

RADBOUD UNIVERSITY NIJMEGEN



MASTER'S THESIS MATHEMATICS

AN INVARIANT OF FINITE NONCOMMUTATIVE GEOMETRIC SPACES

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[Abstract]

We construct an invariant of finite noncommutative geometric spaces based on the work of Connes in [1]. A finite noncommutative geometric space is described by a finite spectral triple (M, H, D) , where $M = \bigoplus_{\alpha} M_{d_{\alpha}}(\mathbb{C})$ is a matrix algebra acting on a finite-dimensional Hilbert space H , and D a symmetric operator. We will associate an algebra N to the operator D and consider the pair of algebras (M, N) acting on the same Hilbert space H . Our invariant $\text{Spec}_N(M)$, the *relative spectrum*, is a set of projections that measures the position of the algebra M relative to N . It describes the position of the irreducible representations of M in H relative to the eigenspaces of D . Together with the spectrum of the operator D and the set of positive integers $\{d_{\alpha}\}_{\alpha}$ we obtain, modulo unitary equivalence, a complete invariant. In the commutative case, it is possible to switch the roles of the algebras M and N in the construction, which gives rise to a *dual invariant* $\text{Spec}_M(N)$. We give a direct relation between $\text{Spec}_N(M)$ and $\text{Spec}_M(N)$. We also relate our invariant to *decorated graphs* of finite spectral triples, as described in [2].

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1 Introduction

Noncommutative geometric spaces ([3],[5]) are a generalization of Riemannian geometric spaces. We can think of a finite noncommutative geometric space as a finite space whose points have some internal noncommutative structure together with a notion of distance between the points. In order to find an algebraic description of such spaces we start from the Gelfand duality between a topological space and its algebra of functions. By the finite version of this duality, a commutative *matrix algebra* describes a finite space. As a generalization, a noncommutative matrix algebra will now be thought of as the algebra of continuous functions on a finite noncommutative space. An algebraic description of a finite space together with a metric structure is given by the commutative matrix algebra M that arises from the Gelfand duality, represented on a finite-dimensional Hilbert space H , and a symmetric operator D . Such a triple (M, H, D) is called a *finite spectral triple*. Dropping the condition of commutativity in a finite spectral triple then gives a description of finite noncommutative geometric spaces.

Although, we restrict ourselves in this thesis to finite noncommutative spaces, also not necessarily finite noncommutative spaces can be described by a similar spectral triple. Connes showed in [6] for example that a spectral triple with commutative algebra and certain conditions necessarily arises from a compact oriented Riemannian manifold and as a consequence this suggests that a noncommutative Riemannian manifold is defined by dropping the commutativity of the algebra. As an important application in physics, we note that a suitable choice for the spectral triple gives rise to the standard model [7]. This model describes all elementary particles and their electromagnetic, weak and strong interactions.

In this thesis we construct an invariant of finite noncommutative geometric spaces inspired by Connes in [1]. This invariant measures the relative position of the algebra M and the algebra associated to the operator D . Finding an invariant yields a description of finite noncommutative geometric spaces in another way than the description by finite spectral triples, which is in some examples more convenient.

The first part of this thesis consists of some preliminaries, so that we understand the notion of a finite spectral triple. We then first construct an invariant of finite commutative spectral triples assuming a nondegenerate spectrum of the operator D . For such a triple (M, H, D) , we will view the position of the irreducible representations of the algebra M in the Hilbert space H relative to the eigenspaces of the operator D . We will do this by associating an algebra N to the operator D and then considering the pair of algebras (M, N) acting on the same Hilbert space. Our invariant $\text{Spec}_N(M)$, the *relative spectrum*, then gives the position of the irreducible representations of M in H relative to those of N . The relative spectrum $\text{Spec}_N(M)$ is a set consisting of projections expressed by complex matrices whose entries are labeled by $\text{Spec}(N)$. Together with $\sigma(D)$, the spectrum of the operator D , we then obtain, modulo unitary equivalence, a complete invariant, i.e. there is, modulo unitary equivalence, a one-to-one correspondence:

$$(M, H, D) \xleftrightarrow{(1:1)} (\text{Spec}_N(M), \sigma(D)).$$

After this construction we will generalize this invariant for the finite commutative case, no longer assuming the spectrum of D to be nondegenerate. In this case the relative spectrum $\text{Spec}_N(M)$, that again measures the relative position of the two algebras M and N , is a set of projections expressed by matrices $\{\gamma\}$ labeled by elements of $\text{Spec}(N)$ such that $\gamma_{\lambda\mu} : V_\mu \rightarrow V_\lambda$ with $\lambda, \mu \in \text{Spec}(N)$ is a map between eigenspaces of D , so no longer a complex number. In the end we will see that the construction also works for the noncommutative case, where $M = \bigoplus_\alpha M_{d_\alpha}(\mathbb{C})$ and that the relative spectrum $\text{Spec}_N(M)$ then together with the spectrum of D and the set of positive integers $\{d_\alpha\}_\alpha$ is, modulo unitary equivalence, a complete invariant. Along the way we consider several examples that clarify the construction.

In the commutative case, it is possible to interchange the roles of the algebras M and N in our construction which gives rise to a *dual invariant* $\text{Spec}_M(N)$. We will give a direct relation between the invariant and its dual. The purpose of this is that in some examples calculating the dual is easier than calculating $\text{Spec}_N(M)$. Furthermore we will find a relation between our invariant and a graphical invariant of finite spectral triples [2]. The relation we find here, directly comes from the relation of the relative spectrum and its dual.

This thesis is aimed at master students in mathematics who are familiar with functional analysis and with representation theory of algebras. The most basic definitions we use from functional analysis such as the notion of a Hilbert space are not recalled in this text and can for example be found in [11] or [14]. The notions from representation theory of algebras are stated in the preliminaries.

During my master I followed several courses with subjects in functional analysis, of which some at SISSA in Trieste, Italy. As a kind of unfortunate coincidence I did not have the opportunity to follow a course in noncommutative geometry, neither in Nijmegen, nor in Italy. However since my bachelor project under supervision of Walter D. van Suijlekom, I have known that I would like to learn more about the field of noncommutative geometry. Therefore finding a subject for my master thesis was not very difficult. During my thesis project I discovered that indeed noncommutative geometry is a very interesting subject. When starting this project I knew that it would be the end of my studies, but I also knew it would be the last time spending my time as a mathematical researcher. Now I am finished, I am very happy and grateful that I had this opportunity. I have learned a lot about this enthralling subject, but also about me as a person. I hope you will enjoy reading my thesis.

2 Preliminaries

2.1 Spectral triples

In this subsection we will start with the definition of a unital complex $*$ -algebra and come to the definition of a finite spectral triple. Furthermore we will consider the spectral theorem in the finite case. We will follow the books [2] and [10].

Definition 2.1. A (*complex*) *algebra* is a vector space A (over \mathbb{C}) together with a bilinear associative product $A \times A \rightarrow A$ denoted by $(a, b) \mapsto ab$.

A *unital algebra* is an algebra A with a unit $1 \in A$ satisfying $1a = a1 = a$ for all $a \in A$.

A *$*$ -algebra* is an algebra A with a conjugate linear map $*$: $A \rightarrow A$ such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in A$. This map is called the *involution* map.

We will only consider unital complex algebras and refer to them as algebras.

Example 2.2. The set $C(X)$ of complex valued functions on a finite space X is a $*$ -algebra. We have a pointwise linear structure:

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \text{ and} \\ (\lambda f)(x) &= \lambda(f(x)) \end{aligned}$$

for all $f, g \in C(X), \lambda \in \mathbb{C}$. The multiplication is pointwise

$$fg(x) = f(x)g(x)$$

and the involution is given by conjugation:

$$f^*(x) = \overline{f(x)}.$$

Example 2.3. The set $M_n(\mathbb{C})$ of $n \times n$ matrices with coefficients in \mathbb{C} is a $*$ -algebra. The bilinear product is given by matrix multiplication. The involution map is given by hermitian conjugation:

$$(A^*)_{ij} = \overline{A_{ji}}.$$

Definition 2.4. A (*complex*) *matrix algebra* A is a direct sum

$$A = \bigoplus_{\alpha=1}^K M_{d_\alpha}(\mathbb{C}),$$

for some K and d_α in \mathbb{N} . The involution is given by hermitian conjugation.

A *commutative* matrix algebra A is then a matrix algebra of the form \mathbb{C}^K .

Example 2.5. Let $X = \{1, \dots, N\}$ be a finite space. Then the $*$ -algebra $C(X)$ is a matrix algebra. We can represent a function $f \in C(X)$ as the following matrix:

$$\begin{pmatrix} f(1) & & & \\ & f(2) & & \\ & & \ddots & \\ & & & f(N) \end{pmatrix}.$$

In this way pointwise multiplication becomes matrix multiplication and $C(X) \cong \mathbb{C}^N$.

Example 2.6. Let H be a finite-dimensional inner product space, with inner product $\langle \cdot, \cdot \rangle \rightarrow \mathbb{C}$. The space of linear operators on H denoted by $\mathcal{L}(H)$ is a $*$ -algebra with its product given by composition and its involution given by mapping an operator T to its adjoint T^* .

Definition 2.7. A **-algebra map* (or **-homomorphism*) between two **-algebras* A and B is a linear map $\phi : A \rightarrow B$ such that

- (i) $\phi(a)\phi(b) = \phi(ab)$ and
- (ii) $\phi(a^*) = \phi(a)^*$

for all $a, b \in A$.

Definition 2.8. A *representation* of a finite dimensional **-algebra* A is a pair (H, π) where H is a finite-dimensional (complex) inner product space and π is a **-algebra map*

$$\pi : A \rightarrow \mathcal{L}(H).$$

A representation H is *irreducible* if $H \neq 0$ and the only subspaces in H that are left invariant under π , the action of A , are $\{0\}$ and H . In other words we have that the only subrepresentations of H are $\{0\}$ and H .

A representation is called *faithful* if the map π is injective.

Since an inner product space and a Hilbert space are the same in the finite-dimensional case, we will use the term Hilbert space instead of inner product space.

Example 2.9. For $M_n(\mathbb{C})$ is $H = \mathbb{C}^n$ a representation on which $M_n(\mathbb{C})$ acts by left matrix multiplication. It is clear that this representation is irreducible. A reducible representation for example is given by $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with an action given as follows:

$$M_n(\mathbb{C}) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{L}(\mathbb{C}^n \oplus \mathbb{C}^n).$$

Definition 2.10. Two representations (H, π) and (H', π') of a **-algebra* A are *unitarily equivalent* if there exists a unitary map $U : H \rightarrow H'$ such that

$$\pi'(a) = U\pi(a)U^*$$

for all $a \in A$.

Lemma 2.11. *Each finite-dimensional faithful representation of a **-algebra* A can be written as the direct sum of irreducible subrepresentations.*

Proof. Any representation H of A has an irreducible subrepresentation $H_1 \subset H$. Namely, take a subrepresentation $H_1 \subset H$ with minimal dimension. Using the faithfulness we conclude that H_1^\perp is a subrepresentation of H and $H = H_1 \oplus H_1^\perp$. We continue by taking an irreducible subrepresentation H_2 of H_1^\perp . Then $H_1^\perp = H_2 \oplus H_2^\perp$. Since H is finite dimensional we can continue in this way and find a decomposition of H into irreducible representations. \square

Theorem 2.12. *Let $M = \bigoplus_{\alpha=1}^K M_{d_\alpha}(\mathbb{C})$ be a matrix algebra. Then we have that $\mathbb{C}^{d_1}, \dots, \mathbb{C}^{d_K}$ are the irreducible representations of A .*

Proof. A proof of this can be found in [10], Theorem 3.3.1. \square

Definition 2.13. The *structure space* \widehat{A} of a **-algebra* A is the set of all unitary equivalence classes of irreducible representations of A .

Example 2.14. From Lemma 2.11 and Theorem 2.12 it follows that, modulo a basis transformation, a representation (H, π) of a matrix algebra $M = \bigoplus_{\alpha} M_{d_\alpha}(\mathbb{C})$ is of the following form:

$$H = \mathbb{C}^n, \quad \pi(a) = \bigoplus_{\alpha} m_{\alpha} a_{\alpha},$$

where m_α is the multiplicity of \mathbb{C}^{d_α} , $n = \sum_{\alpha=1}^K m_\alpha d_\alpha$ and $a = (a_\alpha)_\alpha \in M$.

The structure space of M is given by $\widehat{M} = \{1, \dots, K\}$, where each integer $\alpha \in \widehat{M}$ corresponds to the equivalence class of representations of M on \mathbb{C}^{d_α} . Labeling the latter equivalence class with d_α gives an identification $\widehat{M} \cong \{d_1, \dots, d_K\}$.

Definition 2.15. A *finite spectral triple* is a triple (A, H, D) consisting of a unital $*$ -algebra A which is represented faithfully on a finite-dimensional Hilbert space H , together with a symmetric operator $D : H \rightarrow H$. A finite *commutative* spectral triple is a finite spectral triple, where the $*$ -algebra A is commutative.

Remark 2.16. The representation of the algebra A on the Hilbert space H is implicitly assumed in the definition of a finite spectral triple and given a finite spectral triple (A, H, D) we view the elements of A as operators on H .

Theorem 2.17. *Let A be a unital $*$ -algebra that acts faithfully on a Hilbert space H . Then A is a matrix algebra*

$$A \cong \bigoplus_{\alpha=1}^K M_{d_\alpha}(\mathbb{C}).$$

Proof. The representation $\pi : A \rightarrow \mathcal{L}(H)$ is injective by assumption and hence A is a $*$ -subalgebra of $\mathcal{L}(H)$. Because we have $\mathcal{L}(H) \cong M_{\dim H}(\mathbb{C})$, its only $*$ -subalgebras are themselves matrix algebras. .

A complete proof can be found in [10], Theorem 3.5.4. □

Because of the above theorem we will write a finite spectral triple as (M, H, D) with M a matrix algebra.

Example 2.18. We have that $(M_n(\mathbb{C}), \mathbb{C}^n, D)$ is a finite spectral triple with an action by left matrix multiplication and D a hermitian matrix. The space \mathbb{C}^n is equipped with the standard inner product.

Definition 2.19. Two finite spectral triples (M, H, D) and (M', H', D') are called *unitarily equivalent*, if there exists a unitary operator $U : H \rightarrow H'$ and an isomorphism $v : M \rightarrow M'$ such that

- (i) $U\pi(a)U^* = \pi'(v(a))$ for $a \in M$ and
- (ii) $UDU^* = D'$.

