THE STRUCTURE OF GAUGE THEORIES IN ALMOST COMMUTATIVE GEOMETRIES

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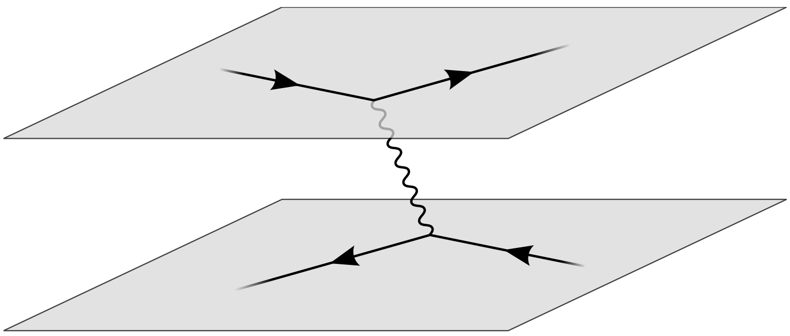

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A Master's Thesis in Physics & Astronomy.

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On the cover: an impression of electrodynamics on the two-point space.

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ABSTRACT

Within the framework of noncommutative geometry, the concept of almost commutative geometries can be used to describe models in physics. We shall study the properties of real even almost commutative geometries in their most general form. In particular, we show that any such almost commutative geometry describes a gauge theory, minimally coupled to a Higgs field, and we show how to recover the gauge group from the spectral triple. In the 4-dimensional case, we derive the general form of the bosonic Lagrangian, and show that it is conformally invariant up to the appearance of a dilaton field. We then apply the general framework to three examples, which step by step take us towards the description of the Standard Model of high energy physics. The first example of electrodynamics shows that it is also possible to describe abelian gauge theories in noncommutative geometry. In the second example of the Glashow-Weinberg-Salam model, we in particular show that the Higgs mechanism provides a spontaneous breaking of both the gauge symmetry and the conformal symmetry. In the third example, we can then easily extend the results to describe the full Standard Model.

PREFACE

Seven years and four months, that is how much time has passed between following my first lecture at the university and finalizing my thesis. Though a study in physics should nominally take only five years, I have tried my best to extend this period, and with success. I am glad that I am able to say that the entire delay was time well-spent. I have spent many hours of my time in useful work for several students' organizations. Although this led to a delay of my graduation, I consider it to be a valuable contribution to my personal academic development.

However, there comes a point when it is time to finish and move on, and the time arrived to finish my Master. About one year ago, I needed to choose a research area for my Master's thesis. Although I am a student in physics, I have always had an interest in mathematics. I am very grateful for the opportunity to do my Master's research right on the borders between physics and mathematics.

I chose to do my Master's research in the field of noncommutative geometry, an area of mathematics that at the time was completely unknown to me. Therefore, a large portion of my time was spent on reading books and papers and reproducing known results. The original plan was to produce a cosmological model from noncommutative geometry. However, as time went by, other interesting topics in noncommutative geometry revealed themselves to me, and I frequently changed directions. Essentially, I had to finish my thesis before I could say what it would be about.

Now my thesis is finished. During the past year I have not only familiarized myself with (a part of) noncommutative geometry, but I have also had the chance to start developing my own ideas on some topics. Perhaps, the latter can be considered the most important result of my thesis.

ACKNOWLEDGMENT

Of course, writing this thesis is not something I could have done all by myself. I would like to thank my supervisor, Walter van Suijlekom, for all his help and advice, for allowing me the flexibility to move freely from one topic to the other, and for pointing me in the right direction when needed.

I thank my parents for their ongoing encouragement. I also thank everyone in our office and the rest of the department of high energy physics for providing a pleasant working environment, where hard work and fun really seem to go hand in hand.

SAMENVATTING VOOR LEKEN

summary for laymen (in dutch)

Gedurende het afgelopen jaar werd mij regelmatig door bekenden de vraag gesteld: "Waar gaat je scriptie eigenlijk over?" In veel gevallen werd deze vraag vrij snel gevolgd door: "Of begrijp ik daar toch niks van?" Het eenvoudigste antwoord dat ik op deze vragen kan geven is: "Nee, dat begrijp je inderdaad niet. . . " Vaak waagde ik vervolgens toch een halfslachtige poging om uit te leggen waar ik mee bezig was, en ik vertelde dan zoiets als:

> "Ik bestudeer een deelgebied van de wiskunde dat *niet-commutatieve meetkunde* heet. Ik ben natuurkundestudent, en zit hiermee op het grensvlak van de wiskunde en de natuurkunde. Een belangrijke toepassing van de niet-commutatieve meetkunde is dat je het Standaard Model van de natuurkunde hieruit kunt afleiden. Dit Standaard Model beschrijft alle elementaire deeltjes en hun onderlinge interacties, behalve de zwaartekracht. Bijkomend voordeel van de niet-commutatieve meetkunde is dat je, naast het Standaard Model, ook de zwaartekracht erbij cadeau krijgt. Zelf bestudeer ik wat voor andere modellen men zou kunnen afleiden uit de niet-commutatieve meetkunde, en wat voor eigenschappen deze modellen in het algemeen hebben."

Ik heb echter zelden het idee gehad dat mensen ook daadwerkelijk veel wijzer zijn geworden van deze korte uitleg. Bij deze wil ik daarom de uitdaging aangaan om te proberen in lekentaal samen te vatten waar mijn scriptie ongeveer over gaat. Voor ik echt begin, wil ik je complimenteren met het feit dat je het hebt aangedurfd om deze scriptie open te slaan, en dat je samen met mij de uitdaging wilt aangaan om er iets van te begrijpen. Ik kan je niet beloven dat je na het lezen van deze samenvatting alles perfect begrijpt, maar ik zal mijn best doen om ervoor te zorgen dat je er in ieder geval iets van zult opsteken.

meetkunde en zwaartekracht

Laten we ons eerst een idee vormen van wat meetkunde is. De meetkunde geeft een beschijving van de ruimte. Hiermee beschrijft men bijvoorbeeld wat de afstand tussen twee punten is, of wat het verschil tussen recht en krom is. Ruimte en tijd zijn nauw met elkaar verbonden, en het samengevoegde concept ruimte-tijd kunnen we geheel meetkundig beschrijven.

De notie van ruimtelijke kromming verstoort ons intuïtieve idee van wat recht en krom is. Een steen die we in de lucht werpen zal een boogvormige baan volgen en verderop weer op de grond terecht komen. Deze boogvormige baan is in onze ogen duidelijk krom, maar kan niettemin ook beschreven worden als een 'rechtlijnige' beweging op een gekromde ruimte. Op deze wijze kan de zwaartekracht worden beschreven als meetkundige theorie. De aantrekkingskracht van de aarde kromt de ruimte, en zorgt er zo voor dat de omhoog geworpen steen vanzelf weer naar beneden komt.

De zwaartekracht is echter niet de enige kracht in de natuur die we kennen. Bekend uit het dagelijkse leven zijn elektriciteit en magnetisme. Daarnaast bestaan bovendien de sterke en zwakke kernkrachten, die zorgen voor stabiele atoomkernen of voor radioactief verval. Een omschrijving van alle elementaire deeltjes (de fundamentele bouwstenen van alle materie), inclusief het effect van de drie genoemde interacties op deze deeltjes, wordt gegeven door het Standaard Model van de natuurkunde.

Dit Standaard Model bezit een bepaalde symmetrie, genaamd ijksymmetrie. Deze ijksymmetrie levert een wiskundig elegante omschrijving van de interacties tussen elementaire deeltjes. Het feit dat materie massa heeft, verstoort echter de symmetrie. Een theoretische oplossing die de symmetrie herstelt, is de introductie van een nieuw deeltje, het Higgs-deeltje, in het Standaard Model. Dit theoretisch voorspelde deeltje is echter het enige onderdeel van het Standaard Model dat nog niet is waargenomen. Op dit moment wedijveren de deeltjesversnellers van Fermilab (Chicago) en CERN (Genève) om de ontdekking hiervan.

De eenvoudige meetkunde is niet in staat om, naast de zwaartekracht, ook de interacties van het Standaard Model te beschrijven. Voor deze overige interacties is het nodig om, aan ieder punt in de ruimte-tijd, extra interne vrijheidsgraden toe te voegen. Deze interne vrijheidsgraden beschrijven welke deeltjes er bestaan, en welke interacties ze met elkaar kunnen hebben. Hier komt de niet-commutatieve meetkunde om de hoek kijken. Zij maakt de meetkunde algemener, door naast de ruimte-tijd ook deze interne vrijheidsgraden te beschrijven. Via deze combinatie van ruimte-tijd met interne vrijheidsgraden, beschrijft de niet-commutatieve meetkunde een natuurkundig model. De precieze vorm van dit natuurkundige model hangt af van de vorm van de interne vrijheidsgraden. Bij een geschikte keuze van deze interne vrijheidsgraden blijkt dat het bijbehorende natuurkundige model precies het Standaard Model oplevert (inclusief het Higgs-deeltje), ditmaal echter gekoppeld aan de meetkundige beschrijving van de zwaartekracht.

MIJN SCRIPTIE

In mijn scriptie beschrijf ik eveneens natuurkundige modellen, die door de combinatie van ruimte-tijd met bepaalde interne vrijheidsgraden uit de niet-commutatieve meetkunde verkregen worden. Ik doe dit echter zonder te kiezen welke interne vrijheidsgraden er zijn. Dit levert een algemene beschrijving op van natuurkundige modellen in niet-commutatieve meetkunde. Ook zonder te weten welke interne vrijheidsgraden er zijn, is het nog steeds mogelijk om de algemene eigenschappen van zulke natuurkundige modellen goed te beschrijven. In het bijzonder toon ik aan dat deze modellen altijd een ijksymmetrie bezitten. Ook het Higgs-deeltje komt in alle gevallen vanzelf tevoorschijn. Daarmee biedt mijn scriptie goede inzichten in het soort natuurkundige modellen dat met behulp van de niet-commutatieve meetkunde beschreven kan worden.

Voortbouwend op deze algemene beschrijving van natuurkundige modellen, is het niet heel moeilijk om concrete voorbeelden van modellen te construeren. Hiervoor hoeven we enkel een concrete keuze te maken voor de interne vrijheidsgraden. Eén van de voorbeelden die ik geef, beschrijft de Electrodynamica, of, simpel gezegd, elektriciteit en magnetisme. De Electrodynamica heeft een bijzondere ijksymmetrie, omdat deze commutatief is. Zonder hier uit te leggen wat commutatief betekent, moge het duidelijk zijn dat niet-commutatief het tegenovergestelde is. In de vakliteratuur bestond het vermoeden dat het onmogelijk was om vanuit de niet-commutatieve meetkunde een dergelijke commutatieve ijksymmetrie te beschrijven, en dus dat het onmogelijk was om de Electrodynamica te beschrijven. Aangezien ik in mijn scriptie wel de Electrodynamica heb kunnen beschrijven, heb ik dit vermoeden uit de literatuur ontkracht.

MOTIVATIE

Tot slot nog een vraag die weleens gesteld wordt: "Maar wat heb je daar nou aan?" Zijn er nuttige toepassingen van dit onderzoek? Als iemand zijn wiskundige modellen toepast op economische vraagstukken, dan zal iedereen begrijpen dat deze wiskunde nuttig kan zijn voor de maatschappij. Maar hoe zit dit bij de toepassing van wiskundige modellen op de theoretische natuurkunde? In mijn ogen is dat wel degelijk een nuttige toepassing van de wiskunde. Jij zou waarschijnlijk echter de voor de hand liggende vervolgvraag stellen: "En wat is dan het nut van de theoretische natuurkunde?" Hierop kan ik slechts het ietwat onbevredigende antwoord geven: "Dat is afwachten." Het eerste doel van de theoretische natuurkunde is namelijk het zo goed mogelijk begrijpen hoe de natuur in elkaar zit. Ik ben van mening dat een verbeterd begrip van de werkelijkheid in zichzelf al waardevol is.

Desalniettemin zal een verbeterd begrip van de werkelijkheid, na verloop van tijd, vanzelf tot nuttige toepassingen leiden. Een sterk voorbeeld hiervan is de relativiteitstheorie, die begin vorige eeuw door Albert Einstein is bedacht. Deze theorie leert ons dat tijdsmetingen relatief zijn. De tijd die een klok aangeeft hangt af van de sterkte van de zwaartekracht en de snelheid waarmee de klok beweegt. Dit heeft dus effect op de klokken waarmee GPS-satellieten zijn uitgerust. Zonder een goed begrip van de relativiteitstheorie is navigatie op basis van GPS-satellieten daarom simpelweg niet mogelijk. Maar als men aan Einstein zelf zou hebben gevraagd wat het nut is van zijn werk, zou hij heus niet aan satellietnavigatie gedacht hebben.

Om deze reden is het mijns inziens geheel gerechtvaardigd om vooruitgang van de wetenschap zelf als doel te beschouwen. Ik ben ervan overtuigd dat een beter begrip van de natuurwetten vanzelf zijn nut zal bewijzen.

Ik hoop je met deze samenvatting een idee te hebben gegeven waar mijn scriptie over gaat. Het is aan jou om te oordelen of dat goed gelukt is. Hoe dan ook wens ik je nog veel leesplezier.

Koen van den Dungen
Januari 2010

CONTENTS

1 INTRODUCTION

The framework of noncommutative geometry [10] provides a generalization of ordinary Riemannian geometry, in two steps. The first step is to obtain an algebraic description of the topology and geometry of a space. The starting point is the Gelfand duality between a topological space and its algebra of functions. By this duality, any unital commutative C^*-algebra describes a topological space. This duality has been expanded to find an algebraic description of the geometry of a space. It has been shown that a Riemannian spin manifold can be fully described by the canonical spectral triple $(C^\infty(M), L^2(M,S), \not{D})$, consisting of the algebra of functions, the Hilbert space of square integrable sections of the spinor bundle, and the canonical Dirac operator [11, 12]. The second step, and the main idea of noncommutative geometry, is to then take this algebraic description in terms of spectral triples as the starting point, and drop the commutativity of the algebra. This noncommutative spectral triple can then be considered to describe a 'noncommutative' space.

The field of noncommutative geometry has interesting physical applications. In physics there are two main theories, which each describe a separate part of physics. First, there is Einstein's theory of General Relativity, which describes gravity by the curvature of space-time. Second, there is the Standard Model of high energy phsyics, which describes all elementary particles, and their electromagnetic, weak and strong interactions.

Using the canonical triple, which describes a Riemannian space, one can find a theory of gravity. In order to obtain other interactions as well, one somehow needs to add internal degrees of freedom to the model. In noncommutative geometry, these internal degrees of freedom can be described by a finite spectral triple.

The main subject of this thesis is the concept of almost commutative geometries. An almost commutative geometry is given by the product of the canonical spectral triple (describing Riemannian space-time) and a finite spectral triple (describing the internal degrees of freedom). One of the main successes of noncommutative geometry (at least from the physicists' point of view) is that for a suitable choice of the finite spectral triple, the resulting almost commutative geometry allows to derive the Standard Model coupled to gravity [8].

We will study the general properties of real even almost commutative geometries. The main focus will be the formulation of gauge theories in noncommutative geometry. A precise definition of the gauge group will be given. We will also prove that any almost commutative geometry describes a gauge theory.

Once the general framework has been established, we discuss three important examples in physics. We start with electrodynamics, and via the Glashow-Weinberg-Salam model we work our way towards the Standard Model. Though the first two examples serve as stepping stones for a good understanding of the Standard Model, they are also interesting by themselves. It has long been thought impossible to describe abelian gauge theories within the framework of noncommutative geometry, but with our description of electrodynamics we show that it is very well possible. The Glashow-Weinberg-Salam model provides an interesting example of the Higgs mechanism in an almost commutative geometry, where not only the gauge symmetry, but also the conformal symmetry, is spontaneously broken.

This thesis has been divided into several parts. In Part I we first gather some preliminary material on differential geometry and spin geometry, which will be needed later on. Next, we will describe how the notions of symmetry and invariance are used to describe theories of physics. We will briefly explain the Lagrangian formalism and gauge theory. We will also give a description of gravity, including higher-order gravity terms, and introduce conformal symmetry.

In Part II, we will introduce the general framework of noncommutative geometry in terms of spectral triples, and show how a gauge group can be recovered from unitary transformations of spectral triples. Next, we will study the general structure of almost commutative geometries, for arbitrary finite spectral triples. The main emphasis will be to show how any almost commutative geometry will give rise to a gauge theory. We will derive the general form of the Lagrangian, and also show that this Lagrangian is invariant (up to a dilaton field) under conformal transformations.

Finally, in Part III, we will apply the general machinery to three examples, in which we work our way towards the Standard Model. The first example is the simplest one, but nonetheless an important one. It has long been thought impossible to describe abelian gauge theories within the framework of noncommutative geometry. In Chapter 7 we show that it is not impossible, and we provide a description of electrodynamics as an almost commutative geometry. In Chapter 8 we expand this model to also incorporate the weak interactions. This provides a good example of the application of the Higgs mechanism in an almost commutative geometry. Because of the coupling of the Higgs field to the scalar curvature, the Higgs mechanism yields a spontaneous breaking of not only the gauge symmetry, but also of the conformal symmetry. These two examples provide a good preparation for Chapter 9, in which we will finally discuss the full Standard Model.

Part I

GEOMETRY, SYMMETRY & PHYSICS

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

— Hermann Weyl

2

DIFFERENTIAL GEOMETRY

In this chapter we have gathered several topics in (or related to) differential geometry. We do not have the intention to provide a decent introduction to these topics. Instead, we merely mention what is needed later on in this thesis.

There is a lot of literature available on differential geometry, we only mention what we have been able to use. Many topics on differential geometry can be found in [16]. A good introduction to Lie groups is given in [15]. An introduction to principal fibre bundles (to be used in gauge theory) is given by [2]. The discussion of Pontryagin and Euler classes (as well as the introduction to Grassmann integrals and Pfaffians) is primarily based on [30]. An introduction to differential operators and generalized Laplacians is given in [1], and the theorems on the heat expansion and the Seeley-DeWitt coefficients are taken from [18].

2.1 LIE GROUPS AND LIE ALGEBRAS

In this section we give the definitions and some properties of Lie groups and Lie algebras. A good introduction to these topics can be found in [15, Ch.1].

2.1 Definition. A *Lie group* is a group G that at the same time is a smooth finite-dimensional manifold, such that the multiplication $\mu\colon G \times G \to G\colon (x,y) \mapsto xy$ and the inversion $\iota\colon G \to G\colon x \mapsto x^{-1}$ are smooth.

2.2 Definition. A (real or complex) *Lie algebra* $\mathfrak{g}$ is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) endowed with an antisymmetric (real or complex) bilinear mapping $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, which satisfies the Jacobi identity

$$\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] = 0 \quad \forall X,Y,Z \in \mathfrak{g}. \tag{2.1}$$

For a Lie group G, we will denote the tangent space at the identity element 1 of G by $\mathfrak{g} := T_1 G$. It is shown in [15, Theorem 1.1.4] that this tangent space, as our notation already suggests, indeed satisfies the definition of a Lie algebra.

2.3 Definition. For each $x \in G$, we will define the adjoint mapping $\operatorname{Ad} x\colon G \to G$ as the conjugation by x given by $(\operatorname{Ad} x)(y) = xyx^{-1}$. The map Ad induces a linear map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$.

2.4 Example. For an N-dimensional vector space V, let $\mathbf{L}(V,V)$ denote the vector space of all linear mappings $V \to V$. The most basic example of a Lie group is the *general linear group* $G = \mathbf{GL}(V)$ of invertible linear transformations of V, given by

$$\mathbf{GL}(V) := \{A \in \mathbf{L}(V,V) \mid \det A \neq 0\}. \tag{2.2}$$

As a manifold, it inherits the smooth structure from its embedding in $\mathbf{L}(V,V) = M_N(\mathbb{C}) \cong \mathbb{R}^{N^2}$. The Lie algebra of $\mathrm{GL}(V)$ is given by $\mathfrak{g} = T_{\mathrm{Id}}\mathbf{GL}(V) = \mathbf{L}(V,V)$. The adjoint action of G on $\mathfrak{g}$ is given by

$$\operatorname{Ad} x\colon Y \mapsto x \circ Y \circ x^{-1}, \tag{2.3}$$

for $x \in \mathbf{GL}(V)$ and $Y \in \mathfrak{g}$. The adjoint map ad is then given by

$$(\operatorname{ad} X)(Y) = X \circ Y - Y \circ X, \tag{2.4}$$

which is the commutator of X and Y in $\mathfrak{g}$.

Not only can we define the Lie algebra of a Lie group, it is also possible to locally recover the Lie group from a Lie algebra. For this purpose we can define the exponential map $\exp\colon \mathfrak{g} \to G$ as in [15, Definition 1.3.3].

2.5 Definition. For each $X \in \mathfrak{g}$, the element $\exp X \in G$ is defined as $h(1)$, where h is the unique differentiable homomorphism $(\mathbb{R}, +) \to (G, \cdot)$ such that $\frac{dh}{dt}(0) = X$.

If V is a finite-dimensional real or complex vector space, and $X \in \mathbf{L}(V, V)$, then the power series

$$e^X := \sum_{k=0}^{\infty} \frac{1}{k!} X^k \tag{2.5}$$

converges in $\mathbf{L}(V, V)$. The map $X \to e^X$ is equal to the exponential mapping $\exp\colon \mathbf{L}(V, V) \to \mathbf{GL}(V)$. We have the well-known formula

$$\det(e^X) = e^{\operatorname{Tr} X}. \tag{2.6}$$

2.1.1 *Representations*

2.6 Definition. A *group representation* of a group G on a vector space V is a group homomorphism $R\colon G \to \mathrm{GL}(V)$.

2.7 Definition. A *Lie algebra representation* of a Lie algebra $\mathfrak{g}$ in an associative algebra $\mathcal{A}$ is a map $r\colon \mathfrak{g} \to \mathcal{A}$ such that $r([X, Y]) = [r(X), r(Y)]$ for all $X, Y \in \mathfrak{g}$, where we have defined the commutator in $\mathcal{A}$ by $[a, b] = ab - ba$ for all $a, b \in \mathcal{A}$.

Now let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, and let $R\colon G \to \mathrm{GL}(V)$ be a representation of G on the finite-dimensional vector space V. We then obtain a representation r of $\mathfrak{g}$ in the associative algebra $\mathcal{A} = \mathrm{End}(V)$ by defining

$$r(X) := \left.\frac{d}{dt}\right|_{t=0} R\big(e^{tX}\big). \tag{2.7}$$

2.1.2 *Lie subgroups and quotients*

We now provide some useful definitions and results on Lie subgroups and Lie subalgebras, based on [15, §1.10].

2.8 Definition. A *normal subgroup* N of a group G is a subgroup such that $gng^{-1} \in N$ for all $n \in N$ and $g \in G$.

2.9 Definition. A *Lie subgroup* H of a Lie group G is a subgroup of G that is also a submanifold of G.

2.10 Definition. A *matrix Lie group* is a closed subgroup of $\mathrm{GL}(V)$, for some finite-dimensional vector space V. By [36, Theorem 3.42], a matrix Lie group is indeed a Lie group.

2.11 Definition. A *Lie subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. A Lie subalgebra is called an *ideal* if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

It follows that a Lie subalgebra is itself a Lie algebra. In particular, the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G becomes a Lie subalgebra of the Lie algebra $\mathfrak{g}$ of G.

2.12 Definition. Let H be a subgroup of G. We define the *quotient* G/H as the set

$$G/H := \{xH \mid x \in G\}, \tag{2.8}$$

where for any $x \in G$ the *left coset* xH is given by

$$xH := \{xh \mid h \in H\}. \tag{2.9}$$

We can define an equivalence relation $\sim$ in G, by letting $x \sim y$ if $y \in xH$, i.e. if $y = xh$ for some $h \in H$. The resulting equivalence class $[x] = [xh]$ is then equal to the coset xH. Hence the elements of G/H are given by the equivalence classes $[x]$.

2.13 Proposition ([15, Corollary 1.11.5])**.** *If H is a closed normal subgroup of G, the* quotient *G/H is a Lie group.*

2.14 Definition. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. We define the *quotient space* $\mathfrak{g}/\mathfrak{h}$ as the quotient of vector spaces:

$$\mathfrak{g}/\mathfrak{h} := \{X + \mathfrak{h} \mid X \in \mathfrak{g}\}, \tag{2.10}$$

where for any $X \in \mathfrak{g}$ the set $X + \mathfrak{h}$ is simply given by

$$X + \mathfrak{h} := \{X + Y \mid Y \in \mathfrak{h}\}. \tag{2.11}$$

We now define an equivalence relation $\sim$ in $\mathfrak{g}$, by letting $X \sim Z$ if $Z \in X + \mathfrak{h}$, i.e. if $Z = X + Y$ for some $Y \in \mathfrak{h}$. The resulting equivalence class $[X]$ is then equal to the set $X + \mathfrak{h}$. Hence the elements of $\mathfrak{g}/\mathfrak{h}$ are given by the equivalence classes $[X]$.

2.15 Proposition. *If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{h}$ forms a Lie algebra.*

Proof. We need to check that the Lie bracket is well-defined. Take $X, Y \in \mathfrak{g}$. The Lie bracket on $\mathfrak{g}/\mathfrak{h}$ is given by $[X + \mathfrak{h}, Y + \mathfrak{h}] = [X, Y] + \mathfrak{h}$. If we take different representatives X', Y' such that $X + \mathfrak{h} = X' + \mathfrak{h}$ and $Y + \mathfrak{h} = Y' + \mathfrak{h}$, then there are $U, V \in \mathfrak{h}$ such that $X' = X + U$ and $Y' = Y + V$. Then we obtain for the Lie bracket that $[X', Y'] = [X + U, Y + V] = [X, Y] + [U, Y] + [X, V] + [U, V]$, where the latter three terms are in $\mathfrak{h}$ because $\mathfrak{h}$ is an ideal Lie subalgebra. So we obtain $[X', Y'] + \mathfrak{h} = [X, Y] + \mathfrak{h}$, and the Lie bracket is indeed well-defined. □

2.2 FIBRE BUNDLES

2.16 Definition. Let M be a smooth manifold. We define a *fibre bundle* over the base space M as a smooth manifold E together with a smooth surjective map $\pi \colon E \to M$ (called the projection onto M), such that for each $x \in M$ the inverse image $\pi^{-1}(x)$ is diffeomorphic to some manifold F, which is called the *fibre* of E. Moreover, for every $x \in M$ there must exist a neighbourhood $U \subset M$ and a diffeomorphism $h_U \colon \pi^{-1}(U) \to U \times F$ such that $\pi_1 \circ h_U = \pi$ on $\pi^{-1}(U)$, where we have denoted $\pi_1 \colon U \times F \to U$ for the projection on the first factor. The pair (U, h_U) is called a *local trivialization* of E. The fibre bundle is called *trivial* if there exists a global trivialization $h \colon E = \pi^{-1}(M) \to M \times F$.

2.17 Definition. A fibre bundle $E \xrightarrow{\pi} M$ is called a vector bundle if the fibre F has the structure of a vector space, and if on every local trivialization the map $h_U\big|_{\pi^{-1}(x)} \colon \pi^{-1}(x) \to x \times F$ is an isomorphism of vector spaces. The dimension of F is called the *rank* of the vector bundle.

2.18 Definition. For two local trivializations (U, h_U) and (V, h_V), with $U \cap V \neq 0$, of a vector bundle $E \xrightarrow{\pi} M$, we have the diffeomorphism $h_V \circ h_U^{-1}$ from $U \cap V \times F$ to itself, which yields the transition from (U, h_U) to (V, h_V). We can write this diffeomorphism as $h_V \circ h_U^{-1}(x,f) = \big(x, g_{UV}(x)f\big)$, where g_{UV} is a smooth map from $U \cap V$ to the group $\mathrm{End}(F)$. These maps g_{UV} are called the *transition functions* of the vector bundle. If $g_{UV}(x) \in G \subset \mathrm{End}(F)$ for a subgroup G of $\mathrm{End}(F)$ for all local trivializations with $U \cap V \neq 0$, then G is called the *structure group* of the vector bundle E.

2.19 Definition. Let $E \xrightarrow{\pi} M$ be a fibre bundle. We define a *local section* on an open set $U \subset M$ as a smooth mapping $s\colon U \to E$ such that $\pi \circ s = \mathrm{Id}_U$. We denote $\Gamma(U,E)$ for the space of sections on U. A section is called a *global section* if it is defined on $U = M$, and we denote $\Gamma(M,E) = \Gamma(E)$ for the space of global sections.

Notice that $\Gamma(E)$ is a module over the algebra $C^\infty(M)$. The action of $C^\infty(M)$ is just given by pointwise scalar multiplication:

$$(\sigma f)(x) := \sigma(x)f(x), \tag{2.12}$$

for $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$.

2.20 Definition. Let E be a vector bundle with fibre V, and let $r = \dim V$ be the rank of E. A *local frame* over U is a set of local sections $\{e_1, \ldots, e_r\} \subset \Gamma(U,E)$ that are linearly independent in each point $x \in U$.

2.2.1 *Principal fibre bundles*

2.21 Definition. The group action of G on a space X is called *free* if for all $x \in X$, $gx = x$ implies $g = e$. The action is called transitive if $Gx = X$ for any $x \in X$.

2.22 Definition. Let G be a Lie group. A *principal G-bundle* P is a fibre bundle $P \xrightarrow{\pi} M$ with fibre G, and with a free and transitive right action of G on P, such that for a local trivialization (U, h_U) of P the map h_U intertwines the right action of G on P with the natural right action of G on $U \times G$; if $h_U(p) = (x,f)$, then $h_U(pg) = (x, fg)$.

For a principal G-bundle $P \xrightarrow{\pi} M$, it follows that the projection π is G-invariant: $\pi(pg) = \pi(p)$, since on a local trivialization we have for $h_U(p) = (x,f)$ that $\pi(pg) = \pi_1(x,fg) = \pi_1(x,f) = \pi(p)$.

2.2.2 *Associated vector bundles*

Let P be a principal G-bundle over M and suppose we have a representation of G on the vector space V. Then there is a natural right action of G on $P \times V$ given by $(p,v)g := (pg, g^{-1}v)$. We define $P \times_G V$ to be the quotient of $P \times V$ with respect to this right action. In other words, $P \times_G V$ consists of the equivalence classes $[pg, v] = [p, gv]$. By [32, Theorem 7], we can state:

2.23 Proposition. *The* associated vector bundle $E := P \times_G V$ *is a vector bundle over M with fibre V.*

2.24 Example. If $P = M \times G$ is a trivial principal G-bundle, the associated vector bundle $E = M \times V$ is also trivial.

2.2.3 *Bundle morphisms*

2.25 Definition. Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ be two fibre bundles over the same base space M. A *bundle homomorphism* $f\colon E_1 \to E_2$ is a smooth map such that $\pi_2 \circ f = \pi_1$. This means that f maps the fibre $(E_1)_x$ into the fibre $(E_2)_x$ for each $x \in M$.

If E_1 and E_2 are vector bundles, we further require that $f\colon (E_1)_x \to (E_2)_x$ is a linear map for each $x \in M$.

If $E_1 = P_1$ and $E_2 = P_2$ are two principal G-bundles, we further require that f is compatible with the group action of G, i.e. $f(pg) = f(p)g$.

A bundle homomorphism $f\colon E \to E$ is called a bundle endomorphism, and we denote by $\mathrm{End}(E)$ the set of all bundle endomorphisms of E.

2.26 Definition. If a bundle homomorphism $f\colon E_1 \to E_2$ is a diffeomorphism, the fibre bundles E_1 and E_2 are said to be *equivalent* or *isomorphic*, and f is called a *bundle isomorphism*. A bundle isomorphism $f\colon E \to E$ is called a *bundle automorphism* of the bundle E.

2.3 CONNECTIONS

Let M be an n-dimensional differentiable manifold. The set of *continuous functions* on M is denoted by $C(M)$. At each $x \in M$ the set of tangent vectors forms the vector space T_xM. These tangent spaces are joined together to a vector bundle, which is called the tangent bundle TM. The sections of this bundle form the set of *continuous vector fields* $\mathfrak{X}(M) := \Gamma(M, TM)$. These vector fields are derivations on the set of *smooth functions* $C^\infty(M)$ on M. In local coordinates we have a local basis ∂_μ of TM.

The cotangent bundle is the bundle with as fibres the dual spaces T_x^*M of the tangent spaces, and its sections are the *continuous* 1*-forms* $\mathcal{A}^1(M) := \Gamma(M, T^*M)$. In local coordinates we have a local basis dx^μ of T^*M, which is dual to the basis ∂_μ of TM. By taking the (completely antisymmetric) exterior product of k elements of $\mathcal{A}^1(M)$, we obtain $\mathcal{A}^k(M) := \Gamma(M, \bigwedge^k T^*M)$. Taking the direct sum over all k (including for $k = 0$ the set $\mathcal{A}^0(M) := C^\infty(M)$) then yields the exterior algebra $\mathcal{A}^\bullet(M) = \Gamma(M, \bigwedge^\bullet T^*M)$ of differential forms.

The *exterior derivative* is the unique linear antiderivation $d\colon \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ satisfying the Leibniz rule $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$ for $\alpha \in \mathcal{A}^k(M)$ and $\beta \in \mathcal{A}^l(M)$, and which for $f \in C^\infty(M)$ gives a 1-form df satisfying $df(X) = X(f)$ for $X \in \mathfrak{X}(M)$.

A tensor of type (k, l) is an element of the tensor product $T_xM^{\otimes k} \otimes T_x^*M^{\otimes l}$. These again join together to form a bundle, and its sections are the *continuous tensor fields* $T_l^k(M) := \Gamma(M, TM^{\otimes k} \otimes T^*M^{\otimes l})$. Such a tensor field can be viewed as a $C(M)$-multilinear map $T^*M^{\otimes k} \otimes TM^{\otimes l} \to C(M)$.

2.27 Definition. Let $E \xrightarrow{\pi} M$ be a smooth vector bundle with smooth sections $\Gamma(E)$. A *connection* ∇^E is a linear map

$$\nabla^E\colon \Gamma(E) \to \Gamma(E \otimes T^*M), \tag{2.13}$$

such that the Leibniz rule

$$\nabla^E(\sigma f) = \nabla^E(\sigma) f + \sigma \otimes df \tag{2.14}$$

is satisfied for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$. Equivalently, we can regard ∇^E as a map $\mathfrak{X}(M) \otimes \Gamma(E) \to \Gamma(E)$. By inserting a vector field $X \in \mathfrak{X}(M)$, we then get a map $\nabla^E_X\colon \Gamma(E) \to \Gamma(E)$ with the Leibniz rule $\nabla^E_X(\sigma f) = \nabla^E_X(\sigma) f + \sigma X(f)$. This map is also called a *covariant derivative*.

2.28 Lemma. *The difference between two connections on a vector bundle E is a* 1*-form with values in* $\mathrm{End}(E)$.

PROOF. Take two connections $\nabla, \nabla'\colon \Gamma(E) \to \Gamma(E \otimes T^*M)$. We need to check that their difference is $C^\infty(M)$-linear. Indeed, using the Leibniz rule, we find for $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$ that

$$(\nabla - \nabla')(\sigma f) = \nabla(\sigma) f + \sigma \otimes df - \nabla'(\sigma) f - \sigma \otimes df = (\nabla - \nabla')(\sigma) f. \qquad \square$$

Let E be a vector bundle E of rank n. On a locally trivializing neighbourhood U in M, we have an orthonormal frame $\{e_1, \dots, e_n\}$ of $\Gamma(U, E)$. Any section of E on U can then be identified with an n-tuple of smooth functions $(s_1, \dots, s_n)$ by writing $s = \sum_{i=1}^n s_i e_i$.

We have the exterior derivative $d\colon C^\infty(M) \to \mathcal{A}^1(M)$. We can extend this derivative to arbitrary local sections by defining $d\colon \Gamma(U, E) \to \Gamma(U, E \otimes T^*M)$ to be given locally as $ds = (ds_1, \dots, ds_n)$. It is a straighforward calculation to show that d satisfies the Leibniz rule. Consequently, we have the following lemma.

2.29 Lemma. *The exterior derivative $d\colon \Gamma(U, E) \to \Gamma(U, E \otimes T^*M)$ defines a local connection on E.*

2.30 Proposition. *Any connection ∇^E on a vector bundle E can locally be expressed as $\nabla^E = d + \omega$, where d is the exterior derivative and ω is a local* End(E)*-valued* 1*-form, called the* connection 1-form*. Conversely, any local* End(E)*-valued* 1*-form ω defines a local connection on E by $\nabla^E = d + \omega$.*

Proof. By Lemma 2.29, d is a local connection. By applying Lemma 2.28, we can then define $\omega := \nabla^E - d$. For the converse, we use that d satisfies the Leibniz rule and that the connection form $\omega \in \Gamma(U, \mathrm{End}(E) \otimes T^*M) = \Gamma(U, \mathrm{End}(E)) \otimes \mathcal{A}^1(U) = \mathrm{End}_{C^\infty(U)}(\Gamma(U, E)) \otimes \mathcal{A}^1(U)$ is $C^\infty(U)$-linear, so $d + \omega$ satisfies the Leibniz rule and hence defines a local connection. □

2.31 Proposition. *Let (U, ϕ_U) and (V, ϕ_V) be two local trivializations of M such that $U \cap V \neq \emptyset$. Let ∇^E be a connection on a vector bundle E over M. By Proposition 2.30 we can locally write $\nabla^E = d + \omega_U$ on U and $\nabla^E = d + \omega_V$ on V. The connection forms ω_U and ω_V are related on $U \cap V$ by*

$$\omega_V = g_{UV} \omega_U g_{UV}^{-1} - d g_{UV} g_{UV}^{-1}. \tag{2.15}$$

Proof. Let s be a section of a vector bundle E over M. We shall write $s_U = (s_1, \dots, s_n)$ on U and $s_V = (s'_1, \dots, s'_n)$ on V. On $U \cap V$, we have $s_V = g_{UV} s_U$ for the transition function g_{UV}, and we must also have $g_{UV} \nabla^E s_U = \nabla^E s_V$. Thus the following two expressions are equal:

$$g_{UV}(d + \omega_U) s_U = g_{UV} ds_U + g_{UV} \omega_U s_U,$$
$$(d + \omega_V) s_V = (d + \omega_V)(g_{UV} s_U) = d g_{UV} s_U + g_{UV} ds_U + \omega_V g_{UV} s_U.$$

This yields the identity $g_{UV} \omega_U = d g_{UV} + \omega_V g_{UV}$. Rewriting this expression yields the desired result. □

For principal fibre bundles, there is a different approach to introduce connections. There are several different but equivalent definitions, and a detailed discussion can be found in [2, §1.2]. Our definition given below is based on [2, Definition 1.2.3]. For the purpose of this thesis we will only deal with the case of a principal G-bundle for a matrix Lie group G, which simplifies the definition.

2.32 Definition. Let G be a matrix Lie group and let P be a principal G-bundle. A *connection* on P assigns to each local trivialization (U, h_U) a $\mathfrak{g}$-valued 1-form A_U on U. For two local trivializations (U, h_U) and (V, h_V) with $U \cap V \neq \emptyset$ for which $g_{UV}\colon U \cap V \to G$ is the transition function from U to V, we require that on $U \cap V$ we have the transformation property

$$A_V = g_{UV} A_U g_{UV}^{-1} - d g_{UV} g_{UV}^{-1}. \tag{2.16}$$

One should note that this transformation property is precisely the same as the relation between the local connection forms on different trivializations of a vector bundle (cf. Proposition 2.31). Of course this is no coincidence, and the following proposition relates the connection on the principal bundle to a connection on an associated vector bundle.

2.33 Proposition. *For an associated vector bundle $E = M \times V$ of the trivial principal bundle $P = M \times G$, the connection form A on P defines a connection $\nabla^E = d + A$ on E.*

PROOF. The representation of G on V induces a representation of $\mathfrak{g}$ on V as in Section 2.1.1, which yields the embedding of $\mathfrak{g}$ in $\mathrm{End}(V)$. The connection form A is by definition an element of $\Gamma(T^*M \otimes \mathfrak{g})$, and is thus an $\mathrm{End}(E)$-valued 1-form on M. By Proposition 2.30, $d + A$ defines a connection on $E = M \times V$. $\square$

2.4 RIEMANNIAN MANIFOLDS

2.34 Definition. A manifold M is called *pseudo-Riemannian* if it is equipped with a symmetric tensor field g of type $(0, 2)$, called the *pseudo-Riemannian metric*, such that the bilinear form $g_x\colon T_xM \otimes T_xM \to \mathbb{R}$ is non-degenerate for every $x \in M$, i.e. for $v \in T_xM$ we have

$$g_x(v, w) = 0 \text{ for all } w \in T_xM \;\Rightarrow\; v = 0. \tag{2.17}$$

M is a *Riemannian manifold* if g_x is also positive definite, i.e.

$$g_x(v, v) \geq 0 \text{ for all } v \in T_xM, \text{ and } g_x(v, v) = 0 \Rightarrow v = 0, \tag{2.18}$$

and now g is called a *Riemannian metric*.

On a local basis ∂_μ of $\mathfrak{X}(M)$, we define the components of g by $g_{\mu\nu} := g(\partial_\mu, \partial_\nu)$. The *trace over* g of a tensor product of two 1-forms α, β and a section s of some bundle E is defined by

$$\mathrm{Tr}_g(\alpha \otimes \beta \otimes s) := g(\alpha, \beta)s. \tag{2.19}$$

Using a local basis ∂_μ of $\mathfrak{X}(M)$ and its dual basis dx^μ on $\mathcal{A}^1(M)$, we can write a vector field as $X = X^\mu\partial_\mu = dx^\mu(X)\partial_\mu$, where $X^\mu = dx^\mu(X) \in C^\infty(M)$. Since ∇^E_X is $C^\infty(M)$-linear in X, we can locally write $\nabla^E_X = dx^\mu(X)\nabla^E_{\partial_\mu}$, or $\nabla^E = dx^\mu \otimes \nabla^E_{\partial_\mu}$.

2.35 Proposition. *For the tangent bundle TM there is a unique connection $\nabla\colon \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is torsion-free:*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \tag{2.20}$$

and compatible with the metric g:

$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Z\left(g(X, Y)\right) \tag{2.21}$$

This connection is called the Levi-Civita *connection.*

2.36 Definition. On an orthonormal local basis ∂_μ the Christoffel symbols ${\Gamma^\rho}_{\mu\nu}$ are defined by

$$\nabla_{\partial_\mu}\partial_\nu = {\Gamma^\rho}_{\mu\nu}\partial_\rho. \tag{2.22}$$

If the metric g is Riemannian, i.e. if each g_x is positive definite, then there are mutually inverse isomorphisms between $\mathfrak{X}(M)$ and $\mathcal{A}^1(M)$:

$$\begin{matrix} \mathfrak{X}(M) \to \mathcal{A}^1(M);\; X \mapsto X^\flat, \\ \mathcal{A}^1(M) \to \mathfrak{X}(M);\; \alpha \mapsto \alpha^\sharp, \end{matrix} \quad \text{given by} \quad \begin{matrix} X^\flat(Y) := g(X, Y), \\ \alpha(Y) =: g(\alpha^\sharp, Y). \end{matrix} \tag{2.23}$$

This can be used to define a *metric on* $\mathcal{A}^1(M)$ by

$$g^{-1}(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp) \tag{2.24}$$

for $\alpha, \beta \in \mathcal{A}^1(M)$. On a local basis dx^μ of $\mathcal{A}^1(M)$ dual to the basis ∂_μ of $\mathfrak{X}(M)$, we define the components of g^{-1} by $g^{\mu\nu} := g^{-1}(dx^\mu, dx^\nu) = dx^\mu\big((dx^\nu)^\sharp\big)$. From this we conclude that $(dx^\nu)^\sharp = g^{\nu\rho}\partial_\rho$. We then derive

$$g^{\mu\nu}g_{\nu\rho} = g^{\mu\nu}g(\partial_\nu, \partial_\rho) = g\big((dx^\mu)^\sharp, \partial_\rho\big) = dx^\mu(\partial_\rho) = {\delta^\mu}_\rho, \tag{2.25}$$

so the matrix of $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$, which justifies the notation of g^{-1} for the metric on $\mathcal{A}^1(M)$. By a slight abuse of notation, the metric g^{-1} is often written simply as g as well.

2.37 Definition. Using the Levi-Civita connection on TM, we can define an *associated connection* on the cotangent bundle T^*M, also denoted ∇ and referred to as the *Levi-Civita connection on* 1*-forms*, given by

$$(\nabla\alpha)(X) := d(\alpha(X)) - \alpha(\nabla X), \tag{2.26}$$

or $(\nabla_Z\alpha)(X) := Z(\alpha(X)) - \alpha(\nabla_Z X)$.

Locally, we can use the same Christoffel symbols of the Levi-Civita connection on TM to write $\nabla dx^\rho = -\Gamma^\rho{}_{\mu\nu} dx^\mu \otimes dx^\nu$. If we choose instead an orthonormal basis θ^a of 1-forms over a chart $U \subset M$, we can write locally

$$\nabla\theta^a = -\tilde{\Gamma}^a{}_{\mu b} dx^\mu \otimes \theta^b. \tag{2.27}$$

2.5 CURVATURE

2.38 Definition. The *curvature* $\Omega^E \in \mathcal{A}^2(M, \mathrm{End}(E))$ of a connection ∇^E is defined by

$$\Omega^E(X,Y) := \nabla^E_X\nabla^E_Y - \nabla^E_Y\nabla^E_X - \nabla^E_{[X,Y]} \in \mathrm{End}(E), \tag{2.28}$$

for $X, Y \in \mathfrak{X}(M)$. We immediately see that $\Omega^E(X,Y) = -\Omega^E(Y,X)$. The components of the curvature Ω^E are given by $\Omega^E_{\mu\nu} = \Omega^E(\partial_\mu, \partial_\nu) \in \mathrm{End}(E)$.

2.39 Definition. A *Euclidean vector bundle* is a real vector bundle $E \to M$, endowed with a $C^\infty(M,\mathbb{R})$-bilinear symmetric positive definite form $g^E \colon \Gamma(E) \times \Gamma(E) \to C^\infty(M,\mathbb{R})$.

2.40 Definition. Denote $E^{\mathbb{C}}$ for the complexification of E, and write $\mathcal{E} := \Gamma(E^{\mathbb{C}})$. We extend g^E to a $C^\infty(M)$-bilinear form $g^E \colon \mathcal{E} \times \mathcal{E} \to C^\infty(M)$. By setting $(s|t) = g^E(s^*, t)$ for $s, t \in \mathcal{E}$, we get a $C^\infty(M)$-valued *hermitian pairing* which satisfies

- $(s|t)$ is linear in t;
- $(t|s) = \overline{(s|t)} \in C^\infty(M)$;
- $(s|s) \geq 0$, with $(s|s) = 0 \Rightarrow s = 0$ in $\mathcal{E}$;
- $(s|ta) = (s|t)a$ for all $s, t \in \mathcal{E}$ and $a \in C^\infty(M)$.

2.41 Definition. We define a scalar product on $\mathcal{E}$ by

$$\langle s|t\rangle := \int_M (s|t)\nu_g \tag{2.29}$$

for $s, t \in \mathcal{E}$, where $(s|t)$ is the hermitian pairing on $\mathcal{E}$ as in Definition 2.40.

2.42 Definition. If $\mathcal{E}$ is equipped with a $C^\infty(M)$-valued hermitian pairing $(\,|\,)$, we say that a connection ∇^E on $\mathcal{E}$ is *hermitian* if

$$(\nabla^E s|t) + (s|\nabla^E t) = d(s|t). \tag{2.30}$$

2.43 Lemma. *If a vector bundle E over M is endowed with a metric g^E and has a hermitian connection ∇^E, then the curvature is skew-symmetric:*

$$g^E(s, \Omega^E(X,Y)t) = -g^E(t, \Omega^E(X,Y)s) \tag{2.31}$$

for $X, Y \in \mathfrak{X}(M)$ and for real sections $s, t \in \mathcal{E}$. In other words, we see that we have $\Omega^E \in \mathcal{A}^2(M, \mathfrak{so}(E))$.

Proof. For real sections s, t, the fact that ∇^E is hermitian implies that

$$g^E(\nabla_X s, t) + g^E(s, \nabla_X t) = X\left(g^E(s,t)\right). \tag{2.32}$$

We then obtain

$$\begin{aligned}
&g^E(s, \Omega^E(X,Y)t) + g^E(t, \Omega^E(X,Y)s)\\
&\quad = g^E(s, \nabla^E_X\nabla^E_Y t) - g^E(s, \nabla^E_Y\nabla^E_X t) - g^E(s, \nabla^E_{[X,Y]} t)\\
&\qquad + g^E(t, \nabla^E_X\nabla^E_Y s) - g^E(t, \nabla^E_Y\nabla^E_X s) - g^E(t, \nabla^E_{[X,Y]} s)\\
&\quad = Xg^E(s, \nabla^E_Y t) - g^E(\nabla^E_X s, \nabla^E_Y t) - Yg^E(s, \nabla^E_X t) + g^E(\nabla^E_Y s, \nabla^E_X t)\\
&\qquad + Xg^E(t, \nabla^E_Y s) - g^E(\nabla^E_X t, \nabla^E_Y s) - Yg^E(t, \nabla^E_X s) + g^E(\nabla^E_Y t, \nabla^E_X s)\\
&\qquad - g^E(s, \nabla^E_{[X,Y]} t) - g^E(\nabla^E_{[X,Y]} s, t)\\
&\quad = XYg^E(s,t) - YXg^E(s,t) - [X,Y]g^E(s,t) = 0.
\end{aligned}$$

□

2.5.1 *Riemannian curvature*

In the special case of the Levi-Civita connection ∇ on $\mathfrak{X}(M)$, we obtain the *Riemannian curvature tensor* $R \in \mathcal{A}^2(M, \mathrm{End}(TM))$. R can be seen as a tensor of type $(1,3)$. By a slight abuse of notation, we define a tensor R of type $(0,4)$, which we will also call the *Riemannian curvature tensor*, by $R(V,W,X,Y) = g(V, R(X,Y)W)$. In local coordinates, we define the components of R by $R_{\mu\nu\rho\sigma} := g(\partial_\mu, R(\partial_\rho, \partial_\sigma)\partial_\nu)$.

