac operators D_b' as described in the spectral triple case. This means their closure $D_b = \overline{(D_b)_0}$ are densely defined, closed and self adjoint.

7.1.3 $\overline{D}$ is self adjoint and regular

In the previous two sections, we first proved that the $(D_b)_0$ exist. We then showed these are all essentially self-adjoint. This means D is symmetric. This implies two things. It means the localisations D_b exist and are self-adjoint. Secondly, it implies the closure $\overline{D}$ exists. In Theorem 6.2.2, we need a closed symmetric operator, and it's localisations. This means we have to look at the localisations of $\overline{D}$.

Lemma 7.1.12. We have $D_b = \overline{D}_b$

Proof. Lemma 2.1 in [KL12] tells us that $D^* = \overline{D}^*$. This gives us

$$D \subset \overline{D} \subset D^* = \overline{D}^*.$$

This means $\overline{D} : \mathrm{Dom}(\overline{D}) \to X$ is densely defined, symmetric and closed. So the localisations $(\overline{D}_b)_0$ exist, are symmetric (because $\overline{D}$ is) and in fact extend $(D_b)_0$ by definition. First note $\mathrm{Dom}((D_b)_0) \subset \mathrm{Dom}((\overline{D}_b)_0)$ and for any $s \in \mathrm{Dom}((D_b)_0)$ we have

$$\begin{aligned}(\overline{D}_b)_0[s] &= [\overline{D}s]\\ &= [Ds]\\ &= (D_b)_0[s]\end{aligned}$$

As D_b is the closure of $(D_b)_0$ we see that $D_b \subset \overline{D}_b$. Using lemma 2.1 of [KL12] again, we see that $\overline{D}_b$ is also symmetric. If we apply the adjoint to the relation $D_b \subset \overline{D}_b$ and use the above and the fact that D_b is self adjoint we get

$$\overline{D}_b \subset (\overline{D}_b)^* \subset D_b,$$

so we conclude $\overline{D}_b = D_b$. □

This now finally allows us to show $\overline{D}$ is regular and self-adjoint.

Proposition 7.1.13. $\overline{D}$ is a self-adjoint regular operator.

Proof. We know $\overline{D}$ is symmetric (proposition 7.1.4) and closed. Above we have proved the localisations D_b are closed and self-adjoint. Using theorem 6.2.2 we conclude $\overline{D}$ is self adjoint and regular. □

7.2 Compact resolvent

Now that we know the vertical Dirac operator is regular and self-adjoint, we also know the resolvent exists, and is a bounded adjointable map from X to X. Similarly to proving $\overline{D}$ is regular and self-adjoint, proving the resolvent of $\overline{D}$ is compact actually is pretty involved.

At first one might expect that there exists a theorem similar to the regular and self-adjointness. It is true that if the global resolvent of $\overline{D}$ is compact, every localisation will be compact. Sadly enough, the converse does not hold. We need an extra property, essentially that the localisations of the resolvent vary continuously in operator norm over the base space B.

Due to the time-constraints for writing this thesis, we were only able to prove the result about the compact resolvent for case where $E = M \times \mathbb{C}^l$ is a trivial bundle. We strongly suspect the general case to be true as well. Some ideas and complications for proving the general result can be found in section 8. This means from now on, we will take $E = M \times \mathbb{C}^l$ to be a trivial bundle with the trivial inner product: point-wise multiplication.

Proving the resolvent is compact, entails the following steps. First we prove the Hilbert $C(B)$ module X can be rewritten as a continuous bundle of Hilbert spaces over B, denoted by $\mathcal{H}$. For every point $b \in B$, the fibre will be $L^2(M_b, \mathbb{C}^l)$. We then show the compact operators on $\mathcal{H}$ are isomorphic (as a C^*-algebra) to the continuous sections of a continuous bundle of C^*-algebra's over B. The C^* algebra above a point b will be $\mathcal{K}(L^2(M_b, E_b))$, and the bundle will be denoted by $\mathcal{K}(\mathcal{H})$.

These equivalences will reduce the problem of proving the resolvent is compact to proving a certain section of $\mathcal{K}(\mathcal{H})$ is continuous. This we can do, using the fact D is a first order elliptic differential operator in combination with the resolvent identity.

7.2.1 Equivalence to a bundle of Hilbert spaces

We set $E = M \times \mathbb{C}^l$ as a trivial hermitian vector bundle. The Hilbert $C(B)$ module obtained by fibre-wise integration can be viewed as a bundle of Hilbert spaces. Recall that for some cover $\{U_i\}$ of B and $\{V_j\}$ of Z we have the following charts

$$\psi_{i,j} := \psi_M \circ (\alpha, \beta) : \psi_M^{-1}(U_i \times V_j) \overset{\psi_M}{\to} U_i \times V_j \overset{(\alpha,\beta)}{\to} \alpha(U_i) \times \beta(V_j) \subset \mathbb{R}^n \times \mathbb{R}^k$$

giving rise to local coordinates $(x_1, \ldots, x_n, \ldots, x_{n+1} \ldots x_{n+k})$, with the associated vector fields $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n+k}})$, where the last k are a local frame for the vertical tangent bundle. These coordinates also define a chart for M_b via

$$\psi_{b,j} : \psi_M^{-1}(\{b\} \times V_j) \overset{\psi_M}{\to} \{b\} \times V_j \overset{(\alpha,\beta)}{\to} \{\alpha(b)\} \times \beta(V) \cong \beta(V) \subset \mathbb{R}^k.$$

We have chosen the cover U_i such that the fibre bundle M over B trivializes. Pick a fixed $b_i \in U_i$. Using the charts and trivialisations of the fibre bundle, we can identify M_b with M_{b_i} for all $b \in U_i$.

Proposition 7.2.1. For any U_i over which the bundle $M \to B$ trivializes, and a fixed $b_i \in B$ there exists a unitary map between the Hilbert spaces $L^2(M_b, \mathbb{C}^l)$ and $L^2(M_{b_i}, \mathbb{C}^l)$.

Proof. First, there is a canonical way of moving a point in a fibre M_b to $M_{b'}$ using the trivialisation. Let us write $\rho_{b,b'} : M_b \overset{\cong}{\to} M_b'$ by sending m to $\phi^{-1}(b', \mathrm{pr}_2(\phi(m))$. So we first move m to $b \times Z$, which is isomorphic to $b' \times Z$, after which we use ψ_M^{-1} to move to $M_{b'}$. By definition of a fibre bundle this is a homeomorphism.

Recall the $\det(g_b)$ is a smooth map from $\psi_M^{-1}(\{b\} \times V_j)$ to $\mathbb{R}$ for every patch $\psi_M^{-1}(\{b\} \times V_j)$ over M_b defined by $\det(g_b)(m) = \det\left(\left\langle \frac{\partial}{\partial x_{p+n}}, \frac{\partial}{\partial x_{q+n}} \right\rangle (m)\right)$.

Pick a fixed $b_i \in U_i$. For any patch $\phi^{-1}(\{b_i\} \times V_j)$, we can look at the real valued function

$$\frac{\det(g_b)}{\det(g_{b_i})}^{\frac{1}{4}} : \psi_M^{-1}(\{b_i\} \times V_j) \to \mathbb{R}$$

$$m \mapsto \frac{\det(g_b)(\rho_{b_i,b}(m))}{\det(g_{b_i})(m)}^{\frac{1}{4}}$$

We show this map agrees on intersections $\psi_M^{-1}(\{b_i\} \times V_j) \cap \psi_M^{-1}(\{b_i\} \times V_{j'})$, making $\frac{\det(g_b)}{\det(g_{b_i})}^{\frac{1}{4}}$ into a smooth map on M_{b_i}.