Remark 2.20. In Example 2.5 we have associated a matrix algebra $C(X)$ to a finite space X . This algebra behaves naturally with respect to maps between topological spaces and $*$ -algebras i.e. if we have a map $\phi : X_1 \rightarrow X_2$ of finite discrete spaces, then the corresponding map $\phi^* : C(X_2) \rightarrow C(X_1)$ is given by the pullback

$$\phi^* f = f \circ \phi \in C(X_1) \text{ with } f \in C(X_2),$$

which is a $*$ -homomorphism. The following question arises: given a matrix algebra M , does there exist a finite discrete space X such that $C(X) \cong M$? The answer is no, since $C(X)$ is commutative but M is not necessarily commutative. To solve this we can restrict to commutative matrix algebras. Given a commutative algebra M then its structure space $\widehat{M} \cong \{1, \dots, K\}$ is the right candidate. We see that there is a duality between finite spaces and commutative matrix algebras. This is just a finite-dimensional version of the *Gelfand duality* stated in [2], Theorem 4.28.

Another way to solve this is to allow more morphisms between matrix algebras. We then find a duality between finite spaces and Morita equivalence classes of matrix algebras. The definition of

Morita equivalence can be found in [2], Definition 2.12. In Section 2.2 of [2] we see that the data of a finite metric space is captured in a finite commutative spectral triple and that this metric space can also be reconstructed from that commutative spectral triple. A finite noncommutative spectral triple then describes in the same way a metric on the finite noncommutative space \widehat{M} .

2.2 Spectral theorem

In this subsection we will define the functional calculus of a symmetric operator D on a finite-dimensional Hilbert space. Therefore we first need the Spectral theorem.

Theorem 2.21 (Spectral theorem (finite version), [15], Theorem 6.16). *Let H be a finite-dimensional Hilbert space and D be a symmetric operator in $\mathcal{L}(H)$. Then there exists an orthonormal basis of H consisting of eigenvectors.*

Proof. According to Theorem 6.14 of [15] we can find an orthonormal basis v_1, \dots, v_n for H such that the operator D written in this basis is a matrix T which is upper triangular. Then v_1 is clearly an eigenvector of D . Suppose that v_1, \dots, v_{k-1} are eigenvectors of D . We then have that

$$D(v_k) = T_{1k}v_1 + \dots + T_{jk}v_j + \dots + T_{kk}v_k$$

and for $j < k$

$$T_{jk}(v_j) = \langle v_j, D(v_k) \rangle = \langle D(v_j), v_k \rangle = \langle \lambda_j v_j, v_k \rangle = 0$$

where λ_j is the eigenvalue corresponding to the eigenvector v_j . We conclude that $D(v_k) = T_{kk}v_k$ and hence v_k is an eigenvector. By induction v_1, \dots, v_n are eigenvectors of D . \square

This immediately implies the following:

Corollary 2.22. *Let H be a finite-dimensional Hilbert space. A symmetric operator $D \in \mathcal{L}(H)$ can be written as*

$$D = \sum_i \lambda_i e^i,$$

where λ_i are the eigenvalues and e^i the (spectral) projections onto the eigenspace corresponding to λ_i .

Definition 2.23. The *spectrum* $\sigma(D)$ of an operator $D \in \mathcal{L}(H)$ for a finite-dimensional Hilbert space H is the space of eigenvalues with multiplicities. The spectrum is called *nondegenerate* if each $\lambda \in \sigma(D)$ has multiplicity one.

Definition 2.24. Let H be a finite-dimensional Hilbert space and $D = \sum_i \lambda_i e^i \in \mathcal{L}(H)$ be a symmetric operator. The *functional calculus* of D is defined as the map

$$\begin{aligned} C(\sigma(D)) &\rightarrow \mathcal{L}(H) \text{ such that} \\ f &\mapsto f(D) := \sum_i f(\lambda_i) e^i. \end{aligned}$$

We can also take $f \in C(\mathbb{R})$ since all the eigenvalues of a symmetric operator are real numbers.

Lemma 2.25. *Let H be a finite-dimensional Hilbert space and let D be a symmetric operator in $\mathcal{L}(H)$. The set N defined by*

$$N := \{f(D) | f \in C(\mathbb{R})\}$$

is a commutative unital $$ -algebra.*

Proof. Write $D = \sum_i \lambda_i e^i$, where λ_i are the eigenvalues of D and e^i the projections onto the eigenspace corresponding to λ_i . The linear structure is as follows:

$$\sum_i f(\lambda_i) e^i + \sum_i f'(\lambda_i) e^i = \sum_i (f + f')(\lambda_i) e^i \text{ and}$$

$$\mu \left(\sum_i f(\lambda_i) e^i \right) = \sum_i (\mu f)(\lambda_i) e^i,$$

for $f, f' \in C(\mathbb{R}), \mu \in \mathbb{C}$.

The multiplication between two elements of N is given as follows:

$$\sum_i f(\lambda_i) e^i \cdot \sum_i f'(\lambda_i) e^i = \sum_i (f f')(\lambda_i) e^i = \sum_i f(\lambda_i) f'(\lambda_i) e^i.$$

Commutativity is immediately clear with this structure. The involution map is given by:

$$\left(\sum_i f(\lambda_i) e^i \right)^* = \sum_i f^*(\lambda_i) e^i = \sum_i \overline{f(\lambda_i)} e^i.$$

□

3 The invariant of finite commutative spectral triples with non-degenerate spectrum

In this section we will construct the invariant of finite commutative spectral triples in case of nondegenerate spectrum. The idea of this construction is based on the invariant of the relative position of two finite-dimensional commutative von Neumann algebras M and N acting on the same Hilbert space constructed in [1] by Connes. From Lemma 2.25 we can associate a pair of algebras (M, N) , acting on the same finite-dimensional space H , to a finite spectral triple (M, H, D) and do a similar construction as in [1]. Although in [1] is started with the case where M is acting with multiplicity one on the Hilbert space, we do not make this assumption.

3.1 The construction of the invariant of finite commutative spectral triples with nondegenerate spectrum

Let (M, H, D) be a finite commutative spectral triple with H a Hilbert space of dimension n . Let $\sigma(D) = \{\lambda_1, \dots, \lambda_n\}$ be the spectrum of D and by assumption each λ_i has multiplicity one. Since M is commutative we have

$$M \cong \mathbb{C}^K,$$

for a certain $K \in \mathbb{N}$. Each copy \mathbb{C} of M corresponds to a set $E_\alpha \subset H$ such that

$$\pi(e_M^\alpha)E_\alpha = E_\alpha.$$

Here π is the action of M on H and $\{e_M^1, \dots, e_M^K\}$ is the set of one-dimensional idempotents of M that generate M and act on H with multiplicity m_α . We see that E_α is an m_α -dimensional space isomorphic to $\mathbb{C} \otimes \mathbb{C}^{m_\alpha}$ and we have a decomposition of H into irreducible representations of M :

$$H = \bigoplus_{\alpha=1}^K E_\alpha.$$

As a consequence $\sum_{\alpha=1}^K m_\alpha = n$.

We write the operator D as

$$D = \sum_{\lambda \in \sigma(D)} \lambda e_N^\lambda,$$

where e_N^λ are the orthogonal projections onto the eigenspaces V_λ . Note that by assumption all the e_N^λ project onto a one-dimensional space. We take the $*$ -algebra N such as defined in Lemma 2.25

$$N = \{f(D) | f \in C(\mathbb{R})\} = \left\{ \sum_{\lambda} f(\lambda) e_N^\lambda | f \in C(\mathbb{R}) \right\} \cong \mathbb{C}^n.$$

The action ρ of N on H is defined by:

$$\rho\left(\sum_{\lambda} f(\lambda) e_N^\lambda\right)(x) = \sum_{\lambda} f(\lambda) e_N^\lambda(x),$$

for $f \in C(\mathbb{R})$ and $x \in H$. Let $\{\eta_\lambda\}_{\lambda \in \sigma(D)} \in H$ be an orthonormal basis of eigenvectors of D so that

$$\rho(e_N^\lambda)\eta_\lambda = \eta_\lambda.$$

We see that the irreducible representations of N in H are just the eigenspaces of D .

We want to know the position of the spaces $E_\alpha \subset H$ relative to the eigenspaces V_λ of the operator D . We will establish this by defining a unitary isomorphism between H and $l^2(\text{Spec}(N)) \cong \mathbb{C}^n$.

Remark 3.1. We have $\text{Spec}(N) = \{e_N^\lambda\}_\lambda$ is the set of spectral projections and $l^2(\text{Spec}(N))$ is the space of complex sequences labeled by elements in $\text{Spec}(N)$. We will refer to a label e_N^λ just with λ .

Let $\phi : H \rightarrow l^2(\text{Spec}(N))$ be a unitary isomorphism such that $\phi(\eta_\lambda) = \varepsilon_\lambda$, where $\{\varepsilon_\lambda\}_\lambda$ is the canonical basis of $l^2(\text{Spec}(N))$. Then

$$\phi(x)(\lambda) = \langle \eta_\lambda, x \rangle \tag{1}$$

for all $x \in H$. The action ρ of N on H has become the diagonal action on $l^2(\text{Spec}(N))$. Furthermore the decomposition of irreducible representations of M on $l^2(\text{Spec}(N))$ is given by

$$l^2(\text{Spec}(N)) = \bigoplus_{\alpha=1}^K \phi(E_\alpha).$$

Since a basis of normalised eigenvectors of D is unique up to a phasefactor $z_\lambda \in \mathbb{C}$ with $|z_\lambda| = 1$ for each λ , the isomorphism ϕ is unique up to an action of

$$\mathcal{U}(N) := \underbrace{U(1) \times \dots \times U(1)}_n.$$

We conclude that each copy \mathbb{C} of M corresponds to a space $\phi(E_\alpha) \subset l^2(\text{Spec}(N))$ and that the subspaces $\phi(E_\alpha) \subset l^2(\text{Spec}(N))$ describe the positions of the spaces E_α relative to the eigenspaces V_λ . In order to view the subspaces $\phi(E_\alpha) \subset l^2(\text{Spec}(N))$ in a nicer way, we will give an equivalent description of the spaces $\phi(E_\alpha)$ in terms of orthogonal projections. This requires some definitions and a lemma.

Definition 3.2 ([13], Section 1.5). Let the *Grassmannian*, $\text{Gr}(m, H)$, be the set of all m -dimensional subspaces of a n -dimensional Hilbert space H .

Notation 3.3. Denote with P_m^+ the set of $n \times n$ matrices such that for $\gamma \in P_m^+$:

- (i) γ is a positive semidefinite matrix of rank m
- (ii) $\gamma^2 = \gamma$
- (iii) $\gamma^* = \gamma$.

In fact an element of P_m^+ is an *orthogonal projection* onto a m -dimensional subspace of H . Note that the rank of a projection equals its trace.

Remark 3.4. Since the eigenvalues of an orthogonal projection can only be 0 or 1, the corresponding matrix is always positive semidefinite.

Lemma 3.5. *For each element in $\text{Gr}(m, H)$ there exist a unique element in P_m^+ .*

Proof. Let E be a m -dimensional subspace of a n -dimensional Hilbert space H . Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of E . Then E is represented by the matrix

$$\tilde{E} = \begin{pmatrix} e_{11} & \dots & e_{m1} \\ \vdots & & \vdots \\ e_{1n} & \dots & e_{mn} \end{pmatrix}.$$

Define the map

$$\psi : \text{Gr}(m, H) \rightarrow P_m^+, \quad \psi(E) = \tilde{E}\tilde{E}^*.$$

The map ψ is independent of the basis chosen for E . Namely, if we choose another orthonormal basis $\{e'_1, \dots, e'_m\}$ which gives a representation \tilde{E}' of E , we have $\tilde{E} = \tilde{E}'u$ with $u \in U(m)$, a unitary $m \times m$ matrix. Hence

$$\psi(E) = \tilde{E}\tilde{E}^* = \tilde{E}'uu^*\tilde{E}'^* = \tilde{E}'\tilde{E}'^*.$$

Clearly $\tilde{E}\tilde{E}^*$ is of rank m and $(\tilde{E}\tilde{E}^*)^* = \tilde{E}\tilde{E}^*$. Furthermore we have $(\tilde{E}^*\tilde{E})_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$ and hence $(\tilde{E}\tilde{E}^*)^2 = \tilde{E}(\tilde{E}^*\tilde{E})\tilde{E}^* = \tilde{E}\tilde{E}^*$. So $\psi(E) \in P_m^+$. Hence the map ψ is welldefined. It is clear that $\psi(E)$ is exactly the orthogonal projection onto the space E .

We will show that ψ is an isomorphism by finding its inverse. Take m linear independent columns $\{\gamma_1, \dots, \gamma_m\}$ of an element $\gamma \in P_m^+$ and take $G = \text{Span}\{\gamma_1, \dots, \gamma_m\}$. We clearly have that the inverse map is given by

$$\psi^{-1} : P_m^+ \rightarrow \text{Gr}(m, H), \quad \psi^{-1}(\gamma) = G.$$

Note that to obtain an orthonormal basis of G we orthonormalize $\{\gamma_1, \dots, \gamma_m\}$. \square

From this lemma we can find for each m_α -dimensional subspace $\phi(E_\alpha) \subset l^2(\text{Spec}(N))$ an element $\gamma_\alpha \in P_{m_\alpha}^+(N)$. Here N refers to the fact that γ_α is a matrix labeled by elements of $\text{Spec}(N)$. This labeling appears since each vector of $\phi(E_\alpha)$ is labeled by elements of $\text{Spec}(N)$. The order in which we take the eigenvalues of D determines the order of the labeling in the γ_α . The action of the unitary group $\mathcal{U}(N)$ then becomes the adjoint action:

$$(\text{Ad}(u)\gamma)_{\lambda\mu} = u_\lambda \gamma_{\lambda\mu} \bar{u}_\mu$$

where $u \in \mathcal{U}(N)$.

Following Connes in [1], Definition 2.5, we introduce $\text{Spec}_N(M)$.

Definition 3.6. We define the *relative spectrum* of M relative to N as

$$\text{Spec}_N(M) = \{\gamma_\alpha | \alpha \in \{1, \dots, K\}\} \subset \bigcup_{\alpha} P_{m_\alpha}^+(N),$$

which is defined up to the adjoint action of $\mathcal{U}(N)$.

Since each of the $\phi(E_\alpha)$ are orthogonal to each other we see from lemma 3.5 that $\gamma_\alpha \gamma_{\alpha'} = 0$. Since also $\bigoplus_{\alpha=1}^K \phi(E_\alpha) = l^2(\text{Spec}(N))$ we obtain that

$$\sum_{\alpha=1}^K \gamma_\alpha = \mathbb{I},$$

where \mathbb{I} is the identity map.

We conclude the following proposition:

Proposition 3.7 ([1], Proposition 2.6). *Let (M, H, D) , be a finite commutative spectral triple with nondegenerate spectrum of D . Then the relative spectrum of M relative to N consists of K elements and*

$$S = \text{Spec}_N(M) \subset \bigcup_{i=1}^{\infty} P_i^+(N),$$

such that

- (i) $\gamma\gamma' = 0$ for all $\gamma \neq \gamma' \in S$
- (ii) $\sum_S \gamma_{\lambda\mu} = \delta_{\lambda\mu}$.
- (iii) $\sum_{\gamma \in S} m(\gamma) = n$, where $m(\gamma) = i$ if $\gamma \in P_i^+(N)$ and n is the dimension of H .