The Riemannian curvature tensor R has several symmetries, which we shall now explore. Since R is a 2-form, the endomorphism $R(X,Y)$ is skewsymmetric in X and Y. From Lemma 2.43 we know that $R(V,W,X,Y)$ is also antisymmetric in V and W, which implies that $R \in \mathcal{A}^2(M, \mathfrak{so}(TM))$. Using the fact that the Levi-Civita connection is torsion-free, one can show by explicit calculation that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$. Using the symmetry properties we have obtained so far, one can also show that $R(V,W,X,Y) = R(X,Y,V,W)$. In local coordinates, we thus have the following relations:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \tag{2.33}$$

as well as the (first) *Bianchi identity*

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0. \tag{2.34}$$

Using all these properties of R one can also prove the *second Bianchi identity*

$$\nabla_Z R(V,W,X,Y) + \nabla_X R(V,W,Y,Z) + \nabla_Y R(V,W,Z,X) = 0. \tag{2.35}$$

Using the compact notation $X^\rho{}_{;\mu} := \nabla_\mu X^\rho$, this gives locally

$$R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\sigma\lambda;\rho} + R_{\mu\nu\lambda\rho;\sigma} = 0. \tag{2.36}$$

Taking a trace over the first and third indices of the Riemannian curvature tensor, we get the *Ricci tensor*, whose components are $R_{\nu\sigma} := g^{\mu\rho}R_{\mu\nu\rho\sigma}$. This Ricci tensor is symmetric: $R_{\nu\sigma} = R_{\sigma\nu}$. The trace of the Ricci tensor is the *scalar curvature* $s := g^{\nu\sigma}R_{\nu\sigma} = g^{\nu\sigma}g^{\mu\rho}R_{\mu\nu\rho\sigma} \in C^\infty(M)$.

Note that in this convention for the Riemannian curvature tensor, the scalar curvature of a sphere is positive. In other literature, one often finds the definition $R(V,W,X,Y) = g(R(X,Y)V,W)$, which by Lemma 2.43 effectively changes the sign of the Riemannian curvature tensor.

Now, let us evaluate the curvature of the associated connection on 1-forms. Using the definition of Eq. (2.26), we find that

$$\begin{aligned}(R(Y,Z)\alpha)(X) &:= \Big((\nabla_Y\nabla_Z - \nabla_Z\nabla_Y - \nabla_{[Y,Z]})\alpha\Big)(X) \\ &= Y(\nabla_Z\alpha)(X) - (\nabla_Z\alpha)(\nabla_Y X) - Z(\nabla_Y\alpha)(X) + (\nabla_Y\alpha)(\nabla_Z X) \\ &\quad - [Y,Z]\alpha(X) + \alpha(\nabla_{[Y,Z]}X) \\ &= \alpha(\nabla_Z\nabla_Y X) - \alpha(\nabla_Y\nabla_Z X) + \alpha(\nabla_{[Y,Z]}X) \\ &= -\alpha\left(R(Y,Z)X\right). \end{aligned} \tag{2.37}$$

2.44 Lemma. *In local coordinates, we have for a 1-form $\alpha = \alpha_\mu dx^\mu$ the formula*

$$R^\mu{}_{\nu\rho\sigma}\alpha_\mu = \left(\nabla_\sigma\nabla_\rho - \nabla_\rho\nabla_\sigma\right)\alpha_\nu. \tag{2.38}$$

Proof. We need to rewrite Eq. (2.37) into local coordinates. First we note that, by writing the fields in local coordinates and applying Eq. (B.14), we have

$$R(Y,Z)X = X^\nu Y^\rho Z^\sigma R(\partial_\rho,\partial_\sigma)\partial_\nu = X^\nu Y^\rho Z^\sigma R^\mu{}_{\nu\rho\sigma}\partial_\mu.$$

Using Eq. (2.37), writing $X^\nu = dx^\nu(X)$, using $\alpha(\partial_\mu) = \alpha_\mu$ and canceling the components of Y and Z, we find for the 1-form $\alpha = \alpha_\mu dx^\mu$ that

$$R(\partial_\rho,\partial_\sigma)\alpha_\nu = -R^\mu{}_{\nu\rho\sigma}\alpha_\mu.$$

Since $[\partial_\rho,\partial_\sigma] = 0$, we have $R(\partial_\rho,\partial_\sigma) = \nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho$, and we obtain the desired result. □

2.6 LAPLACIANS

2.45 Definition. If $E \to M$ is a smooth vector bundle with a connection ∇^E on $\mathcal{E} = \Gamma(M,E)$, we can consider the connection $\nabla^{E'} := \nabla \otimes 1 + 1 \otimes \nabla^E$ on the tensor product $\mathcal{E}' = \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$, where ∇ is the Levi-Civita connection on $\mathcal{A}^1(M)$. Their composition is an operator $\nabla^{E'} \circ \nabla^E$ from $\mathcal{E}$ to $\mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$. Using the metric to take the trace over the first two factors, we define the *Laplacian* by

$$\Delta^E := -\mathrm{Tr}_g\big(\nabla^{E'} \circ \nabla^E\big) \colon \mathcal{E} \to \mathcal{E}. \tag{2.39}$$

In particular, when $E = M \times \mathbb{C}$ is the trivial line bundle (for which $\Gamma(E) = C^\infty(M)$), we get the *scalar Laplacian* Δ.

2.46 Lemma. *In local coordinates, the Laplacian equals*

$$\Delta^E = -g^{\mu\nu}\left(\nabla^E_{\partial_\mu}\nabla^E_{\partial_\nu} - \Gamma^\rho{}_{\mu\nu}\nabla^E_{\partial_\rho}\right). \tag{2.40}$$

Proof.

$$\begin{aligned}\Delta^E &= -\mathrm{Tr}_g(\nabla^{E'} \circ \nabla^E) \\ &= -\mathrm{Tr}_g\left(\left((dx^\mu \otimes \nabla_{\partial_\mu}) \otimes 1 + 1 \otimes (dx^\nu \otimes \nabla^E_{\partial_\nu})\right) \circ \left(dx^\rho \otimes \nabla^E_{\partial_\rho}\right)\right) \\ &= -\mathrm{Tr}_g\left(dx^\mu \otimes (-\Gamma^\rho{}_{\mu\nu}dx^\nu) \otimes \nabla^E_{\partial_\rho} + dx^\rho \otimes dx^\nu \otimes \nabla^E_{\partial_\nu}\nabla^E_{\partial_\rho}\right),\end{aligned}$$

where we have used $\nabla_{\partial_\mu}dx^\rho = -\Gamma^\rho{}_{\mu\nu}dx^\nu$. Taking the trace over g means $\mathrm{Tr}_g(dx^\mu \otimes dx^\nu) = g(dx^\mu, dx^\nu) = g^{\mu\nu}$, so we obtain (by using the symmetry of g) that

$$\Delta^E = g^{\mu\nu}\Gamma^\rho{}_{\mu\nu}\nabla^E_{\partial_\rho} - g^{\rho\nu}\nabla^E_{\partial_\nu}\nabla^E_{\partial_\rho} = -g^{\mu\nu}\left(\nabla^E_{\partial_\mu}\nabla^E_{\partial_\nu} - \Gamma^\rho{}_{\mu\nu}\nabla^E_{\partial_\rho}\right). \qquad \square$$

Note that the scalar Laplacian is then given in local coordinates by

$$\Delta = -g^{\mu\nu}\left(\partial_\mu\partial_\nu - \Gamma^\rho{}_{\mu\nu}\partial_\rho\right). \tag{2.41}$$

For a function $f \in C^\infty(M)$, we use the same compact notation $f_{;\mu} = \nabla_\mu f = \partial_\mu f$. For the scalar Laplacian, one then often writes

$$f_{;\mu}{}^{;\mu} := -\Delta f. \tag{2.42}$$

2.7 integration on oriented manifolds

In this section, we will assume that the n-dimensional manifold M is orientable. Equivalently, there exists an n-form $\nu \in A^n(M)$ that vanishes nowhere on M.

For functions $f\colon \mathbb{R}^n \to \mathbb{R}$ with compact support in $\mathbb{R}^n$, we can calculate the integral $\int_{\mathbb{R}^n} f$. If A and B are two open subsets of $\mathbb{R}^n$, and $\phi\colon A \to B$ is a diffeomorphism, then the integral of a function $f\colon B \to \mathbb{R}$ with compact support in B can be rewritten as an integral over A by the formula

$$\int_B f = \int_A |\det(D_j\phi^i)| f \circ \phi, \tag{2.43}$$

where $D_j\phi^i$ is the Jacobi matrix of ϕ.

2.47 Definition. A differential form $\theta \in A^n(M)$ is called an *elementary form* if there is a chart (U, h) for which the support of θ lies in U.

2.48 Definition. For an elementary form θ, given locally by $\theta|_U = f dx^1 \wedge \cdots \wedge dx^n$, we define the integral

$$\int_M \theta := \int_{h(U)} f \circ h^{-1}. \tag{2.44}$$

Since M is an oriented manifold, Eq. (2.43) shows that this integral is well-defined on the overlap of two charts. Note that the integral is additive:

$$\int_M (\theta_1 + \theta_2) = \int_M \theta_1 + \int_M \theta_2, \tag{2.45}$$

for two elementary forms θ_1 and θ_2 on the same chart. Every n-form θ with compact support on M can be written as a finite sum of elementary forms θ_i. We can then use the additive property of the integral to define the integral of θ.

2.49 Definition. For an n-form θ with compact support on M, we define the integral

$$\int_M \theta := \sum_i \int_M \theta_i, \tag{2.46}$$

where we have written $\theta = \sum_i \theta_i$ as a finite sum of elementary forms θ_i. This integral is linear:

$$\int_M (c_1\theta_1 + c_2\theta_2) = c_1 \int_M \theta_1 + c_2 \int_M \theta_2, \tag{2.47}$$

for two forms θ_1 and θ_2 with compact support and $c_1, c_2 \in \mathbb{R}$.

We shall now state without proof the well-known Stokes' theorem.

2.50 Theorem (Stokes)**.** *Let M be an oriented n-dimensional manifold. Let $D \subset M$ be an area with smooth boundary, for which the closure $\overline{D}$ is compact. For a differential form $\omega \in A^{n-1}(M)$ we have*

$$\int_D d\omega = \int_{\partial D} j^*\omega, \tag{2.48}$$

where $j\colon \partial D \to M$ is the inclusion of the boundary of D in M, and where ∂D is endowed with the standard orientation.

2.51 Corollary. *If M is a compact manifold without boundary, we have for every $(n-1)$-form ω that $\int_M d\omega = 0$.*

2.7.1 *Integration on Riemannian manifolds*

2.52 Definition. For $g_{\mu\nu}$ the matrix of the metric g, we write $|g| := |\det[g_{\mu\nu}]|$. The *Riemannian volume form*, for a given orientation and (pseudo-Riemannian) metric g, is defined locally by

$$\nu_g := \sqrt{|g|} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \mathcal{A}^n(M). \tag{2.49}$$

2.53 Definition. For a vector field $A \in \mathfrak{X}(M)$, we define the *divergence* $\nabla \cdot A \in C^\infty(M)$ of A as the Lie derivative of the volume form, i.e.

$$\nabla \cdot A \; \nu_g := \mathcal{L}_A(\nu_g). \tag{2.50}$$

By Cartan's formula (see, for instance, [16, §4.2]), we have $\mathcal{L}_A = di_A + i_A d$, where $i_A \colon \mathcal{A}^1(M) \to C^\infty(M)$ is given by $i_A(\alpha) = \alpha(A)$. Since $d\nu_g = 0$, we then have

$$\nabla \cdot A \; \nu_g = di_A(\nu_g). \tag{2.51}$$

2.54 Proposition. *For a vector field A, written locally as $A = A^\mu \partial_\mu$, we have*

$$\nabla \cdot A = \nabla_\mu A^\mu. \tag{2.52}$$

Proof. By using $d\left(\sqrt{|g|}A^\mu\right) = \partial_\rho \left(\sqrt{|g|}A^\mu\right) dx^\rho$ and $d^2 = 0$, we find that

$$\begin{aligned} di_A(\nu_g) &= d\left(\sqrt{|g|} \sum_{\mu=1}^{n} (-1)^{\mu-1} A^\mu dx^1 \wedge \cdots \wedge \widehat{dx^\mu} \wedge \cdots \wedge dx^n\right) \\ &= \sum_{\mu=1}^{n} \partial_\rho \left(\sqrt{|g|}A^\mu\right) (-1)^{\mu-1} dx^\rho \wedge dx^1 \wedge \cdots \wedge \widehat{dx^\mu} \wedge \cdots \wedge dx^n, \end{aligned}$$

where the circumflex over dx^μ means (as usual) that this factor should be omitted. Because of the antisymmetric nature of the wedge product (which yields $dx^\rho \wedge dx^\rho = 0$), only the terms for which $\mu = \rho$ survive. Since $dx^\mu \wedge dx^1 \wedge \cdots \wedge dx^{\mu-1} = (-1)^{\mu-1} dx^1 \wedge \cdots \wedge dx^\mu$, we obtain

$$di_A(\nu_g) = \partial_\mu \left(\sqrt{|g|}A^\mu\right) \frac{1}{\sqrt{|g|}} \nu_g.$$

We can conclude that the divergence of A can locally be written as

$$\nabla \cdot A = \partial_\mu \left(\sqrt{|g|}A^\mu\right) \frac{1}{\sqrt{|g|}}. \tag{2.53}$$

Eq. (B.26) gives us the derivative of $\sqrt{|g|}$. By using Eq. (B.2) and Eq. (B.3), this derivative can be written in terms of the Christoffel symbols of the Levi-Civita connection on TM as

$$\partial_\mu \sqrt{|g|} = \frac{1}{2}\sqrt{|g|} g^{\rho\sigma} \partial_\mu g_{\rho\sigma} = \sqrt{|g|} \Gamma^\rho{}_{\rho\mu},$$

so the local formula for the divergence is

$$\nabla \cdot A = \partial_\mu A^\mu + \Gamma^\rho{}_{\rho\mu} A^\mu.$$

From Eq. (B.7), we finally conclude that $\nabla \cdot A = \nabla_\mu A^\mu$. □

2.55 Theorem (Divergence theorem)**.** *If M is a compact manifold without boundary, we have for every vector field $A \in \mathfrak{X}(M)$ that*

$$\int_M \nabla \cdot A \; \nu_g = 0. \tag{2.54}$$

PROOF. We consider the global $(n-1)$-form α corresponding to the vector field A, given by $\alpha := i_A(\nu_g)$. From Definition 2.53 we see that $\nabla \cdot A\ \nu_g = d\alpha$. Applying Corollary 2.51 then proves the proposition. □

2.56 Corollary. *If M is a compact manifold without boundary, we have for every function $f \in C^\infty(M)$ that*

$$\int_M \Delta f\ \nu_g = 0. \tag{2.55}$$

PROOF. We define a vector field $A = A^\mu \partial_\mu$ by $A^\mu = \partial^\mu f = g^{\mu\nu}\partial_\nu f$. Using the divergence theorem and writing out the integrand yields

$$\begin{aligned} 0 &= \int_M \nabla \cdot A\ \nu_g = \int_M \nabla_\mu A^\mu\ \nu_g = \int_M \nabla_\mu (g^{\mu\nu}\partial_\nu f)\nu_g \\ &= \int_M g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho{}_{\mu\nu}\partial_\rho) f\ \nu_g = -\int_M \Delta f\ \nu_g, \end{aligned}$$

where in the last step we have used the local formula of Eq. (2.41) for the scalar Laplacian. □

2.8 GRASSMANN INTEGRALS AND PFAFFIANS

In this section, we will give a short introduction to Grassmann variables, and use this to find the relation between the Pfaffian and the determinant of an antisymmetric matrix. A more detailed introduction can be found in for instance [30, §1.5].

A Grassmann variable is an anticommuting variable. For a set of Grassmann variables θ_i, we have $\theta_i\theta_j = -\theta_j\theta_i$, and thus in particular, $\theta_i^2 = 0$.

2.57 Definition. On anticommuting *Grassmann variables* θ_j, we define an integral by

$$\int 1 d\theta_j = 0, \qquad \int \theta_j d\theta_j = 1. \tag{2.56}$$

If we have a Grassmann vector θ consisting of N components, we define the integral over $D[\theta]$ as the integral over $d\theta_1 \cdots d\theta_N$. Suppose we have two Grassmann vectors η and θ of N components. We then define the integration element as $D[\eta,\theta] = d\eta_1 d\theta_1 \cdots d\eta_N d\theta_N$.

The determinant of an $N \times N$-matrix A is given by the formula

$$\det(A) = \frac{1}{N!}\sum_{\sigma,\tau\in\Pi_N} (-1)^{|\sigma|+|\tau|} A_{\sigma(1)\tau(1)} \cdots A_{\sigma(N)\tau(N)}, \tag{2.57}$$

where Π_N denotes the set of all permutations of $1, 2, \ldots, N$.

2.58 Lemma. *For an $N \times N$-matrix A, we obtain the formula*

$$\int e^{\theta^T A \eta} D[\eta,\theta] = \det A. \tag{2.58}$$

PROOF. The only terms in the expansion of the exponential that give a non-zero contribution to the integral are the terms that contain each Grassmann variable θ_j and η_j. Thus the only contribution comes from the term

$$\frac{1}{N!}\Big(\sum_{i,j}\theta_i A_{ij}\eta_j\Big)^N = \frac{1}{N!}\sum_{\sigma,\tau\in\Pi_N} A_{\sigma(1)\tau(1)} \cdots A_{\sigma(N)\tau(N)}\theta_{\sigma(N)}\eta_{\tau(N)} \cdots \theta_{\sigma(1)}\eta_{\tau(1)},$$

where we have used that the terms $\theta_j\eta_j$ commute with each other and have square zero. The calculation of the integral gives us

$$\int \theta_{\sigma(N)}\eta_{\tau(N)} \cdots \theta_{\sigma(1)}\eta_{\tau(1)} D[\eta,\theta] = (-1)^{|\sigma|}(-1)^{|\tau|}.$$

Using Eq. (2.57), we then obtain

$$\int \frac{1}{N!}(-1)^N \Big(\sum_{i,j} \theta_i A_{ij} \eta_j\Big)^N D[\eta,\theta] = \det(A). \qquad \square$$

2.59 Definition. The *Pfaffian* of an antisymmetric $2l \times 2l$-matrix A is given by

$$\mathrm{Pf}(A) = \frac{(-1)^l}{2^l l!} \sum_{\sigma \in \Pi_{2l}} (-1)^{|\sigma|} A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2l-1)\sigma(2l)}. \tag{2.59}$$

2.60 Lemma. *For an antisymmetric $2l \times 2l$-matrix A, we obtain the formula*

$$\int e^{\frac{1}{2}\eta^T A \eta} D[\eta] = \mathrm{Pf}(A). \tag{2.60}$$

Proof. As in Lemma 2.58, the only terms in the expansion of the exponential that give a non-zero contribution to the integral are the terms that contain each Grassmann variable η_j. Thus the only contribution comes from the term

$$\frac{1}{l!}\Big(\frac{1}{2}\sum_{i,j} \eta_i A_{ij} \eta_j\Big)^l = \frac{1}{2^l l!} \sum_{\sigma \in \Pi_{2l}} A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(N-1)\sigma(N)} \eta_{\sigma(1)} \eta_{\sigma(2)} \cdots \eta_{\sigma(N-1)} \eta_{\sigma(N)}.$$

The calculation of the integral gives us

$$\int \eta_{\sigma(1)}\eta_{\sigma(2)} \cdots \eta_{\sigma(N-1)}\eta_{\sigma(N)} D[\eta] = \int (-1)^l \eta_{\sigma(N)}\eta_{\sigma(N-1)} \cdots \eta_{\sigma(2)}\eta_{\sigma(1)} D[\eta] = (-1)^l (-1)^{|\sigma|}.$$

Combining this and comparing with Definition 2.59 gives

$$\int \frac{1}{l!}\Big(\frac{1}{2}\sum_{i,j} \eta_i A_{ij} \eta_j\Big)^l D[\eta] = \mathrm{Pf}(A). \qquad \square$$

2.61 Lemma. *The determinant of a $2l \times 2l$ skewsymmetric matrix A is the square of the Pfaffian:*

$$\det A = \mathrm{Pf}(A)^2. \tag{2.61}$$

Proof. From Lemma 2.60 we know that the Pfaffian is given by the Grassmann integral $\mathrm{Pf}(A) = \int e^{\frac{1}{2}\eta^T A \eta} D[\eta]$, where η is a $2l$-component Grassmann vector. The square of this integral has the form

$$\mathrm{Pf}(A)^2 = \int e^{\frac{1}{2}\eta^T A \eta} D[\eta] \int e^{\frac{1}{2}\eta'^T A \eta'} D[\eta'] = \int e^{\frac{1}{2}\eta^T A \eta + \frac{1}{2}\eta'^T A \eta'} D[\eta] D[\eta'],$$

where we have used that $e^{\frac{1}{2}\eta'^T A \eta'}$ is an even polynomial and thus commutes with all Grassmann variables. We make the change of variables

$$\theta_j = \frac{1}{\sqrt{2}}(\eta_j + i\eta'_j), \qquad \zeta_k = \frac{1}{\sqrt{2}}(\eta_k - i\eta'_k).$$

Using the anticommutation of η'_j and η_k, as well as the antisymmetry of A_{jk}, we see that $\eta'_j A_{jk} \eta_k = \eta_k A_{kj} \eta'_j$, so that we obtain

$$\theta_j A_{jk} \zeta_k = \frac{1}{2}(\eta_j A_{jk} \eta_k + i\eta'_j A_{jk} \eta_k - i\eta_j A_{jk} \eta'_k + \eta'_j A_{jk} \eta'_k) = \frac{1}{2}\eta_j A_{jk} \eta_k + \frac{1}{2}\eta'_j A_{jk} \eta'_k.$$

We also note that

$$d\theta_j d\zeta_j = \frac{1}{2}\left(d\eta_j d\eta_j + i d\eta'_j d\eta_j - i d\eta_j d\eta'_j + d\eta'_j d\eta'_j\right) = -i d\eta_j d\eta'_j,$$

so we can rewrite the integration element as

$$\begin{aligned} D[\eta]D[\eta'] = d\eta_1 \cdots d\eta_{2l} d\eta'_1 \cdots d\eta'_{2l} &= (-1)^l d\eta_1 d\eta'_1 \cdots d\eta_{2l} d\eta'_{2l} \\ &= (-1)^l i^{2l} d\theta_1 d\zeta_1 \cdots d\theta_{2l} d\zeta_{2l} = D[\theta, \zeta]. \end{aligned}$$

We thus obtain

$$\mathrm{Pf}(A)^2 = \int e^{\frac{1}{2}\eta_j A_{jk}\eta_k + \frac{1}{2}\eta'_j A_{jk}\eta'_k} D[\eta]D[\eta'] = \int e^{\theta_j A_{jk}\zeta_k} D[\theta, \zeta] = \det A. \qquad \square$$

We have evaluated the Grassmann integral over a function of the form $e^{\theta^T A \eta}$. The matrix A can be considered as a bilinear form. We first considered the case where θ and η are independent variables, and found that we obtain the determinant of A. In the second case, we took $\theta = \eta$, and now we obtained the Pfaffian of A. Finally, the square of this Pfaffian equals the determinant of A. So, by simply considering one instead of two independent Grassmann variables, we are in effect taking the square root of a determinant.

2.9 PONTRYAGIN AND EULER CLASSES

In this section we define the Pontryagin class and the Euler class. We will show that the integral over M of the Pfaffian of the Riemannian curvature will yield the Euler characteristic of M. A good introduction to these topics can be found in [30, Section 11.4].

Let E be a real k-dimensional vector bundle over an n-dimensional manifold M. Assuming that we have a hermitian connection ∇^E on E, we can choose an orthonormal frame for E. By Lemma 2.43, the curvature $\Omega^E \in \mathcal{A}^2(M, \mathfrak{so}(E))$ of this connection can then be seen as an antisymmetric matrix whose entries are 2-forms.

2.62 Definition. The *total Pontryagin class* of the curvature is defined by

$$p(\Omega^E) := \det\left(I + \frac{\Omega^E}{2\pi}\right). \tag{2.62}$$

Since the transpose of Ω^E equals $-\Omega^E$, we have $\det\left(I + \frac{\Omega^E}{2\pi}\right) = \det\left(I - \frac{\Omega^E}{2\pi}\right)$, and it follows that $p(\Omega^E)$ is an even function of Ω^E. We expand the Pontryagin class as

$$p(\Omega^E) = 1 + p_1(\Omega^E) + p_2(\Omega^E) + \ldots \tag{2.63}$$

where $p_j(\Omega^E)$ is a polynomial of order $2j$. This expansion is cut off for either $2j > k = \dim E$ (the highest order arising from the determinant) or $4j > n = \dim M$ (since Ω^E is a 2-form).

2.63 Lemma. *If $k = \dim E$ is even, we find that*

$$p_{k/2}(\Omega^E) = \left(\frac{1}{2\pi}\right)^k \det(\Omega^E). \tag{2.64}$$

PROOF. A skew-symmetric matrix A can be block diagonalized as

$$A \to \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \lambda_{k/2} \\ & & & -\lambda_{k/2} & 0 \end{pmatrix}.$$

The Pontryagin class is invariant under such a transformation, because

$$\det(I + BAB^{-1}) = \det(B(I+A)B^{-1}) = \det(B)\det(I+A)\det(B)^{-1} = \det(I+A).$$

If we take $A = \Omega^E/2\pi$, we then see that the Pontryagin class is given by

$$p(\Omega^E) = \det\left(I + \frac{\Omega^E}{2\pi}\right) = \prod_{i=1}^{k/2}\left(1 + \frac{\lambda_i^2}{4\pi^2}\right),$$

where the λ_i are the eigenvalues of Ω^E. The highest order term is then given by

$$p_{k/2}(\Omega^E) = \prod_{i=1}^{k/2}\frac{\lambda_i^2}{4\pi^2} = \left(\frac{1}{2\pi}\right)^k \det(\Omega^E). \qquad \square$$

2.64 Definition. Let M be a $2l$-dimensional orientable Riemannian manifold with Riemannian curvature R of the tangent bundle TM. We define the *Euler class* e of M by

$$e(A)e(A) = p_l(A), \tag{2.65}$$

where both sides are to be understood as functions of a $2l \times 2l$-matrix A and not of the curvature R, since $p_l(R)$ vanishes identically. However, R can be written as a matrix-valued 2-form, and the above procedure applied to that matrix does define a $2l$-form (i.e. a volume form) $e(M) := e(R)$.

We state here without proof the following theorem (see, for instance, [30, §11.4.2]).

2.65 Theorem (Gauss-Bonnet-Chern)**.** *The integral of the Euler class over a compact orientable manifold M without boundary yields the Euler characteristic:*

$$\int_M e(M) = \chi(M). \tag{2.66}$$

2.66 Proposition. *For a $2l$-dimensional Riemannian manifold M, endowed with the Levi-Civita connection ∇, we obtain that $\mathrm{Pf}(R) = (2\pi)^l e(M)$, and hence that*

$$\int_M \mathrm{Pf}(R) = (2\pi)^l \chi(M). \tag{2.67}$$

Proof. The Euler class $e(M)$ is defined by $e(A)e(A) = p_l(A)$. By Lemma 2.63, we have $p_l(R) = \left(\frac{1}{2\pi}\right)^{2l}\det(R)$. By Lemma 2.61 we have $\det(R) = \mathrm{Pf}(R)^2$. From this we conclude that $(2\pi)^l e(M) = \mathrm{Pf}(R)$. The last statement then follows from the Gauss-Bonnet-Chern Theorem 2.65. $\square$

2.67 Corollary. *In the case* $\dim M = 2l = 4$, *we have*

$$\int_M R^*R^*\nu_g = 8\pi^2\chi(M), \tag{2.68}$$

where R^*R^* is given by

$$R^*R^* = s^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{2.69}$$

Proof. We have $R \in \mathcal{A}^2(M, \mathrm{End}(TM))$, so for two vector fields X, Y we have an endomorphism $R(X,Y) \in \mathrm{End}(TM)$. We write $R^\alpha{}_\beta \in \mathcal{A}^2(M)$ for the components of the matrix of this endomorphism, such that $R^\alpha{}_\beta(X,Y)Z^\beta\partial_\alpha = R(X,Y)Z$ for a vector field $Z = Z^\alpha\partial_\alpha$ on a local basis ∂_α of TM. The determinant of R over $\mathrm{End}(TM)$ is given by $\det(R) = \det(R^\alpha{}_\beta)$, which can be rewritten as $\det(R) = \det(g^{\alpha\gamma}R_{\gamma\beta}) = |g|^{-1}\det(R_{\gamma\beta})$. Note that the components $R_{\gamma\beta}$ should not be confused with the

Ricci tensor. For the Pfaffian we then obtain $\mathrm{Pf}(R) = \mathrm{Pf}(R^{\alpha}{}_{\beta}) = \sqrt{|g|}^{-1}\mathrm{Pf}(R_{\alpha\beta})$. This Pfaffian can now be written as

$$\mathrm{Pf}(R) = \frac{(-1)^2}{2^2 2!}\sqrt{|g|}^{-1}\sum_{\sigma}(-1)^{|\sigma|}R_{\sigma(1)\sigma(2)} \wedge R_{\sigma(3)\sigma(4)} = \frac{1}{8}\delta^{\alpha\beta\gamma\delta}R_{\alpha\beta} \wedge R_{\gamma\delta},$$

where, using Eq. (B.31), the sign of the permutation and the factor $\sqrt{|g|}^{-1}$ have been absorbed into $\delta^{\alpha\beta\gamma\delta}$. We write $R_{\alpha\beta} = R_{\alpha\beta\mu\nu}dx^{\mu} \wedge dx^{\nu}$ and $R_{\gamma\delta} = R_{\gamma\delta\rho\sigma}dx^{\rho} \wedge dx^{\sigma}$ to obtain

$$\mathrm{Pf}(R) = \frac{1}{8}\delta^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta\rho\sigma}dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma}.$$

We note that $dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} = \delta^{\mu\nu\rho\sigma}\nu_g$, where ν_g is the Riemannian volume form defined in Definition 2.52. Using Eq. (B.32), we can then write

$$\mathrm{Pf}(R) = \frac{1}{2} *R^{\gamma\delta\rho\sigma}R_{\gamma\delta\rho\sigma}\nu_g.$$

We introduce the common notation $R^*R^* := *R^{\gamma\delta\rho\sigma}R_{\gamma\delta\rho\sigma}$. So, to conclude we have

$$\mathrm{Pf}(R) = \frac{1}{2}R^*R^*\nu_g,$$

where by Eq. (B.38) we have

$$R^*R^* = s^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$

The statement then follows from Proposition 2.66. □

2.10 DIFFERENTIAL OPERATORS AND THE HEAT EXPANSION

The main goal of this section is to present two important theorems, which we will need later in Chapter 6 to calculate the spectral action of almost commutative spaces. The first of these theorems states that there exists a heat expansion for a generalized Laplacian. The second theorem gives explicit formulas for the coefficients of this expansion. We first start with a few definitions.

2.68 Definition. The algebra of *differential operators* on E, denoted by $\mathcal{D}(M,E)$, is the subalgebra of $\mathrm{End}(\Gamma(M,E))$ generated by elements of $\Gamma(M,\mathrm{End}(E))$ acting by multiplication on $\Gamma(M,E)$, and the covariant derivatives ∇^E_X, where ∇^E is any connection on E and X ranges over all vector fields on M. The algebra $\mathcal{D}(M,E)$ has a natural filtration, defined by letting

$$\mathcal{D}_i(M,E) = \Gamma(M,\mathrm{End}(E)) \cdot \mathrm{span}\{\nabla^E_{X_1} \cdots \nabla^E_{X_j};\ j \leq i\}. \tag{2.70}$$

We call an element of $\mathcal{D}_i(M,E)$ an i-th order differential operator.

An important example of a second order differential operator is the Laplacian Δ^E of a connection ∇^E on E (cf. Definition 2.45). We can generalize the notion of Laplacians by defining generalized Laplacians. One way of introducing these generalized Laplacians is by requiring the symbol of a second order differential operator to yield the square of a norm on the sections of E. In [1, Prop. 2.5] it is shown that the following definition is equivalent.

2.69 Definition. We define a *generalized Laplacian* H on a vector bundle E as a second order differential operator of the form $\Delta^E - F$, where Δ^E is the Laplacian of some connection ∇^E on E and F is a section of the bundle $\mathrm{End}(E)$.

2.70 Definition ([1, section 3.3])**.** A (general) *Dirac operator* on a $\mathbb{Z}_2$-graded vector bundle E is a first order differential operator of odd parity on E:

$$D\colon \Gamma(M,E^{\pm}) \to \Gamma(M,E^{\mp}), \tag{2.71}$$

such that D^2 is a generalized Laplacian.

2.10.1 *The heat expansion*

2.71 Theorem ([18], §1.7)**.** *For a generalized Laplacian H on E we have the following expansion in t, known as the* heat expansion:

$$\operatorname{Tr}\left(e^{-tH}\right) \sim \sum_{k\geq 0} t^{\frac{k-n}{2}} a_k(H), \tag{2.72}$$

where n is the dimension of the manifold, the trace is taken over the Hilbert space $L^2(M,E)$ and the coefficients of the expansion are given by

$$a_k(H) := \int_M a_k(x,H)\sqrt{|g|}d^4x. \tag{2.73}$$

The coefficients $a_k(x,H)$ are called the Seeley-DeWitt coefficients.

We also state here without proof Theorem 4.8.16 from Gilkey [18]. Note that the conventions used by Gilkey for the Riemannian curvature R are such that $g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}$ is negative for a sphere, in contrast to our own conventions. Therefore we have replaced $s = -R$. Furthermore, we have used that $f_{;\mu}{}^{;\mu} = -\Delta f$ for $f \in C^\infty(M)$.

2.72 Theorem ([18], Theorem 4.8.16)**.** *For a generalized Laplacian $H = \Delta^E - F$ the Seeley-DeWitt coefficients are given by*

$$a_0(x,H) = (4\pi)^{-\frac{n}{2}}\operatorname{Tr}(\mathrm{Id}) \tag{2.74}$$

$$a_2(x,H) = (4\pi)^{-\frac{n}{2}}\operatorname{Tr}\left(\frac{s}{6} + F\right) \tag{2.75}$$

$$\begin{aligned} a_4(x,H) = (4\pi)^{-\frac{n}{2}}\frac{1}{360}\operatorname{Tr}\big(&-12\Delta s + 5s^2 - 2R_{\mu\nu}R^{\mu\nu} \\ &+ 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 60sF + 180F^2 \\ &- 60\Delta F + 30\Omega^E_{\mu\nu}(\Omega^E)^{\mu\nu}\big), \end{aligned} \tag{2.76}$$

where the traces are now taken over the fibre E_x. Here s is the scalar curvature of the Levi-Civita connection ∇, Δ is the scalar Laplacian and Ω^E is the curvature of the connection ∇^E corresponding to Δ^E. All $a_k(x,H)$ with odd k vanish.

3 SPIN GEOMETRY

In this chapter we give an introduction to spin geometry. We start in the first two sections with the description of Clifford algebras and spin groups. Next, we use this formalism to describe the Clifford bundle and the spinor bundle on a spin manifold M. After introducing the spin connection, we can define the canonical Dirac operator $\not{D}$ on the spinor bundle. This spinor bundle and the Dirac operator will be main ingredients in the definition of the canonical spectral triple in Chapter 5. For more details on the topics of this chapter, we refer to [17, 19, 34].

3.1 CLIFFORD ALGEBRAS

3.1.1 *Real Clifford algebras*

Suppose we have an n-dimensional real vector space V equipped with a symmetric bilinear form g.

3.1 Definition. The *real Clifford algebra* over V, denoted $Cl(V)$, is the associative algebra generated by V, where the Clifford product (denoted by $\cdot$) is subject to the relation

$$u \cdot v + v \cdot u = 2g(u, v) \tag{3.1}$$

for all $u, v \in V$.

In the special case of the zero bilinear form, the corresponding algebra is just the exterior algebra $\bigwedge^\bullet V$. In general, the Clifford algebra has the same underlying vector space as the exterior algebra, but with a modified product operation.

3.2 Definition. The *exterior multiplication* ϵ on $\bigwedge^\bullet V$ is defined by

$$\epsilon(v)(u_1 \wedge \cdots \wedge u_k) := v \wedge u_1 \wedge \cdots \wedge u_k. \tag{3.2}$$

The *contraction* by g (or *interior multiplication*) is defined by

$$\iota(v)(u_1 \wedge \cdots \wedge u_k) := \sum_{j=1}^{k} (-1)^{j-1} g(v, u_j) u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k, \tag{3.3}$$

where the circumflex on u_j means that this factor is omitted.

3.3 Proposition. *We define* $\sigma \colon Cl(V) \to \bigwedge^\bullet V$ *by* $\sigma(v_1 \cdots v_k) := c(v_1) \cdots c(v_k)\, 1$, *where* $c(v) := \epsilon(v) + \iota(v)$. *This map* σ *gives an* isomorphism of vector spaces $Cl(V) \cong \bigwedge^\bullet V$.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis for V. Then the set

$$\{e_{i_1} \cdot e_{i_2} \cdots e_{i_k} \mid 1 \leq k \leq n, i_1 < i_2 < \ldots < i_k\}$$

forms a basis for $Cl(V)$ (as a vector space), while

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq k \leq n, i_1 < i_2 < \ldots < i_k\}$$

forms a basis for $\bigwedge^\bullet V$. Both vector spaces evidently have equal dimension 2^n. Since the e_i are orthonormal, σ maps $e_{i_1} \cdot e_{i_2} \cdots e_{i_k}$ to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$. For σ to be well-defined, we need to check that $c(u)c(v) + c(v)c(u) = 2g(u, v)$. Because

$u \wedge v = -v \wedge u$, we have $\epsilon(u)\epsilon(v) + \epsilon(v)\epsilon(u) = 0$. The factors $(-1)^j$ in the definition of $\iota(v)$ make sure that also $\iota(u)\iota(v) + \iota(v)\iota(u) = 0$. Therefore we have

$$c(u)c(v) + c(v)c(u) = \epsilon(u)\iota(v) + \iota(v)\epsilon(u) + \epsilon(v)\iota(u) + \iota(u)\epsilon(v).$$

The action of $\epsilon(u)\iota(v) + \iota(v)\epsilon(u)$ is given by

$$(\epsilon(u)\iota(v) + \iota(v)\epsilon(u))(u_1 \wedge \cdots \wedge u_k) = \sum_{j=1}^{k} (-1)^{j-1} g(v, u_j) u \wedge u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k$$
$$- \sum_{j=1}^{k} (-1)^{j-1} g(v, u_j) u \wedge u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k + g(v, u) u_1 \wedge \cdots \wedge u_k,$$

so we conclude that $\epsilon(u)\iota(v) + \iota(v)\epsilon(u) = g(v, u)$. Hence, by symmetry,

$$c(u)c(v) + c(v)c(u) = 2g(u, v). \qquad \square$$

3.4 Lemma. *For $u, v, w \in V$, we find*

$$\sigma(uvw) = u \wedge v \wedge w + g(u, v)w - g(u, w)v + g(v, w)u. \tag{3.4}$$

Proof. We simply write this out:

$$\begin{aligned} \sigma(uvw) &= c(u)c(v)c(w)\,1 = c(u)c(v)w = c(u)\,(v \wedge w + g(v, w)) \\ &= u \wedge v \wedge w + g(u, v)w - g(u, w)v + g(v, w)u. \end{aligned} \qquad \square$$

3.5 Proposition. *If A is a unital real algebra, and $f\colon V \to A$ is a real-linear map satisfying $f(v)^2 = g(v, v)1_A$ for all $v \in V$, then f extends uniquely to an algebra homomorphism $\widetilde{f}\colon Cl(V) \to A$.*

Proof. $\widetilde{f}(v_1 \cdots v_k) = f(v_1) \cdots f(v_k)$ must hold for this algebra homomorphism, and this defines $\widetilde{f}$ uniquely by linearity. For $\widetilde{f}$ to be well-defined, we need to check that $\widetilde{f}(u \cdot v + v \cdot u - 2g(u, v)) = 0$ for all $u, v \in V$. Indeed,

$$\begin{aligned} \widetilde{f}(u \cdot v + v \cdot u) &= \widetilde{f}((u + v)^2 - u^2 - v^2) = f(u + v)^2 - f(u)^2 - f(v)^2 \\ &= g(u + v, u + v)1_A - g(u, u)1_A - g(v, v)1_A = 2g(u, v)1_A. \end{aligned} \qquad \square$$

3.6 Example. If $h\colon V \to V$ satisfies $g(hu, hv) = g(u, v)$ for all $u, v \in V$, then by taking $A = Cl(V)$ it extends to the *Bogoliubov automorphism* θ_h of $Cl(V)$.

3.7 Definition. The *grading automorphism* (or $\mathbb{Z}_2$-grading) χ of $Cl(V)$ is defined as $\chi := \theta_{-1}$.

The grading automorphism splits $Cl(V)$ into an even subalgebra and an odd subspace: $Cl(V) = Cl^+(V) \oplus Cl^-(V)$, where $Cl^{\pm}(V)$ denotes the $(\pm)$-eigenspace of χ. The even subalgebra $Cl^+(V)$ consists of all linear combinations of Clifford products of an even number of elements of V.

3.1.2 *Complexification*

We shall often be concerned with the complex case. Denote by $V^{\mathbb{C}}$ the complexification $V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$. A real-linear operator A on V simply extends to a complex-linear operator on $V^{\mathbb{C}}$ by $A(u + iv) := Au + iAv$. Similarly the symmetric bilinear form g on V extends by $g(u_1 + iv_1, u_2 + iv_2) = g(u_1, u_2) + ig(u_1, v_2) + ig(v_1, u_2) - g(v_1, v_2)$. The complexification of the exterior algebra is then $\bigwedge^\bullet V \otimes_{\mathbb{R}} \mathbb{C} \cong \bigwedge^\bullet V^{\mathbb{C}}$. The exterior and interior multiplication are still defined by Eq. (3.2) and Eq. (3.3).

3.8 Definition. The *complex Clifford algebra* over V, denoted $\mathbb{C}l(V)$, is the associative algebra generated by $V^{\mathbb{C}}$, where the Clifford product (denoted by $\cdot$) is subject to the relation

$$u \cdot v + v \cdot u = 2g(u,v) \tag{3.5}$$

for all $u, v \in V^{\mathbb{C}}$.

3.9 Proposition. *There is a canonical* isomorphism of vector spaces $\mathbb{C}l(V) \cong \bigwedge^\bullet V^{\mathbb{C}}$. *Under this isomorphism, left Clifford multiplication by $v \in V^{\mathbb{C}}$ is given by*

$$v\cdot = \epsilon(v) + \iota(v). \tag{3.6}$$

Proof. The proof is analogous to the proof of Proposition 3.3. □

3.10 Definition. The grading automorphism in the complex case is defined in the same way as in Definition 3.7, using the complex analogue of Proposition 3.5. Write $n = 2m$ or $n = 2m+1$ for n even or odd, respectively. The *chirality element* γ of $\mathbb{C}l(V)$ is

$$\gamma := (-i)^m e_1 \cdots e_n, \tag{3.7}$$

for an orthonormal basis $e_1, \dots, e_n$. For a fixed orientation, γ is independent of the chosen basis.

For n even, $\gamma \cdot v \cdot \gamma = -v$, so by Definition 3.7 $\gamma \cdot a \cdot \gamma = \chi(a)$ for $a \in \mathbb{C}l(V)$. For n odd, $\gamma \cdot v \cdot \gamma = v$ and hence $\gamma \cdot a \cdot \gamma = a$.

3.1.3 *Classification of the Clifford algebras*

We will now describe the (complex) Clifford algebras more concretely, by giving an isomorphism between the Clifford algebra and a matrix algebra (for even dimensions) or a direct sum of two matrix algebras (for odd dimensions).

3.11 Lemma. *There is a periodicity isomorphism*

$$\mathbb{C}l(\mathbb{R}^{n+2}) \cong \mathbb{C}l(\mathbb{R}^n) \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{3.8}$$

Proof. Take $V = \mathbb{R}^{n+2}$ with an orthonormal basis $e_1, \dots, e_{n+2}$, $A = \mathbb{C}l(\mathbb{R}^n) \otimes_{\mathbb{C}} M_2(\mathbb{C})$ and $f\colon V \to A$ given, for $j = 1, \dots, n$, by

$$\begin{aligned} f(e_j) &:= e_j \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ f(e_{n+1}) &:= 1 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ f(e_{n+2}) &:= 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

All $f(e_j)$ anticommute because all e_j do, in fact (for $i = 1, \dots, n+2$) all $f(e_i)$ anticommute and satisfy $f(e_i)^2 = 1$. By (the complex analogue of) Proposition 3.5, f extends to an algebra homomorphism $\tilde{f}\colon \mathbb{C}l(V) \to A$. As complex vector spaces, $\mathbb{C}l(V)$ and A have equal dimension 2^{n+2}, so $\tilde{f}$ is in fact an isomorphism. □

3.12 Proposition. *If $N = 2^m$, then*

$$\begin{aligned} \mathbb{C}l(\mathbb{R}^{2m}) &\cong M_N(\mathbb{C}), \\ \mathbb{C}l(\mathbb{R}^{2m+1}) &\cong M_N(\mathbb{C}) \oplus M_N(\mathbb{C}). \end{aligned} \tag{3.9}$$

Proof. It is clear that $\mathbb{C}l(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C} \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ and $\mathbb{C}l(\mathbb{R}^2) \cong M_2(\mathbb{C})$, so the first isomorphism holds for $m = 1$ and the second holds for $m = 0$. In the first case, suppose $\mathbb{C}l(\mathbb{R}^{2m}) \cong M_N(\mathbb{C})$. The periodicity isomorphism of Lemma 3.11 then yields

$$\mathbb{C}l(\mathbb{R}^{2(m+1)}) \cong \mathbb{C}l(\mathbb{R}^{2m}) \otimes M_2(\mathbb{C}) \cong M_N(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_{2N}(\mathbb{C}).$$

By induction, the first isomorphism holds for all $m \in \mathbb{N}^\times$. Similarly, in the second case suppose that $\mathbb{C}l(\mathbb{R}^{2m+1}) \cong M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$. The periodicity isomorphism now gives

$$\begin{aligned}\mathbb{C}l(\mathbb{R}^{2(m+1)+1}) &\cong \mathbb{C}l(\mathbb{R}^{2m+1}) \otimes M_2(\mathbb{C}) \cong \big(M_N(\mathbb{C}) \oplus M_N(\mathbb{C})\big) \otimes M_2(\mathbb{C}) \\ &\cong M_{2N}(\mathbb{C}) \oplus M_{2N}(\mathbb{C}),\end{aligned}$$

and by induction, the second isomorphism holds for all $m \in \mathbb{N}$. □

3.13 Lemma. *The centre of the complex Clifford algebra $\mathbb{C}l(V)$ equals $\mathbb{C}$ for n even or $\mathbb{C} \oplus \mathbb{C}\gamma$ for n odd. In both cases, the even subset of the centre equals $\mathbb{C}$.*

Proof. In both cases, $\mathbb{C}$ is obviously in the centre. For n even, the isomorphism from Proposition 3.12 shows that the centre is 1-dimensional, and hence equal to $\mathbb{C}$. For n odd, the chirality element γ is also in the centre. Now the isomorphism from Proposition 3.12 shows that the centre is 2-dimensional, and hence equals the span of 1 and γ. □

3.1.4 *Spinor space*

In Proposition 3.12 we have obtained an isomorphism between the Clifford algebra and matrix algebras. We can regard a matrix algebra $M_N(\mathbb{C})$ as the space of endomorphisms of an N-dimensional complex vector space $\mathbb{S}$. If V is an n-dimensional vector space, it is isomorphic to $\mathbb{R}^n$. Therefore we have the following result:

3.14 Corollary. *For an n-dimensional complex vector space V, write $n = 2m$ or $n = 2m+1$. Define $N = 2^m$, and let $\mathbb{S}$, $\mathbb{S}^+$ and $\mathbb{S}^-$ be N-dimensional complex vector spaces. Then we have isomorphisms*

$$\mathbb{C}l(V) \cong \begin{cases} \mathrm{End}(\mathbb{S}), & n \text{ even}, \\ \mathrm{End}(\mathbb{S}^+) \oplus \mathrm{End}(\mathbb{S}^-), & n \text{ odd}. \end{cases} \tag{3.10}$$

3.15 Definition. For n even, we call $\mathbb{S}$ the *spin space*. For n odd, we define the total *spin space* as $\mathbb{S} := \mathbb{S}^+ \oplus \mathbb{S}^-$.