Write $\frac{\partial}{\partial x_{p+n}}$ for the last k local coordinates on $\psi_M^{-1}(U_i \times V_j)$ and $\frac{\partial}{\partial x_{p'+n}}$ for the last k local coordinates on $\psi_M^{-1}(U_i \times V_{j'})$. Recall that the last k coordinates form a local basis for the vertical tangent bundle, and in particular a local basis for each fibre M_b with $b \in B$. A coordinate transform has the form

$$\frac{\partial}{\partial x_{p'+n}} = \sum_p \frac{\partial x_{p+n}}{\partial x_{p'+n}} \frac{\partial}{\partial x_{p+n}}.$$

All the metrics on the fibres M_b can be written as: $(g_b)_{p,q} = \left\langle \frac{\partial}{\partial x_{p+n}}, \frac{\partial}{\partial x_{q+n}} \right\rangle$ (and primed indices for the metric on $\phi^{-1}(\{b\} \times V_{j'})$. We see

$$(g_b)_{p',q'} = \sum_{p,q} \frac{\partial x_{p+n}}{\partial x_{p'+n}} \frac{\partial x_{q+n}}{\partial x_{q'+n}} (g_b)p, q$$

We can write $A_{p,p'} = \frac{\partial x_{p+n}}{\partial x_{p'+n}}$ for a smooth matrix valued function on $\psi_M^{-1}(U_i \times V_j) \cap \psi_M^{-1}(U_i \times V_{j'}$. Also, we can write, for any b: $(g_b) = A^T (g_b)' A$, which means $\det(g_b) = \det((g_b)') \det(AA^T)$. It is important to note that $A(\rho_{b,b'}(m)) = A(m)$ for all $b, b' \in U_i$. A only 'sees' the Z part of the patch $\phi^{-1}(U_i \times Z)$. This means

$$\frac{\det(g_b(\rho_{b_i,b}(m)))}{\det(g_{b_i})} = \frac{\det((g_b)'(\rho_{b_i,b}(m))) \det(AA^T(\rho_{b_i,b}(m)))}{\det((g_{b_i})') \det(AA^T(m))}$$
$$= \frac{\det((g_b)')}{\det((g_{b_i})')}$$

which means $\frac{\det(g_b)}{\det(g_{b_i})}^{\frac{1}{4}}$ is a well-defined real-valued map on the whole of M_{b_i}. Using this, we can define a unitary map form $L^2(M_b, \mathbb{C}^l)$ to $L^2(M_{b_i}, \mathbb{C}^l)$.

$$\chi_{i,b} : L^2(M_b, \mathbb{C}^l) \to L^2(M_{b_i}, \mathbb{C}^l)$$
$$f \mapsto \chi_{i,b}(f)$$

with , for $m \in M_{b_i}$, define $(\chi_{i,b}(f))(m) = f(\rho_{b_i,b}(m))\frac{\det(g_b)}{\det(g_{b_i})}^{\frac{1}{4}}(m)$. This map is injective and surjective. The following shows it is an isometry

$$\begin{aligned}
\langle \chi_{i,b}(s), \chi_{i,b}(t)\rangle &= \int_{\{b_i\}\times Z} \left\langle \sum_j \chi_{i,b}(\rho_j s), \sum_{j'} \chi_{b,i}(\rho_{j'} t)\right\rangle dV_{g_{b_i}} \\
&= \sum_j \int_{\{b_i\}\times V_j} |\rho_j|^2 \left\langle s(m_i)\frac{\det(g_b)(m_i)}{det(g_{b_i})(m)}^{\frac{1}{4}}, t(m_i)\frac{\det(g_b)(m_i)}{det(g_{b_i})(m)}^{\frac{1}{4}}\right\rangle (\det g_{b_i})^{\frac{1}{2}} dx_{1+n}\wedge\ldots\wedge dx_{n+k} \\
&= \sum_j \int_{\{b\}\times V_j} |\rho_j|^2 \langle s(m), t(m)\rangle \det(g_b)^{\frac{1}{2}} dx_{1+n}\wedge\ldots\wedge dx_{n+k} \\
&= \langle s, t\rangle
\end{aligned}$$

□

We know that any two separable Hilbert spaces, there exists an isometry. In the story above we have constructed a finite number of model fibres, one for each open subset U_i which cover B. We now just pick b_0 as the single model fibre, and identify $\mathcal{H}_{b_i}$ with $\mathcal{H}_{b_0}$ via such an isometry. We from now on write $\chi_{b,i}$ for the unitary map from $L^2(M_b, \mathbb{C}^l)$ to $L^2(M_{b_0}, \mathbb{C}^l)$ for $b \in U_i$. Essentially we just multiply with a smooth function. This means $\chi_{b,i}$ also maps the Sobolev space $L^2_1(M_b, \mathbb{C}^l)$ into $L^2_1(M_{b_0}, \mathbb{C}^l)$.

This result about identifying the spaces $L^2(M_b, \mathbb{C}^l)$ locally gives us an alternative way to view the space X, namely as a continuous bundle of Hilbert spaces denoted by $\mathcal{H}$.

Proposition 7.2.2. $\mathcal{H}$ defined by $\mathcal{H} = \amalg_{b\in B} L^2(M_b, \mathbb{C}^l)$ is a locally trivial continuous bundle of Hilbert spaces over B. The fibre is $L^2(M_{b_0}, \mathbb{C}^l)$ and the structure group the unitary operators on $L^2(M_{b_0}, \mathbb{C}^l)$, with operator norm.

Proof. We just set $\mathcal{H} = \amalg_{b\in B} L^2(M_b, \mathbb{C}^l)$ with the obvious projection to B. For convenience let us write $\mathcal{H}_b = L^2(M_b, \mathbb{C}^l)$ for each fibre. This projection, written as π in this proof, is continuous. Let $\{U_i\}$ be the cover that trivialises the fibre bundle $p : M \to B$. For such a cover, we know using Proposition 7.2.1 that we have unitary maps $\chi_{i,b} : L^2(M_b, \mathbb{C}^l) \overset{\rightarrow}{\cong} L^2(M_{b_0}, \mathbb{C}^l)$ for a single b_0 in B. We define the following trivialisations

$$\begin{aligned}
\psi_{\mathcal{H},i} : \amalg_{b\in U_i}\mathcal{H}_b &\to U_i \times \mathcal{H}_{b_0} \\
(b, f \in \mathcal{H}_b) &\mapsto (b, \chi_{b,i}(f)).
\end{aligned}$$

These maps commute with the projections. For continuity, we have canonical injections $\phi_b : L^(M_b, \mathbb{C}^l) \to \mathcal{H}$, and by the universal property of the disjoint union $\psi_{\mathcal{H},i}$ is continuous if and only if $\psi_{\mathcal{H},i} \circ \phi_b$ are continuous for all $b \in U_i$. But $\psi_{\mathcal{H},i} \circ \phi_b = \chi_{b,i}$, which we know to be continuous.

Let us look at the transition function. The intersection $U_i \cap U_j$ has two trivialisations given by $\psi_{\mathcal{H},i}$ and $\psi_{\mathcal{H},j}$. The map $\psi_{\mathcal{H},i} \circ \psi^{-1}_{\mathcal{H},j}$ gives for every $b \in U_i \cap U_j$ the morphism $\chi_{i,b} \circ \chi_{j,b}$ which lies in the topological group of unitary maps on $\mathcal{H}_{b_0}$ (with the operator norm). We conclude $\mathcal{H}$ is a fibre bundle with fibre $\mathcal{H}_{b_0}$ and as structure group the unitary operators on $\mathcal{H}_{b_0}$ under operator norm. □

The space of smooth sections of $\mathcal{H}$ has more structure. It will turn out to be a Hilbert $C(B)$ module, isomorphic to X.
Proposition 7.2.3. The space of smooth sections of $\mathcal{H}$, denote by $\Gamma(B, \mathcal{H})$ is a Hilbert $C(B)$ module.