The relative spectrum, $\text{Spec}_N(M)$, modulo the adjoint action of $\mathcal{U}(N)$ is an invariant of (M, H, D) .

Remark 3.8. As mentioned before, requirements (i) and (iii) imply (ii). Also requirements (i) and (ii) imply (iii). Therefore we can leave out requirement (iii).

Proposition 3.9. Let (M, H, D) and (M', H', D') be two unitarily equivalent triples. Then we have

$$\text{Spec}_N(M) \cong \text{Spec}_{N'}(M').$$

Moreover $\sigma(D) = \sigma(D')$.

Proof. By definition there is a unitary isomorphism $U : H \rightarrow H'$ and an isomorphism $v : M \rightarrow M'$ such that:

- (i) $U\pi(a)U^* = \pi'(v(a))$ for all $a \in M$ and
- (ii) $UDU^* = D'$.

We immediately see that $\sigma(D) = \sigma(D')$. We choose a basis $\{\eta_\lambda\}_\lambda$ in H consisting of eigenvectors of D . Then we choose $\{U\eta_\lambda\}_\lambda$ as a basis for H' , which are eigenvectors of D' . Using this bases we obtain maps $\phi : H \rightarrow l^2(\text{Spec}(N))$ and $\phi' : H' \rightarrow l^2(\text{Spec}(N'))$. Then the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{U} & H' \\ \phi \downarrow & & \downarrow \phi' \\ l^2(\text{Spec}(N)) & \xrightarrow{\tilde{U}} & l^2(\text{Spec}(N')) \end{array}$$

The map \tilde{U} from $l^2(\text{Spec}(N))$ to $l^2(\text{Spec}(N'))$ is the identity map, where only the labeling changes according to $\text{Spec}(N') = \{Ue_N^\lambda U^*\}_{e_N^\lambda \in \text{Spec}(N)}$.

Let $\{E_\alpha\}_\alpha$ be the set of irreducible representations of M in H . Because of property (i) and the fact that v sends idempotents to idempotents we have that the set of irreducible representations of M' in H' is given by $\{E'_\alpha\}_\alpha = \{UE_\alpha\}_\alpha$. We compute a representative for $\text{Spec}_N(M)$ and for $\text{Spec}_{N'}(M')$:

$$\{\gamma_\alpha\}_\alpha = \left\{ \widetilde{\phi(E_\alpha)} \left(\widetilde{\phi(E_\alpha)} \right)^* \right\}_\alpha = \left\{ \phi'(UE_\alpha) \left(\phi'(UE_\alpha) \right)^* \right\}_\alpha = \{\gamma'_\alpha\}_\alpha,$$

which is an equality as set of matrices. Hence $\text{Spec}_N(M) \cong \text{Spec}_{N'}(M')$, where a map between the two is given by a change of labeling. \square

The relative spectrum $\text{Spec}_N(M)$ together with the spectrum of D is, modulo unitary equivalence, a complete invariant of finite commutative spectral triples. This is stated in the following theorem.

Theorem 3.10. There is a one-to-one correspondence between finite commutative spectral triples (M, H, D) with nondegenerate spectrum of D , modulo unitary equivalence, and pairs (S, Λ) where

$$S \subset \bigcup_{i=0}^{\infty} P_i^+(N),$$

such that

- (i) $\gamma\gamma' = 0$ for all $\gamma \neq \gamma' \in S$ and
- (ii) $\sum_S \gamma_{\lambda\mu} = \delta_{\lambda\mu}$

is a finite set defined up to the adjoint action of $\mathcal{U}(N)$ and $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ whose number of elements and order corresponds to the labeling of the $\gamma \in S$.

Proof. We have already seen that (S, Λ) with $S = \text{Spec}_N(M)$ and $\Lambda = \sigma(D)$ is an invariant of a finite commutative spectral triple. We are left with reconstructing a finite spectral triple from (S, Λ) and showing that it is exactly the inverse of our invariant. Let (S, Λ) be a pair such as stated in the theorem. We will now construct a finite spectral triple from it. It is clear that the algebra is given by

$$M' = \mathbb{C}^{|S|}.$$

An element $\gamma \in S$ is the orthogonal projection onto some $m(\gamma)$ -dimensional subspace, where $m(\gamma) = i$ if $\gamma \in P_i^+(N)$. Then, with $E_\gamma = \text{Im } \gamma$, we define a Hilbert space

$$H' = \bigoplus_{\gamma \in S} E_\gamma,$$

which has dimension $\sum_{\gamma \in S} m(\gamma) = n$. We take $\{E_\gamma\}_{\gamma \in S}$ as the irreducible representations of M' in H' and therefore the action of M' on H' is defined. We take the following symmetric operator

$$D' = \sum_{\lambda \in \Lambda} \lambda \sum_{\gamma \in S} \gamma_{\lambda\lambda},$$

which is just a diagonal matrix.

We claim that the constructed triple (M', H', D') is the inverse of our invariant. Therefore we need to check two things.

1. The pair (S, Λ) is indeed the invariant of (M', H', D') .
2. If (M, H, D) is a finite spectral triple and (S, Λ) its invariant, then (M', H', D') is unitarily equivalent to (M, H, D) .

Proof of statement 1. In the construction of the invariant of (M', H', D') we choose a unitary isomorphism $\phi' : H' \rightarrow l^2(\text{Spec}(N'))$ which is in this case the identity map, by identifying $l^2(\text{Spec}(N'))$ with H' . Hence a representative for $\text{Spec}_{N'}(M')$ is $\{\tilde{E}_\gamma \tilde{E}_\gamma^*\}_{\gamma \in S} = S$. Furthermore we have $\sigma(D) = \Lambda$.

Proof of statement 2. Clearly we must have $M \cong M'$. In the construction of the invariant of (M, H, D) we choose a unitary isomorphism $\phi : H \rightarrow l^2(\text{Spec}(N))$. We have irreducible representations E_α of M in H and a representative for the invariant is then given by

$$\left\{ \gamma_\alpha = \widetilde{\phi(E_\alpha)} \left(\widetilde{\phi(E_\alpha)} \right)^* \right\}_\alpha,$$

which equals the set S by assumption. Furthermore we have $\{\phi(E_\alpha)\}_\alpha = \{E_\gamma\}_{\gamma \in S}$. Then (M, H, D) and (M', H', D') are unitarily equivalent using the unitary operator $\phi : H \rightarrow l^2(\text{Spec}(N))$ viewed as map from H to H' such as in the proof of statement 1. Note that $\sigma(D) = \sigma(D') = \Lambda$ by assumption and $\phi D \phi^*$ is a diagonal matrix with the eigenvalues of D on its diagonal and therefore we have $\phi D \phi^* = D'$. \square

Remark 3.11. Whenever we do not consider pairs (S, Λ) , but only subsets S , we have a complete invariant of the pair of algebras (M, N) . Since it is possible to have operators with different eigenvalues but with the same eigenspaces, we need to know the eigenvalues including their order to reconstruct the operator D .

We will often refer to the invariant as $\text{Spec}_N(M)$, without mentioning the set Λ .

Remark 3.12. The triple (M', H', D') that we reconstructed from a pair (S, Λ) depends on the representative $\{\gamma\}_{\gamma \in S}$ that is taken. Another representative is given by $\{U\gamma U^*\}_{\gamma \in S}$ where

$$U = \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix},$$

where $u_i \in U(1)$. If we reconstruct the spectral triple from the latter representative we obtain the same operator D' and $\{U\tilde{E}_\gamma\}_{\gamma \in S}$ as the irreducible representations of M' in the Hilbert space. This triple is then unitarily equivalent to (M', H', D') using the unitary operator $U : H \rightarrow H$ and noting that $UD'U^* = D'$.

3.2 Examples

Example 3.13. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

The operator D has eigenvalues $\lambda, -\lambda$ and 0 with corresponding normalised eigenvectors

$$\eta_\lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \eta_{-\lambda} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \eta_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The minimal projections of M are equal to

$$e_M^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } e_M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so the irreducible representations of M in H are given by:

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

With Equation (1) we compute the following representative for $\text{Spec}_N(M)$:

$$\gamma_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in P_2^+(N) \text{ and}$$

$$\gamma_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in P_1^+(N).$$

Consider the same triple but with a different action of M on H :

$$\pi'(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

Since this triple is not unitarily equivalent to the previous one, we expect another invariant. The minimal projections of M are now given by

$$e_M^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } e_M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and then the irreducible representations of M in H are

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We then find as a representative for $\text{Spec}_N(M)$ the following set:

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in P_2^+(N) \text{ and} \\ \gamma_2 &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in P_1^+(N). \end{aligned}$$

From Remark 3.12 we easily see that indeed this representative gives another equivalence class.

Example 3.14. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0, 1\}$,

$$\begin{aligned} M &= \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \pi(z_1, z_2) &= \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}, z_1, z_2 \in \mathbb{C}. \end{aligned}$$

The operator D has eigenvalues $\lambda, -\lambda, 1$ and 0 with corresponding normalised eigenvectors

$$\eta_\lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \eta_{-\lambda} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \eta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \eta_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Considering the minimal projections of M we obtain the following irreducible representations of M in H

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We obtain as a representative for $\text{Spec}_N(M) \subset P_2^+(N)$ the following set:

$$\begin{aligned}\gamma_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and} \\ \gamma_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Consider the same finite spectral triple but with the following action of M on H :

$$\pi'(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

We will now obtain another invariant since this triple is not unitarily equivalent with the previous one. The irreducible representations of M on H are now given by

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

This gives the following representative for the set $\text{Spec}_N(M) \subset P_2^+(N)$:

$$\begin{aligned}\gamma_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and} \\ \gamma_2 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

which indeed gives, using the same argument as in the previous example, not the same equivalence class.

3.3 Viewing the invariant as spheres

The sets $\phi(E_\alpha)$ are in one-to-one correspondence with the sets

$$\phi(E_\alpha) \cap S_{\mathbb{C}}^N = S_\alpha,$$

where

$$S_{\mathbb{C}}^N = S(l^2(\text{Spec}(N))) := \{z \in l^2(\text{Spec}(N)) \mid \sum_{\lambda \in \text{Spec}(N)} |z_\lambda|^2 = 1\}.$$

Each idempotent of M corresponds to $S_\alpha \cong S^{2m_\alpha - 1} \subset S_{\mathbb{C}}^N$. These sets are still mutually orthogonal to each other. It is clear that $S = \{S_\alpha \mid \alpha \in \{1, \dots, K\}\}$ together with the spectrum of the

operator D is a complete invariant. Namely, the dimensions of the spheres give the multiplicities of the corresponding idempotents and the way these spheres are embedded in $S_{\mathbb{C}}^N$ gives the action of M on $l^2(\text{Spec}(N))$. The operator D is given by the diagonal matrix with the given eigenvalues on its diagonal. Note that the set S is defined up to an action of $\mathcal{U}(N)$. If we restrict ourselves to the real numbers and take a low dimension, we can make some pictures of the invariant.

Example 3.15. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{R} \oplus \mathbb{R}, H = \mathbb{R}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\pi(x_1, x_2) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_2 \end{pmatrix}, x_1, x_2 \in \mathbb{R}.$$

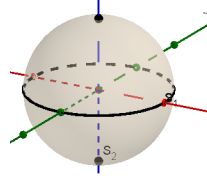
This is the same triple as in Example 3.13, but then on \mathbb{R}^3 . Using Equation (1) we see that we get the following representative for our invariant:

$$s_1 = \phi \left(\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right) \cap S_{\mathbb{R}}^N = \text{Span} \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\} \cap S_{\mathbb{R}}^N,$$

which equals a circle in the (x, y) -plane with radius 1 and

$$s_2 = \phi \left(\text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right) \cap S_{\mathbb{R}}^N = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \cap S_{\mathbb{R}}^N = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

The invariant $S = \{s_1, s_2\}$ looks like:



Consider the following action of M on H :

$$\pi'(x_1, x_2) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_1 \end{pmatrix}, x_1, x_2 \in \mathbb{R}.$$

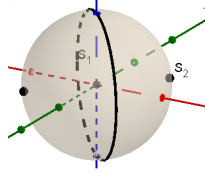
Now we obtain a representative

$$s_1 = \phi \left(\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right) \cap S_{\mathbb{R}}^N = \text{Span} \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \cap S_{\mathbb{R}}^N,$$

which equals a circle in the $(x = y)$ -plane with radius 1 and

$$s_2 = \phi \left(\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right) \cap S_{\mathbb{R}}^N = \left\{ \text{Span} \left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right) \right\} \cap S_{\mathbb{R}}^N = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}.$$

The invariant $S = \{s_1, s_2\}$ then looks like:



Since in this example we restrict to the real case, we have that $\mathcal{U}(N) = \{1, -1\} \times \{1, -1\} \times \{1, -1\}$. Therefore we see that in the first triple of this example there is only one representative. In the second triple the only other representative is obtained by a rotation of 90° around the z -axis. Hence it is clear that the two invariants are as expected not the same.

3.4 The CKM matrix

We will now take a closer look at the case that M acts with multiplicity 1 on the Hilbert space H . In this case we have that $m_\alpha = 1$ for every α . Therefore we obtain n one-dimensional subspaces E_α with $\alpha \in \{1, \dots, n\}$. In the computation of the invariant we choose an orthonormal basis of each E_α , which is in this case just one normalised vector $e_\alpha \in E_\alpha$. This vector is unique up to a multiplication of an element of $U(1)$. Furthermore we also choose a basis $\{\eta_\lambda\}_\lambda \in H$ consisting of normalised eigenvectors. Using this notations and following (8) in [1] we define the CKM matrix.

Definition 3.16. The *CKM matrix* C of a finite spectral triple is given by:

$$C_{\lambda\alpha} = \langle \eta_\lambda, e_\alpha \rangle,$$

which is an $n \times n$ matrix labeled by λ and α .