3.16 Definition. In both cases, the isomorphism of Corollary 3.14 yields a representation of the Clifford algebra on the spin space, given by $c\colon \mathbb{C}l(V) \to \mathrm{End}(\mathbb{S})$. Since $\mathrm{End}(\mathbb{S}) \cong \mathbb{S} \otimes \mathbb{S}^*$, we can also regard this as a map $\hat{c}\colon \mathbb{C}l(V) \otimes \mathbb{S} \to \mathbb{S}$, which we will call *Clifford multiplication*.

We also note here that in the even case, $\mathbb{C}l(V)$ has one irreducible representation given by c. In the odd case, the representation c splits into two irreducible representations on $\mathbb{S}^+$ and $\mathbb{S}^-$.

3.2 SPIN GROUPS

3.2.1 *The real case*

We will now continue by examining the relation between symmetry operations (reflections and rotations) on the vector space V and Clifford multiplication in $Cl(V)$.

We will start in the real case, and define subgroups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ of $Cl(V)$, and these will turn out to be nontrivial double covers of the orthogonal group $O(V)$ and the special orthogonal group $SO(V)$, respectively.

3.17 Proposition. *The reflection of a vector $v \in V$ with respect to the hyperplane orthogonal to a unit vector $r \in V$ is represented in the Clifford algebra by $-r \cdot v \cdot r$.*

Proof. By Eq. (3.1), we have $-r \cdot v \cdot r = (v \cdot r - 2g(v,r)) \cdot r = v - 2g(v,r)r$. □

For such a unit vector r we of course have $r \cdot r = 1$, and the previous proposition has shown that the product $-r \cdot v \cdot r$ is again a vector in V. The following definition uses these two properties to generalize the set of reflections.

3.18 Definition. The group $\mathrm{Pin}(V)$ is defined as the subgroup (with respect to the Clifford product) of elements $a \in Cl(V)$ which satisfy $a \cdot a^! = 1$ and $a \cdot V \cdot a^! \subseteqq V$, where the antiautomorphism ! is defined as $(v_1 \cdots v_k)^! = v_k \cdots v_1$.

Note that all unit vectors in V are elements of $\mathrm{Pin}(V)$, as well as all Clifford products of unit vectors. Conversely, suppose $a = v_1 \cdots v_k$ is an element of $\mathrm{Pin}(V)$. Then $a \cdot a^! = 1$ implies $g(v_1, v_1) \cdots g(v_k, v_k) = 1$. If we define $r_i = \frac{v_i}{\sqrt{g(v_i,v_i)}}$, we see that we can rewrite $a = v_1 \cdots v_k = \sqrt{g(v_1,v_1) \cdots g(v_k,v_k)} r_1 \cdots r_k = r_1 \cdots r_k$. So we can always write a as the Clifford product of *unit* vectors in V.

3.19 Proposition. *There is an exact sequence (of group homomorphisms)*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Pin}(V) \to O(V) \to 1. \tag{3.11}$$

Proof. The homomorphism $\rho\colon \mathrm{Pin}(V) \to O(V)$ is given by the action $\rho(a)v = \chi(a) \cdot v \cdot a^!$ for $a \in \mathrm{Pin}(V)$ and $v \in V$. If $a = r_1 \cdots r_k$ is a product of unit vectors, $\rho(a) = \rho(r_1) \cdots \rho(r_k)$ is a composition of reflections. Since $O(V)$ is generated by all reflections, ρ is surjective. We proceed to determine its kernel. Assume that $\rho(a)$ is the identity, so $\rho(a)v = \chi(a) \cdot v \cdot a^! = v$. This implies that $\chi(a) \cdot v = v \cdot a$. Let $e_1, \ldots, e_n$ be an orthonormal basis for V. Choose $v = e_i$ and write $a = a_+ + a_- + b_+ \cdot e_i + b_- \cdot e_i$, where $a_\pm$ and $b_\pm$ do not involve e_i, a_+ and b_+ are even elements of $Cl(V)$ and a_- and b_- are odd elements of $Cl(V)$. Then $\chi(a) = a_+ - a_- - b_+ \cdot e_i + b_- \cdot e_i$. We have

$$\chi(a) \cdot e_i = a_+ \cdot e_i - a_- \cdot e_i - b_+ + b_- = e_i \cdot a_+ + e_i \cdot a_- - b_+ + b_-$$

and

$$e_i \cdot a = e_i \cdot a_+ + e_i \cdot a_- + e_i \cdot b_+ \cdot e_i + e_i \cdot b_- \cdot e_i = e_i \cdot a_+ + e_i \cdot a_- + b_+ - b_-.$$

Since both expressions must be equal, we see that $b_+ = b_- = 0$. This holds for all i, so we conclude that a is a constant. Then $a \cdot a^! = a^2 = 1$ gives $a = \pm 1$. Hence $\mathrm{Ker}(\rho) \cong \mathbb{Z}/2\mathbb{Z}$. □

3.20 Definition. The *spin group* $\mathrm{Spin}(V)$ is the subgroup of all even elements of $\mathrm{Pin}(V)$.

The group $SO(V)$ consists of all rotations on V. Each rotation can be written as the composition of two reflections. It is no surprise that this Spin group is related to $SO(V)$.

3.21 Proposition. *There is an exact sequence (of group homomorphisms)*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V) \to SO(V) \to 1. \tag{3.12}$$

Proof. The composition of an even number of reflections yields a rotation, so the homomorphism ρ (as defined in the proof of Proposition 3.19) restricts on $\mathrm{Spin}(V)$ to $SO(V)$. Since $SO(V)$ is generated by all rotations, it is surjective. As in the proof of Proposition 3.19, $\mathrm{Ker}(\rho) \cong \mathbb{Z}/2\mathbb{Z}$. □

3.2.2 *The complex case*

The construction of the spin group from the complex Clifford algebra is slightly more complicated. First we shall define the concepts of conjugation and unitary elements of $\mathbb{C}l(V)$.

3.22 Definition. Recall that the 'product reversal' antiautomorphism ! of $\mathbb{C}l(V)$ is given by $(v_1 \cdots v_k)^! = v_k \cdots v_1$. Denote by $\overline{v}$ the complex conjugate of $v \in V^{\mathbb{C}}$. This extends to an antilinear conjugation $a \mapsto \overline{a}$ of $\mathbb{C}l(V)$. We define the *involution* $*$ on $\mathbb{C}l(V)$ by $a \mapsto a^* := (\overline{a})^!$. We define *charge conjugation* by $\kappa(a) := \chi(\overline{a})$.

3.23 Definition. An element $u \in \mathbb{C}l(V)$ is called a *unitary element* if $u^*u = 1$. Any unitary vector $w \in V^{\mathbb{C}}$ is of the form $w = \lambda v$, for $v \in V$ a unit vector and $|\lambda| = 1$.

3.24 Definition. The group $\text{Spin}^c(V)$ is defined as the subgroup of $\mathbb{C}l(V)$ generated by all even unitary elements.

3.25 Proposition. *There is an exact sequence (of group homomorphisms)*

$$1 \to \mathbb{T} \to \text{Spin}^c(V) \to SO(V) \to 1, \tag{3.13}$$

where $\mathbb{T}$ is the circle group of unitary scalars $\{\lambda \in \mathbb{C}; |\lambda| = 1\}$.

Proof. The proof is similar to Proposition 3.19. The homomorphism $\rho\colon \text{Spin}^c(V) \to SO(V)$ is now given by $\rho(u)v = u \cdot v \cdot u^{-1}$ for $u \in \text{Spin}^c(V)$ and $v \in V$. (Since $\text{Spin}^c(V)$ only contains even elements, we do not need the grading automorphism.) Now $u^{-1} = u^* = (\overline{u})^!$, so $\rho(u)$ is an even composition of reflections, and hence is a rotation in $SO(V)$. If $u \in \text{Ker}(\rho)$, meaning $v = u \cdot v \cdot u^{-1}$ for all $v \in V$, then $v \cdot u = u \cdot v$, so u commutes with all $v \in V$. Using Lemma 3.13, any $u \in \text{Ker}(\rho)$ must be a unitary scalar. □

There is another important homomorphism $\nu\colon \text{Spin}^c(V) \to \mathbb{T}$, given by $\nu(u) := uu^!$. For a unitary vector $w = \lambda v$ (with v a unit vector in V), we have $\nu(w) = \lambda^2 \in \mathbb{T}$. We are now ready to give an alternative definition for the spin group of V.

3.26 Definition. The *spin group* $\text{Spin}(V)$ is defined as the kernel of ν in $\text{Spin}^c(V)$.

3.27 Proposition. *Definition 3.20 and Definition 3.26 are equivalent.*

Proof. For $a \in \text{Spin}(V)$ by Definition 3.20, a is an even element of the real group $\text{Pin}(V)$. Since $\overline{a} = a$, we have $a^* = a^!$, so $a \cdot a^* = a \cdot a^! = 1$. Hence a is an even unitary in $\mathbb{C}l(V)$ and also complies with Definition 3.26.

Conversely, for $a \in \text{Spin}(V)$ by Definition 3.26, write $a = w_1 \cdots w_k$ as a product of unitary vectors. We can write $w_j = \lambda_j v_j$, and define $\lambda = \lambda_1 \cdots \lambda_k$. This yields $a = \lambda v_1 \cdots v_k$. Because $a \in \text{Ker}(\nu)$, we have $\lambda = \pm 1$, so $a = \pm v_1 \cdots v_k$ is an even product of real unit vectors in V. Then $a \cdot a^* = 1$ becomes $a \cdot a^! = 1$, and $a \cdot V \cdot a^! \subset V$ follows from Proposition 3.17. Hence a complies with Definition 3.20. □

Since the spin group is a subgroup of the Clifford algebra, the representation c from Definition 3.16 restricts to a representation of the spin group. Furthermore, the elements of the spin group are invertible and hence give automorphisms of $\mathbb{S}$.

3.28 Definition. The homomorphism $c\colon \mathbb{C}l(V) \to \text{End}(\mathbb{S})$ can be restricted to the subgroup $\text{Spin}(V)$ to obtain the *spin representation*

$$\mu\colon \text{Spin}(V) \to \text{Aut}(\mathbb{S}). \tag{3.14}$$

3.2.3 *The Lie algebra of* Spin(V)

Recall the map σ from Proposition 3.3, and write $Q\colon \bigwedge^\bullet V \to Cl(V)$ for its inverse.

3.29 Proposition. $Q(\bigwedge^2 V)$ *forms a* Lie algebra.

Proof. $Q(\bigwedge^2 V)$ consists of all $a \in Cl(V)$ for which $\sigma(a) \in \bigwedge^2 V$. We know that $\sigma(uv) = c(u)c(v)1 = u \wedge v + g(u,v)$, so we have

$$Q(u \wedge v) = uv - g(u,v) = \frac{1}{2}(uv - vu), \quad \text{for all } u, v \in V.$$

Hence all $a \in Q(\bigwedge^2 V)$ must be a linear combination of such vectors. We need to check that $Q(\bigwedge^2 V)$ is closed under commutation. Take $a = uv - g(u,v)$ and $b = wx - g(w,x)$, then

$$\begin{aligned}[a,b] := (ab - ba) &= uvwx - wxuv = uvwx + wuxv - 2g(u,x)wv \\ &= uvwx - wuvx - 2g(u,x)wv + 2g(x,v)wu \\ &= uvwx + uwvx - 2g(u,x)wv + 2g(x,v)wu - 2g(w,u)vx \\ &= -2g(u,x)wv + 2g(x,v)wu - 2g(w,u)vx + 2g(w,v)ux \\ &= -2g(u,x)(wv - g(w,v)) + 2g(x,v)(wu - g(w,u)) \\ &\quad - 2g(w,u)(vx - g(v,x)) + 2g(w,v)(ux - g(u,x)).\end{aligned}$$

This is a linear combination of elements of the form $uv - g(u,v)$, so the commutator is again an element of $Q(\bigwedge^2 V)$. □

3.30 Definition. The Lie algebra of $SO(V)$ consists of the skewsymmetric operators

$$\mathfrak{so}(V) := \{A \in \mathrm{End}_{\mathbb{R}}(V) \mid g(y, Ax) = -g(Ay, x)\ \forall x, y \in V\}. \tag{3.15}$$

3.31 Definition. We define the *adjoint map* $\mathrm{ad} : Q(\bigwedge^2 V) \to \mathfrak{so}(V)$ by $(\mathrm{ad}\, a)(x) = [a, x]$ for $a \in Q(\bigwedge^2 V)$ and $x \in V$.

We have seen that the elements of $Q(\bigwedge^2 V)$ are of the form $a = \frac{1}{2}(uv - vu)$, so

$$\begin{aligned}2[a,x] &= uvx - vux - xuv + xvu \\ &= -uxv + 2g(v,x)u + vxu - 2g(u,x)v + uxv - 2g(u,x)v - vxu + 2g(v,x)u \\ &= 4g(v,x)u - 4g(u,x)v \in V.\end{aligned}$$

So $\mathrm{ad}\, a \in \mathrm{End}_{\mathbb{R}}(V)$, and furthermore

$$\begin{aligned}g(y, (\mathrm{ad}\, a)(x)) &= g(y, 2g(v,x)u - 2g(u,x)v) = 2g(v,x)g(y,u) - 2g(u,x)g(y,v) \\ &= -g(2g(v,y)u - 2g(u,y)v, x) = -g((\mathrm{ad}\, a)(y), x),\end{aligned}$$

so the map ad indeed maps $Q(\bigwedge^2 V)$ into $\mathfrak{so}(V)$.

3.32 Proposition. *The map* $\mathrm{ad}\colon Q(\bigwedge^2 V) \to \mathfrak{so}(V)$ *is a Lie algebra isomorphism.*

Proof. First we must check that ad is a Lie algebra homomorphism, i.e. $\mathrm{ad}\,[a,b] = [\mathrm{ad}\, a, \mathrm{ad}\, b]$:

$$\begin{aligned}(\mathrm{ad}\,[a,b])(x) &= [[a,b],x] = -[[b,x],a] - [[x,a],b] = [a,[b,x]] - [b,[a,x]] \\ &= (\mathrm{ad}\, a)\big((\mathrm{ad}\, b)(x)\big) - (\mathrm{ad}\, b)\big((\mathrm{ad}\, a)(x)\big) = [\mathrm{ad}\, a, \mathrm{ad}\, b](x).\end{aligned}$$

If $\mathrm{ad}\, b = 0$, i.e. $[b,x] = 0$ for all $x \in V$, then b is a central element of $Cl(V,g)$. Since $Q(\bigwedge^2 V)$ consists only of even elements, b must be a scalar by Lemma 3.13. Because there are no pure scalars in $Q(\bigwedge^2 V)$, we conclude that $b = 0$ so ad is injective. Furthermore, $\dim \mathfrak{so}(V) = \frac{1}{2}n(n-1) = \dim \bigwedge^2 V$, so ad is in fact bijective and hence an isomorphism. □

3.33 Proposition. *The inverse of* ad *is given by the map* $\mathrm{ad}^{-1}\colon \mathfrak{so}(V) \to Q(\bigwedge^2 V)$ *defined by*

$$\mathrm{ad}^{-1}(A) := \frac{1}{4}\sum_{j,k=1}^{n} g(e_j, Ae_k)e_je_k = \frac{1}{2}\sum_{j<k} g(e_j, Ae_k)e_je_k. \tag{3.16}$$

Proof. We simply calculate

$$\begin{aligned}\big(\mathrm{ad}\,\mathrm{ad}^{-1}(A)\big)(e_l) &= \frac{1}{4}\sum_{j,k=1}^{n} g(e_j, Ae_k)[e_je_k, e_l] = \frac{1}{2}\sum_{j,k=1}^{n} g(e_j, Ae_k)(\delta_{kl}e_j - \delta_{jl}e_k)\\ &= \frac{1}{2}\sum_{j=1}^{n} g(e_j, Ae_l)e_j - \frac{1}{2}\sum_{k=1}^{n} g(e_l, Ae_k)e_k = \sum_{i=1}^{n} g(e_i, Ae_l)e_i = Ae_l,\end{aligned}$$

where in the fourth step we used the skewsymmetry of A. Also, for $a = \frac{1}{2}(uv - vu) \in Q(\bigwedge^2 V)$, we have

$$\begin{aligned}\mathrm{ad}^{-1}(\mathrm{ad}\, a) &= \frac{1}{4}\sum_{j,k=1}^{n} g(e_j, 2g(v, e_k)u - 2g(u, e_k)v)e_je_k\\ &= \frac{1}{2}\sum_{j,k=1}^{n} \big(g(u, e_j)g(v, e_k)e_je_k - g(v, e_j)g(u, e_k)e_je_k\big) = \frac{1}{2}(uv - vu) = a.\end{aligned}$$

□

3.34 Lemma. *For $b \in Q(\bigwedge^2 V)$ and $x \in V$, we have the formula*

$$(\mathrm{ad}\, b)^r(x) = \sum_{k=0}^{r} \binom{r}{k} b^k x(-b)^{r-k}. \tag{3.17}$$

Proof. The case $r = 1$ immediately follows from the definition of ad. Suppose the statement holds for r, then we have for $r + 1$ that

$$\begin{aligned}(\mathrm{ad}\, b)^{r+1}(x) &= [b, (\mathrm{ad}\, b)^r(x)] = [b, \sum_{k=0}^{r} \binom{r}{k} b^k x(-b)^{r-k}]\\ &= \sum_{k=0}^{r} \binom{r}{k} \left(b^{k+1}x(-b)^{r-k} - b^k x(-b)^{r-k+1}\right)\\ &= \sum_{k=1}^{r+1} \binom{r}{k-1} b^k x(-b)^{r+1-k} - \sum_{k=0}^{r} \binom{r}{k} b^k x(-b)^{r+1-k}\\ &= \sum_{k=0}^{r+1} \binom{r+1}{k} b^k x(-b)^{r+1-k},\end{aligned}$$

where we have used that $\binom{r+1}{k} = \binom{r}{k} + \binom{r}{k-1}$. Hence, by induction, the statement holds for all $r \in \mathbb{N}$. □

3.35 Proposition. *The Lie algebra of the spin group* $\mathrm{Spin}(V)$ *is* $Q(\bigwedge^2 V)$.

Proof. For $b \in Q(\bigwedge^2 V)$, define

$$\exp b := \sum_{k \geq 0} \frac{b^k}{k!}. \tag{3.18}$$

Take $u = \exp b$, then for $x \in V$ we have by the previous lemma

$$uxu^{-1} = \sum_{k,l \geq 0} \frac{1}{k!l!} b^k x(-b)^l = \sum_{r \geq 0} \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} b^k x(-b)^{r-k} = \sum_{r \geq 0} \frac{1}{r!}(\mathrm{ad}\, b)^r(x) \in V.$$

Since $b^! = -b$ in $Q(\bigwedge^2 V)$, we have $\nu(u) = uu^! = \exp b \exp(-b) = 1$. From this we conclude that $u \in \mathrm{Spin}(V)$. By recalling the homomorphism ρ from Proposition 3.19, we can write this as $\rho(u) = \exp(\operatorname{ad} b) \in SO(V)$, and these $\rho(u)$ cover all of $SO(V)$. There is a unique $A \in \mathfrak{so}(V)$ such that $b = \operatorname{ad}^{-1}(A)$. Choose an orthonormal basis such that $g(e_i, Ae_1) = 0$ and $g(e_i, Ae_2) = 0$ for all $2 < i \leq n$. Then we have

$$b = \frac{1}{2} g(e_1, Ae_2) e_1 e_2 + \frac{1}{4} \sum_{j,k=3}^{n} g(e_j, Ae_k) e_j e_k,$$

so clearly b commutes with $e_1 e_2$. Since $(e_1 e_2)^2 = -1$, we see that $\exp(\pi e_1 e_2) = \cos \pi + e_1 e_2 \sin \pi = -1$. This means that for each $u = \exp b \in \mathrm{Spin}(V)$, we also have $-u \in \mathrm{Spin}(V)$, because $-\exp b = \exp(\pi e_1 e_2) \exp(b) = \exp(\pi e_1 e_2 + b)$. We conclude that $\exp(Q(\bigwedge^2 V))$ is a subset of $\mathrm{Spin}(V)$ that doubly covers all of $SO(V)$, and hence equals $\mathrm{Spin}(V)$. □

3.36 Definition. $Q(\bigwedge^2 V)$ is a subset of the Clifford algebra $\mathbb{C}\mathrm{l}(V)$, which is represented on the spinor space $\mathbb{S}$ by Clifford multiplication. We define the *infinitesimal spin representation* $\dot{\mu} \colon \mathfrak{so}(V) \to \mathrm{End}(\mathbb{S})$ by the composition

$$\dot{\mu}(A) := c(\operatorname{ad}^{-1} A). \tag{3.19}$$

By the previous proposition we have $\mu(\exp b) = \exp(\dot{\mu}(\operatorname{ad} b))$ for $b \in Q(\bigwedge^2 V)$, which justifies the name.

3.37 Proposition. *The infinitesimal spin representation satisfies*

$$[\dot{\mu}(A), c(v)] = c(Av), \tag{3.20}$$

for all $A \in \mathfrak{so}(V)$ and $v \in V$.

PROOF. It is enough to show that $[\operatorname{ad}^{-1} A, v] = Av \in \mathbb{C}\mathrm{l}(V)$. By choosing an orthonormal basis e_i of V, we see that

$$\begin{aligned} [\operatorname{ad}^{-1} A, v] &= \frac{1}{4} \sum_{j,k=1}^{n} g(e_j, Ae_k)[e_j e_k, v] = \frac{1}{4} \sum_{j,k=1}^{n} g(e_j, Ae_k) \big(2g(e_k, v) e_j - 2g(e_j, v) e_k \big) \\ &= \frac{1}{2} \sum_{k=1}^{n} g(e_k, v) Ae_k + \frac{1}{2} \sum_{j=1}^{n} g(e_j, v) Ae_j = Av, \end{aligned}$$

because of the skewsymmetry of A. □

3.3 MODULES

3.3.1 *Clifford bundle*

3.38 Definition. The *Clifford bundle* is the bundle of complex Clifford algebras $\mathbb{C}\mathrm{l}(M) \to M$ generated by the cotangent bundle $T^*M \to M$ with g^{-1} as its Euclidean structure. In other words, $\mathbb{C}\mathrm{l}(M) := \mathbb{C}\mathrm{l}(T^*M)$.

3.39 Definition. From here on we will write $A := C(M)$ and (for $n = \dim M$)

$$B := \begin{cases} \Gamma(M, \mathbb{C}\mathrm{l}(T^*M)), & n \text{ even}, \\ \Gamma(M, \mathbb{C}\mathrm{l}^+(T^*M)), & n \text{ odd}. \end{cases} \tag{3.21}$$

We will denote $\mathcal{A}$ and $\mathcal{B}$ for the restriction of A and B to the smooth functions and sections, respectively.

3.3.2 *Spinor module*

From Corollary 3.14 we see that the Clifford algebra $\mathbb{C}l(T_x^*M)$ has a representation on the spinor space $\mathbb{S}$. These Clifford algebras join to form the vector bundle $\mathbb{C}l(T^*M)$. One might wonder whether the vector space $\mathbb{S}$ can also be turned into a vector bundle S, such that its sections $\Gamma(M,S)$ form a module over the algebra B defined in Definition 3.39. This is not always possible, but depends on certain topological conditions. We will require that the Dixmier-Douady class $\delta(\underline{B})$ and the second Stiefel-Whitney class $\kappa(B)$ both vanish identically. A manifold that satisfies these conditions is called a spin manifold. For a detailed discussion, we refer to [19, §9.2].

3.40 Definition. Let M be a compact boundaryless orientable even-dimensional spin manifold. We define the *spinor module* as the B-A-bimodule $\mathcal{S} := \Gamma(M,S)$ such that at each $x \in M$, the vector space $S_x \cong \mathbb{S}$ is an irreducible representation of the simple algebra $B_x \cong \mathbb{C}l(T_x^*M)$.

The Hilbert space $L^2(M,S)$ of L^2-spinors on M is defined as the completion of $\left\{s \in \mathcal{S} = \Gamma(M,S) \,;\, \int_M (s|s)\nu_g < \infty\right\}$ in the norm $\|\psi\| := \sqrt{\langle\psi|\psi\rangle}$, where the scalar product is defined in Definition 2.41.

3.41 Proposition ([19, §9.2])**.** *There exists an antilinear endomorphism C of $\mathcal{S}$ such that*

- $C(\psi a) = C(\psi)\bar{a}$, *for* $\psi \in \mathcal{S}$, $a \in A$;
- $C(b\psi) = \chi(\bar{b})C(\psi)$, *for* $\psi \in \mathcal{S}$, $b \in B$;
- C *is* antiunitary *in the sense that* $(C\phi|C\psi) = (\psi|\phi)$, *for* $\phi, \psi \in \mathcal{S}$;
- $C^2 = \pm 1$ *on* $\mathcal{S}$ *whenever* M *is connected.*

On the Hilbert-space completion $L^2(M,S)$ of $\mathcal{S}$, the operator C becomes an antiunitary operator, and it is called the charge conjugation*.*

3.42 Definition. Recall the Clifford multiplication $\hat{c}\colon \mathbb{C}l(V) \otimes \mathbb{S} \to \mathbb{S}$ from Definition 3.16. We now have a module of sections, $\mathcal{S}$, which carries an action of $\mathcal{B}$ (defined in Definition 3.39). We can extend the Clifford multiplication to this bundle by setting

$$(\hat{c}(b \otimes \psi))\,(x) = \hat{c}\,(b(x) \otimes \psi(x)) \tag{3.22}$$

for $b \in \mathcal{B}$ and $\psi \in \mathcal{S}$. As before, we can write $\hat{c}(b \otimes \psi) = c(b)\psi$. Using the inclusion $\mathcal{A}^1(M) \hookrightarrow \mathcal{B}$, this defines a map $\hat{c}\colon \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \to \mathcal{S}$. In the odd-dimensional case, $\mathcal{B}$ consists only of even elements, and the inclusion is given by $c(\alpha) := c(\alpha\gamma)$ for $\alpha \in \mathcal{A}^1(M)$ and γ the chirality element of Definition 3.10. On a local basis dx^μ of $\mathcal{A}^1(M)$, we define the abbreviation $\gamma^\mu := c(dx^\mu)$.

3.4 THE SPIN CONNECTION

This section is based on [34, §2.6]. From here on we will take M to be a differentiable manifold. Instead of the algebras A and B, we will now consider their restrictions to the smooth case, $\mathcal{A}$ and $\mathcal{B}$. Similarly we will replace $\Gamma(M,S)$ by the set of smooth sections $\Gamma^\infty(M,S)$. Thus the spinor module $\mathcal{S} = \Gamma^\infty(M,S)$ will denote the $\mathcal{A}$-module of smooth spinors. Recall from Section 2.3 the definition of a hermitian connection on the smooth sections of a vector bundle.

3.43 Definition. A *spin connection* on a spinor module $\mathcal{S}$ is any hermitian connection $\nabla^S\colon \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$ which is compatible with the action of $\mathcal{B}$:

$$\nabla^S\,(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi, \tag{3.23}$$

where ∇ is the Levi-Civita connection on $\mathcal{A}^1(M)$, $\alpha \in \mathcal{A}^1(M)$ and $\psi \in \mathcal{S}$. Moreover, for real $X \in \mathfrak{X}(M)$, we require that each $\nabla_X\colon \mathcal{S} \to \mathcal{S}$ commutes with C.

3.44 Proposition. *For a spin manifold M, the spin connection ∇^S exists and is unique.*

Proof. Recall from Eq. (2.27) the functions $\widetilde{\Gamma}^a{}_{\mu b}$, given on a chart $U \subset M$. The metric compatibility of ∇ yields

$$g^{-1}(\nabla_{\partial_\mu}\theta^a, \theta^b) + g^{-1}(\theta^a, \nabla_{\partial_\mu}\theta^b) = \partial_\mu(\delta^{ab}) = 0,$$

which implies $\widetilde{\Gamma}^a{}_{\mu b} + \widetilde{\Gamma}^b{}_{\mu a} = 0$. This means that for fixed μ, the functions $\widetilde{\Gamma}^a{}_{\mu b}$ form the components of a skewsymmetric matrix. We can write

$$\widetilde{\Gamma} \in \mathcal{A}^1(U, \mathfrak{so}(T^*M)) \cong \mathcal{A}^1(U) \otimes \mathfrak{so}(\mathbb{R}^n).$$

On the local chart U, we then have $\nabla = d - \widetilde{\Gamma}$ for the Levi-Civita connection on 1-forms.

On the chart U, we can replace $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ by C_0^∞, $C_0^\infty(U, \mathbb{C}l(\mathbb{R}^n))$ and $C_0^\infty(U, \mathbb{S})$ respectively. We extend the action of $\widetilde{\Gamma}$ on 1-forms to Clifford products of 1-forms by the Leibniz rule $\widetilde{\Gamma}(\alpha_1 \cdots \alpha_k) := \sum_{j=1}^k \alpha_1 \cdots \widetilde{\Gamma}\alpha_j \cdots \alpha_k$, which turns $\widetilde{\Gamma}$ into a derivation on $C_0^\infty(U, \mathbb{C}l(\mathbb{R}^n))$. Since we have $\widetilde{\Gamma} \in \mathcal{A}^1(U) \otimes \mathfrak{so}(\mathbb{R}^n)$, we can apply the infinitesimal spin representation $\dot{\mu}$ on the $\mathfrak{so}(\mathbb{R}^n)$ part of $\widetilde{\Gamma}$, and define

$$\nabla^S := d - \dot{\mu}(\widetilde{\Gamma}). \tag{3.24}$$

By using the commutation property Eq. (3.20) in the third step, we check that for $\psi \in \Gamma(U, S)$ and $v \in \Gamma(U, \mathbb{C}l(M))$ we have the Leibniz rule

$$\begin{aligned}\nabla^S(c(v)\psi) &= d(c(v)\psi) - \dot{\mu}(\widetilde{\Gamma})c(v)\psi \\ &= c(dv)\psi + c(v)d\psi - [\dot{\mu}(\widetilde{\Gamma}), c(v)]\psi - c(v)\dot{\mu}(\widetilde{\Gamma})\psi \\ &= (c(dv) - c(\widetilde{\Gamma}v))\psi + c(v)(d\psi - \dot{\mu}(\widetilde{\Gamma})\psi) = c(\nabla v)\psi + c(v)\nabla^S\psi,\end{aligned}$$

so Eq. (3.24) is compatible with the action of $\mathcal{B}$. For a local orthonormal basis θ^a, we can write

$$\dot{\mu}(\widetilde{\Gamma}_\mu) = \frac{1}{4}\sum_{a,b} g(\theta^a, \widetilde{\Gamma}^c{}_{\mu b}\theta^b)c(\theta^a)c(\theta^b) = \frac{1}{4}\sum_{a,b}\widetilde{\Gamma}^a{}_{\mu b}c(\theta^a)c(\theta^b).$$

Since $C(b\psi) = \chi(\bar{b})C(\psi)$, we see that C commutes with $c(\theta^a)c(\theta^b)$, and since the Christoffel symbols are real, C commutes with ∇^S, so Eq. (3.24) indeed defines a spin connection.

Suppose we have another spin connection ∇'. The difference $\nabla^S - \nabla'$ commutes with all $c(v)$, so it is a multiplication by a scalar. Since this scalar commutes with C, it must be real, but by Eq. (2.30) this scalar must be purely imaginary. Hence it is zero, and the local definition of Eq. (3.24) is unique. This also means that the locally defined connections must coincide on the overlap of two charts, so that ∇^S is also uniquely defined globally. □

3.45 Lemma. *The curvature Ω^S of the spin connection is given by $\dot{\mu}(R)$.*

Proof. The proof is based on [19, Prop. 9.9]. Using the local expression $\nabla = d - \widetilde{\Gamma}$, i.e. $\nabla_X = X - \widetilde{\Gamma}(X)$, the Riemannian curvature R can be written

$$\begin{aligned}R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} &= [X - \widetilde{\Gamma}(X), Y - \widetilde{\Gamma}(Y)] - [X,Y] + \widetilde{\Gamma}([X,Y]) \\ &= -X(\widetilde{\Gamma}(Y)) + Y(\widetilde{\Gamma}(X)) + [\widetilde{\Gamma}(X), \widetilde{\Gamma}(Y)] + \widetilde{\Gamma}([X,Y]) \\ &= (-d\widetilde{\Gamma} + \widetilde{\Gamma} \wedge \widetilde{\Gamma})(X,Y).\end{aligned}$$

A similar calculation shows that for $\nabla^S = d - \dot{\mu}(\widetilde{\Gamma})$, the curvature equals

$$\Omega^S = -d\dot{\mu}(\widetilde{\Gamma}) + \dot{\mu}(\widetilde{\Gamma}) \wedge \dot{\mu}(\widetilde{\Gamma}) = \dot{\mu}(R).$$ □

3.46 Lemma. *The scalar curvature can be expressed locally as*

$$s = -2\gamma^{\mu}\gamma^{\nu}[\nabla^S_{\partial_\mu}, \nabla^S_{\partial_\nu}]. \tag{3.25}$$

Proof. The spin curvature $\Omega^S \in \mathcal{A}^2(M, \mathrm{End}S)$ is defined as in Definition 2.38, so on a local basis ∂_μ we have

$$\Omega^S(\partial_\mu, \partial_\nu) = \left[\nabla^S_{\partial_\mu}, \nabla^S_{\partial_\nu}\right]$$

because $\left[\partial_\mu, \partial_\nu\right] = 0$ and hence $\nabla_{[\partial_\mu,\partial_\nu]} = 0$. From Lemma 3.45 we have the relation $\Omega^S(\partial_\mu, \partial_\nu) = \dot{\mu}(R(\partial_\mu, \partial_\nu))$, so by Eq. (3.16) we see that

$$\Omega^S(\partial_\mu, \partial_\nu) = \frac{1}{4}g(\partial_\rho, R(\partial_\mu, \partial_\nu)\partial_\sigma)c(dx^\rho)c(dx^\sigma) = \frac{1}{4}R_{\rho\sigma\mu\nu}\gamma^\rho\gamma^\sigma = -\frac{1}{4}R_{\sigma\rho\mu\nu}\gamma^\rho\gamma^\sigma. \tag{3.26}$$

Therefore the right-hand-side of Eq. (3.25) equals $\frac{1}{2}R_{\sigma\rho\mu\nu}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$. By applying the isomorphism Q onto the result of Lemma 3.4 and choosing $u = dx^\mu$, $v = dx^\nu$ and $w = dx^\rho$, we know that

$$\gamma^\mu\gamma^\nu\gamma^\rho = Q(dx^\mu \wedge dx^\nu \wedge dx^\rho) + g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu.$$

This factor gets multiplied by $R_{\sigma\rho\mu\nu}$. A cyclic permutation of μ, ν, ρ has no effect on $Q(dx^\mu \wedge dx^\nu \wedge dx^\rho)$, so this term contributes zero by the Bianchi identity of Eq. (2.34). Because of the antisymmetry $R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}$, the term $g^{\mu\nu}\gamma^\rho$ also contributes zero, while the terms $-g^{\mu\rho}\gamma^\nu$ and $g^{\nu\rho}\gamma^\mu$ contribute equally. To summarize, we can replace $\gamma^\mu\gamma^\nu\gamma^\rho$ by $-2g^{\mu\rho}\gamma^\nu$, and we get

$$-2\gamma^\mu\gamma^\nu[\nabla^S_{\partial_\mu}, \nabla^S_{\partial_\nu}] = R_{\rho\sigma\mu\nu}g^{\mu\rho}\gamma^\nu\gamma^\sigma.$$

We now recognize the definition of the Ricci tensor, and use its symmetry to obtain

$$R_{\rho\sigma\mu\nu}g^{\mu\rho}\gamma^\nu\gamma^\sigma = R_{\sigma\nu}\gamma^\nu\gamma^\sigma = \frac{1}{2}R_{\sigma\nu}(\gamma^\sigma\gamma^\nu + \gamma^\nu\gamma^\sigma) = R_{\sigma\nu}g^{\sigma\nu} = s. \qquad \square$$

3.5 THE DIRAC OPERATOR

3.47 Definition. Let M be a compact Riemannian spin manifold with spinor module $\mathcal{S}$. The (canonical) *Dirac operator* $\not{D}$ on $\mathcal{S}$ is defined by

$$\not{D} := -i\,\hat{c} \circ \nabla^S. \tag{3.27}$$

where ∇^S is the spin connection on $\mathcal{S}$ and $\hat{c}\colon \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \to \mathcal{S}$ is given by Definition 3.42.

Note that in Definition 2.70 we have already given a general definition of a Dirac operator. The operator $\not{D}$ indeed also satisfies this general definition. Because of the Clifford multiplication $\hat{c}$ we see that $\not{D}$ is an odd operator. It will be shown below in Proposition 3.50 that furthermore $\not{D}^2$ is a generalized Laplacian as in Definition 2.69.

3.48 Proposition. *The Dirac operator satisfies the* commutation relation

$$[\not{D}, a] = -i\,c(da) \tag{3.28}$$

for all $a \in \mathcal{A}$.

PROOF. For $\psi \in \mathcal{S}$ we have, by using the Leibniz rule for ∇^S,

$$\begin{aligned}[\not{D}, a]\psi &= -i\hat{c}(\nabla^S(a\psi)) + ia\hat{c}(\nabla^S\psi) = -i\hat{c}(\nabla^S(a\psi) - a\nabla^S\psi) \\ &= -i\hat{c}(da \otimes \psi) = -ic(da)\psi. \end{aligned}$$ □

3.49 Lemma. *In local coordinates, we have*

$$\not{D} = -i\gamma^\mu \nabla^S_\mu. \tag{3.29}$$

PROOF. The spin connection is given locally by $\nabla^S = dx^\mu \otimes \nabla^S_\mu$. Applying the Clifford multplication $\hat{c}$ on dx^μ then gives the local formula for $\not{D}$. □

3.5.1 *The Lichnerowicz formula*

3.50 Proposition (Lichnerowicz)**.** *Let M be a compact Riemannian spin manifold with spinor module $\mathcal{S}$. Then*

$$\not{D}^2 = \Delta^S + \frac{1}{4}s \tag{3.30}$$

as an operator on $\mathcal{S}$.

PROOF. We use local coordinates, where $\not{D} = -i\gamma^\mu \nabla^S_{\partial_\mu}$. We can then calculate the square of the Dirac operator as

$$\begin{aligned}\not{D}^2 &= -\gamma^\mu \nabla^S_{\partial_\mu} \gamma^\nu \nabla^S_{\partial_\nu} = -\gamma^\mu\gamma^\nu \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} - \gamma^\mu c(\nabla_{\partial_\mu} dx^\rho)\nabla^S_{\partial_\rho} \\ &= -\gamma^\mu\gamma^\nu \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} + \gamma^\mu\gamma^\nu \Gamma^\rho{}_{\mu\nu}\nabla^S_{\partial_\rho},\end{aligned}$$

where on the first line we used the Leibniz rule of Eq. (3.23), and on the second line that $\nabla_{\partial_\mu} dx^\rho = \Gamma^\rho{}_{\mu\nu} dx^\nu$. Using the Clifford relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, we have for the first term

$$\begin{aligned}-\gamma^\mu\gamma^\nu \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} &= -\frac{1}{2}\gamma^\mu\gamma^\nu \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} - \frac{1}{2}\gamma^\nu\gamma^\mu \nabla^S_{\partial_\nu}\nabla^S_{\partial_\mu} \\ &= -\frac{1}{2}\gamma^\mu\gamma^\nu \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} - \frac{1}{2}\gamma^\nu\gamma^\mu \left(\nabla^S_{\partial_\nu}\nabla^S_{\partial_\mu} + \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} - \nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu}\right) \\ &= -g^{\mu\nu}\nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu[\nabla^S_{\partial_\mu}, \nabla^S_{\partial_\nu}],\end{aligned}$$

where on the second line the first and third terms combine by the Clifford relation, and the second and fourth terms form a commutator. From Eq. (2.20) we know that $\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$, so for the second term in $\not{D}^2$ we get

$$\gamma^\mu\gamma^\nu\Gamma^\rho{}_{\mu\nu}\nabla^S_{\partial_\rho} = \frac{1}{2}\gamma^\mu\gamma^\nu\Gamma^\rho{}_{\mu\nu}\nabla^S_{\partial_\rho} + \frac{1}{2}\gamma^\nu\gamma^\mu\Gamma^\rho{}_{\mu\nu}\nabla^S_{\partial_\rho} = g^{\mu\nu}\Gamma^\rho{}_{\mu\nu}\nabla^S_{\partial_\rho}.$$

Their sum then equals

$$\not{D}^2 = -g^{\mu\nu}\left(\nabla^S_{\partial_\mu}\nabla^S_{\partial_\nu} - \Gamma^\rho{}_{\mu\nu}\nabla^S_{\partial_\rho}\right) - \frac{1}{2}\gamma^\mu\gamma^\nu[\nabla^S_{\partial_\mu}, \nabla^S_{\partial_\nu}].$$

By Lemma 2.46 the first term equals Δ^S, and by Lemma 3.46 the second term equals $\frac{1}{4}s$. This concludes the proof. □

4 LAGRANGIAN THEORY

The basic method for describing a field theory in physics is the Lagrangian formalism. It is a quite remarkable fact that all fundamental physical theories are described by Lagrangians. In this chapter we will give a short introduction to the formalism of Lagrangian theory, and show how the action principle yields the Euler-Lagrange equations, or equations of motion. In the next section we introduce the concept of gauge theory, which is a Lagrangian theory endowed with an additional gauge symmetry. For this first part of the chapter, we have made good use of [2, 32], and for more information on these topics we refer to these references. In the last sections, we discuss Lagrangians that describe theories of gravity. We also introduce conformal transformations, and describe the conformally invariant Weyl gravity. For these sections, good use has been made of [35].

4.1 THE LAGRANGIAN FORMALISM

4.1 Definition. Let M be an m-dimensional manifold. A *field* or *particle field* ψ is defined as a section of a vector bundle $E \to M$. If the rank of the vector bundle E is r, the field is said to have r components, in which case we can write locally $\psi = \psi^a e_a$ in terms of a local frame e_a of E.

4.2 Definition. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, we denote by ∂_α the partial derivative $\partial_\alpha := (\partial_1)^{\alpha_1} \cdots (\partial_m)^{\alpha_m}$. A *local form* of order K is the tensor product of a differential form on M with a smooth function (polynomial) in the coordinates x^μ and $\partial_\alpha \psi(x)$, where $|\alpha| := \sum_{i=1}^m \alpha_i \leq K$ for some finite positive integer K. The algebra of local forms is denoted by $\text{Loc}(E)$.

4.3 Definition. A *K-order Lagrangian density* $\mathcal{L}$ is a local form of degree $m = \dim(M)$ and of order K. A *local functional* $F[\psi]$ of the fields is the integral of a Lagrangian density over M, i.e. $F[\psi] = \int_M \mathcal{L}\big(x, \partial_\alpha \psi(x)\big)$. The algebra (over $\mathbb{C}$) generated by local functionals is denoted by $\mathcal{F}([E])$.

Note that it is also possible to define a global Lagrangian density by using a partition of unity. See, for instance, [32, §6.1] for more details.

4.4 Definition. For a functional $F[\psi]$, we define the *functional derivative* $\frac{\delta F[\psi]}{\delta \psi(x)}$ as the distribution given by

$$\int_M f(x) \frac{\delta F[\psi]}{\delta \psi(x)} d^m x = \left.\frac{d}{dt}\right|_{t=0} F[\psi + tf]. \tag{4.1}$$

For a local functional of the form $F[\psi] = \int_M J(y)\psi(y) d^m y$, this yields

$$\frac{\delta}{\delta \psi(x)} F[\psi] := J(x). \tag{4.2}$$

This functional derivative need not yield a proper function of x, in general it can be any distribution.

4.1.1 *The action principle*

4.5 Definition. Let $\mathcal{L}$ be a K-order Lagrangian density of the set of fields ψ_i. A *variation* of the Lagrangian density is defined by the substitution $\psi_i \to \psi_i + tf_i$, for

$t \in \mathbb{R}$ and a set of fields f_i that satisfy $\partial_\alpha f_i|_{\partial M} = 0$ for $|\alpha| \leq K-1$. The variation is called an *infinitesimal variation* if the parameter t is infinitesimally small.

4.6 Definition. An *action* S is a local functional $S \in \mathcal{F}([E])$ for some vector bundle E over an m-dimensional manifold M, given by $S[\psi] = \int_M \mathcal{L}(x, \partial_\alpha \psi_i) d^m x$ for a set of fields ψ_i. We shall require that $\mathcal{L}(x, \partial_\alpha \psi_i)$ is an integrable function over M, and that $\mathcal{L}$ (as well as its functional derivative) is sufficiently smooth.

We can state the *action principle*, or the *principle of least action*, as follows:

> The physical solution of the fields ψ_i is given by a stationary solution of the action under infinitesimal variations of the fields.

The mathematical formulation of the action principle is that the functional derivative of the action with respect to the fields must vanish:

$$\frac{\delta S[\psi]}{\delta \psi_i(x)} = 0. \tag{4.3}$$

4.7 Proposition. *The action principle translates into the Euler-Lagrange equations*

$$\sum_{|\alpha| \leq K} (-1)^{|\alpha|} \partial_\alpha \left(\frac{\delta \mathcal{L}}{\delta(\partial_\alpha \psi_i)} \right) = 0 \qquad \forall \psi_i. \tag{4.4}$$

Proof. From Eq. (4.1) we see that the action principle implies that

$$\left.\frac{d}{dt}\right|_{t=0} S[\psi + tf] = \int_M \left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(x, \partial_\alpha(\psi_i + tf_i)) d^m x = 0.$$

The variation is given by the set f_j. We take $f_j = 0$ for $j \neq i$. The action then only depends on t through $\psi_i + tf_i$. The total derivative with respect to t is given by

$$\frac{d}{dt} = \sum_{|\alpha| \leq K} \frac{d\big(\partial_\alpha(\psi_i + tf_i)\big)}{dt} \frac{\delta}{\delta(\partial_\alpha \psi_i)} = \sum_{|\alpha| \leq K} \partial_\alpha f_i \frac{\delta}{\delta(\partial_\alpha \psi_i)}.$$

Since the Lagrangian is required to be sufficiently smooth, we may assume that the derivative $\frac{d}{dt}$ commutes with the integration over M, so we obtain

$$\left.\frac{d}{dt}\right|_{t=0} S[\psi + tf] = \int_M \left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(x, \partial_\alpha(\psi_i + tf_i)) d^m x = \int_M \sum_{|\alpha| \leq K} \partial_\alpha f_i \frac{\delta \mathcal{L}}{\delta(\partial_\alpha \psi_i)} d^m x.$$

We can use integration by parts to obtain

$$0 = \int_M f_i \sum_{|\alpha| \leq K} (-1)^{|\alpha|} \partial_\alpha \left(\frac{\delta \mathcal{L}}{\delta(\partial_\alpha \psi_i)} \right) d^m x,$$

where a minus sign arises for each integration by parts, and the boundary condition $\partial_\alpha f_i|_{\partial M} = 0$ for $|\alpha| \leq K-1$ ensures that all boundary terms vanish. Since this equation must hold for arbitrary variations f_i, we conclude that for each ψ_i we have

$$\sum_{|\alpha| \leq K} (-1)^{|\alpha|} \partial_\alpha \left(\frac{\delta \mathcal{L}}{\delta(\partial_\alpha \psi_i)} \right) = 0. \qquad \square$$

4.2 GAUGE THEORY

We will now briefly review the formalism of gauge theory. We do not give a detailed introduction, but merely provide the material that will be needed later on in this thesis. For a more detailed introduction, see for instance [2, §3.2] or [32, §4.7].

Throughout this section, we will consider a differentiable manifold M of dimension m. We will take G to be a Lie group, and denote its Lie algebra by $\mathfrak{g}$. We will denote by P a principal G-bundle over M, and we let π be the projection $P \to M$.

4.8 Definition. We define a *gauge transformation* of P to be a bundle automorphism $\phi\colon P \to P$.

Because of the way we defined bundle morphisms in Definition 2.25, the following two important properties immediately follow from the definition of a gauge transformation:

$$\pi\big(\phi(p)\big) = \pi(p), \quad \text{for all } p \in P, \text{ and} \tag{4.5a}$$
$$\phi(pg) = \phi(p)g, \quad \text{for all } p \in P, g \in G. \tag{4.5b}$$

4.9 Proposition. *A gauge transformation ϕ can be written uniquely as $\phi(p) = p \cdot \gamma(p)$ for a map $\gamma\colon P \to G$ for which*

$$\gamma(p \cdot g) = g^{-1} \cdot \gamma(p) \cdot g. \tag{4.6}$$

Proof. The fact that $\pi\big(\phi(p)\big) = \pi(p)$ shows that we can write $\phi(p) = p \cdot \gamma(p)$. Since $\phi(pg) = \phi(p)g$, we then must have

$$p \cdot g \cdot \gamma(p \cdot g) = p \cdot \gamma(p) \cdot g,$$

and hence $\gamma(p \cdot g) = g^{-1} \cdot \gamma(p) \cdot g$. □

4.10 Definition. We define the *gauge group* $\mathcal{G}(P)$ of P as the set of all maps $\gamma\colon P \to G$ satisfying $\gamma(p \cdot g) = g^{-1} \cdot \gamma(p) \cdot g$.

