Master's Thesis
Noncommutative geometry of Yang–Mills Fields

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ABSTRACT

In the first part of this thesis we describe Yang–Mills theory on a Riemannian spacetime by studying the algebra of smooth functions from the spacetime to $M_n(\mathbb{C})$. For this a spectral triple is defined and it is shown how the spectral action principle gives a Yang-Mills Lagrangian. We compare this model of Yang–Mills theory with with the description of Yang–Mills theory in terms of principal bundles ([1], [2]). It turns out that both descriptions coincide, though the the principal bundle is topologically trivial. In the second part of this thesis we will generalise the previous spectral triple to get a model for Yang–Mills theories that are described by non-trivial principal bundles. Instead of $M_n(\mathbb{C})$-valued functions we now study sections of some nontrivial algebra bundle with copies of $M_n(\mathbb{C})$ as fibres. We will also show that this algebra of sections has enough structure to reconstruct the algebra bundle. From this algebra bundle we construct a topological non-trivial principal bundle. When we apply the spectral action principle to our new triple we get a Yang–Mills Lagrangian where the gauge field can be interpreted as a gauge potential for the just constructed principal bundle.

CONTENTS

1 INTRODUCTION

Noncommutative geometry is an area of mathematics invented by Connes [6]. It can be considered as a generalisation of ordinary geometry and it is a very large area with applications not only in many other fields of mathematics but also in theoretical high energy physics. The main idea of noncommutative geometry isthat we can trade spaces for algebras. This idea already existed in ordinary geometry where one can, for example, trade a topological space for a C^*-algebra. More precisely, if X is a compact Hausdorff space, then the algebra $C(X)$ of complex-valued continuous functions is a commutative unital C^*-algebra. Conversely, any unital commutative C^*-algebra is of the form $C(X)$ for some compact Hausdorff space X. This duality, also known as Gelfand's representations (see [27], section 2.1), between unital abelian C^*-algebras and compact Hausdorff spaces implies that all information of the topological structure of X is contained in $C(X)$. Therefore, we may as well describe the topological space X by the algebra $C(X)$. Since the algebra $C(X)$ is always commutative, topological spaces are always described by a commutative C^*-algebras. Such spaces are therefore also called "commutative" spaces. One can ask what would happen if the C^*-algebra is taken to be noncommutative.

However, just a topological space is not enough to do physics. Let us determine which ingredients we need for a physical theory. First of all, we need a spacetime manifold M in which all the physics takes place. It follows from the Gelfand representation that (only) the topological structure of a spacetime M can be recovered from the algebra of functions on M. Doing physics in a spacetime M also means that we can describe the trajectories of particles once their initial conditions are given. That is, we need to introduce particles and we need to know how they propagate through spacetime.

To introduce (fermionic) particles in our space-time M, a spin structure or a spinor bundle is needed on M. Denote S for the spinor bundle associated to the spin structure. In dimension 4, the fibres of the spinor bundle are copies of $\mathbb{C}^4$ and the inner product on the Hilbert-space of square-integrable spinor $L^2(M,S)$ is given by

$$\langle\psi_1,\psi_2\rangle = \int_M \overline{\psi}_1\psi_2 d^4x \quad \psi_1,\psi_2 \in L^2(M,S), \tag{1.1}$$

where the bar denotes complex conjugation. The Hilbert space $L^2(M,S)$ of square-integrable spinors is the second ingredient for our spectral. It represents the fermionic particles. The algebra $C^\infty(M)$ acts on the spinors by pointwise multiplication

$$f\psi(x) = f(x)\psi(x), \quad (x\in M), \quad \forall f \in C^\infty(M), \psi \in L^2(M,S). \tag{1.2}$$

The Dirac-operator $\not{D}$ is the final ingredient and it is locally given by

$$i\gamma^\mu\nabla^S_\mu, \tag{1.3}$$

where ∇^S_μ is a covariant derivative action on the spinors. Together these three ingredients form the canonical spectral triple

$$(C^\infty(M), L^2(M,S), \not{D}), \tag{1.4}$$

consisting of an algebra, a Hilbert space and a differential operator. This canonical triple provides much more information of the geometry than the algebra alone. For instance, Connes has shown in [7] that one can not only reconstruct the space M as a topological space, but one can also also recover the geodesic distance of the manifold. Indeed, one can check that the geodesic distance between two points x, y is given by

$$\sup\{|f(x) - f(y)|; f \in C^{\infty}(M) \text{ such that } \|[\not{D}, f]\| \leq 1\}. \tag{1.5}$$

Also other properties of the spacetime manifold M can be recovered from the canonical triple. Thus, the canonical triple contains all the information to do physics and it is possible to express the geometric properties of the spacetime in terms of the ingredients in this triple. Therefore, we can trade the spacetime, which is a *space*, for a canonical triple, which consists of *algebraic* structures..

In noncommutative geometry the above situation is generalised to so-called noncommutative spaces. Based on the above remarks, the starting point is a triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of an unital algebra $\mathcal{A}$ represented faithfully on the Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator D on $\mathcal{H}$. Together with some compatibility conditions the triple $(\mathcal{A}, \mathcal{H}, D)$ is called a spectral triple. This definition of a spectral triple is a generalisation of the canonical triple. The key point now is that in noncommutative geometry the algebra $\mathcal{A}$ is allowed to be noncommutative, therefore describing noncommutative spaces.

The spectral triple contains enough information to calculate Lagrangians by applying the so-called spectral action principle. For instance, Chamsedinne and Connes constructed a triple in 1997 ([5]) that gives an action that contains both the Einstein-Hilbert action and a $su(n)$-Yang-Mills Lagrangian. It is this triple that we will study in the first part of this thesis (Chapters 3 and 4).

For the description of gauge theories, and in particular Yang-Mills theory, there also exists a well-known description terms of a principal bundles (see [1] and [2]). Let P be a principal G-bundle (G a Lie-group) describing a gauge theory. Then, for example, the gauge potential is given by a $\mathfrak{g}$-valued one-form on P, where $\mathfrak{g}$ is the Lie-algebra of G.

The $su(n)$-gauge potential occurring in the Yang–Mills Lagrangian of the Chamseddine–Connes triple can also be interpreted as a gauge field for a principal $PSU(n)$-bundle P. Here the bundle P, however, is trivial. This triple therefore describes a Yang–Mills gauge theory that is equivalent to a Yang-Mills theory described by a principal $PSU(n)$-bundle P as long as the bundle P is trivial. The main goal of this thesis is to generalise the above situation by defining a spectral triple that describes Yang–Mills theory in terms of a non-trivial $PSU(n)$-bundle. This is what we will do in the second part of this thesis (Chapters 5 – 7).

In Chapter 2 we pay attention to some necessary preliminaries. In each section I will also refer to some books that may provide more information on the treated topics. The first sections will mostly contain definitions of different kinds of algebraic structures we will often encounter. After that we will turn to to discuss some concepts that are necessary to introduce the spinor bundle and the Dirac operator. The spinor bundle and Dirac operator are important ingredients of the canonical triple and they will be introduced at the end of this chapter. In this chapter also a little bit is said about the description of gauge theories in terms of principal bundles. The preliminary chapter is not meant to give the reader all the necessary background to understand the thesis. For instance, the reader is assumed to have some knowledge of linear algebra, topology and differential geometry. I would like to mention here that more information on topology can be found in [9] and in the first chapter of [30]. For foundations of differential geometry see [19].

In Chapter 3 we introduce the basics of noncommutative geometry. Furthermore, spectral triples and the spectral action principle are discussed. We will also show how the spectral action can be calculated using the heat expansion. At last we will say something about the group of inner transformations of the spectral algebra $\mathcal{A}$. This group is denoted by $\mathrm{Inn}(A)$ and is interpreted as the internal symmetry group (or gauge group).

As sort of an example we will calculate the spectral action (using the heat expansion) of the Chamseddine–Connes spectral triple (see [5]). As said earlier, the heat expansion of the spectral action contains not only the Einstein-Hilbert action, it also contains a $su(n)$-Yang–Mills Lagrangian. Moreover, we will discuss in this chapter how this noncommutative geometric description of gauge theory is related to the description of gauge theory in terms of (principal) bundles. Furthermore, we will show why the bundle is topologically trivial.

In Chapter 5 we will take the first step towards the main goal of this thesis. We will introduce a new spectral algebra $\mathcal{A}$ that generalises the old one. Furthermore, we will show how one can construct a principal fibre bundle P from the spectral algebra $\mathcal{A}$. This bundle P will be important for us later. It needs to be mentioned that additional conditions on the algebra $\mathcal{A}$ are necessary to carry out this construction.

To complete the new spectral triple a Hilbert space and Dirac-operator need to be defined. This is done in Chapter 6. To show that the spectral action once again gives both a Einstein-Hilbert action and a $su(n)$-Yang–Mills Lagrangian the spectral action is calculated in Chapter 7. This calculation shows that the correct Yang–Mills Lagrangian is obtained for a ad P-valued bosonic field. Finally, we will show that this gauge field can also be interpreted as a gauge potential belonging to the principal bundle P we constructed in Chapter 5, and that it transforms correctly under inner transformations.

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2 PRELIMINARIES

In this chapter I will shortly treat some preliminaries where not all readers may be familiar with. First we will discuss some algebraic structures like Clifford algebras, then different kind of fibre bundles and connections on them. These concepts are needed in order to define spinor bundles and Dirac-operators at the end of this chapter. At the beginning of each section there is a list of references to books that contain more information on the subject.

2.1 MODULES

Remark 2.1.1. Most of what is done in this section is based on section 2.A in [15]. More information on modules can also be found in [22].

A module is a generalisation of a vector space in the sense that the set of scalars is a ring instead of a field. Since multiplication need not be commutative in rings, we distinguish between two kind of modules.

Definition 2.1.2. *Let R be a ring. A right R-module is a set $\mathcal{E}$ with a binary operation $+ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and a scalar multiplication $\cdot : \mathcal{E} \times R \to \mathcal{E}$, such that $(\mathcal{E}, +)$ forms an abelian group and such that for all r,r_1,$r_2 \in R$ and for all m,m_1,$m_2 \in \mathcal{E}$ the following holds:*

$$\begin{aligned} (m_1 + m_2) \cdot r &= m_1 \cdot r + m_2 \cdot r, \\ m \cdot (r_1 + r_2) &= m \cdot r_1 + m \cdot r_2, \\ m \cdot (r_1 \cdot r_2) &= (m \cdot r_1) \cdot r_2, \end{aligned}$$

The definition of a left R-module is a verbatim repetition of the one for a right R-module, except for the fact that the scalar multiplication is defined to be a map from $R \times \mathcal{E}$ to $\mathcal{E}$.

If the ring R has an identity element, then it is called unital. In the rest of this section it is assumed that the rings are unital.

Definition 2.1.3. *Let $\mathcal{E}, \mathcal{F}$ be two right R-modules. A right R-module homomorphism $\phi : \mathcal{E} \to \mathcal{F}$ is an additive map such that $\phi(mr) = \phi(m)r$ for all $r \in R$ and $m \in \mathcal{E}$. As in the case of vector spaces, we will also call ϕ a linear map. The set of all linear maps from $\mathcal{E}$ to $\mathcal{F}$ is called $Hom_R(\mathcal{E}, \mathcal{F})$, or $End_R(\mathcal{E})$ when $\mathcal{F} = \mathcal{E}$. The map ϕ is called an isomorphism if it is also bijective.*

There are some important notions concerning R-modules which will be discussed shortly. The first notion is a special kind of R-module which makes the module look more like a vector space.

Definition 2.1.4. *A right module $\mathcal{E}$ over R is called free if it has an R-basis, that is, a set of generators T such that the relation $t_1 r_1 + t_2 r_2 + \cdots t_m r_m = 0$ ($t_j \in T$, $a_j \in R$) implies that $r_1 = \cdots r_m = 0$. A right module $\mathcal{E}$ is called finitely generated if it has a finite generating set.*

This definition implies that in a free module $\mathcal{E}$ there exists a generating set such that all elements of $\mathcal{E}$ can be uniquely written as a linear combination of the generators.

The simplest example of a free right R-module with n generators is $R^n := R \oplus R \oplus \cdots \oplus R$. Scalar multiplication and addition occur component-wise and the standard basis is given by the set:

$\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots(0,\ldots,1)\}$. Any finitely generated free right R-module $\mathcal{E}$ is of the form $\mathcal{E} \cong R^n$ for some $n \in \mathbb{N}$ which can be shown by matching bases.

We now come to a very important definition.

Definition 2.1.5. *A right R-module $\mathcal{P}$ is said to be projective if, given any surjective R-linear map of right R-modules $\phi : \mathcal{E} \to \mathcal{G}$ and any R-linear map $\eta : \mathcal{P} \to \mathcal{G}$, there is an R-linear map $\psi : \mathcal{P} \to \mathcal{E}$ such that $\phi\psi = \eta$. This is also shown in the following diagram:*

$$\begin{tikzcd} & \mathcal{E} \arrow[d, two heads, "\phi"] \\ \mathcal{P} \arrow[ur, dotted, "\exists!\psi"] \arrow[r, "\eta"] & \mathcal{G} \end{tikzcd}$$

Clearly, every finitely generated free module is projective.

Lemma 2.1.6 ([15])**.** *Let R be a ring and let $\mathcal{P}$ be a right R-module. Then $\mathcal{P}$ is finitely generated and projective if and only if it is a direct summand of a free right R-module $\mathcal{F} = R^k$ where k is the number of generators of $\mathcal{P}$.*

Proof. We will first prove the forward implication. Denote the generators of $\mathcal{F}$ by c_j $(j = 1, \ldots, k)$ and the generators of $\mathcal{P}$ by p_j. Define an R-linear map $\phi : \mathcal{F} \to \mathcal{P}$ by sending c_j to p_j. This map is surjective so that there exists an R-linear map $\chi : \mathcal{P} \to \mathcal{F}$ such that $\phi \circ \chi = 1$. Then $\mathcal{F} = \mathrm{im}\chi \oplus \ker\phi$. Indeed, any $v \in \mathcal{F}$ can be written as $v = \chi(\phi(v)) + (v - \chi(\phi(v))$ and for $v \in \ker\phi \cap \mathrm{im}\chi$ it follows that $v = \chi(p)$ for some $p \in \mathcal{P}$, so that $p = \phi(\chi(p)) = \phi(v) = 0$ and hence $v = 0$. Finally, $\mathrm{im}\chi$ is isomorphic to $\mathcal{P}$ since χ is injective.

Conversely, suppose that there exist an R-module $\mathcal{Q}$ and a finitely generated free R-module $\mathcal{F}$ such that $\mathcal{F} = \mathcal{P} \oplus \mathcal{Q}$. Let $\phi : \mathcal{E} \twoheadrightarrow \mathcal{G}$ and $\eta : \mathcal{P} \to \mathcal{G}$ be given R-linear maps. Extend η to the R-linear map on $\mathcal{F}$ that vanishes on $\mathcal{Q}$. Since any free module is projective, there exist an R-linear map $\psi : \mathcal{F} \to \mathcal{E}$ such that $\phi \circ \psi = \eta$. Restriction of ψ to $\mathcal{P}$ gives an R-linear map $\mathcal{P} \to \mathcal{E}$ such that $\phi \circ \psi = \eta$. $\square$

There are more constructions that can be made with modules. Let's mention the most important ones.

Definition 2.1.7. *If $\mathcal{E}$ is a right S-module and a left R-module, then $\mathcal{E}$ is called a R, S-bimodule if*

$$r(ms) = (rm)s \quad \forall r \in R, s \in S \text{ and } m \in \mathcal{E}. \tag{2.1}$$

When $S = R$, then $\mathcal{E}$ is called an R-bimodule.
If $\mathcal{E}$ is a right R-module and $\mathcal{F}$ a left R-module, we can construct $\mathcal{E} \otimes_R \mathcal{F}$, which is the abelian group generated by all simple tensors of the form $m \otimes n$ with $m \in \mathcal{E}$ and $n \in \mathcal{F}$, subject to the conditions that the tensors are linear in both entries:

$$\begin{aligned} m_1 \otimes n + m_2 \otimes n &= (m_1 + m_2) \otimes n, \\ m \otimes n_1 + m \otimes n_2 &= m \otimes (n_1 + n_2), \end{aligned}$$

where m, m_1, $m_2 \in \mathcal{E}$, n, n_1, $n_2 \in \mathcal{F}$; and are compatible with the scalar multiplication:

$$mr \otimes n = m \otimes rn \quad \text{for each} \quad r \in R, m \in \mathcal{E}, n \in \mathcal{F}.$$

This space is a left S-module if $\mathcal{E}$ is an S-R-bimodule and it is a right T-module if $\mathcal{F}$ is an R-T bimodule. When both cases hold $\mathcal{E} \otimes_R \mathcal{F}$ is an S-T-bimodule.

It can be shown that $(\mathcal{E} \otimes_R \mathcal{F}) \otimes_S \mathcal{G} \cong \mathcal{E} \otimes_R (\mathcal{F} \otimes_S \mathcal{G})$ whenever these two expressions are defined.

2.2 ALGEBRAS AND HILBERT SPACES

In this section we will point out the definitions of Hilbert-spaces and C^*-algebras. These two concepts are closely related since the algebra of bounded linear operators on a Hilbert space is a C^*-algebra. The theory of Hilbert spaces and C^*-algebras is huge, but this section is only meant to give the reader an idea of what these structures are. We will also generalise the notion of a Hilbert space to the notion of a Hilbert C^*-module. For a short overview of the theory of Hilbert-spaces, functional analysis (and also of topology) I refer to [30], more information on C^*-algebras can be found in [18],[27], [34] and for Hilbert C^*-modules see [20]. All of the material discussed below can be found in these books.

In this section all fields $\mathbb{F}$ will be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.2.1. *An algebra A is a vector space $(+, \mathbb{F})$ with an associative multiplicative operation $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ which is compatible with the addition and scalar multiplication, that is: for all $\lambda \in \mathbb{F}$, a, b, $c \in \mathcal{A}$ the following equations hold:*

$$\begin{aligned} a(bc) &= (ab)c, \\ \lambda(ab) &= (\lambda a)b = a(\lambda b), \\ a(b+c) &= ab + ac, \\ (a+b)c &= ac + b. \end{aligned}$$

Definition 2.2.2. *A graded algebra is a direct sum decomposition into vector spaces:*

$$\mathcal{A} = \oplus_{n \in \mathcal{N}} \mathcal{A}_n \tag{2.2}$$

such that

$$ab \in \mathcal{A}_{k+l} \quad \forall a \in \mathcal{A}_k, b \in \mathcal{A}_l. \tag{2.3}$$

Elements of $\mathcal{A}_n$ are called the homogeneous elements of degree n.

Definition 2.2.3. *A norm on a vector space $(V, \mathbb{F})$ is a map $||\cdot|| : V \to \mathbb{R}^+$ for which the following conditions hold:*

$$\begin{aligned} ||v|| = 0 \quad &\iff \quad v = 0, \quad (v \in V), \\ ||\lambda v|| \quad &= \quad |\lambda| v, \quad (v \in V), \\ ||v + w|| \quad &\leq \quad ||v|| + ||w||, \quad (v, w \in V), \quad \textit{(triangle inequality)}. \end{aligned}$$

Definition 2.2.4. *A vector space $(V, \mathbb{F})$ is called an inner product space $(V, \langle\cdot,\cdot\rangle)$ if there exists a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ that satisfies:*

$$\begin{aligned} \langle v, v\rangle &\geq 0 \qquad \textit{and} \quad \langle v, v\rangle = 0 \Leftrightarrow v = 0, \\ \langle v + w, u\rangle &= \langle v, u\rangle + \langle w, u\rangle, \\ \langle v, \lambda w\rangle &= \lambda\langle v, w\rangle, \\ \langle w, v\rangle &= \overline{\langle v, w\rangle}, \end{aligned}$$

for all v, $w \in V$ and $\lambda \in \mathbb{F}$. The space $(V, \langle\cdot,\cdot\rangle)$ is also called a pre-Hilbert space.

It is easily checked that an inner product space is also a normed space by setting:

$$\|v\| = \sqrt{\langle v, v\rangle}, \quad (v \in V), \tag{2.4}$$

where the triangle equality follows from the Cauchy-Schwartz inequality

$$|\langle v, w\rangle| \leq ||v|| \cdot ||w||, \quad (v, w \in V), \tag{2.5}$$

that holds for arbitrary inner products.

Definition 2.2.5. *A Banach space is a normed vector space which is complete under the given norm. When a pre-Hilbert space is complete under the given inner product, it is called a Hilbert space. Note that a Hilbert space is automatically a Banach space.*

Definition 2.2.6. *A Banach algebra $\mathcal{A}$ is a Banach space, i.e. a vector space which is complete under a given norm $||\cdot||$, with an associative multiplication $(a,b) \mapsto ab$ that is compatible with the vector space structure, and for which the following relation holds:*

$$||ab|| \leq ||a|| \cdot ||b||. \tag{2.6}$$

This means that the multiplication rule is continuous in each argument.

Definition 2.2.7. *An involution on a Banach algebra $\mathcal{A}$ is an isometric (i.e. norm-conserving) anti-linear map $a \mapsto a^*$ satisfying the following conditions:*

$$\begin{aligned} (a^*)^* &= a, \\ (a+b)^* &= a^* + b^*, \\ (\lambda a)^* &= \bar{\lambda} a^*, \\ (ab)^* &= b^* a^*, \end{aligned}$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{F}$. The element a^* is called the adjoint of a and if $a^* = a$, the element a is said to be self-adjoint. If the algebra has a unit, an element $u \in \mathcal{A}$ is called unitary if $uu^* = u^*u = 1$. The group of all unitary elements is denoted by $\mathcal{U}(\mathcal{A})$.

Definition 2.2.8. *A* complex *Banach algebra A with involution is called a C^*-algebra if for each element $a \in A$ the equality*

$$||a^*a|| = ||a||^2 \tag{2.7}$$

is satisfied.

Note that this equality implies that $||a^*|| = ||a||$. Indeed, combining the inequalities

$$||a||^2 = ||a^*a|| \leq ||a^*|| \cdot ||a||$$

and

$$||a^*||^2 = ||aa^*|| \leq ||a|| \cdot ||a^*||,$$

we immediately see that $||a|| = ||a^*||$.

Let A, B be Banach algebras. A homomorphism $\phi : A \to B$ of Banach algebras is a linear map such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$. If both A and B have an involution, then $\phi : A \to B$ is a $*$-homomorphism if it is a homomorphism of Banach algebras and if it maps a^* to $\phi(a)^*$ for all $a \in A$. We will mostly consider C^*-algebras.

We state here without prove the following theorem (see [27], Theorem 2.1.7).

Theorem 2.2.9. *Any $*$-homomorphism from a Banach algebra to a C^*-algebra is necessarily norm-decreasing (i.e. $\|\phi(a)\| \leq \|a\|$ for all a).*

This implies that a $*$-homomorphism from a Banach algebra to a C^*-algebra is automatically continuous. If we have a $*$-isomorphism $\phi : A \to B$ between two C^*-algebras A and B, then ϕ is necessarily isometric (=norm-preserving).

Example 2.2.10. One of the most important examples of a C^*-algebra is the algebra of bounded (linear) operators on a Hilbert space. A linear map a on a Hilbert space is called a bounded operator if

$$\|a\| = \sup\{\|av\|_{\mathcal{H}}; h \in \mathcal{H}, ||h||_{\mathcal{H}} = 1\} \tag{2.8}$$

is finite. It can be shown that a linear map a is bounded if and only if it is a continuous. The set of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. The value $\|a\|$ is also called the operator norm of a and one can show that $\mathcal{B}(\mathcal{H})$ is a Banach algebra under this norm. Moreover, $\mathcal{B}(\mathcal{H})$ even becomes a C^*-algebra when the adjoint of an element $a \in \mathcal{B}(\mathcal{H})$ is taken to be the unique operator a^* for which the equality $(a^*x, y) = (x, ay)$ holds for all $x, y \in \mathcal{H}$.

A bounded operator $a : \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if $a(S)$ is relatively compact in $\mathcal{H}_2$, where S is the unit ball in $\mathcal{H}_1$. The compact operators form a C^*-sub-algebra of $\mathcal{B}(\mathcal{H})$ and this algebra is denoted by $\mathcal{K}(\mathcal{H})$.

A representation of an algebra A on a Hilbert space $\mathcal{H}$ is a linear and multiplicative map $\phi : A \to \mathcal{B}(\mathcal{H})$. If A has an involution we also require that $\phi(a^*) = \phi(a)^*$ for all $a \in A$. If ϕ is an injective map, the representation is said to be faithful.

Remark 2.2.11. Let $\mathcal{H}$ be a Hilbert-space and let a be a linear map defined on a subspace $\mathcal{D}(a) \subset \mathcal{H}$. The space $\mathcal{D}(a)$ is called the domain of the operator a. If $\|a\| = \sup\{\|av\|_{\mathcal{H}}; h \in \mathcal{D}(a), ||h||_{\mathcal{H}} = 1\}$ is infinite, then $\|a\|$ is said to be an unbounded operator with domain $\mathcal{D}(\mathcal{A})$. In this thesis we will always assume that an unbounded operator is defined on a dense subspace of $\mathcal{H}$.

Example 2.2.12. Let Ω be a locally compact Hausdorff space. Then the abelian algebra $C_0(\Omega)$ of all continuous complex-valued functions that vanish at infinity, is a C^*-algebra under the supremum norm

$$\|f\|_\infty = \sup\{|f(x)|, x \in \Omega\}, \quad (f \in C_0(\Omega)). \tag{2.9}$$

The adjoint of a function $f \in C_0(\Omega)$ is taken to be the function f^* given by $f^*(x) = \overline{f(x)}$. It follows from the Gelfand-representation that every abelian C^*-algebra A is of the form $C_0(\Omega)$ for some unique locally compact Hausdorff space Ω. If A is unital, then Ω is compact and vice versa (see [27], section 2.1).

Let $\mathcal{A}$ be a C^*-algebra. A self-adjoint element $a \in A$ is said to be positive if it is of the form $a = b^*b$ for some $b \in \mathcal{A}$. We introduce a partial ordering on the set of self-adjoint elements of A by setting $a \geq b$ if and only if $a - b$ is a positive element. With this definition we can now define Hilbert C^*-modules.

Definition 2.2.13. *Let $\mathcal{A}$ be a C^*-algebra. A (right) pre-Hilbert $\mathcal{A}$-module $\mathcal{E}$ (also called an right inner product $\mathcal{A}$-module) is a complex vector space that is a right $\mathcal{A}$-module – with compatible scalar multiplication $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $\lambda \in \mathbb{C}, a \in \mathcal{A}, x \in \mathcal{E}$) – together with a map*

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$$

for which the following conditions hold:

$$\begin{aligned} \langle x, \alpha y + \beta z \rangle &= \alpha \langle x, y \rangle + \beta \langle x, z \rangle, \\ \langle x, ya \rangle &= \langle x, y \rangle a, \\ \langle y, x \rangle &= \langle x, y \rangle^*, \\ \langle x, x \rangle &\geq 0 \quad \textit{and} \quad \langle x, x \rangle = 0 \Rightarrow x = 0, \end{aligned} \tag{2.10}$$

or all $x, y, z \in \mathcal{E}, \alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$.

In general, if $\mathcal{A}$ is an involutive algebra and $\mathcal{E}$ is a right $\mathcal{A}$-module, we speak of an $\mathcal{A}$-valued hermitian structure on $\mathcal{E}$ if there is a positive definite pairing $(\cdot,\cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying the rules $(x, y+z) = (x,y) + (x,z)$ and $(x, ya) = (x,y)a$ for all $x, y, z \in \mathcal{E}$, $a \in \mathcal{A}$.

For pre-Hilbert modules there is also a version of the Cauchy-inequality.

Proposition 2.2.14 ([20], Proposition 1.1)**.** *If $\mathcal{E}$ is a pre-Hilbert $\mathcal{A}$-module then*

$$\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\| \langle y, y\rangle.$$

for all $x, y \in \mathcal{E}$.

Proof. Without loss of generality we may assume that $\|\langle x, x\rangle\| = 1$. For $a \in \mathcal{A}$, we have

$$\begin{aligned} 0 \leq \langle xa - y, xa - y\rangle &= a^*\langle x, x\rangle a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle \\ &\leq a^* a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle. \end{aligned}$$

The last line comes from the fact that if c is a positive element of $\mathcal{A}$, then $a^*ca \leq \|c\|a^*a$. If we put $a = \langle x, y\rangle$, we get

$$0 \leq \langle y, x\rangle\langle x, y\rangle - \langle y, x\rangle\langle x, y\rangle - \langle y, x\rangle\langle x, y\rangle + \langle y, y\rangle,$$

and we are done. □

This proposition allows us to define a norm on the inner product $\mathcal{A}$-module $\mathcal{E}$.

Definition 2.2.15. *Define a norm $\|\cdot\|_{\mathcal{E}}$ on $\mathcal{E}$ by setting*

$$\|x\|_{\mathcal{E}} = \|\langle x, x\rangle\|^{\frac{1}{2}} \tag{2.11}$$

for $x \in \mathcal{E}$.

This is indeed a norm since $\|x\|_{\mathcal{E}} \geq 0$ for all $x \in \mathcal{E}$, and if $\|x\|_{\mathcal{E}} = 0$, then $\langle x, x\rangle = 0$ which implies that $x = 0$. Furthermore, $\|\lambda x\|_E^2 = \|\langle \lambda x, \lambda x\rangle\| = \|\lambda\bar{\lambda}\langle x, x\rangle\| = |\lambda|^2\|x\|_E^2$. For the triangle equality, we note that Proposition 2.2.14 implies that $\|\langle x, y\rangle\|^2 \leq \|\langle x, x\rangle\|\|\langle y, y\rangle\|$, so that $\|\langle x, y\rangle\| \leq \|x\|_{\mathcal{E}}\|y\|_{\mathcal{E}}$. Therefore, $\|x+y\|_{\mathcal{E}}^2 = \|\langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle\| \leq \|x\|_{\mathcal{E}}^2 + \|y\|_{\mathcal{E}}^2 + 2\|x\|_{\mathcal{E}}\|y\|_{\mathcal{E}}$, and we get the triangle equality by taking square roots on both sides.

If the pre-Hilbert $\mathcal{A}$-module $\mathcal{E}$ is closed under this norm induced, then $\mathcal{E}$ is called a Hilbert-C^*-module. Note that according to this definition a Hilbert space is a Hilbert $\mathbb{C}$-module. If $\mathcal{E}$ satisfies all conditions in equation (2.10) except for the second part of the last condition, then $\mathcal{E}$ is called a semi-inner-product $\mathcal{A}$-module.

We conclude this section by stating a very important proposition, but first we need to generalise the concept of a bounded operator on a Hilbert space.

Definition 2.2.16. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{E}$, $\mathcal{F}$ (right) Hilbert-$\mathcal{A}$-modules. A map $T : \mathcal{E} \to \mathcal{F}$ is adjointable if there is a map $T^* : \mathcal{F} \to \mathcal{E}$, such that*

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \quad \forall x \in \mathcal{E}, y \in \mathcal{F}. \tag{2.12}$$

A map $T : \mathcal{E} \to \mathcal{F}$ is bounded if

$$\|T\| := sup\{\|Tx\|_{\mathcal{F}} : \|x\|_{\mathcal{E}} \leq 1\}, \tag{2.13}$$

is finite. It can be proved that any adjointable map is linear and bounded (see [20]). The vector space of all adjointable maps $T : \mathcal{E} \to \mathcal{F}$ is denoted by $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ and if $\mathcal{F} = \mathcal{E}$ we denote $End_{\mathcal{A}}(\mathcal{E})$. Moreover, equation (2.13) defines a norm on $End_{\mathcal{A}}(\mathcal{E})$.

2.3 ALGEBRA AUTOMORPHISMS

Inner algebra automorphisms (see below) are important when we study the internal symmetry group belonging to a given spectral triple. The important result in this section is the short exact sequence (2.14). This section is based on the first part of [21], section 8.3.

Consider a real or complex involutive algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$. Associated to this algebra is the group of automorphisms Aut($\mathcal{A}$), which is defined by the set of all bijective endomorphisms ϕ of $\mathcal{A}$ with composition of maps as multiplication rule.[1]. One can check that this is indeed a group with the identity map $\mathrm{Id}_{\mathcal{A}} : a \mapsto a$ as unit element. For u a unitary element the map

$$\alpha_u : \mathcal{A} \to \mathcal{A}, \quad a \mapsto uau^*.$$

is an automorphism of $\mathcal{A}$. Indeed, α_u is a homomorphism since $\alpha_u(ab) = uabu^* = uau^*ubu^* = \alpha_u(a)\alpha_u(b)$ and $\alpha_u(a^*) = ua^*u^* = (uau^*)^* = (\alpha_u(a))^*$. It is bijective since the inverse is given by α_{u^*}: $\alpha_{u^*} \circ \alpha_u(a) = u^*uau^*u = a$ for any $a \in \mathcal{A}$. Hence $\alpha_u \in \mathrm{Aut}(A)$. An automorphisms $\alpha \in \mathrm{Aut}(A)$ that is of the form Ad u for some $u \in \mathcal{U}(\mathcal{A})$, is called an inner automorphisms. The group of all inner automorphisms is denoted by Inn($\mathcal{A}$). One can check that it is indeed a subgroup of Aut(A) because $u = 1$ gives the identity map $\mathrm{Id}_{\mathcal{A}}$, α_{u^*} is the inverse of α_u and for any $u, v \in \mathcal{U}(\mathcal{A})$ we have $\alpha(u) \circ \alpha(v)(a) = uvav^*u^* = (uv)a(uv)^* = \alpha_{uv}$. It is also a normal subgroup. For this note that any automorphism $\phi \in \mathrm{Aut}(\mathcal{A})$ preserves the group of unitaries, because

$$\phi(u)\phi(u)^* = \phi(u)\phi(u^*) = \phi(uu^*) = \phi(1_{\mathcal{A}}) = 1_{\mathcal{A}}, \quad (u \in \mathcal{U}(\mathcal{A})).$$

Then, for any $\phi \in \mathrm{Aut}(\mathcal{A})$ and any $u \in \mathcal{U}(\mathcal{A})$:

$$\alpha_{\phi(u)}(a) = \phi(u)a\phi(u)^* = \phi(u)\phi(\phi^{-1}(a))\phi(u)^* = \phi(u(\phi^{-1}(a))u^*) = \phi \circ \alpha_u \circ \phi^{-1}(a), \quad (a \in \mathcal{A}).$$

Hence $\phi \circ \alpha_u \circ \phi^{-1}$ is also an element of Inn($\mathcal{A}$). We will define the outer automorphisms by the quotient $\mathrm{Out}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$. By the above remarks we have derived the short exact sequence of groups:

$$1_{\mathcal{A}} \to \mathrm{Inn}(\mathcal{A}) \to \mathcal{A} \to \mathrm{Out}(\mathcal{A}) \to 1_{\mathcal{A}} \tag{2.14}$$

Note that for any commutative algebra the group of inner automorphisms is the trivial group so that in that case $\mathrm{Out}(\mathcal{A}) = \mathcal{A}$.