Lemma 3.17. *The CKM matrix C is unitary.*

Proof. Since $\{e_\alpha\}_\alpha$ and $\{\eta_\lambda\}_\lambda$ are orthogonal bases in H we have:

$$\sum_\lambda \bar{C}_{\lambda\alpha} C_{\lambda\alpha'} = \sum_\lambda \langle e_\alpha, \eta_\lambda \rangle \langle \eta_\lambda, e_{\alpha'} \rangle = \langle e_\alpha, e_{\alpha'} \rangle = \delta_{\alpha\alpha'}$$

and in the same way

$$\sum_\alpha C_{\lambda\alpha} \bar{C}_{\lambda'\alpha} = \delta_{\lambda\lambda'}.$$

□

From the CKM matrix we obtain our invariant $\text{Spec}_N(M)$ in the following way:

Proposition 3.18. *If (M, H, D) is a finite commutative spectral triple, where M acts with multiplicity one on H and D has nondegenerate spectrum, then the relative spectrum $\text{Spec}_N(M)$ is obtained via the CKM matrix in the following way:*

$$\text{Spec}_N(M) = \{\gamma_\alpha\}_\alpha \text{ with } (\gamma_\alpha)_{\lambda\mu} = C_{\lambda\alpha} \bar{C}_{\mu\alpha}.$$

Proof. As we have seen the invariant is computed as follows:

$$\gamma_\alpha = \phi(e_\alpha) \phi(e_\alpha)^*,$$

where ϕ is a unitary isomorphism $H \rightarrow l^2(\text{Spec}(N))$. Then

$$(\gamma_\alpha)_{\lambda\mu} = \phi(e_\alpha)(\lambda) \overline{\phi(e_\alpha)(\mu)} = \langle \eta_\lambda, e_\alpha \rangle \overline{\langle \eta_\mu, e_\alpha \rangle} = C_{\lambda\alpha} \bar{C}_{\mu\alpha}.$$

□

Although we have restricted to the case of multiplicity one, the CKM matrix still depends on the choice of the e_α and also on the choice of the η_λ . The CKM matrix therefore gives not yet a complete invariant for finite spectral triples, but only when looking modulo the choice of both bases. That means that the CKM matrix C and UCU' where U and U' are diagonal matrices as in Remark 3.12 are both representatives for the invariant.

Remark 3.19. The Cabibbo-Kobayashi-Maskawa (CKM) matrix expresses the mismatch of the bases of the mass eigenstates of the up quarks and the down quarks. The case of two generations was treated by N. Cabibbo in [16] and the case of three generations by M. Kobayashi and T. Maskawa in [17]. The CKM matrix gave rise to a mathematical treatment of the problem in arbitrary dimension. In Section 2 of [1], Connes proved for example that the representations of a pair of commutative von Neumann algebras (M, N) where M and N act with multiplicity one on the same Hilbert space, are classified up to unitary equivalence by the CKM matrix. So in the case of Connes, the CKM matrix is, in contrast to our case, a complete invariant. Here it is possible to choose a certain entry $C_{\eta e} = \langle \eta, e \rangle$ such that it is positive. Then there is a unique way of choosing the vectors e_α and η_λ such that the row $(C_{\eta_\alpha})_\alpha$ and column $(C_{\lambda e})_\lambda$ are positive. The rest of the CKM matrix is then also determined. Another choice of η and e does not effect the CKM matrix.

We will first give an example of the CKM matrix and then we will eliminate the choice of the e_α .

Example 3.20. Let $H = l^2(G)$ be our Hilbert space, where G is a finite abelian group, which we take in this example equal to $\mathbb{Z}/3\mathbb{Z}$. Consider the convolution algebra $M = C^*(G)$, Where

$$(a * b)(g) = \frac{1}{3} \sum_{x \in G} a(x)b(g - x)$$

for every a and $b \in M$. Note that M is commutative and isomorphic to a matrix algebra by Theorem 2.17. Consider the operator

$$D : H \rightarrow H, \quad Dx(g_i) = \lambda_i g_i,$$

for some distinct $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $x \in H$. Then $N = l^\infty(G)$, the set of multiplication operators. Both M and N are subalgebras of H . Take the following normalised eigenvectors in H

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \eta_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For M we have the following idempotents:

$$e_M^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, e_M^2 = \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} \end{pmatrix} \text{ and } e_M^3 = \begin{pmatrix} 1 \\ e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix}.$$

To obtain the CKM matrix we choose vectors

$$e_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} \end{pmatrix} \text{ and } e_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix}.$$

Then we compute

$$C = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \end{pmatrix}.$$

A representative of the invariant consists then of the following three matrices in $P_1^+(N)$:

$$\begin{aligned}\gamma_1 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^* = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\ \gamma_2 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} \end{pmatrix}^* = \frac{1}{3} \begin{pmatrix} 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & 1 & e^{-\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} & 1 \end{pmatrix}, \\ \gamma_3 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix}^* = \frac{1}{3} \begin{pmatrix} 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} & 1 & e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} & 1 \end{pmatrix}.\end{aligned}$$

In the case of multiplicity one, the idempotents of M correspond exactly to the columns of the CKM matrix C . A column of this matrix $\langle \eta_\bullet, e_\alpha \rangle = (\langle \eta_\lambda, e_\alpha \rangle)_\lambda$ is an element of $S_{\mathbb{C}}^N$. Since such a column is unique up to a multiplication of $U(1)$, we can view the idempotents not anymore as one dimensional subspaces in $l^2(\text{Spec}(N))$, or as circles S^1 in $S_{\mathbb{C}}^N$, but as points in

$$\mathbb{P}_N = S_{\mathbb{C}}^N / U(1).$$

Of course this set of points is still defined up to an action of the group $U(N)$. Hence the idempotents viewed as subset of \mathbb{P}_N together with the spectrum of D is a complete invariant of a finite commutative spectral triple, where M is acting with multiplicity one.

Example 3.21. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}.$$

Take the following normalised eigenvectors of D :

$$\eta_1 = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } \eta_{-\lambda} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

We have idempotents

$$e_M^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_M^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ which give the following two vectors: } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We calculate the CKM matrix

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

A representative for the invariant as points in \mathbb{P}_N is then given by the following two points

$$p_1 = \left[\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right] \text{ and } p_2 = \left[\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right].$$

Remark 3.22. Of course we can define the CKM matrix for other multiplicities of M . In case of constant multiplicity m we can identify the invariant with points in

$$S_{\mathbb{C}}^N / U(m).$$

In this case an idempotent of M corresponds to m columns of the CKM matrix. The choice of these columns is unique up to an action of $U(m)$.

In case of arbitrary multiplicities there is much more ambiguity in choosing the CKM matrix and therefore we can not identify the invariant with a similar set.

4 The invariant of finite commutative spectral triples

In the previous section we assumed the spectrum of the operator D to be nondegenerate. In this section we will construct the invariant of finite commutative spectral triples with a spectrum that is not necessarily nondegenerate. In this case we can not use the space $l^2(\text{Spec}(N))$ anymore. We need another space, based on Section 3.4 of [1], for which we first need some definitions.

4.1 Definitions

Definition 4.1. Let X be a finite set. A *vector bundle* over X is a map $\pi : E \rightarrow X$ such that for all $x \in X$ we have that

$$E_x := \pi^{-1}(x) \subset E$$

is a finite-dimensional vector space. The set E is called the *total space* of the bundle and is a finite-dimensional vector space. The set X is called its *base space* and the map π is called its *projection*.

Definition 4.2. A *section* of a vector bundle $\pi : E \rightarrow X$ is a map $\sigma : X \rightarrow E$ such that

$$\pi \circ \sigma = \mathbb{I}_X.$$

The set of all sections of a vector bundle is then equal to the Hilbert space $l^2(X, E)$, which is the set of sequences labeled by elements of X and $\sigma(x) \in E_x$ for $\sigma \in l^2(X, E)$.

Remark 4.3. The full definition of a vector bundle, where X is a manifold, can be found in [12] page 58-59.

Example 4.4. Let N be an algebra acting on a Hilbert space H . Let V_λ be its irreducible representations in H and $\text{Spec}(N) = \{e_N^1, \dots, e_N^q\}$ the set of minimal idempotents. Then

$$\pi : V \rightarrow \text{Spec}(N)$$

is a vector bundle, where $V = \bigcup_\lambda V_\lambda$ which is a disjoint union and

$$\pi(\eta_\lambda)(\mu) = \eta_\lambda \delta_{\lambda\mu}$$

for each $\eta_\lambda \in V_\lambda$.

Definition 4.5. Let $\pi : E \rightarrow X$ and $\pi' : E' \rightarrow X$ be two vector bundles. A *homomorphism* $h : E \rightarrow E'$ is a linear map such that for $x \in X$ its restrictive $h|_{E_x}$ to E_x is a linear map to E'_x . A homomorphism from E to itself is called an *endomorphism*.

4.2 The construction of the invariant of finite commutative spectral triples

Let (M, H, D) be a finite commutative spectral triple. In the previous section we viewed the minimal idempotents of M as subspaces of $l^2(\text{Spec}(N))$. In this case we would like to view them as subspaces of $l^2(\text{Spec}(N), V)$, where $\pi_V : V \rightarrow \text{Spec}(N)$ is a vector bundle with $V = \bigcup_\lambda V_\lambda$ and $\{V_\lambda\}_\lambda$ the eigenspaces of D . The Hilbert space $l^2(\text{Spec}(N), V)$ is then the space of sections of V . Note that in this case $N \cong \mathbb{C}^q$ with $q \leq n$.

Take again for each α the m_α -dimensional subspace $E_\alpha \subset H$ such that $\pi(e_M^\alpha)E_\alpha = E_\alpha$. Define a unitary isomorphism

$$\phi : H \rightarrow l^2(\text{Spec}(N), V), \quad \phi(\eta_{\lambda,j})(\mu) = \varepsilon_{\lambda,j} \delta_{\lambda\mu}$$

for $\{\eta_{\lambda,j}\}_j$ an orthonormal basis in V_λ and $\mu \in \text{Spec}(N) = \{e_N^1, \dots, e_N^q\}$. Here $\{\varepsilon_{\lambda,j}\}_{\lambda,j}$ is the canonical basis in $l^2(\text{Spec}(N), V)$. For $x \in H$ we have

$$\phi(x)(\lambda) = \sum_j \langle \eta_{\lambda,j}, x \rangle \varepsilon_{\lambda,j}. \quad (2)$$

This isomorphism is unique up to an action of a unitary on each of the subspaces V_λ . So the isomorphism is unique up to an action of the unitary group of endomorphisms of V , denoted by $\mathcal{U}\text{End}(V)$.

Each copy \mathbb{C} of M now corresponds to $\phi(E_\alpha) \subset l^2(\text{Spec}(N), V)$ and $\{\phi(E_\alpha)\}_\alpha$ gives the position of M relative to N . Using the same map ψ as in Lemma 3.5 we see that the m_α -dimensional space $\phi(E_\alpha)$ corresponds to an element

$$\gamma_\alpha := \widetilde{\phi(E_\alpha)}(\widetilde{\phi(E_\alpha)})^* = \begin{pmatrix} \phi(e_1)(\lambda_1) & \dots & \phi(e_{m_\alpha})(\lambda_1) \\ \vdots & & \vdots \\ \phi(e_1)(\lambda_q) & \dots & \phi(e_{m_\alpha})(\lambda_q) \end{pmatrix} \begin{pmatrix} \phi(e_1)(\lambda_1) & \dots & \phi(e_{m_\alpha})(\lambda_1) \\ \vdots & & \vdots \\ \phi(e_1)(\lambda_q) & \dots & \phi(e_{m_\alpha})(\lambda_q) \end{pmatrix}^*,$$

where e_1, \dots, e_{m_α} is an orthonormal basis for E_α . From the lemma it is clear that the choice of this basis does not matter. The entries of the matrix $\widetilde{\phi(E_\alpha)}$ are in this case not just elements in \mathbb{C} , but they are vectors, since we have $\phi(e_i)(\lambda) \in V_\lambda$. Furthermore is $(\phi(e_i)(\lambda))^*$ an element of V_λ^* given by the map

$$V_\lambda \rightarrow V_\lambda^*, \quad v \mapsto \langle v, \cdot \rangle_{V_\lambda}$$

for $v \in V_\lambda$. So we have

$$(\gamma_\alpha)_{\lambda\mu} = \sum_{i=1}^{m_\alpha} \phi(e_i)(\lambda) \langle \phi(e_i)(\mu), \cdot \rangle_{V_\mu},$$

where $\lambda, \mu \in \text{Spec}(N)$. We have

$$((\gamma_\alpha)_{\lambda\mu})(v) = \sum_{i=1}^{m_\alpha} \underbrace{\phi(e_i)(\lambda)}_{\in V_\lambda} \underbrace{\langle \phi(e_i)(\mu), v \rangle}_{\in \mathbb{C}} \in V_\lambda,$$

for $v \in V_\mu$. Hence $(\gamma_\alpha)_{\lambda\mu}$ are operators

$$(\gamma_\alpha)_{\lambda\mu} : V_\mu \rightarrow V_\lambda$$

and

$$\gamma_\alpha \in \text{End}(l^2(\text{Spec}(N), V)) \cong \bigoplus_{\lambda, \mu} V_\lambda \otimes V_\mu^*.$$

We conclude that for each minimal idempotent e_M^α of M we find a matrix γ_α , such that $\{\gamma_\alpha\}_\alpha$ gives the relative position of the irreducible representations of M in H relative to the eigenspaces of D . These elements γ_α have similar properties as elements of the set $P_{m_\alpha}^+(N)$ from the previous section. We have

(a) $\gamma_\alpha^* = \gamma_\alpha$, where

$$((\gamma_\alpha)_{\lambda\mu})^* = \sum_{i=1}^{m_\alpha} \phi(e_i)(\mu) \langle \phi(e_i)(\lambda), \cdot \rangle = (\gamma_\alpha)_{\mu\lambda}.$$

(b) $\gamma_\alpha^2 = \gamma_\alpha$, since

$$\begin{aligned}
((\gamma_\alpha)_{\lambda\mu}^2)(v) &= \sum_{\omega \in \text{Spec}(N)} ((\gamma_\alpha)_{\lambda\omega} \circ (\gamma_\alpha)_{\omega\mu})(v) \\
&= \sum_{\omega \in \text{Spec}(N)} \sum_{i=1}^{m_\alpha} \phi(e_i)(\lambda) \left\langle \phi(e_i)(\omega), \sum_{k=1}^{m_\alpha} \phi(e_k)(\omega) \langle \phi(e_k)(\mu), v \rangle \right\rangle \\
&= \sum_{i=1}^{m_\alpha} \sum_{k=1}^{m_\alpha} \phi(e_i)(\lambda) \langle \phi(e_k)(\mu), v \rangle \sum_{\omega \in \text{Spec}(N)} \langle \phi(e_i)(\omega), \phi(e_k)(\omega) \rangle \\
&= \sum_{i=1}^{m_\alpha} \phi(e_i)(\lambda) \langle \phi(e_i)(\mu), v \rangle \\
&= (\gamma_\alpha)_{\lambda\mu}(v).
\end{aligned}$$

They also satisfy the following properties:

- (c) Since $\phi(E_\alpha)$ is orthogonal to $\phi(E_{\alpha'})$ we see through a similar computation as in (b) that $\gamma_\alpha \gamma_{\alpha'} = 0$.
- (d) The operator γ_α is of rank m_α , since $\sum_{\lambda \in \text{Spec}(N)} \text{Tr}((\gamma_\alpha)_{\lambda\lambda}) = m_\alpha$. Since $\bigoplus_\alpha \phi(E_\alpha) = l^2(\text{Spec}(N), V)$ we also have

$$\sum_\alpha (\gamma_\alpha)_{\lambda\mu} = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{I}_\mu & \text{if } \lambda = \mu \end{cases}$$

where \mathbb{I}_μ is the identity on V_μ .