Proof. We first need to specify what the operations are on this space. We just define everything point-wise, that is for $s, t \in \Gamma(B, \mathcal{H})$ set $\langle s, t \rangle(b) = \langle s(b), t(b) \rangle_{\mathcal{H}_b}$, which is a map from B to $\mathbb{C}$. To see $\langle s, t \rangle$ is a continuous function For the module action by $C(B)$ define for $s \in \Gamma(B, \mathcal{H})$ and $f \in C(B)$: $(s \cdot f)(b) := s(b) \cdot f(b)$ where the right side is scalar multiplication in $\mathcal{H}_b$. To see this again results in a continuous section consider, pick a fixed point $b' \in B$, then $b' \in U_i$ for some open U_i over which the bundle trivializes, and checking for continuity of $s \cdot f$ in b' is the same as checking for continuity of $(s \cdot f)' = \mathrm{pr}_2 \circ \psi_{\mathcal{K}(\mathcal{H}),i} \circ (s \cdot f)$ as a map from U_i to $\mathcal{K}(\mathcal{H}_{b_0})$. We see, for any point b close to b' (so still in U_i) that

$$\begin{aligned}
\|(s \cdot f)'(b') - (s \cdot f)'(b)\| &= \|\chi_i(s(b')f(b'))\chi_i^{-1} - \chi_i(s(b)f(b))\chi_i^{-1}\| \\
&= \|f(b')\chi_i(s(b'))\chi_i^{-1} - f(b)\chi_i(s(b))\chi_i^{-1}\| \\
&\leq \|(f(b') - f(b))\chi_i(s(b'))\chi_i^{-1}\| + \|f(b)\left(\chi_i(s(b'))\chi_i^{-1} - \chi_i(s(b))\chi_i^{-1}\right)\|.
\end{aligned}$$

As f is a continuous map over B, and $\chi_i(s(b'))\chi_i^{-1}$ is a compact map we see the first part is arbitrarily small. For the second part, we know $f(b)$ is bounded, and s is a continuous section.

All other properties regarding the inner product are easily verified to be true, as everything is defined point-wise, and the integral is also positive definite.

We still need to prove $\Gamma(B, \mathcal{H})$ is complete in the metric induces by $\|s\|^2 = \|\langle s, s \rangle\|_\infty$. Suppose we have a Cauchy sequence s_n. In particular this means $s_n(b)$ is a Cauchy sequence for every $b \in B$. Every fibre $\mathcal{H}_b$ is a Hilbert space, and therefore complete, so a limit exists for every sequence $s_n(b)$. This means $s(b) := \lim_n s_n(b)$ is a section. We need to prove it's continuous and the limit of s_n.

For any $\epsilon > 0$, we have an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we know $\|s_n - s_m\| \leq \epsilon$, which means in particular that $\|s_n(b) - s_m(b)\| \leq \epsilon \quad \forall b \in B$. We see

$$\begin{aligned}
\|s(b) - s_m(b)\| &\leq \|s(b) - s_n(b)\| + \|s_n(b) - s_m(b)\| \\
&\leq \|s(b) - s_n(b)\| + \epsilon,
\end{aligned}$$

for all $m, n \geq N$ and $b \in B$. Taking the limit over n on both sides results in

$$\|s(b) - s_m(b)\| \leq \epsilon,$$

for all $m \geq N$ and $b \in B$. So we know $\sup_{b \in B} \|s(b) - s_m(b)\| \leq \epsilon$ for all $m \geq N$.

We only need to prove s is a continuous section. Pick a fixed $b_0 \in B$ and $\epsilon > 0$. First, from the above, there exists an $N \in \mathbb{N}$ such that $\sup_{b \in B} \|s(b) - s_N(b)\| \leq \epsilon$. We know s_N is continuous, so there exists an open set W around b_0 such that for all $b \in W$ we know $\|s_N(b_0) - s_N(b)\| \leq \epsilon$. We conclude

$$\begin{aligned}
\|s(b_0) - s(b)\| &\leq \|s(b_0) - s_N(b_0)\| + \|s_N(b_0) - s_N(b)\| + \|s_N(b) - s(b)\| \\
&\leq 3\epsilon,
\end{aligned}$$

for all $b \in W$. This proves s is continuous and the limit of the sequence s_n. $\square$

Proposition 7.2.4. The Hilbert $C(B)$ modules X and $\Gamma(B, \mathcal{H})$ are isomorphic.

Proof. Note that $\Gamma^\infty((,M), M \times \mathbb{C}^l) = C^\infty(M, \mathbb{C}^l)$ is a dense subspace of X. We have a map

$$\begin{aligned} \phi : C^\infty(M, \mathbb{C}^l) &\to \Gamma(B, \mathcal{H}) \\ s &\mapsto \phi(s) \end{aligned}$$

where $\phi(s)$ is defined by $\phi(s)(b)(m) = s(m)$ for $m \in M_b$ and $b \in B$. In other words, $\phi(s)(b)$ is the map in $L^2(M_b, \mathbb{C}^l)$ given by restricting s to M_b.

ϕ is injective, because if $\phi(s) = 0$, it just means s restricted to every M_b is zero, which in turn means $s = 0$.

Note that $\|s\| = \|\phi(s)\|$ for $s \in C^\infty(M, \mathbb{C}^l)$, so ϕ is bounded and extends to an injective isometry

$$\phi : X \to \Gamma(B, \mathcal{H}).$$

We only need to prove the range of ϕ is dense, which we do by proving $\phi(C^\infty(M, \mathbb{C}^l)$ is dense in $\Gamma(B, \mathcal{H})$. Pick any $s \in \Gamma(B, \mathcal{H})$ and $\epsilon > 0$. Note $\{f(b) \mid f \in \phi(C^\infty(M, \mathbb{C}^l)\}$ is dense in $\mathcal{H}_b$. So for every $b \in B$, there exists $f_b \in \phi(C^\infty(M, \mathbb{C}^l)$ such that $\|f_b(b) - s(b)\| \leq \frac{\epsilon}{2}$. Because $f_b - s$ is a smooth section of $\mathcal{H}$, we see that for every b there exists an Y_b such that $\|f_b(b') - s(b')\| \leq \epsilon$ for all $b' \in Y_b$. These Y_b cover B, so we take finite subcover Y_{b_n} and a *smooth* partition of unity ρ_n subordinate to this cover. Define $f = \sum_n \rho_n f_{b_n}$. Note that $rho_n f_{b_n} \in \phi(C^\infty(M, \mathbb{C}^l)$ because ρ_n is smooth. We conclude $\| \sum_n \rho_n f_{b_n} - s\| \leq \epsilon$. $\square$

The above isomorphism means that instead of looking at compact operators on X, we can look at compact operators on $\Gamma(B, \mathcal{H})$ as $\mathcal{K}(X) \cong \mathcal{K}(\Gamma(B, \mathcal{H}))$. At first this might not look like a big improvement, but it is possible to describe the compact maps on $\Gamma(B, \mathcal{H})$ as the continuous sections of a bundle of C^*-algebra's. Let us introduce the bundle $\mathcal{K}(\mathcal{H})$.

Proposition 7.2.5. $\mathcal{K}(\mathcal{H}) = \amalg_{b \in B} \mathcal{K}\left(L^2(M_b, \mathbb{C}^l)\right)$ is a bundle of C^*-algebra's over B. The model fibre is $\mathcal{K}(\mathcal{H}_{b_0})$ and the structure group is again the unitary operators on $\mathcal{H}_{b_0}$, this time acting by conjugation.