We will consider this short exact sequence in section 3.4 where we introduce the inner transformations of a spectral triple.

2.4 FIBRE BUNDLES

In this thesis fibre and vector bundles play a major role. We will use vector bundles to construct spectral triples and also gauge theories are very well-described in terms of principal fibre bundles or vector bundles. There are a lot of books on fibre bundles like [17], [19]. Books that enter the subject from a more physical point of view are [10] and [28]. In this section I have tried to make a short summary of those topics that are imporant for us in this thesis.

Definition 2.4.1. *Let E, M be smooth manifolds. A surjective smooth map $\pi : E \to M$ is called a $\mathbb{F}$-vector bundle ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) if for each point $x \in M$ the inverse image $\pi^{-1}(x)$ has the structure of a finite dimensional*

1 It is required for an endomorphism ϕ that $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$. Since ϕ is surjective, it also maps 1_A to itself.

vector space over $\mathbb{F}$. Moreover, for every $x \in M$ there exists a neighbourhood U, a finite-dimensional vector space F over $\mathbb{F}$ and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "proj_1"] \\ U & \end{tikzcd}$$

commutes, and the map ϕ_U maps $\pi^{-1}(x)$ linearily onto $\{x\} \times F$ for every $x \in U$.[2] *Then E is called the total space, M the base space and $E_x = \pi^{-1}(x)$ the fibre over x. The pairs (U, ϕ_U) are called local trivialisations of E and U is called a locally trivialising neighbourhood.*

For $\phi_U : \pi^{-1}(u) \to U \times F$ and $\phi_V : \pi^{-1}(V) \to V \times F$ two local trivialisations with $U \cap V \neq 0$, the diffeomorphism $\phi_v \phi_u^{-1}$ from $U \cap V \times F$ to itself can be we written as $\phi_v \phi_u^{-1}(x, f) = (x, g_{uv}(x) f)$ where g_{uv} is a smooth map from U to the group $\mathrm{End}(F)$. The maps g_{uv} are called transition functions and they satisfy the "co-cycle conditions"

$$\begin{aligned} g_{uu}(x) &= \text{identity}, \quad (x \in U), \\ g_{uv}(x) &= g_{vu}(x)^{-1}, \quad (x \in U \cap V), \\ g_{uv}(x) g_{vw}(x) g_{wu}(x) &= \mathrm{Id} \quad (x \in U \cap V \cap W). \end{aligned}$$

Remark 2.4.2. There exists a similar definition of a group bundle $\pi : E \to M$. For the group bundle the same definition as for a vector bundle is used with the exception that the fibres $\pi^{-1}(x)$ are Lie-groups. Moreover, F is taken to be a Lie-group and the local trivialisations (U, ϕ_U) are now required to give group isomorphisms between $\pi^{-1}(x)$ and F. If the fibres of E only carry the structure of a manifold (no further structure), we just speak of a fibre bundle $\pi : E \to M$.

Example 2.4.3. A simple example of a fibre bundle is $\pi : E \to M$ with $E = M \times H$ and $\pi : E \to M$ the projection on the first factor. Since this bundle is globally trivialised, the bundle is called trivial.

Example 2.4.4. Let M be the 2-sphere S^2. For $x \in S^2$ we take the fibre of x to be the tangent surface of the sphere at the point x. These tangent spaces can be glued together to form a non-trivial real vector bundle over S^2 with fibre $\mathbb{R}^2$. This bundle is called the tangent bundle TS^2 of S^2. More generally, for any smooth manifold M one can construct the tangent bundle TM, where the fibre of x is precisely the tangent space $T_x(M)$ of x. Similarly, the cotangent bundle can be defined. This is a vector bundle over M where the fibres are the cotangent space $T_x^*(M)$ $(x \in M)$.

Definition 2.4.5. *Let $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ be two vector bundles over a manifold M. Then the Whitney sum of the bundles E_1 and E_2 is defined by $E := E_1 \oplus E_2 = \{(p, q) \in E_1 \oplus E_2 | \pi_1(p) = \pi_2(q)$. The projection map $\eta : E \to M$ is defined by $\pi(p, q) = \pi_1(p)$. The fibre E_x ($x \in M$) is in natural bijection with $(E_1)_x \oplus (E_2)_x$ giving E_x the structure of a vector space.*

Similarly, one can introduce a tensor product bundle $E := E_1 \otimes E_2$ where the fibre E_x is the tensor product of the fibres $(E_1)_x$ and $(E_2)_x$.

Definition 2.4.6. *Let E_1 and E_2 be two fibre bundles over M with projections π_1 and π_2 respectively. A homomorphism $f : E_1 \to E_2$ is a smooth map such that the diagram*

$$\begin{tikzcd} E_1 \arrow[rr, "f"'] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

2 if $\mathbb{F} = \mathbb{C}$, the differentiable structure on $\mathbb{C}^k$ is obtained by considering it as a real $2k$-dimensional vector space.

commutes. In fact, this means that f maps the fibre $(E_1)_x$ into the fibre $(E_2)_x$ for each $x \in M$. If E_1, E_2 are vector bundles we add the requirement that $f : (E_1)_x \to (E_2)_x$ is a linear map for each $x \in M$. In general, if f is a diffeomorphism, the fibre bundles E_1, E_2 are said to be equivalent.

Definition 2.4.7. *Let $\pi : E \to M$ be a fibre bundle. Then a smooth mapping $s : M \to E$ is called a global section if $\pi s = Id_M$. If such a section only exists on a open set $U \subset M$, s is called a local section. If E is a vector bundle and $(s_1, \dots s_n)$ are sections such that $(s_1(x), \dots s_n(x))$ form a basis of E_x for all x in some open neighbourhood U, then $(s_1, \dots s_n)$ is called a local frame on U (of the bundle E).*

Example 2.4.8. With these definitions a (contravariant) vector field on a manifold M is nothing else than a smooth section $Y : M \to TM$. In the same way we can identify the covariant vector fields with the sections of the cotangent bundle T^*M.

If $\pi : E \to M$ is a vector bundle, the space of sections $\Gamma(M, E)$ is a module over $C^\infty(M)$. The module structure is given by

$$(\cdot, \cdot) : \Gamma(M, E) \times C^\infty(M) \to \Gamma(M, E), \quad (s, f) \mapsto sf, \text{ where } sf(x) = s(x)f(x), (x \in M).$$

The following lemma will be useful many times.

Lemma 2.4.9. *For a given manifold M and vector bundles E_1, E_2 over M, the space $\Gamma(M, E_1 \otimes E_2)$ is isomorphic with $\Gamma(M, E_1) \otimes_{C^\infty(M)} \Gamma(M, E_2)$ as $C^\infty(M)$-modules.*

Proof. We will only prove that this holds for trivial bundles. Any $s \in \Gamma(M, E_1)$ is of the form $s = \sum_j f_j s_j$ where $(s_1, \dots, s_k)$ is a frame of the bundle E_1, and the f_j are elements of $C^\infty(M)$. Therefore, the sections $(s_1, \dots s_k)$ generate $\Gamma(M, E_1)$ freely as a $C^\infty(M)$-module. The same is true for a frame $(t_1, \dots t_l)$ of E_2. Then $\Gamma(M, E_1 \otimes E_2)$ is generated freely by $\{s_i \otimes t_j : i = 1, \dots, r; j = 1, \dots, l\}$ as a $C^\infty(M)$-module since for every $x \in M$, the set $\{s_i(x) \otimes t_j(x)\}$ is a basis for $(E_1)_x \otimes (E_2)_x$.

For the general case, see for instance [15], propostion 2.6. □

From now on we will tacitly use this isomorphism and identify $\Gamma(M, E_1 \otimes E_2)$ with $\Gamma(M, E_1) \otimes_{C^\infty(M)} \Gamma(M, E_2)$ as $C^\infty(M)$-modules.

We now turn to another class of fibre bundles.

Definition 2.4.10 (see [3], Definition 1.1.1)**.** *A principal fibre bundle (P, M, G, π) consists of smooth manifolds P, M and a Lie group G together with a smooth surjective projection map $\pi : P \to M$, where the Lie group G has a free smooth right action on P and $\pi^{-1}(\pi(p)) = \{pg : g \in G\}$. If $x \in M$, then $\pi^{-1}(x)$ is called the fibre above x. Furthermore, for each $x \in M$ there exists an open set $U \ni x$ and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times G$ of the form $\phi_U(p) = (\pi(p), s_u(p))$, where $s_U : P \to G$ has the property $s_U(pg) = s_U(p)g$ for all $g \in G$, $p \in \pi^{-1}(U)$. The map ϕ_U is called a local trivialisation (or, in physics, a choice of gauge).*

Let $\phi_U : \pi^{-1}(U) \to U \times G$ and $T_V : \pi^{-1}(V) \to V \times G$ be two local trivialisations of a principal fibre bundle $\pi : P \to M$. The transition function from U to V is the map $g_{uv} : U \cap V \to G$ defined, for $x \in \pi(p) \in U \cap V$, by $g_{uv}(x) = s_v(p)s_u(p)^{-1}$. This is independent of the choice of $p \in \pi^{-1}(x)$ and the transition functions also satisfy the cocycle "conditions" (2.15).

Definition 2.4.11. *Two principal bundles (P_1, M, G, π_1), (P_2, M, G, π_2) are called isomorphic if there exists a diffeomorphism $f : P_1 \to P_2$ such that the diagram*

$$\begin{tikzcd} P_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & P_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

commutes, and $f(pg) = f(p)g$, i.e. f is compatible with the action of G.

Proposition 2.4.12. *A principal bundle (P, M, G, π) is a trivial bundle if and only if it admits a global section $M \to P$.*

Proof. Let $\sigma : U \to P$ be a given local section. Then define the smooth map $\phi_U : \pi^{-1}(U) \to U \times G$ by $\phi_U(\sigma(x)g) = (\pi(p), g)$. We verify that indeed $\phi_U(\sigma(x)gh) = (\pi(p), gh)$ so that ϕ_U is a local trivialisation. Conversely, let $\phi_U : \pi^{-1}(U) \to U \times G$ be a given local trivialisation, then define a section $\sigma : U \to \pi^{-1}(U)$ by $\sigma(x) = \phi_u^{-1}(x, e)$. Thus, we see that local sections correspond to local trivialisations. In particular, this implies that a global section exists if and only if the bundle is trivial. □

Example 2.4.13. Let M be a smooth manifold of dimension n. Denote by P_x the set of all bases of the tangent space $T_x(M)$. An example of a principal fibre bundle over M is the frame bundle $F(M)$. The fibres of $F(M)$ are the P_x glued together in a suitable way and the group $GL(n)$ acts on P_x by change of basis. If M is a (oriented) Riemannian manifold then denote $(S)O_x$ for the set of all (oriented) orthonormal bases of $T_x(M)$. Then the subset of $F(M)$ where the fibres consist only of (oriented) orthonormal bases, glue together to form a principal $(S)O(n)$-bundle $(S)O(M)$. See also [19], Chapter 1, Example 5.7.

Proposition 2.4.14 ([19], p.54). *Let $\pi : P \to M$ be a (smooth) principal G-bundle over a space M and let F be a manifold on which G acts smoothly on the left. Define a free left action ρ of $g \in G$ on the product $P \times F$ by*

$$(p, f) \mapsto (pg, g^{-1}f) \quad \textit{where} \quad p \in P, \quad f \in F. \tag{2.15}$$

Define $E = P \times_G F := (P \times F)/G$ to be the quotient space (the space of orbits) of this action. Since the action of G on P conserves fibres, we see that the projection $P \times F \xrightarrow{\pi} M$ induces a mapping

$$\pi_\rho : P \times_G F \to M \quad \pi_\rho[(p, f)] = \pi(p) \tag{2.16}$$

Then $\pi_E : P \times_G G$ is a fibre bundle with fibre F.

Proof. A differentiable structure on $P \times_G F$ will be defined soon. First, for each $x \in M$, define $E_x := \pi_E^{-1}(x)$ to be the fibre of x. Every point of x has a neighbourhood U and a map ϕ such that $\phi : \pi^{-1}(U) \to U \times G$ is a diffeomorphism. If $\phi(p) = (\pi(p), g)$, then the element (p, f) is mapped to $(\pi(p), g, f)$ and $(ph, h^{-1} \cdot f)$ is mapped to $(\pi(p), gh, h^{-1} \cdot f)$ under the isomorphism $\phi \times \mathrm{Id}$. So, the induced equivalence relation on $U \times G \times F$ is given by $[(x, g, f)] = \{(x, gh, h^{-1}f) | h \in G\}$, ($x \in M, g \in G, f \in F$), where $[u]$ denotes the equivalence class of the element u. Call this equivalence relation R. It follows that $\pi_E^{-1}(U) \cong (U \times G \times F)/R \cong U \times F$, where the last isomorphism is given by $[x, \mathrm{id}_G, f] \mapsto (x, f)$ for each $x \in U, f \in F$. This way local trivialisations of P induce local trivialisations of E. We introduce a differentiable structure in E by the requirement that $\pi_E^{-1}(U)$ is a open submanifold of E that is diffeomorpic with $U \times F$. The projection π_E is then a smooth map from E to M.

□

Remark 2.4.15. Let $(\tilde{U}_i, \tilde{\phi}_{\tilde{U}_i})$ be locally trivialising neighbourhoods of the bundle P and denote $\tilde{g}_{ij}$ for the transition functions from $\tilde{U}_i$ to $\tilde{U}_j$. Assume that we have a smooth left action $\rho : G \to \mathrm{Aut}(F)$ so that we can construct $E = P \times_G F$. The local trivialisations $(\tilde{U}_i, \tilde{\phi}_{\tilde{U}_i})$ induce local trivialisations (U_i, ϕ_{U_i}) of the bundle E in the way explained in the above proposition. The transition functions of E between each pair U_i, U_j are then given by $\rho(g_{ij})$.

If E is a fibre bundle that can be constructed from a principal bundle P by the above construction, then E is called an associated bundle of P.

Example 2.4.16. For M a smooth manifold of dimension n, the tangent bundle TM is an associated bundle of the frame bundle $F(M)$. Here $GL(n)$ acts on $\mathbb{R}^n$ in the obvious way. If (M, g) is a oriented Riemannian manifold, then the tangent bundle TM is not only an associated bundle of $SO(M)$ but the metric g on TM can also be recovered from $SO(M)$. This is done by requiring that two elements p, q in a fibre $T_x(M)$ $(x \in M)$ are orthonormal to each other if and only if they are orthonormal to each other under a local trivialisation $\phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ of TM, where $\mathbb{R}^n$ is endowed with the standard inner product. Since the transition functions take values in $SO(n)$, this is independent of the choice of the local trivialisation.

Let E be a fibre bundle with structure group G. The following theorem shows how a principal G-bundle P can be constructed from E so that E is an associated bundle of P.

Theorem 2.4.17 (Reconstruction Theorem)**.** *Let M be a compact manifold and G be a Lie-group and $\{U_i\}_{i\in I}$ be a open covering of M. Suppose that for each for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there is a smooth map*

$$g_{ij} : U_i \cap U_j \to G, \tag{2.17}$$

and these maps have the property that , if $U_i \cap U_j \cap U_k \neq \emptyset$, then

$$g_{ij}(x)g_{jk}(x)g_{ki}(x) = e \tag{2.18}$$

for all $x \in U_i \cap U_j \cap U_k$. Then there exists a unique principal G-bundle P over M with has the $\{U_i\}$ as trivialising neighbourhoods and the g_{ij} as transition functions.

Proof. The proof of this theorem can be found in [19], Chapter 1, Propostion 5.2. □

Corollary 2.4.18. Let M be a compact manifold and let V be a real or complex vector space. Let G be be a Lie-group acting on V (action denoted by ρ) and let $\{U_i\}_{i\in I}$ be a open covering of M. Suppose that for each for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there is a smooth map

$$g_{ij} : U_i \cap U_j \to G, \tag{2.19}$$

and these maps have the property that , if $U_i \cap U_j \cap U_k \neq \emptyset$, then

$$g_{ij}(x)g_{jk}(x)g_{ki}(x) = e \tag{2.20}$$

for all $x \in U_i \cap U_j \cap U_k$. Then there exists a unique vector bundle E with fibre V, over M with has the $\{U_i\}$ as trivialising neighbourhoods and the $\rho(g_{ij})$ as transition functions.

Proof. Use Theorem 2.4.17 to construct a principal G-bundle. Then construct $E = P \times_G V$. □

2.5 CONNECTIONS

In this section we will define the notion of connections. Connections can be defined on both principal bundles and vector bundles. Although the definitions seem quite different, they are in fact related in the sense that connections on principal bundles induce connections on associated vector bundles and vice versa. Also the material in this section is based on the books mentioned in the previous section, namely [10], [17], [19] and [28].

2.5.1 *Connections on principal bundles*

In gauge theories connections on a principal G-bundle are interpreted as gauge potentials. The first definition of a connection on a principal bundle is the usual one. In the case that G is a matric Lie group we will present another (equivalent) definition of a connection since that one is more conventient for the purposes of this thesis.

Definition 2.5.1. *Let P be a principal G bundle. A connection on P is a $\mathfrak{g}$-valued 1-form ω on P, with $\mathfrak{g}$ the Lie-algebra belonging to the Lie-group G, such that the following conditions hold.*

- *For $A \in \mathfrak{g}$ define A^* to be the vector field at P given by*

  $$A_p^* = \frac{d}{dt} p \exp(tA)|_{t=0}, \tag{2.21}$$

  then $\omega(A_p^*) = A$.

- *$R_g^* \omega = Ad_{g^{-1}}(\omega)$ for all $g \in G$.*

Here R_g^* is the pull-back of the smooth map $R_g : P \to P$, $R_g(p) = pg$.

For a matrix Lie group G, we have the following equivalent definition (see [3]).

Definition 2.5.2. *Let G be a matrix Lie-group and let P be a principal G-bundle. A connection assigns to each local trivialisation $\phi_U : \pi^{-1}(U) \to U \times G$ a $\mathfrak{g}$-valued one-form ω_u on U. If ϕ_V is another local trivialisation and $g_{uv} : U \cap V \to G$ is the transition function from (U, ϕ_U) to (V, ϕ_V), then we require the following transformation rule*

$$\omega_v = g_{uv}^{-1} dg_{uv} + g_{uv}^{-1} \omega_u g_{uv}. \tag{2.22}$$

If P is a principal G-bundle and ω a connection 1-form on P as defined in Definition 2.5.1, then we construct $\mathfrak{g}$-valued 1-forms on the locally trivialising neighbourhoods U_i of P as follows. Associated to a locally trivialising neighbourhood U_i is a local section $s_i : U \to P$ on U, see Proposition 2.4.12. By pulling back ω through s_i we get a $\mathfrak{g}$-valued 1-form $\omega_i := s_i^* \omega$ on U_i. Given such 1-forms ω_i and ω_j one can show that on $U_i \cap U_j$ these are related by the transformation rule (2.22). Conversely, one can show that a family of 1-forms ω_i on locally trivilialising neighbourhoods U_i that related by the transformation rule (2.22), define a connection 1-form on the bundle B. The local 1-forms ω_i are precisely the gauge potentials one encounters in physics.

2.5.2 *Connections on vector bundles*

In this subsection we will define connections on vector bundles. In physics connections on vector bundles are also known as covariant derivatives.

Definition 2.5.3. *A connection on a complex vector bundle $E \to M$, with fibre $F \cong \mathbb{C}^k$ for some $k \in \mathbb{N}$, is a smooth linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that the Leibniz rule*

$$\nabla(fs) = df \otimes s + f(\nabla s), \tag{2.23}$$

holds for all smooth functions f on M and all smooth sections s of E.

Let ∇, ∇' be two connections on E, then $(\nabla - \nabla')(fs) = \nabla(fs) - \nabla'(fs) = f(\nabla s) - f(\nabla' s) = f(\nabla - \nabla')s$. Therefore $\nabla - \nabla'$ is a $C^\infty(M)$-linear map from $\Gamma(M, E)$ to $\Gamma(M, TM^* \otimes E)$, that is $\nabla - \nabla'$ is a one-form ω on M with values in the endomorphism bundle $\text{End}(E) := E \otimes E^*$. Let E be a vector bundle of rank n. On some local trivialising neighbourhood U, any section of E can be identified with an n-tuple of smooth functions $f = (f_1, \ldots, f_n)$ by writing $s : U \to \mathbb{C}^n$, $s = \sum_{i=1}^n f_i e_i$, where $\{e_i\}$ is the canonical basis of $\mathbb{C}^n$. Then the exterior derivative, defined by $d(f_1, \ldots, f_n) = (df_1, \ldots, df_n)$ is a connection on U. Therefore, any connection ∇ on the vector bundle E can locally expressed as:

$$d + \omega, \quad \omega \in \Omega^1(M) \otimes \text{End}(\mathbb{C}^n). \tag{2.24}$$

Now, let (ϕ_V, V) be another local trivialisation such that $U \cap V \neq \emptyset$. Write $s = f' = (f'_1, \ldots, f'_n)$ and $\nabla = d + \omega_v$ in the coordinates of V. We compute

$$(d + \omega_v)f' = (d + \omega_v)(g_{uv}f) = dg_{uv} \cdot f + g_{uv}df + \omega_v g_{uv} f. \tag{2.25}$$

This should be the same as evaluating the ∇s on U and then transform it to V using the transition function g_{uv}:

$$dg_{uv} \cdot f + g_{uv}df + \omega_v g_{uv} f = g_{uv}df + g_{uv}\omega_u f \tag{2.26}$$

so that we get the transformation law:

$$\omega_u = g_{uv}^{-1}(dg_{uv}) + g_{uv}^{-1}\omega_v g_{uv}, \tag{2.27}$$

This leads to the following equivalent definition of a connection

Definition 2.5.4. *Let M be a smooth manifold and $\pi : E \to M$ a vector bundle with local trivialisations (U_i, ϕ_i). A connection on E is given by a set End$(\mathbb{C})$-valued 1-form ω_i on U_i such that the ω_i obey the following transformation law:*

$$\omega_i = g_{ij}^{-1}dg_{ij} + g_{ij}^{-1}\omega_j g_{ij}, \tag{2.28}$$

where the g_{ij} are the transition functions.

2.6 FINITE-DIMENSIONAL CLIFFORD ALGEBRAS AND SPIN REPRESENTATIONS

In this section we want to define the spin group Spin(n) for every natural number n. For our purposes it is the easiest to introduce the $Spin(n)$-group as a subgroup of the invertible elements of the complexified Clifford algebra $\mathbb{C}l(\mathbb{R}^n)$. Moreover, we will introduce a representation of $\mathbb{C}l(\mathbb{R}^n)$ that, when restricted to $Spin(n)$, gives a representation of the $Spin(n)$-group. The text in this section is based on the books [11], [23], [15].

2.6.1 *Finite-dimensional Clifford Algebras*

In this subsection we introduce the Clifford algebras for real finite dimensional vector spaces endowed with a symmetric bilinear form.

Definition 2.6.1. *Let V be a vector space over an arbitrary field $\mathbb{F}$. The tensor algebra TV is defined by*

$$TV = \oplus_{k \in \mathcal{N}} T^k V \tag{2.29}$$

where $T^0V = \mathbb{F}$, $T^1V = V$ and $T^kV = V^{\otimes^k} = V \otimes ... \otimes V$ k times). This is a graded algebra with the elements of T^kV as the homogeneous elements of order k.

There are some important factor algebras of the tensor algebra which we will now discuss shortly. The first one is the symmetric tensor algebra. For this define the two-sided ideal of TV:

$$I = \text{span}\{TV \otimes (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes TV; v_1, v_2 \in V\} \tag{2.30}$$

Then $SV = TV/I$ is a commutative algebra, called the symmetric tensor algebra of V. Similarly we can define the anti-symmetric algebra by $AV = TV/J$ where

$$J = \text{span}\{TV \otimes (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes TV; v_1, v_2 \in V\} \tag{2.31}$$

Since the generators of the ideals I, J are homogeneous of order 2, a $\mathbb{Z}$-grading is induced on SV and AV. Denote S_k, A_k for the vector space of homogeneous elements of order k in SV, AV respectively.

Now, if V is of dimension n and $\{e_1, \dots, e_n\}$ is an orthonormal basis of V, then

$$e_{i_1} \otimes e_{i_2} \cdots e_{i_k}, \quad (i_1, \dots, i_k \in \{1, \dots, n\}) \tag{2.32}$$

is a basis of $T^k V$. If we take the quotient to J and denote the product $(v_1 + J)\hat{\otimes}(v_2 + j_2)$ in AV simply by $v_1 \otimes v_2$ where v_1, v_2 are in T, then $A^k V$ is spanned by

$$e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n. \tag{2.33}$$

These elements are also independent, so that

$$\text{dim} A^k V = \binom{n}{k}, \quad k \leq n. \tag{2.34}$$

Of course, $\text{dim} A^k = 0$ if $k > n$. The dimension of the exterior algebra of V is therefore 2^n.

We now give the definition of the Clifford algebra for a real vector space V endowed with s symmetric bilinear form g. *From now on all vector spaces V are real and finite-dimensional unless otherwise stated.*

Definition 2.6.2. *Let V be a given vector space and g be a symmetric bilinear form on V. Then we define the Clifford algebra of (V, g) to be the following quotient space of the tensor algebra:*

$$Cl(V, g) = TV/\{TV \otimes (v \otimes w + w \otimes v - 2g(v, w) \otimes TV\}, \quad (v, w \in\in V). \tag{2.35}$$

Note that the Clifford algebra reduces to the exterior algebra when $g(v, w) = 0$ for all v, $w \in V$. The $\mathbb{Z}$-grading of the tensor algebra passes to a $\mathbb{Z}_2$-grading on the Clifford-algebra. The elements of even degree form a sub-algebra $Cl(V, g)^+$. The set of odd elements is denoted by $Cl(V, g)^-$. The product in the Clifford algebra will be denoted by $\cdot$ or by juxtaposition. Note that V can be considered as a subset of $\mathbb{C}l(V, g)$ in a natural way.

Let $\{v_1, \dots, v_n\}$ be a given basis of the real vector space (V, g). Then the products of the form $v_{i_1} \cdot \dots \cdot v_{i_k}$ with $k \in \mathbb{N}$, $0 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, form a basis of $\text{Cl}(V, g)$. Therefore, for any basis $\{v_1, \dots, v_n\}$ of (V, g) the Clifford algebra $\text{Cl}(V, g)$ is generated as an algebra by the elements $(v_1, \dots, v_n)$ subject to the conditions $v_i v_j + v_j v_i = g(v_i, v_j) =: g_{ij}$ for all $0 \leq i, j \leq n$. This description of a Clifford algebra is mostly used in physics.

Remark 2.6.3. Although, the exterior algebra of V and the Clifford algebra of V are not isomorphic as algebras, they are naturally isomorphic as vector spaces by matching bases:

$$\Lambda(V) \to \text{Cl}(V, g) \quad v_1 \otimes \cdots \otimes v_k \mapsto v_1 \cdot v_2 \cdot \cdots \cdot v_k \tag{2.36}$$

Definition 2.6.4. *Let V be a vector space with symmetric bilinear form g. On Cl(V,g) we define the grading operator χ which is determined by $\chi(a) = a$ when $a \in Cl^+(V,g)$ and $\chi(a) = -a$ when $a \in Cl^-(V,g)$. The operator χ is extended to Cl(V,g) by linearity.*

We can complexify the Clifford algebra $\mathrm{Cl}(V)$ by defining the complexified Clifford algebra as

$$\mathbb{C}l(V,g) = \mathrm{Cl}(V,g) \otimes_{\mathbb{R}} \mathbb{C}, \tag{2.37}$$

which is now considered to be a complex space. If $a = (b, \lambda) \in \mathbb{C}l(V,g)$, we define $\overline{a} := (b, \overline{\lambda})$ where the bar on λ denotes complex conjugation.

In the rest of this section we will mostly consider the Clifford algebra of $(\mathbb{R}^n, \tilde{g})$, where $\tilde{g}$ is the standard inner product on $\mathbb{R}^n$. This Clifford algebra is simply denoted by $\mathrm{Cl}(\mathbb{R}^n)$. For the complexified Clifford algebra we will denote $\mathbb{C}l(\mathbb{R}^n)$.

It is surprisingly easy to determine the structure of $\mathbb{C}l(\mathbb{R}^n)$. In this proposition $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{R}^n$.

Proposition 2.6.5 ([15], Lemma 5.5)**.** *There is a non-canonical isomorphism of algebras*

$$\mathbb{C}l(\mathbb{R}^n) \simeq \begin{cases} M_{2^{n/2}}(\mathbb{C}), & n \text{ even} \\ M_{2^{(n-1)2}} \oplus M_{2^{(n-1)/2}}, & n \text{ odd}. \end{cases} \tag{2.38}$$

Denote $\mathbb{S}(n) = \mathbb{C}^{2^{[n/2]}}$, For n is even the isomorphism $\gamma : \mathbb{C}l(\mathbb{R}^n) \to End(\mathbb{S}(n))$ is called the Clifford representation. If n is odd the Clifford representation is defined to be the map $\gamma : \mathbb{C}l(\mathbb{R}^n) \to End(\mathbb{S}(n))$ given by the above isomorphism for n is odd composed with the projection on the first factor. The action of an element $\mathbb{C}l(\mathbb{R}^n)$ on $\mathbb{S}(n)$ is also called Clifford multiplication.

Proof. We will prove this by induction. We remark that $\mathbb{R} \oplus \mathbb{R} \simeq \mathrm{Cl}(\mathbb{R})$ by the identification $\mathbb{R} \oplus \mathbb{R} \ni (a,b) \mapsto \frac{1}{2}(a+b) + \frac{1}{2}(a-b)e_1 \in \mathrm{Cl}(\mathbb{R})$, which is clearly a linear and bijective map. The map also preserves multiplication as one can check. Therefore, $\mathbb{C}l(\mathbb{R}^1) = (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$. For $n = 2$ we define the algebra-isomorphism $\mathbb{C}l(\mathbb{R}^2) \to M_2(\mathbb{C})$ by identifying $1 \mapsto \mathrm{Id}_2$ and $e_i \mapsto \sigma_i$ with σ_i the i^{th} Pauli matrix for $i = 1, 2$. Note that $e_1 e_2$ is then automatically mapped to σ_3. These four matrices form a basis for $M_2(\mathbb{C})$ and they also satisfy the same commutation relations as the generators for the Clifford algebra. Thus, $\mathbb{C}l(\mathbb{R}^2) \simeq M_2(\mathbb{C})$.
We are now done if we can show that

$$\mathbb{C}l(\mathbb{R}^{n+2}) \simeq \mathbb{C}l(\mathbb{R}^n) \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{2.39}$$

An algebra isomorphism between these spaces is the map $\phi : \mathbb{C}l(\mathbb{R}^{n+2}) \to \mathbb{C}l(\mathbb{R}^n) \otimes_{\mathbb{C}} M_2(\mathbb{C})$ induced by

$$\phi(e_{n+1}) = 1 \otimes \sigma_2, \quad \phi(e_{n+2}) = 1 \otimes \sigma_3, \quad \phi(e_k) = e_k \otimes \sigma_1, \quad (k \leq n). \tag{2.40}$$

where $\sigma_{1,2,3}$ are the Pauli-spin matrices. That the map ϕ is a bijection can be checked by showing that products of $\phi(e_j)$ produce all the basis elements of $\mathbb{C}l(\mathbb{R}^n) \otimes_{\mathbb{C}} M_2(\mathbb{C})$ and counting dimensions. Using the anti-commutation relations of the Pauli-matrices, one can check that all the $\phi(e_j)$ anti-commute and that $\phi(e_j)^2 = 1$ so that the correct product rules are satisfied as well. This proves the isomorphism (2.39). $\square$

Let V again be an arbitrary real finite-dimensional vector space with symmetric bilinear form g. Assume that there is an orientation given on (V, g). From now on, we will assume that $\{e_1, \ldots, e_n\}$ is an orthonormal basis that is compatible with the given orientation.

Definition 2.6.6. *Write $n = 2m$ or $n = 2m + 1$ according as n is even or odd. Let $\{e_1, \cdots, e_n\}$ be an oriented orthonormal basis of the vector space V. The chirality element of $\mathbb{C}l(V, g)$ is defined by*

$$\gamma^5 := (-i)^m e_1 e_2 \cdots e_n. \tag{2.41}$$

If $\{e'_1, \cdots, e'_n\}$ is another oriented orthonormal basis of V, then $e'_j = \sum_{i=1}^n h_{ji} e_i$, where h is an orthonormal matrix of determinant $+1$, so $(-i)^m e'_1 \cdots e'_n = det(h) \cdot \gamma^5 = \gamma^5$. Therefore, γ^5 does not depend on the chosen oriented orthonormal basis. Since $n(n-1)/2 = m(2m \mp 1)$ (depending on whether n is even or odd), we have that $(\gamma^5)^* = (-1)^m (-1)^{n(n-1)/2} \gamma^5 = \gamma^5$ and $(\gamma^5)^2 = (\gamma^5)^* \gamma^5 = e_n \cdots e_1 e_1 \cdots e_n = 1$.

Note that γ^5 anti-commutes or commutes with each element e_j according as n is even or odd, and therefore γ^5 anti-commutes or commutes with each element $v \in V$. From the isomorpism (2.38) we see that the centre of even Clifford algebras is a one-dimensional vector space and thus consists of scalars. In the case that n is odd we see, again from (2.38), that the dimension of the centre of $\mathbb{C}l(V, g)$ is two-dimensional so it is generated by $1, \gamma^5$.

2.6.2 *The spin group*

We will now define the spin group $Spin(n)$ where n is a natural number using Clifford-algebras. We follow the steps carried out in [15], section 5.2, to introduce the spin-group. First, we introduce some notation. Let $v_1, \ldots, v_k$ be vectors in V, then we define the following operation: $(v_1, \ldots v_k)^! = v_k \ldots v_1$, and extend it by linearity to the whole Clifford algebra. Then for any $a \in \mathbb{C}l(V, g)$, a^* is defined as $a^* := (\bar{a})^!$.

The set of invertible elements of $\mathbb{C}l(V, g)$ form a group $\mathbb{C}l(V, g)^\times$ under the Clifford multiplication. There are some important subgroups of $\mathbb{C}l(V, g)^\times$. We call an element $u \in \mathbb{C}l(V, g)$ unitary if $uu^* = u^*u = 1$. If v_1, v_2 are two unitary vectors of $V^{\mathbb{C}} (= V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV)$, then their product v_1, v_2 is a unitary element of $\mathbb{C}l(V, g)^+$. The group generated by all such (even) unitaries is called the spin^c-group of $\mathbb{C}l(V, g)$ and is denoted by Spin$^c(V)$.