Using properties (a) and (b) we introduce the set $P_m(N)$, which is an extension of $P_m^+(N)$.

Definition 4.6. Let V_1, \dots, V_q be finite dimensional vector spaces. The set $P_m(N)$ is then defined as the set of $q \times q$ matrices with a certain labeling corresponding to the V_λ such that for each $\gamma \in P_m(N)$ we have

- (i) $\gamma_{\lambda\mu} : V_\mu \rightarrow V_\lambda$,
- (ii) the rank of γ equals m and
- (iii) properties (a) and (b) are satisfied for γ .

Note that N in the definition of $P_m(N)$ refers to the labeling.

We see that our elements γ_α are elements of $P_{m_\alpha}(N)$. The action of $\mathcal{U} \text{End}(V)$ on $\bigcup_\alpha P_{m_\alpha}(N)$ becomes the adjoint action:

$$(\text{Ad}(u)\gamma)_{\lambda\mu} = u_\lambda \gamma_{\lambda\mu} u_\mu^*,$$

where u_λ is a unitary endomorphism of V_λ .

Definition 4.7. We define the *relative spectrum* of M relative to N as

$$\text{Spec}_N(M) = \{\gamma_\alpha | \alpha \in \{1, \dots, K\}\} \subset \bigcup_\alpha P_{m_\alpha}(N),$$

which is defined up to the adjoint action of $\mathcal{U} \text{End}(V)$.

We conclude the following theorem:

Theorem 4.8. *There is a one-to-one correspondence between finite commutative spectral triples, modulo unitary equivalence, and pairs (S, Λ) where*

$$S \subset \bigcup_{i=0}^{\infty} P_i(N),$$

such that

- (i) $\gamma\gamma' = 0$ for all $\gamma \neq \gamma' \in S$ and
(ii) $\sum_S \gamma_{\lambda\mu} = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{I}_\mu & \text{if } \lambda = \mu \end{cases}$

is a finite set defined up to the adjoint action of $\mathcal{U}\text{End}(V)$ with $V = \bigcup_\lambda V_\lambda$ and $\Lambda = \{\lambda_1, \dots, \lambda_q\} \subset \mathbb{R}$ whose number of elements and order corresponds to the labeling of the $\gamma \in S$.

Proof. It is clear that the pair (S, Λ) is an invariant of finite commutative spectral triples with $S = \text{Spec}_N(M)$. The spectrum of D gives the set Λ , which is just the spectrum without multiplicities.

From a pair (S, Λ) such as stated in the theorem we reconstruct its spectral triple. We take the algebra

$$M' = \mathbb{C}^{|S|}$$

and, with

$$E_\gamma = \text{Im } \gamma = \bigoplus_{\lambda} \left(\bigcup_{\mu} \text{Im } \gamma_{\lambda\mu} \right),$$

we define a Hilbert space

$$H' = \bigoplus_{\gamma \in S} E_\gamma.$$

Furthermore we have that our bundle $V = \bigcup_\lambda V_\lambda$ as disjoint union. The symmetric operator D' is given by

$$D' = \sum_{\lambda \in \Lambda} \lambda \sum_{\gamma \in S} \gamma_{\lambda\lambda},$$

which is a diagonal matrix. The constructed triple (M', H', D') is then with statement 1 and 2 of the proof of Theorem 3.10, with $l^2(\text{Spec}(N))$ replaced by $l^2(\text{Spec}(N), V)$, the inverse of our invariant (S, Λ) . Hence the invariant is complete. \square

We will refer to the invariant of a finite commutative spectral triple as the relative spectrum $\text{Spec}_N(M)$ without mentioning the set Λ .

Corollary 4.9. *Let $\text{Spec}_N(M)$ be the invariant of a finite commutative spectral triple (M, H, D) . Let $M' \cong \mathbb{C}^n$ acting on H with multiplicity one together with the same operator D . Then we have that*

$$\sum_{\gamma \in \text{Spec}_N(M)} \gamma = \text{Spec}_N(M').$$

Proof. We see that $\text{Spec}_N(M')$ consists of just one element and by property (ii) of Theorem 4.8 it has to be equal to the identity map \mathbb{I} on $l^2(\text{Spec}(N), V)$. From that property we also conclude that $\sum_{\gamma \in \text{Spec}_N(M)} \gamma = \mathbb{I}$. \square

Remark 4.10. Whenever the spectrum of D is nondegenerate, the construction in this section coincides with the one of the previous section, since the canonical bases of $l^2(\text{Spec}(N))$ and $l^2(\text{Spec}(N), V)$ are the same. Furthermore we have $\mathcal{U}\text{End}(V) = \mathcal{U}(N)$.

Remark 4.11. It is easier to view the map ϕ as a map from H to H in the following way:

$$\phi = \begin{pmatrix} \eta_{\lambda_1,1} \\ \vdots \\ \eta_{\lambda_1,k} \\ \vdots \\ \eta_{\lambda_q,1} \\ \vdots \\ \eta_{\lambda_q,k'} \end{pmatrix},$$

with $\{\eta_{\lambda,j}\}_j$ a basis for V_λ and $\lambda \in \{\lambda_1, \dots, \lambda_q\}$. Then $\phi(x)(\lambda)$ is just a vector in H with nonzero components at the positions $(\lambda, 1), \dots, (\lambda, k)$. We then view $\gamma_{\lambda\mu}$ also as a map from H to H by letting it be zero on its complement.

We then have for all $x \in H$:

$$\begin{aligned} \gamma_\alpha(x) &= \sum_{\lambda,\mu} (\gamma_\alpha)_{\lambda\mu} \\ &= \sum_{\lambda,\mu} \sum_i \phi(e_{\alpha,i})(\lambda) \langle \phi(e_{\alpha,i})(\mu), x \rangle_H \\ &= \sum_{\lambda,\mu} \sum_i \phi \left(\sum_j \langle \eta_{\lambda,j}, e_{\alpha,i} \rangle \eta_{\lambda,j} \right) \left\langle \phi \left(\sum_{j'} \langle \eta_{\mu,j'}, e_{\alpha,i} \rangle \eta_{\mu,j'} \right), x \right\rangle \\ &= \sum_i \phi(e_{\alpha,i}) \langle e_{\alpha,i}, \phi^*(x) \rangle \end{aligned}$$

and hence as a matrix we have:

$$\gamma_\alpha = \phi \left(\sum_i e_{\alpha,i} e_{\alpha,i}^* \right) \phi^*.$$

4.3 Examples

Example 4.12. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} M &= \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \\ \pi(z_1, z_2) &= \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, z_1, z_2 \in \mathbb{C}. \end{aligned}$$

The operator D has eigenvalues λ and $-\lambda$ with multiplicities 2 and 1 respectively. The eigenspaces are given by

$$\begin{aligned} V_\lambda &= \text{Span}\{\eta_{\lambda,1}, \eta_{\lambda,2}\} := \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\} \text{ and} \\ V_{-\lambda} &= \text{Span}\{\eta_{-\lambda}\} := \text{Span} \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

Furthermore we have irreducible representations of M in H given by

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

as we have seen in Example 3.13.

We calculate using Equation (2)

$$\begin{aligned}\phi\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) &= \phi\left(\frac{1}{\sqrt{2}}\eta_{\lambda,2} + \frac{1}{\sqrt{2}}\eta_{-\lambda}\right) = \left(\frac{1}{\sqrt{2}}\epsilon_{\lambda,2}, \frac{1}{\sqrt{2}}\epsilon_{-\lambda}\right), \\ \phi\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) &= \phi\left(\frac{1}{\sqrt{2}}\eta_{\lambda,2} - \frac{1}{\sqrt{2}}\eta_{-\lambda}\right) = \left(\frac{1}{\sqrt{2}}\epsilon_{\lambda,2}, -\frac{1}{\sqrt{2}}\epsilon_{-\lambda}\right), \\ \phi\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) &= \phi(\eta_{\lambda,1}) = (\epsilon_{\lambda,1}, 0),\end{aligned}$$

which are elements in $l^2(\{\lambda, -\lambda\}, V)$.

We compute a representative for the invariant $\text{Spec}_N(M)$:

$$\begin{aligned}\gamma_1 &= \begin{pmatrix} \epsilon_{\lambda,2}\langle\epsilon_{\lambda,2}, \cdot\rangle & 0 \\ 0 & \epsilon_{-\lambda}\langle\epsilon_{-\lambda}, \cdot\rangle \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* & 0 \\ 0 & 1 \cdot \bar{1} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix} \in P_2(N) \text{ and} \\ \gamma_2 &= \begin{pmatrix} \epsilon_{\lambda,1}\langle\epsilon_{\lambda,1}, \cdot\rangle & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 \end{pmatrix} \in P_1(N).\end{aligned}$$

Consider the same triple but with an action of M on H given by

$$\pi'(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

Recall that the irreducible representations of M on H are

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Through a similar computation we now obtain the following representative for $\text{Spec}_N(M)$:

$$\begin{aligned}\gamma_1 &= \begin{pmatrix} \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* & \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} & \frac{1}{2} \end{pmatrix} \in P_2(N) \text{ and} \\ \gamma_2 &= \begin{pmatrix} \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* & -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix} & \frac{1}{2} \end{pmatrix} \in P_1(N).\end{aligned}$$

Since we clearly can not find for example a unitary 2×2 matrix U such that

$$U \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} U^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we notice that the invariants are as expected not the same.

Using Remark 4.11 we see that the invariant of, for example, the last finite spectral triple can be calculated easier: The map $\phi : H \rightarrow H$ equals

$$\phi = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Then

$$\gamma_1 = \phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}^* \phi^* = \left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right)$$

and γ_2 follows in the same way.

5 The dual of the invariant

In this section we will interchange the roles of our algebras M and N and construct the dual invariant. We will find a direct relation between the invariant and its dual.

5.1 The relation between the invariant and its dual

If (M, H, D) is a finite spectral triple we can switch the roles of M and N in our construction. The obtained set is then again a complete invariant which we will call the *dual* of $\text{Spec}_N(M)$ denoted by $\text{Spec}_M(N)$.

We would like to find a direct relation f between $\text{Spec}_N(M)$ and $\text{Spec}_M(N)$ such that the following diagram commutes:

$$\begin{array}{ccc} (M, N) & \xrightarrow{\text{Spec}} & \text{Spec}_N(M) \\ \tau \downarrow & & \downarrow f \\ (N, M) & \xrightarrow{\text{Spec}} & \text{Spec}_M(N) \end{array}$$

In this diagram the map Spec coincides with the construction in the previous section and τ is the map that interchanges the roles of the two algebras M and N . The purpose of finding the map f is that in some examples calculating $\text{Spec}_M(N)$ is easier than calculating $\text{Spec}_N(M)$.

Remark 5.1. The space of pairs of commutative unital $*$ -algebras forms a category. Sets of the form stated in Theorem 4.8 form also a category. Demanding the diagram to be commutative, amounts to Spec being a functor between the two categories.

We will state the theorem that explains the relation $f : \text{Spec}_N(M) \rightarrow \text{Spec}_M(N)$.

Theorem 5.2. *Let*

$$\text{Spec}_N(M) = \{\gamma_\alpha | \alpha \in \{1, \dots, K\}\}$$

be a set with the properties of Theorem 4.8. Define

$$(\tilde{\gamma}_\lambda)_{\alpha\beta} := \sum_{\omega} (\gamma_\alpha)_{\omega\lambda} \sum_{\kappa} (\gamma_\beta)_{\lambda\kappa}, \quad (3)$$

for each λ that occurs in the labeling. The map $f : \text{Spec}_N(M) \rightarrow \text{Spec}_M(N)$ such that $\text{Spec} \circ \tau = f \circ \text{Spec}$ is then given as follows:

$$f(\text{Spec}_N(M)) = f(\{\gamma_\alpha\}_\alpha) = \{\tilde{\gamma}_\lambda\}_\lambda = \text{Spec}_M(N).$$

Proof. Each $(\gamma_\alpha)_{\lambda\mu} : V_\mu \rightarrow V_\lambda$ can be extended to a map of dimension $\sum_\lambda \dim V_\lambda$ by letting it be zero on its orthogonal complement. The definition of $(\tilde{\gamma}_\lambda)_{\alpha\beta}$ thus makes sense. Note also that $(\tilde{\gamma}_\lambda)_{\alpha\beta} : \text{Im } \gamma_\beta \rightarrow \text{Im } \gamma_\alpha$.

Let (M, H, D) be the finite spectral triple with invariant $\text{Spec}_N(M)$ reconstructed as in Theorem 4.8. Let $\{E_\alpha = \text{Im } \gamma_\alpha\}_\alpha$ be the set of irreducible representations of M in H and $\{e_{\alpha,i}\}_i$ be an orthonormal basis for E_α . The set $\{V_\lambda\}_\lambda$ consists of the eigenspaces of D and let $\{\eta_{\lambda,j}\}_j$ be orthonormal bases for the V_λ . A representative for $\text{Spec}_N(M)$ is then

$$(\gamma_\alpha)_{\lambda\mu}(v) = \sum_i \phi(e_{\alpha,i})(\lambda) \langle \phi(e_i)(\mu), v \rangle,$$

for each α . Viewing ϕ as a map from H to H and considering the reconstructed spectral triple (M, H, D) , we have that ϕ is just the identity map so we have

$$(\gamma_\alpha)_{\lambda\mu}(w) = \sum_i \sum_j \langle \eta_{\lambda,j}, e_{\alpha,i} \rangle \eta_{\lambda,j} \sum_{j'} \langle e_{\alpha,i}, \eta_{\mu,j'} \rangle \langle \eta_{\mu,j'}, w \rangle$$

for all $w \in H$.