Proof. We again use the same cover U_i. Recall that we already have trivialisations

$$\psi_{\mathcal{H},i} : \amalg_{b \in U_i} \mathcal{H}_b \to U_i \times \mathcal{H}_{b_0}$$

For U_i we now define

$$\psi_{\mathcal{K}(\mathcal{H}),i} : \amalg_{b \in U_i} \mathcal{K}(\mathcal{H}_b) \to U_i \times \mathcal{K}(\mathcal{H}_{b_0})$$

defined by $\psi_{\mathcal{K}(\mathcal{H}),i}(b, T \in \mathcal{K}(\mathcal{H}_b)) = \psi_{\mathcal{H},i} \circ T \circ \psi_{\mathcal{H},i}$. Unpacking this definition just means we map $T \in \mathcal{K}(\mathcal{H}_b)$ to $\chi_{i,b} \circ T \circ \chi_{i,b}^{-1}$. As $\chi_{i,b}$ and it's inverse are bounded maps, we know $\chi_{i,b} \circ T \circ \chi_{i,b}^{-1} \in \mathcal{K}(\mathcal{H}_{b_0}$. We need to prove $\psi_{\mathcal{K}(\mathcal{H}),i}$ is a continuous map.

On every fibre, we map $\mathcal{K}(\mathcal{H}_b)$ to $\mathcal{K}(\mathcal{H}_{b_0})$ via the map $\chi_{i,b} \circ \cdot \circ \chi_{i,b}^{-1}$. Because $\chi_{i,b}$ is a unitary map, this map between C^*-algebras is a $*$-isomorphism.

On the intersection $U_i \cap U_j$, the trivialisations $\psi_{\mathcal{K}(\mathcal{H}),i} \circ \psi_{\mathcal{K}(\mathcal{H}),j}$ for a single b is a morphism on $\mathcal{K}(\mathcal{H}_{b_0})$ given by $\chi_{j,b}^{-1}\chi_{i,b} \circ \cdot \circ \chi_{j,b}\chi_{i,b}^{-1}$. Note that $\chi_{j,b}^{-1}\chi_{i,b}$ is a unitary map on $\mathcal{K}(\mathcal{H}_{b_0})$ with inverse $\chi_{j,b}\chi_{i,b}^{-1}$, which means the transition functions take values in the $*$-isomorphisms of $\mathcal{K}(\mathcal{H}_{b_0})$. □

The bundle $\mathcal{K}(\mathcal{H})$ again has additional structure. We want to think about $\mathcal{K}(X)$, which is a C^* algebra. Let us prove the continuous sections of $\mathcal{K}(\mathcal{H})$ is also a C^*-algebra.

Lemma 7.2.6. The space of smooth sections of the bundle $\mathcal{K}(\mathcal{H})$ denoted by $\Gamma(B, \mathcal{K}(\mathcal{H}))$, is a C^*-algebra.

Proof. All operations are defined point-wise, using the C^*-algebra structure of each fibre $\mathcal{K}(\mathcal{H}_b)$. The norm is given by taking the supremum norm over B, which exists as B is compact. We only have to show these point-wise operations result in smooth sections of the bundle. To verify this, note that checking for a section s to be continuous, is equivalent to checking the map $\psi_{\mathcal{K}(\mathcal{H}),i} \circ s : U_i \to \mathcal{K}(\mathcal{H}_0)$ is continuous for all subsets U_i over which the bundle trivialises. Here $\psi_{\mathcal{K}(\mathcal{H}),i}$ is a C^* algebra isomorphism on every fibre.

For example, the involution for $s \in \Gamma(B, \mathcal{K}(\mathcal{H}))$ given by $s^*(b) = s(b)^*$, is again a continuous section because taking a fixed $b' \in U_i$ we know

$$\begin{aligned}\|(\psi_{\mathcal{K}(\mathcal{H}),i} \circ s^*)(b') - (\psi_{\mathcal{K}(\mathcal{H}),i} \circ s^*)(b)\| &= \|(\psi_{\mathcal{K}(\mathcal{H}),i} \circ s)^*(b') - (\psi_{\mathcal{K}(\mathcal{H}),i} \circ s)^*(b)\| \\ &= \|(\psi_{\mathcal{K}(\mathcal{H}),i} \circ s)(b') - (\psi_{\mathcal{K}(\mathcal{H}),i} \circ s)(b)\|\end{aligned}$$

where in the second line we use that $\psi_{\mathcal{K}(\mathcal{H}),i}$ is a fibre-wise C^*-algebra isomorphism, and in the third line we use the C^*-identity on $\mathcal{K}(\mathcal{H}_0)$. This proves s^* is again a continuous section. The scalar multiplication and algebra operations give continuous sections in the same way.

For the C^* identity

$$\begin{aligned}\|ss^*\| &= \sup_{b \in B} \|s(b)s(b)^*\| \\ &= \sup_{b \in B} \|s(b)\|^2 \\ &= \|s\|^2\end{aligned}$$

The only thing left is completeness under the supremum norm. Suppose we have a Cauchy sequence $s_n \in \Gamma(B, \mathcal{K}(\mathcal{H}))$. As we use the supremum norm, we know that in every point $b \in B$, $s_n(b)$ is a Cauchy sequence. Every fibre is a C^*-algebra, so the limits exist. Setting $s(b) = \lim_{n\to\infty} s_n(b)$ gives us a continuous section which is the limit of s_n. The proof is is the same as found in 7.2.1. □

The compact operators on $\mathcal{H}$ can now be directly calculated, namely

Proposition 7.2.7. $\Gamma(B, \mathcal{K}(\mathcal{H}))$ and $\mathcal{K}(\Gamma(B, \mathcal{H}))$ are isomorphic as C^* algebras.

Proof. We define a map between the two

$$\kappa : \mathcal{K}(\Gamma(B, \mathcal{H})) \to \Gamma(B, \mathcal{K}(\mathcal{H}))$$
$$T \mapsto \kappa(T),$$

where $\kappa(T)(b)$ is the map in $\mathcal{K}(\mathcal{H}_b)$ defined in the following way. Note that $T \in \mathcal{K}(\Gamma(B, \mathcal{H}))$, so there exist $T_j = \sum \theta_{f_j, g_j}$, with $f_j, g_j \in \Gamma(B, \mathcal{H})$. We first set $\kappa(\theta_{f,g})(b) = \theta_{f(b),g(b)}$, which lies in $\operatorname{span}\{\theta_{x,y} \mid x, y \in \mathcal{H}_b\}$ for every $b \in B$. We then extend κ linearly over finite sums. Let us first show $\kappa(\theta_{f,g})$ lies in $\Gamma(B, \mathcal{K}(\mathcal{H}))$. Recall that checking continuity of $\kappa(\theta_{f,g})$ in $b' \in U_i$ is equivalent to checking the continuity for $\psi_{\mathcal{K}(\mathcal{H}),i} \circ \kappa(\theta_{f,g}) : U_i \to \mathcal{K}(\mathcal{H}_0)$. Unpacking the definitions, we see

$$\begin{aligned}
\psi_{\mathcal{K}(\mathcal{H}),i} \circ \kappa(\theta_{f,g})(b) &= \psi_{\mathcal{K}(\mathcal{H}),i}(\theta_{f(b),g(b)}) \\
&= \chi_i \theta_{f(b),g(b)} \chi_i^{-1} = \theta_{\chi_i f(b), \chi_i g(b)} \\
&= \theta_{(\psi_{\mathcal{H},i} \circ f)(b), (\psi_{\mathcal{H},i} \circ g)(b)}.
\end{aligned}$$