4.11 Lemma. *On a local trivialization (U, h_U), a gauge transformation ϕ can be written as $(x, g) \mapsto (x, t_U(x)g)$, for a smooth map $t_U\colon U \to G$.*

Proof. The action of the gauge transformation ϕ on $U \times G$ is given by $h_U \circ \phi \circ h_U^{-1}$. Eq. (4.5a) shows that this can be written as

$$h_U \circ \phi \circ h_U^{-1}(x, g) = (x, \phi_U(x)(g)). \tag{4.7}$$

Because of Eq. (4.5b), we have $\phi_U(x)(g) = \phi_U(x)(e)g$. We define the map $t_U\colon U \to G$ by $t_U(x) := \phi_U(x)(e)$, so we can write $\phi_U(x)(g) = t_U(x)g$. □

4.12 Proposition. *On a local trivialization (U, h_U), a gauge transformation ϕ transforms a local connection form A_U on P as*

$$A_U \to t_U A_U t_U^{-1} - dt_U t_U^{-1}. \tag{4.8}$$

Proof. We first realize that the map $h_U \circ \phi\colon\ P|_U \to U \times G$ defines a new local trivialization of P. From Lemma 4.11 we see that the transition function of this change of trivialization is given by the map t_U. From Definition 2.32, we then find the transformation of A_U on U. □

4.13 Proposition. *Suppose we have a representation $G \to \mathrm{GL}(V)$ on a finite-dimensional vector space V, so we can construct the associated vector bundle $E = P \times_G V$. On a local trivialization (U, h_U), a gauge transformation ϕ transforms the local particle field ψ as*

$$\psi(x) \to t_U(x)\psi(x). \tag{4.9}$$

Proof. In Lemma 4.11 we have seen that locally we have $(x, g) \mapsto (x, t_U(x)g)$. The local trivialization $E|_U = P|_U \times_G V \to U \times V$ is given by $[(x, g), v] \mapsto (x, gv) = (x, w) \in U \times V$, where we write $w := gv$. The action of the gauge transformation on $U \times V$ is then given by $(x, w) \mapsto (x, t_U(x)w)$. □

4.14 Proposition. *The gauge group of a trivial principal bundle $P = M \times G$ is $C^\infty(M, G)$.*

Proof. The gauge group $\mathcal{G}(P)$ is defined in Definition 4.10 to be the group of maps $\gamma\colon P \to G$ satisfying $\gamma(p \cdot g) = g^{-1} \cdot \gamma(p) \cdot g$. Inserting $p = (x, e)$ into this relation we obtain $\gamma(x, g) = g^{-1} \cdot \gamma(x, e) \cdot g$. Hence we see that γ is completely determined by the image $\gamma(x, e)$ in G. The map $\gamma \mapsto \gamma(\cdot, e) \in C^\infty(M, G)$ then gives the isomorphism $\mathcal{G}(P) \cong C^\infty(M, G)$. □

In Definition 2.32 we have defined a connection form on a trivial principal G-bundle $P = M \times G$ as a global $\mathfrak{g}$-valued 1-form A on M. From Proposition 2.33, we know that $\nabla = d + A$ defines a connection on the trivial associated vector bundle $E = M \times V$. In the context of gauge theory, this connection form A is called a *gauge field*.

We have now gathered all the elements we need to formulate a gauge theory.

4.15 Definition. A *gauge theory* is a Lagrangian theory, where the fields are given by both the gauge fields (i.e. local connection forms) of the principal G-bundle P, and the particle fields of an associated vector bundle $E = P \times_G V$, where G has a representation on V. The Lagrangian density of a gauge theory is required to be invariant under the gauge transformations given by Propositions 4.12 and 4.13.

4.3 GRAVITY

In this section we will explore different possibilities for gravitational theories. We will use the Lagrangian formalism of Section 4.1. Let M be a 4-dimensional (pseudo)-Riemannian manifold. The metric g will be considered as the dynamical quantity of the purely gravitational theory. Thus we shall study Lagrangians of the field $g_{\mu\nu}$, and subsequently apply the action principle to obtain the equations that the metric must satisfy. In order to increase the readability of this section, most of the explicit calculations have been moved to Appendix B.

We will start with the decription of Einstein gravity including a cosmological constant. Next, we will describe higher-order gravity, where the action depends quadratically on the Riemannian curvature tensor. An interesting example is Weyl gravity, for which the action is invariant under conformal transformations. These conformal transformations are discussed in the next section. We will conclude this chapter with the most general form of a gravitational action that depends at most quadratically on the Riemannian curvature tensor.

4.3.1 *Einstein gravity*

Einstein gravity is described by the *Einstein-Hilbert action*

$$S_{\text{EH}}[g] = \int_M \mathcal{L}_{\text{EH}}[g] d^4x := \int_M s\sqrt{|g|} d^4x. \tag{4.10}$$

Note that the Lagrangian density $\mathcal{L}_{\text{EH}}[g]$ contains the factor $\sqrt{|g|}$. We consider an infinitesimal variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, and require that the action is invariant under such an infinitesimal variation:

$$S_{\text{EH}}[g] = S_{\text{EH}}[g + \delta g]. \tag{4.11}$$

Since the scalar curvature s equals $g^{\mu\nu}R_{\mu\nu}$, we can rewrite the action as

$$S_{\text{EH}}[g] := \int_M g^{\mu\nu} R_{\mu\nu} \sqrt{|g|} d^4x, \tag{4.12}$$

where the Ricci tensor $R_{\mu\nu}$ is given by Eq. (B.20). The variation of the Lagrangian $\mathcal{L}_{\text{EH}}[g] = g^{\mu\nu} R_{\mu\nu}\sqrt{|g|}$ is given by

$$\delta\mathcal{L}_{\text{EH}}[g] = \delta g^{\mu\nu} R_{\mu\nu}\sqrt{|g|} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{|g|} + g^{\mu\nu} R_{\mu\nu}\delta\sqrt{|g|}. \tag{4.13}$$

By Eqs. (B.50) and (B.51), the first and last terms equal

$$\delta g^{\mu\nu} R_{\mu\nu} \sqrt{|g|} + g^{\mu\nu} R_{\mu\nu} \delta \sqrt{|g|} = \left(R_{\mu\nu} - \frac{1}{2} s g_{\mu\nu} \right) \delta g^{\mu\nu} \sqrt{|g|}. \tag{4.14}$$

By Eq. (B.62), the middle term is equal to the divergence of the vector field v given by Eq. (B.61). We conclude that the variation of the Einstein-Hilbert action is given by

$$\delta S_{\text{EH}}[g] = \int_M (R_{\mu\nu} - \frac{1}{2} s g_{\mu\nu}) \delta g^{\mu\nu} \sqrt{|g|} d^4x + \int_M \nabla \cdot v \sqrt{|g|} d^4x. \tag{4.15}$$

By the Theorem 2.55, the last term only contributes a boundary term. This boundary term vanishes, because by Definition 4.5 $g^{\mu\nu}$ as well as its first derivatives are held fixed on the boundary. The requirement that the variation of the action is zero for an arbitrary infinitesimal variation of $g^{\mu\nu}$, then yields *Einstein's equations* in vacuum:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} s g_{\mu\nu} = 0, \tag{4.16}$$

where we have introduced the *Einstein tensor* $G_{\mu\nu}$. By multiplying this with $g^{\mu\nu}$, we find $G^{\mu}{}_{\mu} = -s = 0$, so it follows that the scalar curvature vanishes identically in vacuum.

4.3.2 *Cosmological constant*

We can also add a term to the action containing the *cosmological constant* Λ_C:

$$S_C[g] := \int_M \Lambda_C \sqrt{|g|} d^4x. \tag{4.17}$$

Since the cosmological constant is independent of the metric g, applying the action principle on $S_C[g]$ simply yields

$$\Lambda_C \delta \sqrt{|g|} = -\frac{1}{2} \Lambda_C \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} = 0. \tag{4.18}$$

Since this equation holds for arbitrary variations $\delta g^{\mu\nu}$, this implies that the cosmological constant Λ_C simply must equal zero. However, if we take the combined action

$$S_{\text{EH}}[g] + S_C[g] = \int_M (s + \Lambda_C) \sqrt{|g|} d^4x, \tag{4.19}$$

this leads to a modification of Einstein's equations into

$$R_{\mu\nu} - \frac{1}{2} s g_{\mu\nu} - \frac{1}{2} \Lambda_C g_{\mu\nu} = 0. \tag{4.20}$$

In this case, multiplying with $g^{\mu\nu}$ yields $s = -2\Lambda_C$, so the scalar curvature in vacuum no longer vanishes, but is determined by the cosmological constant.

4.3.3 *Higher-order gravity*

Let us now have a look at higher-order terms for the action, that depend quadratically on the Riemann tensor. We consider the following possibilities:

$$S_1[g] := \int_M s^2 \sqrt{|g|} d^4x, \tag{4.21a}$$

$$S_2[g] := \int_M R_{\mu\nu} R^{\mu\nu} \sqrt{|g|} d^4x, \tag{4.21b}$$

$$S_3[g] := \int_M R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \sqrt{|g|} d^4x. \tag{4.21c}$$

These are not the only possible Lorentz invariant Lagrangians that depend quadratically on the Riemann tensor. In principle, one could also consider terms like $R_{\mu\nu\rho\sigma}R^{\mu\rho\nu\sigma}$, where two indices have been interchanged, or even terms of the form $\epsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}R^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\gamma\delta}R^{\alpha\beta\gamma\delta}R^{\mu\nu\rho\sigma}$, containing the completely antisymmetric Levi-Civita symbol ϵ. However, by using the (anti)symmetry relations Eq. (2.33) and the Bianchi identity Eq. (2.34), any interchange of indices can be reduced to the form Eq. (4.21c). The terms involving the Levi-Civita symbol turn out to only contribute boundary terms, as is shown in the case of R^*R^* in Section B.4.4, which will be discussed below.

The variations of the actions in Eqs. (4.21) have been calculated in the Appendix, Section B.4. We write $S_i[g] = \int_M \mathcal{L}_i[g]d^4x$. From Eqs. (B.67), (B.72) and (B.81) respectively, we have (ignoring boundary terms)

$$\delta\mathcal{L}_1[g] = \big(2sR_{\mu\nu} - 2s_{;\mu;\nu} + 2g_{\mu\nu}s^{;\beta}{}_{;\beta} - \frac{1}{2}s^2 g_{\mu\nu}\big)\sqrt{|g|}\delta g^{\mu\nu}, \tag{4.22a}$$

$$\begin{aligned}\delta\mathcal{L}_2[g] = \Big(&- R^{\alpha}{}_{\mu;\nu;\alpha} - R^{\alpha}{}_{\nu;\mu;\alpha} + R_{\mu\nu;\beta}{}^{;\beta} + \frac{1}{2}s^{;\beta}{}_{;\beta}g_{\mu\nu} \\ &+ 2R_{\mu\alpha}R^{\alpha}{}_{\nu} - \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}\Big)\sqrt{|g|}\delta g^{\mu\nu},\end{aligned} \tag{4.22b}$$

$$\begin{aligned}\delta\mathcal{L}_3[g] = \Big(&2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}{}_{;\beta;\alpha} + 2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}{}_{;\alpha;\beta} + 2R_{\mu\alpha\rho\sigma}R_{\nu}{}^{\alpha\rho\sigma} \\ &- \frac{1}{2}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}g_{\mu\nu}\Big)\sqrt{|g|}\delta g^{\mu\nu}.\end{aligned} \tag{4.22c}$$

From Corollary 2.67 we see that (on a compact oriented manifold without boundary)

$$R^*R^*\sqrt{|g|} = \mathcal{L}_1[g] - 4\mathcal{L}_2[g] + \mathcal{L}_3[g], \tag{4.23}$$

and hence that

$$S_1[g] - 4S_2[g] + S_3[g] = 8\pi^2\chi(M). \tag{4.24}$$

The Euler characteristic $\chi(M)$ is independent of the metric, and hence this particular linear combination must be insensitive to variations of the metric. So we find that $\delta(R^*R^*\sqrt{|g|})$ must vanish identically. A direct computation that $\delta(R^*R^*\sqrt{|g|})$ indeed vanishes is given in Section B.4.4. We then have

$$\delta\big(\mathcal{L}_3[g]\big) = \delta\big(4\mathcal{L}_2[g] - \mathcal{L}_1[g]\big), \tag{4.25}$$

and therefore there is no reason to consider the case of $S_3[g]$.

4.4 CONFORMAL SYMMETRY

4.4.1 *Conformal transformations*

We take M to be an m-dimensional Riemannian manifold, with a Riemannian metric g given locally by $g_{\mu\nu}$. We will consider conformal transformations of the metric, which is equivalent to a rescaling of the coordinates on the manifold. Quantities that are invariant under conformal transformations are thus scale-invariant. In this section we will show that, from the Riemannian curvature tensor, we can derive the conformally invariant Weyl tensor. Subsequently we will use this tensor to study conformal gravity. More information on conformal transformations can be found for instance in [35, Appendix C].

4.16 Definition. A *conformal transformation* of the metric is given by $g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega \in C^\infty(M, \mathbb{R}^+)$ is a smooth, strictly positive function. Note that this transformation does not change the coordinates x^μ of M, but only the geometry.

4.17 Lemma. *Let $\widetilde{\nabla}$ denote the Levi-Civita connection corresponding to the transformed metric $\widetilde{g}_{\mu\nu}$. We then have*

$$\widetilde{\nabla}_\mu \omega_\nu = \nabla_\mu \omega_\nu - \big(\delta^\rho_\nu \nabla_\mu(\ln\Omega) + \delta^\rho_\mu \nabla_\nu(\ln\Omega) - g_{\mu\nu}\nabla^\rho(\ln\Omega)\big)\omega_\rho, \tag{4.26}$$

for the Levi-Civita connection ∇ corresponding to the metric $g_{\mu\nu}$ and a 1-form $\omega = \omega_\mu\theta^\mu$.

Proof. By Lemma 2.28, the difference between two connections defines a tensor field. We will define the tensor $C^\rho{}_{\mu\nu}$ by

$$C^\rho{}_{\mu\nu}\omega_\rho := \nabla_\mu\omega_\nu - \widetilde{\nabla}_\mu\omega_\nu.$$

If we take $\omega_\mu = \nabla_\mu f = \widetilde{\nabla}_\mu f$ for some function $f \in C^\infty(M)$, we find

$$\nabla_\mu\nabla_\nu f = \widetilde{\nabla}_\mu\widetilde{\nabla}_\nu f + C^\rho{}_{\mu\nu}\nabla_\rho f.$$

∇ and $\widetilde{\nabla}$ are both torsionless, and hence $\nabla_\mu\nabla_\nu f$ and $\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu f$ are both symmetric in μ and ν. Therefore also $C^\rho{}_{\mu\nu}$ must be symmetric in μ and ν. We use the metric compatibility of $\widetilde{\nabla}$ to obtain

$$0 = \widetilde{\nabla}_\mu\widetilde{g}_{\nu\sigma} = \nabla_\mu\widetilde{g}_{\nu\sigma} - C^\rho{}_{\mu\nu}\widetilde{g}_{\rho\sigma} - C^\rho{}_{\mu\sigma}\widetilde{g}_{\nu\rho},$$

which, for $C_{\sigma\mu\nu} := C^\rho{}_{\mu\nu}\widetilde{g}_{\rho\sigma}$, yields

$$\nabla_\mu\widetilde{g}_{\nu\sigma} = C_{\sigma\mu\nu} + C_{\nu\mu\sigma}.$$

We can then calculate (using the symmetry of $C^\rho{}_{\mu\nu}$) the particular combination

$$\nabla_\mu\widetilde{g}_{\nu\sigma} + \nabla_\nu\widetilde{g}_{\mu\sigma} - \nabla_\sigma\widetilde{g}_{\mu\nu} = C_{\sigma\mu\nu} + C_{\nu\mu\sigma} + C_{\sigma\nu\mu} + C_{\mu\nu\sigma} - C_{\nu\sigma\mu} - C_{\mu\sigma\nu} = 2C_{\sigma\mu\nu},$$

and have thus obtained an explicit formula for $C^\rho{}_{\mu\nu}$. Using the metric compatibility of ∇ we have

$$\nabla_\mu\widetilde{g}_{\nu\sigma} = \nabla_\mu\big(\Omega^2 g_{\nu\sigma}\big) = 2\Omega\nabla_\mu\Omega g_{\nu\sigma}.$$

Hence, we finally obtain

$$\begin{aligned} C^\rho{}_{\mu\nu} = \widetilde{g}^{\rho\sigma}C_{\sigma\mu\nu} = \Omega^{-2}g^{\rho\sigma}C_{\sigma\mu\nu} \\ = \delta^\rho_\nu\nabla_\mu(\ln\Omega) + \delta^\rho_\mu\nabla_\nu(\ln\Omega) - g_{\mu\nu}\nabla^\rho(\ln\Omega). \end{aligned} \tag{4.27}$$

Comparing this to the definition of $C^\rho{}_{\mu\nu}$ yields the desired formula. □

4.18 Proposition. *We write $\widetilde{R}^\mu{}_{\nu\rho\sigma}$ for (the components of) the Riemannian curvature tensor corresponding to the transformed metric $\widetilde{g}_{\mu\nu}$. Similarly we write $\widetilde{R}_{\nu\sigma}$ and $\widetilde{s}$ for the transformed versions of the Ricci tensor and the scalar curvature, respectively. We then obtain*

$$\begin{aligned}
\widetilde{R}^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} &+ \delta^\mu_\sigma\nabla_\rho\nabla_\nu(\ln\Omega) - g_{\sigma\nu}\nabla_\rho\nabla^\mu(\ln\Omega) \\
&- \delta^\mu_\rho\nabla_\sigma\nabla_\nu(\ln\Omega) + g_{\rho\nu}\nabla_\sigma\nabla^\mu(\ln\Omega) \\
&- \delta^\mu_\sigma\nabla_\rho(\ln\Omega)\nabla_\nu(\ln\Omega) + \delta^\mu_\rho\nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega) \\
&+ \delta^\mu_\sigma g_{\rho\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega) - \delta^\mu_\rho g_{\sigma\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega) \\
&- g_{\rho\nu}\nabla_\sigma(\ln\Omega)\nabla^\mu(\ln\Omega) + g_{\sigma\nu}\nabla_\rho(\ln\Omega)\nabla^\mu(\ln\Omega), \\
\widetilde{R}_{\nu\sigma} = R_{\nu\sigma} &- (m-2)\nabla_\sigma\nabla_\nu(\ln\Omega) - g_{\sigma\nu}\nabla_\mu\nabla^\mu(\ln\Omega) \\
&+ (m-2)\nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega) - (m-2)g_{\sigma\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega), \\
\widetilde{s} = \Omega^{-2}&\Big(s - 2(m-1)\nabla^\beta\nabla_\beta(\ln\Omega) - (m-1)(m-2)\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\Big).
\end{aligned}$$

PROOF. We start by calculating

$$\begin{aligned}
\widetilde{\nabla}_\rho\widetilde{\nabla}_\sigma\omega_\nu &= \widetilde{\nabla}_\rho\left(\nabla_\sigma\omega_\nu - C^\mu{}_{\sigma\nu}\omega_\mu\right)\\
&= \nabla_\rho\nabla_\sigma\omega_\nu - C^\mu{}_{\rho\sigma}\nabla_\mu\omega_\nu - C^\mu{}_{\rho\nu}\nabla_\sigma\omega_\mu - \nabla_\rho(C^\mu{}_{\sigma\nu}\omega_\mu)\\
&\quad + C^\beta{}_{\rho\sigma}C^\mu{}_{\beta\nu}\omega_\mu + C^\beta{}_{\rho\nu}C^\mu{}_{\sigma\beta}\omega_\mu\\
&= \nabla_\rho\nabla_\sigma\omega_\nu - C^\mu{}_{\rho\sigma}\nabla_\mu\omega_\nu - C^\mu{}_{\rho\nu}\nabla_\sigma\omega_\mu - C^\mu{}_{\sigma\nu}\nabla_\rho\omega_\mu\\
&\quad + \Big(-\nabla_\rho C^\mu{}_{\sigma\nu} + C^\beta{}_{\rho\sigma}C^\mu{}_{\beta\nu} + C^\beta{}_{\rho\nu}C^\mu{}_{\sigma\beta}\Big)\omega_\mu.
\end{aligned}$$

Using Lemma 2.44, we find

$$\widetilde{R}^\mu{}_{\nu\rho\sigma}\omega_\mu = \Big(R^\mu{}_{\nu\rho\sigma} + \nabla_\rho C^\mu{}_{\sigma\nu} - \nabla_\sigma C^\mu{}_{\rho\nu} - C^\beta{}_{\rho\nu}C^\mu{}_{\sigma\beta} + C^\beta{}_{\sigma\nu}C^\mu{}_{\rho\beta}\Big)\omega_\mu.$$

Inserting Eq. (4.27), we obtain

$$\begin{aligned}
\widetilde{R}^\mu{}_{\nu\rho\sigma} &= R^\mu{}_{\nu\rho\sigma} + \delta^\mu_\sigma\nabla_\rho\nabla_\nu(\ln\Omega) - g_{\sigma\nu}\nabla_\rho\nabla^\mu(\ln\Omega) - \delta^\mu_\rho\nabla_\sigma\nabla_\nu(\ln\Omega)\\
&\quad + g_{\rho\nu}\nabla_\sigma\nabla^\mu(\ln\Omega) - \delta^\mu_\sigma\nabla_\rho(\ln\Omega)\nabla_\nu(\ln\Omega) + \delta^\mu_\rho\nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega)\\
&\quad + \delta^\mu_\sigma g_{\rho\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega) - \delta^\mu_\rho g_{\sigma\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\\
&\quad - g_{\rho\nu}\nabla_\sigma(\ln\Omega)\nabla^\mu(\ln\Omega) + g_{\sigma\nu}\nabla_\rho(\ln\Omega)\nabla^\mu(\ln\Omega).
\end{aligned}$$

By contracting over the indices μ and ρ, we find

$$\begin{aligned}
\widetilde{R}_{\nu\sigma} &= R_{\nu\sigma} + \nabla_\sigma\nabla_\nu(\ln\Omega) - g_{\sigma\nu}\nabla_\mu\nabla^\mu(\ln\Omega) - m\nabla_\sigma\nabla_\nu(\ln\Omega) + \nabla_\sigma\nabla_\nu(\ln\Omega)\\
&\quad - \nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega) + m\nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega) - g_{\sigma\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\\
&\quad - mg_{\sigma\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega) - \nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega) + g_{\sigma\nu}\nabla_\mu(\ln\Omega)\nabla^\mu(\ln\Omega)\\
&= R_{\nu\sigma} - (m-2)\nabla_\sigma\nabla_\nu(\ln\Omega) - g_{\sigma\nu}\nabla_\mu\nabla^\mu(\ln\Omega)\\
&\quad + (m-2)\nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega) - (m-2)g_{\sigma\nu}\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega).
\end{aligned}$$

Contracting this transformed Ricci tensor with $\widetilde{g}^{\nu\sigma} = \Omega^{-2}g^{\nu\sigma}$ yields

$$\widetilde{s} = \Omega^{-2}\Big(s - 2(m-1)\nabla^\beta\nabla_\beta(\ln\Omega) - (m-1)(m-2)\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\Big). \qquad \square$$

4.19 Proposition. *The Weyl tensor $C^\mu{}_{\nu\rho\sigma}$ is conformally invariant.*

PROOF. We use Eq. (B.41) to obtain the transformed Weyl tensor, which is given by

$$\begin{aligned}
\widetilde{C}^\mu{}_{\nu\rho\sigma} = \widetilde{R}^\mu{}_{\nu\rho\sigma} &- \frac{1}{(m-1)(m-2)}\widetilde{s}\widetilde{g}^{\mu\alpha}(\widetilde{g}_{\alpha\sigma}\widetilde{g}_{\nu\rho} - \widetilde{g}_{\alpha\rho}\widetilde{g}_{\nu\sigma})\\
&- \frac{1}{m-2}\widetilde{g}^{\mu\alpha}(\widetilde{g}_{\alpha\rho}\widetilde{R}_{\nu\sigma} - \widetilde{g}_{\alpha\sigma}\widetilde{R}_{\nu\rho} - \widetilde{g}_{\nu\rho}\widetilde{R}_{\alpha\sigma} + \widetilde{g}_{\nu\sigma}\widetilde{R}_{\alpha\rho}).
\end{aligned}$$

Using the results obtained in the previous proposition, we find after rearranging the terms that

$$\begin{aligned}
\widetilde{C}^\mu{}_{\nu\rho\sigma} &= C^\mu{}_{\nu\rho\sigma} - \left(\frac{2}{m-2} - \frac{2(m-1)}{(m-1)(m-2)}\right)\Big(\delta^\mu_\sigma g_{\rho\nu} - \delta^\mu_\rho g_{\sigma\nu}\Big)\nabla^\beta\nabla_\beta(\ln\Omega)\\
&\quad - \left(2 - \frac{2(m-2)}{m-2}\right)\Big(\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\sigma g_{\rho\nu}\Big)\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\\
&\quad - \left(1 - \frac{m-2}{m-2}\right)\Big(-\delta^\mu_\sigma\nabla_\rho\nabla_\nu(\ln\Omega) + g_{\sigma\nu}\nabla_\rho\nabla^\mu(\ln\Omega) + \delta^\mu_\rho\nabla_\sigma\nabla_\nu(\ln\Omega)\\
&\quad - g_{\rho\nu}\nabla_\sigma\nabla^\mu(\ln\Omega) + \delta^\mu_\sigma\nabla_\rho(\ln\Omega)\nabla_\nu(\ln\Omega) - \delta^\mu_\rho\nabla_\sigma(\ln\Omega)\nabla_\nu(\ln\Omega)\\
&\quad + g_{\rho\nu}\nabla_\sigma(\ln\Omega)\nabla^\mu(\ln\Omega) - g_{\sigma\nu}\nabla_\rho(\ln\Omega)\nabla^\mu(\ln\Omega)\Big).
\end{aligned}$$

Hence we see that all extra terms cancel each other, and we conclude that indeed $\widetilde{C}^\mu{}_{\nu\rho\sigma} = C^\mu{}_{\nu\rho\sigma}$. $\square$

4.4.2 *Conformal gravity*

4.20 Definition. We define the *Weyl action* by

$$S_W[g] := \int_M C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}d^4x. \tag{4.29}$$

From Eq. (B.43) we see that this gravitational action can be written as the linear combination $S_W = \frac{1}{3}S_1 - 2S_2 + S_3$, where the S_i have been defined in Eq. (4.21). We will show in the next proposition that this Weyl action is conformally invariant, and for this reason it is also called the action of *conformal gravity*.

4.21 Proposition. *In the case* $\dim(M) = m = 4$, *the Weyl action is conformally invariant.*

PROOF. By Proposition 4.19 we know that the Weyl tensor is conformally invariant. However, since the metric does transform under conformal transformations, this invariance depends on the position of the indices. We can calculate that

$$\widetilde{C}_{\mu\nu\rho\sigma} = \widetilde{g}_{\mu\alpha}\widetilde{C}^{\alpha}{}_{\nu\rho\sigma} = \Omega^2 g_{\mu\alpha}C^{\alpha}{}_{\nu\rho\sigma} = \Omega^2 C_{\mu\nu\rho\sigma}.$$

Similarly, since $\widetilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}$, we have

$$\widetilde{C}^{\mu\nu\rho\sigma} = \Omega^{-6}C^{\mu\nu\rho\sigma}.$$

We use the definition of the determinant $|g|$ of the metric, given by Eq. (B.28), to see that $|\widetilde{g}| = \Omega^8|g|$, and hence that $\sqrt{|\widetilde{g}|} = \Omega^4\sqrt{|g|}$. Combining this we find that

$$\widetilde{C}_{\mu\nu\rho\sigma}\widetilde{C}^{\mu\nu\rho\sigma}\sqrt{|\widetilde{g}|} = \Omega^2 C_{\mu\nu\rho\sigma}\Omega^{-6}C^{\mu\nu\rho\sigma}\Omega^4\sqrt{|g|} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}. \qquad \square$$

The variation of the Weyl action is calculated in the Appendix, Section B.4.5. We know from Eq. (B.96) that it is given by

$$\delta\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}\right) = \left(4C_{\mu\rho\nu\sigma}{}^{;\rho;\sigma} + 2C_{\mu\rho\nu\sigma}R^{\rho\sigma}\right)\sqrt{|g|}\delta g^{\mu\nu}. \tag{4.30}$$

4.5 THE GRAVITATIONAL LAGRANGIAN

Let us now summarize the results of this chapter by considering the most general gravitational Lagrangian that depends at most quadratically on the Riemannian curvature. Such an action can be written as a linear combination of the form

$$aS_{EH}[g] + bS_C[g] + cS_1[g] + dS_2[g] + eS_3[g], \tag{4.31}$$

for arbitrary constants a, b, c, d, e and where the actions are given by Eqs. (4.11), (4.17) and (4.21). The overall constant is irrelevant, so we are free to choose $a = 1$. The constant b can be absorbed into the cosmological constant Λ_C. From Eq. (4.24) we know that the action S_3 is a linear combination of S_1, S_2 and the Euler characteristic. We can therefore write the general action as

$$S_{EH}[g] + S_C[g] + c'S_1[g] + d'S_2[g], \tag{4.32}$$

for new constants c' and d'. We can rewrite this further by using the Weyl action S_W of Eq. (4.29). By Eq. (B.43) we have

$$S_W[g] = \frac{1}{3}S_1[g] - 2S_2[g] + S_3[g]. \tag{4.33}$$

Using again Eq. (4.24), this can be rewritten as

$$S_W[g] = -\frac{2}{3}S_1[g] + 2S_2[g] + 8\pi^2\chi(M) \tag{4.34}$$

Since the variation of the Euler characteristic must vanish identically, it is irrelevant, and therefore we can replace the action S_2 by a linear combination of S_1 and S_W. This leads us to the following result.

4.22 Proposition. *The most general gravitational Lagrangian that depends at most quadratically on the Riemannian curvature is given by the combination*

$$S_{EH}[g] + S_C[g] + \alpha S_1[g] + \beta S_W[g] = \int_M \left(s + \Lambda_C + \alpha s^2 + \beta C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \sqrt{|g|} d^4x,$$

for arbitrary constants α and β.

Part II

NONCOMMUTATIVE GEOMETRY

Mathematics takes us into the region of absolute necessity, to which not only the actual word, but every possible word, must conform.

— Bertrand Russell

5 INTRODUCTION TO NONCOMMUTATIVE GEOMETRY

In this chapter we will give an introduction to the main concepts of noncommutative geometry. We will start with the motivating example, the canonical spectral triple, which describes a Riemannian spin manifold. We will briefly list the key steps that were taken to arrive at this example. In the next few sections we will give the definition of a spectral triple, and develop some useful tools. Subsequently, we describe the spectral action and the fermionic action. These actions are crucial for the applications of spectral triples as models in physics. We end the chapter with a discussion of gauge symmetry in spectral triples, and we show how a gauge group and gauge transformations are obtained from a spectral triple.

There are two books that have been of great help in writing this chapter. Much material of this chapter has been taken from [13]. A good introduction to noncommutative geometry can be found in [19].

5.1 A SHORT MOTIVATION

In this section we will briefly review the basic steps that lead us from topology and geometry to the spectral triples of noncommutative geometry. We will not go into detail, and for more information on these topics we refer to [19].

The framework of noncommutative geometry provides a generalization of ordinary Riemannian geometry, by replacing topological and geometrical objects by algebraic objects. The first step is to replace a topological space by its algebra of continuous functions. For a compact Hausdorff space X, the algebra of continuous functions $C(X)$ turns out to be a commutative unital C^*-algebra. Let $M(A)$ denote the set of characters of a Banach algebra A, i.e. the set of nonzero algebra homomorphisms $\mu\colon A \to \mathbb{C}$. It can be shown that every character is automatically continuous, and that $M(A)$ is a topological space (see [19, §1.2]). The *Gelfand transform* of $a \in A$ is the function $\hat{a}\colon M(A) \to \mathbb{C}$ given by $\hat{a}(\mu) := \mu(a)$.

5.1 Theorem (Gelfand-Naimark [19, Theorem 1.4])**.** *If A is a commutative C^*-algebra, the Gelfand transformation is an isometric $*$-isomorphism of A onto $C_0(M(A))$.*

Here, C_0 denotes the algebra of continuous functions that "vanish at infinity." This theorem implies that any Hausdorff space X can equivalently be described by its algebra of functions $C_0(X)$. Any commutative C^*-algebra corresponds to such a space. The basic idea of noncommutative geometry is to generalize the notion of topological spaces by dropping the commutativity of the C^*-algebra. From this perspective, we will consider a noncommutative C^*-algebra to describe a general "noncommutative space."

A major ingredient in differential geometry is given by vector bundles over differentiable manifolds. The next key step is to find an algebraic description of these vector bundles. This description arises naturally by looking at the continuous sections $\Gamma(E)$ of a vector bundle $E \to M$. The sections $\Gamma(E)$ are easily seen to form a module over the algebra of functions $C(M)$.

5.2 Theorem (Serre-Swan [19, Theorem 2.10])**.** *The functor Γ from vector bundles over a compact space M to finitely generated projective modules over $C(M)$ is an equivalence of categories.*

In order to be able to speak of noncommutative geometry, we still need a way to express the geometry on a topological space in algebraic terms. The last key step is thus to rewrite the notion of distance between two points in algebraic terms. The usual geodesic distance between the points x and y is given by

$$d_g(x,y) = \inf \int_\gamma ds, \tag{5.1}$$

where $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and the infimum is taken over all paths from x to y. Let us now consider the (canonical) Dirac operator $\not{D}$ on a Riemannian spin manifold M, defined in Definition 3.47. Let $\mathcal{H} = L^2(M,S)$ be the Hilbert space of square integrable spinors, with the algebra $\mathcal{A} = C^\infty(M)$ acting on $\mathcal{H}$ as multiplication operators by the pointwise product $(a\psi)(x) := a(x)\psi(x)$. It is shown in [13, Prop. 1.119] that the formula

$$d_{\not{D}}(x,y) = \sup\left\{|a(x) - a(y)| \colon a \in \mathcal{A}, \|[\not{D}, a]\| \leq 1\right\}, \tag{5.2}$$

is equal to the geodesic distance. In this way, a generalized notion of geometry is obtained by focusing on the Dirac operator instead of the metric. To conclude this section, we see that the geometry of a Riemannian spin manifold M can be completely described by an algebra, a Hilbert space (which forms a module over the algebra) and a Dirac operator, given by

$$\left(C^\infty(M), L^2(M,S), \not{D}\right). \tag{5.3}$$

5.2 spectral triples

In this section we shall follow the definitions of [13, Ch. 1, §10].

5.3 Definition. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint (in general unbounded) operator D with compact resolvent (i.e. $(1 + D^2)^{-1/2}$ is a compact operator) such that all commutators $[D, a]$ are bounded for $a \in \mathcal{A}$.

The operator D is called the *Dirac operator* of the spectral triple. However, it should be noted that this is a generalization of Dirac operators, and D need not be a Dirac operator in the sense of Definition 2.70. We will show later in Proposition 6.25, that in the case of an almost commutative geometry the (fluctuated) Dirac operator does satisfy Definition 2.70.

The actual algebra that we are interested in is not $\mathcal{A}$, but its embedding $\pi(\mathcal{A})$ as a subalgebra of the bounded operators $B(\mathcal{H})$. We will usually not make a distinction between $\mathcal{A}$ and $\pi(\mathcal{A})$, and simply consider $\mathcal{A}$ as a subalgebra of $B(\mathcal{H})$.

The algebra $\mathcal{A}$ can be either real or complex, and furthermore, for a complex algebra, the representation $\pi \colon \mathcal{A} \to B(\mathcal{H})$ on the Hilbert space can be either real- or complex-linear. If we have a real-linear representation $\pi \colon \mathcal{A} \to B(\mathcal{H})$ of a complex algebra $\mathcal{A}$, the algebra $\pi(\mathcal{A})$ becomes a real algebra. In this case, we will then treat $\mathcal{A}$ as a real algebra instead of a complex algebra. From here on, if we say that $\mathcal{A}$ is a complex algebra, this implicitly means that π is complex-linear and hence $\pi(\mathcal{A})$ is also a complex algebra.

5.4 Definition. A spectral triple is *even* if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}/2$-grading γ which commutes with any $a \in \mathcal{A}$ and anticommutes with D. γ decomposes the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into its two eigenspaces, where $\mathcal{H}^\pm := \{\psi \in \mathcal{H} \mid \gamma\psi = \pm\psi\}$.

5.5 Definition. An (even) spectral triple has a *real structure* if there is an antilinear isomorphism $J \colon \mathcal{H} \to \mathcal{H}$ with $J^2 = \varepsilon$, $JD = \varepsilon' DJ$ and, if the spectral triple is even,

$J\gamma = \varepsilon''\gamma J$. The signs ε, ε' and ε'' determine the *KO-dimension* n modulo 8 of the spectral triple, according to the following table.

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

Moreover, the action of $\mathcal{A}$ satisfies the commutation rule

$$[a, b^0] = 0 \qquad \forall a, b \in \mathcal{A}, \tag{5.4}$$

where we have defined the right action b^0 of b by

$$b^0 := Jb^*J^{-1}. \tag{5.5}$$

The operator D satisfies the so-called *order one condition*

$$\big[[D, a], b^0\big] = 0 \qquad \forall a, b \in \mathcal{A}. \tag{5.6}$$

In Section 5.1 we have already considered the data $\big(C^\infty(M), L^2(M, S), \not{D}\big)$ as the description of a Riemannian spin manifold. Let us return to this motivating example for spectral triples, and also assign it with a grading and a real structure.

5.6 Definition. We now take M to be a compact even-dimensional spin manifold, and define the *canonical triple* by

$$\big(\mathcal{A}, \mathcal{H}, D\big) = \big(C^\infty(M), L^2(M, S), \not{D}\big), \tag{5.7}$$

where $\not{D}$ is the canonical Dirac operator of Definition 3.47. We have a $\mathbb{Z}/2$-grading given by $\gamma_M := c(\gamma)$, where c is the Clifford representation and γ is defined in Definition 3.10. Furthermore we have an antilinear isometry J_M, which is the charge conjugation operator on M (cf. Proposition 3.41).

5.7 Proposition. *The canonical triple defines a real even spectral triple of KO-dimension* $m = \dim M$.

Proof. For a detailed proof, we refer the reader to [19, Theorem 9.20]. □

On the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, we can write every operator on $\mathcal{H}$ as a 2×2-matrix. Let us evaluate the form of the operators discussed above. We have by definition of the decomposition that $\gamma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$. We obtain the following result:

5.8 Proposition. *For an even spectral triple, we can write* $D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$ *and* $a = \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix}$ *for any* $a \in \mathcal{A}$.

Proof. Since D is selfadjoint, it must have the form $\begin{pmatrix} s^+ & d \\ d^* & s^- \end{pmatrix}$ for selfadjoint operators $s^\pm$ on $\mathcal{H}^\pm$. We calculate

$$\begin{aligned} 0 = D\gamma + \gamma D &= \begin{pmatrix} s^+ & d \\ d^* & s^- \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} s^+ & d \\ d^* & s^- \end{pmatrix} \\ &= \begin{pmatrix} s^+ & -d \\ d^* & -s^- \end{pmatrix} + \begin{pmatrix} s^+ & d \\ -d^* & -s^- \end{pmatrix} = \begin{pmatrix} 2s^+ & 0 \\ 0 & -2s^- \end{pmatrix}. \end{aligned}$$

Since γ must anticommute with D, the above must vanish identically, so we conclude that $s^+ = 0$ and $s^- = 0$.

We write $a = \begin{pmatrix} a^+ & a^1 \\ a^2 & a^- \end{pmatrix}$ for an element $a \in \mathcal{A}$, and calculate

$$0 = a\gamma - \gamma a = \begin{pmatrix} a^+ & a^1 \\ a^2 & a^- \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} a^+ & a^1 \\ a^2 & a^- \end{pmatrix}$$
$$= \begin{pmatrix} a^+ & -a^1 \\ a^2 & -a^- \end{pmatrix} - \begin{pmatrix} a^+ & a^1 \\ -a^2 & -a^- \end{pmatrix} = \begin{pmatrix} 0 & -2a^1 \\ 2a^2 & 0 \end{pmatrix}.$$

Since this commutator must vanish, we conclude that $a^1 = a^2 = 0$. □

5.2.1 *Finite spectral triples*

5.9 Definition. A real even finite spectral triple is given by the data

$$(\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F)\,, \tag{5.8}$$

for a finite dimensional Hilbert space $\mathcal{H}_F$. The operators D_F, γ_F and J_F satisfy the relations $J_F^2 = \varepsilon$, $J_F D_F = \varepsilon' D_F J_F$ and $J_F \gamma_F = \varepsilon'' \gamma_F J_F$.

Recall from Proposition 5.8 that we can write $D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$ and $a = \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix}$ for any $a \in \mathcal{A}$ on the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. We shall now evaluate in the finite case, what general form the operator J_F can have for the different even KO-dimensions.

5.10 Proposition. *For a real even finite spectral triple, we can write*

KO-dimension 0: $J_F = \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix} C$ for symmetric $j_\pm \in U(\mathcal{H}^\pm)$;

KO-dimension 2: $J_F = \begin{pmatrix} 0 & j \\ -j^T & 0 \end{pmatrix} C$ for $jj^* = j^*j = \mathbb{1}$;

KO-dimension 4: $J_F = \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix} C$ for anti-symmetric $j_\pm \in U(\mathcal{H}^\pm)$;

KO-dimension 6: $J_F = \begin{pmatrix} 0 & j \\ j^T & 0 \end{pmatrix} C$ for $jj^* = j^*j = \mathbb{1}$.

Proof. Let the operator C denote complex conjugation. Then any antiunitary operator J_F can be written as UC, where U is some unitary operator on $\mathcal{H}_F$. We then have $J_F^* = CU^* = U^T C$, and $J_F J_F^* = UU^* = \mathbb{1}$.

The different possibilities for the choice of J_F are characterized by $J_F^2 = UCUC = U\overline{U} = \epsilon$ and $J_F \gamma_F = \epsilon'' \gamma_F J_F$. We shall have a look at each possibility separately.

KO-DIMENSION 0

In this case we have $J_F \gamma_F = \gamma_F J_F$, which implies we can write $J_F = \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix} C$, where $j_\pm \in \mathcal{H}_F^\pm$. Its adjoint is given by $J_F^* = \begin{pmatrix} j_+^T & 0 \\ 0 & j_-^T \end{pmatrix} C$. Since $J_F J_F^* = 1$ must hold, we find $j_\pm \in U(\mathcal{H}_F^\pm)$. The requirement that in this case we have $J_F^2 = \mathbb{1}$, implies $j_\pm \overline{j_\pm} = \mathbb{1}_{\mathcal{H}_F^\pm}$. So we must have that $j_\pm = j_\pm^T$.

KO-DIMENSION 2

In this case we have $J_F \gamma_F = -\gamma_F J_F$, which implies that we can write $J_F = \begin{pmatrix} 0 & j_1 \\ j_2 & 0 \end{pmatrix} C$, for operators $j_1 \colon \mathcal{H}_F^- \to \mathcal{H}_F^+$ and $j_2 \colon \mathcal{H}_F^+ \to \mathcal{H}_F^-$. Its adjoint is given by $J_F^* = \begin{pmatrix} 0 & j_2^T \\ j_1^T & 0 \end{pmatrix} C = J_F^T$. Since $J_F J_F^* = \mathbb{1}$, we find that $j_1 j_1^* = j_2 j_2^* = \mathbb{1}$. The requirement $J_F^2 = -\mathbb{1}$ gives $j_1 \overline{j_2} = j_2 \overline{j_1} = -\mathbb{1}$, which yields $j_2 = -j_1^T$.

KO-DIMENSION 4
: As for KO-dimension 0, we have $J_F = \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix} C$ for $j_\pm \in U(\mathcal{H}_F^\pm)$. The requirement that in this case we have $J_F^2 = -\mathbb{1}$, implies $j_\pm \overline{j_\pm} = -\mathbb{1}_{\mathcal{H}_F^\pm}$. So we must have that $j_\pm = -j_\pm^T$.

KO-DIMENSION 6
: As for KO-dimension 2, we have $J_F = \begin{pmatrix} 0 & j_1 \\ j_2 & 0 \end{pmatrix} C$ with $j_1 j_1^* = j_2 j_2^* = \mathbb{1}$. Now the requirement $J^2 = \mathbb{1}$ gives $j_1 \overline{j_2} = j_2 \overline{j_1} = \mathbb{1}$, which yields $j_2 = j_1^T$. □

5.11 Remark. Note that we have not yet used all aspects of the definition of a spectral triple. There are still two commutation rules that are required to be satisfied, namely

$$\begin{aligned} [a, b^0] &= 0 \qquad \forall a, b \in \mathcal{A}_F, \\ \big[[D_F, a], b^0\big] &= 0 \qquad \forall a, b \in \mathcal{A}_F, \end{aligned}$$

where $b^0 := J_F b^* J_F^*$. Furthermore, we must have $J_F D_F = D_F J_F$ for even KO-dimensions. We will not examine the precise implications of these commutation rules, but one should be aware that these rules impose further restrictions on the operators D_F and J_F. Later on in Proposition 7.2, we will use these restrictions to show that the two-point space does not simultaneously allow a real structure J_F and a non-zero Dirac operator D_F.

5.3 SUBGROUPS AND SUBALGEBRAS

5.12 Definition. We define two subalgebras of $\mathcal{A}$ by

$$\mathcal{A}_J := \{a \in \mathcal{A} \mid aJ = Ja\} = \{a \in \mathcal{A} \mid a^0 = a^*\}, \tag{5.9}$$

$$\widetilde{\mathcal{A}}_J := \{a \in \mathcal{A} \mid aJ = Ja^*\} = \{a \in \mathcal{A} \mid a^0 = a\}. \tag{5.10}$$

The definition of $\mathcal{A}_J$ is taken from [13, Prop. 1.125]. We have provided a similar but different definition for $\widetilde{\mathcal{A}}_J$, since this subalgebra will turn out to be very useful for the description of the gauge group in Section 5.5.2.

5.13 Proposition. *For a complex algebra $\mathcal{A}$, the following two statements hold.*

1. *The subalgebra $\mathcal{A}_J$ is an involutive commutative* real *subalgebra of the center of $\mathcal{A}$.*
2. *The subalgebra $\widetilde{\mathcal{A}}_J$ is an involutive commutative* complex *subalgebra of the center of $\mathcal{A}$.*

PROOF.
1. By construction, $\mathcal{A}_J$ is a real subalgebra of $\mathcal{A}$. One has $(JaJ^{-1})^* = Ja^*J^{-1}$ for all $a \in \mathcal{A}$. If $a \in \mathcal{A}_J$, one then has $JaJ^{-1} = a$ and hence $Ja^*J^{-1} = (JaJ^{-1})^* = a^*$, so $a^* \in \mathcal{A}_J$ and we have shown that $\mathcal{A}_J$ is involutive. For any $a, b \in \mathcal{A}$ we have $[b, a^0] = 0$ by Eq. (5.4). For $a \in \mathcal{A}_J$, we have $a^0 = Ja^*J^{-1} = a^*$, so $[b, a^*] = 0$ for any $a^* \in \mathcal{A}_J$ and $b \in \mathcal{A}$. Hence $\mathcal{A}_J$ is contained in the center of $\mathcal{A}$.
2. We easily see that $aJ = Ja^*$ implies $a = Ja^*J^* = a^0$ and vice versa. Since we must have $[a, b^0] = 0$ for any $a, b \in \mathcal{A}$, we have $[a, b] = 0$ for any $a \in \mathcal{A}$ and $b \in \widetilde{\mathcal{A}}_J$, so $\widetilde{\mathcal{A}}_J$ is contained in the center of $\mathcal{A}$. The requirement $a = a^0$ is complex linear, and also implies that $a^* = (a^0)^* = (a^*)^0$, so we have $a^* \in \widetilde{\mathcal{A}}_J$ for $a \in \widetilde{\mathcal{A}}_J$. Finally, we check that for $a, b \in \widetilde{\mathcal{A}}_J$, we find $(ab)^0 = b^0 a^0 = ba = ab$, so $ab \in \widetilde{\mathcal{A}}_J$. □

5.14 Definition. The *unitary group* $U(\mathcal{A})$ of a unital, involutive algebra $\mathcal{A}$ is defined by

$$U(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = 1\}. \tag{5.11}$$

For a finite-dimensional algebra $\mathcal{A}_F$ with a representation $\pi\colon \mathcal{A}_F \to B(\mathcal{H}_F)$ on a finite Hilbert space $\mathcal{H}_F$, we can define the determinant of an element $a \in \mathcal{A}_F$ as $\det(\pi(a))$. Usually, we will not explicitly mention the representation π and simply write $\det(a)$. We can then define the *special unitary group* $SU(\mathcal{A}_F)$ by

$$SU(\mathcal{A}_F) = \{u \in U(\mathcal{A}_F) \mid \det(u) = 1\}. \tag{5.12}$$

The condition $\det(u) = 1$ is also known as the *unimodularity condition*.