We now compute $(\tilde{\gamma}_\lambda)_{\alpha\beta}$ using equation (3), where we use the representative γ_α we found above. First we calculate

$$\begin{aligned} \sum_\omega (\gamma_\alpha)_{\omega\lambda}(w) &= \sum_\omega \sum_i \sum_j \langle \eta_{\omega,j}, e_{\alpha,i} \rangle \eta_{\omega,j} \sum_{j'} \langle e_{\alpha,i}, \eta_{\lambda,j'} \rangle \langle \eta_{\lambda,j'}, w \rangle \\ &= \sum_i \sum_j e_{\alpha,i} \langle e_{\alpha,i}, \eta_{\lambda,j} \rangle \langle \eta_{\lambda,j}, w \rangle. \end{aligned}$$

This gives us

$$\begin{aligned} (\tilde{\gamma}_\lambda)_{\alpha\beta}(w) &= \left(\sum_\omega (\gamma_\alpha)_{\omega\lambda} \sum_\kappa (\gamma_\beta)_{\lambda\kappa} \right) (w) \\ &= \sum_i \sum_j e_{\alpha,i} \langle e_{\alpha,i}, \eta_{\lambda,j} \rangle \left\langle \eta_{\lambda,j}, \sum_{i'} \sum_{j'} \langle \eta_{\lambda,j'}, e_{\beta,i'} \rangle \eta_{\lambda,j'} \langle e_{\beta,i'}, w \rangle \right\rangle \\ &= \sum_{i,i'} \sum_{j,j'} e_{\alpha,i} \langle e_{\alpha,i}, \eta_{\lambda,j} \rangle \langle \eta_{\lambda,j'}, e_{\beta,i'} \rangle \langle e_{\beta,i'}, w \rangle \langle \eta_{\lambda,j}, \eta_{\lambda,j'} \rangle \\ &= \sum_{i,i'} \sum_j e_{\alpha,i} \langle e_{\alpha,i}, \eta_{\lambda,j} \rangle \langle \eta_{\lambda,j}, e_{\beta,i'} \rangle \langle e_{\beta,i'}, w \rangle, \end{aligned}$$

On the other hand, if we switch the roles of the algebras M and N for the finite spectral triple (M, H, D) we obtain, using the same choice of bases, the following representative for the invariant $\text{Spec}_N(M)$:

$$(\gamma_\lambda)_{\alpha\beta} = \sum_{i,i'} \sum_j \phi'(e_{\alpha,i}) \langle e_{\alpha,i}, \eta_{\lambda,j} \rangle \langle \eta_{\lambda,j}, e_{\beta,i'} \rangle \langle \phi'(e_{\beta,i'}), w \rangle,$$

where $\phi' : H \rightarrow H$ sends $\{e_{\alpha,i}\}_i$, the bases for the E_α 's, to the standard basis in H and this map coincides with a map $\phi' : H \rightarrow l^2(\text{Spec}(M), E)$, with $E = \bigcup_\alpha E_\alpha$. We conclude that $(\tilde{\gamma}_\lambda)_{\alpha\beta} : E_\beta \rightarrow E_\alpha$ and $(\gamma_\lambda)_{\alpha\beta} : \phi'(E_\beta) \rightarrow \phi'(E_\alpha)$ are the same as $m_\alpha \times m_\beta$ matrix. Therefore we have $(\tilde{\gamma}_\lambda)_{\alpha\beta} = (\gamma_\lambda)_{\alpha\beta}$.

Hence equation (3) indeed gives the invariant $\text{Spec}_M(N)$ in such a way that

$$\text{Spec} \circ \tau = f \circ \text{Spec}.$$

□

Corollary 5.3. *With the notation of Theorem 5.2 we have the following relation*

$$\gamma_\lambda = \sum_\alpha (\gamma_\alpha)_{\lambda\lambda}.$$

Proof. We view $(\gamma_\lambda)_{\alpha\beta} : \phi'(E_\beta) \rightarrow \phi'(E_\alpha)$ as a map of dimension $\sum_\alpha \dim E_\alpha$ by letting it be

zero on the orthogonal complement. Then

$$\begin{aligned}
\gamma_\lambda &= \sum_{\alpha, \beta} (\gamma_\lambda)_{\alpha\beta} \\
&= \sum_{\alpha, \beta} (\tilde{\gamma}_\lambda)_{\alpha\beta} && \text{(as a matrix using the proof of Theorem 5.2)} \\
&= \sum_j \eta_{\lambda, j} \langle \eta_{\lambda, j}, \cdot \rangle \\
&= \sum_\alpha (\gamma_\alpha)_{\lambda\lambda}.
\end{aligned}$$

□

In the following corollary is stated how to reconstruct the finite spectral triple from $\text{Spec}_M(N)$.

Corollary 5.4. *Given the dual invariant $\text{Spec}_M(N) = \{\gamma_\lambda\}_\lambda$ we reconstruct the finite commutative spectral triple (M, H, D) as follows:*

$$\left(M = \mathbb{C}^K, H = \bigoplus_\alpha E_\alpha, D = \sum_{\lambda \in \Lambda} \lambda \gamma_\lambda \right),$$

where K is the number of elements that occurs in the labeling of the elements in $\text{Spec}_M(N)$.

Proof. Note first that Λ is the same set, as the one corresponding to $\text{Spec}_N(M)$ and the spaces E_α appear in the definition of $\text{Spec}_M(N) \subset \bigcup_{i=1}^\infty P_i(N)$. From Corollary 5.3 we see that (M, H, D) is the same triple as the one constructed in Theorem 4.8. □

We can also conclude that the inverse map $f^{-1} : \text{Spec}_M(N) \rightarrow \text{Spec}_N(M)$ such that $\text{Spec} \circ \tau^{-1} = f^{-1} \circ \text{Spec}$, with $\tau^{-1} = \tau$ is obtained in the same way as f .

Remark 5.5. As we have seen, the pair of algebras (N, M) together with the spectrum of D corresponds to the triple (M, H, D) . We can ask ourselves if it is possible to determine a corresponding triple of the form (N, H, \tilde{D}) with $\{f(\tilde{D}) | f \in C(\mathbb{R})\} = M$, which then would be the dual of the finite spectral triple (M, H, D) . From Remark 3.11 we see that, since we do not know what the eigenvalues of \tilde{D} should be, it is not possible to determine the operator \tilde{D} in a unique way.

5.2 Examples

Example 5.6. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where M acts with multiplicity 1. The eigenvectors and eigenspaces of D are already given in Example 4.12. Furthermore we have

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^3 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The map $\phi : H \rightarrow H$ is given by

$$\phi = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

A representative for the invariant $\text{Spec}_N(M)$ is then given by:

$$\begin{aligned} \gamma_1 &= \phi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^* \phi^* = \phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi^* = \left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right), \\ \gamma_2 &= \phi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^* \phi^* = \phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi^* = \left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \text{ and} \\ \gamma_3 &= \phi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^* \phi^* = \phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \phi^* = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right). \end{aligned}$$

We compute its dual $\text{Spec}_M(N)$ using the formula in Equation (3):

$$(\gamma_\lambda)_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

which is a map $(\gamma_\lambda)_{11} : \phi(E_M^1) \rightarrow \phi(E_M^1)$ such that:

$$\phi \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \phi \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

So as a one-dimensional map

$$(\gamma_\lambda)_{11} = \frac{1}{2}.$$

We have

$$(\gamma_\lambda)_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{pmatrix},$$

which is a map $(\gamma_\lambda)_{12} : \phi(E_M^2) \rightarrow \phi(E_M^1)$ such that:

$$\phi \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \phi \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

So as a one dimensional map

$$(\gamma_\lambda)_{12} = \frac{1}{2}.$$

In this way we obtain as a representative for $\text{Spec}_M(N)$:

$$\gamma_\lambda = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma_{-\lambda} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It might be easier in the computation to use Corollary 5.3. We now get:

$$\gamma'_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \gamma'_{-\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here γ'_λ is the identity map on $\phi(V_\lambda)$, but we need the identity map on V_λ . Hence we obtain a representative for $\text{Spec}_M(N)$ as follows:

$$\gamma_\lambda = \phi^* \gamma'_\lambda \phi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma_{-\lambda} = \phi^* \gamma'_{-\lambda} \phi = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will now calculate $\text{Spec}_M(N)$ directly from the pair (N, M) to confirm that the two calculations give the same result. The map $\phi' : H \rightarrow H$ that coincides with the map $\phi' : H \rightarrow l^2(\text{Spec}(M))$ is just the identity. We obtain as a representative for $\text{Spec}_M(N)$:

$$\begin{aligned} \gamma_\lambda &= V_\lambda V_\lambda^* = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and} \\ \gamma_{-\lambda} &= V_{-\lambda} V_{-\lambda}^* = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}^* = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

which is indeed the same.

Remark 5.7. Note that the last method of computation using Corollary 5.3 only works in the case that M has multiplicity 1. In that case the map ϕ' is the identity. In other cases the map ϕ' is not necessary the identity and we obtain $\text{Spec}_M(N)$ by an extra calculation, namely we have to calculate $\phi' \gamma_\lambda \phi'^*$.

Example 5.8. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} M &= \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \\ \pi(z_1, z_2) &= \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, z_1, z_2 \in \mathbb{C}. \end{aligned}$$

The map ϕ is the same as in the previous example. Recall from Example 4.12 that

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

As we have seen in Example 4.12 a representative for the invariant $\text{Spec}_N(M)$ is given by

$$\gamma_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \text{ and } \gamma_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right).$$

We calculate the dual:

$$(\gamma_\lambda)_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

which is a map $(\gamma_\lambda)_{11} : \phi(E_M^1) \rightarrow \phi(E_M^1)$ such that:

$$\phi \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} 0 & 2 \\ \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \phi \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and hence

$$(\gamma_\lambda)_{11} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

In the same way we have

$$(\gamma_\lambda)_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{pmatrix},$$

which is a map $(\gamma_\lambda)_{12} : \phi(E_M^2) \rightarrow \phi(E_M^1)$ such that:

$$\phi \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \phi \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and hence

$$(\gamma_\lambda)_{12} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

If we proceed this we obtain as a representative for $\text{Spec}_M(N)$:

$$\gamma_\lambda = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \right) \text{ and } \gamma_{-\lambda} = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \right).$$

When we calculate $\text{Spec}_M(N)$ directly from the pair (N, M) we will obtain exactly this representative if we take the same choice of basis with $\phi' : H \rightarrow H$ given by

$$\phi' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 5.9. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

We take the eigenspaces with corresponding orthonormal bases:

$$V_\lambda = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \right\}, V_{-\lambda} = \text{Span} \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } V_0 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The map $\phi : H \rightarrow H$ is then given by the following matrix

$$\phi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore we have that the irreducible representations of M in H are

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We obtain as a representative for $\text{Spec}_N(M)$:

$$\gamma_1 = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}.$$

We compute the dual in the same way as we did before and we obtain the following representative for $\text{Spec}_M(N)$:

$$\begin{aligned} \gamma_\lambda &= \begin{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \\ \gamma_{-\lambda} &= \begin{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \\ \gamma_0 &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \end{aligned}$$

We end this section with some corollaries for specific cases of M and N .

Corollary 5.10. *Let (M, H, D) be a finite spectral triple such that M and N act on H with equal multiplicities. Then the invariant of the triple (N, H, \tilde{D}) , where \tilde{D} is the operator with the same eigenvalues as D and eigenspaces equal to the irreducible representations of M in H , equals the dual of $\text{Spec}_N(M)$. We have $\text{Spec}_N(M) \cong \text{Spec}_M(N)$.*

Proof. Let $\{E_\alpha\}_\alpha$ be the irreducible representations of M in H and $\{V_{\lambda_i}\}_i$ be the eigenspaces of D . It is clear that the invariant of (N, H, \tilde{D}) equals $\text{Spec}_M(N)$, noting that Λ in $\text{Spec}_M(N)$ and $\text{Spec}_N(M)$ is the same. We can take a unitary isomorphism $U : H \rightarrow H$ such that

$$V_{\lambda_i} \xrightarrow{\sim} E_\alpha$$

and an isomorphism

$$\varphi : N \rightarrow M, \quad \varphi(e_N^i) = e_M^\alpha,$$

Then clearly

- (i) $U\rho(e_n^i)U^* = \pi(e_M^\alpha)$ and
- (ii) $U\tilde{D}U^* = D$.

Hence (M, H, D) is unitary equivalent to (N, H, \tilde{D}) . □

Example 5.11. In Example 5.8 we see that the multiplicities of M and N coincide and indeed $\text{Spec}_M(N)$ and $\text{Spec}_N(M)$ are isomorphic. Namely, another representative for $\text{Spec}_M(N)$ is given by

$$\{U\gamma_\lambda U^*, U\gamma_{-\lambda} U^*\},$$

with

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The representatives for $\text{Spec}_M(N)$ and $\text{Spec}_N(M)$ are then the same as matrices. We see that a change of labeling gives the isomorphism.

Corollary 5.12. *Let (M, H, D) be a finite spectral triple such that both M and N act on H with multiplicity one. In the construction of $\text{Spec}_N(M)$ we calculate the CKM matrix C_N and for $\text{Spec}_M(N)$ we calculate the CKM matrix C_M . We have*

$$C_N = C_M^*.$$

Proof. This follows immediately by writing out the definitions of the two CKM matrices. □

6 A graphical invariant of finite commutative spectral triples

We will introduce another invariant of finite commutative spectral triples which is a graph. We do this by following section 2.3 of [2]. Then we will relate this graphical invariant to our invariant $\text{Spec}_N(M)$.

6.1 Decorated graphs

Definition 6.1. A *graph* Γ consists of a set V of *vertices* and E of *edges*, which are pairs of vertices. An edge $e = (v, v)$ is allowed and is called a *loop*. The graph is called *directed* if the edges have a direction.

Let (M, H, D) be a finite commutative spectral triple. We have that

$$M \cong \mathbb{C}^K$$

and we decompose the Hilbert space as direct sum of irreducible representations of M in H :

$$H = \bigoplus_{\alpha=1}^K E_{\alpha} \cong \bigoplus_{\alpha=1}^K \mathbb{C} \otimes \mathbb{C}^{m_{\alpha}}.$$

We will now construct a graph from it. We draw a node for each copy of \mathbb{C} labeled with an α . Multiple nodes at the same position correspond with multiplicities of the irreducible representations \mathbb{C} in H . Thus at each position α there are $m_{\alpha} = \dim E_{\alpha}$ nodes.



In this figure we see for example that $E_1 \subset H$ is just a copy of \mathbb{C} and $E_j \subset H$ is a copy of $\mathbb{C} \oplus \mathbb{C}$. Corresponding to the above decomposition of H we can write the operator D as a sum of matrices:

$$D_{\alpha\beta} : E_{\alpha} \rightarrow E_{\beta}.$$

Since D is symmetric we have $D_{\alpha\beta} = D_{\beta\alpha}$. We express the $D_{\alpha\beta}$ in the graph as directed edges between the nodes α and β .



In this graph we see that the edges between i and j , and i and K represent non-zero operators $D_{ij} : E_i \rightarrow E_j$ i.e. $\mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{C}^2$ (multiplicity 2) and $D_{iK} : \mathbb{C} \rightarrow \mathbb{C}$. Their adjoints give the operators D_{ji} and D_{Ki} which results in edges with an opposite direction.