For $a \in \mathbb{C}l(V, g)^\times$ consider the map $\phi(a) : \mathbb{C}l(V, g) \to \mathbb{C}l(V, g), \quad b \mapsto \chi(a)ba^{-1}$. For any vector $x \in V$ and v a unit vector: $\phi(v)x = (xv - 2g(v, x))v = x - 2g(v, x)v$, so $\phi(v)$ preserves the subspace V of $\mathbb{C}l(V, g)$ and its action on V is just the reflection in the hyper-surface orthogonal to v.

In $V^{\mathbb{C}}$ any unit vector w is equal to $w = \lambda v$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ so that $\phi(w) = \phi(v)$. These reflections generate the orthonormal group $O(V, g)$ of (V, g). The products of an even number of reflections form the rotation subgroup $SO(V, g)$. Then for each $u \in$ Spin$^c(V)$, the map $\phi(u) : x \mapsto \chi(u)xu^{-1} = uxu^{-1}$ is a rotation of V, so we may regard ϕ as a surjective homomorphism from Spin$^c(V)$ to $SO(V)$. We determine the kernel: assume $u \in$ kerϕ. Then u commutes with all $v \in V$, so it is an even element in the centre of $\mathbb{C}l(V, g)$ and is therefore a scalar. Hence, ker$\phi \simeq U(1)$ and we have the following exact sequence:

$$1 \to U(1) \to \mathrm{Spin}^c(V) \to SO(V) \to 1. \tag{2.42}$$

We can restrict this homomorphism further to the subgroup of unitary elements for which also holds that $u^! u = 1$. Such elements are called *real*. This group is called the spin group Spin(V). Since any element $v \in$ Spin(V) can be written as $\lambda \cdot u$ where $u \in$ spin^c and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the restriction of ϕ to Spin(V) is still surjective. The kernel of $\phi_{\mathrm{Spin}(V)}$is determined by all scalars x for which $x^2 = 1$. Hence, ker$(\phi) = \{\pm 1\}$, and we have the following exact sequence:

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to SO(V) \to 1. \tag{2.43}$$

We now define the spin groups.

Definition 2.6.7. *The $Spin(n)$-group is the group of even, unitary elements and real elements of the Clifford algebra $\mathbb{C}l(\mathbb{R}^n)$. The $Spin(n)$-group is a double covering of the $SO(n)$.*

In Proposition 2.6.5 we have developed the Clifford representations

$$\gamma : \mathbb{C}l(\mathbb{R}^n) \to \text{End}(\mathbb{S}(n)), \tag{2.44}$$

where $\mathbb{S} = \mathbb{C}^{2^{[n/2]}}$. Since the group Spin$(n)$ sits inside the invertible elements of the Clifford algebra, we obtain the spin representation

$$\gamma : \text{Spin}(n) \to \text{Aut}(\mathbb{S}). \tag{2.45}$$

In odd dimensions this representation is irreducible. In even dimension, however, it is not. Since $\gamma(\gamma^5)^2 = 1$, the space $\mathbb{S}$ splits into a positive $\mathbb{S}^+$ and negative piece $\mathbb{S}^-$. These are invariant subspaces of the spin representation. This is shown as follows: let $s \in \mathbb{S}^+(\mathbb{S}^-)$ and let v be a vector in $\mathbb{C}l(V, g)$. Then $\gamma(\gamma^5)s = \pm s$. Now, $\gamma(\gamma^5)\gamma(v)s = -\gamma(v)\gamma(\gamma^5)s = \mp\gamma(v)s$, so that $\gamma(v)s$ is an element of $\mathbb{S}^+(\mathbb{S}^-)$. Thus, $\gamma(v)$ interchanges S^+ and S^-. Since Spin(n) only consists of even elements of $\mathbb{C}l(V, g)$ the representation c of Spin(n) maps $\mathbb{S}^+$ to $\mathbb{S}^+$ and $\mathbb{S}^-$ to $\mathbb{S}^-$. The spaces $\mathbb{S}^+$ and $\mathbb{S}^-$ *are* invariant subspaces for the spin representation.

Summarising, we have a Clifford representation of $\mathbb{C}l(\mathbb{R}^n)$ on the spinor space $\mathbb{S}(n)$ induced by the isomorphisms in Proposition 2.6.5. The spin group $Spin(n)$ is a subset of the invertible elements. By restricting the Clifford representation to the spin group we get the spin group representation

$$\gamma : \text{Spin}(n) \to \text{Aut}(\mathbb{S}). \tag{2.46}$$

It is important to notice that the Clifford representation is not canonical. One could easily construct another isomorphism between $\mathbb{C}l(\mathbb{R}^n)$ and $M_{2^{n/2}}(\mathbb{C})$ or $M_{2^{(n-1)2}}$ depending on whether n is even or odd. This would give another Clifford and spin representation. In the rest of this thesis we will work with the representation established in Proposition 2.6.5.

To conclude this section we prove a very important formula. It is this formula that allows us to construct the Dirac-operator in Section 2.7. We denote λ for the double covering homomorphism $Spin(n) \to SO(n)$.

Proposition 2.6.8. *Given $g \in Spin(n)$, $v \in V$, then*

$$g \cdot (v \cdot \psi) = \lambda(g)v \cdot (g \cdot \psi) \tag{2.47}$$

for all $\psi \in S$. Here $\cdot$ denotes Clifford multiplication.

Proof. We just check the formula.

$$g \cdot (v \cdot \psi) = \gamma(g)\gamma(v)\gamma(g^{-1})\gamma(g)\psi = \gamma(gvg^{-1})\gamma(g)\psi = \gamma(\lambda(g)v)\gamma(g)\psi = \lambda(g)v \cdot (g \cdot \psi). \tag{2.48}$$

This proves the proposition. □

2.7 SPINOR BUNDLE AND DIRAC-OPERATOR

In this section M will be a oriented Riemannian manifold of dimension n. We will introduce the spinor bundle and the so-called Dirac-operator on this spinor bundle. These structures are introduced in order to describe spinors on manifolds other than $\mathbb{R}^4$. Not all manifolds allow such a description and manifolds that do will be called spin manifolds ([23]). As first shown by Lichnerowicz ([24]) the Dirac-operator of arbitrary manifolds has properties similar to the Dirac-operator in Minkowski-space, namely it is a first order differential operator that is more or less a square-root of the Laplacian.

2.7.1 *Spinor bundle*

In this section we will introduce the spinor bundle for Riemannian manifolds ([11]).

Definition 2.7.1. *A spin structure on an oriented Riemannian manifold (M,g) is a principal $Spin(n)$ bundle $Spin(M)$ which double covers $SO(M)$, and for which the diagram*

$$\begin{tikzcd}
Spin(M) \arrow[d, "2:1"'] & \times & Spin(n) \arrow[r] \arrow[d, "2:1"', "\lambda"] & Spin(M) \arrow[d, "2:1"] \\
SO(M) & \times & SO(n) \arrow[r] & SO(M)
\end{tikzcd}$$

commutes. Here $SO(M)$ is the $SO(n)$-principal bundle associated to the oriented tangent bundle TM with metric g. Such a spin structure is not allowed on all manifolds. We state here without proof that a spin structure on a manifold M is allowed if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ vanishes (see [23], Theorem II.1.7).

Definition 2.7.2. *Since $Spin(n)$ has a representation on the spinor space $\mathbb{S}(n)$, we can construct the associated complex vector bundle (cf. Proposition 2.4.14)*

$$S = Spin(M) \times_{Spin(n)} \mathbb{S}(n). \tag{2.49}$$

This vector bundle is called the spinor bundle. Since the isomorphism γ preserves the star operation and the elements of $spin(n)$ are unitary, the fibers S_x carry an inner product that, on a local trivialization, is nothing else than the standard inner product on $\mathbb{C}^{2^{[n/2]}}$. An inner product on $\Gamma(M,S)$ can then be defined by

$$\langle s,t\rangle = \int_M \langle s(x), t(x)\rangle_{S_x} dx \quad (s,t \in \Gamma(M,S)). \tag{2.50}$$

This turns $\Gamma(M,S)$ into a pre-Hilbert-space. The closure of $\Gamma(M,S)$ with respect to the norm induced by the inner product is a Hilbert space and its elements are called the (square-integrable) spinors.

Remark 2.7.3. If $(U_i, \phi_{U_i})_{i\in I}$ is a set of locally trivialising neighbourhoods for the bundle $Spin(M)$ and g_{ij} the transition functions, then $SO(M)$ has the same locally trivialising neighbourhoods but with transition functions $\lambda(g_{ij})$. That is, (see also [12])

$$TM = Spin(M) \times_{Spin(n)} \mathbb{R}^n, \tag{2.51}$$

where $Spin(n)$ acts on $\mathbb{R}$ via the spin-homomorphism $\lambda : spin(n) \to SO(n)$. The bundles TM and S therefore have the same trivialising neighbourhoods $\{U_i\}_{i\in I}$. Furthermore, the transition functions of TM are given by $\lambda(g_{ij})$, those of S by $\gamma(g_{ij})$, where γ is the spin representation. According to Cororollary 2.4.18 these sets of trivialising neighbourhoods and transition functions completely determine the bundle TM and S.

2.7.2 *Dirac-operator*

In this subsection we will introduce the Dirac-operator. The construction of the Dirac-operator consists of two stepts. First, there is something like a Clifford multiplication of 1-forms on spinors, that is there is an action $\gamma : \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M,S) \to \Gamma(M,S)$. Secondly, we need to introduce a connection ∇^S on S.

Let (M, g) be a compact Riemannian spin-manifold and let S be the spinor bundle. For every $x \in M$, the metric g induces an inner product on the tangent space $T_x^*(M)$. So we can take complexified Clifford-algebra of the vector space $(T_x^*(M), g_x)$. Using the spin representation an action $\gamma : \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, S) \to \Gamma(M, S)$ can now be defined. Take $\omega \in \Omega^1(M)$, $\psi \in \Gamma(M, S)$. We will define the action of ω on ψ by defining it on the local trivialisations $\{U_i\}$ as in Remark 2.7.3.

On such a local trivialisation U_i, the 1-form ω can be considered as a smooth map $\omega_i : U_i \to \mathbb{C}l(\mathbb{R}^n)$ and ψ as a map $\psi_i : U \to M_n(\mathbb{C})$. On U_i, set

$$\gamma(\omega_i \otimes \psi_i)(x) = \gamma(\omega_i(x))\psi_i(x). \tag{2.52}$$

If we choose another local trivialisation on some neighbourhood U_j with $U_i \cap U_j \neq \emptyset$, then

$$\gamma(\omega_j \otimes \psi_j)(x) = \gamma(\omega_j(x))\psi_j(x) = \gamma(\lambda(g_{ij}) \cdot \omega_i(x))\gamma(g_{ij}(x))\psi_i(x) = \gamma(g_{ij})\gamma(\omega_i(x))\psi_i(x), \tag{2.53}$$

where we have used Proposion 2.6.8 in the last step. This implies that the action is independent of the choice of the local trivialisation. Thus we get an action

$$\gamma : \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, S) \to \Gamma(M, S). \tag{2.54}$$

The action of $\Omega^1(M)$ on $\Gamma(M, S)$ will also be called Clifford multiplication. It is important to realise that this action can be defined because of the transformation rule in Propostion 2.6.8. This is the first step towards the definition of the Dirac-operator. The second step is to introduce a connection on the spinor bundle S.

Definition 2.7.4. *The Levi-Cevita connection ∇^g on the tangent space TM induces a hermitian connection ∇^S on the spinor bundle S. (see for instance [23]). This connection, ∇^S, is called the spin-connection of M. It satisfies the Leibniz rule (see [15], Chapter 9):*

$$\nabla^S(\gamma(\omega)\psi) = \gamma(\nabla^g\omega)\psi + \gamma(\omega)(\nabla^S\psi), \quad (\omega \in \Omega^1(M), \psi \in \Gamma(M, S)). \tag{2.55}$$

Here $\gamma(\omega)$ acts on $\Omega^1(M) \otimes \Gamma(M, S)$ as $\gamma(\omega)(\alpha \otimes \psi) = \alpha \otimes (\gamma(\omega)\psi)$, $(\alpha, \omega \in \Omega^1(M), \psi \in \Gamma(M, S))$.

It can be shown that the curvature tensor $\Omega_{\mu\nu}$ of ∇^S is given by ([15], Chapter 9)

$$\Omega_{\mu\nu} = \frac{1}{4} R_{\rho\sigma\mu\nu}\gamma^\rho\gamma^\sigma, \tag{2.56}$$

where $R_{\rho\sigma\mu\nu}$ is the Riemann curvature tensor of the Levi-Cevita connection ∇^g on TM.

Remark 2.7.5. If E is a hermitian complex vector bundle over M with hermitian connection ∇^E, we can construct the tensor product bundle $S \otimes E$. Then $\nabla^{S\otimes E} = \nabla^S \otimes 1 + 1 \otimes \nabla^E$ is a connection on $S \otimes E$. Furthermore, since E and S are both hermitian vector bundles, the fibre $S_x \otimes E_x$ $(x \in M)$ carries a natural inner product. Then an inner product on $\Gamma(M, S \otimes E)$ is defined by

$$\langle \psi, \eta \rangle = \int_M \langle \psi(x), \eta(x) \rangle_{S_x \otimes B_x} dx, \quad (\psi, \eta \in \Gamma(M, S \otimes E)), \tag{2.57}$$

turning $\Gamma(M, S \otimes E)$ into a pre-Hilbert-space. With respect to this inner product on the fibres, the connection $\nabla^{S\otimes E} = \nabla^S \otimes 1 + 1 \otimes \nabla^E$ is again hermitian. The closure of $\Gamma(M, S \otimes E)$ is denoted by $L^2(M, S \otimes E)$. Hereby, we have identified the $C^\infty(M)$-module $\Gamma(M, S \otimes E)$ with the $C^\infty(M)$-modules $\Gamma(M, S) \otimes_{C^\infty(M)} \Gamma(M, E)$.

We now give the definition of the Dirac-operator.

Definition 2.7.6 (Dirac-operator)**.** *The Dirac-operator on M is defined by the composition*

$$D = i\gamma \circ \nabla^S. \tag{2.58}$$

On some local coordinate neighbourhoods this action can be written in a form that is more familiar to physicists:

$$D = i\gamma^\mu \nabla_\mu, \tag{2.59}$$

where $\gamma^\mu = \gamma(dx^\mu)$.

If E is a complex vector bundle and ∇^E a hermitian connection on E, the operator

$$D_F = i\gamma \circ (\nabla^S \otimes 1 + 1 \otimes \nabla^E), \tag{2.60}$$

on the bundle $F = S \otimes E$ is called a twisted Dirac-operator. Here the action of γ is extended to $\Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, S \otimes E)$ by $\gamma(\omega \otimes \psi \otimes s) = \gamma(\omega)\psi \otimes s$, for $\omega \in \Omega^1(M), \psi \in \Gamma(M,S), s \in \Gamma(M,E)$.

Before we proceed to show that the Dirac-operator is more or less the square-root of the Laplacian we mention the following theorem.

Theorem 2.7.7. *Let S be the spinor bundle over a compact spin-manifold M and let E be a hermitian vector bundle over M. Denote ∇^S for the spin-connection and let ∇^E be any hermitian connection on E. Then the following statements hold.*

1. *The Dirac-operator D on $\Gamma(M,S)$ is formally self-adjoint. In particular, the closure of D in the Hilbert-space $L^2(M,S)$ is a self-adjoint operator. More generally, D_E is a formally self-adjoint operator on $\Gamma(M,S) \otimes_{C^\infty(M)} \Gamma(M,E)$ and the closure of D_E is a self-adjoint operator on $L^2(M, S \otimes E)$.*
2. *The Dirac-operator and generalized Dirac-operator have compact resolvent.*

Proof. The proof of these statements can be found in [15], Chapter 9. □

There is an important formula, also known as the Lichnerowicz formula, that shows that the Dirac operator is almost the square root of the Laplacian operator. However, there is an extra term that involves the curvature of the manifold. If the curvature vanishes, the Dirac operator is indeed a square root of the connection Laplacian.

Proposition 2.7.8 (Lichnerowicz formula, [24],[15], Proposition 9.16)**.** *Let D be a Dirac operator and let $(\nabla^S)^*\nabla^S$ be the appropriate Laplacian for any Dirac bundle S. Then*

$$D^2 = (\nabla^S)^*\nabla^S + \frac{R}{4} \tag{2.61}$$

with R the curvature tensor and $(\nabla^S)^*\nabla^S$ denotes the connection Laplacian, which in local coordinates is given by $-g^{\mu\nu}(\nabla^S_\mu \nabla^S_\nu - \Gamma^\beta_{\mu\nu}\nabla_\beta)$ (see [15], p.262).

Proof. We start by writing out what D^2 is, and for simplicity we omit the superscript $'S'$ for the spin-connection:

$$\begin{aligned}
D^2 &= -\gamma^\mu \nabla_\nu(\gamma^\nu \nabla_\nu) \\
&= -\gamma^\mu \gamma^\mu \nabla_\mu \nabla_\nu - \gamma^\mu \gamma(\nabla_\mu dx^\nu)\nabla_\nu \\
&= -\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu + \gamma^\mu \gamma^\nu \Gamma^\beta_{\mu\nu} \nabla_\beta \\
&= -\frac{1}{2}\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu - \frac{1}{2}\gamma^\nu \gamma^\mu \nabla_\nu \nabla_\mu + \frac{1}{2}\gamma^\mu \gamma^\nu \Gamma^\beta_{\mu\nu}\nabla_\beta + \frac{1}{2}\gamma^\nu \gamma^\mu \Gamma^\beta_{\mu\nu}\nabla_\beta \\
&= -\frac{1}{2}\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu - g^{\mu\nu}\nabla_\mu \nabla_\nu + \frac{1}{2}\gamma^\mu \gamma^\nu \nabla_\nu \nabla_\mu + g^{\mu\nu}\Gamma^\beta_{\mu\nu}\nabla_\beta \\
&= \nabla^*\nabla - \frac{1}{2}\gamma^\mu \gamma^\nu [\nabla_\mu, \nabla_\nu],
\end{aligned} \tag{2.62}$$

where in the third step we have used that the Christoffel symbol Γ is symmetric in the last two indices (torsion-free) and in the fourth step that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. If we use that $[\nabla_\mu, \nabla_\nu] = \frac{1}{4} R_{\rho\sigma\mu\nu}\gamma^\rho\gamma^\sigma$ (cf. (2.56), then

$$D^2 - \nabla^*\nabla = -\frac{1}{8} R_{\rho\sigma\mu\nu}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \frac{1}{8} R_{\sigma\rho\mu\nu}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{2.63}$$

Now, if we use equation (A.5), we get the following expression:

$$D^2 - \nabla^*\nabla = \frac{1}{8} R_{\sigma\rho\mu\nu}\Big(Q(\gamma^\mu\gamma^\nu\gamma^\rho)\gamma^\sigma + g^{\mu\nu}\gamma^\rho\gamma^\sigma + g^{\rho\nu}\gamma^\mu\gamma^\sigma - g^{\rho\mu}\gamma^\nu\gamma^\sigma\Big) \tag{2.64}$$

By the first Bianchi identity it follows that the first term vanishes and the second term vanishes because of the skew symmetry of $R_{\mu\nu\rho\sigma}$ in the last two indices. The second and third give equal contributions:

$$\begin{aligned} D^2 - \nabla^*\nabla &= \frac{1}{8}(R_{\sigma\mu}\gamma^\mu\gamma^\sigma + R_{\sigma\nu}\gamma^\nu\gamma^\sigma) \\ &= \frac{1}{4} R_{\sigma\mu}\gamma^\mu\gamma^\sigma = \frac{1}{4} R_{\sigma\mu} g^{\sigma\mu} = \frac{R}{4}, \end{aligned} \tag{2.65}$$

where we have used that $R_{\mu\nu} := R^\rho_{\mu\rho\nu}$ is symmetric in its indices. This proves the proposition. □

In the same way as above one can calculate the square of the operator D_F. Since the curvature of $\nabla^S \otimes 1 + 1 \otimes \nabla^E$ is the sum of the curvature of ∇^S and ∇^E, it is not surprising that we get the following formula (see [15] or [23]).

Proposition 2.7.9 (The generalised Lichnerowicz formula). *Let D_E be a twisted Dirac-operator on the bundle $S \otimes E$ over a compact spin manifold. Write $\nabla^F = \nabla^S + \otimes 1_E + 1_S \otimes \nabla^E$ and let $R(E)$ be the curvature of ∇^E. Then*

$$D^2 = (\nabla^F)^*\nabla^F + \frac{R}{4} + R_E \tag{2.66}$$

where $R_E = \sum_{\mu\nu} \gamma^\mu\gamma^\nu \otimes R(E)_{\mu\nu}$ and $(\nabla^F)^*\nabla^F$ is the connection Laplacian of ∇^F.

2.8 GAUGE THEORIES AND FIBRE BUNDLES

In sections 2.4 we introduced principal fibre bundles $P \to M$, along with the definition of a gauge potential as a $\mathfrak{g}$-valued one-form ω on the principal bundle P. In this section we will show how these concepts can be used to describe gauge theories. This description is very well-developed (see [1], [2]) but here we will only treat those ideas that will be needed later on. Most of the definitions and proofs in this section are based on [3], Chapters 1–3.

Definition 2.8.1. *Suppose a Lie group G acts on some manifold F on the left, that is, for each $g \in G$ there is a map $L_g : F \to F$ (we write $L_g f = g \cdot f$) such that $e \cdot f = f$, $(g_1 g_2) \cdot f) = g_1 \cdot (g_2 \cdot F)$ for all $f \in F$ and the map $(g, f) \mapsto g \cdot f$ is C^∞. Then, we denote by $C(P, F)$ the space of all smooth maps $\tau : P \to F$ such that $\tau(pg) = g^{-1} \cdot \tau(p)$. We say that τ is a G-equivariant map from P to F.*

In fact, the space $C(P, F)$ is naturally isomorphic to the space of sections $\Gamma(M, P \times_G F)$ where $P \times_G F$ is the associated bundle of P induced by the left action of G on the manifold F (cf. Proposition 2.4.14). Recall that $P \times_G F$ consists of all equivalence classes $[p, f]$ under the equivalence relation $(pg, f) \sim (p, g \cdot f)$. If $\tau : P \to F$ is a G-equivariant map, we define a section $\sigma : M \to P \times_G F$ by

$$\sigma(x) = [p, \tau(p)], \quad p \in \pi^{-1}(x). \tag{2.67}$$

Note that σ is well-defined since $[pg, \tau(pg)] = [p, g \cdot \tau(pg)] = [p, g \cdot (g^{-1} \cdot \tau(p))] = [p, \tau(p)]$. Conversely, for a given $\sigma \in \Gamma(M, P \times_G F)$ equation (2.67) defines a G-equivariant mapping $\tau : P \to F$. In the case $F = G$ with G acting on itself in the adjoint (that is, $g \cdot h = ghg^{-1}$), $C(P, G)$ has a group structure where the multiplication is defined by $\tau \circ \tau'(p) = \tau(p)\tau'(p)$. This leads to the following definition.

Definition 2.8.2. *The fibre bundle $\pi_G : P \times_G G$ where G acts on itself in the adjoint representation, is called the adjoint bundle and is denoted by Ad P. Its space of sections is isomorphic to $C(P, G)$.*

That there exists a natural group structure on the fibres of the bundle Ad P can been seen as follows. From the fact that the transition functions of Ad P have values in Ad G, it follows that $\tilde{g}_{uv}(x)$ is a(n) (inner) group automorphisms for every $x \in U \cap V$. Therefore the fibres of Ad P can be endowed with a group structure by carrying it over from the local trivialisation, that is: for $x \in U$ and (U, ϕ_U) a local trivialisation, define for $p, q \in \pi_G^{-1}(x)$:

$$pq = \phi^{-1}(\phi(p)\phi(q)). \tag{2.68}$$

This is independent of the choice of local trivialisation since the transition functions are group automorphisms. Thus, we have proved the following proposition.

Proposition 2.8.3. *Let P be a principal G-bundle. The adjoint bundle Ad P is a group bundle with fibre G.*

The space of sections $\Gamma(M, \text{Ad } P)$ also carries a group structure when the multiplication is defined by $(\sigma \cdot \sigma')(x) = \sigma(x)\sigma'(x)$. Then the bijection between $C(P, G)$ and $\Gamma(M, \text{Ad}P)$, is even a group isomorphism. Let $\tau_1, \tau_2 \in C(P, G)$ be given and let $\sigma_1, \sigma_2 \in \Gamma(M, P \times_G G)$ be the corresponding elements of τ_1, τ_2 respectively. Then $[p, (\tau_1\tau_2)(p)] = [p, \tau_1(p)\tau_2(p)] = [p, \tau_1(p)][p, \tau_2(p)] = \sigma_1(x)\sigma_2(x) = (\sigma_1\sigma_2)(x)$. Thus, $\tau_1\tau_2$ is identified with $\sigma_1\sigma_2$ and we have a group homomorphism.

Definition 2.8.4. *An automorphism of a principal fibre bundle $\pi : P \to M$ is a diffeomorphism $f : P \to P$ such that $f(pg) = f(p)g$ for all $g \in G$, $p \in P$. Note that f defines a well-defined diffeomorphism $\bar{f} : M \to M$ given by $\bar{f}(\pi(p)) = \pi(f(p))$. A gauge transformation of a principal fibre bundle is an automorphism $f : P \to P$ such that $\bar{f} = 1_M$. We denote $GA(P)$ for the group of gauge transformations.*

Remark 2.8.5. If ω is a connection 1-form on P, then it can be shown that for any $f \in GA(P)$ the 1-form $f^*\omega$ is also a connection. Let σ be a local section of the bundle P on some open neighbourhood $U \subset M$. Then $\sigma^*\omega$ was interpreted as the physical gauge potential on U. Under a gauge transformation $f \in GA(P)$ the gauge potential $\sigma^*\omega$ transforms to $\sigma^*(f^*\omega)$. This transformation corresponds with a gauge transformation in physics.

Proposition 2.8.6 ([3], theorem 3.2.2)**.** *There is a natural isomorphism $C(P, G) \cong GA(P)$.*

Proof. If $\tau \in C(P, G)$, then define $f : P \to P$ by $f(p) = p\tau(p)$. Indeed, $f(pg) = pg\tau(pg) = pgg^{-1}\tau(p) = pgg^{-1}\tau(p)g = p\tau(p)g = f(p)g$ and $bar f = 1_M$. It follows that $f \in GA(P)$. Conversely, let $f \in GA(P)$ be given, then define τ by the relation $f(p) = p\tau(p)$. Since the action of G on the fibres is free and transitive, this map is well-defined. Now, $\tau(pg)$ is determined by $f(pg) = pg\tau(pg)$. Since $f(pg) = f(p)g = p\tau(p)g$, it follows that $g\tau(pg) = \tau(p)g = gg^{-1} \cdot \tau(p)$ which implies $\tau(pg) = g^{-1} \cdot \tau(p)$. Hence $\tau \in C(P, G)$. Moreover, the map $f \mapsto \tau$ is a homomorphism: if $f_1, f_2 \in GA(P)$, with $f_1(p) = p\tau_1 p$ and $f_2(p) = p\tau_2(p)$ where $\tau_{1,2} \in C(P, G)$, then $(f_1 \circ f_2)(p) = f_1(f_2(p)) = f_1(p\tau_2(p)) = f_1(p)\tau_2(p) = p\tau_1(p)\tau_2(p) = p(\tau_1 \circ \tau_2)(p)$. □

So we have seen that $GA(P) \cong C(P, G) \cong \Gamma(M, \text{Ad } (P))$. That is, in formulating gauge theory in terms of a principal bundle P the gauge group is isomorphic to the sections of the adjoint bundle AdP, which is a group bundle with fibre G. There is a lot more that can be shown, but this last fact is what we will need when we discuss a gauge theory formulation using the machinery in noncommutative geometry.

3 NON-COMMUTATIVE GEOMETRY: SPECTRAL TRIPLES

In this chapter we will give an introduction in the way non-commutative geometry gives a formalism to derive Lagrangian(s) (densities) for a noncommutative space. One starts with an algebra, represented on a Hilbert space and a first order differential operator on the Hilbert space. Together with some compatibility conditions these objects form a so-called spectral triple. A principle called the spectral action principle allows one to calculate the Lagrangian (or action) from the spectral triple. All the information of the theory is thus contained in the spectral triple.

In section 3.1 we will discuss the canonical triple for a compact smooth spin-manifold M and explain the role of the different ingredients. The form of the canonical triple (that is commutative) will lead us to the definition of s spectral triple. This will be done in section 3.2. In section 3.3 we will treat the spectral action principle and show how the action can be calculated using the so-called heat expansion. In section 3.4 we will discuss the internal symmetry group of a spectral triple and we conclude this chapter with the study of internal fluctuations of the metric in section 3.5.

3.1 CANONICAL TRIPLE

In this section we shortly recall the definition of the canonical triple for a compact Riemannian spin-manifold M (see [32], section 4.2). Before we will construct the canonical triple, let us remember which ingredients we need for a physical theory. First of all, we need a space-time manifold M in which all the physics takes place. It follows from the Gelfand representation that the topology on the space M can be reconstructed from the algebra of complex-valued smooth functions $C^\infty(M)$ as the set of pure states on $\mathcal{A}$. We take the algebra $C^\infty(M)$ as the first ingredient of the triple.

Secondly, we need to introduce (fermionic) particles (or spinors) to our space-time M. To describe spinors a spin structure is needed on M. Denote S for the associated spinor bundle of this spin structure. In dimension 4, the fibres of the spinor bundle are copies of $\mathbb{C}^4$ and the inner product on the square-integrable section of $L^2(M,S)$ is (locally) given by

$$\langle\psi_1,\psi_2\rangle = \int \overline{\psi}_1\psi_2 d^4x \quad \psi_1,\psi_2 \in L^2(M,S), \tag{3.1}$$

where the bar denotes complex conjugation. The Hilbert space $L^2(M,S)$ is the second ingredient for our spectral triple. Elements of $L^2(M,S)$ represent the fermionic particles. The algebra $C^\infty(M)$ acts on $L^2(M,S)$ by fibre-wise multiplication

$$(f\psi)(x) := f(x)\psi(x), \quad (f\in C^\infty(M), \psi\in L^2(M,S)). \tag{3.2}$$

The Dirac-operator $\not{D}$ is the final ingredient for the spectral triple and it is locally given by

$$i\gamma^\mu\nabla^S_\mu, \tag{3.3}$$

where ∇^S_μ is the spin-connection that acts on the spinors. In physics the connection ∇^S_μ is also known as the covariant derivative.

Together these three ingredients form the canonical spectral triple

$$(C^\infty(M), L^2(M,S), \not{D}), \tag{3.4}$$

consisting of an algebra, a Hilbert space and a differential operator. This spectral triple contains a lot of information about the geometry of the space-time. As we have already remarked the algebra $C^\infty(M)$ describes the topological structure of the space-time manifold M because of the Gelfand-representation. If we also take into account the Hilbert-space and the Dirac-operator, the geodesic distance between two points is determined by (see [7], [15], Proposition 9.12)

$$\sup\{|\delta_x(f) - \delta_y(f)|; f \in C^\infty(M) \text{ such that } \|[\not{D}, f]\| \leq 1\}, \tag{3.5}$$

where δ_x, δ_y are pure states of $C^\infty(M)$. Similarly other properties of the spacetime manifold M can be recoverd from the canonical triple, like the dimension and the differential forms ([32]).

In the following section we will give a definition of a spectral triple in noncommutative geometry that generalises the canonical triple (3.4).

3.2 SPECTRAL TRIPLES

In this section we give the definition of a general spectral triple. The definitons we use are taken from [21], sections 5.4 and 5.7.

Definition 3.2.1. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ represented faithfully on the Hilbert space $\mathcal{H}$, together with a densily defined, self-adjoint operator $D = D^*$ on $\mathcal{H}$ with the following properties:*

- *The resolvent $(D - \lambda)^{-1}$ is a compact operator on $\mathcal{H}$.*
- *$[D, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$.*

The triple is said to be even if there exists an operator Γ on $\mathcal{H}$ with the properties

$$\Gamma^* = \Gamma, \quad \Gamma^2 = 1, \quad \Gamma D + D\Gamma = 0, \quad \Gamma a - a\Gamma = 0. \tag{3.6}$$

If such an operator does not exist, then the triple is said to be odd. Note that we allow the algebra to be noncommutative!

The properties of Γ imply that $\mathcal{H}$ splits into the eigenspaces $E_\pm$ of Γ belonging to the eigenvalues ± 1.

Example 3.2.2. The triple $(C^\infty(M), L^2(M, S), \not{D})$ in the previous section is an example of a spectral triple. Theorem 2.7.7 states that $\not{D}$ is a self-adjoint operator that has compact resolvent. For every $f \in C^\infty(M)$ the operator $[\not{D}, f]$ is bounded. Indeed, $[\not{D}, f]\psi = i\gamma^\mu\partial_\mu(f\psi) - if\gamma^\mu\partial_\mu\psi = i\gamma^\mu(\partial_\mu f)\psi$ ($\psi \in L^2(M, S)$), and $\partial_\mu f$ is bounded as a smooth function on a compact space.

Furthermore, in dimension $n = 4$ there exists a grading operator γ^5 on the space $L^2(M, S)$. With respect to some orthonormal frame $\{e_1, \dots, e_4\}$ it is given by

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4. \tag{3.7}$$

The triple $(C^\infty(M), L^2(M, S), \not{D}, \gamma^5)$ is an example of an even spectral triple.

There are spectral triples that have additional structure. For instance, spectral triples can have a real structure. This structure is needed in Chapter 4 to derive the Yang–Mills Lagrangian.

Definition 3.2.3. *A real structure of KO-dimension $n \in \mathbb{Z}_8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that*

$$J^2 = \varepsilon; \quad JD = \varepsilon' DJ \quad ; J\Gamma = \varepsilon'' J\Gamma \text{ (even case)}. \tag{3.8}$$

where the numbers ε, ε', ε'' are a function of n mod 8 given by

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Moreover, the action of $\mathcal{A}$ satisfies the commutation rule

$$[a, b^0] = 0 \text{ for all } a, b \in \mathcal{A} \tag{3.9}$$

where

$$b^0 = Jb^* J^{-1} \text{ for all } b \in \mathcal{A}, \tag{3.10}$$

and the operator D satisfies the order one condition

$$[[D, a], b^0] = 0 \text{ for all } a, b \in \mathcal{A}. \tag{3.11}$$

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ endowed with a real structure J is called a real spectral triple.

Condition (3.9) turns $\mathcal{H}$ into an $\mathcal{A}$-bimodule if we define

$$\psi a = Ja^* J\psi, \quad (a \in \mathcal{A}, \psi \in \mathcal{H}). \tag{3.12}$$

This is indeed a right action of $\mathcal{A}$ on $\mathcal{H}$ and condition (3.9) guarantees that the right action commutes with the left action.