For convenience write $v(b) = (\psi_{\mathcal{H},i} \circ f)(b)$, $w(b) = (\psi_{\mathcal{H},i} \circ g)(b)$, which by definition are continuous. We see for $\eta \in \mathcal{H}_{b_0}$ that

$$\begin{aligned}
\theta_{v(b'),w(b')}(\eta) - \theta_{v(b),w(b)}(\eta) &= v(b)' \langle w(b'), \eta \rangle - v(b) \langle w(b), \eta \rangle \\
&= v(b') \langle w(b'), \eta \rangle - v(b) \langle w(b'), \eta \rangle + v_i(b) \langle w(b'), \eta \rangle - v(b) \langle w(b), \eta \rangle \\
&= (v(b') - v(b)) \langle w(b'), \eta \rangle + v(b) \langle w(b') - w(b), \eta \rangle,
\end{aligned}$$

which means for the norm

$$\begin{aligned}
\|\theta_{v(b'),w(b')}(\eta) - \theta_{v(b),w(b)}(\eta)\| &\leq \|v(b') - v(b)\| |\langle w(b'), \eta \rangle| + \|v(b)\| |\langle w(b') - w(b), \eta \rangle| \\
&\leq \|v(b') - v(b)\| \|w(b')\| \|\eta\| + \|v(b)\| \|w(b') - w(b)\| \|\eta\|,
\end{aligned}$$

which means the operator norm of $\theta_{v(b'),w(b')} - \theta_{v(b),w(b)}$ is bounded by $\|v(b') - v(b)\| \|w(b')\| + \|v(b)\| \|w(b') - w(b)\|$, which can be made arbitrarily small as both v and w are continuous and bounded. This proves that $\kappa(\theta_{f,g})$ is a continuous section.

Recall that T_j can we written as $T_j = \sum \theta_{f_j, g_j}$, with $f_j, g_j \in \Gamma(B, \mathcal{H})$. Linearly extending κ gives $\kappa(T_j)$ which is a continuous section for every j as the continuous sections are closed under addition.

Let us show the $\kappa(T_j)$ form a Cauchy sequence, so we can take the limit. We see for an $h \in \Gamma(B, \mathcal{H})$ that

$$\begin{aligned}
T_j(h)(b) &= \sum_j \theta_{f_j, g_j}(h)(b) \\
&= \sum_j f_j(b) \langle g_j(b), h(b) \rangle
\end{aligned}$$

Unpacking what it means for T_j to be Cauchy gives

$$\begin{aligned}
\|T_n - T_m\| &= \sup_{h \in \Gamma(B,\mathcal{H})} \{\|(T_n - T_m)h\| \mid \|h\| \leq 1\} \\
&= \sup_{h \in \Gamma(B,\mathcal{H})} \{\sup_{b \in B} \{\sum_n f_n(b) \langle g_n(b), h(b) \rangle + \sum_m f_m(b) \langle g_m(b), h(b) \rangle\} \mid \sup_{b \in B} \|h(b)\| \leq 1\}.
\end{aligned} \tag{5}$$

Take any $\epsilon > 0$, then there exists an N such that for all $n, m \geq N$ we know $\|T_n - T_m\| \leq \epsilon$. We see

$$\begin{aligned}
\|\kappa(T_n) - \kappa(T_m)\| &= \sup_{b \in B}\{\| \sum_n \theta_{f_n(b), g_n(b)} + \sum_m \theta_{f_m(b), g_m(b)} \|\} \\
&= \sup_{b \in B}\{ \sup_{\eta \in \mathcal{H}_b} \{\| \sum_n f_n(b) \langle g_n(b), \eta \rangle + \sum_m f_m(b) \langle g_m(b), \eta \rangle \| \mid \|\eta\| \leq 1\}\} \\
&\leq \epsilon,
\end{aligned}$$

because of equation 5 above. This means the limit of $\kappa(T_j)$ exists (as the sections are complete), and we can define $\kappa(T) = \lim_{j \to \infty} \kappa(T_j)$ which is a continuous section of the bundle $\mathcal{K}(\mathcal{H}) \to B$.

We prove κ is an isometry. This is straightforward. κ is an isometry on $\mathrm{span}\{\theta_{f,g} \mid f, g \in \Gamma(B, \mathcal{H})\}$. We then see

$$\begin{aligned}
\|\kappa(T)\| &= \lim_{j \to \infty} \|\kappa(T_j)\| \\
&= \lim_{j \to \infty} \|T_j\| \\
&= \|T_j\|
\end{aligned}$$

because in both the Hilbert $C(B)$ module and the C^* algebra, the norm is a continuous map.

We can now prove κ is injective. Suppose $\kappa(T) = 0$. By definition this means we have $T_j \to T$ with $T_j \in \mathrm{span}\{\theta_{f,g} \mid f, g \in \Gamma(B, \mathcal{H})\}$. We know $\kappa(T_j) \to \kappa(T) = 0$. This means $\|T_j\| = \|\kappa(T_j)\| \to 0$ which proves the $T = 0$.

An isometry has closed range. This means that showing that the image of κ is dense is enough to prove κ is a isometric isomorphism. We know the set $A = \mathrm{span}\{\kappa(\theta_{f,g}) \mid f, g \in \Gamma(B, \mathcal{H})\}$ lies in the image of κ. But for every $s \in A$, we know $s(b)$ lies in a dense subset of $\mathcal{K}\mathcal{H}_b$ because $\kappa(\theta_{f,g})(b) = \theta_{f(b),g(b)} \in \mathcal{H}_b$.

Pick a fixed $t \in \Gamma(B, \mathcal{K}(\mathcal{H}))$, and an $\epsilon > 0$. It is now enough to find an $s \in A$ such that $\|s - t\| \leq \epsilon$.

For every $b \in B$, we can find $s_b \in A$ such that $\|s_b(b) - t(b)\| \leq \frac{\epsilon}{2}$. Note that $s_b - t$ is a continuous section, so for every $b \in B$, there exists some open subset Y_b such that $\|s_b(b') - t(b')\| \leq \epsilon$ for all $b' \in Y_b$. These Y_b cover B, so we can find a finite sub-cover as B is compact. We write Y_{b_n} for this sub-cover, with points b_n. Let us take a partition of unity subordinate to this cover, denoted by $\rho_n : B \to \mathbb{R}$.

We know s_{b_n} has the form $\sum_j \kappa(\theta_{f_j, g_j})$. We want to multiply s_{b_n} by the partition of unity ρ_n

but still obtain something inside A. But see

$$\begin{aligned}\rho_n(b)s_{b_n}(b) &= \sum_j \rho_n(b)\kappa(\theta_{f_j,g_j})(b)\\ &= \sum_j \rho_n(b)\theta_{f_j(b),g_j(b)}\\ &= \sum_j \theta_{(\sqrt{\rho_n}f_j)(b),(\sqrt{\rho_n}g_j)(b)}.\end{aligned}$$

As $\sqrt{\rho_n}f_j$ is still a continuous section we conclude that $\rho_n s_{b_n}$ still lies in A. We claim that $\sum_n \rho_n s_{b_n}$ lies close to t.

$$\|\sum_n \rho_n(b)s_{b_n}(b) - t(b)\| \leq \sum_n \rho_n(b)\|s_{b_n}(b) - t(b)\|,$$

For such a b, either b lies in Y_n and $\|s_{b_n}(b) - t(b)\| \leq \epsilon$, or b does not lie in Y_n and $\rho_n(b) = 0$. So the second part in the right-hand of the following equation is zero

$$\begin{aligned}\|\sum_n \rho_n(b)s_{b_n}(b) - t(b)\| &\leq \sum_{n|b\in Y_n} \rho_n(b)\|s_{b_n}(b) - t(b)\| + \sum_{n|b\notin Y_n} \rho_n(b)\|s_{b_n}(b) - t(b)\|\\ &\leq \sum_{n|b\in Y_n} \rho_n(b)\epsilon\\ &\leq \epsilon,\end{aligned}$$

for every $b \in B$, so the supremum is also bounded by ϵ. We conclude that κ has a dense range, and is an $*$-isomorphism. □