If the algebra $\mathcal{A}$ is represented on an infinite dimensional Hilbert space $\mathcal{H}$, there is no clear notion of a determinant, and hence we are unable to define the special unitary group of $\mathcal{A}$.

5.3.1 *The adjoint action*

Similar to Definition 2.3, we define the adjoint action of $u \in U(\mathcal{A})$ by

$$\mathrm{Ad}\,(u)P = uPu^* \tag{5.13}$$

for an operator P on $\mathcal{H}$. Similar to Eq. (2.4), we can also define the adjoint map $\mathrm{ad} : \mathfrak{u}(\mathcal{A}) \to \mathrm{End}(B(\mathcal{H}))$ by

$$(\mathrm{ad}\,X)(P) = [X, P], \tag{5.14}$$

for a bounded operator P on $\mathcal{H}_F$.

Let us consider a finite-dimensional algebra $\mathcal{A}_F$, that is represented on an N-dimensional Hilbert space $\mathcal{H}_F$. The group $U(\mathcal{A}_F)$ is then a subgroup of the matrix Lie group $B(\mathcal{H}_F) = \mathrm{GL}(\mathcal{H}_F) = M_N(\mathbb{C})$. Using the exponential map of Definition 2.5, any element A of the Lie algebra $\mathfrak{u}(\mathcal{A}_F)$ of $U(\mathcal{A}_F)$ defines an element $u = \exp(A) \in U(\mathcal{A}_F)$. This unitary element satisfies $u^* = u^{-1}$, and for A this implies that $\exp\big(A^*\big) = u^* = u^{-1} = \exp(-A)$, hence $A^* = -A$. If $u \in SU(\mathcal{A}_F)$, we have $1 = \det\big(\exp(A)\big) = \exp\big(\mathrm{Tr}(A)\big)$. Hence, for elements $A \in \mathfrak{su}(\mathcal{A}_F)$, we can also state the *unimodularity condition* as

$$\mathrm{Tr}(A) = 0. \tag{5.15}$$

For a real spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F, J_F)$, the operator J_F provides a right action of $a \in \mathcal{A}_F$ on $\mathcal{H}_F$ by $a^0 = J_F a^* J_F^*$, as in Eq. (5.5). Using this right action, we can define maps $\widetilde{\mathrm{Ad}} : U(\mathcal{A}_F) \to \mathrm{End}(\mathcal{H}_F)$ and $\widetilde{\mathrm{ad}} : \mathfrak{u}(\mathcal{A}_F) \to \mathrm{End}(\mathcal{H}_F)$ by

$$\begin{aligned} (\widetilde{\mathrm{Ad}}\,u)\xi &:= u\xi u^* = u(u^*)^0\xi, \\ (\widetilde{\mathrm{ad}}\,A)\xi &:= A\xi - \xi A = (A - A^0)\xi, \end{aligned} \tag{5.16}$$

for $\xi \in \mathcal{H}_F$. By inserting $a^0 = J_F a^* J_F^*$, we obtain

$$\begin{aligned} &\widetilde{\mathrm{Ad}}\,u = uJuJ^*, \\ &\widetilde{\mathrm{ad}}\,A = A - JA^*J^{-1} = A + JAJ^{-1}, \qquad \text{for } A^* = -A. \end{aligned} \tag{5.17}$$

If we would replace $A = iB$, we could then define

$$\widetilde{\mathrm{ad}}\,B := -i\widetilde{\mathrm{ad}}\,(iB) = B - JBJ^{-1}, \qquad \text{for } B^* = B. \tag{5.18}$$

These maps look very similar to the adjoint maps Ad and ad, but are nevertheless different, since $\widetilde{\mathrm{Ad}}\,u$ and $\widetilde{\mathrm{ad}}\,A$ act on the Hilbert space $\mathcal{H}_F$, whereas $\mathrm{Ad}\,u$ and $\mathrm{ad}\,A$ act on operators on $\mathcal{H}_F$. The maps $\widetilde{\mathrm{Ad}}\,u$ and $\widetilde{\mathrm{ad}}\,A$ are often also called the *adjoint representations* of u and A, respectively, on the Hilbert space $\mathcal{H}_F$.

5.4 THE ACTION FUNCTIONAL

In this section we will first define the inner fluctuations of a spectral triple. These inner fluctuations arise from considering Morita equivalences between algebras. For a detailed discussion, we refer to [13, Ch. 1, §10.8]. In this section, we will simply give the definition. Subsequently we will define both the spectral action and the fermionic action.

5.15 Definition. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we define the set

$$\Omega_D^1 := \Big\{ \sum_j a_j[D, b_j] \mid a_j, b_j \in \mathcal{A} \Big\}. \tag{5.19}$$

For a real spectral triple (endowed with an antilinear operator J) we define the *fluctuated Dirac operator* by

$$D_A := D + A + \varepsilon' J A J^{-1}, \tag{5.20}$$

for a selfadjoint $A = A^* \in \Omega_D^1$. The elements A are called the *inner fluctuations* of the spectral triple.

Note that in the case $\mathcal{A} = C^\infty(M)$ and $D = \not{D}$ (cf. Definition 3.47) we have by Proposition 3.48 that $\Omega^1_{\not{D}}$ is given by the Clifford representation of the 1-forms $\mathcal{A}^1(M)$. The elements of Ω_D^1 for a general Dirac operator D are therefore regarded as a generalization of 1-forms. They will be interpreted as gauge potentials or gauge fields. This interpretation will be justified for the case of almost commutative geometries in Section 6.2.

5.16 Proposition. *The inner fluctuations of the canonical spectral triple vanish.*

PROOF. We take $a, b \in C^\infty(M)$ and calculate elements of the form $A = a[\not{D}, b]$. By using the local formula $\not{D} = i\gamma^\mu \nabla_\mu^S$ we find

$$A = i\gamma^\mu a \partial_\mu b =: \gamma^\mu A_\mu.$$

Since A must be selfadjoint, $A_\mu = ia\partial_\mu b$ must be a real function in $C^\infty(M)$. Since J_M commutes with $\not{D} = i\gamma^\mu \nabla_\mu^S$ and anticommutes with i, we know that J_M must anticommute with γ^μ. J_M commutes with A_μ since A_μ is real. Hence we conclude

$$\not{D}_A = \not{D} + A + J_M A J_M^{-1} = \not{D} + A - A J_M J_M^{-1} = \not{D}. \qquad \square$$

The above proposition shows that we do not obtain a gauge field from the canonical spectral triple. Associated to the algebra $\mathcal{A} = C^\infty(M)$ is the unitary group $C^\infty(M, U(1))$, so one might have expected that this would yield a $U(1)$ gauge field. Indeed, we found that $A_\mu(x) \in \mathbb{R} = i\,\mathfrak{u}(1)$, but in the combination $A_\mu - J_M A_\mu J_M^{-1}$ this field disappears because it commutes with J_M, and therefore there is no gauge field in this case.

5.17 Lemma. *For an inner fluctuation A and an element $a \in \mathcal{A}$, we have the relations $[A, JaJ^*] = 0$ and $[J^*AJ, a] = 0$.*

PROOF. The inner fluctuation A is of the form $\sum_j a_j[D, b_j]$. Because of the commutation rules (5.4) and (5.6), we immediately find $[A, JaJ^*] = 0$. We then note that $J^*[A, JaJ^*]J = [J^*AJ, a]$ and conclude that then also $[J^*AJ, a] = 0$. $\square$

5.18 Definition. The spectral action of a real spectral triple is defined by

$$S_B := \mathrm{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right), \tag{5.21}$$

where f is a positive even function, Λ is a cut-off parameter and D_A is the fluctuated Dirac operator.

The spectral action accounts only for the purely bosonic part of the action. For the terms involving fermions and their coupling to the bosons, we need something extra. The precise form of the fermionic action depends on the KO-dimension of the spectral triple. For the purpose of this thesis, we will only consider the case of KO-dimension 2 and give the fermionic action for this case. By Definition 5.5, we have the relations

$$J^2 = -1, \qquad JD = DJ, \qquad J\gamma = -\gamma J. \tag{5.22}$$

We use the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ by the grading γ. Following [13, Ch. 1, §16.2-3], the relations above yield a natural construction of an antisymmetric form on $\mathcal{H}^+$.

5.19 Proposition. *Let $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ be a real even spectral triple of KO-dimension 2. The expression*

$$\mathfrak{A}_D(\xi, \xi') = \langle J\xi, D\xi' \rangle \tag{5.23}$$

for $\xi, \xi' \in \mathcal{H}^+$ defines an antisymmetric bilinear form on $\mathcal{H}^+$, where $\langle\ ,\ \rangle$ is the inner product on $\mathcal{H}$.

Proof. The inner product is antilinear in the first variable, and since J is also antilinear, $\mathfrak{A}_D$ is a bilinear form. We check that it is antisymmetric:

$$\begin{aligned}\mathfrak{A}_D(\xi, \xi') = \langle J\xi, D\xi' \rangle &= -\langle J\xi, J^2 D\xi' \rangle = -\langle JD\xi', \xi \rangle \\ &= -\langle DJ\xi', \xi \rangle = -\langle J\xi', D\xi \rangle = -\mathfrak{A}_D(\xi', \xi),\end{aligned}$$

where we have used the relations of Eq. (5.22) and the fact that J is antiunitary, i.e. $\langle J\xi, J\xi' \rangle = \langle \xi', \xi \rangle$ for all $\xi, \xi' \in \mathcal{H}$. Furthermore, we can restrict $\mathfrak{A}_D$ to $\mathcal{H}^+$ without automatically getting zero, since we have $\gamma JD = JD\gamma$. For $\xi = \gamma\xi, \xi' = \gamma\xi' \in \mathcal{H}^+$, we have

$$\langle J\xi, D\xi' \rangle = \langle J\gamma\xi, D\xi' \rangle = -\langle \gamma J\xi, D\xi' \rangle = -\langle J\xi, \gamma D\xi' \rangle = \langle J\xi, D\gamma\xi' \rangle = \langle J\xi, D\xi' \rangle. \square$$

5.20 Definition. We define the set of *classical fermions* corresponding to $\mathcal{H}^+$,

$$\mathcal{H}^+_{\text{cl}} := \{\tilde{\xi} \mid \xi \in \mathcal{H}^+\}, \tag{5.24}$$

as the set of Grassmann variables $\tilde{\xi}$ for $\xi \in \mathcal{H}^+$. Assuming that the Hilbert space is separable and has a basis $\{e_j\}$, we can write $\xi = \sum_j \xi_j e_j$. The Grassmann variable $\tilde{\xi}$ is then obtained by making every component ξ_j into a Grassmann variable $\tilde{\xi}_j$, so $\tilde{\xi} = \tilde{\xi}_j e_j$. If the Hilbert space is of the form $L^2(M, E)$ for a vector bundle $E \to M$ of rank k, then there locally exists a frame $\{e_1, \ldots, e_k\}$ such that $\xi(x) = \sum_{j=1}^k \xi_j(x) e_j(x)$. In that case we obtain the Grassmann variable $\tilde{\xi}(x) = \sum_{j=1}^k \tilde{\xi}_j(x) e_j(x)$.

5.21 Definition. For a real even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ of KO-dimension 2 we define the full *action functional* by

$$S := S_B + S_F := \operatorname{Tr}\left(f\left(\frac{D_A}{\Lambda}\right)\right) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi} \rangle, \tag{5.25}$$

for $\tilde{\xi} \in \mathcal{H}^+_{\text{cl}}$. The factor $\frac{1}{2}$ in front of the *fermionic action* S_F has been chosen for future convenience.

5.22 Remark. One should note that we have incorporated two restrictions in the fermionic action S_F. The first is that we restrict ourselves to even vectors in $\mathcal{H}^+$, instead of considering all vectors in $\mathcal{H}$. The second restriction is that we do not consider the inner product $\langle J\tilde{\xi}', D_A\tilde{\xi} \rangle$ for two independent vectors ξ and ξ', but instead use the same vector ξ on both sides of the inner product. Each of these restrictions reduces the number of degrees of freedom in the fermionic action by a factor 2, yielding a factor 4 in total. It is precisely this approach that solves the problem of fermion doubling pointed out in [28] (see also [13, Ch. 1, §16.3]). We shall discuss this in more detail in Chapter 7, where we calculate the fermionic action for electrodynamics.

5.5 GAUGE SYMMETRY IN SPECTRAL TRIPLES

5.5.1 *Unitary transformations*

We denote by $\mathrm{Aut}(\mathcal{A})$ the group of algebra automorphisms of $\mathcal{A}$ (i.e. invertible algebra homomorphisms $\mathcal{A} \to \mathcal{A}$). An automorphism α is called an *inner* automorphism if it is induced by a unitary element $u \in U(\mathcal{A})$ such that $\alpha(a) = uau^*$. The group of inner automorphisms $\alpha_u : a \mapsto uau^*$ is denoted by $\mathrm{Inn}(\mathcal{A})$.

5.23 Lemma. *There is an exact sequence of groups*

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1, \tag{5.26}$$

where we define the set of outer *automorphisms* $\mathrm{Out}(\mathcal{A})$ *as the quotient* $\mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$.

PROOF. For $\beta \in \mathrm{Aut}(A)$ and $\alpha_u \in \mathrm{Inn}(\mathcal{A})$, we find that

$$\beta \circ \alpha_u \circ \beta^{-1}(a) = \beta\big(u\beta^{-1}(a)u^*\big) = \beta(u)a\beta(u)^* = \alpha_{\beta(u)}.$$

This means that $\mathrm{Inn}(\mathcal{A})$ is a normal subgroup of $\mathrm{Aut}(\mathcal{A})$, and hence it is the kernel of some map $\phi \colon \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A})$. □

5.24 Proposition. *The group of inner automorphisms* $\mathrm{Inn}(\mathcal{A})$ *is isomorphic to* $U(\mathcal{A})/Z$, *where* Z *is the subgroup of* $U(\mathcal{A})$ *that commutes with* $\mathcal{A}$.

PROOF. The map $\phi \colon U(\mathcal{A}) \to \mathrm{Inn}(\mathcal{A}) \colon u \mapsto \alpha_u$ is clearly surjective, but it is not injective. The kernel is given by $\mathrm{Ker}(\phi) = \{u \in U(\mathcal{A}) \mid uau^* = a,\ \forall a \in \mathcal{A}\}$. The relation $uau^* = a$ implies $ua = au$ for all $a \in \mathcal{A}$. We thus see that $\mathrm{Ker}(\phi) = Z$. The map ϕ induces a map from the quotient $U(\mathcal{A})/Z$ to $\mathrm{Inn}(\mathcal{A})$. This induced map is bijective and hence gives the isomorphism $\mathrm{Inn}(\mathcal{A}) \cong U(\mathcal{A})/Z$. □

We would like to study the notion of 'symmetry' in spectral triples. The starting point is to define an equivalence of spectral triples. The symmetry is then revealed when it turns out that the bosonic and fermionic action functionals of a spectral triple are identical for equivalent spectral triples. We take our definition of equivalent spectral triples from [27, §6.9], but make a slight modification by incorporating the algebra isomorphism α.

5.25 Definition. Two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$, with the associated representations $\pi_j \colon \mathcal{A}_j \to B(\mathcal{H}_j)$ for $j = 1, 2$, are *unitarily equivalent* if there exists a unitary operator $U \colon \mathcal{H}_1 \to \mathcal{H}_2$, called the *intertwining operator*, such that $UD_1U^* = D_2$, and $U\pi_1(a)U^* = \pi_2(\alpha(a))$ for any $a \in \mathcal{A}_1$, where α is an algebra isomorphism $\mathcal{A}_1 \to \mathcal{A}_2$. If the two triples are even with grading operators γ_1 and γ_2, one also requires that $U\gamma_1U^* = \gamma_2$. If the two triples are real with real structures J_1 and J_2, one also requires that $UJ_1U^* = J_2$.

Note that for a discussion of the equivalence of spectral triples, it is good to explicitly mention the representation of the algebra on the Hilbert space, since the intertwining operator affects this representation. Let us now consider two basic examples of intertwining operators.

5.26 Proposition. *The following two spectral triples are equivalent to the spectral triple* $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ *with representation* $\pi\colon \mathcal{A} \to B(\mathcal{H})$*:*

1. $(\mathcal{A}, \mathcal{H}, UDU^*, \gamma, UJU^*)$ *with representation* $\pi \circ \alpha_u$ *for* $U = \pi(u)$ *with* $u \in U(\mathcal{A})$*;*
2. $(\mathcal{A}, \mathcal{H}, UDU^*, \gamma, J)$ *with representation* $\pi \circ \alpha_u$ *for* $U = \pi(u)J\pi(u)J^*$ *with* $u \in U(\mathcal{A})$*.*

Proof. 1. We only need to check that $U\pi(a)U^* = \pi \circ \alpha_u(a)$ and $U\gamma U^* = \gamma$. The latter relation is evident since the grading operator γ commutes with the algebra. We also see that

$$U\pi(a)U^* = \pi(u)\pi(a)\pi(u)^* = \pi(uau^*) = \pi \circ \alpha_u(a).$$

2. First, we easily see that $UU^* = uJuJ^*(uJuJ^*)^* = uJuJ^*Ju^*J^*u^* = 1$ and similarly $U^*U = 1$, so U is indeed a unitary operator. The relation $U\gamma U^* = \gamma$ holds since $\pi(u)J\pi(u)J^*\gamma = (\epsilon'')^2\gamma\pi(u)J\pi(u)J^*$. Since $J\pi(u)J^*$ commutes with $\pi(a)$, we find that

$$\begin{aligned} U\pi(a)U^* &= \pi(u)J\pi(u)J^*\pi(a)J\pi(u)^*J^*\pi(u)^* \\ &= \pi(u)\pi(a)J\pi(u)J^*J\pi(u)^*J^*\pi(u)^* \\ &= \pi(u)\pi(a)\pi(u)^* = \pi(uau^*) = \pi \circ \alpha_u(a). \end{aligned}$$

Using $J^* = \epsilon J$, we check that

$$\begin{aligned} J' = UJU^* &= \pi(u)J\pi(u)J^*JJ\pi(u)^*J^*\pi(u)^* \\ &= \pi(u)J\pi(u)J\pi(u)^*J^*\pi(u)^* = \pi(u)J\pi(u)\pi(u)^*J\pi(u)^*J^* \\ &= \pi(u)JJ\pi(u)^*J^* = \epsilon J^* = J. \end{aligned}$$

□

5.5.2 *The gauge group*

In the first case of Proposition 5.26, the intertwining operator U is given by left multiplication with an element of the unitary subgroup $U(\mathcal{A})$. In the second case, the action of the operator U on a vector $\xi \in \mathcal{H}$ can be written as $U\xi = u\xi u^*$, since we identify JuJ^* with the right action of u^*. This case is especially interesting because we see that the intertwining operator has no effect on J. The group generated by all operators of the form $U = uJuJ^*$ characterizes all equivalent spectral triples $(\mathcal{A}, \mathcal{H}, UDU^*, \gamma, J)$, in which only the Dirac operator is affected by the unitary transformation. This group shall be interpreted as the gauge group, and this interpretation will later be justified by Theorem 6.22. Now, let us first derive this gauge group from the spectral triple.

5.27 Definition. The *gauge group* $\mathcal{G}(\mathcal{A})$ of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ is defined by

$$\mathcal{G}(\mathcal{A}) := \{U = uJuJ^* \mid u \in U(\mathcal{A})\}.$$

5.28 Proposition. *There is a short exact sequence of groups*

$$1 \to H \to U(\mathcal{A}) \to \mathcal{G}(A) \to 1, \qquad (5.27)$$

where $H = U(\mathcal{A}) \cap \widetilde{\mathcal{A}}_J$*, and* $\widetilde{\mathcal{A}}_J$ *is defined in Definition 5.12. In other words, the gauge group* $\mathcal{G}(\mathcal{A})$ *is isomorphic to the quotient* $U(\mathcal{A})/H$*.*

Proof. The map $\phi\colon U(\mathcal{A}) \to \mathcal{G}(A)$ is given by $u \mapsto uJuJ^*$, and is by definition surjective. ϕ is a group homomorphism, since the commutation relation $[a, JbJ^*] = 0$ implies that $\phi(b)\phi(a) = bJbJ^*aJaJ^* = baJbaJ^* = \phi(ba)$. This map has kernel $H := \mathrm{Ker}(\phi) = \{u \in U(\mathcal{A}) \mid uJuJ^* = 1\}$. The relation $uJuJ^* = 1$ is equivalent to $uJ = Ju^*$, and we note that this is the defining relation of the commutative subalgebra $\widetilde{\mathcal{A}}_J$ of Definition 5.12. Hence we have $H = U(\mathcal{A}) \cap \widetilde{\mathcal{A}}_J$. □

From Proposition 5.13 we know that $\widetilde{\mathcal{A}}_J$ is a subalgebra of the center of $\mathcal{A}$. Hence the group H of the previous proposition is contained in the subgroup Z of $U(\mathcal{A})$. Propositions 5.24 and 5.28 then imply that in general, the gauge group $\mathcal{G}(\mathcal{A})$ is larger than the group of inner automorphisms $\mathrm{Inn}(\mathcal{A})$.

5.29 Corollary. *If $H = U(\mathcal{A}) \cap \widetilde{\mathcal{A}}_J$ is* equal *to Z, we have in fact* $\mathrm{Inn}(\mathcal{A}) \cong \mathcal{G}(\mathcal{A})$.

5.5.3 *Gauge transformations*

In Proposition 5.26 we have seen that an element $U \in \mathcal{G}(\mathcal{A})$ transforms the Dirac operator as $D \to UDU^*$. Let us now consider the effect of this transformation on the fluctuated Dirac operator $D_A = D + A + \epsilon' JAJ^*$. In the following proposition we will show that the transformed operator UD_AU^* can also be written as a fluctuated Dirac operator $D_{A'}$, for a new fluctuation A'. This only works because we have restricted $U(\mathcal{A})$ to the gauge group $\mathcal{G}(\mathcal{A})$, to make sure that the conjugation operator J remains unchanged. The resulting transformation on the inner fluctuation $A \to A'$ shall be interpreted in physics as the gauge transformation of the gauge field.

5.30 Proposition. *Under a transformation given by $U = uJuJ^* \in \mathcal{G}(\mathcal{A})$, the inner fluctuation A transforms as*

$$A \to uAu^* + u[D, u^*] \in \Omega^1_D. \tag{5.28}$$

Proof. In Proposition 5.26 (sub 2) we have shown that the operator UDU^* yields a Dirac operator equivalent to D. We apply this to the fluctuated Dirac operator $D_A = D + A + \epsilon' JAJ^*$. Using the commutation rules $[a, b^0] = 0$, $\big[[D, a], b^0\big] = 0$ and $JD = \epsilon' DJ$, we calculate that

$$\begin{aligned} UDU^* &= uJuJ^*DJu^*J^*u^* = \epsilon' uJuDu^*J^*u^* = \epsilon' uJ(D + u[D, u^*])J^*u^* \\ &= uDu^* + \epsilon' JJ^*uJu[D, u^*]J^*u^* = D + u[D, u^*] + \epsilon' Ju[D, u^*]J^*. \end{aligned}$$

Using the commutation relation $[A, JuJ^*] = 0$ from Lemma 5.17, we see that

$$UAU^* = uJuJ^*AJu^*J^*u^* = uAu^*$$

and

$$\begin{aligned} U\epsilon' JAJ^*U^* &= \epsilon' uJuJ^*JAJ^*Ju^*J^*u^* = \epsilon' uJuAu^*J^*u^*JJ^* \\ &= \epsilon' uJJ^*u^*JuAu^*J^* = \epsilon' JuAu^*J^*. \end{aligned}$$

Combining these three relations, we find that

$$UD_AU^* = D_{A'} \quad , \quad \text{for } A' := uAu^* + u[D, u^*].$$ □

5.5.4 *Invariance of the action functional*

5.31 Proposition. *The spectral action S_B is invariant under unitary transformations of $\mathcal{G}(\mathcal{A})$.*

Proof. The transformation of the fluctuated Dirac operator is given by $D_A \to UD_AU^*$ for $U \in \mathcal{G}(\mathcal{A})$, so the spectral action transforms as

$$\mathrm{Tr}\left(f\left(\frac{D_A}{\Lambda}\right)\right) \to \mathrm{Tr}\left(f\left(\frac{UD_AU^*}{\Lambda}\right)\right).$$

The trace depends only on the discrete spectrum of the fluctuated Dirac operator D_A. The unitary transformation has no effect on this spectrum. Namely, if we let

ψ_n be the eigenvectors of D_A with eigenvalues λ_n, then the vectors $\psi'_n := U\psi_n$ are the eigenvectors of $D'_A := UD_AU^*$ with the same eigenvalues λ_n:

$$D'_A\psi'_n = UD_AU^*U\psi_n = UD_A\psi_n = U\lambda_n\psi_n = \lambda_n U\psi_n = \lambda_n\psi'_n.$$

For the spectral action, we thus obtain

$$\mathrm{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) = \sum_n f\Big(\frac{\lambda_n}{\Lambda}\Big) = \mathrm{Tr}\left(f\Big(\frac{UD_AU^*}{\Lambda}\Big)\right). \qquad \square$$

5.32 Proposition. *The fermionic action S_F is invariant under unitary transformations of $\mathcal{G}(\mathcal{A})$.*

Proof. The transformation of the fluctuated Dirac operator is given by $D_A \to UD_AU^*$ for $U \in \mathcal{G}(\mathcal{A})$, whereas the conjugation operator remains unchanged since $UJU^* = J$ (cf. Proposition 5.26, sub 2). From the unitarity of U we then easily see that

$$\langle J\widetilde{\xi}, D_A\widetilde{\xi}\rangle \to \langle JU\widetilde{\xi}, UD_AU^*U\widetilde{\xi}\rangle = \langle UJ\widetilde{\xi}, UD_A\widetilde{\xi}\rangle = \langle UJ\widetilde{\xi}, UD_A\widetilde{\xi}\rangle = \langle J\widetilde{\xi}, D_A\widetilde{\xi}\rangle. \qquad \square$$

6 ALMOST COMMUTATIVE GEOMETRIES

In this chapter we shall consider the product of the canonical spectral triple with an arbitrary real even finite spectral triple. Such a product is called an almost commutative geometry, a term which we borrow from [20–24]. For simplicity, we restrict ourselves to the case of a 4-dimensional spin manifold M, for which the canonical spectral triple is given by the data (cf. Definition 5.6)

$$\big(C^\infty(M), L^2(M,S), \not{D}, \gamma_5, J_M\big), \tag{6.1}$$

where we now denote $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$.

We will start by deriving the general form of the inner fluctuations of an almost commutative geometry. Next, we will show how one can recover the notion of a gauge theory from the spectral triple of an almost commutative geometry. Furthermore, we will calculate the heat expansion of the spectral action for an almost commutative geometry. In the last section, we will show that the spectral action is conformally invariant.

6.1 ALMOST COMMUTATIVE GEOMETRIES

6.1 Definition. An *almost commutative geometry* is given by the product of the canonical spectral triple of a 4-dimensional spin manifold and a finite spectral triple, and has the form

$$(\mathcal{A}, \mathcal{H}, D) = \Big(C^\infty(M, \mathcal{A}_F), L^2(M,S)\otimes\mathcal{H}_F, \not{D}\otimes\mathbb{1} + \gamma_5\otimes D_F\Big), \tag{6.2}$$

together with a grading $\gamma = \gamma_5 \otimes \gamma_F$ and an antilinear isomorphism $J = J_M \otimes J_F$. By defining the trivial vector bundle $E := M \times \mathcal{H}_F$, we can also write the Hilbert space as $\mathcal{H} = L^2(M, S\otimes E)$.

6.2 Proposition. *An almost commutative geometry, for which the finite spectral triple has even KO-dimension n, defines a spectral triple of KO-dimension* $4+n \mod 8$.

Proof. The canonical spectral triple is of KO-dimension 4, so we have the relations $J^2 = -\varepsilon$ and $J\gamma = \varepsilon''\gamma J$. Since $J(\not{D}\otimes\mathbb{1}) = (\not{D}\otimes\mathbb{1})J$, we are required to demand that also $J(\gamma_5\otimes D_F) = (\gamma_5\otimes D_F)J$, which yields $\varepsilon' = 1$. This requirement is satisfied by every *even* finite spectral triple. The KO-dimension $4+n \mod 8$ is in agreement with the table from Definition 5.5. □

6.3 Proposition. *The inner fluctuations of an even almost commutative geometry take the form*

$$A = \gamma^\mu \otimes A_\mu + \gamma_5 \otimes \phi, \tag{6.3}$$

for selfadjoint operators $A_\mu \in i\mathcal{A}$[1] *and* $\phi \in \Gamma\big(\mathrm{End}(E)\big)$.

Proof. The Dirac operator $D = \not{D}\otimes\mathbb{1} + \gamma_5\otimes D_F$ consists of two terms, and hence we can also split $A = a[D,b]$ in two terms. The first term is given by

$$a[\not{D}\otimes\mathbb{1}, b] = i\gamma^\mu \otimes a\partial_\mu b =: \gamma^\mu \otimes A_\mu,$$

where $A_\mu := ia\partial_\mu b \in i\mathcal{A}$ must be selfadjoint. The second term yields

$$a[\gamma_5\otimes D_F, b] = \gamma_5 \otimes a[D_F, b] =: \gamma_5\otimes\phi,$$

for selfadjoint $\phi := a[D_F, b]$. □

1 Note that $i\mathcal{A} = \mathcal{A}$ for complex algebras only.

6.4 Proposition. *The fluctuated Dirac operator of an even almost commutative geometry takes the form*

$$D_A = \not{D} \otimes \mathbb{1} + \gamma^\mu \otimes B_\mu + \gamma_5 \otimes \Phi = i\gamma^\mu \nabla^E_\mu + \gamma_5 \otimes \Phi, \tag{6.4}$$

where $B_\mu := A_\mu - J_F A_\mu J_F^{-1}$ *and* $\Phi := D_F + \phi + J_F \phi J_F^{-1}$ *are in* $\Gamma\big(\mathrm{End}(E)\big)$, *and* ∇^E *is a connection on the bundle* $S \otimes E$ *given by*

$$\nabla^E_\mu = \nabla^S_\mu \otimes \mathbb{1} - i\mathbb{1} \otimes B_\mu. \tag{6.5}$$

Proof. The fluctuated Dirac operator is given by $D_A = D + A + JAJ^{-1}$, where from Proposition 6.3 we know that $A = \gamma^\mu \otimes A_\mu + \gamma_5 \otimes \phi$. We then calculate

$$\gamma^\mu \otimes A_\mu + J\gamma^\mu \otimes A_\mu J^{-1} = \gamma^\mu \otimes \big(A_\mu - J_F A_\mu J_F^{-1}\big) =: \gamma^\mu \otimes B_\mu,$$

where we have defined $B_\mu \in \Gamma\big(\mathrm{End}(E)\big)$. Using the definition of ∇^E we have obtained the first term $i\gamma^\mu \nabla^E_\mu$. For the second term $\gamma_5 \otimes \Phi$, we see from the definition of $\Phi \in \Gamma\big(\mathrm{End}(E)\big)$ that

$$\gamma_5 \otimes D_F + \gamma_5 \otimes \phi + J(\gamma_5 \otimes \phi)J^{-1} = \gamma_5 \otimes \Phi. \qquad \square$$

6.5 Lemma. *If the finite spectral triple is even, the field* ϕ *satisfies* $\phi\gamma_F = -\gamma_F\phi$ *and the field* Φ *satisfies* $\Phi\gamma_F = -\gamma_F\Phi$ *and* $\Phi J_F = J_F \Phi$.

Proof. These relations follow directly from the definitions of ϕ and Φ and the commutation relations for D_F. $\square$

6.6 Lemma. *The trace over* $\mathcal{H}_F$ *of* aD_F *vanishes identically for any* $a \in \mathcal{A}_F$.

Proof. We use the cyclic property of the trace and the fact that the grading commutes with the algebra and anticommutes with the Dirac operator, to find

$$\mathrm{Tr}\big(aD_F\big) = \mathrm{Tr}\big(aD_F{\gamma_F}^2\big) = \mathrm{Tr}\big(\gamma_F a D_F \gamma_F\big) = \mathrm{Tr}\big(-aD_F{\gamma_F}^2\big) = -\mathrm{Tr}\big(aD_F\big). \qquad \square$$

6.7 Proposition. *The traces of the fields* B_μ, ϕ *and* Φ *over the finite Hilbert space* $\mathcal{H}_F$ *vanish identically.*

Proof. We use the cyclic property of the trace to find

$$\mathrm{Tr}_{\mathcal{H}_F}\big(B_\mu\big) = \mathrm{Tr}_{\mathcal{H}_F}\big(A_\mu - J_F A_\mu J_F^{-1}\big) = \mathrm{Tr}_{\mathcal{H}_F}\big(A_\mu - A_\mu J_F^{-1} J_F\big) = 0.$$

For the field ϕ we find, using the cyclic property of the trace, that

$$\mathrm{Tr}_{\mathcal{H}_F}\big(\phi\big) = \mathrm{Tr}_{\mathcal{H}_F}\big(a[D_F, b]\big) = \mathrm{Tr}_{\mathcal{H}_F}\big([b,a]D_F\big).$$

Applying Lemma 6.6 shows that this trace also vanishes. It then automatically follows that $\Phi = D_F + \phi + J_F \phi J_F^{-1}$ is also traceless. $\square$

6.8 Definition. Let us evaluate the relations between the connection, its curvature and their adjoint actions. We define the operator D_μ as the adjoint action of the connection ∇^E_μ, i.e. $D_\mu = \mathrm{ad}\,\big(\nabla^E_\mu\big)$. In other words, we have

$$D_\mu \Phi = [\nabla^E_\mu, \Phi] = \partial_\mu \Phi - i[B_\mu, \Phi]. \tag{6.6}$$

We shall define the curvature $F_{\mu\nu}$ of the gauge field B_μ by

$$F_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu, B_\nu]. \tag{6.7}$$

6.9 Lemma. *The curvature of the connection ∇^E defined in Eq. (6.5) is given by*

$$\Omega^E_{\mu\nu} = \left[\nabla^E_\mu, \nabla^E_\nu\right] = \Omega^S_{\mu\nu} \otimes \mathbb{1} - i\mathbb{1} \otimes F_{\mu\nu}. \tag{6.8}$$

We also have the relation[2]

$$\left[D_\mu, D_\nu\right] = -i\mathrm{ad}\left(F_{\mu\nu}\right). \tag{6.9}$$

Proof. From Definition 2.38, we calculate

$$\begin{aligned}
\Omega^E_{\mu\nu} &= \nabla^E_\mu \nabla^E_\nu - \nabla^E_\nu \nabla^E_\mu \\
&= (\nabla^S_\mu \otimes \mathbb{1} - i\mathbb{1} \otimes B_\mu)(\nabla^S_\nu \otimes \mathbb{1} - i\mathbb{1} \otimes B_\nu) \\
&\quad - (\nabla^S_\nu \otimes \mathbb{1} - i\mathbb{1} \otimes B_\nu)(\nabla^S_\mu \otimes \mathbb{1} - i\mathbb{1} \otimes B_\mu) \\
&= \Omega^S_{\mu\nu} \otimes \mathbb{1} - i\mathbb{1} \otimes \partial_\mu B_\nu + i\mathbb{1} \otimes \partial_\nu B_\mu - \mathbb{1} \otimes [B_\mu, B_\nu],
\end{aligned}$$

and we then insert Eq. (6.7). This yields the desired formula for $\Omega^E_{\mu\nu}$. The relation $\left[D_\mu, D_\nu\right] = -i\mathrm{ad}\left(F_{\mu\nu}\right)$ can be verified explicitly, but also follows directly from the definition of D_μ and the Jacobi identity:

$$\begin{aligned}
[D_\mu, D_\nu]\Phi &= \mathrm{ad}\left(\nabla^E_\mu\right)\mathrm{ad}\left(\nabla^E_\nu\right)\Phi - \mathrm{ad}\left(\nabla^E_\nu\right)\mathrm{ad}\left(\nabla^E_\mu\right)\Phi \\
&= \left[\nabla^E_\mu, [\nabla^E_\nu, \Phi]\right] - \left[\nabla^E_\nu, [\nabla^E_\mu, \Phi]\right] \\
&= \left[[\nabla^E_\mu, \nabla^E_\nu], \Phi\right] = \left[\Omega^E_{\mu\nu}, \Phi\right] = \mathrm{ad}\left(\Omega^E_{\mu\nu}\right)\Phi.
\end{aligned}$$

Since $\Omega^S_{\mu\nu}$ commutes with Φ, we obtain $\left[D_\mu, D_\nu\right] = -i\mathrm{ad}\left(F_{\mu\nu}\right)$. □

6.2 Gauge symmetry of almost commutative geometries

In this section we will show how the notion of gauge theory naturally arises in an almost commutative geometry. A valuable source of inspiration has been the work of Boeijink [3, 4], which generalizes the Einstein-Yang-Mills system to topologically non-trivial gauge configurations. In this section we follow a similar approach by describing the gauge theory of an arbitrary almost commutative geometry. However, in our case we shall only deal with topologically trivial gauge theories. The main result of this section is Theorem 6.22, which shows that the data in the spectral triple of an almost commutative geometry allow us to completely reconstruct a gauge theory on the manifold M.

6.10 Lemma. *For an almost commutative geometry, the subalgebras $\mathcal{A}_J$ and $\widetilde{\mathcal{A}}_J$ are given by*

$$\mathcal{A}_J = C^\infty\left(M, (\mathcal{A}_F)_{J_F}\right), \qquad \widetilde{\mathcal{A}}_J = C^\infty\left(M, (\widetilde{\mathcal{A}}_F)_{J_F}\right). \tag{6.10}$$

Proof. The definitions of $(\mathcal{A}_F)_{J_F}$ and $(\widetilde{\mathcal{A}}_F)_{J_F}$ are exactly those of Definition 5.12, but now for the finite case. Since $J = J_M \otimes J_F$, and the effect of J_M on a function on M is simply complex conjugation, we obtain that the requirements $aJ = Ja$ or $aJ = Ja^*$ must be satisfied pointwise, i.e. $a(x)J_F = J_F a(x)$ or $a(x)J_F = J_F a(x)^*$, respectively, for $a(x) \in \mathcal{A}_F$. □

6.11 Proposition. *In the case of an almost commutative geometry, the unitary group is given by $U(\mathcal{A}) = C^\infty(M, U(\mathcal{A}_F))$, with the corresponding Lie algebra given by $\mathfrak{u}(\mathcal{A}) = C^\infty(M, \mathfrak{u}(\mathcal{A}_F))$, where $\mathfrak{u}(\mathcal{A}_F) = \{X \in \mathcal{A}_F \mid X^* = -X\}$. Furthermore, the Lie algebra of the* unimodular *group $SU(\mathcal{A}_F)$ is given by $\mathfrak{su}(\mathcal{A}_F) = \{X \in \mathcal{A}_F \mid X^* = -X, \mathrm{Tr}(X) = 0\}$.*

2 Note that this relation simply reflects the fact that ad is a Lie algebra homomorphism.

PROOF. The conjugation on $\mathcal{A} = C^\infty(M, \mathcal{A}_F)$ is given by pointwise conjugation on $\mathcal{A}_F$. The requirement $uu^* = u^*u = \mathbb{1}$ must hold for each $x \in M$, which gives $u(x)u(x)^* = u(x)^*u(x) = \mathbb{1}$. Hence, $u \in U(\mathcal{A}) \Leftrightarrow u(x) \in U(\mathcal{A}_F)$.

An element u of the unitary group $U(\mathcal{A}_F)$ can be written as $u = \exp(X)$ for some element X of the Lie algebra $\mathfrak{u}(\mathcal{A}_F)$. The condition $uu^* = 1$ then becomes $\exp(X)\exp(X^*) = 1$, which yields $X^* = -X$. Hence we find that $\mathfrak{u}(A_F) = \{X \in \mathcal{A}_F \mid X^* = -X\}$. For $X, Y \in \mathfrak{u}(\mathcal{A}_F)$, we see that $[X,Y]^* = [Y^*, X^*] = [-Y, -X] = -[X,Y]$, so $\mathfrak{u}(\mathcal{A}_F)$ is closed under the commutator, and hence forms a real Lie algebra.

For $u = \exp(X) \in SU(\mathcal{A}_F)$, the condition $\det(u) = 1$ becomes $1 = \det(\exp(X)) = \exp(\text{Tr}(X))$, which yields $\text{Tr}(X) = 0$. Since the trace is cyclic, the trace of a commutator always vanishes, and hence $\mathfrak{su}(A_F) = \{X \in \mathcal{A}_F \mid X^* = -X, \text{Tr}(X) = 0\}$ is closed under the commutator and forms a real Lie algebra. □

6.12 Definition. We define two subsets of $\mathcal{A}_F$ by

$$H_F := U(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F} \subset U(\mathcal{A}_F) \subset \mathcal{A}_F, \tag{6.11}$$

$$\mathfrak{h}_F := \mathfrak{u}(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F} \subset \mathfrak{u}(\mathcal{A}_F) \subset \mathcal{A}_F. \tag{6.12}$$

Note that the group H_F is the finite counterpart of the group H from Proposition 5.28. Let us evaluate the structure of this group in more detail.

6.13 Lemma. *1. H_F forms a normal Lie subgroup of $U(\mathcal{A}_F)$;*

2. *$\mathfrak{h}_F$ forms an ideal real Lie subalgebra of $\mathfrak{u}(\mathcal{A}_F)$;*

3. *$\mathfrak{h}_F$ is the Lie algebra of the Lie group H_F;*

4. *We can rewrite $\mathfrak{h}_F = \mathfrak{u}(\mathcal{A}_F) \cap i(\mathcal{A}_F)_{J_F}$.*

PROOF. 1. Two elements $u, v \in H_F$ commute, so we have for their product $(uv)^0 = v^0u^0 = vu = uv$ and $uv(uv)^* = uvv^*u^* = 1$. Hence H_F forms a subgroup, and by Definition 2.10, it is in fact a Lie subgroup. Since H_F is contained in the center of $\mathcal{A}_F$ by Proposition 5.13, the condition $uvu^* = v$ for $v \in H_F$ and $u \in U(\mathcal{A}_F)$ is evidently satisfied, and hence H_F is a normal subgroup.

2. By construction, $\mathfrak{h}_F$ is a real subspace of $\mathfrak{u}(\mathcal{A}_F)$. It inherits the product structure from $\mathcal{A}_F$, but it is not closed under this product, so it is not a subalgebra. However, the product on $\mathfrak{h}_F$ is commutative, so all commutators vanish, and hence $\mathfrak{h}_F$ does form a real Lie subalgebra. Take $X \in \mathfrak{u}(\mathcal{A}_F)$ and $Y \in \mathfrak{h}_F$, then their commutator vanishes because $(\widetilde{\mathcal{A}}_F)_{J_F}$ is a subalgebra of the center of $\mathcal{A}_F$ by Proposition 5.13. Hence $[\mathfrak{u}(\mathcal{A}_F), \mathfrak{h}_F] = \{0\} \subset \mathfrak{h}_F$, so $\mathfrak{h}_F$ forms an ideal.

3. We shall write $u = \exp(X) \in H_F$. From Proposition 6.11, we find that the Lie algebra $\text{Lie}(H_F)$ of H_F is contained in $\mathfrak{u}(\mathcal{A}_F)$. Since also $u \in (\widetilde{\mathcal{A}}_F)_{J_F}$, the condition $uJ = Ju^*$ implies $\exp(X)J = J\exp(X^*)$, which yields $XJ = JX^*$, and hence $X \in (\widetilde{\mathcal{A}}_F)_{J_F}$. Therefore we can conclude that $\text{Lie}(H_F) = \mathfrak{h}_F$.

4. For $X \in \mathfrak{u}(\mathcal{A}_F)$, we have $X^* = -X$, so the condition $XJ = JX^*$ implies $XJ = -JX$. We then see that $iXJ = JiX$, which means that $iX \in (\mathcal{A}_F)_{J_F}$ or $X \in i(\mathcal{A}_F)_{J_F}$. □

6.14 Proposition. *The gauge group $\mathcal{G}(\mathcal{A})$ of an almost commutative geometry is given by $C^\infty(M, \mathcal{G}(\mathcal{A}_F))$, where $\mathcal{G}(\mathcal{A}_F) = U(\mathcal{A}_F)/H_F$ is the gauge group of the finite triple.*

PROOF. By Lemma 6.13, we know that H_F is a normal subgroup of $U(\mathcal{A}_F)$, so their quotient $\mathcal{G}(\mathcal{A}_F)$ is well-defined. We have seen in Proposition 5.28 that $H = U(\mathcal{A}) \cap \widetilde{\mathcal{A}}_J$, which by Lemma 6.10 and Proposition 6.11 equals $C^\infty(M, H_F)$. We thus obtain that $\mathcal{G}(\mathcal{A}) = U(\mathcal{A})/H = C^\infty(M, U(\mathcal{A}_F))/C^\infty(M, H_F) = C^\infty(M, \mathcal{G}(\mathcal{A}_F))$. □

6.15 Proposition. *The Lie algebra of the quotient $\mathcal{G}(\mathcal{A}_F) = U(\mathcal{A}_F)/H_F$ is given by the quotient $\mathfrak{g}_F = \mathfrak{u}(\mathcal{A}_F)/\mathfrak{h}_F$.*

PROOF. By Lemma 6.13, both quotients are well-defined. The quotient $U(\mathcal{A}_F)/H_F$ is given by the equivalence classes $[u]$ for $u \in U(\mathcal{A}_F)$, where $[uh] = [u]$ for all $h \in H_F$ (cf. Definition 2.12). We can write $u = e^X$ for $X \in \mathfrak{u}(\mathcal{A}_F)$ and $h = e^Y$ for $Y \in \mathfrak{h}_F$. Thus we obtain an equivalence class of $X \in \mathfrak{u}(\mathcal{A}_F)$ by $e^{[X]} = [u] = [uh] = e^{[X+Y]}$ for all $Y \in \mathfrak{h}_F$. We then recognize from Definition 2.14 that the equivalence relation $[X+Y] = [X]$ indeed defines the quotient $\mathfrak{u}(\mathcal{A}_F)/\mathfrak{h}_F$. □

6.16 Proposition. *The field B_μ found in Proposition 6.4 is given by the adjoint action of a gauge field A_μ for the gauge group $\mathcal{G}(\mathcal{A}_F)$ with Lie algebra $\mathfrak{g}_F$.*

PROOF. Since $J_F A_\mu J_F^{-1} = J A_\mu J^{-1}$, we can also write $B_\mu = \text{ad}\,(A_\mu)$, where ad has been defined in Eq. (5.18). The demand that A_μ is selfadjoint is equivalent to $(iA_\mu)^* = -iA_\mu$. Since A_μ is of the form $ia\partial_\mu b$ for $a, b \in \mathcal{A}$ (see Proposition 6.3), we see that iA_μ is an element of the algebra $\mathcal{A}$ (also if $\mathcal{A}$ is only a real algebra). Thus by Proposition 6.11, we have $A_\mu(x) \in i\,\mathfrak{u}(A_F)$.

The only way in which A_μ affects the results is through the action of $A_\mu - J_F A_\mu J_F^{-1}$. If we take $A'_\mu = A_\mu - a_\mu$ for some $a_\mu \in \mathfrak{h}_F = (\mathcal{A}_F)_{J_F} \cap i\,\mathfrak{u}(\mathcal{A}_F)$ (which commutes with J_F), we see that $A'_\mu - J_F A'_\mu J_F^{-1} = A_\mu - J_F A_\mu J_F^{-1}$. Therefore we can, without any loss of generality, assume that $A_\mu(x)$ is an element of the quotient $i\mathfrak{g}_F = i\Big(\mathfrak{u}(\mathcal{A}_F)/\mathfrak{h}_F\Big)$. By Proposition 6.15, $\mathfrak{g}_F$ is the Lie algebra of the gauge group $\mathcal{G}(\mathcal{A}_F)$. □

6.17 Lemma. *If $\mathcal{A}_F$ is a complex algebra endowed with a complex algebra representation $\pi\colon \mathcal{A}_F \to B(\mathcal{H}_F)$, the group $U(1)$ is a normal subgroup of H_F. If, on the other hand, π is a real algebra representation, we only obtain that $\{1, -1\}$ is a normal subgroup of H_F.*

PROOF. $\mathcal{A}_F$ is unital, and we have $\pi(1) = \mathbb{1} \in B(\mathcal{H}_F)$. For $\lambda \in \mathbb{C}$ we then have $\pi(\lambda) = \lambda\mathbb{1}$. Since J_F is antilinear, we see that $\mathbb{C} \in (\widetilde{\mathcal{A}}_F)_{J_F}$. Restricting to unitary elements of $\mathcal{A}_F$ implies that $\lambda \in U(1)$. Hence we conclude that $U(1)$ is a subgroup of H_F. Because H_F is commutative, $U(1)$ is then automatically a normal subgroup of H_F.