3.3 SPECTRAL ACTION

In the previous section we have given the definition of a spectral triple as a generalisation of the canonical triple mentioned in section 3.1. However, we have not yet discussed how one gets physics from a spectral triple. This is done by the spectral action principle that we discuss in this section. In Chapter 4 we will derive the Yang–Mills Lagrangian using this spectral action principle (see [5] for more details on the spectral action principle).

Definition 3.3.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The spectral action of the triple $(\mathcal{A}, \mathcal{H}, D)$ is given by*

$$Tr\left(f(D/\Lambda)\right), \tag{3.13}$$

where Λ is a real cutt-off number and f is a suitable (even) function from $\mathbb{R}$ to $\mathbb{R}$ that cut-offs all eigenvalues of D with absolute value greater than Λ.

To actually calculate the spectral action for a general triple we need to expand the spectral action (3.13) in Λ using the so-called heat expansion. We do this in the next subsection.

3.3.1 *Heat expansion*

In this subsection we will use the heat equation to expand the spectral action in Λ. This expansion is known as the heat expansion and details on this subject can be found in [13], sections 1.7 and 4.8.

Let M be an m-dimensional manifold and let E be a vector bundle over M. Assume that $P : \Gamma(M, E) \to \Gamma(M, E)$ is a differential operator on the bundle E that locally can be written as

$$P = -(g^{\mu\nu} I \partial_\mu \partial_\nu + A^\mu \partial_\mu + B), \tag{3.14}$$

where I is the identity matrix, and A^μ and B are endomorphisms of the bundle E. For differential operators as P in (3.14) we have the following expansion in t (see [13], Section 1.7):

$$\mathrm{Tr}(e^{-tP}) \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} \int_M a_n(x, P) \sqrt{g} d^4x, \tag{3.15}$$

where m is the dimension of the manifold M. Equation (3.15) is known as the heat expansion. The coefficients $a_n(x, P)$ are called the Seeley-DeWitt coefficients.

There is a Lemma (see [13], Lemma 4.8.1) that states that for an operator of the same form as P there is a unique connection ∇ on E and a unique endomorphism of the bundle E, say F, such that $P = \nabla^*\nabla - F$ where $\nabla^*\nabla$ is the connection Laplacian of the connection ∇. For these second-order operators we have the following proposition.

Proposition 3.3.2 ([13], Theorem 4.8.16). *For an operator $P = \nabla^*\nabla - F$ as above, the Seeley-DeWitt coefficients are given by the following expressions:*

$$\begin{aligned} a_0(x, P) &= (4\pi)^{-m/2} Tr(Id) \\ a_2(x, P) &= (4\pi)^{-m/2} Tr(-\frac{R}{6} Id + F) \\ a_4(x, P) &= (4\pi)^{-m/2} \frac{1}{360} Tr \left(-12 R_{;\mu}{}^{;\mu} + 5R^2 - 2R^{\mu\nu} R_{\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 60RF \right. \\ &+ \left. 180F^2 + 60F_{;\mu}{}^{;\mu} + 30\Omega_{\mu\nu}\Omega^{\mu\nu} \right), \end{aligned} \tag{3.16}$$

where $\Omega_{\mu\nu}$ is the curvature of the differential operator ∇. The odd coefficients are zero (see [13], Theorem 4.8.16).

The expansion (3.15) can be used on the Laplace-transform of $f(D/\Lambda)$ to expand the spectral action (3.13). We will do this now. We follow the steps in [8], Chapter 11, omitting all the technical details.

We start with a function $k : \mathbb{R} \to \mathbb{R}$ such that $k(u^2) = f(u)$ for all $u \in \mathbb{R}$. The function k can be written as a Laplace transform:

$$k(u) = \int_0^\infty e^{-su} h(s) ds. \tag{3.17}$$

Note that $k(0) = \int_0^\infty h(s)ds$. We will use this fact later. We can also write

$$k(\Delta) = \int_0^\infty e^{-s\Delta} h(s) ds, \tag{3.18}$$

for a (possibly unbounded) self-adjoint operator Δ. If we now take the trace of the left and right term we get:

$$\mathrm{Tr} k(\Delta) = \int_0^\infty \mathrm{Tr}(e^{-s\Delta}) h(s) ds. \tag{3.19}$$

Since $k(u^2) = f(u)$, we can write $f(D/\Lambda) = k(D^2/\Lambda^2)$. This yields

$$\mathrm{Tr}\,(f(D/\Lambda)) = \mathrm{Tr}(k(D^2/\Lambda^2) = \int_0^\infty \mathrm{Tr}(e^{-\frac{s}{\Lambda^2}D^2})h(s)ds. \tag{3.20}$$

Now, we can use the expansion (3.15) for the exponential since (check equation (2.62)) D is locally of the form (3.14):

$$\begin{aligned} f(D/\Lambda) &= \int_0^\infty \mathrm{Tr}\left(e^{-\frac{s}{\Lambda^2}D^2}\right) h(s)ds \sim \sum_{n\geq 0}\int_0^\infty \left(\frac{s}{\Lambda^2}\right)^{\frac{n-m}{2}} a_n(D^2)h(s)ds \\ &= \sum_{n\geq 0} \Lambda^{m-n} a_n(D^2) \int_0^\infty s^{\frac{n-m}{2}} h(s)ds, \end{aligned} \tag{3.21}$$

where we have written $a_n(P) = \int_M a_n(x,P)\sqrt{g}d^4x$. We use the following fact: for $\alpha < 0$ one has $s^\alpha = \frac{1}{\Gamma(-\alpha)}\int_0^\infty e^{-sv}v^{-\alpha-1}dv$. Multiplying both sides by $h(s)$ and integrating over s then gives:

$$\begin{aligned} \int_0^\infty s^\alpha h(s)ds &= \frac{1}{\Gamma(-\alpha)}\int_0^\infty\int_0^\infty h(s)e^{-sv}v^{-\alpha-1}dsdv = \frac{1}{\Gamma(-\alpha)}\int_0^\infty k(v)v^{-\alpha-1}dv \\ &= \frac{2}{\Gamma(-\alpha)}\int_0^\infty f(u)u^{-2\alpha-1}du = \frac{2}{\Gamma(-\alpha)}f_{-2\alpha}, \end{aligned} \tag{3.22}$$

where we have substituted $v = u^2$ and defined $f_{-2\alpha} = \int_0^\infty f(u)u^{-2\alpha-1}du$ for $\alpha < 0$. In the case $m = 4$ the above expression is valid for $n = 0, 2$ terms. If we do the heat expansion until $n = 4$ we get

$$\begin{aligned} \mathrm{Tr}\,(f(D/\Lambda)) &\sim f(0)\Lambda^0 a_4(D^2) + \sum_{n=0,2}\Lambda^{4-n}a_n(D^2)\frac{1}{\Gamma(\frac{4-n}{2})}\int_0^\infty k(v)v^{\frac{4-n}{2}-1}dv \\ &= f(0)\Lambda^0 a_4(D^2) + 2\Lambda^2 a_2(D^2)f_2 + 2\Lambda^4 f_4 a_0(D^2), \end{aligned} \tag{3.23}$$

where we have used that $\Gamma(n) = (n-1)!$, $f(0) = k(0) = \int_0^\infty h(s)ds$.

Thus, in 4 dimensions equation (3.23) is the expansion of the spectral action

$$\mathrm{Tr}\,(f(D/\Lambda)), \tag{3.24}$$

in terms of Λ up to order $n = 4$. Therefore, to calculate the spectral action in 4 dimension the only thing left to do is to calculate the Seeley-DeWitt coefficients using Proposition 3.3.2.

The spectral action principle gives the bosonic part of the Lagrangian. For the fermions we have the following action (see [5]).

Definition 3.3.3. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The fermionic action is given by*

$$\langle \psi, D\psi \rangle, \tag{3.25}$$

where D is the Dirac operator and ψ is in $\mathcal{H}$ and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathcal{H}$.

3.4 INNER TRANSFORMATIONS

In gauge theories there is a notion of a gauge group which is interpreted as the group of internal symmetries of the theory. Also for a spectral triple there exists a notion of an internal symmetry group (see [5]; [8], section 9.9; [21], section 8.3). In this section we will consider these internal symmetries.

Consider a compact Riemannian manifold M of dimension 4 and let $(C^\infty(M), L^2(M,S), \not{D})$ be the canonical triple of M. The group of symmetries for the Einstein-Hilbert action is the group of diffeomorphisms on M: Diff$(M) \cong$ Aut$(C^\infty(M))$.

If we also add the standard model Lagrangian, the symmetry group is much bigger, since it also contains the gauge group of the standard model: $\mathcal{G}_{SM}$. Here $\mathcal{G}_{SM}$ is the group of smooth maps from M to the gauge group $U(1) \times SU(2) \times SU(3)$, [8]. This will give a richer symmetry group to the Lagrangian. The group $\mathcal{G}_{SM}$ are known as the internal symmetries of the theory. It is related to Diff(M) in the following way.

Definition 3.4.1. *Given two groups N and H and let $\theta : H \to Aut(N)$ be a group homomorphism. The semi-direct product $G := N \rtimes H$ is defined to be a group with underlying set $\{(n,h)|n \in N, h \in H\}$ and group operation $((n,h)(n',h') := (n\theta(h)n', hh')$. One can verify that this is a group (the inverse of (n,h) being given by $(\theta(h^{-1})(n^{-1}), h^{-1})$), that H is a subgroup of G and that N is a normal subgroup of G.*

There exist an action θ of the group Diff(M) on $\mathcal{G}_{SM}$ by defining

$$\theta : \text{Diff}(M) \to \text{Aut}(\mathcal{G}_{SM}), \big(\theta(\phi)(A)\big)(x) = A(\phi^{-1}(x)), \quad (\phi \in \text{Diff}(M), A \in \mathcal{G}_{SM}). \tag{3.26}$$

The map θ is a homomorphism so it follows from Definition 3.4.1 that we can use θ to define the semidirect product $\mathcal{G}_{SM} \rtimes$ Diff(M). The group $\mathcal{G}_{SM} \rtimes$ Diff(M) is the full symmetry group of the theory. It is related to the inernal symmetry group by the following exact sequence of groups:

$$1 \to \mathcal{G}_{SM} \to \mathcal{G}_{SM} \rtimes \text{Diff}(M) \to \text{Diff}(M) \to 1. \tag{3.27}$$

If we compare this with the short exact sequence (2.14), we see that $\mathcal{G}_{SM}$ and Inn(A) play the same role in the exact sequence. Thus, if we interpret Aut$(\mathcal{A})$ as the full symmetry group of the triple, just as Aut$(C^\infty(M))$ is the symmetry group of the canonical triple, the group of internal automorphism Inn$(\mathcal{A})$ should be interpreted as the internal symmetry group (compare the exact sequences (2.14) and (3.27)).

To emphasise the interpretation of Inn$(\mathcal{A})$ as the gauge group we put it into a definition.

Definition 3.4.2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple The internal symmetry group or gauge group is given by the group of inner automorphisms of the algebra $\mathcal{A}$*

3.5 INNER FLUCTUATIONS OF THE METRIC

In this section we study inner fluctuations of the Dirac-operator in spectral triples. These inner fluctuations come from the existence of Morita-equivalences. For more information on Morita equivalence see [31].

Morita equivalence is a notion of equivalence, weaker than isomorphy, between rings or algebras. However, when two given algebras $\mathcal{A}$, $\mathcal{B}$ are both commutative, they are Morita-equivalent if and only if they are isomorphic. This is different in the noncommutative case where there are Morita-equivalent algebras that are not isomorphic. What is important for us is that two rings R, S are Morita equivalent if (and only if) the categories of right modules R-Mod and S-Mod are equivalent as abelian categories. That is, Morita equivalent algebras have the same representation theory. In this section we will start with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and we will construct new spectral triples that come frome Morita-equivalent algebras. In particular, we will find spectral triples of the form $(\mathcal{A}, \mathcal{H}, D + \text{bounded self-adjoint operator})$. Since the Dirac-operator describes the geometretrical properties of the noncommutative space, this extra term is interpreted as an inner fluctuations of the metric. At the end of this chapter we discuss the (physical) interpretation of the inner fluctuations.

3.5.1 *Inner fluctuations*

Remark 3.5.1. This subsection is an elaboration of [8], section 10.8.

Let $\mathcal{A}$ be a unital $*$-algebra represented faithfully on a Hilbert space $\mathcal{H}$. One can show that an algebra $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$ if and only if

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}), \tag{3.28}$$

for some finitely generated projective (right) module $\mathcal{E}$ over $\mathcal{A}$. Note that $\mathcal{B}$ is again a unital involutive algebra which is required if B is to be taken as the spectral algebra in a triple. The unit is given by the identity-endomorphism of $\mathcal{E}$ and the involution is defined as the map that assigns to each operator $b \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ its adjoint operator (which is a well-defined involution because of the definition of $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ for general modules (cf. 2.2.16).

Now, construct the space $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$. Since $\mathcal{E}$ is projective there exists a hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $\mathcal{E}$ (cf. [21], section 4.3). This induces an inner product on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ by

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle_{\mathcal{H}'} = \langle \xi_1, \langle \eta_1, \eta_2 \rangle_{\mathcal{A}} \xi_2 \rangle_{\mathcal{H}}, \quad (\eta_{1,2} \in \mathcal{E}, \xi_{1,2} \in \mathcal{H}). \tag{3.29}$$

Note that this inner product is well-defined with respect to the $\mathcal{A}$-linearity of the tensor product. The action of $\mathcal{B}$ on $\mathcal{H}'$ is given by

$$\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H}'), \quad \pi(b)(\eta \otimes \xi) = (b\eta) \otimes \xi, \quad (b \in \mathcal{B}, \eta \in \mathcal{E}, \xi \in \mathcal{H}). \tag{3.30}$$

We will check that $\pi(b^*) = \pi(b)^*$. For $\eta_1, \eta_2 \in \mathcal{E}$ and $\xi_1, \xi_2 \in \mathcal{H}$ we calculate:

$$\begin{aligned} \langle \pi(b^*)(\eta_1 \otimes \xi_1), \eta_2 \otimes \xi_2 \rangle_{\mathcal{H}'} &= \langle \xi_1, \langle b^* \eta_1, \eta_2 \rangle_{\mathcal{A}} \xi_2 \rangle \\ &= \langle \xi_1, \langle \eta_1, b\eta_2 \rangle_{\mathcal{A}} \xi_2 \rangle_{\mathcal{H}} = \langle \eta_1 \otimes \xi_1, \pi(b)(\eta_2 \otimes \xi_2) \rangle_{\mathcal{H}'} = . \end{aligned}$$

To get a spectral triple we would like to define a Dirac operator D' on $\mathcal{H}'$. A first guess might be the operator D' which acts as

$$D'(\eta \otimes \xi) = \eta \otimes D\xi, \quad (\eta \in \mathcal{E}, \xi \in \mathcal{H}), \tag{3.31}$$

but this is not well-defined because $D'(\eta a \otimes \xi)$ should be equal to $D'(\eta \otimes a\xi)$ which is not always the case, since in general $[D, a] \neq 0$:

$$D'(\eta a \otimes \xi) = (\eta a \otimes D\xi) = \eta \otimes aD\xi \neq \eta \otimes D(a\xi) = D'(\eta \otimes a\xi). \tag{3.32}$$

To fix this problem, a *hermitian connection* is needed on $\mathcal{E}$.

Definition 3.5.2. *A connection on a right $\mathcal{A}$-module $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ that satisfies the Leibniz rule*

$$\nabla(\eta a) = (\nabla \eta) a + \eta \otimes da, \quad \forall \eta \in \mathcal{E}, a \in \mathcal{A},$$

where in the first term the tensor product sign is omitted since a acts from the right on the $\Omega^1_D(\mathcal{A})$-part of $\nabla\eta$. Here $da := [D,a]$ and

$$\Omega^1_D := \{\sum_J a_j[D,b_j] | a_j, b_j \in \mathcal{A}\}$$

is a bimodule over $\mathcal{A}$ (see also [21], Chapter 6 for more details on $\Omega^1_D(\mathcal{A})$).
The connection ∇ is called hermitian if

$$(\eta,\nabla\tau) - (\nabla\eta,\tau) = d(\eta,\tau), \quad (\eta,\tau \in \mathcal{E}). \tag{3.33}$$

Here $(\cdot,\cdot)$ denotes the hermitian structure on $\mathcal{E}$ and we have understood that $(\eta\otimes\omega,\tau) = \omega^*(\eta,\tau)$ and $(\eta,\tau\otimes\omega) = (\eta,\tau)\omega$ for all $\eta,\tau\in\mathcal{E}$, $\omega\in\Omega^1_D(\mathcal{A})$. [1]

The Dirac operator D' on $\mathcal{H}'$ is now defined as

$$D'(\eta\otimes\xi) = \eta\otimes D\xi + (\nabla\eta)\xi, \quad (\eta\in\mathcal{E},\xi\in\mathcal{H}), \tag{3.34}$$

which is well defined as an operator on $\mathcal{H}' = \mathcal{E}\otimes_{\mathcal{A}}\mathcal{H}$ since

$$\begin{aligned} D'(\eta\otimes a\xi) &= \eta\otimes D(a\xi) + (\nabla\eta)a\xi = \eta\otimes a(D\xi) + \eta\otimes[D,a]\xi + (\nabla\eta)a\xi \\ &= \eta a\otimes D\xi + (\nabla\eta)a\xi + \eta\otimes da\cdot\xi = D'(\eta a\otimes\xi). \end{aligned}$$

In particular, we can consider the case where $\mathcal{E} = \mathcal{A}$, since any algebra is Morita equivalent to itself. The algebra $\mathcal{B}$ can then be identified with $\mathcal{A}$ via the isomorphism $\mathcal{B}\ni b\mapsto b(1)\in\mathcal{A}$. The Hilbert space is $\mathcal{H}' = \mathcal{A}\otimes_{\mathcal{A}}\mathcal{H}\simeq\mathcal{H}$ by identification of $1\otimes\xi\mapsto\xi$. For the new Dirac operator we find, with $a\in\mathcal{A},\xi\in\mathcal{H}$:

$$\begin{aligned} D'(a\xi) &= a(D\xi) + (\nabla a)\xi = a(D\xi) + \nabla(1\cdot a) = a(D\xi) + (\nabla(1)\cdot a)\xi + da\cdot\xi \\ &= a(D\xi) + [D,a]\xi + (\nabla(1)\cdot a)\xi = D(a\xi) + \nabla(1)(a\xi), \end{aligned} \tag{3.35}$$

so we have

$$D' = D + \nabla(1) = D + A, \tag{3.36}$$

where $A = \nabla(1)\in\Omega^1_D(\mathcal{A})$. Since D' should be self adjoint we demand that $A^* = A$ is a self adjoint element of $\Omega^1_D(\mathcal{A})$. We have now constructed a new spectral triple $(\mathcal{A},\mathcal{H},D+A)$.

When we have a real spectral triple, that is, there exists a real structure J in the sense of Definition 3.2.3, we can do the same trick again. Define a right action of $\mathcal{A}$ on $\mathcal{H}$. For $a\in\mathcal{A}$ and $\xi\in\mathcal{H}$, set

$$\xi a = Ja^*J^*\xi \quad (a\in A,\xi\in\mathcal{H}), \tag{3.37}$$

turning $\mathcal{H}$ in a right $\mathcal{A}$-module also. Since $[a,Jb^*J^*] = 0$ for all $a,b\in A$, it follows that $\mathcal{H}$ is a $\mathcal{A}$-bimodule.

Definition 3.5.3. *Let $\mathcal{F}$ be a $\mathcal{B}-\mathcal{A}$-module. We define the conjugate $\mathcal{A}$-$\mathcal{B}$-module $\overline{\mathcal{F}}$. As an additive group the module $\overline{\mathcal{F}}$ is $\mathcal{F}$. For $\eta\in\mathcal{F}$ we denote $\overline{\eta}$ for the corresponding element in $\overline{\mathcal{F}}$. The left $\mathcal{A}$-action and the right $\mathcal{B}$-action are given by*

$$\begin{aligned} a\cdot\overline{\eta} &= \overline{\eta a^*}, \quad (a\in\mathcal{A},\eta\in\mathcal{F}), \\ \overline{\eta}\cdot b &= \overline{b^*\eta}, \quad (b\in\mathcal{B},\eta\in\mathcal{F}). \end{aligned}$$

1 It can be shown that any finitely generated projective module allows a hermitian connection. See for example [21], section 7.3.

If the right $\mathcal{A}$-module $\mathcal{F}$ has an $\mathcal{A}$-valued hermitian structure, then the left $\mathcal{A}$-module $\overline{\mathcal{F}}$ has an $\mathcal{A}$-valued hermitian structure given by

$$\langle \overline{\eta_1}, \overline{\eta_2} \rangle_{\mathcal{A}} = \langle \eta_1, \eta_2 \rangle_{\mathcal{A}}, \quad (\eta_1, \eta_2 \in \mathcal{F}). \tag{3.38}$$

The appropiate properties are readily checked.[2]

As $\mathcal{H}$ has now a right $\mathcal{A}$-module structure as well, consider the $\mathcal{B}$-module

$$\mathcal{H}'' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}}, \tag{3.39}$$

The action of $\mathcal{B}$ on $\mathcal{H}''$ is defined by

$$b(\eta \otimes \xi \otimes \overline{\tau}) = b\eta \otimes \xi \otimes \overline{\tau}, \tag{3.40}$$

Furthermore, we define an operator J'' on $\mathcal{H}''$ by

$$J''(\eta \otimes \xi \otimes \overline{\tau}) = \tau \otimes J\xi \otimes \overline{\eta}, \quad (\eta \in \mathcal{E}, \xi \in \mathcal{H}, \overline{\tau} \in \overline{\mathcal{E}}). \tag{3.41}$$

The operator J'' is well defined as we can check:

$$\begin{aligned} J''(\eta \otimes \xi \otimes a\overline{\tau}) &= \tau a^* \otimes J\xi \otimes \overline{\eta} = \tau \otimes a^* \varepsilon J^* J^* J\xi \otimes \overline{\eta} \\ &= \tau \otimes J^2 a^* J^* \xi \otimes \overline{\eta} = \tau \otimes J(\xi a) \otimes \overline{\eta} = J''(\eta \otimes \xi a \otimes \overline{\tau}), \end{aligned}$$

and

$$\begin{aligned} J''(\eta \otimes a\xi \otimes \overline{\tau}) &= \tau \otimes J(a\xi) \otimes \overline{\eta} = \tau \otimes J(aJ^* J\xi) \otimes \overline{\eta} \\ &= \tau \otimes (J\xi) a^* \otimes \overline{\eta} = \tau \otimes J\xi \otimes \overline{\eta a} = J''(\eta a \otimes \xi \otimes \overline{\tau}). \end{aligned}$$

To get a well-defined Dirac-operator on $\mathcal{H}''$ we need to introduce a hermitian connection on the *left* $\mathcal{A}$-module $\overline{\mathcal{E}}$.

Definition 3.5.4. *A connection on a* left $\mathcal{A}$-*module* $\mathcal{F}$ *is a linear map* $\nabla : \mathcal{F} \to \Omega^1_D \mathcal{A} \otimes \mathcal{F}$ *that satisfies*

$$\nabla(a\tau) = da \otimes \tau + a(\nabla\tau), \tag{3.42}$$

where $da = [D, a]$.

Just as in the previous case the Dirac operator on $\mathcal{H}''$ must be of the form

$$D''(\eta \otimes \xi \otimes \overline{\tau}) = (\nabla_{\mathcal{E}}\eta)\xi \otimes \overline{\tau} + \eta \otimes D\xi \otimes \overline{\tau} + \eta \otimes \xi(\overline{\nabla}_{\overline{\mathcal{E}}}\overline{\tau}), \quad \eta \in \mathcal{E}, \xi \in \mathcal{H}, \overline{\tau} \in \overline{\mathcal{E}}) \tag{3.43}$$

to have $\mathcal{A}$-linearity. Here $\nabla_{\mathcal{E}}$, $\nabla_{\overline{\mathcal{E}}}$ are connections on $\mathcal{E}, \overline{\mathcal{E}}$ respectively.

Take $\mathcal{E} = \mathcal{A}$. This again gives $\mathcal{B} \cong \mathcal{A}$ and $\mathcal{H}'' = \mathcal{H}$. For J'' to be a real structure for $(\mathcal{A}, \mathcal{H}, D'')$ it must satisfy the condition $D''J'' = \varepsilon' J'' D''$. More explicitly, this equation reads

$$\begin{aligned} D''J'' &= (\nabla_{\mathcal{E}} 1) \otimes J_M \xi \otimes \overline{1} + 1 \otimes DJ_M \xi \otimes \overline{1} + 1 \otimes \xi \otimes (\nabla_{\overline{\mathcal{E}}} \overline{1}) \\ J''D'' &= (\nabla_{\overline{\mathcal{E}}} \overline{1})^* \otimes J_M \xi \otimes \overline{1} + 1 \otimes J_M D\xi \otimes \overline{1} + 1 \otimes \xi \otimes (\nabla_{\mathcal{E}} \overline{1})^*. \end{aligned} \tag{3.44}$$

The condition $D''J'' = \varepsilon' J'' D''$ is then fulfilled if $\nabla_{\overline{\mathcal{E}}} \overline{1} = \varepsilon' (\nabla_{\mathcal{E}} 1)^*$. In the same way as above we can get an expression for the Dirac operator D'':

$$D''(1 \cdot \xi \cdot 1) = (\nabla_{\mathcal{E}} 1)\xi + (D\xi) + \varepsilon' \xi (\nabla_{\overline{\mathcal{E}}} \overline{1}) = D + A + \varepsilon' JAJ^*,$$

2 The conditions for a hermitian structure on a left $\mathcal{A}$-module $\mathcal{E}$ are the same as for a right module with the exception that $(\eta_1, a\eta_2) = (\eta_1, \eta_2)a^*$ instead of $(\eta_1, \eta_2 a) = (\eta_1, \eta_2)a$, for $a \in \mathcal{A}$, $\eta_{1,2} \in \mathcal{E}$.

where $A = (\nabla_{\mathcal{E}} 1)$ is a self-adjoint element of $\Omega^1_D\mathcal{A}$. So, the most general form of the Dirac operator D'' obtained from inner fluctuations relative to $\mathcal{A}$ and then relative to $A^0 = JAJ^*$, such that $J''D'' = \varepsilon' D''J''$ is:

$$D'' = D + A + \varepsilon' JAJ^*, \quad \text{with } A \in \Omega^1_D(\mathcal{A}) \text{ self-adjoint.} \tag{3.45}$$

From now on, we will denote D_A for the operator $D + A + \varepsilon' JAJ^*$.

Proposition 3.5.5. *The triple $(\mathcal{A}, \mathcal{H}, D_A, J, \gamma^5)$ is again a real and even spectral triple.*

Proof. The operator $D + A + \varepsilon' JAJ^*$ is still self-adjoint since D and A are self-adjoint operators. Furthermore, for all $a \in \mathcal{A}$ the operator $[D + A + JAJ^*, a]$ is bounded since $[D, a]$, A and J are all bounded operators. We will now explicitly show that $D + A + JAJ^*$ has compact resolvent. Let $\lambda \in \mathbb{C} - \mathbb{R}$ be given, then

$$(D + A - \lambda)^{-1} = (D - \lambda)^{-1}((D + A - \lambda)(D - \lambda)^{-1})^{-1} = (D - \lambda)^{-1}(1 + A(D - \lambda)^{-1})^{-1}. \tag{3.46}$$

Since $(D - \lambda)^{-1}$ is a compact operator, it suffices to show that $(1 + A(D - \lambda)^{-1})^{-1}$ is bounded because $\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$. Now, $(1 + A(D - \lambda)^{-1}) : \mathcal{H} \to \mathcal{H}$ is a bounded and invertible operator $\mathcal{H} \to \mathcal{H}$. In particular, $(1 + A(D - \lambda)^{-1})$ is a closed operator. Hence, its inverse is a closed operator ([30], Proposition 5.1.7) and therefore bounded by the closed graph theorem ([30], Theorem 2.2.7). Hence, $(D + A - \lambda)^{-1}$ is compact for all $z \in \mathbb{C} - \mathbb{R}$.

It remains to check the compatibility conditions for the real structure J. We have already checked that $D_A J = \varepsilon' J D_A$. Finally, the commutator $[[D_A, a], b^0] = 0$ for all $a, b \in \mathcal{A}$ since $[[D, c], d^0] = 0$ for all $c, d \in \mathcal{A}$ and A is of the form $\sum_j a_j[D, b_j]$, where a_j, b_j are in $\mathcal{A}$. □

Remark 3.5.6 (see also [21], section 8.3)**.** In the commutative case, that is, $\mathcal{A}$ is supposed to be a commutative algebra, the internal pertubation $A + \varepsilon' JAJ^*$ vanishes. Namely, in the commutative case left and right multiplication are the same operation so $JaJ^* = a^*$. This, in turn, implies that $[a, [D, b]] = 0$ for all $a, b \in \mathcal{A}$. Using these equalities and the fact that A is a self-adjoint operator of the form $\sum_j a_j[D, b_j]$, we get

$$\begin{aligned} \varepsilon' JAJ^* &= \varepsilon' Ja[D, b]J^* = \varepsilon' JaJ^* J[D, b]J^* = \varepsilon' a^* J[D, b]J^* \\ &= \varepsilon' a^*(\varepsilon' DJbJ^* - \varepsilon' JbJ^* D) = a^*(Db^* - b^* D) = [D, b^*]a^* = -A^* = -A. \end{aligned} \tag{3.47}$$

Thus, inner fluctuations only arise when the spectral algebra is noncommutative.

Fluctations of a fluctuated Dirac-operator are also fluctuations of the initial Dirac-operator. The following proposition makes this formal.

Proposition 3.5.7 (see also [8], section 10.8)**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and let $D' = D + A$, for some $A \in \Omega^1_D$ with $A = A^*$. Then for any $B \in \Omega^1_{D'}$, with $B = B^*$, one has*

$$D' + B = D + A', \quad A' = A + B \in \Omega^1_D. \tag{3.48}$$

If, in addition the spectral triple has a real structure J. Then for any $B \in \Omega^1_{D'}$ with $B = B^*$, one has

$$D' + B + \varepsilon' JBJ^* = D + A' + \varepsilon' JA'J^*, \quad A' = A + B \in \Omega^1_D. \tag{3.49}$$

Proof. In both cases it suffices to check that $\Omega^1_{D'} \subset \Omega^1_D$. Let's start with the second part of the proposition. We have $a[D', b] = a[D, b] + a[A, b] + \varepsilon' a[JAJ^*, b]$ where $a, b \in \mathcal{A}$ and $A \in \Omega^1_D$. The last term is zero since J is a real structure. So, we can prove both statements at once by showing that $a[D, b] + a[A, b] \in \Omega^1_D$, but this is immediate because Ω^1_D is a $\mathcal{A}$-bimodule. □

Let $(\mathcal{A}, \mathcal{H}, \pi, D, J)$ be a real spectral triple, where, for convenience we have explicitly mentioned the representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. In section 3.4 we interpreted the inner automorphisms group of the algebra $\mathcal{A}$ as the internal symmetry group of the triple. Under such an inner transformation $\alpha_u : a \mapsto uau^*$, for u a unitary, we get an equivalent triple $(\mathcal{A}, \mathcal{H}, \pi \circ \alpha_u, UDU^{-1}J)$, where $U = uJuJ^*$ is the intertwining operator. Let's see how the perturbation A transforms under inner transformations. Let $u \in \mathcal{U}(\mathcal{A})$ be given, then (here we frequently use that $[a, b^0] = [b[D, a], JcJ^*] = 0$ for all $a, b, c \in \mathcal{A}$)

$$\begin{aligned} uJuJ^*(D + A + \varepsilon' JAJ^*)Ju^*J^*u^* &= \varepsilon' uJuDu^*J^*u^* + uJuJ^*Ju^*J^*Au^* + \varepsilon' uJuJ^*JAJ^*Ju^*J^*u^* \\ &= \varepsilon' uJ(D + u[D, u^*])J^*u^* + uAu^* + \varepsilon' uJuAu^*J^*u^*JJ^* \\ &= uDu^* + JJ^*uJu[D, u^*]J^*u^* + uAu^* + \varepsilon' uJuAJ^*u^*Ju^*J^* \\ &= D + u[D, u^*] + \varepsilon' JuJ^*uJ[D, u^*]J^*u^* + uAu^* + \varepsilon' uJuJ^*u^*JAu^*J^* \\ &= D + u[D, u^*] + \varepsilon' Ju[D, u^*]J^*uJJ^*u^* + uAu^* + \varepsilon' JuJ^*uu^*JAu^*J^* \\ &= D + u[D, u^*] + \varepsilon' Ju[D, u^*]J^* + uAu^* + \varepsilon' JuAu^*J^* \end{aligned}$$

where we have also used that A is self-adjoint and $DJ = \varepsilon' JD$. Therefore, under an inner fluctuation induced by a unitary $u \in \mathcal{A}$, the perturbation term A transforms as

$$A \mapsto uAu^* + u[D, u^*], \tag{3.50}$$

which is again in $\Omega^1_D(\mathcal{A})$.

Remark 3.5.8. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Note that under a inner transformation D transforms as

$$D \mapsto D + u[D, u^*] + Ju[D, u^*]J^*. \tag{3.51}$$

We see that under an inner transformation a non-perturbed Dirac-operator transforms to a perturbed Dirac-operator with perturbation $A = u[D, u^*]$.

We interpret the perturbation A as a gauge potential. The inner automorphism can then be interpreted as gauge transformations. Under such a gauge transformation α_u ($u \in \mathcal{U}(\mathcal{A})$) the gauge potential A transforms as $A \mapsto uAu^* + u[D, u^*]$. Since the spectral action depends on the (spectrum of the) Dirac-operator D, adding a perturbation term to the Dirac-operator, will affect the action. Of course, under a gauge transformation the physics must not change. It is therefore necessary to check if the bosonic and fermionic action are invariant under the internal transformations.

Proposition 3.5.9. *The spectral action*

$$S = Tr_{\mathcal{H}}\left(f(D_A/\Lambda)\right) \tag{3.52}$$

is invariant under the inner automorphisms of the algebra $\mathcal{A}$.

Proof. Denote $U = \mathrm{Ad}(u) = uJuJ^*$ for $u \in \mathcal{U}(\mathcal{A})$. Then U is a unitary operator on $\mathcal{H}$ since $UU^* = uJuJ^*(uJuJ^*)^* = uJuJ^*(Ju^*J^*u^*) = 1$ and likewise $U^*U = 1$. So we have

$$S = \mathrm{Tr}_{\mathcal{H}}\left(f(D_A/\Lambda)\right) = \mathrm{Tr}_{\mathcal{H}}\left(f(UD_AU^*/\Lambda)\right) = \mathrm{Tr}_{\mathcal{H}}\left(f(U^*UD_A/\Lambda)\right) = \mathrm{Tr}_{\mathcal{H}}\left(f(D_A/\Lambda)\right), \tag{3.53}$$

Thus, S is invariant under inner transformations. □

To conclude this chapter, we will show that also the fermionic action is also invariant under inner transformations.