Definition 6.2. A *decorated graph* is a finite directed graph $\Gamma = (V, E)$ where

- (i) the vertices $v \in V$ are labeled by $n(v) \in \mathcal{A}$, for some index set \mathcal{A} such that $\{n(v)\}_{v \in V} = \mathcal{A}$ and
- (ii) the directed edges $e = (v, v') \in E$ are labeled by operators

$$D_e : \mathbb{C} \rightarrow \mathbb{C}.$$

Note that $|V| \geq |\mathcal{A}|$ and clearly we have

$$D_{ij} = \sum_{\substack{e=(v,v') \\ n(v)=i \\ n(v')=j}} D_e.$$

It is clear that this graph is an invariant of a finite commutative spectral triple. Unitary equivalent spectral triples give the same decorated graphs. The invariant is also complete since given a decorated graph $\Gamma = (V, E)$, we obtain the spectral triple by:

$$\left(M = \mathbb{C}^{|\mathcal{A}|}, H = \mathbb{C}^{|V|}, D = \sum_{e \in E} D_e \right)$$

The number of nodes with the same label corresponds to the multiplicity of the corresponding irreducible representation.

We thus have obtained the following theorem:

Theorem 6.3. *There is a one-to-one correspondence between finite commutative spectral triples modulo unitary equivalence and decorated graphs.*

Remark 6.4. This classification of finite spectral triples is based on the so-called Krajewski diagrams in [8].

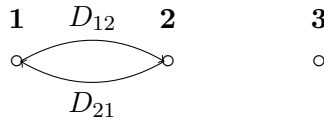
Example 6.5. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \mathbb{C}^3, \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The irreducible representations of M in H are given by

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_M^3 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We have that $D_{12} = D_{21} = \lambda$ is a one-dimensional map and D_{11} and D_{22} are zero and hence we obtain the following decorated graph:



Example 6.6. Take the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C}, \mathbb{C}^3, \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

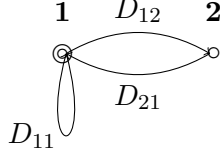
The irreducible representations of M in H are given by

$$E_M^1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } E_M^2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We then have that

$$D_{11} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, D_{12} = (\lambda \ 0), D_{21} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \text{ and } D_{22} = 0.$$

We obtain the following decorated graph:



In fact D_{12} splits into two edges which correspond to maps

$$D_{e_1} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \rightarrow \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } D_{e_2} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \rightarrow \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

where D_{e_1} is a one-dimensional map equal to λ and $D_{e_2} = 0$. In the same way we split D_{21} and D_{11} . We obtain in this way the same number of edges as in the graph above.

6.2 Decorated graphs and the invariant $\text{Spec}_N(M)$

Since there is a one-to-one correspondence between finite commutative spectral triples and decorated graphs there is a correspondence between the invariant $\text{Spec}_N(M)$ and decorated graphs. In the next theorem we will find a direct relation g between the invariant $\text{Spec}_N(M)$ and a decorated graph, such that the following diagram commutes:

$$\begin{array}{ccc} (M, H, D) & \xrightarrow{\text{Spec}} & \text{Spec}_N(M) \\ \mathcal{G} \downarrow & & \swarrow g \\ \Gamma = (V, E) & & \end{array}$$

Here \mathcal{G} denotes the correspondence of commutative finite spectral triples and decorated graphs from the previous subsection.

Theorem 6.7. *Let*

$$\text{Spec}_N(M) = \{\gamma_\alpha | \alpha \in \{1, \dots, K\}\},$$

be a set with the properties of Theorem 4.8 and $\Lambda = \{\lambda_1, \dots, \lambda_q\} \subset \mathbb{R}$. Take the decorated graph $\tilde{\Gamma}$ with $m_\alpha = \text{Tr } \gamma_\alpha$ vertices with label α such that there are $\sum_{\alpha=1}^K m_\alpha$ vertices. Define

$$\tilde{D}_{\alpha\beta} : \sum_{\lambda \in \Lambda} \lambda \sum_{\omega} (\gamma_{\omega\lambda})_\beta \sum_{\kappa} (\gamma_{\lambda\kappa})_\alpha,$$

which results in the edges \tilde{D}_e of $\tilde{\Gamma}$. Then the relation g such that $g \circ \text{Spec} = \mathcal{G}$ is given by

$$g(\text{Spec}_N(M)) = \tilde{\Gamma}.$$

Proof. Let (M, H, D) be the finite spectral triple reconstructed as in Theorem 4.8, with $D = \sum_{\lambda \in \Lambda} \lambda \sum_{\alpha} (\gamma_{\alpha})_{\lambda\lambda}$. To construct the corresponding decorated graph Γ we see that we have to draw the nodes in the same way as claimed in the theorem we are proving. To draw the edges we need to split D into maps

$$D_{\alpha\beta} : \text{Im } \gamma_{\alpha} \rightarrow \text{Im } \gamma_{\beta}.$$

Indeed we have $\tilde{D}_{\alpha\beta} : \text{Im } \gamma_{\alpha} \rightarrow \text{Im } \gamma_{\beta}$ and we are left with checking that $\tilde{D} = \sum_{\alpha, \beta} \tilde{D}_{\alpha\beta} = \sum_{\alpha, \beta} D_{\alpha\beta} = D$. Take an orthonormal basis $\{\eta_{\lambda, j}\}_j$ in each V_{λ} and take an orthonormal basis $e_{\alpha, i}$ in each $\text{Im } \gamma_{\alpha}$. We compute

$$\begin{aligned} \tilde{D}(v) &= \sum_{\alpha, \beta} \tilde{D}_{\alpha\beta}(v) \\ &= \sum_{\alpha, \beta} \sum_{\lambda} \lambda \sum_{\omega} (\gamma_{\beta})_{\omega\lambda} \sum_{\kappa} (\gamma_{\alpha})_{\lambda\kappa}(v) \\ &= \sum_{\alpha, \beta} \sum_{\lambda} \lambda \sum_{i, i'} \sum_j e_{\alpha, i} \langle e_{\alpha, i}, \eta_{\lambda, j} \rangle \langle \eta_{\lambda, j}, e_{\beta, i'} \rangle \langle e_{\beta, i'}, v \rangle \quad (\text{by the proof of Theorem 5.2}) \\ &= \sum_{\lambda} \lambda \sum_j \eta_{\lambda, j} \langle \eta_{\lambda, j}, v \rangle \\ &= \sum_{\lambda \in \Lambda} \lambda \sum_{\alpha} (\gamma_{\alpha})_{\lambda\lambda}(v) \quad (\text{by Corollary 5.3}) \\ &= D(v). \end{aligned}$$

with $v \in H$. We conclude that $\tilde{\Gamma} = \Gamma$. □

This theorem results in a corollary that gives the direct relation g' between the dual invariant $\text{Spec}_M(N)$ and decorated graphs, such that the following diagram commutes:

$$\begin{array}{ccc} (M, H, D) & \xrightarrow{\text{Spec}} & \text{Spec}_M(N) \\ \mathcal{G} \downarrow & & \swarrow g' \\ \Gamma = (V, E) & & \end{array}$$

Corollary 6.8. *Let $\text{Spec}_M(N) = \{\gamma_{\lambda}\}_{\lambda}$ with $(\gamma_{\lambda})_{\alpha\beta} : E_{\beta} \rightarrow E_{\alpha}$ be the dual of the invariant given in Theorem 6.7. Draw the decorated graph Γ' with $m_{\alpha} = \dim E_{\alpha}$ vertices with label α . Define*

$$D'_{\alpha\beta} = \sum_{\lambda \in \Lambda} \lambda (\gamma_{\lambda})_{\alpha\beta},$$

which results in the edges D'_e of Γ' . Then the relation g' such that $g' \circ \text{Spec} = \mathcal{G}$ is given by

$$g'(\text{Spec}_M(N)) = \Gamma'.$$

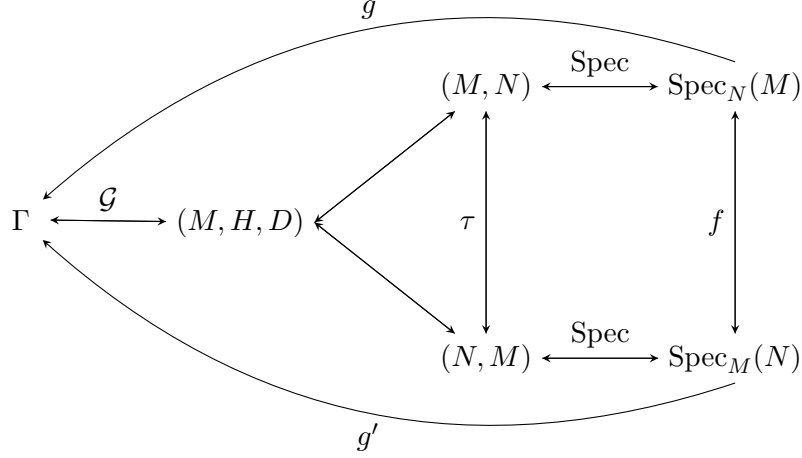
Proof. From Theorem 5.2 we see we have that $\tilde{D}_{\alpha\beta}$ as defined in Theorem 6.7 equals $D'_{\alpha\beta}$. Hence

$$g(\text{Spec}_M(N)) = g'(f(\text{Spec}_M(N))) = g'(\text{Spec}_M(N))$$

and $\Gamma' = \tilde{\Gamma}$. □

Remark 6.9. To find a relation from a decorated graph to the invariant $\text{Spec}_M(N)$ it is necessary to extract the eigenvalues of D from the data of the graph. The only way to find them is to compute D and diagonalize it. Therefore a relation from a decorated graph to the invariant $\text{Spec}_M(N)$ can only be obtained via the spectral triple (M, H, D) itself.

We summarize the relations between our invariants in the following diagram:



Recall that in (M, N) , (N, M) , $\text{Spec}_N(M)$ and $\text{Spec}_M(N)$ also the data of the spectrum of D is included.

Example 6.10. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, z_1, z_2 \in \mathbb{C}.$$

As we have seen in Example 4.12 a representative for $\text{Spec}_N(M)$ is

$$\gamma_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \text{ and } \gamma_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right).$$

Then

$$\begin{aligned} D_{11} &= \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} + (-\lambda) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix} - \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

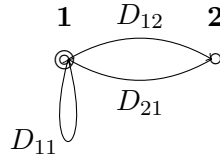
which is a map from $\phi(E_M^1)$ to itself rewritten as

$$D_{11} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix},$$

where ϕ is given in Example 4.12. In the same way we obtain

$$D_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\lambda & \frac{1}{2}\lambda \\ 0 & -\frac{1}{2}\lambda & -\frac{1}{2}\lambda \end{pmatrix},$$

which we rewrite as $D_{12} = (\lambda \ 0)$ Furthermore $D_{22} = 0$, so we obtain the same decorated graph as in Example 6.6:



The dual invariant $\text{Spec}_M(N)$ is computed in Example 5.8. From Corollary 6.8 and the fact that the map $\phi' : H \rightarrow l^2(\text{Spec}_M(N))$ viewed as map from H to H is given by

$$\phi' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we see that the graph we obtain from the dual is indeed the same graph as above. If we consider the representation

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, z_1, z_2 \in \mathbb{C},$$

we have seen in Example 4.12 that a representative for $\text{Spec}_N(M)$ is

$$\gamma_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \text{ and } \gamma_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

We then obtain

$$D_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

which we rewrite as a map from $\phi(E_m^1)$ to itself:

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}.$$

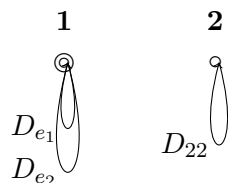
We split up D_{11} in a map

$$D_{e_1} : \phi \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \rightarrow \phi \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \text{ and } D_{e_2} : \phi \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \rightarrow \phi \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right),$$

that both are equal to the one-dimensional map λ . We have that

$$D_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

which corresponds to the one-dimensional map λ and $D_{12} = D_{21} = 0$. Hence the decorated graph looks as follows:



7 The invariant of finite spectral triples

In this section we will compute the invariant $\text{Spec}_N(M)$ of finite spectral triples where the algebra M is not necessarily commutative.

7.1 The construction of the invariant of finite spectral triples

Let (M, H, D) be a finite spectral triple with H a n -dimensional Hilbert space. According to Theorem 2.17 the algebra M is a matrix algebra of the form

$$M \cong \bigoplus_{\alpha=1}^K M_{d_\alpha}(\mathbb{C}).$$

We have that

$$\bigoplus_{\alpha=1}^K M_{d_\alpha}(\mathbb{C}) \cong \bigoplus_{\alpha=1}^K \mathcal{L}(\mathbb{C}^{d_\alpha}),$$

where \mathbb{C}^{d_α} are the irreducible representations of M and m_α its corresponding multiplicity in H . Furthermore $\sum_{\alpha=1}^K m_\alpha d_\alpha = n$.

Modulo a basis transformation, an action π of M on H is by Example 2.14 of the following form

$$\pi((a_\alpha)_\alpha) = \bigoplus_{i=1}^K m_\alpha a_\alpha \in \mathcal{L}(H),$$

for each $(a_\alpha)_\alpha \in M$. Here $m_\alpha a_\alpha$ means m_α copies of a_α . Each component $M_{d_\alpha}(\mathbb{C})$ of M is not generated anymore by one minimal idempotent, but we can still define the sets E_α that correspond to the components $M_{d_\alpha}(\mathbb{C})$. Namely, take $\{E_\alpha | \alpha \in \{1, \dots, K\}\}$ the irreducible representations of M in H , such that each E_α corresponds to the irreducible representation $\mathbb{C}^{d_\alpha} \otimes \mathbb{C}^{m_\alpha} \subset H$. We have that the dimension of E_α is $m_\alpha d_\alpha$ and $\bigoplus_\alpha E_\alpha = H$. If we use the unitary isomorphism $\phi : H \rightarrow l^2(\text{Spec}(N), V)$ as defined in Section 4, we obtain the sets $\phi(E_\alpha)$ that give the relative positions of the irreducible representations of M in H relative to the eigenspaces of D . For each $\phi(E_\alpha)$ we compute, using Lemma 3.5, the corresponding projection γ_α , which is now an element of $P_{m_\alpha d_\alpha}(N)$. The invariant $\text{Spec}_N(M)$ is then defined as follows:

Definition 7.1. The *relative spectrum* of M relative to N is the set

$$\text{Spec}_N(M) = \{\gamma_\alpha | \alpha \in \{1, \dots, K\}\} \subset \bigcup_{\alpha} P_{m_\alpha d_\alpha}(N),$$

which is defined up to the adjoint action of $\mathcal{U}\text{End}(V)$.

The relative spectrum $\text{Spec}_N(M)$ is then a complete invariant together with $\sigma(D)$, the spectrum of D , and $\{d_1, \dots, d_K\} \cong \widehat{M}$.