For a real representation π, the situation changes, because the formula $\pi(\lambda) = \lambda\mathbb{1}$ only needs to hold for $\lambda \in \mathbb{R}$, and we then see that $\mathbb{R} \in (\widetilde{\mathcal{A}}_F)_{J_F}$. Restricting to unitary (i.e. in this case orthogonal) elements then gives $\lambda = \pm 1$. □

6.18 Proposition. *If $\mathcal{A}_F$ is a complex algebra endowed with a complex algebra representation $\pi\colon \mathcal{A}_F \to B(\mathcal{H}_F)$, the gauge group is isomorphic to*

$$\mathcal{G}(\mathcal{A}_F) \cong SU(\mathcal{A}_F)/SH_F, \tag{6.13}$$

where we have defined $SH_F = \{g \in H_F \mid \det g = 1\}$.

PROOF. An element of the quotient $\mathcal{G}(\mathcal{A}_F) = U(\mathcal{A}_F)/H_F$ is given by the equivalence class $[u]$ of some element $u \in U(\mathcal{A}_F)$, subject to the equivalence relation $[u] = [uh]$ for all $h \in H_F$. Similarly, the quotient $SU(\mathcal{A}_F)/SH_F$ consists of the classes $[v]$ for $v \in SU(\mathcal{A}_F)$ with the equivalence relation $[v] = [vg]$ for all $g \in SH_F$. We first show that this quotient is well-defined, i.e. that SH_F is a normal subgroup of $SU(\mathcal{A}_F)$. We thus need to check that $vgv^{-1} \in SH_F$ for all $v \in SU(\mathcal{A}_F)$ and $g \in SH_F$. We already know that $vgv^{-1} \in H_F$, because H_F is a normal subgroup of $U(\mathcal{A}_F)$ (cf. Lemma 6.13). We then also see that $\det(vgv^{-1}) = \det g = 1$, so $vgv^{-1} \in SH_F$, and the quotient $SU(\mathcal{A}_F)/SH_F$ is indeed well-defined.

There exists a $\lambda_u \in U(1)$ such that ${\lambda_u}^N = \det u$, where N is the dimension of the finite Hilbert space $\mathcal{H}_F$. Since $U(1)$ is a subgroup of $U(\mathcal{A}_F)$ by Lemma 6.17, we then see that ${\lambda_u}^{-1}u \in SU(\mathcal{A}_F)$. We can then define the group homomorphism $\varphi\colon \mathcal{G}(\mathcal{A}_F) \to SU(\mathcal{A}_F)/SH_F$ by $\varphi([u]) = [{\lambda_u}^{-1}u]$. We need to check that φ is well-defined, i.e. that $\varphi([u])$ is independent of the choice of the representative $u \in U(\mathcal{A}_F)$, as well as independent of the choice of λ_u. Suppose we also have λ'_u such that ${\lambda'_u}^N = \det u$. We then must have ${\lambda_u}^{-1}\lambda'_u \in \mu_N$, where μ_N is the multiplicative group of the N-th roots of unity. Since $U(1)$ is a subgroup of H_F, we see that μ_N is a subgroup of SH_F, so $[{\lambda_u}^{-1}u] = [{\lambda'_u}^{-1}u]$ and the image of φ is indeed independent of the choice of λ_u. Next, for any $h \in H_F$, we also check that

$$\varphi([u]) = [{\lambda_u}^{-1}u] = [{\lambda_u}^{-1}u{\lambda_h}^{-1}h] = [(\lambda_u\lambda_h)^{-1}uh] = \varphi([uh]),$$

where we have used that $g = {\lambda_h}^{-1}h \in SH_F$ (because, by Lemma 6.17, $U(1)$ is a subgroup of H_F) and that $(\lambda_u\lambda_h)^N = \det uh$.

Since $SU(\mathcal{A}_F) \subset U(\mathcal{A}_F)$, the homomorphism φ is clearly surjective. Now suppose $\varphi([u_1]) = \varphi([u_2])$ for some $u_1, u_2 \in U(\mathcal{A}_F)$. This means that ${\lambda_{u_1}}^{-1}u_1 = {\lambda_{u_2}}^{-1}u_2 g$ for some $g \in SH_F$. We then obtain that $u_1 = u_2 h$ for an element $h = \lambda_{u_1}{\lambda_{u_2}}^{-1}g \in H_F$, so $[u_1] = [u_2]$ and φ is also injective. Hence φ is a group isomorphism. □

6.19 Remark. The significance of Proposition 6.18 is that, in the case of a complex algebra with a complex representation, an equivalence class of the quotient $\mathcal{G}(\mathcal{A}_F) = U(\mathcal{A}_F)/H_F$ can always be represented (though not uniquely) by an element of $SU(\mathcal{A}_F)$. In that sense, all elements of $\mathcal{G}(\mathcal{A}_F)$ naturally satisfy the unimodularity condition. In the case of an algebra with a real representation, this is not true. For this reason one needs to impose the unimodularity condition on the inner fluctuations by hand in the derivation of the standard model from noncommutative geometry (see Remark 9.4).

6.20 Proposition. *Under transformations of the group $\mathcal{G}(\mathcal{A})$ of Proposition 5.28, the fields A_μ and ϕ of Proposition 6.3 transform as*

$$\begin{aligned} A_\mu &\to uA_\mu u^* + iu\partial_\mu u^*, \\ \phi &\to u\phi u^* + u[D_F, u^*]. \end{aligned} \tag{6.14}$$

PROOF. From Proposition 5.30 we know that an element $U = uJuJ^* \in \mathcal{G}(\mathcal{A})$ transforms an inner fluctuation as $A \to uAu^* + u[D, u*]$. Writing out $A = \gamma^\mu \otimes A_\mu + \gamma_5 \otimes \phi$ (by Proposition 6.3) and $D = i\gamma^\mu\nabla^S_\mu \otimes \mathbb{1} + \gamma_5 \otimes D_F$, and by using that $[\nabla^S_\mu, u^*] = \partial_\mu u^*$, we thus obtain the transformation of A_μ and ϕ. □

6.21 Remark. In Proposition 4.12 we have given the transformation of a gauge field in gauge theory. Upon rewriting the hermitian field A_μ as the anti-hermitian field $iA_\mu \in \mathfrak{g}_F$, the above transformation property of the field A_μ corresponds precisely to the gauge transformation (4.8) of the gauge field iA_μ, as desired.

However, the transformation property of the field ϕ is more surprising. In the usual setup in physics, a Higgs field transforms linearly under the gauge group. The transformation for ϕ derived above on the other hand is non-linear. From the framework of noncommutative geometry this is no surprise, since both bosonic fields A_μ and ϕ are obtained from the inner fluctuations of the Dirac operator, and are thereby expected to transform in a similar manner. It might be though that for particular choices of the finite spectral triple, the transformation property of ϕ reduces to a linear transformation. An example of this will be discussed in Chapter 8, where we derive the electroweak sector of the Standard Model as an almost commutative geometry.

6.22 Theorem. *A real even almost commutative geometry $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ describes a gauge theory with gauge group $\mathcal{G}(\mathcal{A}) = C^\infty(M, \mathcal{G}(\mathcal{A}_F))$.*

PROOF. In Proposition 6.16 we have obtained that $iA_\mu(x) \in \mathfrak{g}_F = \mathfrak{u}(\mathcal{A}_F)/\mathfrak{h}_F$. The total algebra is given by $\mathcal{A} = C^\infty(M, \mathcal{A}_F)$, and this is by construction the space of smooth sections of the trivial bundle $M \times \mathcal{A}_F$. Therefore the gauge field A_μ defines a global smooth $\mathfrak{g}_F$-valued 1-form $A = iA_\mu dx^\mu$. Consider the trivial principal bundle $P = M \times \mathcal{G}(\mathcal{A}_F)$. By Definition 2.32, A is a connection form on P.

By Proposition 4.14, we know that the gauge group of $P = M \times \mathcal{G}(\mathcal{A}_F)$ is given by $C^\infty(M, \mathcal{G}(\mathcal{A}_F))$, and by Proposition 6.14 this group is equal to $\mathcal{G}(\mathcal{A})$. This means that the gauge group of the principal bundle P, as given in Definition 4.10, is the same as the gauge group of the spectral triple, as given in Proposition 5.28. We have seen in Section 5.5.4 that the total Lagrangian we obtain from the bosonic and fermionic action functionals is invariant under this gauge group.

Since the representation of $\mathcal{A}_F$ on $\mathcal{H}_F$ induces a representation of $\mathcal{G}(\mathcal{A}_F)$ on $\mathcal{H}_F$, we see from Example 2.24 that $M \times \mathcal{H}_F$ is an associated vector bundle of the principal bundle $P = M \times \mathcal{G}(\mathcal{A}_F)$. We have thus seen that, from the spectral triple of an almost commutative geometry, we can recover all the ingredients of a gauge theory, as given in Definition 4.15. □

6.23 Remark. In the above theorem, we have used the gauge field A_μ to construct a connection on a (trivial) principal $\mathcal{G}(\mathcal{A}_F)$-bundle $P = M \times G$, for which $E = M \times \mathcal{H}_F$ is an associated vector bundle. One should note however that our Hilbert space $\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes E)$ consists of the sections of the total bundle $S \otimes E$, and this total bundle is not an associated vector bundle of P.

6.3 THE HEAT EXPANSION OF THE SPECTRAL ACTION

We wish to obtain an explicit formula for the spectral action of an almost commutative geometry. This explicit formula arises by considering the heat expansion. We will start by calculating a generalized Lichnerowicz formula for the square of the fluctuated Dirac operator. Next we will show how we can use this formula to obtain the heat expansion from the spectral action.

6.3.1 *A generalized Lichnerowicz formula*

6.24 Lemma. *The Laplacian of the connection ∇^E on the module $\Gamma(M, S) \otimes \mathcal{H}_F$ defined in Eq. (6.5) is given by*

$$\Delta^E = \Delta^S \otimes \mathbb{1} + 2i(\mathbb{1} \otimes B^\mu)(\nabla^S_\mu \otimes \mathbb{1}) + ig^{\mu\nu}(\mathbb{1} \otimes \partial_\mu B_\nu) + \mathbb{1} \otimes B_\mu B^\mu - ig^{\mu\nu}\Gamma^\rho{}_{\mu\nu} \otimes B_\rho. \tag{6.15}$$

PROOF. We use the local formula for the Laplacian, given by Lemma 2.46, to find

$$\begin{aligned} \Delta^E &= -g^{\mu\nu}\left(\nabla^E_\mu \nabla^E_\nu - \Gamma^\rho{}_{\mu\nu}\nabla^E_\rho\right) \\ &= \Delta^S \otimes \mathbb{1} - g^{\mu\nu}\Big(-i(\nabla^S_\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\nu) - i(\mathbb{1} \otimes B_\mu)(\nabla^S_\nu \otimes \mathbb{1}) \\ &\qquad - \mathbb{1} \otimes B_\mu B_\nu + i\Gamma^\rho{}_{\mu\nu} \otimes B_\rho\Big) \\ &= \Delta^S \otimes \mathbb{1} + 2i(\mathbb{1} \otimes B^\mu)(\nabla^S_\mu \otimes \mathbb{1}) + ig^{\mu\nu}(\mathbb{1} \otimes \partial_\mu B_\nu) \\ &\quad + \mathbb{1} \otimes B_\mu B^\mu - ig^{\mu\nu}\Gamma^\rho{}_{\mu\nu} \otimes B_\rho. \end{aligned}$$ □

6.25 Proposition. *The square of the flucutated Dirac operator of an almost commutative geometry is a generalized Laplacian of the form (cf. Definition 2.69)*

$$D_A{}^2 = \Delta^E - Q. \tag{6.16}$$

The endomorphism Q is given by

$$Q = -\frac{1}{4}s \otimes \mathbb{1} - \mathbb{1} \otimes \Phi^2 - \frac{1}{2} i\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} - i\gamma^\mu\gamma_5 \otimes D_\mu\Phi, \qquad (6.17)$$

where D_μ and $F_{\mu\nu}$ are defined in Definition 6.8.

Proof. Rewriting the formula for D_A, we have

$$\begin{aligned} {D_A}^2 &= \big(\not{D} \otimes \mathbb{1} + \gamma^\mu \otimes B_\mu + \gamma_5 \otimes \Phi\big)^2 \\ &= \not{D}^2 \otimes \mathbb{1} + \gamma^\mu\gamma^\nu \otimes B_\mu B_\nu + \mathbb{1} \otimes \Phi^2 + (\not{D}\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\mu) \\ &\quad + (\mathbb{1} \otimes B_\mu)(\gamma^\mu \not{D} \otimes \mathbb{1}) + (\not{D} \otimes \mathbb{1})(\gamma_5 \otimes \Phi) + (\gamma_5 \otimes \Phi)(\not{D} \otimes \mathbb{1}) \\ &\quad + (\gamma^\mu \otimes B_\mu)(\gamma_5 \otimes \Phi) + (\gamma_5 \otimes \Phi)(\gamma^\mu \otimes B_\mu). \end{aligned}$$

For the first term we use the Lichnerowicz formula of Proposition 3.50. We rewrite the second term into

$$\begin{aligned} \gamma^\mu\gamma^\nu \otimes B_\mu B_\nu &= \frac{1}{2}\gamma^\mu\gamma^\nu \otimes \big(B_\mu B_\nu + B_\nu B_\mu + [B_\mu, B_\nu]\big) \\ &= \mathbb{1} \otimes B_\mu B^\mu + \frac{1}{2}\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu]. \end{aligned}$$

where we have used the Clifford relation to obtain the second equality. For the fourth and fifth terms we use the local formula $\not{D} = i\gamma^\nu\nabla^S_\nu$ of Lemma 3.49 to obtain

$$\begin{aligned} &(\not{D}\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\mu) + (\mathbb{1} \otimes B_\mu)(\gamma^\mu \not{D} \otimes \mathbb{1}) \\ &\qquad = (i\gamma^\nu\nabla^S_\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\mu) + (\mathbb{1} \otimes B_\mu)(\gamma^\mu i\gamma^\nu\nabla^S_\nu \otimes \mathbb{1}). \end{aligned}$$

Using the identity $[\nabla^S_\nu, c(\alpha)] = c(\nabla_\nu\alpha)$ of Eq. (3.23) we find $[\nabla^S_\nu \otimes \mathbb{1}, (\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\mu)] = c\big(\nabla_\nu(\theta^\mu \otimes B_\mu)\big)$. We thus obtain

$$\begin{aligned} &(\not{D}\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\mu) + (\mathbb{1} \otimes B_\mu)(\gamma^\mu \not{D} \otimes \mathbb{1}) \\ &\quad = i(\gamma^\nu \otimes \mathbb{1})c\big(\nabla_\nu(\theta^\mu \otimes B_\mu)\big) + i(\gamma^\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes B_\mu)(\nabla^S_\nu \otimes \mathbb{1}) \\ &\qquad + i(\mathbb{1} \otimes B_\mu)(\gamma^\mu\gamma^\nu\nabla^S_\nu \otimes \mathbb{1}) \\ &\quad = i(\gamma^\nu \otimes \mathbb{1})c\big(\theta^\mu \otimes (\partial_\nu B_\mu) - {\Gamma^\rho}_{\mu\nu}\theta^\mu \otimes B_\rho\big) + 2i(\mathbb{1} \otimes B^\nu)(\nabla^S_\nu \otimes \mathbb{1}) \\ &\quad = i(\gamma^\nu\gamma^\mu \otimes \mathbb{1})\Big(\mathbb{1} \otimes \partial_\nu B_\mu - {\Gamma^\rho}_{\mu\nu} \otimes B_\rho\Big) + 2i(\mathbb{1} \otimes B^\nu)(\nabla^S_\nu \otimes \mathbb{1}) \\ &\quad = i(\gamma^\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes \partial_\nu B_\mu) - ig^{\mu\nu}{\Gamma^\rho}_{\mu\nu} \otimes B_\rho + 2i(\mathbb{1} \otimes B^\nu)(\nabla^S_\nu \otimes \mathbb{1}). \end{aligned}$$

The sixth and seventh terms are rewritten into

$$\begin{aligned} &(\not{D} \otimes \mathbb{1})(\gamma_5 \otimes \Phi) + (\gamma_5 \otimes \Phi)(\not{D} \otimes \mathbb{1}) = -(\gamma_5 \otimes \mathbb{1})\big[\not{D} \otimes \mathbb{1}, \mathbb{1} \otimes \Phi\big] \\ &\quad = -(\gamma_5 \otimes \mathbb{1})(i\gamma^\mu \otimes \partial_\mu\Phi) = i\gamma^\mu\gamma_5 \otimes \partial_\mu\Phi. \end{aligned}$$

The eighth and ninth terms are rewritten as

$$(\gamma^\mu \otimes B_\mu)(\gamma_5 \otimes \Phi) + (\gamma_5 \otimes \Phi)(\gamma^\mu \otimes B_\mu) = \gamma^\mu\gamma_5 \otimes [B_\mu, \Phi].$$

Summing all these terms then yields the formula

$$\begin{aligned} {D_A}^2 &= (\Delta^S + \frac{1}{4}s) \otimes \mathbb{1} + (\mathbb{1} \otimes B_\mu B^\mu) + \frac{1}{2}\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu] + \mathbb{1} \otimes \Phi^2 \\ &\quad + i(\gamma^\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes \partial_\nu B_\mu) - ig^{\mu\nu}{\Gamma^\rho}_{\mu\nu} \otimes B_\rho + 2i(\mathbb{1} \otimes B^\nu)(\nabla^S_\nu \otimes \mathbb{1}) \\ &\quad + i\gamma^\mu\gamma_5 \otimes \partial_\mu\Phi + \gamma^\mu\gamma_5 \otimes [B_\mu, \Phi]. \end{aligned}$$

Inserting the formula for Δ^E obtained in Lemma 6.24 we obtain

$$D_A{}^2 = \Delta^E + \frac{1}{4}s \otimes \mathbb{1} + \frac{1}{2}\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu] + \mathbb{1} \otimes \Phi^2 + i(\gamma^\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes \partial_\nu B_\mu) \\ - ig^{\mu\nu}(\mathbb{1} \otimes \partial_\mu B_\nu) + i\gamma^\mu\gamma_5 \otimes \partial_\mu\Phi + (\gamma^\mu\gamma_5 \otimes \mathbb{1})\big[\mathbb{1} \otimes B_\mu, \mathbb{1} \otimes \Phi\big].$$

Using Eq. (6.7), we shall rewrite

$$\begin{aligned}
&(\gamma^\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes \partial_\nu B_\mu) - g^{\mu\nu}(\mathbb{1} \otimes \partial_\mu B_\nu) \\
&= (\gamma^\nu\gamma^\mu \otimes \mathbb{1})(\mathbb{1} \otimes \partial_\nu B_\mu) - \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \otimes (\partial_\mu B_\nu) \\
&= \frac{1}{2}\gamma^\mu\gamma^\nu \otimes (\partial_\mu B_\nu) - \frac{1}{2}\gamma^\nu\gamma^\mu \otimes (\partial_\mu B_\nu) \\
&= \frac{1}{2}\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} + \frac{1}{2}i\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu].
\end{aligned}$$

Using Eq. (6.6), we thus finally obtain

$$D_A{}^2 = \Delta^E + \frac{1}{4}s \otimes \mathbb{1} + \mathbb{1} \otimes \Phi^2 + \frac{1}{2}i\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} + i\gamma^\mu\gamma_5 \otimes D_\mu\Phi,$$

from which we can read off the formula for Q. □

6.3.2 *The heat expansion*

We have seen in Proposition 6.25 that the square of the fluctuated Dirac operator of an almost commutative geometry is a generalized Laplacian. Applying Theorem 2.71 on $D_A{}^2$ then yields the heat expansion:

$$\operatorname{Tr}\left(e^{-tD_A{}^2}\right) \sim \sum_{k\geq 0} t^{\frac{k-4}{2}} a_k(D_A{}^2), \tag{6.18}$$

where the Seeley-DeWitt coefficients are given by Theorem 2.72. In the following proposition, we use this heat expansion for $D_A{}^2$ to obtain an expansion of the spectral action.

6.26 Proposition. *For an almost commutative geometry, the spectral action given by Eq. (5.21) can be expanded in powers of Λ in the form*

$$\operatorname{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) \sim a_4(D_A{}^2)f(0) + 2\sum_{\substack{0\leq k<4 \\ k \text{ even}}} f_{4-k}\Lambda^{4-k}a_k(D_A{}^2)\,\frac{1}{\Gamma\big(\frac{4-k}{2}\big)} + O(\Lambda^{-1}), \tag{6.19}$$

where $f_j = \int_0^\infty f(v)v^{j-1}dv$ are the momenta of the function f for $j > 0$.

PROOF. This proof is partly based on [13, Theorem 1.145]. Consider a function $g(u)$ and its Laplace transform

$$g(v) = \int_0^\infty e^{-sv}h(s)ds.$$

We can then formally write

$$g(tD_A{}^2) = \int_0^\infty e^{-stD_A{}^2}h(s)ds.$$

We now take the trace and use the heat expansion of $D_A{}^2$ to obtain

$$\begin{aligned}
\operatorname{Tr}\big(g(tD_A{}^2)\big) &= \int_0^\infty \operatorname{Tr}\big(e^{-stD_A{}^2}\big)h(s)ds \\
&\sim \int_0^\infty \sum_{k\geq 0}(st)^{\frac{k-4}{2}}a_k(D_A{}^2)h(s)ds \\
&= \sum_{k\geq 0} t^{\frac{k-4}{2}}a_k(D_A{}^2)\int_0^\infty s^{\frac{k-4}{2}}h(s)ds.
\end{aligned}$$

The parameter t is considered to be a small expansion parameter. From here on we will therefore drop the terms with $k > 4$. The term with $k = 4$ equals

$$a_4(D_A{}^2)\int_0^\infty s^0 h(s)ds = a_4(D_A{}^2)g(0).$$

We can rewrite the terms with $k < 4$ using the definition of the Γ-function as the analytic continuation of

$$\Gamma(z) = \int_0^\infty r^{z-1}e^{-r}dr \tag{6.20}$$

for $z \in \mathbb{C}$, and by inserting $r = sv$, we see that (for $k < 4$)

$$\Gamma\Big(\frac{4-k}{2}\Big) = \int_0^\infty (sv)^{\frac{4-k}{2}-1}e^{-sv}d(sv) = s^{\frac{4-k}{2}}\int_0^\infty v^{\frac{4-k}{2}-1}e^{-sv}dv.$$

From this we obtain an expression for $s^{\frac{k-4}{2}}$, which we insert into the equation for $\mathrm{Tr}\big(g(tD_A{}^2)\big)$, and then we perform the integration over s to obtain

$$\begin{aligned}\mathrm{Tr}\big(g(tD_A{}^2)\big) \sim\ & a_4(D_A{}^2)f(0)\\ &+\sum_{0\le k<4} t^{\frac{k-4}{2}}a_k(D_A{}^2)\,\frac{1}{\Gamma\big(\frac{4-k}{2}\big)}\int_0^\infty v^{\frac{4-k}{2}-1}g(v)dv + O(\Lambda^{-1}).\end{aligned}$$

Now we choose the function g such that $g(u^2) = f(u)$. We rewrite the integration over v by substituting $v = u^2$ and obtain

$$\int_0^\infty v^{\frac{4-k}{2}-1}g(v)dv = \int_0^\infty u^{4-k-2}g(u^2)d(u^2) = 2\int_0^\infty u^{4-k-1}f(u)du,$$

which by definition equals $2f_{4-k}$. Furthermore, we write $t = \Lambda^{-2}$, and we have

$$\begin{aligned}\mathrm{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) &= \mathrm{Tr}\left(g(\Lambda^{-2}D_A{}^2)\right)\\ &\sim a_4(D_A{}^2)f(0) + 2\sum_{0\le k<4} f_{4-k}\Lambda^{4-k}a_k(D_A{}^2)\,\frac{1}{\Gamma\big(\frac{4-k}{2}\big)} + O(\Lambda^{-1}).\end{aligned}$$

Using $a_k(D_A{}^2) = 0$ for odd k, we conclude

$$\mathrm{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) \sim a_4(D_A{}^2)f(0) + 2\sum_{\substack{0\le k<4\\ k\text{ even}}} f_{4-k}\Lambda^{4-k}a_k(D_A{}^2)\,\frac{1}{\Gamma\big(\frac{4-k}{2}\big)} + O(\Lambda^{-1}). \qquad \square$$

6.4 THE SPECTRAL ACTION OF ALMOST COMMUTATIVE GEOMETRIES

In this section, we will explicitly calculate the heat expansion for the canonical triple and for a general almost commutative geometry.

6.4.1 *Spectral gravity of the canonical triple*

By Proposition 6.26 we have

$$\mathrm{Tr}\left(f\Big(\frac{\not{D}}{\Lambda}\Big)\right) \sim 2f_4\Lambda^4 a_0(\not{D}^2) + 2f_2\Lambda^2 a_2(\not{D}^2) + f(0)a_4(\not{D}^2) + O(\Lambda^{-1}). \tag{6.21}$$

Recall the Lichnerowicz formula from Proposition 3.50, which says $\not{D}^2 = \Delta^S + \frac{1}{4}s$, where Δ^S is the Laplacian of the spin connection ∇^S, and s is the scalar curvature of the Levi-Civita connection. Using this formula, we can calculate the Seeley-DeWitt coefficients from Theorem 2.72.

6.27 Proposition. *For the canonical triple $(C^\infty(M), L^2(M,S), D\!\!\!/)$, the spectral action is given by:*

$$\mathrm{Tr}\left(f\left(\frac{D\!\!\!/}{\Lambda}\right)\right) \sim \frac{1}{4\pi^2}\int_M \mathcal{L}_M(g_{\mu\nu})\sqrt{|g|}d^4x + O(\Lambda^{-1}), \tag{6.22}$$

where the Lagrangian is defined by

$$\mathcal{L}_M(g_{\mu\nu}) := 2f_4\Lambda^4 - \frac{1}{6}f_2\Lambda^2 s + f(0)\Big(\frac{1}{120}\Delta s - \frac{1}{80}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{1440}R^*R^*\Big). \tag{6.23}$$

PROOF. We have $m = \dim M = 4$, and $\mathrm{Tr}(\mathrm{Id}) = \dim S = 2^{m/2} = 4$. Inserting this into Eq. (2.74) gives

$$a_0(D\!\!\!/^2) = \frac{1}{4\pi^2}\int_M \sqrt{|g|}d^4x.$$

From the Lichnerowicz formula we see that $F = -\frac{1}{4}s\,\mathrm{Id}$, so

$$a_2(D\!\!\!/^2) = -\frac{1}{48\pi^2}\int_M s\sqrt{|g|}d^4x.$$

Using $F = -\frac{1}{4}s\,\mathrm{Id}$ we calculate

$$5s^2\mathrm{Id} + 60sF + 180F^2 = \frac{5}{4}s^2\mathrm{Id}.$$

Inserting this into $a_4(D\!\!\!/^2)$ gives

$$a_4(D\!\!\!/^2) = \frac{1}{16\pi^2}\frac{1}{360}\int_M \mathrm{Tr}\big(3\Delta s\,\mathrm{Id} + \frac{5}{4}s^2\mathrm{Id} - 2R_{\mu\nu}R^{\mu\nu}\mathrm{Id} \\ + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\mathrm{Id} + 30\Omega^S_{\mu\nu}\Omega^{S\mu\nu}\big)\sqrt{|g|}d^4x.$$

The curvature Ω^S of the spin connection is defined as in Definition 2.38, its components are $\Omega^S_{\mu\nu} = \Omega^S(\partial_\mu, \partial_\nu)$. As in Eq. (3.26), we have $\Omega^S_{\mu\nu} = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma$. We use this and Eq. (A.4) to calculate the last term of $a_4(D\!\!\!/^2)$:

$$\begin{aligned}\mathrm{Tr}(\Omega^S_{\mu\nu}\Omega^{S\mu\nu}) &= \frac{1}{16}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\kappa}\,\mathrm{Tr}(\gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\kappa) \\ &= \frac{1}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\kappa}\,(g^{\rho\sigma}g^{\lambda\kappa} - g^{\rho\lambda}g^{\sigma\kappa} + g^{\rho\kappa}g^{\sigma\lambda}) \\ &= -\frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},\end{aligned} \tag{6.24}$$

where on the second line because of the antisymmetry of $R_{\mu\nu\rho\sigma}$ in ρ and σ, the first term vanishes and the other two terms contribute equally. We thus obtain

$$a_4(D\!\!\!/^2) = \frac{1}{16\pi^2}\frac{1}{360}\int_M \big(12\Delta s + 5s^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big)\sqrt{|g|}d^4x. \tag{6.25}$$

Using Eq. (B.43) and Eq. (2.69) we calculate:

$$\begin{aligned}&-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^* \\ &= -\frac{1}{20}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{10}R_{\nu\sigma}R^{\nu\sigma} - \frac{1}{60}s^2 + \frac{11}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{44}{360}R_{\nu\sigma}R^{\nu\sigma} + \frac{11}{360}s^2 \\ &= \frac{1}{360}\big(-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8R_{\nu\sigma}R^{\nu\sigma} + 5s^2\big)\end{aligned} \tag{6.26}$$

Therefore we can rewrite Eq. (6.25) and obtain

$$a_4(D\!\!\!/^2) = \frac{1}{16\pi^2}\int_M \Big(\frac{1}{30}\Delta s - \frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\Big)\sqrt{|g|}d^4x.$$

Inserting the obtained formulas for $a_0(D\!\!\!/^2)$, $a_2(D\!\!\!/^2)$ and $a_4(D\!\!\!/^2)$ into Eq. (6.21) proves the proposition. □

6.28 Remark. In Proposition 4.22 we have evaluated the general form of the gravitational Lagrangian. One should note here that the term s^2 is not present in the spectral action of the canonical triple. The only higher-order gravitational term that arises is the conformal gravity term $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$. This feature of the spectral action will later allow us in Section 6.5 to derive the conformal symmetry of the spectral action.

Note that alternatively, using only Eq. (2.69), we could also have written

$$a_4(\not{D}^2) = \frac{1}{16\pi^2}\frac{1}{30}\int_M \left(\Delta s + s^2 - 3R_{\mu\nu}R^{\mu\nu} - \frac{7}{12}R^*R^*\right)\sqrt{|g|}d^4x.$$

If the manifold M is compact without boundary, we can discard the term with Δs by Corollary 2.56. Furthermore, by Corollary 2.67 the term with R^*R^* integrates to Euler's constant, which we will also disregard. From here on, we will consider the Lagrangian

$$\mathcal{L}_M(g_{\mu\nu}) = 2f_4\Lambda^4 - \frac{1}{6}f_2\Lambda^2 s - \frac{1}{80}f(0)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \tag{6.27}$$

or

$$\mathcal{L}_M(g_{\mu\nu}) = 2f_4\Lambda^4 - \frac{1}{6}f_2\Lambda^2 s + \frac{1}{120}f(0)\left(s^2 - 3R_{\mu\nu}R^{\mu\nu}\right). \tag{6.28}$$

6.4.2 *Spectral gravity of almost commutative geometries*

6.29 Proposition. *The spectral action of the fluctuated Dirac operator of an almost commutative geometry is given by*

$$\mathrm{Tr}\left(f\left(\frac{D_A}{\Lambda}\right)\right) \sim \frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{|g|}d^4x + O(\Lambda^{-1}), \tag{6.29}$$

for

$$\mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) := N\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_B(B_\mu) + \mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi). \tag{6.30}$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 6.27, and N is the dimension of the finite Hilbert space $\mathcal{H}_F$. $\mathcal{L}_B$ gives the kinetic term of the gauge field and equals

$$\mathcal{L}_B(B_\mu) := \frac{1}{6}f(0)\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{6.31}$$

$\mathcal{L}_H$ gives the Higgs potential including its interactions plus a boundary term given by

$$\begin{aligned}\mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi) := -2f_2\Lambda^2\mathrm{Tr}(\Phi^2) + \frac{1}{2}f(0)\mathrm{Tr}(\Phi^4) + \frac{1}{6}f(0)\Delta\big(\mathrm{Tr}(\Phi^2)\big)\\ + \frac{1}{12}f(0)s\mathrm{Tr}(\Phi^2) + \frac{1}{2}f(0)\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big).\end{aligned} \tag{6.32}$$

Proof. The proof is very similar to Proposition 6.27, but we now use the formula for ${D_A}^2$ given by Proposition 6.25. The trace over the Hilbert space $\mathcal{H}_F$ yields an overall factor $N := \mathrm{Tr}(\mathbb{1}_{\mathcal{H}_F})$, so we have

$$a_0({D_A}^2) = Na_0(\not{D}^2).$$

The square of the Dirac operator now contains three extra terms. The trace of $\gamma^\mu\gamma_5$ vanishes (cf. Eq. (A.6)). Since $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $F_{\mu\nu}$ is anti-symmetric, the trace of $\gamma^\mu\gamma^\nu F_{\mu\nu}$ also vanishes. Thus we find that

$$a_2({D_A}^2) = Na_2(\not{D}^2) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}(\Phi^2)\sqrt{|g|}d^4x.$$

Furthermore we obtain several new terms from the formula for $a_4(D_A{}^2)$. First we calculate

$$\frac{1}{360}\mathrm{Tr}(60sF) = -\frac{1}{6}s\left(Ns + 4\mathrm{Tr}(\Phi^2)\right).$$

The next contribution arises from the trace over F^2, which (ignoring traceless terms) equals

$$F^2 = \frac{1}{16}s^2 \otimes \mathbb{1} + \mathbb{1} \otimes \Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \otimes F_{\mu\nu}F_{\rho\sigma} + \gamma^\mu\gamma^\nu \otimes (D_\mu\Phi)(D_\nu\Phi) + \frac{1}{2}s \otimes \Phi^2 + \text{ traceless terms.}$$

We use the trace identities of Appendix A to obtain

$$\frac{1}{360}\mathrm{Tr}(180F^2) = \frac{N}{8}s^2 + 2\mathrm{Tr}(\Phi^4) + \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + 2\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) + s\mathrm{Tr}(\Phi^2).$$

Another contribution arises from $-\Delta F$. Again we can simply ignore the traceless terms and obtain

$$\frac{1}{360}\mathrm{Tr}(-60\Delta F) = \frac{1}{6}\Delta\left(Ns + 4\mathrm{Tr}(\Phi^2)\right).$$

The final contribution comes from the term $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$, where the curvature Ω^E is given by Lemma 6.9. We have

$$\Omega^E_{\mu\nu}\Omega^{E\mu\nu} = \Omega^S_{\mu\nu}\Omega^{S\mu\nu} \otimes \mathbb{1} - \mathbb{1} \otimes F_{\mu\nu}F^{\mu\nu} - 2i\Omega^S_{\mu\nu} \otimes F^{\mu\nu}.$$

Using Eq. (3.26), we find

$$\mathrm{Tr}(\Omega^S_{\mu\nu}) = \frac{1}{4}R_{\rho\sigma\mu\nu}\mathrm{Tr}(\gamma^\rho\gamma^\sigma) = \frac{1}{4}R_{\rho\sigma\mu\nu}g^{\rho\sigma} = 0$$

by the anti-symmetry of $R_{\rho\sigma\mu\nu}$, so the trace over the cross-terms in $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$ vanishes. From Eq. (6.24) we then obtain

$$\frac{1}{360}\mathrm{Tr}(30\Omega^E_{\mu\nu}\Omega^{E\mu\nu}) = \frac{1}{12}\left(-\frac{N}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right).$$

Gathering all terms, we obtain

$$\begin{aligned}
a_4(x, D_A{}^2) = \frac{1}{(4\pi)^2}\frac{1}{360}\Bigg(& -48N\Delta s + 20Ns^2 - 8NR_{\mu\nu}R^{\mu\nu} + 8NR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
& - 60s\left(Ns + 4\mathrm{Tr}(\Phi^2)\right) + 360\bigg(\frac{N}{8}s^2 + 2\mathrm{Tr}(\Phi^4) + \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
& + 2\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) + s\mathrm{Tr}(\Phi^2)\bigg) \\
& + 60\Delta\left(Ns + 4\mathrm{Tr}(\Phi^2)\right) - 30\left(\frac{N}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 4\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right)\Bigg) \\
= \frac{1}{(4\pi)^2}\frac{1}{360}\Bigg(& 12N\Delta s + 5Ns^2 - 8NR_{\mu\nu}R^{\mu\nu} - 7NR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
& + 120s\mathrm{Tr}(\Phi^2) + 360\Big(2\mathrm{Tr}(\Phi^4) + 2\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big)\Big) \\
& + 240\Delta\left(\mathrm{Tr}(\Phi^2)\right) + 240\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\Bigg).
\end{aligned}$$

By comparing the first line of the second equality to Eq. (6.25), we see that we can write

$$a_4(x, {D_A}^2) = Na_4(x, \not{D}^2) + \frac{1}{4\pi^2}\Bigg(\frac{1}{12}s\mathrm{Tr}(\Phi^2) + \frac{1}{2}\mathrm{Tr}(\Phi^4) + \frac{1}{2}\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) + \frac{1}{6}\Delta\left(\mathrm{Tr}(\Phi^2)\right) + \frac{1}{6}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\Bigg).$$

Inserting these Seeley-DeWitt coefficients into Eq. (6.21) proves the proposition. □

6.30 Remark. In the previous proposition, we have obtained the Higgs potential

$$\mathcal{L}_{\mathrm{pot}}(\Phi) := -2f_2\Lambda^2\mathrm{Tr}(\Phi^2) + \frac{1}{2}f(0)\mathrm{Tr}(\Phi^4) + \frac{1}{12}f(0)s\mathrm{Tr}(\Phi^2), \tag{6.33}$$

which includes a coupling of the Higgs field to the scalar curvature s. Let the field V be such that $\mathcal{L}_{\mathrm{pot}}(V)$ obtains its minimum on all points of M. We shall call V the *vacuum expectation value* of Φ. Note that in general there is no unique choice for V. We can now apply the Higgs mechanism by defining the new *physical* Higgs field $H := \Phi - V$. In Chapter 8 we discuss this Higgs mechanism in the Glashow-Weinberg-Salam model. We will show that the gauge symmetry is spontaneously broken, so that we can obtain mass terms for the gauge bosons. Because of the coupling of the Higgs field to the scalar curvature, we will adjust the Higgs mechanism to also incorporate a conformal transformation. This will lead to a spontaneous breaking of the conformal symmetry as well.

6.5 CONFORMAL INVARIANCE OF THE SPECTRAL ACTION

In [6], the scale invariance of the spectral action is discussed. In this section, we shall explicitly calculate the conformal transformation of the asymptotic expansion of the spectral action for a general almost commutative geometry, and show that it is invariant up to a kinetic term of a dilaton field.

Under a conformal transformation, we will let the fields B_μ and Φ of Proposition 6.4 transform as

$$\widetilde{B}_\mu = B_\mu, \tag{6.34a}$$

$$\widetilde{\Phi} = \Omega^{-1}\Phi. \tag{6.34b}$$

The spectral action depends on the choice of the cut-off scale Λ, and it is no surprise that a conformal transformation also affects this cut-off scale. We let the new scale be given by

$$\widetilde{\Lambda} = \Omega^{-1}\Lambda. \tag{6.35}$$

6.31 Proposition. *A conformal transformation of the spectral action of an almost commutative geometry (cf. Proposition 6.29) yields (ignoring boundary terms) the Lagrangian*

$$\mathcal{L}(\widetilde{g}_{\mu\nu}, \widetilde{B}_\mu, \widetilde{\Phi}, \widetilde{\Lambda}) = \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi, \Lambda) + Nf_2\Omega^{-2}\Lambda^2\nabla^\beta(\Omega)\nabla_\beta(\Omega)\sqrt{|g|}. \tag{6.36}$$

PROOF. We shall calculate the conformal transformations of the different terms in the Lagrangian separately. We shall ignore topological and boundary terms and use Eq. (6.27) for $\mathcal{L}_M$. We then find that

$$\mathcal{L}_M(\widetilde{g}_{\mu\nu}, \widetilde{\Lambda})\sqrt{|\widetilde{g}|} = 2f_4\Omega^{-4}\Lambda^4\Omega^4\sqrt{|g|} - \frac{1}{6}f_2\Omega^{-2}\Lambda^2\widetilde{s}\Omega^4\sqrt{|g|} - \frac{1}{80}f(0)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|},$$

where we have used the conformal invariance of the Weyl action (cf. Proposition 4.21). Inserting the formula for $\widetilde{s}$ obtained in Proposition 4.18, we obtain several extra terms, and we have

$$\begin{aligned}\mathcal{L}_M(\widetilde{g}_{\mu\nu}, \widetilde{\Lambda})\sqrt{|\widetilde{g}|} &= \mathcal{L}_M(g_{\mu\nu}, \Lambda)\sqrt{|g|} \\ &\quad + f_2\Lambda^2\Big(\nabla^\beta\nabla_\beta(\ln\Omega) + \nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\Big)\sqrt{|g|}.\end{aligned}$$

On the second line, the first term is a total divergence and yields a boundary term, which we will ignore, and the second term can be rewritten as

$$\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega) = \Omega^{-2}\nabla^\beta(\Omega)\nabla_\beta(\Omega).$$

Hence we obtain (ignoring the boundary term)

$$\mathcal{L}_M(\widetilde{g}_{\mu\nu}, \widetilde{\Lambda})\sqrt{|\widetilde{g}|} = \mathcal{L}_M(g_{\mu\nu}, \Lambda)\sqrt{|g|} + f_2\Lambda^2\Omega^{-2}\nabla^\beta(\Omega)\nabla_\beta(\Omega)\sqrt{|g|}.$$

Since the gauge field B_μ does not transform, neither does $F_{\mu\nu}$. However, we do have

$$\widetilde{F}^{\mu\nu} = \widetilde{g}^{\mu\alpha}\widetilde{g}^{\nu\beta}\widetilde{F}_{\alpha\beta} = \Omega^{-4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = \Omega^{-4}F^{\mu\nu}.$$

From this we find that the kinetic term of the gauge field remains invariant under conformal transformations:

$$\mathcal{L}_B(\widetilde{B}_\mu)\sqrt{|\widetilde{g}|} = \frac{1}{6}f(0)\mathrm{Tr}(F_{\mu\nu}\Omega^{-4}F^{\mu\nu})\Omega^4\sqrt{|g|} = \mathcal{L}_B(B_\mu)\sqrt{|g|}.$$

We shall split $\mathcal{L}_H$ into two parts, and we shall write $\mathcal{L}_1$ for the Higgs potential (ignoring the boundary term) and $\mathcal{L}_2$ for the kinetic term and the minimal coupling to the other fields. The Higgs potential $\mathcal{L}_1$ transforms as

$$\begin{aligned}\mathcal{L}_1(\widetilde{\Phi}, \widetilde{\Lambda})\sqrt{|\widetilde{g}|} &= -2f_2\Omega^{-2}\Lambda^2\mathrm{Tr}(\Omega^{-2}\Phi^2)\Omega^4\sqrt{|g|} + \frac{1}{2}f(0)\mathrm{Tr}(\Omega^{-4}\Phi^4)\Omega^4\sqrt{|g|} \\ &= -2f_2\Lambda^2\mathrm{Tr}(\Phi^2)\sqrt{|g|} + \frac{1}{2}f(0)\mathrm{Tr}(\Phi^4)\sqrt{|g|} \\ &= \mathcal{L}_1(\Phi, \Lambda)\sqrt{|g|}.\end{aligned}$$

For the last part of the Lagrangian we have

$$\begin{aligned}\mathcal{L}_2(\widetilde{g}_{\mu\nu}, \widetilde{B}_\mu, \widetilde{\Phi})\sqrt{|\widetilde{g}|} &= \frac{1}{12}f(0)\widetilde{s}\mathrm{Tr}(\Omega^{-2}\Phi^2)\Omega^4\sqrt{|g|} \\ &\quad + \frac{1}{2}f(0)\mathrm{Tr}\big((\widetilde{D}_\mu\Omega^{-1}\Phi)(\widetilde{D}^\mu\Omega^{-1}\Phi)\big)\Omega^4\sqrt{|g|}.\end{aligned}$$

The first term is given by

$$\begin{aligned}\frac{1}{12}f(0)\widetilde{s}\mathrm{Tr}(\Omega^{-2}\Phi^2)\Omega^4\sqrt{|g|} &= \frac{1}{12}f(0)s\mathrm{Tr}(\Phi^2)\sqrt{|g|} \\ &\quad - \frac{1}{2}f(0)\Big(\nabla^\beta\nabla_\beta(\ln\Omega) + \nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega)\Big)\mathrm{Tr}(\Phi^2)\sqrt{|g|}.\end{aligned}$$

We shall rewrite

$$\nabla^\beta\nabla_\beta(\ln\Omega) = \nabla^\beta\Big(\Omega^{-1}\nabla_\beta(\Omega)\Big) = -\Omega^{-2}\nabla^\beta(\Omega)\nabla_\beta(\Omega) + \Omega^{-1}\nabla^\beta\nabla_\beta(\Omega)$$

and

$$\nabla^\beta(\ln\Omega)\nabla_\beta(\ln\Omega) = \Omega^{-2}\nabla^\beta(\Omega)\nabla_\beta(\Omega),$$

and obtain

$$\frac{1}{12}f(0)\tilde{s}\mathrm{Tr}(\Omega^{-2}\Phi^2)\Omega^4\sqrt{|g|} = \frac{1}{12}f(0)s\mathrm{Tr}(\Phi^2)\sqrt{|g|} - \frac{1}{2}f(0)\Omega^{-1}\nabla^\beta\nabla_\beta(\Omega)\mathrm{Tr}(\Phi^2)\sqrt{|g|}.$$

D_μ has been defined in Proposition 6.25 by $D_\mu\Phi = [\nabla^E_\mu, \Phi]$. The transformation of ∇^E is determined by the transformation of ∇ as given in Lemma 4.17, and it only yields new terms which commute with Φ. Therefore we can conclude that $\widetilde{D}_\mu = D_\mu$, and $\widetilde{D}^\mu = \Omega^{-2}D^\mu$. From this we find that

$$\widetilde{D}_\mu\Omega^{-1}\Phi = \Omega^{-1}(D_\mu\Phi) + (\partial_\mu\Omega^{-1})\Phi.$$

We then find that the second term of $\mathcal{L}_2$ decomposes as

$$\frac{1}{2}f(0)\mathrm{Tr}\big((D_\mu\Omega^{-1}\Phi)(D^\mu\Omega^{-1}\Phi)\big)\Omega^2\sqrt{|g|} = \frac{1}{2}f(0)\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big)\sqrt{|g|} + \frac{1}{2}f(0)\Omega(\partial_\mu\Omega^{-1})\mathrm{Tr}\big(D^\mu\Phi^2\big)\sqrt{|g|} + \frac{1}{2}f(0)\Omega^2(\partial_\mu\Omega^{-1})(\partial^\mu\Omega^{-1})\mathrm{Tr}\big(\Phi^2\big)\sqrt{|g|}.$$

Note that $\mathrm{Tr}\big([B^\mu, \Phi^2]\big) = 0$ by the cyclic property of the trace, so that $\mathrm{Tr}\big(D^\mu\Phi^2\big) = \mathrm{Tr}\big(\partial^\mu\Phi^2\big)$. Combining both terms of $\mathcal{L}_2$, we see that

$$\mathcal{L}_2(\tilde{g}_{\mu\nu}, \widetilde{B}_\mu, \widetilde{\Phi})\sqrt{|\tilde{g}|} = \mathcal{L}_2(g_{\mu\nu}, B_\mu, \Phi)\sqrt{|g|} - \frac{1}{2}f(0)\Omega^{-1}\nabla^\beta\nabla_\beta(\Omega)\mathrm{Tr}(\Phi^2)\sqrt{|g|} - \frac{1}{2}f(0)\Omega^{-1}(\partial_\mu\Omega)\mathrm{Tr}\big(\partial^\mu\Phi^2\big)\sqrt{|g|} + \frac{1}{2}f(0)\Omega^{-2}(\partial_\mu\Omega)(\partial^\mu\Omega)\mathrm{Tr}\big(\Phi^2\big)\sqrt{|g|}.$$

By using the fact that $\nabla_\beta f = \partial_\beta f$ for functions $f \in C^\infty(M)$, we note that

$$\nabla^\beta\Big(\Omega^{-1}\nabla_\beta(\Omega)\mathrm{Tr}(\Phi^2)\Big) = -\Omega^{-2}\partial^\beta(\Omega)\partial_\beta(\Omega)\mathrm{Tr}(\Phi^2) + \Omega^{-1}\nabla^\beta\nabla_\beta(\Omega)\mathrm{Tr}(\Phi^2) + \Omega^{-1}\partial_\beta(\Omega)\mathrm{Tr}(\partial^\beta\Phi^2),$$

and thus we can conclude that

$$\mathcal{L}_2(\tilde{g}_{\mu\nu}, \widetilde{B}_\mu, \widetilde{\Phi})\sqrt{|\tilde{g}|} = \mathcal{L}_2(g_{\mu\nu}, B_\mu, \Phi)\sqrt{|g|} - \frac{1}{2}f(0)\nabla^\beta\Big(\Omega^{-1}\nabla_\beta(\Omega)\mathrm{Tr}(\Phi^2)\Big)\sqrt{|g|}.$$

Ignoring this boundary term, we see that $\mathcal{L}_H$ is invariant under conformal transformations. □

6.32 Remark. Let us write the conformal scaling factor as $\Omega = e^\eta$. We can then write the transformation of the Lagrangian as

$$\mathcal{L}(\tilde{g}_{\mu\nu}, \widetilde{B}_\mu, \widetilde{\Phi}, \widetilde{\Lambda}) = \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi, \Lambda) + Nf_2\Lambda^2(\partial^\beta\eta)(\partial_\beta\eta)\sqrt{|g|}. \tag{6.37}$$

So the only effect of the conformal transformation is that we obtain in the Lagrangian the kinetic term $(\partial^\beta\eta)(\partial_\beta\eta)$ of a *dilaton field* η.