Proposition 3.5.10. *The fermionic action*

$$S_F = \langle \psi, D_A \psi \rangle \tag{3.54}$$

is invariant under inner transformations.

Proof. This is just a little calculation.

$$\langle uJu^*J^*\psi, uJuJ^*D_AJu^*J^*u^*(uJuJ^*\psi)\rangle = \langle uJuJ^*\psi, uJuJ^*D_A\psi\rangle = \langle \psi, D_A\psi\rangle, \tag{3.55}$$

where we have used that uJu^*J^* is a unitary map. □

We see that the spectral action and the fermionic action are indeed invariant under the inner transformations. Therefore, the interpretation of the inner automorphisms as the internal symmetries of the theory is completely in line with the action being given by the spectral action formula and equation (3.25).

So, we have seen in this chapter that we can calculate a spectral action for each spectral triple $(\mathcal{A}, \mathcal{H}, D)$. One can fluctuate the Dirac-operator D by adding a perturbation term $A + JAJ^*$ to the Dirac-operator, where A is a self-adjoint element in $\Omega^1_D(\mathcal{A})$. The operator A is interpreted as a gauge potential that transforms according to the rule $A \mapsto uAu^* + u[D, u^*]$ under a gauge transformation α_u. Under such an internal symmetry or gauge transformation the action does not change. In the next chapter we will reproduce the calculation of Chamseddine and Connes [5]. They derived the Yang–Mills Lagrangian by applying the spectral action principle to a suitable spectral triple. They showed that in this case the perturbation A is indeed a gauge potential in the Yang–Mills theory.

4 NONCOMMUTATIVE GEOMETRY AND YANG-MILLS THEORY

In this chapter we will determine the spectral action for a particular almost commutative spectral triple of dimension 4. Chamseddine and Connes ([5] were the first ones to construct this triple and to calculate the spectral action. They showed that one gets, among other terms, the Yang-Mills Lagrangian for an $su(n)$-gauge theory. This derivation will show that all the information needed to derive the Lagrangian comes from the spectral triple. In this chapter we will redo this calculation. Of course, to obtain the correct results, all relies on the choice of a spectral triple. Therefore, we will first focus on the choice of a suitable spectral triple. Then, as an example and to get a feeling of how the machinery works, we will use the spectral action principle to derive the spectral action. We will see that indeed the Lagrangian for Yang–Mills gauge-theory is obtained. In section 4.2 we show how this Yang–Mills gauge theory is related to the description of gauge theories in terms of principal bundle as explained in section 2.8. However, this principal bundle turns out to be trivial. In later chapters, we will look for spectral triples that give Yang–Mills theories that are described by a non-trivial principal bundles.

4.1 CALCULATION

In this subsection we will show how the Einstein–Yang–Mills Lagrangian can be obtained from a suitable noncommutative spectral triple by using the spectral action principle. The calculation done here is just a reproduction of the calculation first carried out by Chamseddine and Connes ([5]; [8], section 11.4). In this section the dimension of the manifold is 4.

First, we will construct the right spectral triple. The spectral algebra is chosen to be

$$\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C}). \tag{4.1}$$

For future purposes it is convenient to identify the algebra $C^\infty(M) \otimes M_n(\mathbb{C})$ with the algebra $C^\infty(M, M_n(\mathbb{C}))$. The following lemma shows that they are naturally isomorphic.

Lemma 4.1.1. *There is the following isomorphism of $*$-algebras:*

$$C^\infty(M) \otimes M_n(\mathbb{C}) \cong C^\infty(M, M_n(\mathbb{C})) \tag{4.2}$$

Proof. We construct an isomorphism. Define a linear map $\phi : C^\infty(M) \otimes M_n(\mathbb{C}) \to C^\infty(M, M_n(\mathbb{C}))$ induced by

$$f \otimes A \mapsto A \cdot f, \text{ where } (A \cdot f)(x) = A \cdot f(x), \quad (x \in M). \tag{4.3}$$

One can check that this is an algebra-homomorphism with trivial kernel, so it is injective. Furthermore, ϕ is also surjective. Indeed, every entry of an element $s \in C^\infty(M, M_n(\mathbb{C}))$ is a smooth function of x. Call f_{ij} the function at entry (i,j) of s, then

$$\phi\left(\sum_{i,j=1}^{n} f_{ij} \otimes e_{ij}\right) = s, \tag{4.4}$$

where $\{e_{ij}\}$ denotes the canonical basis of $M_n(\mathbb{C})$. Hence the surjectivity of ϕ is proved. The involution is preserved if we define $(g \otimes A)^* = \overline{g} \otimes A^*$ for $g \in C^\infty(M)$ and $A \in M_n(\mathbb{C})$, and extend the involution to whole $C^\infty(M) \otimes M_n(\mathbb{C})$ by (anti-)linearity. Here, the element $\overline{g} \in C^\infty(M)$ is the function that maps x to $\overline{g(x}$, $(x \in M)$. □

The rest of spectral triple is obtained by tensoring the canonical triple of a manifold with the general triple for $M_n(\mathbb{C})$ given by the left action of $M_n(\mathbb{C})$ on the Hilbert-space of $n \times n$ matrices that is endowed with the Hilbert-Schmidt-norm. The Dirac operator is taken to be trivial: $D = 0$. Thus, the spectral product is of the form:

- $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$
- $\mathcal{H} = L^2(M, S) \otimes M_n(\mathbb{C})$
- $D = \not{D}_M \otimes 1$ (4.5)
- $J = J_M \xi \otimes T$
- $\gamma = \gamma_5 \otimes 1.$ (4.6)

Here J_M is the usual real structure on $L^2(M, S)$, which is the charge conjugation operator, and T maps a matrix to its adjoint. The adjoint operation T is a real structure for the triple $(C^\infty(M, M_n(\mathbb{C})), M_n(\mathbb{C}), 0)$ of KO-dimension 0. Since J_M is a real structure of KO-dimension 4, the operator J is a real structure of KO-dimension 4 for the triple (4.6).

The spectral triple is again even because γ^5 and 1_n obey equations (3.6). Hence, the given product spectral triple is indeed an even and real spectral triple. We note that the inner product on $\mathcal{H}$ is given by

$$\langle \xi, \eta \rangle = \int_M \mathrm{Tr}(\xi(x)^* \eta(x)) \sqrt{g} d^4 x, \tag{4.7}$$

where we have surpressed the spinor indices.

Remark 4.1.2. A spectral triple that is obtained by taking the tensor product of the canonical triple with a triple that consists of a noncommutative algebra and a Hilbert-space, is called almost commutative. The spectral triple in (4.5) is thus an example of an almost commutative spectral triple.

To simplify calculations and for future purpose it is convenient to consider the Hilbert-space $L^2(M, B) \otimes M_n(\mathbb{C})$ as the space of square-integrable sections of some vector bundle. For this, first note that $\Gamma(M, S) \otimes M_n(\mathbb{C})$ is a dense subspace of $L^2(M, S) \otimes M_n(\mathbb{C})$. Now the $C^\infty(M)$-module algebra $\Gamma(M, S) \otimes M_n(\mathbb{C})$ can be identified with the $C^\infty(M)$-module algebra $\Gamma(M, S) \otimes_{C^\infty(M)} C^\infty(M, M_n(\mathbb{C})) \cong \Gamma(M, S \otimes B)$ where B is a trivial $M_n(\mathbb{C})$-bundle (check Lemma 2.4.9 for the isomorphism). We identify The Hilbert-space $L^2(M, S) \otimes M_n(\mathbb{C})$ with the space $L^2(M, B \otimes S)$. The corresponding inner product on $L^2(M, B \otimes S)$ is given by

$$\langle \psi, \eta \rangle = \int_M \langle \psi(x), \eta(x) \rangle_{S_x \otimes B_x}, \quad (\psi, \eta \in L^2(M, S \otimes B)), \tag{4.8}$$

where $\langle \cdot, \cdot, \rangle_{S_x \otimes B_x}$ denotes the inner product on the fibre $S_x \otimes B_x$. The corresponding operator D on $L^2(M, B)$ is then

$$\not{D} \otimes 1 + 1 \otimes d, \tag{4.9}$$

where d acts on an element $s \in \Gamma(M, B) \cong C^\infty(M, M_n(\mathbb{C})$ by taking the exterior derivative on each of the entries of the matrix-valued function s.[1] The action of the algebra $C^\infty(M, M_n(\mathbb{C}))$ acts on $\Gamma(M, S) \otimes_{C^\infty(M)} \Gamma(M, B)$ by multiplication on the second argument. The following proposition summarises the above results.

1 Here we silently interchange between the modules $\Gamma(M, S) \otimes_{C^\infty(M)} \Gamma(M, B)$ and $\Gamma(M, S \otimes B)$.

Proposition 4.1.3. *The spectral triple in equation (4.5) can be replaced by*

$$
\begin{aligned}
&\bullet\ \mathcal{A} = C^\infty(M, M_n(\mathbb{C}))\\
&\bullet\ \mathcal{H} = L^2(M, S \otimes B)\\
&\bullet\ D = \not{D}_M \otimes 1 + 1 \otimes d \qquad (4.10)\\
&\bullet\ J = J_M \xi \otimes T\\
&\bullet\ \gamma = \gamma^5 \otimes 1,
\end{aligned}
$$

with the action of $\mathcal{A}$ and D on $\mathcal{H}$ as discussed above. Here T now maps a section $s \in \Gamma(M, E)$ to the section s^* given by $s^*(x) = s(x)^*$, ($x \in M$).

In the rest of the chapter we use the form 4.11 for the spectral triple. We will determine the form of the fluctuated Dirac-operator and calculate the spectral action.

First, we will introduce the vierbein e^a_μ defined by the relation.

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab} \tag{4.11}$$

so that $\gamma_a := \gamma_\mu e^\mu_a$ satisfies the commutation relation $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$.

The Dirac operator $\not{D}_M$ is locally of the form

$$\not{D}_M = i\gamma^\mu((\partial_\mu + \omega^S_\mu) \otimes 1 + 1 \otimes \partial_\mu), \tag{4.12}$$

where $\omega^S_\mu = \frac{1}{2}\omega^{ab}_\mu \gamma_{ab}$ with

$$\gamma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \tag{4.13}$$

The inner fluctuations are given by operators of the form

$$A = \sum_j a_j[D, b_j] \quad \text{with} \quad a_j, b_j \in \mathcal{A}. \tag{4.14}$$

with the additional condition that A is a self-adjoint operator[2]. This expression can be rewritten as:

$$A = \sum_j \gamma \circ (a_j[\nabla, b_j] \otimes 1), \tag{4.15}$$

where $\gamma : \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, S) \otimes_{C^\infty(M)} \Gamma(M, B) \to \Gamma(M, S) \otimes_{C^\infty(M)} \Gamma(M, B)$ is given by

$$\gamma(\omega \otimes \psi \otimes s) = \gamma(\omega)\psi \otimes s. \tag{4.16}$$

Here, $\sum_j a_j[\nabla, b_j]$ is as an operator of the form $\sum_j \omega_j \otimes c_j$ where $\omega_j \in \Omega^1(M)$ and $c_j \in \Gamma(M, B)$.

This can locally be written as $A_\mu \gamma^\mu$ for some $A_\mu \in C^\infty(M, M_n(\mathbb{C}))$. The requirement that $A^* = A$ translates to the statement that $A^*_\mu = A_\mu$, that is all the self-adjoint elements of the algebra $\mathcal{A}$.

Well, recall from equation (3.45) that $D_A = D + A + \varepsilon' JAJ^*$. In this case this gives, using the fact that $\varepsilon' = 1$ in 4 dimensions:

$$A + \varepsilon' JAJ^* = \gamma^\mu A_\mu + J\gamma^\mu A_\mu J^*. \tag{4.17}$$

In even dimensions one has

$$J_M \gamma^\mu J^*_M = -\gamma^\mu. \tag{4.18}$$

2 Since $[D, a]$ is bounded for all $a \in \mathcal{A}$, the operator A is bounded

If we also use that left-multiplication by $JA_\mu J^*$ is right multiplication by A_μ^*, equation (4.17) turns into $A + JAJ^* = \gamma^\mu \cdot \mathrm{ad}(A_\mu)$, since A is self-adjoint. Thus we have arrived at the following expression for the fluctuated Dirac-operator:

$$D_A = D + \gamma^\mu \cdot \mathrm{ad}(A_\mu) = ie_\mu^a \gamma_a((\partial_\mu + \omega_\mu) \otimes 1_{n^2} + 1 \otimes (\partial_\mu + \mathbb{A}_\mu)), \tag{4.19}$$

where $\mathbb{A}_\mu = -i\mathrm{ad}\ A_\mu$. From now on we will denote $\mathbb{A}$ for the ad $su(n)$-valued 1-form whose components are $\mathbb{A}_\mu$. So we see that the $su(n)$-valued gauge potential $-iA_\mu$ acts in the *adjoint representation* on the spinors.

We have now determined the form of the fluctuated Dirac-operators. From equation (2.66) we see that D_A^2 is of the same from as the operator P in equation (3.14), so we can use the heat expansion to calculate the spectral action. The following lemma contains all the information needed to calculate the spectral action with respect to a fluctuated Dirac-operator.

Lemma 4.1.4. *For the spectral triple described above, the fluctuated Dirac-operator D_A is of the form $\nabla^*\nabla - E$ where $\nabla = \nabla^S \otimes 1 + 1 \otimes (d + \mathbb{A})$ and E a unique endomorphism of the bundle $S \otimes B$. Furthermore, we have the following expressions:*

$$E = -\frac{1}{4}R \otimes 1_{n^2} - \sum_{\mu<\nu} \gamma^\mu\gamma^\nu \otimes F_{\mu\nu}$$

$$\Omega_{\mu\nu} = \frac{1}{4}R_{\mu\nu}^{ab}\gamma_{ab} \otimes 1_{n^2} + id_4 \otimes F_{\mu\nu},$$

where $\Omega_{\mu\nu}$ is the curvature of the connection ∇.

Proof. From the generalised Lichenowicz-formula (2.66) we immediately read off the form of D_A and also the expression for E:

$$E = -\frac{1}{4}R \otimes 1_{n^2} - R_V = -\frac{1}{4}s \otimes 1_{n^2} - \sum_{\mu<\nu} \gamma^\mu\gamma^\nu \otimes F_{\mu\nu} \tag{4.20}$$

with $F_{\mu\nu}$ the curvature of the 1-form $\mathbb{A}_\mu$, where $\mathbb{A}_\mu$ is obtained from the A_μ after passing to the adjoint representation. It remains to calculate the curvature tensor $\Omega_{\mu\nu}$ of the connection $\nabla^S \otimes 1 + 1 \otimes (d + \mathbb{A})$:[3]

$$\begin{aligned}
\Omega_{\mu\nu} &= [\partial_\mu + \omega_\mu \otimes 1 + 1 \otimes \mathbb{A}_\mu, \partial_\nu + \omega_\nu \otimes 1 + 1 \otimes \mathbb{A}_\nu] \\
&= (\partial_\mu\omega_\nu - d_\nu\omega_\mu + [\omega_\mu, \omega_\nu]) \otimes 1 + 1 \otimes (\partial_\mu\mathbb{A}_\nu - \partial_\nu\mathbb{A}_\mu + [\mathbb{A}_\mu, \mathbb{A}_\nu]) \\
&= \frac{1}{4}R_{\mu\nu}^{ab}\gamma_{ab} \otimes 1 + 1 \otimes F_{\mu\nu}(\mathbb{A}),
\end{aligned}$$

where we have expressed the curvature of the spin connection in terms of $R_{\mu\nu}^{ab}$ and γ_{ab}. Here $R_{\mu\nu}^{ab} := R_{\rho\sigma\mu\nu}e^{a\rho}e^{b\sigma}$ and we have used equation (2.56). □

With equation (4.19) and Lemma 4.20 we are now ready to determine the spectral action of the fluctuated Dirac operator D_A. The calculation of the spectral action is done in the following theorem.

Theorem 4.1.5. *For the product geometry $(\mathcal{A}, \mathcal{H}, D)$ described above, the spectral action is given by*

$$Tr(f(D/\Lambda)) \sim \frac{1}{4\pi^2} \int_M \mathcal{L}(g_{\mu\nu}, A)\sqrt{g}d^4x \tag{4.21}$$

where $\mathcal{L}(g_{\mu\nu}, A)$ is the Lagrangian of the Einstein-Yang-mills system given by

$$\mathcal{L}(g_{\mu\nu}, A) = 2n^2\Lambda^4 f_4 + \frac{n^2}{6}\Lambda^2 f_2 R + \frac{f(0)}{6}\mathcal{L}_{YM}(A) - \frac{n^2 f(0)}{80}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \tag{4.22}$$

3 note that the 1-form $\mathbb{A}$ takes values in $\mathrm{End}(M_n(\mathbb{C}))$ so that $d + \mathbb{A}$ is again a connection (it is even a hermitian connection).

Proof. Recalling the results of subsection 3.3.1 we will only need to calculate the Seeley-DeWitt coefficients a_0, a_2, a_4. Since we work in dimension 4, we can then substitute the expressions for a_0, a_2, a_4 in equation (3.23) to get the expansion of the spectral action.

We will now calculate the Seeley-DeWitt coefficients.

$$a_0(P) = \int_M (4\pi)^{-2}\mathrm{Tr}(\mathrm{Id})\sqrt{g}d^4x = \int_M (4\pi)^{-2}4n^2\sqrt{g}d^4x = \frac{n^2}{4\pi^2}\int_M \sqrt{g}d^4x, \tag{4.23}$$

and secondly:

$$\begin{aligned} a_2(P) &= \int_M (4\pi)^{-2}\mathrm{Tr}\left(-\frac{R}{6}\mathrm{Id} + E\right)\sqrt{g}d^4x \\ &= \int_M (4\pi)^{-2}\left(-4n^2\frac{R}{6} + \frac{1}{4}R4n^2\right)\sqrt{g}d^4x \\ &= \int_M \frac{n^2}{4\pi^2}\frac{R}{12}\sqrt{g}d^4x \\ &= \frac{n^2}{48\pi^2}\int_M R\sqrt{g}d^4x, \end{aligned} \tag{4.24}$$

where we have used that

$$\sum_{\mu<\nu}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu \otimes F_{\mu\nu}\right) = \sum_{\mu,\nu}\mathrm{Tr}(\gamma^\mu\gamma^\nu)\mathrm{Tr}(F_{\mu\nu}) = \sum_{\mu,\nu} 4g^{\mu\nu}\mathrm{Tr}(F_{\mu\nu}) = 0, \tag{4.25}$$

because $F_{\mu\nu}$ is anti-symmetric and $g^{\mu\nu}$ is symmetric in its indices.

Now, for the coefficient $a_4(P)$ we note that the boundary terms $E_{;\mu}{}^{\mu}$ and $R_{;\mu}{}^{\mu}$ vanish since M is a compact manifold. We will first look at the terms containing the Yang–Mills Langrangian $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. We have

$$\begin{aligned} \mathrm{Tr}(E^2) &= \mathrm{Tr}\left(\frac{R^2}{16}\otimes 1_{n^2}\right) + \frac{1}{4}\sum_{\mu,\nu,\rho,\sigma}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) \\ &= \frac{n^2}{4}R^2 + (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) \\ &= \frac{n^2}{4}R^2 - 2\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \\ &= \frac{n^2}{4}R^2 + 2\mathrm{Tr}(F_{\mu\nu}\overline{F}^{\mu\nu}) \end{aligned} \tag{4.26}$$

where the factor 4 comes from the dimension of the spinors and the minus sign comes from $F^{\mu\nu} := -\bar{F}^{\mu\nu}$. We have also used that equation (A.4).

For $\mathrm{Tr}(\Omega_{\mu\nu}\Omega_{\mu\nu})$ we have

$$\begin{aligned} \mathrm{Tr}(\Omega^{\mu\nu}\Omega_{\mu\nu}) &= \mathrm{Tr}\left(\frac{1}{16}R^{ab\mu\nu}\gamma_{ab}R^{cd}_{\mu\nu}\gamma_{cd}\otimes 1_{n^2} + id_4\otimes F^{\mu\nu}F_{\mu\nu}\right) \\ &= \frac{2\cdot 2n^2}{16}\mathrm{Tr}\left(\sum_{a<b}\sum_{c<d}R^{ab\mu\nu}\gamma_{ab}R^{cd}_{\mu\nu}\gamma_{cd}\right) + 4\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \end{aligned}$$

where the cross terms vanish because $\mathrm{Tr}(\gamma_{ab}) = 0$. Since $\mathrm{Tr}(\gamma_{ab}\gamma_{cd}) = 4\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d) = -4\delta_{ac}\delta_{bd}$ when $a < b$ and $c < d$ (see equation (A.9), we get

$$\begin{aligned}\mathrm{Tr}\left(\Omega^{\mu\nu}\Omega_{\mu\nu}\right) &= -\frac{4n^2}{4}\sum_{a<b} R^{ab\mu\nu}R_{ab\mu\nu} + 4\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) = -\frac{n^2}{2}R^{ab\mu\nu}R_{ab\mu\nu} + 4\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \\ &= -\frac{n^2}{2}R^{ab\mu\nu}R_{ab\mu\nu} - 4\mathrm{Tr}(\overline{F}^{\mu\nu}F_{\mu\nu})\end{aligned}$$

where in the first step a factor 4 comes from the spinor dimension.Hence, the factor before the Yang–Mills Lagrangian $\mathrm{Tr}(\bar{F}^{\mu\nu}F_{\mu\nu}$ in $a_4(x,P)$ is $\frac{1}{16\pi^2}(180\cdot 2 - 30\cdot 4) = \frac{1}{24\pi^2}$.

We can now evaluate $a_4(x,P)$ modulo boundary terms:

$$\begin{aligned}a_4(x,P) &= \frac{1}{360\cdot 16\pi^2}\left(\mathrm{Tr}(5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RE + E^2) - \frac{30n^2}{2}R^{ab\mu\nu}R_{ab\mu\nu}\right) \\ &+ \frac{1}{24\pi^2}\mathrm{Tr}(F_{\mu\nu}\bar{F}^{\mu\nu})\end{aligned}$$

If we now again use that $\mathrm{Tr}(E) = n^2R$, recall from equation (4.26) that the Riemann contribution of $\mathrm{Tr}(E^2)$ is $\frac{n^2}{4}R^2$ and calculate that $R^{ab\mu\nu}R_{ab\mu\nu} = R^{\rho\sigma\mu\nu}e^a_\rho e^b_\sigma R_{\tau\gamma\mu\nu}e^\tau_a e^\gamma_b = R^{\rho\sigma\mu\nu}R_{\tau\gamma\mu\nu}g^\tau_\rho g^\gamma_\sigma = R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu}$, we get

$$\begin{aligned}a_4(x,P) &= \frac{n^2}{360\cdot 16\pi^2}\left(20R^2 - 8R_{\mu\nu}R^{\mu\nu} + 8R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60R^2 + 45R^2 - 15R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \\ &+ \frac{1}{24\pi^2}\mathrm{Tr}(F_{\mu\nu}\bar{F}^{\mu\nu}) \\ &= \frac{n^2}{360\cdot 16\pi^2}(5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \frac{1}{24\pi^2}\mathrm{Tr}(F_{\mu\nu}\bar{F}^{\mu\nu}) \qquad (4.27)\end{aligned}$$

We can rewrite this expression of a_4 by using:

$$R^*R^* := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \qquad (4.28)$$

This term is a topological term, called the Pontryagin class. Using the identity

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \qquad (4.29)$$

we can recast $a_4(P)$ into the alternative form

$$a_4(P) = \frac{n^2}{16\pi^2}\int_M (-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*)\sqrt{g}d^4x + \frac{1}{24\pi^2}\int_M L_{YM}(A)\sqrt{g}d^4x. \qquad (4.30)$$

Here $C_{\mu\nu\rho\sigma}$ is the Weyl-tensor. This can be shown by observing, using equations (4.28), (4.29),

$$\begin{aligned}&-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^* \\ =\ &-\frac{1}{20}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{10}R_{\mu\nu}R^{\mu\nu} - \frac{1}{60}R^2 + \frac{11}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{11\cdot 4}{360}R_{\mu\nu}R^{\mu\nu} + \frac{11}{360}R^2 \\ =\ &\frac{1}{360}(-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 5R^2 - 8R_{\mu\nu}R^{\mu\nu}) \qquad (4.31)\end{aligned}$$

From now on, we will drop the topological term R^*R^*.

Inserting the above expressions for a_0, a_2, a_4 from equations (4.23), (4.24), (4.30) into (3.23) yields the following expression:

$$f(D/\Lambda) \sim \int_M \left(f(0)\Lambda^0 \left(-\frac{n^2}{16\pi^2}\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{24\pi^2}L_{YM}(A) \right) \right.$$
$$\left. + 2f_2\Lambda^2\frac{n^2}{48\pi^2}R + 2f_4\Lambda^4\frac{n^2}{4\pi^2} \right) \sqrt{g}d^4x$$

This completes the proof of the proposition. □

We see that the expansion of the spectral action in Theorem 4.1.5 consists of four terms. The term containing f_4 is the cosmological term. The other two terms are the Einstein-Hilbert action (the term with f_2) and the Yang-Mills action $\int_M \mathcal{L}_{YM}$. Therefore, we have come to the remarkable conclusion that the spectral triple in Proposition 4.1.3 not only yields the Lagrangian for $su(n)$-Yang–Mills theory but also the Einstein-Hilbert action when the spectral action is applied.[4]

Remark 4.1.6. The action in Theorem 4.1.5 can be cast into a global form by noting that $F_{\mu\nu}F^{\mu\nu}\varepsilon \sim F \wedge *F$, where F is the curvature tensor of the connection $d + \mathbb{A}$, ε is the volume form of the manifold M and $*F$ is the hodge-dual 2-form of F. The Yang–Mills Lagrangian is then of the form

$$\int_M \mathrm{Tr}(F \wedge *F). \tag{4.32}$$

4.2 NONCOMMUTATIVE GEOMETRY AND GAUGE THEORY

The spectral triple (4.5) and the calculation above in Theorem 4.1.5 is an example of the way Lagrangians (or actions) can be obtained from spectral triples by applying the spectral action principle. We have seen that we get a Yang–Mills Lagrangian with respect to some $su(n)$-valued gauge potential. Moreover, we have seen that this gauge potential acts in the adjoint representation of the spinors. There are some remarkable conclusion that can be drawn from these results with respect to the description of gauge theories in terms of a principal bundle P.

Since the gauge potential $-iA_\mu$ is an $su(n)$-valued 1-form on M it can be interpreted as a connection one 1-form for a trivial principal $SU(n)$- or $PSU(n)$-bundle P (see Definition 2.5.1), since $SU(n)$ and $PSU(n)$ both have Lie-algebra $su(n)$. Moreover, the Yang–Mills Lagrangian is obtained. From these observation we could have $PSU(n)$ Yang–Mills theory as well as $SU(n)$-gauge theory. However, since the particles are also sitting in the adjoint representation (cf. equation (4.19)), the centre of $SU(n)$ cancels out and the symmetry group of the theory is more likely to be $PSU(n)$ rather than $SU(n)$.[5] This gives a remarkable resemblance between the noncommutative geometric description of Yang-Mills theory and the one in terms of a principal bundle P, albeit the bundle P in this case is trivial.

One could ask if one can choose a spectral triple that describes Yang–Mills theory in terms of a nontrivial bundle P. In the rest of this thesis it is precisely this question we will study: *can we choose a spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ in such a way that it describes $PSU(n)$-gauge theory in terms of a nontrivial $PSU(n)$-bundle just like the spectral triple (4.5) describes $PSU(n)$-gauge theory with a trivial $PSU(n)$-bundle P?*

4 For a discussion of the term with the Weyl-tensor $C_{\mu\nu\rho\sigma}$ see [8], section 17.11.
5 When we go to the nontrivial case, we will see that we indeed get a $PSU(n)$-Yang–Mills theory.

Remark 4.2.1. There is another interesting point. The internal symmetry group of the spectral triple is given by $\mathrm{Inn}(\mathcal{A})$ and the gauge group for a principal G-bundle P is $\Gamma(M, \mathrm{Ad}\, P)$. To get a full correspondence between both descriptions of Yang–Mills theory (the one using spectral triples and the one using principal fibre bundle), we want these two groups to be isomorphic. In the case of $G = PSU(n)$ I have not yet been able to find out if both groups are isomorphic or not. If they are isomorphic, we would have another argument to believe that we actually deal with $PSU(n)$-Yang–Mills theory. When $G = SU(n)$ both groups are certainly not isomorphic since $\Gamma(\mathrm{Ad}\, P)$, where P is a principal $SU(n)$-bundle, has nontrivial centre.

5 BUNDLE CONSTRUCTIONS AND FINITELY GENERATED PROJECTIVE MODULES

In the previous chapter we have calculated the spectral action for the real and even spectral triple $(C^\infty(M) \otimes M_n(\mathbb{C}), L^2(M,S) \otimes M_n(\mathbb{C}), \not{D} \otimes 1, J_M \otimes T, \gamma^5 \otimes 1)$ and it turned out that the spectral action yields the Yang–Mills Lagrangian with respect to some $su(n)$-valued gauge field A. The remarkbable observation was that A could also be interpreted as a gauge potential for a principal $PSU(n)$-bundle. albeit this bundle was topologically trivial.

The main goal of this thesis is to generalise the above situation. That is, we try to construct a spectral triple that gives $PSU(n)$-Yang-Mills theory in terms of arbitrary non-trivial bundles. An important observation here is that we started with the algebra $C^\infty(M) \otimes M_n(\mathbb{C})$ which is precisely the space of sections of the trivial bundle over M with fibre $M_n(\mathbb{C})$. This suggests taking $\Gamma(M, B)$, where B is a non-trivial algebra bundle with fibre $M_n(\mathbb{C})$, as the algebra in our new spectral triple. Schematically, the plan is as follows:

- First, we will construct an algebra bundle B where each fibre is isomorphic to $M_n(\mathbb{C})$, but we allow for non-trivial transition functions between B_U and B_V for charts $U, V \subset M$.
- Then, we construct the algebra of sections $\Gamma(M, B)$ of the bundle B. This replaces the previous algebra of sections of the trivial bundle $M \times M_n(\mathbb{C})$.
- We will try to construct, in a natural way, a principal $PSU(n)$-bundle P from the algebra of sections $\Gamma(M, B)$. Here, we would like Inn($\mathcal{A}$) to be isomorphic to $\Gamma(M, \mathrm{Ad}\, P)$.
- In Chapter 6, the spectral triple will be completed by choosing a suitable Hilbert space and Dirac-operator. Furthermore, we will examine which further conditions are needed on the spectral triple in order to make a real structure possible. This real structure should be a a generalisation of the real structure J in the trivial case.
- Finally, in Chapter 7 the inner fluctuations will be determined and we will calculate the spectral action, containing the Yang–Mills action

$$\int_M \mathrm{Tr}(F \wedge \star F), \tag{5.1}$$

with F the curvature of a gauge potential A belonging tothe nontrivial $PSU(n)$-bundle P. This, then, extends the procedure carried out in section 4.1 to the non-trivial case.

5.1 ALGEBRA BUNDLES

In the above scheme we mentioned the word algebra bundle but until now we have not defined what an algebra bundle is. Since we want to be as clear as possible we given the definition of a (real or complex) algebra bundle (with finite-dimensional fibre) here in this section. This definition is in complete analogue with the definition of a vector bundle.

Definition 5.1.1. *Let B, M be smooth manifolds and $\mathbb{F}$ be the field of real or complex numers. An $\mathbb{F}$-algebra bundle over M is a surjective smooth map $\pi : B \to M$ such that*

- *For each point $x \in M$ the fibre $B_x = \pi^{-1}(x)$ of π over x has the structure of a finite-dimensional $\mathbb{F}$-algebra.*
- *For any point $x \in M$ there exists an open neighbourhood U of x, a finite-dimensional $\mathbb{F}$-algebra A and a diffeomorphism ϕ_U such that the following diagram commutes,*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi_U"'] & U \times A \arrow[dl, "proj_1"] \\
U &
\end{tikzcd} .$$

Moreover, $\phi_U | B_x \to A$ is an isomorphism of algebras for each $x \in M$.

We call M the base space, B the total space. The map ϕ_U is called a local trivialisation of B with trivialising neighbourhood U.

Definition 5.1.2. *A smooth map $s : M \to B$ is called a section of B if $\pi \circ s = id_M$. If s is only defined on some neighbourhood $U \subset M$, then s is called a local section.*

Contrary to what we have seen in the case of principal bundles, any algebra bundle (trivial or not) admits a global section, namely the zero section given by

$$\sigma_0 : M \to B, \quad \sigma_0(x) = \phi_U^{-1}(x, 0), \tag{5.2}$$

where $\phi_U : \pi^{-1}(U) \to U \times \mathcal{A}$ is a local trivialisation. This section is well-defined, that is, it does not depend on the choice of a local trivialisation of a neighbourhood of x, since any algebra homomorphism maps zero to zero.[1] This is different in the case of principal G-bundles where global sections exists if and only if the principal G-bundle is trivial.

The space of sections of B defined on some open $U \subset M$ has a natural algebra structures given by:

$$\begin{aligned}
(s_1 + s_2)(x) &= s_1(x) + s_2(x), \\
(s_1 s_2)(x) &= s_1(x) + s_2(x), \\
(\lambda s)(x) &= \lambda s(x),
\end{aligned}$$

where s, $s_{1,2}$ are sections of B on U and $\lambda \in \mathbb{F}$. In particular: $\Gamma(M, B)$ is an algebra. If the fibres of the algebra bundle B also have a star structure, then we speak of a $*$-algebra bundle if, in addition to the requirements above, the map $\phi_U|_{B_x} : B_x \to \{x\} \times A$ satisfies $\phi_U|_{B_x}(a^*) = (\phi_U|_{B_x}(a))^*$, where A is now also an algebra with involution. The star structure on the fibres induces a star operation on the space of sections $\Gamma(M, B)$ where $\sigma^* \in \Gamma(M, B)$ is given by $\sigma^*(x) = \sigma(x)^*$, $(x \in M)$.