The inverse of κ has a particularly easy form.
Lemma 7.2.8. The inverse of κ is given by

$$\begin{aligned}\kappa^{-1} : \Gamma(B, \mathcal{K}(\mathcal{H})) &\to \mathcal{K}(\Gamma(B, \mathcal{H}))\\ s \mapsto \kappa^{-1}(s) \quad & \kappa^{-1}(s)(f)(b) = s(b)f(b)\end{aligned}$$

Proof. We know $A = \text{span}\{\kappa(\theta_{f,g}) \mid f, g \in \Gamma(B, \mathcal{H})\}$ is a dense subset of $\Gamma(B, \mathcal{K}(\mathcal{H}))$. On A we first define κ'^{-1} by $\kappa'^{-1}(s)(f)(b) = s(b)f(b)$ where $s \in A$ and $f \in \Gamma(B, \mathcal{H})$. It is immediate that $\kappa'^{-1}\kappa(\theta_{f,g}) = \theta_{f,g}$. This means $\kappa' - 1$ is a $*$-morphism from A to $\mathcal{K}(\Gamma(B, \mathcal{H})$, and κ^{-1} and κ'^{-1} agree on A. κ'^{-1} is also an isometry, which means κ^{-1} is equal to κ'^{-1}. □

Now that we have written X as a bundle of Hilbert spaces, and proven that compact operators on the continuous sections of this bundle are isomorphic to continuous sections of some bundle C^*-algebras we are finally in a position to prove the resolvent is compact.

7.2.2 Resolvent as a continuous section

We can define the following map from B to $\mathcal{K}(\mathcal{H})$.

Proposition 7.2.9. The map

$$\begin{aligned} R : B &\to \mathcal{K}(\mathcal{H}) \\ b &\mapsto (i + D_b)^{-1} \end{aligned}$$

is a continuous section of the bundle $\mathcal{K}(\mathcal{H})$.

R will turn out the be the resolvent of $\overline{D}$ if we move it back to a operator on X via the above isomorphisms between X and $\Gamma(B, \mathcal{H})$, and between $\mathcal{K}(\Gamma(B, \mathcal{H}))$ and $\Gamma(B, \mathcal{K}(\mathcal{H}))$. The proof makes use of some important properties of the operators of the form $\chi_b D_b \chi_b^{-1}$ (because we work over trivialisations of the bundle $\mathcal{K}(\mathcal{H})$). These properties will be outlined and proven in the next Section 7.3.

Proof. We will use some properties and results about the maps $\chi_b D_b \chi_b^{-1}$ as outlined in Section 7.3. The assignment of $(i+D_b)^{-1}$ to b is a section of the bundle $\mathcal{K}(\mathcal{H})$, as $(i+D_b)^{-1}$ is a compact operator on $L^2(M_b, \mathbb{C}^l)$, which is the fibre above b in $\mathcal{K}(\mathcal{H})$. We need to prove this sections is continuous (in operator norm). We do this by composing R with a trivialisation $\psi_{\mathcal{K}(\mathcal{H}),i}$ and proving this gives a continuous map from U_i to $K(\mathcal{H}_0)$. Fix a $b_1 \in U_i \subset B$. Then for b close to b_1 (also in U_i) we have the following inequalities

$$\begin{aligned} \|R(b) \circ \psi_{\mathcal{K}(\mathcal{H}),i} - R(b_1) \circ \psi_{\mathcal{K}(\mathcal{H}),i}\| &= \|\chi_{b,i}(i + D_b)^{-1}\chi_{b,i}^{-1} - \chi_{b_1,i}(i + D_{b_1})^{-1}\chi_{b_1,i}\| \\ &= \| \left(\chi_{b,i}(i + D_b)^{-1}\chi_{b,i}^{-1}\right) \left(\chi_{b_1,i}D_{b_i}\chi_{b_1,i}^{-1} - \chi_{b,i}D_b\chi_{b,i}^{-1}\right) \left(\chi_{b_1,i}(i + D_{b_1})^{-1}\chi_{b_1,i}\right) \| \end{aligned}$$

using the resolvent identity. We then get

$$\begin{aligned} \|R(b) \circ \psi_{\mathcal{K}(\mathcal{H}),i} - R(b_1) \circ \psi_{\mathcal{K}(\mathcal{H}),i}\| &\leq \|\chi_{b,i}(i + D_b)^{-1}\chi_{b,i}^{-1}\| \\ &\quad \| \left(\chi_{b_1,i}D_{b_i}\chi_{b_1,i}^{-1} - \chi_{b,i}D_b\chi_{b,i}^{-1}\right) \left(\chi_{b_1,i}(i + D_{b_1})^{-1}\chi_{b_1,i}\right) \| \\ &= \alpha \cdot \beta \end{aligned}$$

For α, note that using functional calculus it is easy to see that $\|(i + D_b)^{-1}\| \leq 1$. Using the fact $\chi_{b,i}$ is an isometry, we see

$$\begin{aligned} \alpha &= \|\chi_{b,i}(i + D_b)^{-1}\chi_{b,i}^{-1}\| \\ &\leq 1 \end{aligned}$$

For β, we look at it's two parts. For the left part, we show its a bounded map from W to H with an arbitrarily small norm using Corollary 7.3.3. We see, for b close to b_1 that for a $\xi \in L^2_1(M_{b_0}, \mathbb{C}^l)$ that

$$\|\chi_{b_1,i}D_{b_i}\chi_{b_1,i}^{-1} - \chi_{b,i}D_b\chi_{b,i}^{-1}\xi\| \leq \epsilon\|\xi\|_{L^2_1(M_{b_0},\mathbb{C}^l)}.$$

Now we use Lemma 7.3.2 to give an estimation of the most right part of β. We shows it's a bounded map from H to W. Namely, $(i + D_{b_1})$ is automatically a bounded map in graph norm. Using Lemma 7.3.2, we know the graph norm is equivalent with the Sobolev norm on $L^2_1(M_{b_0}, \mathbb{C}^l)$. This means for any $\eta \in \mathcal{H}_{b_0}$, we see

$$\|\chi_{b_1,i}(i + D_{b_1})^{-1}\chi_{b_1,i}^{-1}\eta\|_{L^2_1(M_{b_0},\mathbb{C}^l)} \leq C\|\eta\|,$$

for some constant C. Combining the above, we see for any $\eta \in \mathcal{H}_{b_0}$, that

$$\| \left(\chi_{b_1,i} D_{b_i} \chi_{b_1,i}^{-1} - \chi_{b,i} D_b \chi_{b,i}^{-1}\right) \left(\chi_{b_1,i}(i + D_{b_1})^{-1} \chi_{b_1,i}^{-1}\right) \eta\| \leq \epsilon C \|\eta\|$$

This means β is a bounded map from $\mathcal{H}_{b_0} = L^2(M_{b_0}, \mathbb{C}^l)$ to itself.

We conclude, for b_1 close to b (as determined by the continuity of $b \mapsto \chi_b D_b \chi_b^{-1}$) that

$$\|R(b) \circ \psi_{\mathcal{K}(\mathcal{H}),i} - R(b_1) \circ \psi_{\mathcal{K}(\mathcal{H}),i}\| \leq 1 \cdot \epsilon C$$

This works over all charts $\psi_{\mathcal{K}(\mathcal{H}),i}$ which means R indeed is a continuous section of $\mathcal{K}(\mathcal{H})$ □

The only thing left to verify is that R indeed is $(i + \overline{D})^{-1}$ on X.

Proposition 7.2.10. Using the isomorphisms between $\Gamma(B, \mathcal{K}(\mathcal{H}))$ and $\mathcal{K}(\Gamma(B, \mathcal{H})) \cong \mathcal{K}(X)$, the continuous section R of $\mathcal{K}(\mathcal{H})$ corresponds to the map $(i + \overline{D})^{-1}$ on X.