Theorem 7.2. *There is a one-to-one correspondence between finite spectral triples (M, H, D) , modulo unitary equivalence, and triples $(S, \Lambda, \widehat{M})$ where*

$$S \subset \bigcup_{i=1}^{\infty} P_i(N)$$

such that

- (i) $\gamma\gamma' = 0$ for all $\gamma \neq \gamma' \in S$

$$(ii) \sum_S \gamma_{\lambda\mu} = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{I}_\mu & \text{if } \lambda = \mu \end{cases},$$

is a finite set defined up to the adjoint action of $\mathcal{U}\text{End}(V)$, $\Lambda = \{\lambda_1, \dots, \lambda_q\} \subset \mathbb{R}$ whose number of elements and order corresponds to the labeling of the $\gamma \in S$ and $\widehat{M} = \{d_1, \dots, d_K\}$ is a set of positive integers whose number of elements and order corresponds to the elements of S and d_α divides $m(\gamma_\alpha)$ where $m(\gamma_\alpha) = i$ if $\gamma_\alpha \in P_i(N)$.

Proof. Given a spectral triple (M, H, D) we see that S is the relative spectrum $\text{Spec}_N(M)$, Λ is obtained from $\sigma(D)$ and \widehat{M} is obtained from the dimensions of the components of M .

Let $(S, \Lambda, \widehat{M})$ be a triple such as stated in the theorem. We reconstruct its corresponding spectral triple. We take

$$M' \cong \bigoplus_{d_\alpha \in \widehat{M}} M_{d_\alpha}(\mathbb{C}).$$

Then, with $E_\gamma = \text{Im } \gamma$, we define a Hilbert space

$$H' = \bigoplus_{\gamma \in S} E_\gamma$$

and we take the symmetric operator

$$D' = \sum_{\lambda \in \Lambda} \lambda \sum_{\gamma \in S} \gamma_{\lambda\lambda}.$$

To determine the representation π' of M' on H' we take $\pi' : M' \rightarrow \bigoplus_{\gamma \in S} \mathcal{L}(E_\gamma)$ such that

$$\pi' : M|_{M_{d_\alpha}(\mathbb{C})} \rightarrow \mathcal{L}(E_{\gamma_\alpha}).$$

Note that this map π' is not necessary of the form stated in Example 2.14, but we can find a basis transformation φ such that the triple $(M', H', \varphi D' \varphi^*)$ with an action as in Example 2.14, is unitarily equivalent with (M', H', D') . Since statement 1 and 2 from the proof of Theorem 3.10 hold for (M', H', D') , is the finite spectral triple (M', H', D') the inverse of our invariant and hence it is complete. \square

If we refer to the invariant as $\text{Spec}_N(M)$ we do not specifically mention the sets Λ and \widehat{M} .

Corollary 7.3. *When $M = M_d(\mathbb{C})$ is a matrix algebra with one component, we obtain for $\text{Spec}_N(M)$ just one element, which equals the identity matrix.*

Remark 7.4. The inclusion of the set of positive integers \widehat{M} in our invariant is necessary to reconstruct the algebra M . When we have an element $\gamma \in P_{m \cdot d}(N)$, we can for example reconstruct a copy $M_{m \cdot d}(\mathbb{C})$ of M acting with multiplicity 1 on H or a copy $M_d(\mathbb{C})$ acting with multiplicity m on H . Whenever we make sure that the number d is included in the invariant we do not have this problem.

7.2 Examples.

Example 7.5. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0, 1\}$,

$$M = M_2(\mathbb{C}), H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in M.$$

The eigenvectors of D are the same as in Example 3.14. The irreducible representation in H is just \mathbb{C}^4 and hence we have as a representative for $\text{Spec}_N(M) \subset P_{2,2}(N) = P_4(N)$ one element:

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 7.6. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0, 1\}$,

$$M = M_2(\mathbb{C}) \oplus \mathbb{C}, H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\pi(a, b) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, a \in M_2(\mathbb{C}), b \in \mathbb{C}.$$

The irreducible representations of M in H are

$$E_1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } E_2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Therefore the calculation of the invariant is exactly the same as in Example 3.14, so we obtain the following representative for $\text{Spec}_N(M)$:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in P_{2,1}(N) = P_2(N) \text{ and } \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in P_{1,2}(N) = P_2(N).$$

If we consider the representation

$$\pi'(a, b) = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, a \in M_2(\mathbb{C}), b \in \mathbb{C},$$

we obtain the spaces

$$E_1 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The calculation of a representative for $\text{Spec}_N(M)$ is the same as in Example 3.14 and results in

$$\gamma_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in P_{2,1}(N) = P_2(N) \text{ and } \gamma_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in P_{1,2}(N) = P_2(N).$$

The reason that we obtain the same matrices as in Example 3.14 is explained in Remark 7.4.

Example 7.7. Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0, 1\}$,

$$M = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}, H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\pi(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, a \in M_2(\mathbb{C}), b, c \in \mathbb{C}.$$

We obtain the irreducible representations of M in H

$$E_1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}, E_{(2,1)} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } E_{(2,2)} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Calculating the invariant results in the following representative consisting of three elements:

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in P_2^+(N),$$

$$\gamma_{(2,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \gamma_{(2,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in P_1^+(N).$$

If we consider the representation

$$\pi'(a, b, c) = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}, a \in M_2(\mathbb{C}), b, c \in \mathbb{C},$$

we obtain the spaces

$$E_1 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}, E_{(2,1)} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } E_{(2,2)} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and the following three elements as a representative for $\text{Spec}_N(M)$:

$$\gamma_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in P_2^+(N),$$

$$\gamma_{(2,1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \gamma_{(2,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in P_1^+(N).$$

Example 7.8. The last example is one that will include multiplicity of the algebra N . Consider the finite spectral triple, for fixed $\lambda \in \mathbb{R} \setminus \{0\}$,

$$M = M_2(\mathbb{C}) \oplus \mathbb{C}, H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi(a, b) = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, a \in M_2(\mathbb{C}), b \in \mathbb{C}.$$

The eigenspaces of D and the map ϕ are given in Example 5.9. We have the following irreducible representations of M in H :

$$E_1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } E_2 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Calculating $\phi \tilde{E}_\alpha (\tilde{E}_\alpha)^* \phi^*$ gives a representative for the invariant $\text{Spec}_N(M) \subset P_2(N)$:

$$\gamma_1 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}.$$

7.3 Decorated graphs and the invariant $\text{Spec}_N(M)$

We would like to define a decorated graph of a finite spectral triple that is not necessarily commutative, i.e. the case where $M \cong \bigoplus_{\alpha=1}^K M_{d_\alpha}(\mathbb{C})$ and

$$H = \bigoplus_{\alpha=1}^K E_\alpha \cong \bigoplus_{\alpha=1}^K \mathbb{C}^{d_\alpha} \otimes \mathbb{C}^{m_\alpha}.$$

We then need a labeling of the vertices by the positive integers d_1, \dots, d_K . We extend our definition of decorated graphs following section 2.3 of [2].

Definition 7.9. An \mathcal{A} -decorated graph is a pair of a finite directed graph $\Gamma = (V, E)$ and a finite set \mathcal{A} of positive integers, with a labeling of the

- (i) vertices $v \in V$ by $n(v) \in \mathcal{A}$, such that $n(V) = \mathcal{A}$ and
- (ii) directed edges $e = (v, v') \in E$ by operators

$$D_e : \mathbb{C}^{n(v)} \rightarrow \mathbb{C}^{n(v')}.$$

Note that we can have several vertices with the same label, i.e. $|V| \geq |\mathcal{A}|$. In fact \mathcal{A} is of the form $\{d_1, \dots, d_K\}$, which means that the same number can occur more than once.

Remark 7.10. This definition coincides in the commutative case with Definition 6.2, since then \mathcal{A} equals the set $\{1, \dots, 1\}$, where each 1 has some index, i.e. $\mathcal{A} = \{1_1, \dots, 1_K\}$.

We extend Theorem 6.3.

Theorem 7.11. *There is a one-to-one correspondence between finite spectral triples modulo unitary equivalence and decorated graphs.*

Proof. Given an \mathcal{A} -decorated graph $\Gamma = (V, E)$, its corresponding triple is given by:

$$\left(M = \bigoplus_{\alpha \in \mathcal{A}} M_{\alpha}(\mathbb{C}), H = \bigoplus_{v \in V} \mathbb{C}^{n(v)}, D = \sum_{e \in E} D_e \right).$$

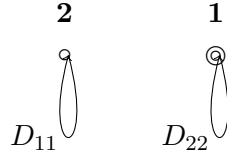
A copy $M_{\alpha}(\mathbb{C})$ acts with multiplicity $|\{v \in V | n(v) = \alpha\}|$ on H . □

Example 7.12. Consider the finite spectral triple from Example 7.6, for fixed $\lambda \in \mathbb{R} \setminus \{0, 1\}$,

$$M = M_2(\mathbb{C}) \oplus \mathbb{C}, H = \mathbb{C}^4, D = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\pi(a, b) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, a \in M_2(\mathbb{C}), b \in \mathbb{C}.$$

We obtain the following decorated graph:

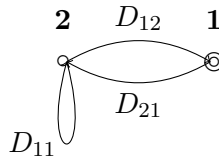


with $D_{11} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ and $D_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, where D_{11} splits into two nonzero edges.

For the same spectral triple with action given by

$$\pi'(a, b) = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, a \in M_2(\mathbb{C}), b \in \mathbb{C},$$

we obtain the following decorated graph:



with $D_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $D_{12} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$.

We have a similar theorem as Theorem 6.7 which gives a relation g between our invariant $\text{Spec}_N(M)$ and \mathcal{A} -decorated graphs such that the following diagram commutes:

$$\begin{array}{ccc} (M, H, D) & \xrightarrow{\text{Spec}} & \text{Spec}_N(M) \\ \mathcal{G} \downarrow & & \swarrow g \\ \Gamma = (V, E) & & \end{array}$$

Theorem 7.13. *Let*

$$\text{Spec}_N(M) = \{\gamma_\alpha | \alpha \in \{1, \dots, K\}\},$$

be a set with the properties of Theorem 7.2, $\Lambda = \{\lambda_1, \dots, \lambda_q\} \subset \mathbb{R}$ and $\widehat{M} = \{d_1, \dots, d_K\}$ a set of positive integers. Take the decorated graph $\tilde{\Gamma}$ with $m_\alpha = \frac{\text{Tr} \gamma_\alpha}{d_\alpha}$ vertices with label d_α such that there are $\sum_{\alpha=1}^K m_\alpha$ vertices. Define

$$\tilde{D}_{\alpha\beta} : \sum_{\lambda \in \Lambda} \lambda \sum_{\omega} (\gamma_\beta)_{\omega\lambda} \sum_{\kappa} (\gamma_\alpha)_{\lambda\kappa},$$

which results in the edges \tilde{D}_e of $\tilde{\Gamma}$. Then the relation g such that $g \circ \text{Spec} = \mathcal{G}$ is given by

$$g(\text{Spec}_N(M)) = \tilde{\Gamma}.$$

Proof. If we may use Equation (3) then the proof of this theorem is identical to the proof of Theorem 6.7. So we want to know if Equation (3) holds in the noncommutative case. Therefore we look at the proof of Theorem 5.2 and see that if we take for (M, H, D) the spectral triple reconstructed in Theorem 7.2 we conclude the same expression for $(\tilde{\gamma}_\lambda)_{\alpha\beta}$. Switching the the roles of the algebras M and N in the construction is still possible and we see that the expression for $(\gamma_\lambda)_{\alpha\beta}$ also holds. Hence Equation (3) holds. \square

Remark 7.14. The proof of the above statement implicitly says that we can define the dual invariant $\text{Spec}_M(N)$ also in the noncommutative case as long as in $\text{Spec}_M(N)$, besides the spectrum of the operator, also the data of \widehat{M} is included. Here we look at the pairs of algebras (M, N) and (N, M) with M noncommutative and N commutative. A pair of noncommutative algebras does not correspond to a finite spectral triple anymore. However, for a pair of noncommutative algebras (M, N) we can still construct the invariant $\text{Spec}_N(M)$, which is then together with the sets of positive integers \widehat{M} and \widehat{N} a complete invariant. Also the dual invariant $\text{Spec}_M(N)$ is then defined. Therefore, if we let go the correspondence between pairs of algebras and finite spectral triples, we have the following commutative diagram for the category of pairs of unital $*$ -algebras acting faithfully on the same Hilbert space:

$$\begin{array}{ccc} (M, N) & \xrightarrow{\text{Spec}} & \text{Spec}_N(M) \\ \tau \downarrow & & \downarrow f \\ (N, M) & \xrightarrow{\text{Spec}} & \text{Spec}_M(N) \end{array}$$

Example 7.15. Consider the finite spectral triple from Example 7.6. We can use the computed representative of $\text{Spec}_N(M)$ to compute the corresponding decorated graph using Theorem 7.13. We then obtain the same graph as in Example 7.12.

8 Outlook

As a possible starting point for further research, one thing came up at the end of this thesis project. A question that naturally arises from our invariant $\text{Spec}_N(M)$ is: given a morphism between two finite spectral triples, what is the relation between the invariants? The definition of a morphism between two spectral triples is not completely obvious in literature. We suggest the following one:

Definition 8.1. A *morphism* between two finite spectral triples (M_1, H_1, D_1) and (M_2, H_2, D_2) is given by a pair (Φ, U) where Φ is a $*$ -homomorphism $\Phi : M_1 \rightarrow M_2$ and U a partial isometry $U : H_1 \rightarrow H_2$ such that

- (i) $U\pi_1(a) = \pi_2(\Phi(a))U$ for $a \in M_1$ and
- (ii) $UD_1 = D_2U$.

We are interested in finding a direct relation h between the two corresponding invariants $\text{Spec}_{N_1}(M_1)$ and $\text{Spec}_{N_2}(M_2)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 (M_1, H_1, D_1) & \xrightarrow{\text{Spec}} & \text{Spec}_{N_1}(M_1) \\
 (\Phi, U) \downarrow & & \downarrow h \\
 (M_2, H_2, D_2) & \xrightarrow{\text{Spec}} & \text{Spec}_{N_2}(M_2)
 \end{array}$$

To get a first idea, one can consider the case where U is a unitary map, which is a much stronger assumption. Since the invariants are determined modulo unitary equivalence we can identify $H_1 = H_2$ and $D_1 = D_2$. We then only need to consider the map Φ . This map is determined by a Bratteli diagram [9]. A direct relation h between $\text{Spec}_N(M_1)$ and $\text{Spec}_N(M_2)$ is then obtained by a rearrangement of the $\gamma_\alpha \in \text{Spec}_N(M_1)$ according to the Bratteli diagram. However, if we only consider the set $\text{Spec}_N(M_1)$ and a Bratteli diagram \mathbb{B} , this rearrangement is not unique. The same problem occurs when we try to find a relation h that goes in the opposite direction, i.e. from $\text{Spec}_N(M_2)$ to $\text{Spec}_N(M_1)$. The main question still remains open.

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