Just as for general fibre bundles, we can study the so-called transition functions. Let $\phi_U : \pi^{-1}(U) \to U \times A$, $\phi_V : \pi^{-1}(V) \to V \times A$ be two local trivialisations with $U \cap V \neq 0$. Then the transition functions $g_{uv} : U \cap V \to \mathrm{Aut}(A)$ are defined by

$$\phi_V \phi_U^{-1}(x, a) = (x, g_{uv}(x)a), \tag{5.3}$$

for $(x, a) \in U \cap V \times A$. That, for fixed x, the map $g_{uv}(x) : A \to A$ is an algebra automorphism of $\mathcal{A}$ follows directly from the fact that $\phi_V|B_x$, $\phi_U|B_x$ are algebra isomorphisms. The transition functions obey the following 'cocycle conditions':

1 Hence, every vector bundle also allows a global section.

$$\begin{aligned} g_{uv}(x)g_{vu}(x) &= \mathrm{Id}_A \quad \forall x \in U \cap V; \\ g_{uu}(x) &= \mathrm{Id}_A \quad \forall x \in U; \\ g_{uv}(x)g_{vw}(x)g_{wu}(x) &= \mathrm{Id}_A \quad \forall x \in U \cap V \cap W. \end{aligned}$$

The definition of transition functions can naturally be extended to $*$-algebra bundles. We speak of a unital $*$-algebra bundle, if each fibre has the structure of a unital algebra.

Definition 5.1.3. *Let B_1, B_2 be two $\mathbb{F}$-($*$-)algebra bundles. A smooth map $\phi : B_1 \to B_2$ is said to be a(n) ($*$-)algebra bundle homomorphism if it is a vector bundle homomorphism and the restriction $\phi - |(B_1)_x : (B_1)_x \to (B_2)_x$ is a(n) ($*$-)algebra homomorphism.*

In the next section we will study these algebra bundles a bit more.

5.2 FROM UNITAL FINITELY GENERATED PROJECTIVE ALGEBRA BUNDLE WITH INVOLUTION TO PRINCIPAL BUNDLES

In this section all algebras are assumed to be over the complex field unless stated otherwise. Let M be a compact manifold. As explained in the introduction of this algebra our new spectral algebra will be the involutive and unital algebra of sections $\Gamma(M, B)$ of a (possibly nontrivial) involutive $*$-algebra bundle with fibre $M_n(\mathbb{C})$. In this section we establish a link between this algebra $\Gamma(M, B)$ and principal $PSU(n)$-bundles. This is done in steps and along the way we will prove some statements that hold for general $*$-algebra bundles over a compact manifold M. The steps are as follows: in subsection 5.2.1 we will construct a bijective correspondence between finitely generated projective $C^\infty(M)$-modules and complex vector bundles over M. This result is already well-known and was first established by Swan ([33]) for topological vector bundles over a normal (topological) space X. In subsection 5.2.2 we will use this result to construct an (involutive) algebra bundle from its (involutive) algebra of sections. In the case that B is a unital $*$-algebra bundle with fibres $M_n(\mathbb{C})$ we will show how a $PSU(n)$-bundle can be constructed from the algebra bundle B. This will be done in subsection 5.2.3.

5.2.1 *Vector bundle reconstruction*

The main goal of this subsection is to reconstruct a vector bundle from its space of sections. Along the way we will prove some other useful propositions. The results in this section are already well-known and the proofs here are based on proofs written in [29], [33]. Some categorical language, words like functor, are often used. In Appendix C some information on categories can be found.

Let $\pi : E \to M$ be a complex vector bundle and denote by $\Gamma(M, E)$ the complex vector space of smooth sections of E. It can be turned into a module over the (commutative) algebra of functions $C^\infty(M)$ by defining the action of $C^\infty(M)$ via the pointwise product:

$$sa(x) := s(x)a(x), \quad \text{for } s \in \Gamma(M, E), a \in C^\infty(M), x \in M. \tag{5.4}$$

If the fibres, and therefore the space of sections, have a star structure then

$$(sa)^*(x) = ((sa)(x))^* = (s(x)a(x))^* = s(x)^*a(x)^* = (s^*a^*)(x), \tag{5.5}$$

so that $(sa)^* = s^*a^*$. Here, for given $a \in C^\infty(M)$, the star operation is defined as $a^*(x) = a(x)^*$ $(x \in M)$.

Let $\tau : E \to E'$ be a bundle map. Then there is a $C^\infty(M)$-linear map $\Gamma\tau : \Gamma(M, E) \to \Gamma(M, E')$, given by $\Gamma\tau(s) = \tau \circ s$. The $C^\infty(M)$-linearity of $\Gamma\tau$ follows from the linearity of each $\tau_x : E_x \to E'_x$. Indeed,

$$\Gamma\tau(sa)(x) = \tau(s(x)a(x)) = \tau(s(x))a(x) = (\Gamma\tau(s))(x)a(x) = (\Gamma(\tau(s))a)(x) \tag{5.6}$$

for all $x \in M$. Thus, $\Gamma\tau(sa) = \Gamma(\tau s)a$ for all $s \in \Gamma(M, E)$, $a \in C^\infty(M)$. Moreover, $\Gamma(\mathrm{id}_E) = 1_{\Gamma(M,E)}$ and $\Gamma(\tau \circ \sigma) = \Gamma\tau \circ \Gamma\sigma$ so that Γ is a functor from the category of vector bundles to the category of $C^\infty(M)$-modules, sending

$$\begin{aligned} E &\to \Gamma(M, E) \\ (\tau : E \to E') &\to (\Gamma\tau : \Gamma(M, E) \to \Gamma(M, E')). \end{aligned}$$

We will show that $\Gamma(M, E)$ is actually a finitely generated projective $C^\infty(M)$-module. This implies that the functor Γ assigns to each vector bundle a finitely generated projective $C^\infty(M)$-module. Conversely, we will show that to any finitely generated projective $C^\infty(M)$-module $\mathcal{P}$ we can assign a vector bundle E such that $\Gamma(M, E) \cong \mathcal{P}$. This makes the functor Γ essentially surjective (see Appendix C).

Theorem 5.2.1 (Swan, [33]). *Let M be a compact manifold. Then any vector bundle E over M is a direct summand of a trivial bundle.*

This will be needed in order to prove the following statements

Proposition 5.2.2 (Swan, [33]). *For any (complex) vector bundle E over a compact space M, the $C^\infty(M)$-module $\Gamma(M, E)$ is finitely generated and projective.*

Proof. There exists a vector bundle $E_0 \to M$ and $k \in \mathbb{N}$ such that $E \oplus E_0 \cong M \times \mathbb{C}^k$ by Theorem 5.2.1. It is straightforward to check that the map

$$\phi : \Gamma(M, E) \times \Gamma(M, E_0) \to \Gamma(M, E \oplus E_0), \quad (s_1, s_2) \mapsto s_1 \oplus s_2, \tag{5.7}$$

is an isomorphism of $C^\infty(M)$-modules. Under the natural identification of $\Gamma(M, E) oplus \Gamma(M, E_0)$ with $\Gamma(M, E) \times \Gamma(M, E_0)$ we get that $\Gamma(M, E) \oplus \Gamma(M, E_0) = \Gamma(M, E \oplus E_0)$. Hence, $\Gamma(M, E_0)$ is a direct summand of a finitely generated free module and thus it is finitely generated projective by Lemma 2.1.6. $\square$

The converse of Proposition 5.2.2, namely that to any finitely generated projective $C^\infty(M)$-module $\mathcal{P}$, there is a vector bundle whose space of sections is isomorphic to $\mathcal{P}$, is a little bit technical. The details are discussed in some lemmae.

Lemma 5.2.3. *Let E be a trivial vector bundle over M. Then $\Gamma(M, E)$ is a free $C^\infty(M)$-module.*

Proof. First, let $E = M \times V$ be a trivial vector bundle. Choose a basis $\{v_1, \dots, v_k\}$ of V. Define the sections $e_i(x) = (x, v_i)$, $(x \in M)$ for $i = 1, \dots, k$. Then $\{e_1, \dots, e_k\}$ is a free basis of the module $\Gamma(M, E)$. Hence, $\Gamma(M, E)$ is free. $\square$

It might be clear what the construction will look like. Let $\mathcal{P}$ be a finitely generated projective $C^\infty(M)$-module. This is equivalent to saying that $\mathcal{P}$ is a direct summand of a finitely generated free $C^\infty(M)$-module. By Lemma 5.2.3 there exists a trivial bundle $\pi : \zeta \to M$ so that $\Gamma(M, \zeta) \cong \mathcal{P} \oplus \mathcal{Q}$ for some $C^\infty(M)$-module $\mathcal{Q}$. Then we define the fibre $E_x := \mathcal{P}_x := \{p(x) | p \in \mathcal{P}\}$. (In the same way $\mathcal{Q}_x$ can be defined.)

Proposition 5.2.4 ([29],Theorem 11.32). *Let $\mathcal{P}$, $\mathcal{Q}$, $\pi : \zeta \to M$ be as above, then $E = \cup E_x$ is a subbundle of ζ and $\Gamma(M, E) \cong \mathcal{P}$ as $C^\infty(M)$-modules.*

Proof. The following lemma plays a crucial role.

Lemma 5.2.5 ([29],Lemma 11.8b). *Let $\pi : E \to M$ be a vector bundle. Suppose s is a section with $s(x) = 0$ for some $x \in M$. Then there exist functions f_i with $f_i(x) = 0$ and sections $s_i \in \Gamma(M, E)$ so that s can be written as a finite sum $s = \sum_i f_i s_i$.*

Proof. Suppose that the bundle E is trivial, then there exist a free basis $\{s_1, \dots, s_k\}$ of $\Gamma(M, E)$. Therefore, a given section can be spanned over this basis as $s = \sum_{i=1}^k f_i s_i$ for certain $f_i \in C^\infty(M)$. The equality $s(x) = 0$ then implies that $f_i(x) = 0$ for all $i = 1, \dots, k$, so in the trivial case the lemma is true. [2].

For the nontrivial case, there exists a neighbourhood $U \subset M$ of x, a finite number of sections $s_i \in \Gamma(U, E)$ and functions $f_i \in C^\infty(U)$ with $f_i(x) = 0$, so that $s_{|U} = \sum_i f_i s_i$ (this is the previous paragraph). Choose a function $f \in C^\infty(M)$ such that supp$f \subset U$ and $f(x) = 1$. Now, extend the functions $f f_i \in C^\infty(U)$ and the sections $f s_i \in \Gamma(U, E)$ to the whole of M by setting them zero outside U. Now, we can write (keeping the notation $f f_i$ and $f s_i$ for their extensions as well)

$$f^2 s = \sum_i (f f_i)(f s_i), \tag{5.8}$$

so that

$$s = (1 - f^2)s + \sum_i (f f_i)(f s_i). \tag{5.9}$$

Here we remark that $(1 - f^2)(x), f_1(x), \dots, f_k(x)$ are indeed zero. □

Lemma 5.2.6. *For any $x \in M$, the fibre ζ_x can be decomposed as $\zeta_x = \mathcal{P}_x \oplus \mathcal{Q}_x$ and the dimensions of $\mathcal{P}_x$ and $\mathcal{Q}_x$ are locally constant.*

Proof. For given $y \in \zeta_x$, choose $s \in \Gamma(M, \zeta)$ such that $s(x) = y$ and write $s = p + q$ for some $p \in \mathcal{P}$ and $q \in \mathcal{Q}$. Then $y = p(x) + q(x) \in \mathcal{P}_x + \mathcal{Q}_x$. Now we show that $\mathcal{P}_x \cap \mathcal{Q}_x = 0$. Suppose that $y \in \mathcal{P}_x \cap Q_x$. Then there exist $p' \in \mathcal{P}$, $q' \in \mathcal{Q}$ with $p'(x) = q'(x) = y$. This gives $(p' - q')(x) = 0$ so that there exist elements $p_i \in \mathcal{P}$, $q_i \in \mathcal{Q}$, $f_i \in \{g \in C^\infty(M) | g(x) = 0\}$ such that $p - q = \sum_i f_i p_i + \sum_i f_i q_i$. Since $\mathcal{P} \cap \mathcal{Q} = 0$, we have $p' = \sum_i f_i p_i$, which implies $p'(x) = y = 0$. Hence $\mathcal{P}_x \cap \mathcal{Q}_x = 0$. This proves the first statement of the lemma.

For the second, let dim$E_x = r$. Choose $s_1, \dots, s_r \in \mathcal{P}$ such that $s_1(x), \dots, s_r(x)$ span $\mathcal{P}_x$. The continuity of the sections implies that $s_1(y), \dots, s_r(y)$ form an independent set in E_y for all y in some neighbourhood of x. Therefore, dim$\mathcal{P}_y \geq$ dim$\mathcal{P}_x$. Similarly, it can be shown that dim$\mathcal{Q}_y \geq$ dim$\mathcal{Q}_x$ in some neighbourhood of x. Since the sum dim$\mathcal{P}_y$ + dim$\mathcal{Q}_y$ is locally constant, it follows that dim$\mathcal{P}_y$ is locally constant.

□

We continue with the proof of Proposition 5.2.4. It is immediate that $E_x \subset \zeta_x$ is a subspace of ζ_x since for $p, q \in E_x \subset \zeta_x$ we have that $(s_1 + s_2)(x) = p + q$ if $s_1(x) = p$ and $s_2(x) = q$. In the same way it follows that E_x is closed under scalar multiplication. The projection map $\pi_E : E \to M$ is just the restriction of $\pi : \zeta \to M$ to E. It remains to check if E is locally trivial. Given $x \in M$, find sections $s_1 \dots s_k \in \mathcal{P}$ such that $s_1(x), \dots, s_k(x)$ span E_x. Such a basis clearly exists. The continuity of the

2 the sections s_i are never zero since $\{s_i\}$ forms a basis for E_y at each point $y \in M$

sections and the fact that the dimension of the fibre E_x is locally constant imply that $s_1(y), \dots, s_k(y)$ form a basis for E_y for all y in some neighbourhood U of x. Then, $\pi^{-1}(U) \cong U \times V$ through the map $E_y \ni e = \sum_i^k \lambda_i s_i(y) \mapsto (y, \lambda_1, \dots, \lambda_k)$. Hence, E is locally trivial.

Is $\Gamma(M, E)$ indeed isomorphic to $\mathcal{P}$ as $C^\infty(M)$-modules? From the above construction it follows that $\mathcal{P} \subset \Gamma(M, E)$. Conversely, let $s \in \Gamma(M, E) \subset \Gamma(M, \zeta)$ be given. Then there are elements $p \in \mathcal{P}$, $q \in \mathcal{Q}$ such that $s = p + q$. Since $\mathcal{P}_x \cap \mathcal{Q}_x = 0$, it follows that $q(x) = 0$ for all $x \in M$. Hence $q = 0$ and $s = p \in \mathcal{P}$. □

We have shown that the functor Γ assigns to each vector bundle a finitely generated projective module $\Gamma(M, E)$. Conversely, the construction in Proposition 5.2.4 shows that to any finitely generated projective $C^\infty(M)$-module $\mathcal{P}$ we can construct a vector bundle E such that $\Gamma(M, E) \cong \mathcal{P}$ as $C^\infty(M)$-modules. In other words, the functor Γ from the category of (complex) vector bundles over M to the category of finitely generated projective $C^\infty(M)$-modules is essentially surjective. It has been proved by Swan [33] (see also Nestruev [29], Theorem 11.29) that the functor Γ is faithful and full.

Proposition 5.2.7 ([33])**.** *The functor Γ, defined as earlier, is faithful and full.*

Proof. First, we show Γ is faithful. Indeed, let E_1, E_2 be two vector bundles over M and suppose that $\phi, \psi : E_1 \to E_2$ are vector bundle homomorphism such that $\Gamma(\phi) = \Gamma(\psi) \Leftrightarrow \phi(s(x)) = \psi(s(x))$ for all $s \in \Gamma(M, E_1)$, $x \in M$. Since any point $e \in E_1$ is represented by $s(x)$ for some $s \in \Gamma(M, E_1)$ and $x \in M$, it follows that $\phi = \psi$.

To show Γ is full, we prove that for given $F : \Gamma(M, E_1) \to \Gamma(M, E_2)$ the bundle homomorphism ϕ defined by

$$\phi(e) = F(s)(x), \tag{5.10}$$

where $s \in \Gamma(M, E_1)$ and $x \in M$ are chosen so that $e = s(x)$, is mapped to F by Γ. The map ϕ is well-defined: let s_1, s_2 be two sections such that $s_1(x) = s_2(x)$. Then $s_1 - s_2 = \sum_i f_i s_i$, where the s_i are sections of E_1 and the f_i are functions on M that vanish in x. Therefore, $F(s_1 - s_2)(x) = 0$. Furthermore, $(\Gamma(\phi)s)(x) = \phi(s(x)) = F(s)(x)$, so that $\Gamma(\phi) = F$. Finally, the map ϕ is smooth on $\pi^{-1}(U)$ for any open set $U \subset M$ over which both E_1 and E_2 are both locally trivial (see also [29]). Therefore ϕ is smooth on M. □

Together with the essential surjectivity Proposition 5.2.7 establishes an equivalence of categories (see Appendix C). This implies that to any finitely generated projective $C^\infty(M)$-module $\mathcal{P}$ there is (up to isomorphism) a unique vector bundle E such that $\Gamma(M, E) \cong \mathcal{P}$. In the next subsection we will particularly be interested in the construction carried out in Proposition 5.2.4. For clarity we summarise this section in the following theorem.

Theorem 5.2.8 (Serre-Swan)**.** *Let M be a compact manifold. There is an equivalence between the category of complex vector bundles over M and the category of finitely generated projective $C^\infty(M)$-modules. In particular this establishes a bijective correspondence of isomorphism classes of vector bundles over M and isomorphism classes of finitely generated projective $C^\infty(M)$-modules: to an isomorphism class of a vector bundle one associates the isomorphism class of its $C^\infty(M)$-module of sections; conversely, to an isomorphism class of any finitely generated projective $C^\infty(M)$-module one associates the class of the vector bundle obtained via the construction in Proposition 5.2.4. These operations are each others inverses.*

5.2.2 *Algebra bundle reconstruction*

Let M be a compact manifold. In the previous section we have shown that the space of sections $\Gamma(M, E)$ of some vector bundle E over M is a finitely generated projective $C^\infty(M)$-module. Conversely, we have shown that to any finitely generated projective $C^\infty(M)$-module $\mathcal{P}$ we can associate a unique vector bundle (up to isomorphism) such that $\Gamma(M,E) \cong \mathcal{P}$ as $C^\infty(M)$-modules. Now, assume that we are dealing with a vector bundle E whose fibres have some extra structure, for instance the structure of an (involutive) algebra. Then $\Gamma(M,E)$ can also be endowed with the structure of an (involutive) algebra. These extra operations are compatible with the $C^\infty(M)$-module structure of $\Gamma(M,E)$, i.e.

$$f(st) = (fs)t = s(ft), \quad (fs)^* = f^*s^*, \quad (f \in C^\infty(M), s,t \in \Gamma(M,E)). \tag{5.11}$$

Now, for the converse one can ask: what if we are given a finitely generated projective $C^\infty(M)$-module $\mathcal{P}$ which in addition carries the structure of an (involutive) algebra whose multiplicative and star operations are compatible with the $C^\infty(M)$-structure as in (5.11). Is it then possible to construct a ($*$-)algebra bundle from this module $\mathcal{P}$ such that its space of smooth sections is isomorphic to P? The answer is affirmative and this construction will be carried out below.

The above mentioned structures are so important in the following that we give a name to them.

Definition 5.2.9. *Let R be a commutative ring. An R-algebra is an R-module $\mathcal{A}$ with an associative multiplication $\mathcal{A} \times \mathcal{A} \mapsto \mathcal{A} : (a,b) \mapsto ab$ which is R-bilinear:*

$$r(ab) = (ra)b = a(rb) \quad \forall a,b \in M, r \in R. \tag{5.12}$$

We call such an R-algebra finitely generated and projective if it is finitely generated and projective as an R-module. A homomorphism between two R-algebras is an R-linear map that preserves multiplication.

Definition 5.2.10. *Let R be a complex commutative $*$-algebra. An involution on an R-algebra $\mathcal{A}$ is a map $* : \mathcal{A} \to \mathcal{A}$ such that for all $r, s \in R$ and $a, b \in \mathcal{A}$ (here $*$ is used to denote the star operation in both R and $\mathcal{A}$) the following equalities hold:*

$$\begin{aligned} (ab)^* &= b^*a^*; \\ (a+b)^* &= a^* + b^*; \\ (ra)^* &= r^*a^*. \end{aligned}$$

If R has a unit, the R-algebra $\mathcal{A}$ is also a complex algebra and $(\lambda a)^* = \overline{\lambda} a^*$ for $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$. A homomorphism between two involutive R-algebras is an R-algebra homomorphism that preserves the star structure.

Example 5.2.11. If E is an algebra bundle, then $\Gamma(M,E)$ is a finitely generated projective $C^\infty(M)$-algebra and if E is an $*$-algebra bundle then $\Gamma(M,E)$ is a finitely generated projective and $C^\infty(M)$-algebra with involution.

If our finitely generated projective $C^\infty(M)$-module $\mathcal{P}$ is actually a $C^\infty(M)$-algebra, we can define in a natural way an associative multiplication on the fibres E_x of the vector bundle E that is obtained from $\mathcal{P}$ via the construction in Propostion 5.2.4. This turns E into an algebra bundle as we will see shortly. Moreover, if $\mathcal{P}$ is also endowed with a star structure, then the fibres can be endowed with a star operation as well, turning E into a $*$-algebra bundle.

To prove this, we we will first explicitly determine the module structure of $\mathcal{P}$ as a direct summand of the free module $F = C^\infty(M)^k$ that is regarded as module of sections of the trivial bundle of rank k. The action of $f \in C^\infty(M)$ on F is just $f(f_1, \dots f_n)) = (ff_1, \dots ff_n)$, where $ff_1(x)$ is $f(x)f_1(x)$, $(x \in M)$. Therefore the module structure of $C^\infty(M)$ on $\mathcal{P}$ is given by

$$fs(x) = f(x)s(x), \quad f \in C^\infty(M), s \in \mathcal{P}. \tag{5.13}$$

This module structure is needed to define the multiplication on the fibres E_x.

Proposition 5.2.12. *For $x \in M$, let $p, q \in E_x$ be given and suppose $s, t \in \mathcal{P}$ are such that $p = s(x)$ and $q = t(x)$. The fibre multiplication given by $pq := st(x)$ is well-defined. Consequently, we have $st(y) = s(y)t(y)$ for all $y \in M$ and $s, t \in \Gamma(M, E)$. In particular, E is an algebra bundle.*

Proof. We need to show that the definition of the fibre product is independent of the choice of sections s, t with $s(x) = p$ and $t(x) = q$. Therefore, let s', t' be two other sections of the bundle E with $s'(x) = p$ and $t'(x) = q$. Then $s_0 = s' - s$ and $t_0 = t' - t$ are sections for which $s_0(x) = t_0(x) = 0$. According to Lemma 5.2.5 s_0 and t_0 can be written as $s_0 = \sum_i f_i s_i$, $t_0 = \sum_i g_i t_i$ where $f_i(x) = g_i(x) = 0$ for every i. This gives

$$s't' - st = (s' - s)t' + s(t' - t) = \sum f_i s_i t' + \sum_i g_i s t_i, \tag{5.14}$$

which evaluated at x gives zero because of the module structure of $\Gamma(M, E)$. This argument shows that $s't'(x) = st(x)$ and the product is well-defined.

Because of the associativity of the product in $\Gamma(M, E)$, the product in the fibres will also be associative. If $\mathcal{P} \cong \Gamma(M, E)$ is unital with unit 1_P, then we can fix a unit in the fibre E_x by setting $1_{E_x} = 1_{\mathcal{P}}(x)$. Therefore, E is an algebra bundle. □

If $\mathcal{P}$ is a finitely generated $C^\infty(M)$-algebra with involution, then in the same way as in the proof of the previous theorem one introduces an involution on the fibres of E.

Proposition 5.2.13. *For given $p \in E_x$, let $s \in \mathcal{P}$ be such that $s(x) = p$. Define $p^* = s^*(x)$. This is a well-defined star structure on the fibre E_x. In particular, E is a $*$-algebra bundle.*

Proof. To prove this we will use the same argument as above. Let s' be another section such that $s'(x) = p$. Then $s_0 = s - s'$ is zero at point x. Therefore, s_0 can be written as a sum $\sum_i f_i s_i$ where s_i are elements of $\mathcal{P}$, f_i are in $C^\infty(M)$ and $f_i(x) = 0$ for all i. This gives

$$s^*(x) - s'^*(x) = (s - s')^*(x) = \sum_i (f_i s_i)^* = \sum_i f_i^*(x) s_i^*(x) = 0, \tag{5.15}$$

so that the star structure is well-defined. That this is indeed a star structure on the fibre E_x compatible with the algebra structure of the fibre, follows immediately from the compatibility conditions (5.11). Example: the fact that $(pq)^* = q^*p^*$ is easy to proof. Choose $s, t \in \mathcal{P}$ such that $s(x) = p$ and $t(x) = q$. Then $(pq)^* = (st(x))^* = (st)^*(x) = t^*s^*(x) = t^*(x)s^*(x) = q^*p^*$. The other properties follows similarly. □

It follows from the discussion at the beginning of this section that the functor Γ can be restricted to a functor from the category of ($*$-)algebra bundles to the category of (involutive) finitely generated projective $C^\infty(M)$-algebras. It follows from Propositions 5.2.12 and 5.2.13 that the restricted functor Γ is still essentially surjective. Moreover, since Γ is a restriction of a faithful functor, it is still faithful. To show that Γ is full, let E_1, E_2 be two ($*$-)algebra bundles and $F : \Gamma(M, E_1) \to \Gamma(M, E_2)$ be a ($*$-star preserving) $C^\infty(M)$-algebra-homomorphism. It is sufficient to check that the map ϕ defined by (5.10) is now a ($*$-)algebra bundle homomorphism. Let $s, t \in \Gamma(M, E)$ be such that $p = s(x)$, $q = t(x)$, then

$$\begin{aligned}
\phi(pq) &= F(st)(x) = F(s)F(t)(x) = F(s)(x) \cdot F(t)(x) = \phi(p)\phi(q), \\
\phi(p^*) &= F(s^*)(x) = (F(s))^*(x) = (F(s)(x))^* = \phi(p)^*.
\end{aligned}$$

Hence, Γ is a full functor.

Remark 5.2.14. If E_1, E_2 are unital ($*$-)algebra bundles and $\phi : E_1 \to E_2$ is a unital ($*$-)algebra bundle homomorphism, then $\Gamma(\phi)$ is a unital ($*$-star preserving) $C^\infty(M)$-algebra-homomorphism. Conversely, if $F : \Gamma(M, E_1) \to \Gamma(M, E_2)$ is a unital ($*$-star preserving) $C^\infty(M)$-algebra-homomorphism, and $\phi : E_1 \to E_2$ is defined by (5.10), then

$$\phi(1_x) = F(1)(x) = 1_x. \tag{5.16}$$

We summarise the results of this subsection in the following theorem.

Theorem 5.2.15 (Serre-Swan)**.** *Let M be a compact manifold. There is an equivalence between the category of ($*$-)algebra bundles over M and the category of finitely generated projective (involutive) $C^\infty(M)$-algebras. In particular this establishes a bijective correspondence of isomorphism classes of ($*$-)algebra bundles over M and isomorphism classes of finitely generated projective (involutive) $C^\infty(M)$-algebras. More specifically, to the isomorphism class of a ($*$-)algebra bundle one associates the isomorphism class of its $C^\infty(M)$-module of sections; conversely, to the isomorphism class of any (involutive) finitely generated projective $C^\infty(M)$-module one associates the class of the ($*$-)algebra bundle obtained via the construction in Propositions 5.2.4, 5.2.12 and 5.2.13. These maps are each others inverses. The same results hold if ($*$-)algebra bundles and finitely generated projective (involutive) $C^\infty(M)$-algebras are replaced by unital ($*$-)algebra bundles and unital finitely generated projective (involutive) $C^\infty(M)$-algebras, respectively.*

5.2.3 *From $*$-algebra bundle to principal bundle*

For a general unital $*$-algebra bundle E we can define the group of unitary elements in a fibre. Let $x \in M$ be given. We define U_x to be the set of all elements $p \in E_x$ such that $pp^* = p^*p = 1_x$. It is easily checked that this is a group. Elements of this group are called unitary elements of the fibre E_x. Since the fibre E_x is $*$-isomorphic to some unital $*$-algebra A, it follows that U_x is isomorphic to $\mathcal{U}(A)$ under this isomorphism. Furthermore, an element $s \in \Gamma(M, E)$ is unitary if and only if $s(x) \in U_x$ for all $x \in M$: indeed, $s(x)s(x)^* = s(x)s^*(x) = ss^*(x) = 1$ for all $x \in M$.

Remark 5.2.16. From this point B denotes an arbitrary *unital* $*$-algebra bundle B with fibre $M_n(\mathbb{C})$ over a compact manifold M.

According to the previous subsections we are able to reconstruct the unital $*$-algebra bundle B from $\Gamma(M, B)$. All the information of B, therefore, is contained in $\Gamma(M, B)$, regarded as a involutive finitely generated unital $C^\infty(M)$-algebra. If we want to describe Yang-Mills theory we need a $PSU(n)$-principal bundle. In this subsection we show how this principal bundle can be obtained from the algebra bundle B.

The following lemma is important. Its proof is based on the proof of Theorem 2.4.8 in [27]. Let $\mathcal{H}$ be a Hilbert space. In this theorem we frequently consider operators of the form $p_{ef} : z \mapsto e\langle z, f\rangle$ on $\mathcal{H}$, where e, f are vectors in $\mathcal{H}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. It is straightforward to check that such operators have norm $\|e\|\|f\|$. Also we will make use of Theorem 2.2.9.

Proposition 5.2.17. *Let $\alpha : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a given $*$-automorphism, then there exists a $u \in U(n)$ such that $\alpha = Ad\ u$.*

Proof. We will show that such u exists. We do this by considering $M_n(\mathbb{C})$ as the algebra of linear operators on $\mathbb{C}^n$. Here, $\mathbb{C}^n$ is endowed with the usual inner product. Let E be an orthonormal basis of $\mathbb{C}^n$ and denote, for $e \in E$, the map $z \mapsto e(e, z)$ ($z \in \mathbb{C}^n$) by p_e. Then for $q_e := \alpha(p_e)$ we have that $q_e\alpha(M_n(\mathbb{C}))q_e = \alpha(p_e M_n(\mathbb{C})p_e) = \mathbb{C}q_e$. This implies that q_e is a one-dimensional projection and therefore of the form $q_e(z) = \tilde{e}(\tilde{e}, z)$ for some unit vector $\tilde{e}$ in $\mathbb{C}^n$. If f is another element of E, then

$\tilde{e}(\tilde{e},\tilde{f})(\tilde{f},z) = q_e q_f(z) = \alpha(p_e p_f) = e(e,f)(f,z)$ for all $z \in \mathbb{C}^n$. Hence, $\tilde{e}$ and $\tilde{f}$ are also orthonormal to each other. It follows that $\{\tilde{e}\}_{e\in E}$ is an orthonormal basis of $\mathbb{C}^n$.

Let p_{ef} be the linear map given by $p_{ef}(z) = e(f,z)$ and set $q_{ef} = \alpha(p_{ef})$. Since $q_{ef} = q_e q_f$, the range of q_{ef} is the span of $\tilde{e}$ and therefore q_{ef} is of the form $q_{ef}(z) = \tilde{e}(y,z)$ for some unit vector $y \in \mathbb{C}$. Since $q_{fe} = q^*_{ef}$ has range $\mathbb{C}\tilde{f}$, $y = \lambda_{ef}\tilde{f}$ for some λ_{ef} with $|\lambda_{ef}| = 1$. Thus $q_{ef} = \lambda_{ef} p_{\tilde{e}\tilde{f}}$.

We will now define a unitary map $u : \mathbb{C} \to \mathbb{C}$ such that Ad $u(e) = \alpha(e)$ for all $e \in E$. Fix an element $f \in E$ and set $\mu_e = \lambda_{ef}$. Define $u(e) = \mu_e \tilde{e}$ for all $e \in E$. Then for $e, g \in E$:

$$\text{Ad } u(p_{eg}) = p_{u(e)u(g)} = \mu_e \overline{\mu}_g p_{\tilde{e},\tilde{g}} = \alpha(p_{eg}), \tag{5.17}$$

where in the last step we used the equality $\alpha(p_{eg}) = q_{eg} = \lambda_{eg} p_{\tilde{e}\tilde{g}} = \lambda_{ef}\lambda_{fg} p_{\tilde{e}\tilde{g}} = \mu_e \overline{\mu_g} p_{\tilde{e}\tilde{g}}$. The equality $\lambda_{eg} = \lambda_{ef}\lambda_{fg}$ follows from $q_{eg} = q_{ef} q_{fg}$. We have now shown that Ad u and α are equal on the rank-one operators. Since these operators span $M_n(\mathbb{C})$ it has been shown that Ad $u = \alpha$. □

We will use this proposition to say something more about the transition functions of B. Let (U, ϕ_U), (V, ϕ_V) be two local trivialisations of B. Proposition 5.2.17 implies that the transition function g_{uv} is a smooth map from U to Ad $U(n)$. Here Ad $U(n)$ is considered as a subgroup of Aut$(M_n(\mathbb{C}))$.[3] For any group G its adjoint group Ad G is isomorphic to the quotient $G/Z(G)$, where $Z(G)$ is the centre of G. If $G = U(N)$, then Ad $G \cong U(N)/Z(U(N)) \cong PSU(n)$, where $PSU(n)$ is the projective unitary group. Thus, the transition functions of B take values in the group $PSU(n)$.

The reconstruction Theorem 2.4.17 says that one can construct a principal G-bundle if the group G, the trivialising neighbourhoods$\{U_i\}_{i\in I}$ and all the transition functions $g_{ij} : U_i \cap U_j \to G$ are given. This allows us to construct a principal bundle P from the bundle B. Since the transition functions of B are $PSU(n)$-valued we can take the trivialising neighbourhoods and transition functions for P to be the same as those for B. We have now specified all the data to construct P. Note that by definition of P the bundle B is an associated bundle of P.[4]

Let us summarise the results in a theorem.