Part III

EXAMPLES OF ALMOST COMMUTATIVE GEOMETRIES

Few things are harder to put up with than the annoyance of a good example.

— Mark Twain

7 ELECTRODYNAMICS

7.1 THE TWO-POINT SPACE

In this section we will provide an example of a commutative spectral triple which will be shown to describe a $U(1)$ gauge theory. In [27, Chapter 9], a proof is given for the claim that the inner fluctuation $A + JAJ^*$ vanishes for commutative algebras. The proof is based on the claim that the left and right action can be identified, i.e. $a = a^0$, for a commutative algebra. Though this claim holds in the case of the canonical spectral triple, it need not be true for arbitrary commutative algebras. The spectral triple given below provides a counter-example.

7.1.1 *A two-point space*

In this section we will discuss one of the simplest possible finite spectral triples, namely that of the two-point space $X = \{x, y\}$. A complex function on this space is simply determined by two complex numbers. The algebra of functions on X is then given by $C(X) = \mathbb{C}^2$.

7.1 Proposition. *The most general form of an* even *spectral triple for the two-point space, with a faithful irreducible representation $\mathcal{A}_F \to B(\mathcal{H}_F)$, is given by the data*

$$(\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F) = \left(\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{7.1}$$

PROOF. Since we require the representation to be faithful, the Hilbert space $\mathcal{H}_F$ must be at least 2-dimensional, and the irreducibility implies that $\mathcal{H}_F$ is at most 2-dimensional. We thus obtain $\mathcal{H}_F = \mathbb{C}^2$.

We must have a $\mathbb{Z}/2$-grading γ_F, and use this to decompose $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$ into the two eigenspaces $\mathcal{H}_F^\pm = \{\psi \in \mathcal{H}_F \mid \gamma_F \psi = \pm\psi\}$. Hence, we can write

$$\gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2}$$

By Proposition 5.8, the self-adjoint Dirac operator must be off-diagonal and the action of an element $a \in \mathcal{A}_F$ on $\psi \in \mathcal{H}_F$ can be written as $a\psi = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. □

In Section 5.1, we have mentioned that for the canonical triple, the geodesic distance is equal to the formula

$$d_{\not{D}}(x, y) = \sup\left\{|a(x) - a(y)| \colon a \in \mathcal{A}, \|[\not{D}, a]\| \leq 1\right\}. \tag{7.3}$$

We will use this formula as a generalized notion of distance, so on our finite spectral triple we can write

$$d_{D_F}(x, y) = \sup\left\{|a(x) - a(y)| \colon a \in \mathcal{A}_F, \|[D_F, a]\| \leq 1\right\}. \tag{7.4}$$

Note that we now have only two distinct points x and y in the space X, and we shall calculate the distance between these points. An element $a \in \mathbb{C}^2 = C(X)$ is given by $\begin{pmatrix} a(x) & 0 \\ 0 & a(y) \end{pmatrix}$, so the commutator with D_F becomes

$$[D_F, a] = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} \begin{pmatrix} a(x) & 0 \\ 0 & a(y) \end{pmatrix} - \begin{pmatrix} a(x) & 0 \\ 0 & a(y) \end{pmatrix} \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} = \big(a(y) - a(x)\big) \begin{pmatrix} 0 & t \\ -\bar{t} & 0 \end{pmatrix}. \tag{7.5}$$

The norm of this commutator is given by $|a(y)-a(x)|\,|t|$, so $\|[D_F,a]\| \leq 1$ implies $|a(y)-a(x)| \leq \frac{1}{|t|}$. We thus obtain that the distance between the two points x and y is given by

$$d_{D_F}(x,y) = \frac{1}{|t|}. \tag{7.6}$$

Next, we want to introduce a real structure on the spectral triple, so we must give an antilinear isomorphism J_F on $\mathbb{C}^2$ which satisfies Definition 5.5. We must have $J_F{}^2 = \epsilon$ and $J_F\gamma_F = \epsilon''\gamma_F J_F$, and we shall consider all possible (even) KO-dimensions separately. We can use Proposition 5.10 to obtain the general form, and then impose the commutation rules of Definition 5.5.

KO-DIMENSION 0
: We have $J_F = \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix} C$ for $j_\pm \in U(1)$. We then calculate that for $b = \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}$ we obtain $b^0 = \begin{pmatrix} j_+ b_+ \overline{j_+} & 0 \\ 0 & j_- b_- \overline{j_-} \end{pmatrix} = b$, and see that this indeed commutes with the left action of $a \in \mathbb{C}^2$. Next, we check the order one condition

$$\begin{aligned} 0 = \left[[D_F,a],b^0\right] &= \left[\begin{pmatrix} 0 & ta_- - a_+t \\ \bar{t}a_+ - a_-\bar{t} & 0 \end{pmatrix}, \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}\right] \\ &= (a_+ - a_-)(b_+ - b_-)D_F. \end{aligned}$$

Since this must hold for all $a,b \in \mathbb{C}^2$, we conclude that we must require $D_F = 0$.

KO-DIMENSION 2
: We have $J_F = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} C$ for $j \in U(1)$. We then calculate that for $b = \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}$ we obtain $b^0 = \begin{pmatrix} jb_-\bar{j} & 0 \\ 0 & jb_+\bar{j} \end{pmatrix} = \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}$, and see that this indeed commutes with the left action of $a \in \mathbb{C}^2$. Next, we check the order one condition

$$\begin{aligned} 0 = \left[[D_F,a],b^0\right] &= \left[\begin{pmatrix} 0 & ta_- - a_+t \\ \bar{t}a_+ - a_-\bar{t} & 0 \end{pmatrix}, \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}\right] \\ &= (a_+ - a_-)(b_- - b_+)D_F. \end{aligned}$$

Again we conclude that we must require $D_F = 0$.

KO-DIMENSION 4
: We have $J_F = \begin{pmatrix} j_+ & 0 \\ 0 & j_- \end{pmatrix} C$ for $j_\pm = -j_\pm^T \in U(1)$, but this implies that $j_\pm = 0$, so the given spectral triple cannot have a real structure in KO-dimension 4.

KO-DIMENSION 6
: We have $J_F = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} C$ for $j \in U(1)$. We then calculate that for $b = \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}$ we obtain $b^0 = \begin{pmatrix} jb_-\bar{j} & 0 \\ 0 & jb_+\bar{j} \end{pmatrix} = \begin{pmatrix} b_- & 0 \\ 0 & b_+ \end{pmatrix}$, just as for KO-dimension 2. Hence again the commutation rules are only satisfied for $D_F = 0$.

To conclude, we have now proven the following proposition:

7.2 Proposition. *The spectral triple of the two-point space, given by Proposition 7.1, can only have a real structure if $D_F = 0$.*

7.3 Remark. Note that we have calculated in Eq. (7.6) that the distance between the points of the two-point space is equal to $\frac{1}{|t|}$, where t is the complex parameter in the finite Dirac operator D_F. Then, for a *real* spectral triple of the two-point space, we have $t = 0$, so the distance between the two points becomes infinite.

7.1.2 *The product space*

In Chapter 6 we have evaluated almost commutative geometries. We will now consider the specific case of the product of the canonical spectral triple with the two-point space, given by the data

$$\Big(C^\infty(M,\mathbb{C}^2), L^2(M,S)\otimes\mathbb{C}^2, \not{D}\otimes\mathbb{1}, \gamma_5\otimes\gamma_F, J_M\otimes J_F\Big), \tag{7.7}$$

where we still need to make a choice for J_F. The algebra of this almost commutative geometry is given by $C^\infty(M,\mathbb{C}^2)\cong C^\infty(M)\oplus C^\infty(M)$. By the Gelfand-Naimark theorem 5.1, this algebra corresponds to the space $N := M\times X\cong M\sqcup M$, which consists of the disjoint union of two identical copies of the space M, and we can write $C^\infty(N)=C^\infty(M)\oplus C^\infty(M)$. We can also decompose the total Hilbert space as $\mathcal{H}=L^2(M,S)\oplus L^2(M,S)$. For $a,b\in C^\infty(M)$ and $\psi,\phi\in L^2(M,S)$, an element $(a,b)\in C^\infty(N)$ then simply acts on $(\psi,\phi)\in\mathcal{H}$ as $(a,b)(\psi,\phi)=(a\psi,b\phi)$.

From Remark 7.3 we know that the distance between the two points x,y in X is infinite. Let p be a point in M, and write (p,x) or (p,y) for the two corresponding points in $N=M\times X$. A function $a\in C^\infty(N)$ is then determined by two functions $a_x,a_y\in C^\infty(M)$, given by $a_x(p):=a(p,x)$ and $a_y(p):=a(p,y)$. Now consider the distance function on N given by

$$d_{\not{D}\otimes\mathbb{1}}(n_1,n_2)=\sup\left\{|a(n_1)-a(n_2)| : a\in\mathcal{A}, \|[\not{D}\otimes\mathbb{1},a]\|\le 1\right\}. \tag{7.8}$$

If n_1 and n_2 are points in the same copy of M, for instance if $n_1=(p,x)$ and $n_2=(q,x)$ for points $p,q\in M$, then their distance is determined by $|a_x(p)-a_x(q)|$, for functions $a_x\in C^\infty(M)$ for which $\|[\not{D},a_x]\|\le 1$. Thus, in this case we obtain that we recover the geodesic distance on M, i.e. $d_{\not{D}\otimes\mathbb{1}}(n_1,n_2)=d_g(p,q)$.

However, if n_1 and n_2 are points in a different copy of M, for instance if $n_1=(p,x)$ and $n_2=(q,y)$, then their distance is determined by $|a_x(p)-a_y(q)|$ for two functions $a_x,a_y\in C^\infty(M)$, such that $\|[\not{D},a_x]\|\le 1$ and $\|[\not{D},a_y]\|\le 1$. These latter requirements however yield no restriction on $|a_x(p)-a_y(q)|$, so in this case the distance between n_1 and n_2 is infinite. We thus find that the space N is given by two disjoint copies of M, that are separated by an infinite distance.

It should be noted that the only way in which the distance between the two copies of M could have been finite, is when the commutator $[D_F,a]$ would be nonzero. This same commutator generates the field ϕ of Proposition 6.3, hence the finiteness of the distance is related to the existence of a Higgs field.

7.1.3 $U(1)$ *Gauge Theory*

Our objective is to calculate the action functional for this almost commutative spectral triple, and we want this action to be different from the spectral action of the canonical triple. In Proposition 6.4, we have found the fields B_μ and Φ. Since the Dirac operator D_F on the two-point space must vanish (cf. Proposition 7.2), the field Φ must also vanish. So the only way to obtain a different spectral action is if the field B_μ does not vanish. By Proposition 6.16 we know that we obtain $B_\mu=A_\mu-J_FA_\mu J_F^*$, where

$$iA_\mu(x)\in\mathfrak{g}_F=\mathfrak{u}(\mathcal{A}_F)/\big((\widetilde{\mathcal{A}}_F)_{J_F}\cap\mathfrak{u}(\mathcal{A}_F)\big).$$

This means that if we want to obtain a non-vanishing gauge field, we must choose J_F such that $\mathfrak{u}(\mathcal{A}_F)$ is not contained in $(\widetilde{\mathcal{A}}_F)_{J_F}$. Hence we must have KO-dimension 2 or 6.

We take the data

$$\left(\mathbb{C}^2,\mathbb{C}^2,0,\gamma_F=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, J_F=\begin{pmatrix}0&C\\C&0\end{pmatrix}\right), \tag{7.9}$$

which define a finite spectral triple of KO-dimension 6. We shall consider the almost commutative geometry given by the product of the canonical spectral triple with the two-point space of the above triple. Since the finite spectral triple has KO-dimension 6, this almost commutative geometry has KO-dimension 2 (cf. Proposition 6.2). This means that we can use Definition 5.21 for the fermionic action, which we will calculate in Section 7.1.4. First, let us derive the gauge group.

7.4 Proposition. *The gauge group $\mathcal{G}(\mathcal{A}_F)$ of the two-point space is given by $U(1)$.*

Proof. First, for $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \in U(\mathbb{C}^2)$ we have $uu^* = u^*u = \mathbb{1}$, which implies $u_1\overline{u_1} = u_2\overline{u_2} = 1$. This means that $u_1, u_2 \in U(1)$, and we have $U(\mathbb{C}^2) = U(1) \oplus U(1)$. Second, the subgroup $H_F = U(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F}$ is determined by the condition that $J_F u^* J_F^* = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \overline{u}_1 & 0 \\ 0 & \overline{u}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} u_2 & 0 \\ 0 & u_1 \end{pmatrix}$ is equal to $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$. This implies that $u_1 = u_2$, so we obtain the subgroup

$$H_F = U(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F} = \left\{ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \mid \lambda \in U(1) \right\} \cong U(1).$$

By Definition 5.27, we then obtain $\mathcal{G}(\mathcal{A}_F) := U(\mathcal{A}_F)/H_F$. Let us construct the short exact sequence

$$1 \to H_F \to U(\mathcal{A}_F) \to U(1) \to 1,$$

by defining the group homomorphism $\varphi\colon U(\mathcal{A}_F) \to U(1)$ as $\varphi(u) := u_1\overline{u_2}$. We immediately see that $\varphi(u) = 1$ if and only if $u_1 = u_2$, so the kernel of φ is indeed the subgroup H_F. We thus see that the gauge group $\mathcal{G}(\mathcal{A}_F)$ is isomorphic to $U(1)$. □

In Proposition 6.29 we have calculated the spectral action of an almost commutative geometry. Before we can apply this to the two-point space, we need to find the exact form of the field B_μ. Since we have $(\widetilde{\mathcal{A}}_F)_{J_F} = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \in \mathcal{A}_F \mid a \in \mathbb{C} \right\} \cong \mathbb{C}$, we find that $\mathfrak{h}_F = \mathfrak{u}(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F} \cong i\mathbb{R}$. From Propositions 6.16 and 6.18 we then see that the gauge field $A_\mu(x) \in i\mathfrak{g}_F = i\big(\mathfrak{u}(\mathcal{A}_F)/(i\mathbb{R})\big) = i\,\mathfrak{su}(\mathcal{A}_F) \cong \mathbb{R}$ becomes traceless. We shall write

$$A_\mu = \frac{1}{2}\begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = \frac{1}{2} Y_\mu \otimes \gamma_F$$

for $Y_\mu(x) \in \mathbb{R} = i\,\mathfrak{u}(1)$. We then find (using $J_F\gamma_F = -\gamma_F J_F$)

$$B_\mu = A_\mu - J_F A_\mu J_F^{-1} = 2A_\mu = Y_\mu \otimes \gamma_F. \tag{7.10}$$

Let us reconsider how we obtained this $U(1)$ gauge field. An arbitrary hermitian field of the form $A_\mu = ia\partial_\mu b$ would be given by $\begin{pmatrix} X_\mu^1 & 0 \\ 0 & X_\mu^2 \end{pmatrix}$, for two $U(1)$ gauge fields $X_\mu^1, X_\mu^2 \in C^\infty(M, \mathbb{R})$. However, the combination $A_\mu - J_F A_\mu J_F^{-1}$ ensures that we can take the quotient of the Lie algebra $\mathfrak{u}(\mathcal{A}_F)$ with $\mathfrak{h}_F$ (cf. Proposition 6.16). This effectively identifies the $U(1)$ gauge fields on the two copies of M, thus leaving us only one $U(1)$ gauge field Y_μ.

From Proposition 6.25, we find that the square of the fluctuated Dirac operator is now given by

$${D_A}^2 = \Delta^E + \frac{1}{4}s \otimes \mathbb{1} + \frac{1}{2} i\gamma^\mu\gamma^\nu \mathcal{F}_{\mu\nu} \otimes \gamma_F, \tag{7.11}$$

where we have defined $\mathcal{F}_{\mu\nu} := \partial_\mu Y_\nu - \partial_\nu Y_\mu$.

7.5 Proposition. *For the spectral triple*

$$\Big(C^\infty(M,\mathbb{C}^2), L^2(M,S)\otimes\mathbb{C}^2, \not{D}\otimes\mathbb{1}, \gamma_5\otimes\gamma_F, J_M\otimes J_F\Big), \tag{7.12}$$

the spectral action of the fluctuated Dirac operator is given by

$$\operatorname{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right)\sim\frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu},Y_\mu)\sqrt{|g|}d^4x+O(\Lambda^{-1}), \tag{7.13}$$

for

$$\mathcal{L}(g_{\mu\nu},Y_\mu):=2\mathcal{L}_M(g_{\mu\nu})+\mathcal{L}_Y(Y_\mu). \tag{7.14}$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 6.27. $\mathcal{L}_Y$ gives the kinetic term of the gauge field and equals

$$\mathcal{L}_Y(Y_\mu):=\frac{1}{3}f(0)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{7.15}$$

PROOF. The trace over the Hilbert space $\mathbb{C}^2$ yields an overall factor $N=2$. The field B_μ is given by Eq. (7.10). Since $B_\mu = Y_\mu\otimes\gamma_F$, we have $F_{\mu\nu}=\mathcal{F}_{\mu\nu}\otimes\gamma_F$. This yields $\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})=2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$. Inserting this into Proposition 6.29 proves the statement. □

7.1.4 *The fermionic action*

Let us take $\{e,\bar{e}\}$ as the set of orthonormal basis vectors of $\mathcal{H}_F$, where e is the basis element of $\mathcal{H}_F^+$ and $\bar{e}$ of $\mathcal{H}_F^-$. Note that on this basis, we have $J_Fe=\bar{e}$, $J_F\bar{e}=e$, $\gamma_Fe=e$ and $\gamma_F\bar{e}=-\bar{e}$.

The total Hilbert space $\mathcal{H}$ is given by $L^2(M,S)\otimes\mathcal{H}_F$. Since we can also decompose $L^2(M,S)=L^2(M,S)^+\oplus L^2(M,S)^-$ by means of γ_5, we obtain that the positive eigenspace $\mathcal{H}^+$ of $\gamma=\gamma_5\otimes\gamma_F$ is given by

$$\mathcal{H}^+=L^2(M,S)^+\otimes\mathcal{H}_F^+\oplus L^2(M,S)^-\otimes\mathcal{H}_F^-. \tag{7.16}$$

An arbitrary vector $\xi\in\mathcal{H}^+$ can then uniquely be written as

$$\xi=\psi_L\otimes e+\psi_R\otimes\bar{e}, \tag{7.17}$$

for two Weyl spinors $\psi_L\in L^2(M,S)^+$ and $\psi_R\in L^2(M,S)^-$. One should note here that this vector ξ is completely determined by one Dirac spinor $\psi:=\psi_L+\psi_R$.

7.6 Proposition. *The fermionic action of the spectral triple*

$$\Big(C^\infty(M,\mathbb{C}^2), L^2(M,S)\otimes\mathbb{C}^2, \not{D}\otimes\mathbb{1}, \gamma_5\otimes\gamma_F, J_M\otimes J_F\Big)$$

is given by

$$S_F=i\langle J_M\widetilde{\psi}_L,\gamma^\mu(\nabla_\mu^S+iY_\mu)\widetilde{\psi}_R\rangle. \tag{7.18}$$

PROOF. The spectral triple has KO-dimension 2, so we can use the definition of the fermionic action S_F from Definition 5.21. The fluctuated Dirac operator is given by Proposition 6.4 and equals

$$D_A=\not{D}\otimes\mathbb{1}+\gamma^\mu\otimes B_\mu.$$

An arbitrary $\xi\in\mathcal{H}^+$ has the form of Eq. (7.17), and then we obtain the following expressions:

$$\begin{aligned}
J\xi &= J_M\psi_L\otimes\bar{e}+J_M\psi_R\otimes e,\\
(\not{D}\otimes\mathbb{1})\xi &= \not{D}\psi_L\otimes e+\not{D}\psi_R\otimes\bar{e},\\
(\gamma^\mu\otimes B_\mu)\xi &= \gamma^\mu\psi_L\otimes Y_\mu e-\gamma^\mu\psi_R\otimes Y_\mu\bar{e}.
\end{aligned}$$

Since the basis $\{e, \bar{e}\}$ of $\mathcal{H}_F$ is orthonormal, we can easily calculate the inner product on $\mathcal{H}_F$. The fermionic action can then be written as

$$\frac{1}{2}\langle J\widetilde{\xi}, D_A\widetilde{\xi}\rangle = \frac{1}{2}\langle J_M\widetilde{\psi}_L, \not{D}\widetilde{\psi}_R\rangle + \frac{1}{2}\langle J_M\widetilde{\psi}_R, \not{D}\widetilde{\psi}_L\rangle + \frac{1}{2}\langle J_M\widetilde{\psi}_L, -\gamma^\mu Y_\mu\widetilde{\psi}_R\rangle + \frac{1}{2}\langle J_M\widetilde{\psi}_R, \gamma^\mu Y_\mu\widetilde{\psi}_L\rangle.$$

Using the fact that $\not{D}$ changes the chirality of a Weyl spinor, and that the subspaces $L^2(M,S)^+$ and $L^2(M,S)^-$ are orthogonal, we can rewrite

$$\frac{1}{2}\langle J_M\widetilde{\psi}_L, \not{D}\widetilde{\psi}_R\rangle + \frac{1}{2}\langle J_M\widetilde{\psi}_R, \not{D}\widetilde{\psi}_L\rangle = \frac{1}{2}\langle J_M\widetilde{\psi}, \not{D}\widetilde{\psi}\rangle.$$

Using that $J_M^* = -J_M$ and $J_M\not{D} = \not{D}J_M$, we see that the form $\langle J_M\chi, \not{D}\psi\rangle$ is antisymmetric:

$$\langle J_M\chi, \not{D}\psi\rangle = \langle J_M^*\not{D}\psi, \chi\rangle = -\langle J_M\not{D}\psi, \chi\rangle = -\langle \not{D}J_M\psi, \chi\rangle = -\langle J_M\psi, \not{D}\chi\rangle.$$

However, since $\widetilde{\chi}$ and $\widetilde{\psi}$ are Grassmann variables, the form $\langle J_M\widetilde{\chi}, \not{D}\widetilde{\psi}\rangle$ is symmetric. Thus we could also write

$$\frac{1}{2}\langle J_M\widetilde{\psi}_L, \not{D}\widetilde{\psi}_R\rangle + \frac{1}{2}\langle J_M\widetilde{\psi}_R, \not{D}\widetilde{\psi}_L\rangle = \langle J_M\widetilde{\psi}_L, \not{D}\widetilde{\psi}_R\rangle.$$

Note that the factor $\frac{1}{2}$ has now disappeared in the result, and this is the reason why we have included this factor in the definition of the fermionic action.

Since $J_M\gamma^\mu = \gamma^\mu J_M$, the form $\langle J_M\chi, \gamma^\mu\psi\rangle$ is symmetric:

$$\langle J_M\chi, \gamma^\mu\psi\rangle = \langle J_M^*\gamma^\mu\psi, \chi\rangle = -\langle J_M\gamma^\mu\psi, \chi\rangle = \langle \gamma^\mu J_M\psi, \chi\rangle = \langle J_M\psi, \gamma^\mu\chi\rangle,$$

and hence the form $\langle J_M\widetilde{\chi}, \gamma^\mu\widetilde{\psi}\rangle$ for Grassmann variables is antisymmetric. This allows us to rewrite

$$\frac{1}{2}\langle J_M\widetilde{\psi}_L, -\gamma^\mu Y_\mu\widetilde{\psi}_R\rangle + \frac{1}{2}\langle J_M\widetilde{\psi}_R, \gamma^\mu Y_\mu\widetilde{\psi}_L\rangle = \langle J_M\widetilde{\psi}_L, -\gamma^\mu Y_\mu\widetilde{\psi}_R\rangle. \qquad \square$$

7.7 Remark. In the above proposition we have calculated the fermionic action, and have shown that this gives rise to the coupling of a $U(1)$ gauge field Y_μ to fermions. One should note that in this action we find a coupling involving *two* Weyl spinors, which can be combined into only *one* Dirac spinor.

7.2 ELECTRODYNAMICS

Inspired by the previous section, which shows that one can use the framework of Noncommutative Geometry to describe a gauge theory of the commutative gauge group $U(1)$, we shall now attempt to describe the full theory of electrodynamics. There are two changes we need to make to the $U(1)$ gauge theory of the previous section. First, we want our fermions to become massive, so we need a finite Dirac operator that is non-zero. Second, from [9, Ch.7, §5.2], we find the Euclidean action for a free Dirac field:

$$S = -\int i\overline{\psi}(\gamma^\mu\partial_\mu - m)\psi d^4x, \tag{7.19}$$

where the fields ψ and $\overline{\psi}$ must be considered as *totally independent variables*. Thus, we require that the fermionic action S_F contains two *independent* Dirac spinors. Both these changes can be obtained by doubling our finite Hilbert space.

We start with the same algebra $C^\infty(M, \mathbb{C}^2)$ that corresponds to the space $N = M \times X \cong M \sqcup M$. The finite Hilbert space will now be used to describe four particles, namely both the left-handed and the right-handed electrons and positrons. We will choose the orthonormal basis $\{e_L, e_R, \overline{e_L}, \overline{e_R}\}$ for $\mathcal{H}_F = \mathbb{C}^4$. Hence, for two vectors $h_F = ae_L + be_R + c\overline{e_L} + d\overline{e_R}$ and $h'_F = a'e_L + b'e_R + c'\overline{e_L} + d'\overline{e_R}$ in $\mathcal{H}_F$, we take the inner product on $\mathcal{H}_F$ to be given by

$$\langle h_F, h'_F \rangle := aa' + bb' + cc' + dd'. \tag{7.20}$$

The subscript L denotes left-handed particles, and the subscript R denotes right-handed particles, and we have $\gamma_F e_L = e_L$ and $\gamma_F e_R = -e_R$.

We will choose J_F such that it interchanges particles with their antiparticles, so $J_F e_R = \overline{e_R}$ and $J_F e_L = \overline{e_L}$. We will choose the real structure such that is has KO-dimension 6, so we have $J_F^2 = \mathbb{1}$ and $J_F \gamma_F = -\gamma_F J_F$. This last relation implies that the element $\overline{e_R}$ is left-handed and $\overline{e_L}$ is right-handed. Hence, the grading γ_F and the conjugation operator J_F are given by

$$\gamma_F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad J_F = \begin{pmatrix} 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \\ C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{pmatrix}. \tag{7.21}$$

The grading γ_F decomposes the Hilbert space $\mathcal{H}_F$ into $\mathcal{H}_L \oplus \mathcal{H}_R$, where the bases of $\mathcal{H}_L$ and $\mathcal{H}_R$ are given by $\{e_L, \overline{e_R}\}$ and $\{e_R, \overline{e_L}\}$, respectively. We can also decompose the Hilbert space into $\mathcal{H}_e \oplus \mathcal{H}_{\bar{e}}$, where $\mathcal{H}_e$ contains the electrons, and $\mathcal{H}_{\bar{e}}$ contains the positrons.

The elements $a \in \mathcal{A}_F = \mathbb{C}^2$ are now represented on the basis $\{e_L, e_R, \overline{e_L}, \overline{e_R}\}$ as

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}. \tag{7.22}$$

Note that this representation commutes with the grading, as it should. We can also easily check that $[a, b^0] = 0$ for $b^0 := J_F b^* J_F^*$, since both the left and the right action are given by diagonal matrices. For now, we will still take $D_F = 0$, and hence the order one condition is trivially satisfied. We have now obtained the following result:

7.8 Proposition. *The data $(\mathbb{C}^2, \mathbb{C}^4, 0, \gamma_F, J_F)$ given above define a real even finite spectral triple of KO-dimension* 6.

By taking the product with the canonical spectral triple of Definition 5.6, our full spectral triple (of KO-dimension 2) under consideration is given by

$$\Big(C^\infty(M, \mathbb{C}^2), L^2(M, S) \otimes \mathbb{C}^4, \not{D} \otimes \mathbb{1}, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\Big). \tag{7.23}$$

As in Section 7.1, the algebra decomposes as $C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M)$, and we now decompose the Hilbert space as $\mathcal{H} = (L^2(M, S) \otimes \mathcal{H}_e) \oplus (L^2(M, S) \otimes \mathcal{H}_{\bar{e}})$. The action of the algebra on $\mathcal{H}$, given by Eq. (7.22), is then such that one component of the algebra acts on $L^2(M, S) \otimes \mathcal{H}_e$, and the other component acts on $L^2(M, S) \otimes \mathcal{H}_{\bar{e}}$. Similarly to the previous section, we obtain a $U(1)$ gauge field

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{pmatrix} \qquad \text{for } Y_\mu(x) \in \mathbb{R}. \tag{7.24}$$

To summarize, our space N consists of two copies of M, which are separated by an infinite distance (cf. Section 7.1.2). On one copy of M, we have the vector bundle $S \otimes (M \times \mathcal{H}_e)$, and on the other copy the vector bundle $S \otimes (M \times \mathcal{H}_{\bar{e}})$. The gauge fields on each copy of M are identified, and thus provide an interaction between the electrons e and positrons $\bar{e}$.

7.2.1 *Non-zero Dirac operator*

Let us now consider the possibilities for adding a non-zero Dirac operator to this spectral triple. Since $D_F\gamma_F = -\gamma_F D_F$, the Dirac operator obtains the form

$$D_F = \begin{pmatrix} 0 & d_1 & d_2 & 0 \\ \overline{d}_1 & 0 & 0 & d_3 \\ \overline{d}_2 & 0 & 0 & d_4 \\ 0 & \overline{d}_3 & \overline{d}_4 & 0 \end{pmatrix}. \tag{7.25}$$

Next, we impose the commutation relation $D_F J_F = J_F D_F$, which yields

$$\begin{pmatrix} d_2C & 0 & 0 & d_1C \\ 0 & d_3C & \overline{d}_1C & 0 \\ 0 & d_4C & \overline{d}_2C & 0 \\ \overline{d}_4C & 0 & 0 & \overline{d}_3C \end{pmatrix} = \begin{pmatrix} C\overline{d}_2 & 0 & 0 & Cd_4 \\ 0 & C\overline{d}_3 & C\overline{d}_4 & 0 \\ 0 & Cd_1 & Cd_2 & 0 \\ C\overline{d}_1 & 0 & 0 & Cd_3 \end{pmatrix}. \tag{7.26}$$

This imposes the relation $d_1 = \overline{d}_4$. The last thing to check is the order one condition. First we calculate

$$\begin{aligned}[D_F, a] &= \begin{pmatrix} 0 & d_1a_1 - a_1d_1 & d_2a_2 - a_1d_2 & 0 \\ \overline{d}_1a_1 - a_1\overline{d}_1 & 0 & 0 & d_3a_2 - a_1d_3 \\ \overline{d}_2a_1 - a_2\overline{d}_2 & 0 & 0 & \overline{d}_1a_2 - a_2\overline{d}_1 \\ 0 & \overline{d}_3a_1 - a_2\overline{d}_3 & d_1a_2 - a_2d_1 & 0 \end{pmatrix} \\ &= (a_1 - a_2)\begin{pmatrix} 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & -d_3 \\ \overline{d}_2 & 0 & 0 & 0 \\ 0 & \overline{d}_3 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.27}$$

The order one condition then becomes

$$\begin{aligned} 0 = \big[[D_F, a], b^0\big] &= (a_1 - a_2)\left[\begin{pmatrix} 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & -d_3 \\ \overline{d}_2 & 0 & 0 & 0 \\ 0 & \overline{d}_3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix}\right] \\ &= (a_1 - a_2)(b_2 - b_1)\begin{pmatrix} 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_3 \\ \overline{d}_2 & 0 & 0 & 0 \\ 0 & \overline{d}_3 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.28}$$

Since this must hold for all $a, b \in \mathbb{C}^2$, we must require that $d_2 = d_3 = 0$. To conclude, the Dirac operator only depends on one complex parameter and is given by

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \overline{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d} \\ 0 & 0 & d & 0 \end{pmatrix}. \tag{7.29}$$

Recall from Proposition 6.4 the definition of the field Φ:

$$\Phi(x) := D_F + a(x)[D_F, b(x)] + J_F a(x)[D_F, b(x)]J_F, \tag{7.30}$$

for $a, b \in C^\infty(M, \mathcal{A}_F)$. As we have seen above, the commutator $[D_F, b] = 0$, so we only have a constant field $\Phi(x) = D_F$. We also note that $[B_\mu, D_F] = 0$, so we see that $D_\mu\Phi = 0$.

As mentioned before, our space N consists of two copies of M, and the distance between these two copies is infinite. Now, we have introduced a non-zero Dirac operator, but it commutes with the algebra, i.e. $[D_F, a] = 0$ for all $a \in \mathcal{A}$, so there is still no Higgs field. Therefore, the distance between the two copies of M is still infinite.

We now add the Dirac operator D_F to the spectral triple by considering the new Dirac operator $D := \not{D} \otimes \mathbb{1} + \gamma_5 \otimes D_F$.

7.9 Proposition. *The spectral action of the spectral triple*

$$\left(C^\infty(M, \mathbb{C}^2), L^2(M,S) \otimes \mathbb{C}^4, \not{D} \otimes \mathbb{1} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\right)$$

is given by

$$\operatorname{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) \sim \frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu}, Y_\mu)\sqrt{|g|}d^4x + O(\Lambda^{-1}), \tag{7.31}$$

for

$$\mathcal{L}(g_{\mu\nu}, Y_\mu) := 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_Y(Y_\mu) + \mathcal{L}_H(g_{\mu\nu}, d). \tag{7.32}$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 6.27. $\mathcal{L}_Y$ gives the kinetic term of the gauge field and equals

$$\mathcal{L}_Y(Y_\mu) := \frac{2}{3}f(0)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{7.33}$$

The Higgs potential (ignoring the boundary term) only gives two constant terms which add to the cosmological constant, plus an extra contribution to the Einstein-Hilbert action:

$$\mathcal{L}_H(g_{\mu\nu}) := -8f_2\Lambda^2|d|^2 + 2f(0)|d|^4 + \frac{1}{3}f(0)s|d|^2. \tag{7.34}$$

Proof. The proof is similar to Proposition 7.5, only now the trace over $\mathcal{H}_F = \mathbb{C}^4$ yields a factor 4. In addition, the finite Dirac operator only yields extra contributions to the cosmological constant and the Einstein-Hilbert action. □

7.2.2 *The fermionic action*

We have written the set of basis vectors of $\mathcal{H}_F$ as $\{e_L, e_R, \overline{e_L}, \overline{e_R}\}$, and the subspaces $\mathcal{H}_F^+$ and $\mathcal{H}_F^-$ are spanned by $\{e_L, \overline{e_R}\}$ and $\{e_R, \overline{e_L}\}$, respectively. The total Hilbert space $\mathcal{H}$ is given by $L^2(M,S) \otimes \mathcal{H}_F$. Since we can also decompose $L^2(M,S) = L^2(M,S)^+ \oplus L^2(M,S)^-$ by means of γ_5, we obtain

$$\mathcal{H}^+ = L^2(M,S)^+ \otimes \mathcal{H}_F^+ \oplus L^2(M,S)^- \otimes \mathcal{H}_F^-. \tag{7.35}$$

A spinor $\psi \in L^2(M,S)$ can be decomposed as $\psi = \psi_L + \psi_R$. Each subspace $\mathcal{H}_F^\pm$ is now spanned by two basis vectors. A generic element of the tensor product of two spaces consists of sums of tensor products, so an arbitrary vector $\xi \in \mathcal{H}^+$ can uniquely be written as

$$\xi = \chi_L \otimes e_L + \chi_R \otimes e_R + \psi_R \otimes \overline{e_L} + \psi_L \otimes \overline{e_R}, \tag{7.36}$$

for Weyl spinors $\chi_L, \psi_L \in L^2(M,S)^+$ and $\chi_R, \psi_R \in L^2(M,S)^-$. Note that this vector $\xi \in \mathcal{H}^+$ is now completely determined by two Dirac spinors $\chi := \chi_L + \chi_R$ and $\psi := \psi_L + \psi_R$.

7.10 Proposition. *The fermionic action of the spectral triple*

$$\left(C^\infty(M, \mathbb{C}^2), L^2(M,S) \otimes \mathbb{C}^4, \not{D} \otimes \mathbb{1} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\right)$$

is given by

$$S_F = i\langle J_M\widetilde{\chi}, \gamma^\mu(\nabla^S_\mu + iY_\mu)\widetilde{\psi}\rangle + \langle J_M\widetilde{\chi}_L, \overline{d}\widetilde{\psi}_L\rangle - \langle J_M\widetilde{\chi}_R, d\widetilde{\psi}_R\rangle. \tag{7.37}$$

PROOF. The proof is similar to Proposition 7.6. The fluctuated Dirac operator is now given by

$$D_A = \not{\partial} \otimes \mathbb{1} + \gamma^\mu \otimes B_\mu + \gamma_5 \otimes D_F.$$

An arbitrary $\xi \in \mathcal{H}^+$ has the form of Eq. (7.36), and then we obtain the following expressions:

$$\begin{aligned}
J\xi &= J_M\chi_L \otimes \overline{e_L} + J_M\chi_R \otimes \overline{e_R} + J_M\psi_R \otimes e_L + J_M\psi_L \otimes e_R, \\
(\not{\partial} \otimes \mathbb{1})\xi &= \not{\partial}\chi_L \otimes e_L + \not{\partial}\chi_R \otimes e_R + \not{\partial}\psi_R \otimes \overline{e_L} + \not{\partial}\psi_L \otimes \overline{e_R}, \\
(\gamma^\mu \otimes B_\mu)\xi &= \gamma^\mu\chi_L \otimes Y_\mu e_L + \gamma^\mu\chi_R \otimes Y_\mu e_R - \gamma^\mu\psi_R \otimes Y_\mu\overline{e_L} - \gamma^\mu\psi_L \otimes Y_\mu\overline{e_R}, \\
(\gamma_5 \otimes D_F)\xi &= \gamma_5\chi_L \otimes \bar{d}e_R + \gamma_5\chi_R \otimes de_L + \gamma_5\psi_R \otimes d\overline{e_R} + \gamma_5\psi_L \otimes \bar{d}\overline{e_L}.
\end{aligned}$$

We decompose the fermionic action into the three terms

$$\frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle = \frac{1}{2}\langle J\tilde{\xi}, (\not{\partial} \otimes \mathbb{1})\tilde{\xi}\rangle + \frac{1}{2}\langle J\tilde{\xi}, (\gamma^\mu \otimes B_\mu)\tilde{\xi}\rangle + \frac{1}{2}\langle J\tilde{\xi}, (\gamma_5 \otimes D_F)\tilde{\xi}\rangle,$$

and then continue to calculate each term separately. The first term is given by

$$\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi}, (\not{\partial} \otimes \mathbb{1})\tilde{\xi}\rangle = \frac{1}{2}\langle J_M\tilde{\chi}_L, \not{\partial}\tilde{\psi}_R\rangle + \frac{1}{2}\langle J_M\tilde{\chi}_R, \not{\partial}\tilde{\psi}_L\rangle \\
+ \frac{1}{2}\langle J_M\tilde{\psi}_R, \not{\partial}\tilde{\chi}_L\rangle + \frac{1}{2}\langle J_M\tilde{\psi}_L, \not{\partial}\tilde{\chi}_R\rangle.
\end{aligned}$$

Using the fact that $\not{\partial}$ changes the chirality of a Weyl spinor, and that the subspaces $L^2(M,S)^+$ and $L^2(M,S)^-$ are orthogonal, we can rewrite this term as

$$\frac{1}{2}\langle J\tilde{\xi}, (\not{\partial} \otimes \mathbb{1})\tilde{\xi}\rangle = \frac{1}{2}\langle J_M\tilde{\chi}, \not{\partial}\tilde{\psi}\rangle + \frac{1}{2}\langle J_M\tilde{\psi}, \not{\partial}\tilde{\chi}\rangle.$$

Using the symmetry of the form $\langle J_M\tilde{\chi}, \not{\partial}\tilde{\psi}\rangle$, we obtain

$$\frac{1}{2}\langle J\tilde{\xi}, (\not{\partial} \otimes \mathbb{1})\tilde{\xi}\rangle = \langle J_M\tilde{\chi}, \not{\partial}\tilde{\psi}\rangle.$$

Note that the factor $\frac{1}{2}$ has again disappeared in the result. The second term is given by

$$\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi}, (\gamma^\mu \otimes B_\mu)\tilde{\xi}\rangle = -\frac{1}{2}\langle J_M\tilde{\chi}_L, \gamma^\mu Y_\mu\tilde{\psi}_R\rangle - \frac{1}{2}\langle J_M\tilde{\chi}_R, \gamma^\mu Y_\mu\tilde{\psi}_L\rangle \\
+ \frac{1}{2}\langle J_M\tilde{\psi}_R, \gamma^\mu Y_\mu\tilde{\chi}_L\rangle + \frac{1}{2}\langle J_M\tilde{\psi}_L, \gamma^\mu Y_\mu\tilde{\chi}_R\rangle.
\end{aligned}$$

In a similar manner as above, we obtain

$$\frac{1}{2}\langle J\tilde{\xi}, (\gamma^\mu \otimes B_\mu)\tilde{\xi}\rangle = -\langle J_M\tilde{\chi}, \gamma^\mu Y_\mu\tilde{\psi}\rangle,$$

where we have now used that the form $\langle J_M\tilde{\chi}, \gamma^\mu Y_\mu\tilde{\psi}\rangle$ is antisymmetric. The third term is given by

$$\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi}, (\gamma_5 \otimes D_F)\tilde{\xi}\rangle = \frac{1}{2}\langle J_M\tilde{\chi}_L, \bar{d}\gamma_5\tilde{\psi}_L\rangle + \frac{1}{2}\langle J_M\tilde{\chi}_R, d\gamma_5\tilde{\psi}_R\rangle \\
+ \frac{1}{2}\langle J_M\tilde{\psi}_R, d\gamma_5\tilde{\chi}_R\rangle + \frac{1}{2}\langle J_M\tilde{\psi}_L, \bar{d}\gamma_5\tilde{\chi}_L\rangle.
\end{aligned}$$

The bilinear form $\langle J_M\tilde{\chi}, \gamma_5\tilde{\psi}\rangle$ is again symmetric, but we now have the extra complication that two terms contain the parameter d, while the other two terms contain $\bar{d}$. Therefore we are left with two distinct terms:

$$\frac{1}{2}\langle J\tilde{\xi}, (\gamma_5 \otimes D_F)\tilde{\xi}\rangle = \langle J_M\tilde{\chi}_L, \bar{d}\tilde{\psi}_L\rangle - \langle J_M\tilde{\chi}_R, d\tilde{\psi}_R\rangle. \qquad \square$$

7.11 Remark. It is interesting to note that the fermions acquire mass terms without being coupled to a Higgs field. However, it seems we obtain a complex mass parameter d, where we would desire a real parameter m. By simply requiring that our result should be similar to Eq. (7.19), we will choose $d := im$, so that

$$\langle J_M\widetilde{\chi}_L, \overline{d}\widetilde{\psi}_L\rangle - \langle J_M\widetilde{\chi}_R, d\widetilde{\psi}_R\rangle = -i\langle J_M\widetilde{\chi}, m\widetilde{\psi}\rangle. \tag{7.38}$$

The results obtained in this section can now be summarized into the following theorem.

7.12 Theorem. *The full Lagrangian of the almost commutative geometry given by the data*

$$\Big(C^\infty(M,\mathbb{C}^2), L^2(M,S)\otimes\mathbb{C}^4, \not{D}\otimes\mathbb{1}+\gamma_5\otimes D_F, \gamma_5\otimes\gamma_F, J_M\otimes J_F\Big) \tag{7.39}$$

as defined in this section, can be written as the sum of a purely gravitational Lagrangian,

$$\mathcal{L}_{grav}(g_{\mu\nu}) = \frac{1}{\pi^2}\mathcal{L}_M(g_{\mu\nu}) + \frac{1}{4\pi^2}\mathcal{L}_H(g_{\mu\nu}), \tag{7.40}$$

and a Lagrangian for electrodynamics,

$$\mathcal{L}_{ED} = i\Big(J_M\widetilde{\chi}, (\gamma^\mu(\nabla^S_\mu + iY_\mu) - m)\widetilde{\psi}\Big) + \frac{1}{6\pi^2}f(0)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{7.41}$$

PROOF. The spectral action S_B and the fermionic action S_F are given by Propositions 7.9 and 7.10. We shall now absorb all numerical constants into the Lagrangians as well. This immediately yields $\mathcal{L}_{\text{grav}}$. To obtain $\mathcal{L}_{\text{ED}}$, we need to rewrite the fermionic action S_F as the integral over a Lagrangian. The inner product $\langle\ ,\ \rangle$ on the Hilbert space $L^2(S)$ is given by Definition 2.41, where the hermitian pairing is given by the pointwise inner product on the fibres. Thus, we write the inner product as

$$\langle\xi,\psi\rangle = \int_M(\xi,\psi)\sqrt{|g|}d^4x.$$

We can then rewrite the fermionic action into

$$S_F = \int_M i\Big(J_M\widetilde{\chi}, (\gamma^\mu(\nabla^S_\mu + iY_\mu) - m)\widetilde{\psi}\Big)\sqrt{|g|}d^4x. \qquad \square$$

7.13 Remark. To conclude this chapter, let us make a final remark on the fermionic degrees of freedom. As mentioned in Remark 5.22, the number of degrees of freedom of the fermion fields in the fermionic action is related to the restrictions that are incorporated into the definition of the fermionic action. These restrictions make sure that in this case we obtain two independent Dirac spinors in the fermionic action.

In quantum field theory, one would consider the functional integral of e^S over the fields. Let us now denote $\mathfrak{A}$ for the bilinear form on $\mathcal{H}^+$ and $\mathfrak{B}$ for the bilinear form on $L^2(M,S)$, given by

$$\mathfrak{A}(\xi,\zeta) := \langle J\xi, D_A\zeta\rangle, \qquad \text{for } \xi,\zeta\in\mathcal{H}^+, \tag{7.42}$$

$$\mathfrak{B}(\chi,\psi) := i\Big\langle J_M\chi, (\gamma^\mu(\nabla^S_\mu + iY_\mu) - m)\psi\Big\rangle, \qquad \text{for } \chi,\psi\in L^2(M,S). \tag{7.43}$$

We have shown in Proposition 7.10 that for $\xi = \chi_L\otimes e_L + \chi_R\otimes e_R + \psi_R\otimes\overline{e_L} + \psi_L\otimes\overline{e_R}$, where we can define two Dirac spinors by $\chi := \chi_L + \chi_R$ and $\psi := \psi_L + \psi_R$, we obtain

$$\frac{1}{2}\mathfrak{A}(\xi,\xi) = \mathfrak{B}(\chi,\psi). \tag{7.44}$$

Using the Grassmann integrals that were calculated in Lemmas 2.58 and 2.60, we then obtain for the bilinear forms $\mathfrak{A}$ and $\mathfrak{B}$ the equality

$$\text{Pf}(\mathfrak{A}) = \int e^{\frac{1}{2}\mathfrak{A}(\widetilde{\xi},\widetilde{\xi})}D[\widetilde{\xi}] = \int e^{\mathfrak{B}(\widetilde{\chi},\widetilde{\psi})}D[\widetilde{\psi},\widetilde{\chi}] = \det(\mathfrak{B}). \tag{7.45}$$

8 THE GLASHOW-WEINBERG-SALAM MODEL

For a suitable choice of the finite spectral triple, the corresponding almost commutative geometry gives rise to the full Standard Model [8, 13]. In this chapter we will reproduce the Glashow-Weinberg-Salam Model, which describes the electroweak interactions for one generation of the leptonic sector of the Standard Model. An important feature of the Standard Model already occurs in this electroweak theory, namely the Higgs mechanism. The main purpose of this chapter is to show how this Higgs mechanism arises from the almost commutative geometry.

Although it is perfectly possible to derive the fermionic action for this model, by exactly the same approach as for Electro Dynamics in Chapter 7, we will refrain from doing so. The Higgs mechanism is given solely in the bosonic part of the Lagrangian, and for now we will therefore only focus on the spectral action. In Chapter 9 we will discuss the full Standard Model, and we shall derive the fermionic action there.

8.1 THE FINITE TRIPLE

We shall consider one generation of electrons, neutrinos and their antiparticles. We write $\mathcal{H}_l = \mathbb{C}^4$ for the space of leptons, given by the basis (ν_R, e_R, ν_L, e_L). The space of antileptons $\mathcal{H}_{\bar{l}} = \mathbb{C}^4$ then has the basis $(\overline{\nu_R}, \overline{e_R}, \overline{\nu_L}, \overline{e_L})$. The total finite Hilbert space is given by $\mathcal{H}_F = \mathcal{H}_l \oplus \mathcal{H}_{\bar{l}}$.