Proof. Recall that $R \in \Gamma(B, \mathcal{K}(\mathcal{H}))$ and we have the following string of isomorphisms

$$\Gamma(B, \mathcal{K}(\mathcal{H})) \overset{\kappa^{-1}}{\to} \mathcal{K}(\Gamma(B, \mathcal{H}) \overset{\phi^{-1} \circ \cdot \circ \phi}{\to} \mathcal{K}(X)$$

Knowing R is a continuous section of $\mathcal{K}(\mathcal{H})$ we know $\kappa^{-1}(R)$ is a compact operator on $\Gamma(B, \mathcal{H})$ defined by $\kappa^{-1}(R)(s)(b) = (i + D_b)^{-1} s(b)$ for any $s \in \Gamma(B, \mathcal{H})$. We can transport $\kappa^{-1}(R)$ to a compact operator on X by conjugating with ϕ. Our goal is to prove this transported compact operator, and the bounded operator $(i + \overline{D})^{-1}$ on X are the same. We do this by proving they agree on the dense subset $C^\infty(M, \mathbb{C}^l)$ of X. Take $f \in C^\infty(M, \mathbb{C}^l)$. Then $\kappa^{-1}(R)(\phi(f))(b)$ is given by $(i + D_b)^{-1}(f_{|M_b})$. Applying ϕ^{-1} to this section gives us the section defined by $m \mapsto (i + D_{p(m)})^{-1}(f_{|M_b})(m)$, which is equal to $(i + \overline{D})^{-1}(f)$. □

As R is a smooth section, the following corollary completes this trivial vector bundle case.

Corollary 7.2.11. $(i + \overline{D})^{-1}$ is a compact map on X.

Proof. This is immediate when we combine propositions 7.2.9 and 7.2.10. □

7.3 Properties of $\chi_b D_b \chi_b^{-1}$

We prove some properties of the following family: $\{\chi_b D_b \chi_b^{-1}\}_{b \in U_i}$, $\chi_b D_b \chi_b^{-1} : L_1^2(M_{b_0}, \mathbb{C}^l) \to L^2(M_{b_0}, \mathbb{C}^l)$ for one of the open sets U_i over which the fibre bundle trivializes. We will write U without the subindex i from now on. The idea to check these properties is taken from the article [KL13], section 8. Using these properties we can continue similarly to proposition 8.7 in [KL13] to complete the proof for the trivial vector bundle. Let us write $W = L_1^2(M_{b_0}, \mathbb{C}^l)$ and $H = L^2(M_{b_0}, \mathbb{C}^l)$. We then define

$$\begin{aligned} \mathcal{D} : U &\to \mathcal{L}(W, H) \\ b &\to \chi_b D_b \chi_b^{-1} \end{aligned}$$

These (A1) and (A2) below essentially show that $\mathcal{D}$ is a continuous map from U_i to $\mathcal{L}(W, H)$, and that the graph norm for $\mathcal{D}(b)$ and Sobolev norm are equivalent.

Lemma 7.3.1 ((A1))**.** The map $\mathcal{D} : U \to \mathcal{L}(W, H)$ is weakly differentiable. This means $b \mapsto \langle \mathcal{D}(b)\xi, \eta\rangle$ is differentiable for all $\xi \in W$ and $\eta \in H$.

The weak derivative $d(\mathcal{D})(b) : W \to H \otimes T_b^*(U)$ is bounded for each $b \in B$ and the supremum $\sup_{b\in U} \|d(\mathcal{D}(b))\|$ is finite

Proof. Take any $\xi \in W$ and $\eta \in H$. For any b, using the fact that χ_b are unitary equivalences we get:

$$\begin{aligned}\langle \mathcal{D}(b)\xi, \eta\rangle &= \left\langle \chi_b^{-1} \left(\chi_b D_b \chi_b^{-1}\xi\right), \chi_b^{-1}\eta\right\rangle \\ &= \left\langle D_b(\chi_b^{-1}\xi), \chi_b^{-1}\eta\right\rangle \\ &= \langle D_b\xi_b, \eta_b\rangle\end{aligned}$$

where $\xi_b = \chi_b^{-1}(\xi)$ and $\eta_b = \chi_b^{-1}(\eta)$ are ξ and η transported to the fibre M_b using χ_b^{-1}. We need to prove the assignment $b \to \langle D_b(\xi_b), \eta_b\rangle$ is differentiable at every point b. We know $\langle D_b(\xi_b), \eta_b\rangle = \left\langle \overline{D}\tilde{\xi}, \tilde{\eta}\right\rangle(b)$, where $\tilde{\xi}$ and $\tilde{\eta}$ are extensions of ξ_b and η_b. For the right side of this equation, we know it is a smooth map in b if both $\overline{D}\tilde{\xi}$ and $\tilde{\eta}$ are smooth in the horizontal direction. We can create such extensions by moving ξ and η around using χ_b^{-1}. We can then find a small open subset $b \in V \subset \overline{V} \subset U$. Now pick a smooth function $\rho : B \to [0, 1]$ which is 1 on V and has a support contained in U. By setting $\tilde{\xi}(m) = \rho(p(m))\xi_{p(m)}(m)$, we obtain an extension $\tilde{\xi}$ of ξ_b for all $b \in V$. This extension is smooth in the horizontal direction by definition of χ_b^{-1}. Let $\tilde{\eta}$ be defined in the same way. We then know that $\left\langle \overline{D}\tilde{\xi}, \tilde{\eta}\right\rangle$ is a smooth map on B and agrees with $\langle D_b(\xi_b), \eta_b\rangle$ for all $b \in V$, which means it is at least differentiable.

We need to calculate the derivative of $\mathcal{D}$. We know D is a first order differential operator given by

$$D = \sum_{i=1}^{k} A_i \frac{\partial}{\partial x_{i+n}} + B$$

for an open subset of the form $\psi_M^{-1}(U \times V)$. U is the open subset of B, V is an open subset of the fibre Z. A_i and B are matrices of smooth maps on $\psi_M^{-1}(U \times V)$. We also know that D_b are first order differential operators which can be written as

$$D_b = \sum_{i=1}^{k} A_{i,b} \frac{\partial}{\partial x_{i+n}} + B_b$$

where $A_{i,b}$ is just A_i restricted to $\psi_M^{-1}(\{b\} \times V)$. This means $A_{i,b}$ and B_b vary smoothly for changing b. The map χ_b just multiplies with a smooth function, so $\chi_b D_b \chi_b^{-1}$ is also a first order differential operator on $\Gamma^\infty\left(M_{b_0}, \mathbb{C}^l\right)$. This means locally we can write

$$\chi_b D_b \chi_b^{-1} = \sum_{i=1}^{k} A'_{i,b} \frac{\partial}{\partial x_{i+n}} + B'_b$$

Where $A'_{i,b}$ and B'_b are matrices of smooth functions on $\psi_M^{-1}(\{b_0\} \times V)$. Because we only multiply with a smooth map, $A'_{i,b}$ and B_b still vary smoothly in b. We know

$$d(A_i) = \sum_{j=1}^{n} \frac{\partial A_i}{\partial x_j} \otimes dx^j$$

and the same holds for B. This means $\chi_b D_b \chi_b^{-1}$ on a smooth section $s \in C^\infty left(\psi_M^{-1}(\{b_0\} \times V), \mathbb{C}^l)$ is given by

$$d(\mathcal{D})(b)(s) = \sum_{i}^{k} d(A_{i,b}) \frac{\partial}{\partial x_{i+n}}(s) + d(B_b)(s)$$

This means the weak derivative is bounded. The supremum of the norms is also bounded because $A'_{b,i}$ and B_b vary smoothly over b. □

In a second lemma, we prove that the Sobolev norm and graph norm of $\chi_b D_b \chi_b^{-1}$ are equivalent.
Lemma 7.3.2 ((A2))**.** For every $b \in B$, there exist constants C_1^b and C_2^b such that

$$C_1^b \|\xi\|_W \leq \|\xi\|_{\chi_b D_b \chi_b^{-1}} \leq C_2^b \|\xi\|_W$$

for all $\xi \in W$.