Theorem 5.2.18. *Let B be a unital $*$-algebra bundle with fibre $M_n(\mathbb{C})$. Given the unital, involutive, finitely generated projective $C^\infty(M)$-algebra $\Gamma(M,B)$ there exists a principal $PSU(n)$-bundle P. Moreover, if $PSU(n)$ acts on $M_n(\mathbb{C})$ as a subgroup of Aut$(M_n(\mathbb{C}))$ as discussed above, then $P \times_{PSU(n)} M_n(\mathbb{C}) \cong B$. In particular, B is an associated bundle of P. The last statement implies that for any $PSU(n)$-bundle, there is a $M_n(\mathbb{C})$-algebra bundle B such that $\Gamma(M,B)$ gives the bundle P when the above constructions are carried out.*

Proof. That B is an associated bundle of P follows from the fact that $P \times_{PSU(n)} M_n(\mathbb{C})$ and B are vector bundles having the same set of local trivialisations and transition functions. Therefore, they must be the same. □

Definition 5.2.19. *Let P be the principal $PSU(n)$-bundle as above. When we restrict the action of $PSU(n)$ on $M_n(\mathbb{C})$ to $su(n)$ then ad $P = P \times_{PSU(n)} su(n)$ is a vector bundle with fibre $su(n)$. This bundle can be considered to be a sub-bundle of B.*

Definition 5.2.20. *The first ingredient for our spectral algebra is the algebra $\Gamma(M,B)$ of sections of an arbitrary $M_n(\mathbb{C})$ unital $*$-algebra bundle.*

3 This can be done since the unitary matrices linearly span $M_n(\mathbb{C})$.

4 At the end of the previous chapter we mentioned that, in the trivial case, we could have $PSU(n)$- as well as $SU(n)$-Yang–Mills theory. Here, in the nontrivial case, when one takes the group to be $SU(n)$, one runs into trouble if one wants to construct a principal $SU(n)$-bundle P from the bundle B. Since the centre of $SU(n)$ is not trivial, $SU(n)$ is not isomorphic to Ad $SU(n)$. Therefore, given a transition function g_{ij} of B, which takes values in Ad$SU(n)$, it is not clear what the $SU(n)$-valued transition function for the bundle P should be. It may be impossible to choose all these transition functions such that the cocycle conditions are always fulfilled so it is not clear if a principal $SU(n)$-bundle can be constructed.

Remark 5.2.21. It is an interesting question to ask whether or not the internal symmetry group $\mathrm{Inn}(mcA)$ of the triple and the gauge group $\Gamma(M, \mathrm{Ad}\, P)$ coincide. Since these groups are both considered to be the symmetry group in, respectively, noncommutative geometry and gauge theory as it is known in terms of principal bundle. I have not been able to show yet if these two groups are indeed ismorphic or not.

We have now achieved the first three goals outlined in the introduction of this chapter. In the next chapter we will look for a suitable Hilbert-space and Dirac-operator for our spectral triple.

6 COMPLETION OF THE SPECTRAL TRIPLE

Let M be a compact manifold and let B be a unital $*$-algebra bundle with fibre $M_n(\mathbb{C})$. Then the unital, involutive, finitely generated projective $C^\infty(M)$-algebra $\Gamma(M,B)$ of sections of B was taken as the first ingredient for our spectral triple. In section 6.1 we will construct a suitable Hilbert space $\mathcal{H}$ on which $\Gamma(M,B)$ can be faithfully represented. This Hilbert spaces is the second ingredient for our spectral triple. The final ingredient, the Dirac operator, is introduced in section 6.2.

6.1 THE HILBERT SPACE

Before we introduce the Hilbert space for our spectral triple, we will consider Hermitian bundles.

6.1.1 *Hermitian bundles and square-integrable sections*

In this subsection we will define the notion of a hermitian bundle. If E is a hermitian bundle over a compact space M we can define an inner product on the space of sections $\Gamma(M,E)$. The closure of $\Gamma(M,E)$ is then a Hilbert space.

Definition 6.1.1. *A hermitian vector bundle is a smooth map $\pi : E \to M$ such that every fibre $E_x := \pi^{-1}(x)$ ($x \in M$) has the structure of a finite-dimensional complex Hilbert space. Moreover, the bundle E is locally trivial in the sense that for any point x there exists a Hilbert space H, an open neighbourhood U of x and a diffeomorphism ϕ_U such that the following diagram commutes,*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"'] & U \times H \arrow[dl, "proj_1"] \\ U & \end{tikzcd}$$

and where $\phi_U|E_x \to H$ is an isomorphism of Hilbert-spaces. We call M the base space and E the total space. The map ϕ_U is called a local trivialisation of E with trivialising neighbourhood U.

The inner product on the fibres E_x defines a Hermitian structure on the space $\Gamma(M,E)$ of smooth sections from M to E by

$$(\cdot,\cdot) : \Gamma(M,E) \times \Gamma(M,E) \to C^\infty(M) \quad (s,t)(x) = \langle s(x), t(x)\rangle_{E_x}. \quad (x \in M).$$

From this definition it is clear that

$$\begin{aligned} (s,t) &= (t,s)^*; \\ (s,ft) &= (s,t)f; \\ (s,s) &\geq 0, \quad (s,s) = 0 \Leftrightarrow s = 0, \end{aligned} \tag{6.1}$$

for all $s,t \in \Gamma(M,E)$, $f \in C^\infty(M)$. We would like to define an inner product on $\Gamma(M,E)$ by setting

$$\langle s,t\rangle = \int_M \langle s(x), t(x)\rangle dx = \int_M (s,t)(x)dx, \quad (s,t \in \Gamma(M,E)), \tag{6.2}$$

and eventually close $\Gamma(M, E)$ to get a Hilbert-space. Now, for non-compact M one runs into trouble since it is necessary that $\langle s(x), t(x)\rangle$ has compact support in order to be integrable. This need not be the case if we take s, t to be arbitrary smooth sections from M to E. This problem can be solved by looking at the subspace of sections of E that have compact support.

Definition 6.1.2. *Let $\pi : E \to M$ be a hermitian bundle with finite dimensional fibre H. Then the space of smooth sections from M to E that have compact support is denoted by $\Gamma_c(M)$. If M is compact, then $\Gamma_c(M) = \Gamma(M)$.*

Actually, $\Gamma_c(M, E)$ is a vector space. Note that $\{x \in M : (s+t)(x) \neq 0\} \subset \{x \in M : s(x) \neq 0\} \cup \{x \in M : t(x) \neq 0\}$ for all $s, t \in \Gamma_c(M, E)$. This implies that $\text{supp}(s+t) \subset \text{supp}(s) \cup \text{supp}(t)$. The facts that $\text{supp}(s) \cup \text{supp}(t)$ is a finite union of compact sets and that $\text{supp}(s+t)$ is a closed subset of $\text{supp}(s) \cup \text{supp}(t)$, imply that $\text{supp}(s+t)$ is a compact set. Moreover, the $C^\infty(M)$-module structure on $\Gamma(M, E)$ restricts to $\Gamma_c(M, E)$.

Proposition 6.1.3. *The following defines an inner product on $\Gamma_c(M, E)$:*

$$\langle s, t\rangle_{\Gamma_c(M,E)} := \int_M \langle s(x), t(x)\rangle dx = \int_M (s, t)(x) dx, \quad (s, t \in \Gamma_c(M, E)). \tag{6.3}$$

Proof. Since s, t have compact support, the smooth function $\langle s, t\rangle$ also has compact support, so that it is an element of $\Gamma_c(M, E)$. Therefore, it can be integrated over M and the inner product rule (6.3) is well-defined. We just check the rules for an inner product: $\forall \lambda, \mu \in \mathbb{C}, s, t, u \in \Gamma_c(M, E)$ we have

$$\begin{aligned} \langle u, \lambda s + \mu t\rangle &= \int_M \langle u(x), \lambda s(x) + \mu t(x)\rangle dx \\ &= \int_M \left(\lambda\langle u(x), s(x)\rangle + \mu\langle u(x), t(x)\rangle\right) dx \\ &= \lambda \int_M \langle u(x), s(x)\rangle dx + \mu \int_M \langle u(x), t(x)\rangle dx = \lambda\langle u, s\rangle + \mu\langle u, t\rangle, \end{aligned}$$

proving the linearity in the second argument. It is obvious that $\langle s, s\rangle \geq 0$ ($s \in \Gamma_c(M, E)$) since $\langle s(x), s(x)\rangle$ is non-negative for every x because of the positive-definiteness of the inner product on the fibres. Moreover, since $\langle s(x), s(x)\rangle$ is a continuous function for every $s \in \Gamma_c(M, E)$, we only have $\langle s, s\rangle = 0$ if s is the zero section. By definition, for any $f \in C^\infty(M)$

$$\int_M f dx = \int_M \text{Re}(f(x)) dx + i \int_M \text{Im}(f(x)) dx, \tag{6.4}$$

so that the property $\langle s, t\rangle = \overline{\langle s, t\rangle}$ is also satisfied. □

Therefore $\Gamma_c(M, E)$ endowed with the inner product (6.3) is a pre-Hilbert space. We now take its closure with respect to the norm induced by this inner product to obtain a Hilbert space. This Hilbert space is denoted by $L^2(M, E)$ and its elements are called the square integrable sections from M to E.

6.1.2 *The Hilbert space for our spectral triple*

In this subsection we return to the case where B is unital $*$-algebra bundle with fibre $M_n(\mathbb{C})$ over a compact manifold M. We will define the second ingredient in our spectral triple: the Hilbert space.

We start by defining an inner product on the fibres of the algebra bundle B. This gives a natural inner product on the fibres of the tensor product bundle $B \otimes S$. The space $L^2(M, B \otimes S)$ will be the Hilbert space in our spectral triple.

First, we define a sesquilinear form on $M_n(\mathbb{C})$:

$$\langle A, B \rangle = \mathrm{Tr}(A^* B), \quad (A, B \in M_n(\mathbb{C})). \tag{6.5}$$

This is actually an inner product: the linearity in its second argument is clear immediately and we also have $\langle B, A \rangle = \mathrm{Tr}(B^* A) = \overline{\mathrm{Tr}(A^* B)} = \overline{\langle A, B \rangle}$. For any $A \in M_n(\mathbb{C})$ the matrix $A^* A$ is positive, so it can be diagonalised and has all its *positive* eigenvalues on the diagonal. Therefore, (6.5) is an inner product on $M_n(\mathbb{C})$, also known as the Hilbert-Schmidt inner product.

Proposition 5.2.17 immediately implies the following proposition.

Proposition 6.1.4. *Let B be the unital $*$-algebra bundle with fibre $M_n(\mathbb{C})$. For all $x \in M$, define an inner product on the fibre B_x by choosing a local trivialisation (U, ϕ_U) around x and setting*

$$\langle p, q \rangle = Tr(\phi_U(p^*)\phi_U(q)), \tag{6.6}$$

for $p, q \in B_x$. This inner product is independent of the choice of the local trivialisation. Consequently, the $*$-algebra bundle B is a hermitian $*$-algebra bundle (with fibre $(M_n(\mathbb{C}), Tr)$).

Proof. All the transition functions of the bundle B are $*$-automorphisms of $M_n(\mathbb{C})$. Propostion 5.2.17 implies that any $*$-automorphism of $M_n(\mathbb{C})$ preserves the Hilbert–Schmidt inner product so that equation (6.6) is independent of the choice of a local trivialisation. ☐

Denote by $L^2(M, B)$ the Hilbert space of square-integrable sections of B.

We will now construct a suitable Hilbert space for our spectral triple. Consider the vector bundle $B \otimes S$. According to Lemma 2.4.9 there is an $C^\infty(M)$-isomorphism between $\Gamma(M, B \otimes S)$ and $\Gamma(M, B) \otimes_{C^\infty(M)} \Gamma(M, S)$. On $\Gamma(M, B) \otimes_{C^\infty(M)} \Gamma(M, S)$ define the following inner product:

$$\langle s \otimes \psi, t \otimes \eta \rangle = \langle \psi, (s, t)\eta \rangle, \quad (s, t \in \Gamma(M, B), \psi, \eta \in \Gamma(M, S)).$$

Note that it is well-defined with respect to the $C^\infty(M)$-linearity of the tensor product. Under the isomorphism between $\Gamma(M, B) \otimes_{C^\infty(M)} \Gamma(M, S)$ and $\Gamma(M, B \otimes S)$ this carries over to an inner product on $\Gamma(M, B \otimes S)$:

$$\langle \psi, \eta \rangle = \int_M \langle \psi(x), \eta(x) \rangle_{B_x \otimes S_x} dx, \quad (\psi, \eta \in \Gamma(M, B \otimes S)), \tag{6.7}$$

where $\langle \cdot, \cdot \rangle_{B_x \otimes S_x}$ is the inner product in the fibre $B_x \otimes S_x$. This inner product is defined by

$$\langle b_1 \otimes s_1, b_2 \otimes s_2 \rangle_{B_x \otimes S_x} = \langle b_1, b_2 \rangle_{B_x} \langle s_1, s_2 \rangle_{S_x}, \quad (s_1, s_2 \in S_x, \quad b_1, b_2 \in B_x).$$

The norm-closure of $\Gamma(M, B \otimes S)$ with respect to the norm induced by this inner product will be denoted by $L^2(M, B \otimes S)$.

Now we introduce an action of $\Gamma(M, B)$ on $L^2(M, B \otimes S)$. First, we will define the action of $\Gamma(M, B)$ on the dense subspace $\Gamma(M, B \otimes S)$. This action is simply given by

$$a(s \otimes \psi) = as \otimes \psi, \tag{6.8}$$

and is well-defined with respect to the $C^\infty(M)$-linearity of the tensor product. The $\Gamma(M, B)$-module structure on $\Gamma(M, B \otimes S)$ then follows easily from the algebra structure of $\Gamma(M, B)$.

Proposition 6.1.5. *If M is compact, the action of each $a \in \Gamma(M, B)$ on $\Gamma(M, S \otimes B)$ defined as above, is a bounded operator on $\Gamma(M, S \otimes B) \subset L^2(M, S \otimes B)$. In particular, it uniquely extends to a bounded operator defined on $L^2(M, B \otimes S)$.*

Proof. Let $a \in \Gamma(M, B)$ be given. Then locally a acts on $\Gamma(M, B)$ just by fibre-wise matrix-multiplication. Since $a(x)$ acts as a linear operator on the finite-dimensional fibre B_x, it is a bounded operator on B_x (with respect to the norm induced by the inner product on B_x). Denote $\|a(x)\|_{\text{op}}$ for the operator norm of $a(x)$. Then $x \mapsto \|a(x)\|_{\text{op}}$ is a continuous function from M to $\mathbb{R}$ and the compactness of M implies that this function has a maximum value, say m. Then

$$\begin{aligned} \|as \otimes \psi\|^2 &= \int_M \langle \psi(x), \langle a(x)s(x), a(x)s(x) \rangle_{B_x} \psi(x) \rangle_{S_x} dx \\ &\leq m^2 \int_M \langle \psi(x), \langle s(x), s(x) \rangle_{B_x} \psi(x) \rangle_{S_x} dx = m^2 \|s \otimes \psi\|^2 \end{aligned} \tag{6.9}$$

This inequality implies that $\|a\|_{\text{op}} \leq m^2$, hence a is bounded on $\Gamma(M, B \otimes S)$ and a can be extended to a bounded operator on $L^2(M, B \otimes S)$. Thus, $\Gamma(M, B)$ is represented in $\mathcal{B}(L^2(M, B \otimes S))$. This representation is faithful. □

We now take $L^2(M, B \otimes S)$ for the Hilbert space of our spectral triple. Moreover, the action in (6.8) represents $\Gamma(M, B)$ on $L^2(M, B \otimes S)$ according to Proposition 6.1.5. Thus, $L^2(M, B \otimes S)$ is a suitable choice for the Hilbert space in our spectral triple. In the next section we will discuss the Dirac operator on $\mathcal{H} = L^2(M, B \otimes S)$.

6.2 DIRAC-OPERATOR

In this section the final ingredient for our spectral triple, the Dirac operator, is introduced. Before we can define the Dirac operator, we need to introduce hermitian connections.

Definition 6.2.1. *Let E be a vector bundle over M and let $(\cdot, \cdot)$ be a hermitian structure. A connection $\nabla : \Gamma(M, E) \to \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, E)$ is called hermitian if is satisfies*

$$(s, \nabla t) + (\nabla s, t) = d(s, t), \quad (s, t \in \Gamma(M, E)), \tag{6.10}$$

where d is the exterior derivative on M. Here we adapt the rules

$$(\omega \otimes s, t) = \omega^*(s, t), \quad (s, \omega \otimes t) = (s, t)\omega, \quad \omega \in \Omega^1(M), s, t \in \Gamma(M, E). \tag{6.11}$$

We can now define the Dirac-operator for our triple.

Definition 6.2.2. *Let ∇^B be a hermitian connection on the space of section $\Gamma(M, B)$ of our algebra bundle B. On the dense subspace $\Gamma(M, B \otimes S) \subset L^2(M, B \otimes S)$ we define a hermitian connection by*

$$\nabla := \nabla_B \otimes 1 + 1 \otimes \nabla_S : \Gamma(M, B \otimes S) \to \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(M, B \otimes S). \tag{6.12}$$

We define the Dirac-operator D_B in our spectral triple to be the closure in $L^2(M, B \otimes S)$ of the operator $i\gamma \circ \nabla$. That is, D_B is a twisted Dirac-operator (see also Definition 2.7.6).

The following proposition shows that we have indeed constructed a spectral triple.

Proposition 6.2.3. *The triple*

$$(\mathcal{A}, \mathcal{H}, D_B) = (\Gamma(M,B), L^2(M, B \otimes S), D_B), \tag{6.13}$$

is a spectral triple. Here $\Gamma(M,B)$ is considered as a finitely generated projective unital $C^\infty(M)$-algebra with involution; $L^2(M, B \otimes S)$ is the closure of $\Gamma(M, B \otimes S)$ with respect to the norm induced by the inner product (6.7), and D_B is the Dirac operator introduced in Definition 6.2.2. The algebra $\Gamma(M,B)$ acts on $L^2(M, B \otimes S)$ as in Proposition 6.1.5.

Proof. Since the operator D_B is precisely the twisted Dirac-operator for the tensor bundle $B \otimes S$, we already know it is self-adjoint (cf.Theorem 2.7.7) Because of Proposition 6.1.5, $\Gamma(M,B)$ is also faithfully represented on $L^2(M, B \otimes S)$. It remains to check the compatibility condition

$$[D_B, a] \quad \in \quad B(\mathcal{H}) \quad \forall a \in \Gamma(M,B),$$

Let $a \in \Gamma(M,B)$ be given. First, let us determine the form of $[D_B, a]$ in local coordinates:

$$\nabla^B_\mu \otimes 1 + 1 \otimes \nabla^S_\mu = (\partial_\mu + \omega^B_\mu) \otimes 1 + 1 \otimes (\partial_\mu + \omega^S_\mu) = \partial_\mu + \omega^B_\mu \otimes 1 + 1 \otimes \omega^S_\mu,$$

where ω^B_μ and ω^S_μ are the (local) connections 1-forms for ∇^B and ∇^S and ∂_μ is understood as $1 \otimes \partial_\mu + \partial_\mu \otimes 1$. Then a straightforward calculation gives

$$[\gamma^\mu(\partial_\mu + \omega^B_\mu \otimes 1 + 1 \otimes \omega^S_\mu), a](s \otimes \psi) = \gamma^\mu \left(\partial_\mu(as) \otimes \psi + \omega^B_\mu(as) \otimes \psi - a\omega^B_\mu s \otimes \psi - a(\partial_\mu s) \otimes \psi \right).$$

By noting that $\partial_\mu(as) - a\partial_\mu s = (\partial_\mu a)s$, we can rewrite this expression as

$$\left(\partial_\mu a) + [\omega^B_\mu, a]\right) s \otimes \gamma^\mu \psi. \tag{6.14}$$

From equation (6.14) we see that for any $s \otimes \psi \in \Gamma(M, B \otimes S)$ the value of $([D_B, a]s \otimes \psi)(x)$ depends only on the value of $s \otimes \psi$ at the point x (and not on its behavior in a neighbourhood of x as is the case with D_B.). Thus, we can look at the restricion of $[D_B, a]$ to the fibre over x. At each point $x \in U$ the operator $[D, a]$ acts on the fibre $B_x \otimes S_x$ as $\left(\partial_\mu a)(x) + [\omega^B_\mu(x), a(x)]\right) \otimes \gamma^\mu(x)$. This is a linear bounded operator. The operator norm of $\left(\partial_\mu a)(x) + [\omega^B_\mu(x), a(x)]\right) \otimes \gamma^\mu(x)$ varies continuously with x on such a local trivialising neighbourhood U. Thus, the function that assigns to each x the operator norm of the linear map on B_x induced by restriction of $[D_B, a]$ to the fibre, is continuous. Since M is compact, this function takes on its maximum value, say m. Then,

$$\|[D_B, a]s \otimes \psi\|^2 = \int_M \langle ([D,a]s \otimes \psi), ([D,a]s \otimes \psi) \rangle dx \leq \int_M m^2 \langle s(x) \otimes \psi(x), s(x) \otimes \psi(x) \rangle dx = m^2 \|s \otimes \psi\|^2.$$

Thus, $(\Gamma(M,B), L^2(M, B \otimes S), D_B)$ is indeed a spectral triple. □

According to the above proposition we have now successfully constructed a spectral triple $(\Gamma(M,B), L^2(M, B \otimes S), D_B)$ with $\Gamma(M,B)$ as an algebra where B is a possibly non-trivial algebra bundle with fibre $M_n(\mathbb{C})$. To determine the inner fluctuations of the Dirac-operator a real structure J is needed. We will show in the next section that additional properties for the connection ∇^B are needed if we want to introduce a real structure that generalises the real structure in the trivial case.

6.3 REAL STRUCTURE J

In this subsection we will construct a real structure for the spectral triple $(\Gamma(M,B), L^2(M, B\otimes S), D_B)$. It will turn out that the connection ∇^B on the bundle B must satisfy additional properties in order to make all the conditions for a real structure to be fulfilled.

Before we define the real structure on $\mathcal{H} = L^2(M, S\otimes B)$, we first look for a real structure on $L^2(M,B)$. Just as in the trivial case we define an operator $T : \Gamma(M,B) \to \Gamma(M,B)$ that sends an element $s \in \Gamma(M,B)$ to s^*. This is clearly an anti-linear map. It is also anti-unitary since

$$\langle Ts, Tt\rangle = \langle s^*, t^*\rangle = \int_M \mathrm{Tr}(st^*(x))dx = \int_M \mathrm{Tr}(t^*s(x))dx = \langle t, s\rangle = \langle t, s\rangle, \quad (s,t\in\Gamma(M,B)). \tag{6.15}$$

Hence, T is an anti-unitary operator on $\Gamma(M,B)$ and it can therefore be extended uniquely to a anti-unitary operator T on $L^2(M,B)$: for $s\in L^2(M,B)$ define $Ts := \lim_n s_n$ where $(s_n)_n$ is a sequence converging to s. This is well-defined. We check that T is also an anti-unitary operator on $L^2(M,B)$. For $s,t\in L^2(M,B)$, let $(s_n)_n$, $(t_n)_n$ be sequences in $\Gamma(M,B)$ converging to s, t respectively, then

$$\langle Ts, Tt\rangle = \lim_n \lim_m (Ts_n, Tt_m) = \lim_n \lim_m \langle t_m, s_n\rangle = \langle t, s\rangle. \tag{6.16}$$

Thus, T is an anti-unitary operator on $L^2(M,B)$.

Proposition 6.3.1. *The operator T is a real structure for the spectral triple $(\Gamma(M,B), L^2(M,B), D=0)$ of KO-dimension 0.*

Proof. As already shown the operator T is an anti-unitary operator on $L^2(M,B)$. The equation $TD = DT$ is trivial as is the condition $[[D,a],b^0] = 0$ for all $a,b\in\Gamma(M,B)$. Furthermore,

$$[a,b^0]s = [a, Tb^*T]s = aT(b^*Ts) - Tb^*T(as) = asb - asb = 0 \quad \text{for all } a,b,s\in\Gamma(M,B). \tag{6.17}$$

Since $\Gamma(M,B)\subset L^2(M,B)$ is dense, the operator $[a,b^0]$ is also the zero operator on $L^2(M,B)$. Thus, $(\Gamma(M,B), L^2(M,B), D=0, T)$ is a real spectral triple of KO-dimension 0. $\square$

We return to the spectral triple $(\Gamma(M,B), L^2(M,B\otimes S), D_B)$.

Definition 6.3.2. *Define the operator J by*

$$J : \Gamma(M,B)\otimes_{C^\infty(M)}\Gamma(M,S) \to \Gamma(M,B)\otimes_{C^\infty(M)}\Gamma(M,S), \qquad J = T\otimes J_M, \tag{6.18}$$

where J_M is the real structure belonging to the canonical spectral triple of M, $(C^\infty(M), L^2(M,S), \not{D})$. Here, again, we have identified the $C^\infty(M)$-modules $\Gamma(M,B\otimes S)$ and $\Gamma(M,B)\otimes_{C^\infty(M)}\Gamma(M,S)$.

Clearly, the operator J defined above is the natural generalisation of the real structure we had in the trivial case for the triple $(C^\infty(M, M_n(\mathbb{C})), L^2(M,S)\otimes M_n(\mathbb{C}), 1\otimes\not{D})$. Since in the non-trivial case the operator D_B no longer acts non-trivially on the matrix part, additional conditions have to be put on the connection ∇^B to ensure that J satisfies all the compatibility conditions with D_B.

Definition 6.3.3. *A hermitian $*$-algebra bundle $\pi : E\to M$ is a complex $*$-algebra bundle such that every fibre has an inner product and for every $x\in M$ there exists a open neighbourhood U, a $*$-algebra A with inner product g, and a diffeomorphism ϕ_U such that $\phi_U : \pi^{-1}(U)\to U\times A$ is a diffeomorphism. Moreover, it is required that the restriction $(\phi_U)_{E_x}$ is a $*$-isomorphism from E_x to $\{x\}\times A$ that preserves the inner product.*

Definition 6.3.4. *Let M be a manifold and E a $*$-algebra bundle over M. A $*$-algebra connection ∇ on E is a connection on E that satisfies the compatibily conditions*

$$\nabla(st) = s\nabla t + (\nabla s)t, \quad (\nabla s)^* = \nabla s^* \quad , \tag{6.19}$$

for all $s, t \in \Gamma(M, E)$. If E is a hermitian $*$-algebra bundle and ∇ is also a hermitian connection, then ∇ is called a hermitian $*$-algebra connection.

The extra condition that needs to be put on ∇^B is that it is a hermitian $*$-algebra connection. We will prove this below in Theorem 6.3.8. But first, we have to answer another question: do there exist any hermitian $*$-algebra connections on the bundle B? We prove here that even for an arbitrary complex (hermitian) $*$-algebra bundle, the answer is affirmative. First, we prove that such connections exist locally.

Let E be a hermitian $*$-algebra bundle over M. Let U be a local trivialisation of E and let A be the fibre above each point $x \in U$. Recall that locally any connection on a vector bundle E is of the form

$$\nabla_\mu = \partial_\mu + \omega_\mu, \tag{6.20}$$

where ω_μ are the components of a 1-form ω defined on U which takes its values in End(A).

On some trivialising neighbourhood (U, ϕ_u) we have that $(ds)^* = ds^*$ and $d(st) = (ds)t + s(dt)$ for all $s, t \in \Gamma(U, E)$, or equivalenty $(\partial_\mu s)^* = \partial_\mu s^*$ and $\partial_\mu(st) = (\partial_\mu s)t + s(\partial_\mu t)$ for all $s, t \in \Gamma(U, E)$. Therefore, a connection ∇_μ on U is compatible with the $*$-algebra structure of $\Gamma(M, E)$ if and only if ω_μ takes values in the Lie algebra Der$^*(A) \subset$ End(A) of $*$-preserving derivations of A. The connection ∇_μ is hermitian on U if and only if ω_μ takes it values in the Lie algebra $u_g(A) \subset$ End(A) of elements that are anti-hermitian with respect to the inner product g on A: that is, $u \in u_g(A)$ if and only if $(a, ub) = -(ua, b)$ for all $a, b \in A$. Combining these properties gives that a connection on U is both hermitian and compatible with the $*$-algebra structure if and only if ω_μ takes values in the Lie-algebra Der$^*(A) \cap u_g(A)$.

Remark 6.3.5. Clearly, the vector space of hermitian $*$-algebra connections on U is certainly not empty. Take as connection on U the exterior derivative d with respect to the local trivialisation (U, ϕ_u).

We will now prove that for any paracompact M any (hermitian) $*$-algebra bundle E allows a hermitian $*$-algebra connection. In the Appendix B there is an alternative proof that is based on the ideas in (Kobayashi & Nomizu, [19], Chapter I Theorem 5.7) and (Godement, [14], p.151).

Theorem 6.3.6. *Every hermitian $*$-algebra bundle E defined over a paracompact space M, allows a hermitian $*$-algebra connection.*

Proof. Let $\{U_i\}$ be a locally finite open covering of M such that E is trivialised over U_i for each i. Then on each U_i there exists a hermitian $*$-algebra connection ∇_i. Now, let $\{f_i\}$ be a partition of unity subordinate to the open covering $\{U_i\}$ (all f_i are real-valued). Then

$$\nabla = \sum_i f_i \nabla_i \tag{6.21}$$

is a hermitian $*$-algebra connection defined on $\Gamma(M, E)$. □

In particular, the theorem holds for the hermitian $*$algebra bundle B. That is, there exists a hermitian $*$-algebra connection ∇ on B. Locally, on some trivialising neighbourhood U, the connection ∇ can be written as

$$\partial_\mu + \omega_\mu, \tag{6.22}$$

where ω_μ takes its values in subspace $\mathrm{Der}^*(M_n(\mathbb{C})) \cap u_{HS}(M_n(\mathbb{C})$ (HS denotes Hilbert-Schmidt inner product). This space is certainly not just the zero space since for each $A_\mu \in C^\infty(U, su(n))$, the linear map

$$\omega_\mu : s \mapsto [A_\mu, s], \quad s \in \Gamma(U, B), \tag{6.23}$$

is an element of this subspace. Conversely, every derivation δ in $\mathrm{Der}^*(M_n(\mathbb{C})) \cap u_{HS}(M_n(\mathbb{C}))$ is of the form $[u, \cdot]$ for some unique u in $su(n)$. One can prove this as follows. Since δ is in $u_{HS}(M_n(\mathbb{C})$ it sends every element in $a \in M_n(\mathbb{C})$ to an element with trace 0 because $\mathrm{Tr}(\delta(a)) = \mathrm{Tr}(1 \cdot (\delta a)) = \langle 1, \delta a \rangle = -\langle \delta 1, a \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the Hilbert-Schmidt inner product on $M_n(\mathbb{C})$. Here we have used that any derivation maps 1 to 0. Since δ preserves the star operation it sends $u(n)$ to $u(n)$. These two facts imply that the restriction of δ to $su(n)$ is a real derivation. Now, $su(n)$ is a simple Lie-algebra so that any derivation on $su(n)$ if of the form $[A,]$ for some $A \in su(n)$ ([16]). Now ad A is equal to δ as a complex derivation on $M_n(\mathbb{C})$, because $M_n(\mathbb{C}) = su(n) \oplus isu(n) \oplus \mathbb{C}\mathrm{Id}$. Hence we have proved:

Lemma 6.3.7. *On a local trivialising neighbourhood every hermitian $*$-algebra connection ∇^B is of the form (6.23) for a* unique *$su(n)$-valued one-form A_μ.*

From now on, the dimension of the manifold is assumed to be **4**. We are now in the position to define a real structure on the triple of Proposition 6.2.3.

Theorem 6.3.8. *The triple $(\Gamma(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$ is a real, even spectral triple of KO-dimension 4. Here, the triple $(\Gamma(M, B), L^2(M, B \otimes S), D_B)$ is as before with the additional condition that ∇^B in $D_B = i\gamma(\nabla^B \otimes 1 + 1 \otimes \nabla^S)$ is a hermitian $*$-algebra connection. Furthermore, J is defined as the natural extension of the bounded operator $T \otimes J_M : \Gamma(M, B \otimes S) \to \Gamma(M, B \otimes S)$, and γ_B is the natural extension of the bounded operator $1 \otimes \gamma^5 : \Gamma(M, B \otimes S) \to \Gamma(M, B \otimes S)$.*

Proof. First of all, J is well-defined with respect to the $C^\infty(M)$-linearity because $J_M f = \bar{f} J_M$ for all $f \in C^\infty(M)$. Since $T^2 = 1$ and $J_M^2 = -1$ we still have that $J^2 = -1$. Moreover $T \otimes J_M$ is anti-unitary, since for all $s, t \in \Gamma(M, B)$, $\psi, \eta \in \Gamma(M, S)$:

$$\begin{aligned} \langle (T \otimes J_M) s \otimes \psi), (T \otimes J_M) t \otimes \eta \rangle &= \langle J_M \psi, (s^*, t^*) J_M \eta \rangle = \langle J_M \psi, J_M \overline{(s^*, t^*)} \eta \rangle \\ &= \langle \overline{(s^*, t^*)} \eta, \psi \rangle = \langle (s, t) \eta, \psi \rangle = \langle t \otimes \eta, s \otimes \psi \rangle, \end{aligned}$$

where we have in the second step that $J_M f = \bar{f} J_M$ for every $f \in C^\infty(M)$, in the third step that J_M is anti-unitary and in the fourth step that $(s, t) = (t^*, s^*)$ (by definition of the hermitian structure as a fibre-wise trace). Thus, J is anti-unitary. To show that $DJ - JD = 0$, we do a local calculation.

$$\begin{aligned} (JD - DJ)(s \otimes \psi) &= (T \otimes J_M)(\nabla_\mu^B s \otimes i\gamma^\mu \psi + s \otimes \not{D} \psi) - D_B(s^* \otimes J_M \psi) \\ &= (\nabla_\mu^B s)^* \otimes (-i) J_M \gamma^\mu \psi + s^* \otimes J_M \not{D} \psi - \nabla_\mu^B s^* \otimes i\gamma^\mu J_M \psi - s^* \otimes \not{D} J_M \psi \\ &= -i \left((\nabla_\mu^B s)^* - \nabla_\mu^B s^* \right) \otimes J_M \gamma^\mu \psi = 0, \end{aligned}$$

since in four dimensions $\{J_M, \gamma^\mu\} = [\not{D}, J_M] = 0$, and the last step is established by the condition that $(\nabla s)^* = \nabla s^*$ for all $s \in \Gamma(M, B)$.

Furthermore, we have

$$\begin{aligned} [a, b^0](s \otimes \psi) &= aJb^*J^*(s \otimes \psi) - Jb^*J^*a(s \otimes \psi) = aJ(b^*s^* \otimes J_M^* \psi) - Jb^*(s^*a^* \otimes J_M^* \psi) \\ &= asb \otimes \psi - asb \otimes \psi = 0, \end{aligned}$$

where $a, b \in \Gamma(M, B)$ and $s \otimes \psi \in \Gamma(M, B) \otimes_{C^\infty(M)} \Gamma(M, S)$. Since $[a, b^0] = 0$ on $\Gamma(M, B) \otimes_{C^\infty(M)} \Gamma(M, S) \cong \Gamma(M, B \otimes S)$, it is zero on the entire Hilbert space $Ł^2(M, B \otimes S)$. Now, it remains to check the order one condition for the Dirac operator. This is straightforward: first note that

$$[[D, a], b^0](s \otimes \psi) = i\gamma([[\nabla, a], b^0](s \otimes \psi)) \quad (a, b, s \in \Gamma(M, B)). \tag{6.24}$$

This is zero because $[[\nabla, a], b^0](s \otimes \psi)$ is zero:

$$\begin{aligned}
([\nabla, a]sb) \otimes \psi - Jb^* J^*([\nabla, a]s \otimes \psi) &= \nabla(asb) \otimes \psi - a\nabla(sb) \otimes \psi - Jb^* J^* \nabla(as) \otimes \psi + Jb^* J^* a(\nabla s) \otimes \psi \\
&= \nabla(asb) \otimes \psi - a\nabla(sb) \otimes \psi - \nabla(as)b \otimes \psi + a(\nabla s)b \otimes \psi \\
&= ((\nabla a)sb + a(\nabla s)b + as(\nabla b) - a(\nabla s)b \\
&\quad - as(\nabla b) - (\nabla a)sb - a(\nabla s)b + a(\nabla s)b) \otimes \psi, \quad (a, b, s \in \Gamma(M, B)). \\
&= 0
\end{aligned}$$

Thus, J fulfils all the necessary conditions. Finally, we check the conditions for γ_B. It is clear that $\gamma_B^2 = 1$ and $\gamma_B^* = \gamma_B$ since these rules hold for γ^5. Moreover, γ_B also anti-commutes with J,D:

$$J\gamma_B + \gamma_B J = T \otimes J\gamma^5 + T \otimes \gamma^5 J = T \otimes [J, \gamma^5] = 0. \tag{6.25}$$

and the same calculation goes through with J replaced by D. Also $[\gamma_B, a] = 0$ for all $a \in \Gamma(M, B)$. $\square$

Theorem 6.3.8 is the main result of this chapter. We have now successfully constructed a real even spectral triple $(\Gamma(M, B), L^2(M, B \otimes S), D_B, J, \gamma_B)$ that is a generalisation of the real even spectral triple $(C^\infty(M, M_n(\mathbb{C})), L^2(M, S) \otimes M_n(\mathbb{C}), \not{D} \otimes 1, J_M \otimes T, \gamma^5 \otimes 1)$. In the next chapter we will determine the spectral action with respect to some perturbation of the Dirac operator D_B.

Remark 6.3.9. The spectral triple $(\Gamma(M, B), L^2(M, B), 0, T, 1)$ is easily seen to be an even spectral triple of KO-dimension 0.

7 SPECTRAL ACTION

In the previous chapters we have constructed the spectral triple

$$(\Gamma(M,B), L^2(M,B\otimes S), D_B, J, \gamma_B). \tag{7.1}$$

In this chapter we will determine the inner fluctuations of the Dirac-operator and calculate the spectral action with respect to the perturbated Dirac-operator. From now on we will omit the subscript B in D_B since this is notationally more convenient in this chapter. The dimension of M is assumed to be 4.

7.1 INNER FLUCTUATIONS

We will determine the inner fluctuations of the triple $(\Gamma(M,B), L^2(M,B\otimes S), D_B, J, \gamma_B)$. In the next section we will calculate the spectral action for the fluctuated Dirac-operators.

Inner fluctuations of the Dirac-operator are given by a perturbation term of the form

$$A = \sum_j a_j[D,b_j] \in \mathcal{B}(L^2(M,B\otimes S), \quad (a_j, b_j \in \mathcal{A} = \Gamma(M,B)), \tag{7.2}$$

with the additional condition that $\sum_j a_j[D,b_j]$ is a self-adjoint operator[1]. This expression for $\sum_j a_j[D,b_j]$ can be rewritten as:

$$\sum_j \gamma \circ (a_j[\nabla, b_j]\otimes 1), \tag{7.3}$$

where $\gamma : \Omega^1(M)\otimes_{C^\infty(M)} \Gamma(M,B\otimes S) \to \Gamma(M,B\otimes S)$ is given by

$$\gamma(\omega\otimes s\otimes\psi) = s\otimes\gamma(\omega)\psi, \quad (\omega\in\Omega^1(M), s\otimes\psi\in\Gamma(M,B\otimes S)). \tag{7.4}$$

Here, $\sum_j a_j[\nabla, b_j]$ is an element of $\Gamma(T^*M\otimes B)$.

Locally, on some trivialising neighbourhood U the expression in (7.2) can then be written as

$$\gamma^\mu A_\mu, \tag{7.5}$$

where A_μ are the components of the 1-form $\sum_j a_j[\nabla, b_j]$ that has values in the bundle B. Since A is self-adjoint the 1-form A_μ can be considered as a real 1-form taking values in the hermitian elements of the fibres of B.

The expression $A + JAJ^*$ is locally written as

$$\gamma^\mu A_\mu - \gamma^\mu J A_\mu J^*, \tag{7.6}$$

since γ^μ anti-commutes with J in 4 dimensions. Writing out the second term gives:

$$(\gamma^\mu J A_\mu J^*)(s\otimes\psi) = sA_\mu\otimes\gamma^\mu\psi, \quad \forall s\otimes\psi\in\Gamma(M,B\otimes S). \tag{7.7}$$

so that on this local patch $A + JAJ^*$ can be written as

$$\gamma^\mu \text{ad } A_\mu. \tag{7.8}$$

1 Since $[D,a]$ is bounded for all $a\in\mathcal{A}$, the operator $\sum_j a_j[D,b_j]$ is bounded

Since $A + JAJ^*$ eliminates the $u(1)$-part of A, it is natural to impose the uni-modularity condition

$$\text{Tr } A = 0. \tag{7.9}$$

Thus, $-iA_\mu$ is a real one-form on M with values in the bundle ad P (see Definition 5.2.19). We denote A^{pert} for this 1-form (which is defined on the whole of M).

The local and global form of $D + A + JAJ^*$ is given by respectively

$$D_A = i\gamma^\mu(\nabla^B_\mu \otimes 1 + 1 \otimes \nabla^S_\mu - i\text{ad } A_\mu \otimes 1) \tag{7.10}$$

and

$$D_A = i\gamma(1 \otimes \nabla^S + \nabla_B \otimes 1 + \text{ad } A^{pert}), \tag{7.11}$$

where A^{pert} is a real 1-form on M with values in the ad P-bundle introduced in Definition 5.2.19.

On some trivialising neighbourhood U_i ($i \in I$) the connection ∇^B can be expressed as $d + \text{ad } A^0_i$ for some *unique* $su(n)$-valued 1-form A^0_i on U_i. Thus, on U_i the fluctuated Dirac-operator can be rewritten as

$$D_A = i\gamma(d + 1 \otimes \omega^s + \text{ad } (A^0_i + A^{pert}) \otimes 1). \tag{7.12}$$

In the nontrivial case, we interpret ad $(A^0_i + A^{pert})$ as the full gauge potential on U_i. It acts in the adjoint representation on the spinors. As we will show below this family of full gauge potentials $\{\text{ad } (A^0_i + A^{pert})\}_{i\in I}$ has the right transformation property for a gauge potential for the principal $PUS(n)$-bundle P that we introduced in Chapter 5.[2]. Moreover, when we calculate the spectral action fo the fluctuated Dirac-operator D_A in the next section we will get the Yang–Mills Lagrangian with respect to this full gauge potential $su(n)$-valued ad $(A^0 + A^{pert})$. To check that the family $(\text{ad } (A^0_i + A^{pert}))_{i\in I}$ really is a gauge potential for the principal $PSU(n)$-bundle P, it must have the correct transformation rules when we switch to between local trivialisations. This is what the following propostion is about.

Proposition 7.1.1. *The local expression ad$(A^0 + A^{pert})$ has the correct transformation rule under a local gauge transformation (that is, a change of local trivialisation). In particular, the full gauge potential ad $(A^0 + A^{pert})$ induces a connection 1-form on the bundle P that we constructed in Chapter 5.*

Proof. If we identify ad$su(n)$ with $su(n)$ (and also write $A^0 + A^{pert}$ for ad $(A^0 + A^{pert})$,) then under a $PSU(n)(\cong \text{Ad } SU(n))$-valued transition function g, the expression $A^0 + A$ is a $su(n)$-valued field transforms as

$$A^0 + A^{pert} \mapsto (g^{-1}A^0 g + g^{-1}(dg)) + g^{-1}A^{pert}g = g^{-1}(dg) + g(A^0 + A^{pert})g^{-1}, \tag{7.13}$$

where the first two terms are the transformation of A^0 under a change of local trivialisation, and the last term is the transformation of A^{pert}. Since P is an associated bundle of B it follows from Definition 2.5.2 that $A^0 + A^{pert}$ induces a $su(n)$-valued connection 1-form on the principal $PSU(N)$-bundle P. □

The above propostion implies that the familiy $\{A^0_i + A^{pert}\}_{i\in I}$ is also a gauge potential for the description of Yang-Mills $PSU(n)$-bundle P. Moreover, the gauge potential $\{A^0_i + A^{pert}\}_{i\in I}$ acts in the adjoint representation on the spinors. The difference with the trivial case is that the full gauge potential of the theory is not given by only the perturbation term of the Dirac-operator. It also has a contribution from the connection ∇^B. This is a result of the non-triviality of the bundle B. Note that the terms $\{A^0_i\}$ are necessary to give the right transformation rules for the family $(A^0_i + .A^{pert})_{i\in I}$ in equation (7.13).

2 see Defintion 2.5.2

Remark 7.1.2. Let's see how the gauge potential $\{A_i^0 + A^{pert}\}_{i\in I}$ transforms under an internal transformation of the spectral triple. Well, if A^{pert} is given, then under an inner transformation induced by an element $u \in U(\Gamma(M,B))$, the operator A^{pert} transforms as $A \mapsto uAu^* + u[\nabla^B, u^*]$. On some local patch U_i this transformation is

$$A_i^0+A \mapsto A_i^0+uA^{pert}u^*+u[d+A_i^0,u^*] = A_i^0+uA^{pert}u^*+u(du^*)+uA_i^0u^*-A_i^0 = u(du^*)+u(A_i^0+A^{pert})u^*,$$

Here we see that under an internal transformation the gauge potential $A_i^0 + A^{pert}$ transforms in the same way as under the gauge transformation in equation (7.13). It seems therefore justified to interpret the group of internal automorphisms of the algebra $\mathcal{A}$ as the gauge group for the theory. However, I do not know if the groups $\mathrm{Inn}(\mathcal{A})$ and $\Gamma(M, \mathrm{Ad}\ P)$ are ismorphic. If they are, it would fully justify the interpretation of $\mathrm{Inn}(\mathcal{A})$ as gauge group.

In the following section we will show that the spectral action yields a Yang–Mills Lagrangians with respect to the field $A^0 + A^{pert}$. In the rest of this chapter we keep the notation A^0 and A^{pert} as defined above.

7.2 SPECTRAL ACTION: CALCULATION

We now state the theorem that we want to proof in this section. It is the generalisation of theorem 4.1.5. The local calculation is almost the same as in the trivial case.

Theorem 7.2.1. *For the spectral triple* $(\Gamma(M,B), L^2(M, B\otimes S), D_B, J, \gamma_B)$*, the spectral action is given by*

$$Tr\left(f(D/\Lambda)\right) \sim \frac{1}{(4\pi)^2}\int_M \mathcal{L}(g_{\mu\nu}, A)\sqrt{g}d^4x, \tag{7.14}$$

where $\mathcal{L}(g^{\mu\nu}, A)$ *is the Lagrangian of the Einstein-Yang-Mills system is given by*

$$L(g^{\mu\nu}, A) = 2N^2\Lambda^4 f_4 + \frac{N^2}{6}\Lambda^2 f_2 R + \frac{f(0)}{6}\mathcal{L}_{YM}(A) - \frac{N^2 f_0}{80}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \tag{7.15}$$

modulo topological and boundary terms. Here $\mathcal{L}_{YM}(A) = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ *where* $F^{\mu\nu}$ *is the curvature tensor with respect to the gauge field* $A^0 + A$*.*

Proof. This is the same calculation as in Theorem 4.1.5. The only missing ingredients for the calculation are the form of $\Omega_{\mu\nu}$ and E for the operator D_A. In this case E and $\Omega_{\mu\nu}$ are given by

$$E = -\frac{1}{4}s \otimes 1_{N^2} + \sum_{\mu<\nu}\gamma^\mu\gamma^\mu \otimes F^{\mu\nu}$$

$$\Omega_{\mu\nu} = \frac{1}{4}R^{ab}_{\mu\nu}\gamma_{ab} \otimes 1_{N^2} + 1_4 \otimes F^{\mu\nu}$$

where $F^{\mu\nu}$ is now the curvature of $A^0 + A^{pert}$. This difference comes from the fact that in the trivial case the connection $\nabla \otimes 1 + 1 \otimes d$ acted trivially on the matrix part of the bundle. Here the connection d has zero curvature. This was possible because the matrix bundle B was trivial. Since in the nontrivial case, we start with a connection $\nabla^B \otimes 1 + 1 \otimes \nabla^S$, where ∇_B cannot be a trivial connection since the bundle B is no longer trivial. This results in a non-zero curvature of the connection ∇^B, so that in the expressions for E and Ω the curvature tensor of the matrix part has a contribution from both the connection ∇_B on B and from the perturbation A^{pert}.

With E and Ω written this way, the calculation of the spectral action is the same as in the trivial case. □

Thus, we have seen that the spectral action principle indeed gives a Yang–Mills Lagrangian where the gauge field is $\{A_i^0 + A^{pert}\}_{i\in I}$ acting in the adjoint representation on the spinors. According to equation (7.13) in Propostion 7.1.1, the family of local expressions $(A_i^0 + A^{pert})_{i\in I}$ induces a gauge potential for the principal $PSU(n)$-bundle P. Hereby we have shown that the spectral triple $(\Gamma(M,B), L^2(M, B\otimes S), D_B, J, \gamma_B)$, with B a nontrivial algebra bundle with fibre $M_n(\mathbb{C})$ gives a nontrivial $PSU(n)$-Yang–Mills theory in the sense that the theory is also described by a nontrivial $PSU(n)$-fibre bundle. Hereby, we have generalised the triple in Chapter 4 to the nontrivial case.

8 CONCLUSION & OUTLOOK

8.1 CONCLUSION

We started, as sort of an example, by studying the real and even spectral triple introduced by Chamseddine and Connes

- $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$,
- $\mathcal{H} = L^2(M, S \otimes B)$,
- $D = \not{D}_M \otimes 1 + 1 \otimes d$,
- $J(\xi \otimes T) = J_M \xi \otimes T^*$,
- $\gamma = \gamma_5 \otimes 1$,

where S was the spinor bundle, $\not{D}$ the Dirac-operator on the spinor spinor bundle and where B was a *trivial* unital $*$-algebra bundle with fibre $M_n(\mathbb{C})$. Applying the spectral action principle to the perturbed Dirac-operator, we got an expression for the action that contained the Yang–Mills Lagrangian with respect to a $su(n)$-valued gauge field $-iA_\mu$. This gauge field $-iA_\mu$ acted in the adjoint representation on the spinors. Comparing this with the description of gauge theories in terms of principal bundles, we see that the gauge potential $-iA_\mu$ can also be interpreted as a gauge field on a principal $PSU(n)$-bundle, although this bundle is topologically trivial.

In this thesis we have successfully generalised the above situation to the nontrivial case. The key observation was that in the above spectral triple the bundle B was trivial and that the spectral algebra $C^\infty(M, M_n(\mathbb{C})$ is precisely the finitely generated projective $C^\infty(M)$-algebra of sections of this bundle B. Replacing the trivial $M_n(\mathbb{C})$-bundle B by a nontrivial bundle was most likely the best starting point to generalise the above situation.

In Chapter 5 we have shown that given, in general, any (unital) involutive finitely generated projective $C^\infty(M)$-algebra $\mathcal{P}$, one can associate (up to isomorphism) a unique (unital) $*$-algebra bundle B for which $\Gamma(M, B) \cong P$ as involutive $C^\infty(M)$-algebras. Moreover, an explicit construction of this bundle was given. When we applied these steps to the unital involutive finitely generated projective $C^\infty(M)$-algebra $\Gamma(M, B)$ where B is a nontrivial unital $*$-algebra bundle with fibre $M_n(\mathbb{C})$, this gave a nontrivial principal $PSU(n)$-bundle. Moreover, B is an associated bundle of P.

In Chapter 6 we completed our spectral triple by choosing a suitable Hilbert-space and Dirac-operator. The fibres of B were endowed with an inner product and the Hilbert-space was chosen to be $L^2(M, B \otimes S)$. The generalisation of the Dirac-operator to the nontrivial case had a remarkable result. Although the operator was still of the form

$$i\gamma(\nabla^B \otimes 1 + 1 \otimes \nabla^S), \tag{8.1}$$

where ∇^S is the spin connection on the bundle S and ∇^B a hermitian connection on the bundle B, the connection ∇^B could no longer be chosen to have zero curvature.[1] This was a consequence of the fact that the bundle B is no longer trivial and therefore affects the form of the spectral action. This gave the spectral triple (see Proposition 6.2.3)

$$(\Gamma(M, B), L^2(M, B \otimes S), D_B). \tag{8.2}$$

1 in the trivial case, we had $\nabla^B = d$, which has zero curvature

Also in Chapter 6 we defined operators

$$J = T \otimes J_M, \qquad \gamma_B = 1 \otimes \gamma^5, \tag{8.3}$$

where T maps the section s to the section s^*, and where J_M, γ^5 are respectively the real structure and grading operator of the canonical triple. Under the conditions that ∇_B was a hermitian $*$-algebra connection, it was proved that the triple

$$(\Gamma(M,B), L^2(M, B \otimes S), D_B, J, \gamma_B) \tag{8.4}$$

was a real and even spectral triple.

In Chapter 7 it was shown that for the spectral triple (8.4) a $su(n)$-gauge potential was found that acted in the adjoint representation on the spinors, just like in the trivial case. Only this time the gauge potential consisted of two parts, because the connection on B was no longer trivial. One part came from the connection on B, the other part from the pertubation of the Dirac-operator. With respect to this full gauge potential the spectral action gave the correct Yang-Mills Lagrangian. Also, the full gauge potential defined a $su(n)$-gauge field (or $su(n)$-connection 1-form) for the principal $PSU(n)$-bundle we had constructed from the algebra $\Gamma(M,B)$.

Summarising, we can say that with (8.4) we have successfully generalised the triple of (8.1) to the nontrivial case. According to the description of gauge theories in terms of bundles, this triple describes a non-trivial $PSU(n)$-Yang Mills theory. Indeed, using the principles of noncommutative geometry we found the correct Lagrangian with respect to an $su(n)$-gauge potential for the bundle P. Moreover, since B was an associated bundle of P, every Yang–Mills theory formulated in terms of a principle $PSU(n)$-bundle P can be described by a spectral triple of the form (8.4).

Finally, we must remark here that for a full correspondence between both descriptions of Yang–Mills theory it is necessary that the groups $\mathrm{Inn}(\Gamma(M,B))$ and $\Gamma(M, \mathrm{Ad}\, P)$ are isomorphic, since both groups are interpreted as the symmetry group of the theory ($\mathrm{Inn}(A)$ for a spectral triple and $\Gamma(M, \mathrm{Ad}\, P)$ for the gauge theory described by the bundle P). I am not sure if these groups are isomorphic. I intend to look after this at a later moment.

8.2 OUTLOOK

There are a few more things that can be said about the spectral triple $(\Gamma(M,B), L^2(M, B \otimes S), D_B, J, \gamma_B)$ in Theorem 7.2.1. We have calculated the spectral action with respect to a twisted Dirac-operator on the bundle $B \otimes S$. We would expect that a Chern-character will appear if a term containing γ^5 is added to the spectral action. This would be absent in the topological trivial case since then the Chern-character vanishes.

Another interesting point is that the construction of the spectral triple $(\Gamma(M,B), L^2(M,B) \otimes_B L^2(M,S), D)$ is similar in flavor to the internal unbounded Kasparov-product in KK-theory ([26]). Not unrelated is also the general approach to almost commutative spectral triples taken by Ćaćić ([4]). Probably our construction is an example of it.

These two points are very interesting and worthwhile to understand better. Since there is no time left to take a closer look at these points during my master's thesis research, these are questions to be studied at a later moment.

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A CALCULATIONS IN CLIFFORD ALGEBRAS

In Chapter 4 the spectral action for the triple $(C^\infty(M, M_n(\mathbb{C})), L^2(M,S)\otimes M_n(\mathbb{C}), \not{D})$ is calculated. In this calculation some identities in Clifford-algebras were used. These identities are derived here.

The Clifford algebra generators $\{\gamma^\mu\}$ obey the following anti-commutation relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{A.1}$$

where $g^{\mu\nu}$ is a Riemannian metric.

It follows that

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}\mathrm{Tr}(\mathrm{Id}) = 4g^{\mu\nu}. \tag{A.2}$$

We can derive, by using the anti-commutation relations, the following identity:

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2\gamma^\mu\gamma^\nu g^{\rho\sigma} - 2\gamma^\mu\gamma^\rho g^{\nu\sigma} + 2\gamma^\nu\gamma^\rho g^{\mu\sigma} - \gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho. \tag{A.3}$$

Using equations (A.2) and (A.3), it directly follows that:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) + \frac{1}{2}\mathrm{Tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{A.4}$$

For $u, v, w \in V$ in the Clifford algebra of V one can also derive the identity:

$$\begin{aligned}
Q(uvw) &:= \frac{1}{6}(uvw + wuv + vwu - vuw - wvu - uwv)\\
&= \frac{1}{6}(uvw + 2g(u,w)v - 2uwv + 2g(u,w)v - 2vuw - wvu)\\
&= \frac{1}{6}(uvw + 2g(u,w)v - 4g(v,w)u + 2uvw + 2g(u,w)v - 4g(u,v)w + 2uvw - wvu)\\
&= \frac{1}{6}(5uvw + 4g(u,w)v - 4g(v,w)u - 4g(u,v)w - wvu)\\
&= \frac{1}{6}(6uvw + 4g(u,w)v - 4g(v,w)u - 4g(u,v)w - 2g(u,v)w + 2g(u,w)v - 2g(v,w)u)\\
&= uvw + g(u,w)v - g(u,v)w - g(v,w)u.
\end{aligned}$$

This gives us the equality:

$$uvw = Q(uvw) + g(v,w)u - g(u,w)v + g(u,v)w. \tag{A.5}$$

We introduce the vierbein e^a_μ defined by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}, \tag{A.6}$$

so that $\gamma_a = \gamma_\mu e^\mu_a$ satisfies the commutation relation $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. Furthermore, we define

$$\gamma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \tag{A.7}$$

These "flat gammas" obey the following trace relations,

$$\mathrm{Tr}(\gamma_a \gamma_b \gamma_c \gamma_d) = 4(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \tag{A.8}$$

which can be obtained from equation (A.4) by substituting δ_{ab} for $g_{\mu\nu}$. Using this relation we can derive for $a < b$ and $c < d$:

$$\mathrm{Tr}(\gamma_a \gamma_b)^* \gamma_c \gamma_d) = \mathrm{Tr}(\gamma_b \gamma_a \gamma_c \gamma_d) = \delta_{ac}\delta_{bd} \tag{A.9}$$

Indeed, since trace identity (A.8) implies that any trace of four gamma-matrices is only non-zero when the four gamma-matrices have two pairs of equal indices, it follows from $a < b$ and $c < d$ that the above trace is only non-zero when $a = c$ and $b = d$. In that case $\mathrm{Tr}(\gamma_b \gamma_a \gamma_c \gamma_d) = 1$. This proves equation (A.9).

B

ALTERNATE PROOF FOR THE EXISTENCE OF HERMITIAN $*$-ALGEBRA CONNECTIONS ON HERMITIAN $*$-ALGEBRA BUNDLES

In this section we will given an alternative proof of Theorem 6.3.6. The ideas in this section come from Kobayashi and Nomizu ([19], Chapter I Theorem 5.7), and from Godement ([14], p.151). The statement in Corollary B.0.7 is the same statement as in Theorem 6.3.6.

Definition B.0.1. *Let M be a manifold. A function f is said to be smooth on a subset $F \subset M$ if for each $x \in F$ there exists a neighbourhood V_x of x and a smooth function f_x on V_x such that $f_x = f$ on $F \cap V_x$. A same definition can be used to make sense of connections and sections defined on arbitrary subsets of M.*

Theorem B.0.2. *Let F be a closed subset of a paracompact manifold M. Then every real-valued smooth function f defined on F can be extended to a real-valued smooth function on M.*

Proof. For each $x \in F$ let f_x be a smooth function on some open neighbourhood $V_x \ni x$ such that $f_x = f$ on $F \cap V_x$. Let $\{U_i\}$ be a locally finite open refinement of the open covering of M consisting of $M - F$ and V_x for all $x \in F$. For each $i \in I$ define a smooth function g_i on U_i as follows: if U_i is contained in some V_x, set

$$g_i = \text{ restriction of } f_x \text{ to } U_i. \tag{B.1}$$

If there is no such V_x, then set

$$g_i = 0. \tag{B.2}$$

Now, let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define

$$g = \sum f_i g_i. \tag{B.3}$$

Since $\{U_i\}$ is locally finite, every point of M has a neighbourhood on which $\sum f_i g_i$ is a finite sum. Thus, g is a smooth function on M and it clearly extends f. □

Corollary B.0.3. Let $F \subset \mathbb{R}^n$ be a closed set. Then any real-valued smooth function f defined on F can be extended to a smooth function on $\mathbb{R}^n$.

Corollary B.0.4. Let M be a paracompact manifold and E a real vector bundle over M. Every point of M has a neighbourhood U such that every section defined on a closed subset contained in U can be extended to U.

Proof. For $x \in M$, take a neighbourhood U of x such that $\pi^{-1}(U) \cong U \times F$ where F is the fibre. Since F is diffeomorphic with $\mathbb{R}^m$ for some $m \in \mathbb{N}$, a section on U can be identified with a set of m real-valued smooth functions $f_1, \ldots, f_m$ defined on U. The result now follows from Corollary B.0.3. □

The following lemma expands the above idea in Corollary B.0.4 to connections.

Lemma B.0.5. *Let M be a paracompact manifold and let B be a hermitian $*$-algebra bundle with fibre A. Every point of M has a neighbourhood U such that every hermitian $*$-algebra connection defined on a closed subset contained in U can be extended to a hermitian $*$-algebra connection on U.*

Proof. Given $x \in M$, choose a neighbourhood U of x such that $\pi^{-1}(U) \cong U \times A$. If a connection ∇ on a closed subset $F \subset U$ is given, it is of the form $\partial_\mu + \omega_\mu$ where ω_μ is a smooth map taking its values in the real vector space $\mathrm{Der}^*(A) \cap u_g(A)$. By B.0.4 each ω_μ can be extended to a smooth map $(\omega_\mu)_U : U \to \mathrm{Der}^*(A) \cap u_g(A)$. The map

$$s \mapsto ds + \omega_U s, \quad U \to \Omega_1(M) \otimes_{C^\infty(U)} \Gamma(U, B), \tag{B.4}$$

is then the extension of the hermitian $*$-algebra connection ∇ defined on F, to a hermitian $*$-algebra connection on U. □

Theorem B.0.6. *Let M be a compact manifold and let B be a hermitian $*$-algebra bundle. Let F be a closed subset of M. Then every hermitian $*$-algebra connection $\nabla : \Gamma(F, B) \to \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(F, B)$ can be extended to a hermitian $*$-algebra connection on M.*

Proof. Let $\{U_i\}_{i \in I}$ be a locally finite open covering of M such that each U_i has the property stated in Lemma B.0.5. Let $\{V_i\}$ be a open refinement of $\{U_i\}$ such that $\bar{V}_i \subset U_i$ for all $i \in I$. For each subset $J \subset I$ set $S_J = \cup_{i \in J} \bar{V}_i$. Let T be the set of all pairs $(\tilde{\nabla}, J)$ where $J \subset I$ and $\tilde{\nabla}$ is a hermitian $*$-algebra connection of E on S_J such that $\tilde{\nabla} = \nabla$ on $F \cap S_J$. The set T is non-empty: take U_i such that $U_i \cap F \neq \emptyset$ and extend the restriction of ∇ on $F \cap \bar{V}_i$ to a hermitian $*$-algebra connection on $\bar{V}_i$.[1] This is possible because $\bar{V}_i \subset U_i$. We can now introduce an order on T by setting $(\tilde{\nabla}', J') \leq (\tilde{\nabla}'', J'')$ if $J' \subset J''$ and $\tilde{\nabla}' = \tilde{\nabla}''$ on $S_{J'}$. Zorn's lemma implies that there exists a maximal element $(\tilde{\nabla}, J)$ of T. Assume now that $I \neq J$ and choose $i \in I - J$. On the closed[2] subset $(F \cup S_J) \cap \bar{V}_i$, that is contained in U_i, we have a well-defined hermitian $*$-algebra connection $\nabla_i = \nabla$ on $F \cap \bar{V}_i$ and $\nabla_i = \tilde{\nabla}$ on $S_J \cap \bar{V}_i$. Extend ∇_i to a a hermitian $*$-algebra connection on $\bar{V}_i$. Let $J' = J \cup \{i\}$ and $\tilde{\nabla}'$ be the connection on $S_{J'}$ defined by $\tilde{\nabla}' = \tilde{\nabla}$ on S_J and $\tilde{\nabla}' = \nabla_i$ on $\bar{V}_i$. Then $(\tilde{\nabla}, J) < (\tilde{\nabla}', J')$. This is in contradiction with the maximality of $(\tilde{\nabla}, J)$. Therefore, $I = J$ and $\tilde{\nabla}$ is the desired connection. □

Corollary B.0.7. Every hermitian $*$-algebra bundle over a compact manifold M has a hermitian $*$-algebra connection.

Proof. In Theorem B.0.6, take $F = \emptyset$. □

---

1 if $F = \emptyset$, T is not empty either because on any local trivialising neighbourhood U_i we can always construct a hermitian $*$-algebra connection and then restrict it the closed subset $V_i \subset U_i$.

2 although $\cup_{i \in J} \bar{V}_i$ may be a infinite union of closed subsets, it is still closed. This follows from the locally finiteness of the open covering $\{U_i\}$. Take $x \in (\cup_{i \in J} \bar{V}_i)^c = \cap (\bar{V}_i)^c$. Now, since the open covering $\{U_i\}$ is locally finite and $\bar{V}_i \subset U_i$, there exists a open neighbourhood U of x such that there are only finite many $i \in J$ with $U \cap \bar{V}_i \neq \emptyset$. Call this set K. Then $\tilde{U} := \cap_{i \in K} U \cap \bar{V}_i^c$ is an open neighbourhood of x and $\tilde{U} \cap \bar{V}_i = \emptyset$ for all $i \in J$. Hence $(\cup_{i \in J} \bar{V}_i)^c$ is open.

C

CATEGORIES

In Chapter 5 equivalences of categories are established. For the reader who is not familiar with catergory theory, we will briefly explain what is meant by a category and by an equivalence of categories. This section is based on [25].

Roughly speaking, a category $\mathcal{C}$ is an abstract collection consisting of a class of objects $\mathrm{ob}(\mathcal{C})$ and a class of morphisms $\hom(\mathcal{C})$ between objects. The formal definition of a category will be given below.

Given a morphism f in a category, we assign to it an object $A = \mathrm{dom}(f)$ in that category, which is called the domain of f, and an object $B = \mathrm{cod}(f)$, which is called the codomain of f. Usually, we will denote a morphism with domain A and codomain B by

$$f : A \to B.$$

An identity morphism for an object X is a morphism $1_X : X \to X$ such that for every morphism $f : A \to B$ we have $1_B \circ f = f = f \circ 1_A$.

For any two arrows $f : A \to B$ and $g : B \to C$, such that $\mathrm{dom}(g) = \mathrm{cod}(f)$, we define the composite morphism $g \circ f : A \to C$.

Definition C.0.8. *A category $\mathcal{C}$ consists of a class of objects $\mathrm{ob}(\mathcal{C})$ and a class of morphisms $\hom(\mathcal{C})$ between objects, such that for every object X of $\mathcal{C}$ there exists an identity morphism $1_X : X \to X$ and for any three arrows f, g and h with $\mathrm{dom}(g) = \mathrm{cod}(f)$ and $\mathrm{dom}(h) = \mathrm{cod}(g)$ the composition is associative, i.e. $(h \circ g) \circ f = h \circ (g \circ f)$.*

Let R be a ring, then the category R-mod is the category with all right R-modules as objects and all linear maps between these right R-modules as morphisms. Another example of a category is the category of groups Grp that has all groups as objects and all group homomorphisms as morphisms.

A category that will be important for us later is the category of complex vector bundles over a manifold M. The objects are all complex vector bundles over M and the morphisms are the vector bundle homomorphisms.

A morphism $f : A \to B$ in a category $\mathcal{C}$ is called invertible if there exists a morphism $f' : B \to A$ in $\mathcal{C}$ such that $f \circ f' = 1_B$ and $f' \circ f = 1_A$. Then f' is called the inverse of f in $\mathcal{C}$. A morphism can have only one inverse morphism. An invertible morphism is also called an isomorphism.

Definition C.0.9. *Two objects A and B are called isomorphic (notation: $A \cong B$) in $\mathcal{C}$ if there exists an invertible morphism $h : A \to B$.*

To speak of an equivalence of categories we need a notion of a *map* T between two categories. The following definition gives the formal definition of such a map.

Definition C.0.10. *For categories $\mathcal{B}$ and $\mathcal{C}$, a (covariant) functor $T : \mathcal{B} \to C$ with domain $\mathcal{B}$ and codomain $\mathcal{C}$ consists of two functions: the object function, which assigns to every object B of $\mathcal{B}$ an object TB in $\mathcal{C}$, and the arrow function, which assigns to each arrow $f : B \to B'$ in $\mathcal{B}$ an arrow $Tf : TB \to TB'$ in $\mathcal{C}$ such that $T(1_B) = 1_{B'}$ and $T(g \circ f) = Tg \circ Tf$ for any two morphisms f and g for which the composition $g \circ f$ is defined.*

Definition C.0.11. *Let $\mathcal{B}, \mathcal{C}$ be two categories. We say that $\mathcal{B}$ and $\mathcal{C}$ are equivalent as categories if there exists a (covariant) functor $T : \mathcal{B} \to \mathcal{C}$ satisfying the following conditions:*

- *The functor T is faithful. A functor T is said to be faithful when to every pair b_1, b_2 of objects and to every pair $f, g : b_1 \to b_2$ of arrows the equality $Tf = Tg$ implies $f = g$.*
- *The functor T is full. A functor is said to be full when to every pair b_1, b_2 of objects of $\mathcal{B}$ and every arrow $g : Tb_1 \to Tb_2$ of C there is an arrow $f : b_1 \to b_2$ of B with $g = Tf$.*
- *The functor T is essentially surjective. Essentially surjective means that each object $c \in \mathcal{C}$ is isomorphic to Tb for some object $b \in \mathcal{B}$.*

Note that these three conditions establish a bijective correspondence between the isomorphism classes of the categories $\mathcal{B}$ and $\mathcal{C}$. When there is an equivalence between categories, this means that there are strong similarities between the mathematical structures concerned.

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