Proof. In general we know that $\chi_b D_b \chi_b^{-1}$ is a bounded operator from $L_1^2(M_{b_0}, \mathbb{C}^l)$ to $L^2(M_{b_0}, \mathbb{C}^l)$, as it's an elliptic linear first order differential operator. This gives us

$$\|\chi_b D_b \chi_b^{-1} \xi\| \leq C_3^b \|\xi\|_W$$

and Rellich's lemma assures that the inclusion is compact so

$$\|\xi\| \leq C_4^b \|\xi\|_W$$

This results in

$$\|\xi\|_{\chi_b D_b \chi_b^{-1}} \leq C_2^b \|\xi\|_W$$

and Gardings inequality gives us:

$$C_1^b \|\xi\|_W \leq \|\xi\|_{\chi_b D_b \chi_b^{-1}}$$

So for every $b \in B$ we get

$$C_1^b \|\xi\|_W \leq \|\xi\|_{\chi_b D_b \chi_b^{-1}} \leq C_2^b \|\xi\|_W$$

□

Note that the above proposition is not uniform in B (as opposed to (A2) in [KL13]). The map $\mathcal{D}$ defined in (A1) is now a continuous map from B to $\mathcal{L}(W, H)$

the lemma 7.3.1 has an important consequence. Namely, it is enough to prove $\mathcal{D} : U \to \mathcal{L}(W, H)$ is a continuous map (in the operator norm from W to H).

Corollary 7.3.3.

$$\begin{aligned} \mathcal{D} : U &\to \mathcal{L}(W, H) \\ b &\mapsto \chi_b D_b \chi_b^{-1} \end{aligned}$$

is a continuous map from U to $\mathcal{L}(W, H)$.

Proof. We refer to [KL13] Remark 8.4 part 2. It shows the above by estimating using geodesics. It makes use of Lemma 7.3.1. □

8 Discussion on the general case

In this thesis we would of course have liked to give a complete proof. Sadly enough, proving the resolvent of the vertical Dirac operator is compact poses some difficulties. The previous section gives a prove in the case $E = M \times \mathbb{C}^l$.

For a general vector bundle E over M, one might expect the same arguments to hold. We expect is to be enough to know the local resolvents are compact, together with the norm of these resolvents varying smoothly over the base space B.

One might be able to extend the trivial vector bundle case to the general case by using the Serre-Swann theorem. It tells us there exists some smooth complex hermitian vector bundle F over M, such $E \oplus F \cong M \times \mathbb{C}^Q$ for some large Q.

Suppose X is the completion of $\Gamma^\infty(M, E)$ and X_{triv} is the completion of $\Gamma^\infty\left(M, M \times \mathbb{C}^Q\right)$. The decomposition $E \oplus F \cong M \times \mathbb{C}^Q$ then gives us partial isometries

$$\begin{aligned} w &: X \to X_{\text{triv}} \\ w^* &: X_{\text{triv}} \to X \end{aligned}$$

where w^*w is the identity on X, and ww^* is a projection to $w(X)$ on X_{triv}. We know $w(X) \cong X$. Note that both w and w^* are bounded adjointable maps. This means proving $(i + \overline{D})^{-1})$ on X is compact is equivalent to proving $w(i + \overline{D})^{-1}w^*$ is compact on X_{triv}.

Pulling the map $w(i + \overline{D})^{-1}w^*$ through the equivalences between $\mathcal{K}(X_{\text{triv}})$ and $\Gamma^\infty(B, \mathcal{K}(\mathcal{H}))$ means we should look at the following section

$$\begin{aligned} R &: B \to \mathcal{K}(\mathcal{H}) \\ b &\mapsto w_b(i + D_b)^{-1}w_b^* \end{aligned}$$

The proof in the trivial case relied on noting $\chi_b D_b \chi_b^{-1}$ behaves like a nice family of operators. Using the fact D is a first order differential operator, we could prove the assignment $b \to \chi_b D_b \chi_b^{-1}$ is a continuous map (in operator norm) from B to $\mathcal{L}(L_1^2(M_{b_0}, \mathbb{C}^l), L^2(M_{b_0}, \mathbb{C}^l)$. We then proved the operator norm and Sobolev norm on $L_1^2(M_b, \mathbb{C}^l)$ are equivalent using the fact D is elliptic. These two facts combined, gave us the possibility to estimate the operator norm of the resolvent as a smooth section, using the resolvent identity.

In the case of looking at $w\overline{D}w^*$, we can still prove it's a first order differential operator. It is no longer elliptic though, only on $w(X)$. One might hope this is still enough, but as a family of operators, every $\chi_b w_b D_b w_b^* \chi_b^{-1}$ is only defined on $\chi_b w_b(L_1^2(M_b, E_b))$ which is strictly smaller than $L_1^2(M_{b_0}, \mathbb{C}^Q)$. So the domain of the family $\{\chi_b w_b D_b w_b^* \chi_b^{-1}\}$ constantly changes for varying b, and we cannot use expressions of the form $\chi_b w_b D_b w_b^* \chi_b^{-1} - \chi_{b'} w_{b'} D_{b'} w_{b'}^* \chi_{b'}^{-1}$ meaning our previous tactic of using the resolvent identity will no longer work. There was no more time to solve these problems.

We still suspect the result to still be true, and make the following claim.

Claim 8.0.1. For a vertical Dirac operator D on a smooth hermitian vector bundle, the resolvent $(i+\overline{D})^{-1}$ is compact.

where it is enough to prove the following claim
Claim 8.0.2.

$$
\begin{aligned}
R : B &\to \mathcal{K}(\mathcal{H}) \\
b &\mapsto w_b(i+D_b)^{-1}w_b^*
\end{aligned}
$$

is a (in operator norm) continuous section of $\mathcal{K}(\mathcal{H})$.

9 Conclusion

Let us repeat the primary goal, and quickly summarise all the proofs.
Theorem 9.0.1. The geometric data as described in section 4.4 , with $E = M \times \mathbb{C}^l$ defines an odd $(C^\infty(M), X, \overline{D})$ unbounded Kasparov module.

Proof. In Section 5.1 we showed X is a Hilbert $C(B)$-module. In Section 5.2 we showed $C^\infty(M)$ defines an unital $*$-sub-algebra of the bounded adjointable operators on X. In section 7.1.3 we showed D is closable and that its closure $\overline{D}$ is self-adjoint and regular. We got up until this point for a general vector bundle E. The compactness is proven in section 7.2, but only for a trivial vector bundle. □

We started with the well-studied geometric data consisting of a vector-bundle with a connection, metric, and Clifford module structure. This allowed us to define a generalized Dirac operator. This operator is a generalisation of common examples like the Spin Dirac operator on Spin manifolds, and the Hodge-de Rahm operator on the tangent bundle. The Dirac operator together with the smooth sections of the vector bundle produce an odd spectral triple.

The second part of this thesis studied the more general case of a vector bundle on top of a smooth fibre bundle. We assume this vector bundle to have a Clifford action just for vertical 1-forms. This allowed us to define an analogy to the generalised Dirac operator: the vertical Dirac operator. It only derives in the direction of the fibres on M. Using fibre-wise integration we are able to define a Hilbert $C(B)$ module X, which together with D defines an unbounded Kasparov module.

The proof relies on the idea that every slice $E_b \to M_b$ together with a localisation of D behaves like a spectral triple. This family of spectral triples, indexed over the base space B, can be captured in one object: an unbounded Kasparov module.

10 References

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