For the algebra, we shall take $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}$, where $\mathbb{H}$ is the real algebra of quaternions. We can write $q \in \mathbb{H}$ as $q = \alpha + \beta j$ for $\alpha, \beta \in \mathbb{C}$. Using this, we can embed $\mathbb{H} \subset M_2(\mathbb{C})$ by setting $q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$. Note that this embedding is real-linear, but not complex-linear, and consequently the representation given below is also only real-linear.

The representation $\pi\colon \mathcal{A}_F \to B(\mathcal{H}_F)$ is given on $\mathcal{H}_l$ for $a = (\lambda, q) \in \mathcal{A}_F$ by

$$\pi(a) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix}. \tag{8.1}$$

We shall write $q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$ for the embedding of $\mathbb{C}$ in $\mathbb{H}$. We can then write $\pi(a) = \begin{pmatrix} q_\lambda & 0 \\ 0 & q \end{pmatrix}$ on $\mathcal{H}_l$. On the space of antileptons we set $\pi(a)\bar{l} = \lambda \bar{l}$ for $\bar{l} \in \mathcal{H}_{\bar{l}}$.

The grading γ_F is chosen such that $\gamma_F l_R = -l_R$, $\gamma_F l_L = l_L$, $\gamma_F \overline{l_R} = \overline{l_R}$ and $\gamma_F \overline{l_L} = -\overline{l_L}$. The antilinear operator J_F is chosen such that $J_F l = \bar{l}$ and $J_F \bar{l} = l$. This means that $J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$ on the decomposition $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_{\bar{l}}$, where C as before stands for complex conjugation.

We are left only with deriving the most general form of the Dirac operator D_F that is consistent with the above definitions. D_F must be selfadjoint, which implies that we can write $D_F = \begin{pmatrix} S & T^* \\ T & S' \end{pmatrix}$ on the decomposition $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_{\bar{l}}$, for selfadjoint S, S'. Since the spectral triple is even, we need that D_F commutes with J_F:

$$0 = [D_F, J_F] = \begin{pmatrix} C(T^T - T) & C(\overline{S} - S') \\ C(\overline{S'} - S) & C(\overline{T} - T^*) \end{pmatrix}. \tag{8.2}$$

This imposes the relations $S' = \overline{S}$ and $T = T^T$, and we find that $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$ for some selfadjoint S and symmetric T. We also require that D_F anticommutes with γ_F, which yields

$$0 = D_F\gamma_F + \gamma_F D_F = \begin{pmatrix} S\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & T^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \overline{S}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T^* \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{S} \end{pmatrix}. \tag{8.3}$$

This further imposes $S\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S$ and $\left[T, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = 0$, which implies that we can write

$$S = \begin{pmatrix} 0 & Y_0^* \\ Y_0 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} T_R & 0 \\ 0 & T_L \end{pmatrix}, \tag{8.4}$$

where T_R and T_L are required to be symmetric. We will consider the restriction $T_R = \begin{pmatrix} Y_R & 0 \\ 0 & 0 \end{pmatrix}$ and $T_L = 0$, i.e. $T\nu_R = Y_R\overline{\nu_R}$, and $Tl = 0$ for all other leptons $l \neq \nu_R$. As will be shown below, this restriction makes sure that the order one condition (5.6) is satisfied. The mass matrix Y_0 can be written as a diagonal matrix by simply requiring that the basis elements of $\mathcal{H}_F$ are mass eigenstates. Hence we take $Y_0 = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}$, for two complex parameters Y_ν and Y_e. We now arrive at the following result.

8.1 Proposition. *The data $(\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ given above define a real even spectral triple of KO-dimension 6.*

Proof. One immediately sees that γ_F commutes with the algebra $\mathcal{A}_F$. We have already shown that $D_F J_F = J_F D_F$ and $[D_F, \gamma_F] = 0$. We also have $J_F^2 = 1$ and $J_F\gamma_F = -\gamma_F J_F$. From the table in Definition 5.5 we then see that we have KO-dimension 6. It remains to check the order one condition $\big[[D_F, a], b\big] = 0$. The action of the algebra on $\mathcal{H}_{\bar{l}}$ is by scalar multiplication, so we find that $[\overline{S}, a] = 0$ on $\mathcal{H}_{\bar{l}}$. On $\mathcal{H}_l$, the right action $a^0 = Ja^*J^* = \lambda$ is also just scalar multiplication, so we obtain that $\left[\left[\begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix}, a\right], b^0\right] = 0$. Since $a\nu_R = \lambda\nu_R$ and also $a\overline{\nu_R} = \lambda\overline{\nu_R}$, the action of a commutes with T, and the order one condition is indeed satisfied. □

8.2 THE GAUGE THEORY

We shall now consider the product of the canonical spectral triple, describing a 4-dimensional spin manifold M, with the finite triple described in the previous section. We will frequently make use of the Pauli matrices σ^a, which are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.5}$$

8.2 Lemma. *The subalgebra $(\widetilde{\mathcal{A}}_F)_{J_F}$ of the algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}$, with the representation on $B(\mathcal{H}_F)$ as above, is given by $(\widetilde{\mathcal{A}}_F)_{J_F} = \mathbb{R}$. The Lie algebra $\mathfrak{h}_F$ of Definition 6.12 is then given by the trivial subset $\mathfrak{h}_F = \{0\}$.*

Proof. The subalgebra $(\widetilde{\mathcal{A}}_F)_{J_F}$ defined in Definition 5.12 is determined by the relation $aJ_F = J_F a^*$. An element $a = (\lambda, q) \in \mathbb{C} \oplus \mathbb{H}$ satisfies this relation if $\lambda = \overline{\lambda} = \alpha = \overline{\alpha}$ and $\beta = 0$, so if $a = (x, x)$ for $x \in \mathbb{R}$. The elements $u \in \mathfrak{u}(\mathcal{A}_F)$ are given by $u = (\lambda, q)$ for $\lambda \in i\mathbb{R}$ and for iq a linear combination of the Pauli matrices of Eq. (8.5). In particular this means that $\overline{\lambda} = -\lambda$. Hence in the cross-section $\mathfrak{h}_F = \mathfrak{u}(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F}$ we find $\alpha = \overline{\alpha} = \lambda = \overline{\lambda} = -\lambda = 0$. □

8.3 Proposition. *The gauge group $\mathcal{G}(\mathcal{A}_F)$ is given by $\big(U(1) \times SU(2)\big)/\{1, -1\}$.*

Proof. The unitary elements of the algebra form the group $U(\mathcal{A}_F) \cong U(1) \times U(\mathbb{H})$. The quaternions are spanned by the identity matrix $\mathbb{1}$ and the anti-hermitian matrices $i\sigma_j$, where σ_j $(j = 1, 2, 3)$ are the Pauli matrices. A quaternion $q = q_0\mathbb{1} + iq_1\sigma_1 + iq_2\sigma_2 + iq_3\sigma_3$ is unitary if and only if $|q|^2 = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2 = 1$. By using the embedding of $\mathbb{H}$ in $M_2(\mathbb{C})$, we find $|q|^2 = \det(q) = 1$, and this yields the isomorphism $U(\mathbb{H}) \cong SU(2)$ (for more details, see, for instance, [15, §1.2.B]). Note that if q is unitary, then so is $-q$.

From Lemma 8.2 we know that $(\widetilde{\mathcal{A}}_F)_{J_F} = \mathbb{R}$. The group $H_F = U(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F}$ is then given by $H_F = \{1, -1\}$. We thus obtain that the gauge group $\mathcal{G}(\mathcal{A}_F)$ is given by the quotient $\big(U(1) \times SU(2)\big)/\{1, -1\}$. □

8.4 Remark. As given in Proposition 6.18, the unimodularity condition is satisfied naturally only for complex algebras. In this case however we only have a real-linear representation of the algebra, so the unimodularity condition is not satisfied. Indeed, in Lemma 8.2 we found that the Lie subalgebra $\mathfrak{h}_F$ is trivial, and hence the gauge field A_μ takes values in the Lie algebra $\mathfrak{g}_F = \mathfrak{u}(\mathcal{A}_F)/\mathfrak{h}_F = \mathfrak{u}(\mathcal{A}_F) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$, which is obviously not unimodular, because of the presence of the $\mathfrak{u}(1)$ part. Note that in this particular case we also would not want the unimodularity condition to be satisfied, because that would mean that our electromagnetic $U(1)$ gauge field would vanish.

Note that, although we obtain the gauge group $\big(U(1) \times SU(2)\big)/\{1, -1\}$, this is very similar to $U(1) \times SU(2)$, since both groups have the same Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$.

8.5 Proposition. *The fields A_μ and ϕ are given on $\mathcal{H}_l$ by*

$$\begin{aligned} A_\mu &= \begin{pmatrix} \Lambda_\mu & 0 & \\ 0 & -\Lambda_\mu & \\ & & Q_\mu \end{pmatrix}, \qquad \text{for } \Lambda_\mu \in \mathbb{R}, \quad Q_\mu \in i\mathfrak{su}(2); \\ \phi &= \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix}, \qquad \text{for } Y = \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}, \quad \phi_1, \phi_2 \in \mathbb{C}. \end{aligned} \tag{8.6}$$

On $\mathcal{H}_{\bar{l}}$ we have $A_\mu = \Lambda_\mu$ and $\phi = 0$.

Proof. We take two elements $a = (\lambda, q)$ and $b = (\lambda', q')$ of the algebra. The inner fluctuations $A_\mu = ia\partial_\mu b$ are obtained from Eq. (8.1) to be $\Lambda_\mu := i\lambda\partial_\mu\lambda'$ on ν_R, $\Lambda'_\mu := i\overline{\lambda}\partial_\mu\overline{\lambda}'$ on e_R, and $Q_\mu := iq\partial_\mu q'$ on (ν_l, e_L). Demanding the selfadjointness of $\Lambda_\mu = \Lambda^*_\mu$ implies $\Lambda_\mu \in \mathbb{R}$, and also automatically yields $\Lambda'_\mu = -\Lambda_\mu$. Furthermore, $Q_\mu = Q^*_\mu$ implies that Q_μ is a real-linear combination of the Pauli matrices, which span $i\,\mathfrak{su}(2)$.

Next, we calculate the field $\phi = a[D_F, b]$. In the proof of Proposition 8.1 we have already noted that the only part of D_F that does not commute with the algebra is given by S. Therefore, we start by calculating the commutator on $\mathcal{H}_l$ given by

$$[S,b] = \left[\begin{pmatrix} 0 & 0 & \overline{Y}_\nu & 0 \\ 0 & 0 & 0 & \overline{Y}_e \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & \overline{\lambda}' & 0 & 0 \\ 0 & 0 & \alpha' & \beta' \\ 0 & 0 & -\overline{\beta}' & \overline{\alpha}' \end{pmatrix}\right]$$
$$= \begin{pmatrix} 0 & 0 & \overline{Y}_\nu(\alpha'-\lambda') & \overline{Y}_\nu\beta' \\ 0 & 0 & -\overline{Y}_e\overline{\beta}' & \overline{Y}_e(\overline{\alpha}'-\overline{\lambda}') \\ Y_\nu(\lambda'-\alpha') & -Y_e\beta' & 0 & 0 \\ Y_\nu\overline{\beta}' & Y_e(\overline{\lambda}'-\overline{\alpha}') & 0 & 0 \end{pmatrix}.$$

By multiplying this with the element a, we obtain

$$\phi = \begin{pmatrix} 0 & 0 & \overline{Y}_\nu\phi_1' & \overline{Y}_\nu\phi_2' \\ 0 & 0 & -\overline{Y}_e\overline{\phi}_2' & \overline{Y}_e\overline{\phi}_1' \\ Y_\nu\phi_1 & -Y_e\overline{\phi}_2 & 0 & 0 \\ Y_\nu\phi_2 & Y_e\overline{\phi}_1 & 0 & 0 \end{pmatrix},$$

where we define[1]

$$\phi_1 = \alpha(\lambda'-\alpha') + \beta\overline{\beta}', \qquad \phi_2 = \overline{\alpha}\overline{\beta}' - \overline{\beta}(\lambda'-\alpha'),$$
$$\phi_1' = \lambda(\alpha'-\lambda'), \qquad \phi_2' = \lambda\beta'.$$

By demanding self-adjointness $\phi = \phi^*$, we obtain $\phi_1' = \overline{\phi}_1$ and $\phi_2' = \overline{\phi}_2$. Hence we find that the field ϕ is completely determined by the complex doublet (ϕ_1, ϕ_2).

In general, an inner fluctuation is given by a sum of terms, of the form $A = \sum_j a_j[D, b_j]$. For such a general inner fluctuation, we simply need te redefine $\Lambda_\mu := \sum_j i\lambda_j\partial_\mu\lambda_j'$ and $Q_\mu := \sum_j iq_j\partial_\mu q_j'$, as well as

$$\phi_1 = \sum_j \alpha_j(\lambda_j'-\alpha_j') + \beta_j\overline{\beta}_j', \qquad \phi_2 = \sum_j \overline{\alpha}_j\overline{\beta}_j' - \overline{\beta}_j(\lambda_j'-\alpha_j'),$$
$$\phi_1' = \sum_j \lambda_j(\alpha_j'-\lambda_j'), \qquad \phi_2' = \sum_j \lambda_j\beta_j'. \qquad \square$$

8.6 Corollary. *The gauge field $B_\mu = A_\mu - J_F A_\mu J_F^*$ is given by*

$$B_\mu\big|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & \\ 0 & -2\Lambda_\mu & \\ & & Q_\mu - \Lambda_\mu\mathbb{1}_2 \end{pmatrix}, \qquad B_\mu\big|_{\mathcal{H}_{\bar{l}}} = \begin{pmatrix} 0 & 0 & \\ 0 & 2\Lambda_\mu & \\ & & \Lambda_\mu\mathbb{1}_2 - \overline{Q}_\mu \end{pmatrix}. \tag{8.7}$$

Note that from the coefficients in front of Λ_μ in the above formulas, we recognize the correct hypercharges of the corresponding particles.

8.7 Proposition. *The transformation property of Proposition 6.20 now becomes*

$$\begin{aligned} \Lambda_\mu &\to \Lambda_\mu - i\overline{\lambda}\partial_\mu\lambda, \\ Q_\mu &\to qQ_\mu q^* - i(\partial_\mu q)q^*, \\ \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} &\to \overline{\lambda}\,q\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + (\overline{\lambda}\,q - 1)\begin{pmatrix}1\\ 0\end{pmatrix}. \end{aligned} \tag{8.8}$$

1 This notation looks very similar to the notation of $\varphi_1, \varphi_2, \varphi_1', \varphi_2'$ that is used in [13, Ch. 1, §15.2], but we have taken $\phi_1 = \varphi_1'$ and $\phi_2 = -\overline{\varphi}_2'$. The reason for this change in notation is that we obtain a prettier formula for the gauge transformation in Proposition 8.7.

Proof. We simply insert the formulas for the fields obtained in Proposition 8.5 into the transformations given by Proposition 6.20. We shall write $u = (\lambda, q) \in U(1) \times SU(2)$. On $\mathcal{H}_{\bar{l}}$ we see that A_μ commutes with λ. On (ν_R, e_R) we see that $\begin{pmatrix} \Lambda_\mu & 0 \\ 0 & -\Lambda_\mu \end{pmatrix}$ also commutes with q_λ. Hence the only effect of the term $uA_\mu u^*$ is to replace Q by qQq^*. Secondly, we see that the term $iu\partial_\mu u^*$ is given by $i\lambda\partial_\mu\overline{\lambda} = -i\overline{\lambda}\partial_\mu\lambda$ on (ν_R, e_R) and on $\mathcal{H}_{\bar{l}}$, and by $iq\partial_\mu q^* = -i(\partial_\mu q)q^*$ on (ν_L, e_L). We thus obtain the desired transformation for Λ_μ and Q_μ.

For the transformation of ϕ, we separately calculate $u\phi u^*$ and $u[D_F, u^*]$. Since $\phi = 0$ on $\mathcal{H}_{\bar{l}}$, we can restrict our calculation of $u\phi u^*$ to $\mathcal{H}_l$ and find

$$u\phi u^* = \begin{pmatrix} q_\lambda & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix} \begin{pmatrix} q_\lambda^* & 0 \\ 0 & q^* \end{pmatrix} = \begin{pmatrix} 0 & q_\lambda Y^* q^* \\ qYq_\lambda^* & 0 \end{pmatrix},$$

which is still selfadjoint. We then calculate that

$$\begin{aligned} qYq_\lambda^* &= \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} Y_\nu\phi_1 & -Y_e\overline{\phi}_2 \\ Y_\nu\phi_2 & Y_e\overline{\phi}_1 \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \overline{\lambda}Y_\nu(\alpha\phi_1 + \beta\phi_2) & \lambda Y_e(\beta\overline{\phi}_1 - \alpha\overline{\phi}_2) \\ \overline{\lambda}Y_\nu(-\overline{\beta}\phi_1 + \overline{\alpha}\phi_2) & \lambda Y_e(\overline{\alpha}\overline{\phi}_1 + \overline{\beta}\overline{\phi}_2) \end{pmatrix}. \end{aligned}$$

Now let us calculate the second term $u[D_F, u^*]$ for $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$. The operator T only acts on ν_R and therefore commutes with the algebra. On the restriction to $\mathcal{H}_{\bar{l}}$, the operator $\overline{S}$ commutes with the algebra. Hence again we can restrict our calculation to $\mathcal{H}_l$. The term $u[S, u^*]$ splits into $uSu^* - S$, and (similarly to $u\phi u^*$) we find

$$uSu^* = \begin{pmatrix} 0 & q_\lambda Y_0^* q^* \\ qY_0 q_\lambda^* & 0 \end{pmatrix}$$

and

$$qY_0q_\lambda^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \overline{\lambda}Y_\nu\alpha & \lambda Y_e\beta \\ -\overline{\lambda}Y_\nu\overline{\beta} & \lambda Y_e\overline{\alpha} \end{pmatrix}.$$

Combining the two contributions to the transformation, we find that the transformation $u\phi u^* + u[S, u^*]$ yields

$$\begin{aligned} Y = \begin{pmatrix} Y_\nu\phi_1 & -Y_e\overline{\phi}_2 \\ Y_\nu\phi_2 & Y_e\overline{\phi}_1 \end{pmatrix} &\to Y' = \begin{pmatrix} Y_\nu\phi_1' & -Y_e\overline{\phi}_2' \\ Y_\nu\phi_2' & Y_e\overline{\phi}_1' \end{pmatrix} \\ &= \begin{pmatrix} \overline{\lambda}Y_\nu(\alpha\phi_1 + \beta\phi_2) & \lambda Y_e(\beta\overline{\phi}_1 - \alpha\overline{\phi}_2) \\ \overline{\lambda}Y_\nu(-\overline{\beta}\phi_1 + \overline{\alpha}\phi_2) & \lambda Y_e(\overline{\alpha}\overline{\phi}_1 + \overline{\beta}\overline{\phi}_2) \end{pmatrix} + \begin{pmatrix} \overline{\lambda}Y_\nu(\alpha - 1) & \lambda Y_e\beta \\ -\overline{\lambda}Y_\nu\overline{\beta} & \lambda Y_e(\overline{\alpha} - 1) \end{pmatrix}. \end{aligned}$$

where we have defined $\phi_1' := \overline{\lambda}(\alpha\phi_1 + \beta\phi_2 + \alpha) - 1$ and $\phi_2' := \overline{\lambda}(-\overline{\beta}\phi_1 + \overline{\alpha}\phi_2 - \overline{\beta})$. Rewriting this in terms of q then proves the proposition. □

8.8 Remark. In Proposition 6.20, we have seen that in general the transformation of the field ϕ is not a linear transformation. In the present model, Proposition 8.7 shows that it can be reduced to an affine transformation of the doublet ϕ_1, ϕ_2. This can be rewritten in the linear form

$$\begin{pmatrix} \phi_1 + 1 \\ \phi_2 \end{pmatrix} \to \overline{\lambda}\, q \begin{pmatrix} \phi_1 + 1 \\ \phi_2 \end{pmatrix}. \tag{8.9}$$

One should note here that, whereas the complex doublet (ϕ_1, ϕ_2) corresponds to the field ϕ, the doublet $(1, 0)$ corresponds to the operator S, which is a part of D_F. We thus see that the combination $S + \phi$ has a linear transformation under the gauge group.

8.3 THE SPECTRAL ACTION

In this section we will calculate the bosonic part of the Lagrangian of the Glashow-Weinberg-Salam Model. We obtain this Lagrangian by inserting the fields B_μ and Φ into the result of Proposition 6.29. We first start with a few lemmas, in which we capture the rather tedious calculations that are needed to obtain the traces of $F_{\mu\nu}F^{\mu\nu}$, Φ^2, Φ^4 and $(D_\mu\Phi)(D^\mu\Phi)$. Using these results, we can then easily find the expansion of the spectral action.

8.9 Lemma. *The trace of the square of the curvature of B_μ is given by*

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 12\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 2\mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}). \tag{8.10}$$

PROOF. Let us define the curvatures of the $U(1)$ and $SU(2)$ gauge fields by

$$\begin{aligned}\Lambda_{\mu\nu} &:= \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu,\\ Q_{\mu\nu} &:= \partial_\mu Q_\nu - \partial_\nu Q_\mu - i[Q_\mu, Q_\nu].\end{aligned} \tag{8.11}$$

Using Corollary 8.6, we then find that the curvature $F_{\mu\nu}$ of B_μ can be written as

$$F_{\mu\nu}\Big|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & \\ 0 & -2\Lambda_{\mu\nu} & \\ & & Q_{\mu\nu} - \Lambda_{\mu\nu}\mathbb{1}_2\end{pmatrix},$$
$$F_{\mu\nu}\Big|_{\mathcal{H}_{\bar{l}}} = \begin{pmatrix} 0 & 0 & \\ 0 & 2\Lambda_{\mu\nu} & \\ & & \Lambda_{\mu\nu}\mathbb{1}_2 - (\overline{Q})_{\mu\nu}\end{pmatrix}.$$

The curvature squared thus becomes

$$F_{\mu\nu}F^{\mu\nu}\Big|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & \\ 0 & 4\Lambda_{\mu\nu}\Lambda^{\mu\nu} & \\ & & Q_{\mu\nu}Q^{\mu\nu} + \Lambda_{\mu\nu}\Lambda^{\mu\nu}\mathbb{1}_2 - 2\Lambda_{\mu\nu}Q^{\mu\nu}\end{pmatrix},$$
$$F_{\mu\nu}F^{\mu\nu}\Big|_{\mathcal{H}_{\bar{l}}} = \begin{pmatrix} 0 & 0 & \\ 0 & 4\Lambda_{\mu\nu}\Lambda^{\mu\nu} & \\ & & (\overline{Q})_{\mu\nu}(\overline{Q})^{\mu\nu} + \Lambda_{\mu\nu}\Lambda^{\mu\nu}\mathbb{1}_2 - 2\Lambda_{\mu\nu}(\overline{Q})^{\mu\nu}\end{pmatrix}.$$

Since $Q_{\mu\nu}$ is traceless, the cross-term $-2\Lambda_{\mu\nu}Q^{\mu\nu}$ will drop out after taking the trace. Note that since Q_μ is selfadjoint we have $\overline{Q}_\mu = Q_\mu^T$. We then see that $Q_{\mu\nu}$ is also selfadjoint, since

$$\begin{aligned}\overline{(Q_{\mu\nu})} &= \partial_\mu\overline{Q}_\nu - \partial_\nu\overline{Q}_\mu + i[\overline{Q}_\mu, \overline{Q}_\nu] = \partial_\mu Q_\nu^T - \partial_\nu Q_\mu^T + i[Q_\mu^T, Q_\nu^T]\\ &= \big(\partial_\mu Q_\nu - \partial_\nu Q_\mu - i[Q_\mu, Q_\nu]\big)^T = (Q_{\mu\nu})^T.\end{aligned}$$

This implies that

$$\mathrm{Tr}\big(\overline{(Q_{\mu\nu})(Q^{\mu\nu})}\big) = \mathrm{Tr}\big((Q_{\mu\nu})^T(Q^{\mu\nu})^T\big) = \mathrm{Tr}\Big(\big(Q^{\mu\nu}Q_{\mu\nu}\big)^T\Big) = \mathrm{Tr}\big(Q_{\mu\nu}Q^{\mu\nu}\big).$$

We thus obtain that

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 12\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 2\mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}). \qquad \square$$

8.10 Lemma. *The traces of Φ^2 and Φ^4 are given by*

$$\mathrm{Tr}\big(\Phi^2\big) = 4a|H|^2 + 2c, \tag{8.12}$$
$$\mathrm{Tr}\big(\Phi^4\big) = 4b|H|^4 + 8e|H|^2 + 2d, \tag{8.13}$$

where H denotes the complex doublet (ϕ_1+1,ϕ_2) and (following [13, Ch. 1, §15.3])

$$\begin{aligned} a &= |Y_\nu|^2+|Y_e|^2, \\ b &= |Y_\nu|^4+|Y_e|^4, \\ c &= |Y_R|^2, \\ d &= |Y_R|^4, \\ e &= |Y_R|^2|Y_\nu|^2. \end{aligned} \tag{8.14}$$

Proof. The field Φ is given by

$$\Phi = D_F + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} J_F^* = \begin{pmatrix} S+\phi & T^* \\ T & \overline{(S+\phi)} \end{pmatrix}.$$

Its square equals

$$\Phi^2 = \begin{pmatrix} (S+\phi)^2+T^*T & (S+\phi)T^*+T^*\overline{(S+\phi)} \\ T(S+\phi)+\overline{(S+\phi)}T & \overline{(S+\phi)}^2+TT^* \end{pmatrix}.$$

The square of the off-diagonal part yields $T^*T = TT^* = |Y_R|^2$ on ν_R and $\overline{\nu_R}$, and zero on $l \neq \nu_R, \overline{\nu_R}$. The component $S+\phi$ is given by $\begin{pmatrix} 0 & Y^*+Y_0^* \\ Y+Y_0 & 0 \end{pmatrix}$. We calculate

$$X := (Y+Y_0)^*(Y+Y_0) = |H|^2 \begin{pmatrix} |Y_\nu|^2 & 0 \\ 0 & |Y_e|^2 \end{pmatrix}.$$

where we have defined the complex doublet $H := (\phi_1+1,\phi_2)$. Similarly, we define $X' := (Y+Y_0)(Y+Y_0)^*$ and note that $\mathrm{Tr}(X) = \mathrm{Tr}(X')$ by the cyclic property of the trace. Since $X = X^*$ and $\mathrm{Tr}(X) = \mathrm{Tr}(X^T)$, we also have $\mathrm{Tr}(\overline{X}) = \mathrm{Tr}(X)$. We thus obtain that

$$\begin{aligned} \mathrm{Tr}\big(\Phi^2\big) &= \mathrm{Tr}(X+X'+\overline{X}+\overline{X}') + 2|Y_R|^2 = 4\mathrm{Tr}(X)+2|Y_R|^2 \\ &= 4(|Y_\nu|^2+|Y_e|^2)|H|^2+2|Y_R|^2. \end{aligned}$$

In order to find the trace of Φ^4, we calculate

$$(X+T^*T)^2 = |H|^4 \begin{pmatrix} |Y_\nu|^4 & 0 \\ 0 & |Y_e|^4 \end{pmatrix} + 2|H|^2 \begin{pmatrix} |Y_R|^2|Y_\nu|^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} |Y_R|^4 & 0 \\ 0 & 0 \end{pmatrix}.$$

We now also obtain a contribution from the off-diagonal part of Φ^2. Any term of the form $(S+\phi)T^*\overline{(S+\phi)}T$ (or a cyclic permutation thereof) vanishes. We do obtain contributions from $\mathrm{Tr}\big((S+\phi)T^*T(S+\phi)\big)$ and three other similar terms, which each yield the contribution $|H|^2|Y_R|^2|Y_\nu|^2$. We thus obtain

$$\begin{aligned} \mathrm{Tr}\big(\Phi^4\big) &= \mathrm{Tr}\big((X+T^*T)^2+(X')^2+(\overline{X}+TT^*)^2+(\overline{X}')^2\big) + 4|H|^2|Y_R|^2|Y_\nu|^2 \\ &= \mathrm{Tr}\big(4X^2+4XT^*T+2(T^*T)^2\big)+4|H|^2|Y_R|^2|Y_\nu|^2 \\ &= 4|H|^4\big(|Y_\nu|^4+|Y_e|^4\big)+8|H|^2|Y_R|^2|Y_\nu|^2+2|Y_R|^4. \end{aligned}$$ □

8.11 Lemma. *The trace of $(D_\mu\Phi)(D^\mu\Phi)$ is given by*

$$\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) = 4a|\widetilde{D}_\mu H|^2, \tag{8.15}$$

where H denotes the complex doublet (ϕ_1+1,ϕ_2), and the covariant derivative $\widetilde{D}_\mu$ on H is defined as

$$\widetilde{D}_\mu H = \partial_\mu H - iQ_\mu^a\sigma^a H + i\Lambda_\mu H. \tag{8.16}$$

PROOF. We need to calculate the commutator $[B_\mu, \Phi]$. We note that B_μ commutes with $\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$. It is thus sufficient to calculate the commutator $[B_\mu, S+\phi]$ on $\mathcal{H}_l$. We shall write $Q_\mu = Q^1_\mu \sigma^1 + Q^2_\mu \sigma^2 + Q^3_\mu \sigma^3$ as a superposition of the Pauli matrices of Eq. (8.5) for real coefficients Q^a_μ. We then obtain by direct calculation

$$[B_\mu, S+\phi] = \begin{pmatrix} 0 & 0 & -\overline{Y}_\nu \overline{\chi}_1 & -\overline{Y}_\nu \overline{\chi}_2 \\ 0 & 0 & -\overline{Y}_e \chi_2 & \overline{Y}_e \chi_1 \\ Y_\nu \chi_1 & Y_e \overline{\chi}_2 & 0 & 0 \\ Y_\nu \chi_2 & -Y_e \overline{\chi}_1 & 0 & 0 \end{pmatrix},$$

where we have defined the new doublet $\chi = (\chi_1, \chi_2)$ by

$$\chi_1 := (\phi_1 + 1)(Q^3_\mu - \Lambda_\mu) + \phi_2(Q^1_\mu - iQ^2_\mu),$$
$$\chi_2 := (\phi_1 + 1)(Q^1_\mu + iQ^2_\mu) + \phi_2(-Q^3_\mu - \Lambda_\mu).$$

We then obtain that

$$\begin{aligned} D_\mu(S+\phi) &= \partial_\mu \phi - i[B_\mu, S+\phi] \\ &= \begin{pmatrix} 0 & 0 & \overline{Y}_\nu(\partial_\mu \overline{\phi}_1 + i\overline{\chi}_1) & \overline{Y}_\nu(\partial_\mu \overline{\phi}_2 + i\overline{\chi}_2) \\ 0 & 0 & -\overline{Y}_e(\partial_\mu \phi_2 - i\chi_2) & \overline{Y}_e(\partial_\mu \phi_1 - i\chi_1) \\ Y_\nu(\partial_\mu \phi_1 - i\chi_1) & -Y_e(\partial_\mu \overline{\phi}_2 + i\overline{\chi}_2) & 0 & 0 \\ Y_\nu(\partial_\mu \phi_2 - i\chi_2) & Y_e(\partial_\mu \overline{\phi}_1 + i\overline{\chi}_1) & 0 & 0 \end{pmatrix}. \end{aligned}$$

We want to calculate the trace of the square of $D_\mu \Phi$, for this reason we only need to calculate the terms on the diagonal of $(D_\mu \Phi)(D^\mu \Phi)$. We thus find

$$\mathrm{Tr}_{\mathcal{H}_l}\Big((D_\mu(S+\phi))(D^\mu(S+\phi))\Big) = 2a\Big(|\partial_\mu \phi_1 - i\chi_1|^2 + |\partial_\mu \phi_2 - i\chi_2|^2\Big),$$

where we have used $a = |Y_\nu|^2 + |Y_e|^2$ as in Eq. (8.14). The column vector H is given by the complex doublet $(\phi_1 + 1, \phi_2)$. We then note that $\partial_\mu \phi - i\chi$ is equal to the covariant derivative $\widetilde{D}_\mu H$, so that we obtain

$$\mathrm{Tr}_{\mathcal{H}_l}\Big((D_\mu(S+\phi))(D^\mu(S+\phi))\Big) = 2a|\widetilde{D}_\mu H|^2.$$

The trace over $\mathcal{H}_{\bar{l}}$ yields exactly the same contribution, so we need to multiply this by 2 and thus obtain the desired result. □

8.12 Proposition. *The spectral action of the spectral triple*

$$\Big(C^\infty(M, \mathbb{C} \oplus \mathbb{H}), L^2(M,S) \otimes (\mathbb{C}^4 \oplus \mathbb{C}^4), \not{D} \otimes \mathbb{1} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\Big)$$

defined in this chapter is given by

$$\mathrm{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) \sim \frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H)\sqrt{|g|}d^4x + O(\Lambda^{-1}), \tag{8.17}$$

for

$$\mathcal{L}(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H) := 8\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_A(\Lambda_\mu, Q_\mu) + \mathcal{L}_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H). \tag{8.18}$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 6.27. $\mathcal{L}_A$ gives the kinetic terms of the gauge fields and equals

$$\mathcal{L}_A(\Lambda_\mu, Q_\mu) := \frac{1}{6}f(0)\Big(12\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 2\mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu})\Big). \tag{8.19}$$

The Higgs potential (ignoring the boundary term) gives

$$\begin{aligned} \mathcal{L}_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H) &:= 2bf(0)|H|^4 + \big(-8af_2\Lambda^2 + 4ef(0)\big)|H|^2 \\ &\quad - 4cf_2\Lambda^2 + df(0) + \frac{1}{3}f(0)as|H|^2 + \frac{1}{6}f(0)cs + 2af(0)|\widetilde{D}_\mu H|^2. \end{aligned} \tag{8.20}$$

PROOF. We will use the general form of the spectral action of an almost commutative spectral triple as calculated in Proposition 6.29. From Lemma 8.9 we immediately find the term $\mathcal{L}_A$. Combining the formulas of $\mathrm{Tr}(\Phi^2)$ and $\mathrm{Tr}(\Phi^4)$ obtained in Lemma 8.10 we find the Higgs potential

$$\begin{aligned} & -2f_2\Lambda^2\mathrm{Tr}(\Phi^2) + \frac{1}{2}f(0)\mathrm{Tr}(\Phi^4) = \\ & \qquad 2bf(0)|H|^4 + \big(-8af_2\Lambda^2 + 4ef(0)\big)|H|^2 - 4cf_2\Lambda^2 + df(0). \end{aligned}$$

Note that the last two constant terms yield a contribution to the cosmological constant term $16f_4\Lambda^4$ that arises from $\mathcal{L}_M$. The coupling of the Higgs field to the scalar curvature s is given by

$$\frac{1}{12}f(0)s\mathrm{Tr}(\Phi^2) = \frac{1}{3}af(0)s|H|^2 + \frac{1}{6}cf(0)s.$$

Here the second term yields a contribution to the Einstein-Hilbert term $-\frac{4}{3}f_2\Lambda^2 s$ of $\mathcal{L}_M$. The last term is the kinetic term of the Higgs field including the minimal coupling to the gauge fields, obtained from Lemma 8.11, which gives

$$\frac{1}{2}f(0)\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) = 2af(0)|\widetilde{D}_\mu H|^2. \qquad \square$$

8.4 THE COUPLING CONSTANTS

So far we have considered the gauge fields Λ_μ and $Q_\mu = Q_\mu^a\sigma^a$. We shall now introduce coupling constants g_1 and g_2 into the model by rescaling these fields as

$$\Lambda_\mu = \frac{1}{2}g_1 B_\mu, \qquad Q_\mu^a = \frac{1}{2}g_2 W_\mu^a. \tag{8.21}$$

Note that we now use the conventional notation B_μ for the $U(1)$ hypercharge field, which should not be confused with the gauge field we introduced in Proposition 6.4. We define the curvatures $B_{\mu\nu}$ and $W_{\mu\nu}$ by setting

$$\Lambda_{\mu\nu} = \frac{1}{2}g_1 B_{\mu\nu}, \qquad Q_{\mu\nu}^a = \frac{1}{2}g_2 W_{\mu\nu}^a. \tag{8.22}$$

Using Eq. (8.11), this yields

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{8.23}$$

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2\epsilon^{abc}W_\mu^b W_\nu^c, \tag{8.24}$$

where we have used the relation $[\sigma^b,\sigma^c] = 2i\epsilon^{abc}\sigma^a$ for the Pauli matrices. We then rewrite the trace of the square of the curvature, given by Lemma 8.9, into

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 3{g_1}^2 B_{\mu\nu}B^{\mu\nu} + {g_2}^2 W_{\mu\nu}^a W^{\mu\nu,a}, \tag{8.25}$$

where we have used the relation $\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}$. Note that the covariant derivative $\widetilde{D}_\mu H$ can now be written as

$$\widetilde{D}_\mu H = \partial_\mu H - \frac{1}{2}ig_2 W_\mu^a\sigma^a H + \frac{1}{2}ig_1 B_\mu H. \tag{8.26}$$

8.4.1 *Electroweak unification*

It would be natural to require that the kinetic terms of the gauge fields, given by the squares of the curvatures, are properly normalized. That is, we require that both these squares of the curvatures have the coefficient $\frac{1}{4}$. This imposes the relations

$$\frac{f(0)}{8\pi^2}{g_1}^2 = \frac{1}{4} \quad \text{and} \quad \frac{f(0)}{24\pi^2}{g_2}^2 = \frac{1}{4}. \tag{8.27}$$

This then means that the coupling constants are related by ${g_2}^2 = 3{g_1}^2$. The values of the coupling constants depend on the energy scale at which they are evaluated, and their scale-dependence is determined by the renormalization group equations. Let Λ_{EW} be the scale at which the equality ${g_2}^2 = 3{g_1}^2$ holds. Our model of the electroweak theory is then naturally defined at this scale Λ_{EW}, and one could use the renormalization group equations to 'run down' our model to lower energies.

8.5 THE HIGGS MECHANISM

When writing down a gauge theory with massive gauge bosons, one encounters the difficulty that the mass terms of these gauge bosons are not gauge invariant. The Higgs field plays a central role in obtaining these mass terms within a gauge theory. The Higgs mechanism provides a mechanism for *spontaneous breaking* of the gauge symmetry. In this section we will show how the Higgs mechanism breaks the $U(1) \times SU(2)$ symmetry and introduces mass terms for the gauge bosons. We discuss two different cases. First, we will deal with the usual approach as in the Glashow-Weinberg-Salam model or the Standard Model. Second, we realise that in our case another contribution to the Higgs potential arises from the gravitational coupling $s|H|^2$. By taking into account this gravitational coupling, the Higgs mechanism breaks both the gauge symmetry and the conformal symmetry.

8.5.1 *The usual approach*

In Proposition 8.12 we have obtained the Higgs potential

$$\mathcal{L}_{\text{pot}}(H) := 2bf(0)|H|^4 - \big(8af_2\Lambda^2 - 4ef(0)\big)|H|^2. \tag{8.28}$$

The minimum of this potential is obtained if the field H satisfies

$$|H|^2 = \frac{\big(2af_2\Lambda^2 - ef(0)\big)}{bf(0)}. \tag{8.29}$$

We shall assume that $2af_2\Lambda^2 > ef(0)$ so that $|H|^2$ is positive. The fields that satisfy this relation are called the vacuum states of the Higgs field. We shall choose a vacuum state $(v, 0)$, where v is a real parameter such that v^2 is given by Eq. (8.29).

We want to simplify the expression for the Higgs potential. First, we note that the potential only depends on the absolute value $|H|$. A transformation of the doublet H by an element $u \in U(1) \times SU(2)$ is written as $H \to uH$. Since a unitary transformation preserves the absolute value, we see that $\mathcal{L}_{\text{pot}}(uH) = \mathcal{L}_{\text{pot}}(H)$ for any $u \in U(1) \times SU(2)$. We can use this *gauge freedom* to transform the Higgs field into a simpler form.

Consider elements of $U(1) \times SU(2)$ of the form $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$ such that $|a|^2 + |b|^2 = 1$. The doublet H can in general be written as (h_1, h_2), for some $h_1, h_2 \in \mathbb{C}$. We then see that we can write

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} |H| \\ 0 \end{pmatrix}, \qquad a = \frac{h_1}{|H|}, \quad b = \frac{h_2}{|H|}. \tag{8.30}$$

This means that we can always use the gauge freedom to write the doublet H in terms of one real parameter. Let us define a new real-valued field h by setting $h(x) := |H(x)| - v$. We then obtain

$$H = u(x) \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}, \qquad u(x) := \begin{pmatrix} a(x) & -\overline{b(x)} \\ b(x) & \overline{a(x)} \end{pmatrix}. \tag{8.31}$$

Inserting this transformed Higgs field into the Higgs potential, we obtain an expression in terms of the real parameter v and the real field $h(x)$:

$$\begin{aligned}\mathcal{L}_{\text{pot}}(h) &= 2bf(0)(v+h)^4 - \big(8af_2\Lambda^2 - 4ef(0)\big)(v+h)^2 \\ &= 2bf(0)(h^4 + 4vh^3 + 6v^2h^2 + 4v^3h + v^4) \\ &\quad - \big(8af_2\Lambda^2 - 4ef(0)\big)(h^2 + 2vh + v^2).\end{aligned} \tag{8.32}$$

Using Eq. (8.29), the value of v^2 is given by

$$v^2 = \frac{\big(2af_2\Lambda^2 - ef(0)\big)}{bf(0)}. \tag{8.33}$$

We then see that in $\mathcal{L}_{\text{pot}}$ the terms linear in h cancel each other. This is of course no surprise, since the change of variables $|H(x)| \to v + h(x)$ means that at $h(x) = 0$ we are in the minimum of the potential, where the first order derivative of the potential with respect to h must vanish. We thus obtain the simplified expression

$$\mathcal{L}_{\text{pot}}(h) = 2bf(0)\Big(h^4 + 4vh^3 + 4v^2h^2 - v^4\Big). \tag{8.34}$$

We now observe that the field $h(x)$ has obtained a mass term and has two self-interactions given by h^3 and h^4. We also have another contribution to the cosmological constant given by $-v^4$.

Massive gauge bosons

Next, let us consider what this procedure entails for the remainder of the Higgs Lagrangian $\mathcal{L}_H$. We first consider the kinetic term of H, including its minimal coupling to the gauge fields, given by

$$\mathcal{L}_{\text{min}}(\Lambda_\mu, Q_\mu, H) := 2af(0)|\widetilde{D}_\mu H|^2. \tag{8.35}$$

The transformation of Eq. (8.31) is a gauge transformation, and to make sure that $\mathcal{L}_{\text{min}}$ is invariant under this transformation, we also need to transform the gauge fields. The field B_μ is unaffected by the local $SU(2)$-transformation $u(x)$. The transformation of $W_\mu = W_\mu^a \sigma^a$ is obtained from Proposition 8.7 and is given by

$$W_\mu \to uW_\mu u^* - \frac{2i}{g_2}(\partial_\mu u)u^*. \tag{8.36}$$

One then easily checks that we obtain the transformation $\widetilde{D}_\mu H \to u\widetilde{D}_\mu H$, so that $|\widetilde{D}_\mu H|^2$ is invariant under such transformations. So we can just insert the doublet $(v+h, 0)$ into Eq. (8.26) and obtain

$$\begin{aligned}\widetilde{D}_\mu H &= \partial_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix} - \frac{1}{2} i g_2 W_\mu^a \sigma^a \begin{pmatrix} v+h \\ 0 \end{pmatrix} + \frac{1}{2} i g_1 B_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix} \\ &= \partial_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix} - \frac{1}{2} i g_2 W_\mu^1 \begin{pmatrix} 0 \\ v+h \end{pmatrix} - \frac{1}{2} i g_2 W_\mu^2 \begin{pmatrix} 0 \\ i(v+h) \end{pmatrix} \\ &\quad - \frac{1}{2} i g_2 W_\mu^3 \begin{pmatrix} v+h \\ 0 \end{pmatrix} + \frac{1}{2} i g_1 B_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix}.\end{aligned} \tag{8.37}$$

We can then calculate its square as

$$\begin{aligned}|\widetilde{D}_\mu H|^2 &= (\widetilde{D}^\mu H)^\dagger (\widetilde{D}_\mu H) \\ &= (\partial^\mu h)(\partial_\mu h) + \frac{1}{4} {g_2}^2 (v+h)^2 (W^{\mu,1} W_\mu^1 + W^{\mu,2} W_\mu^2 + W^{\mu,3} W_\mu^3) \\ &\quad + \frac{1}{4} {g_1}^2 (v+h)^2 B^\mu B_\mu - \frac{1}{2} g_1 g_2 (v+h)^2 B^\mu W_\mu^3.\end{aligned} \tag{8.38}$$

Note that the last term yields a mixing of the gauge fields B_μ and W_μ^3. The electroweak mixing angle θ_w is defined by

$$c_w := \cos\theta_w = \frac{g_2}{\sqrt{{g_1}^2 + {g_2}^2}}, \qquad s_w := \sin\theta_w = \frac{g_1}{\sqrt{{g_1}^2 + {g_2}^2}}. \tag{8.39}$$

Let us now define new gauge fields by

$$\begin{aligned} W_\mu &:= \frac{1}{\sqrt{2}}(W_\mu^1 + iW_\mu^2), & W_\mu^* &:= \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2), \\ Z_\mu &:= c_w W_\mu^3 - s_w B_\mu, & A_\mu &:= s_w W_\mu^3 + c_w B_\mu. \end{aligned} \tag{8.40}$$

We will show that the new fields Z_μ and A_μ become mass eigenstates. The fields W_μ^1 and W_μ^2 already were mass eigenstates, but the fields W_μ and W_μ^* are chosen such that they obtain a definite charge. We can write

$$\begin{aligned} W_\mu^1 &= \frac{1}{\sqrt{2}}(W_\mu + W_\mu^*), & W_\mu^2 &= \frac{-i}{\sqrt{2}}(W_\mu - W_\mu^*), \\ W_\mu^3 &= s_w A_\mu + c_w Z_\mu, & B_\mu &= c_w A_\mu - s_w Z_\mu, \end{aligned} \tag{8.41}$$

and inserting this into the expression for $|\widetilde{D}_\mu H|^2$ yields

$$|\widetilde{D}_\mu H|^2 = (\partial^\mu h)(\partial_\mu h) + \frac{1}{2}{g_2}^2(v+h)^2 W^{\mu *} W_\mu + \frac{1}{4}\frac{{g_2}^2}{{c_w}^2}(v+h)^2 Z^\mu Z_\mu. \tag{8.42}$$

We thus see that the fields W_μ, W_μ^* and Z_μ acquire a mass term (where Z_μ has a larger mass than W_μ, W_μ^*) and that the field A_μ is massless.

Rescaling the Higgs field

We now perform a rescaling of the Higgs field so that the kinetic term is properly normalized. We define a new field

$$H' = \sqrt{\frac{af(0)}{\pi^2}}H, \tag{8.43}$$

so that the kinetic term becomes

$$\int_M \frac{1}{2}|\widetilde{D}_\mu H'|^2 \sqrt{|g|}d^4x. \tag{8.44}$$

We will again perform the transformation Eq. (8.31) to write H' in terms of v' and h'. We shall now rewrite the obtained mass terms and interactions and drop the primes from here on. The Higgs potential (including its full coefficient) becomes

$$\mathcal{L}_{\text{pot}}(h) = \frac{b\pi^2}{2a^2 f(0)}\Big(h^4 + 4vh^3 + 4v^2h^2 - v^4\Big). \tag{8.45}$$

The Higgs kinetic term, the mass terms of the gauge fields and their couplings to the Higgs field h are given by

$$\frac{1}{2}|\widetilde{D}_\mu H|^2 \;=\; \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{4}{g_2}^2(v+h)^2 W^{\mu *} W_\mu + \frac{1}{8}\frac{{g_2}^2}{{c_w}^2}(v+h)^2 Z^\mu Z_\mu. \tag{8.46}$$

We conclude that the gauge fields have acquired the masses

$$M_W = \frac{1}{2}vg_2, \qquad M_Z = \frac{1}{2}v\frac{g_2}{c_w}. \tag{8.47}$$

8.5.2 *Spontaneous breaking of conformal symmetry*

There is one term in the Lagrangian that we have not yet considered, namely the term $s|H|^2$ which couples the Higgs field H to the scalar curvature s. A gauge transformation does not affect the scalar curvature, so the result of the transformation (8.31) yields

$$s|H|^2 = s(v+h)^2 = s(h^2 + 2vh + v^2). \tag{8.48}$$

The presence of the linear term in h shows that v is not really the minimum of the full Higgs potential $\mathcal{L}_H$ of Proposition 8.12. There are two adjustments we need to make to Eq. (8.31). First, we need to replace v with $v(x)$ given by

$$v(x)^2 = \frac{\left(2af_2\Lambda^2 - ef(0) - \frac{1}{12}af(0)s(x)\right)}{bf(0)}, \tag{8.49}$$

where we have explicitly denoted the x-dependence of $s(x)$, and hence of $v(x)$. Second, we remove the x-dependence through a conformal transformation

$$v(x) = \Omega^{-1}(x)v_0. \tag{8.50}$$

This second transformation provides a spontaneous breaking of the conformal symmetry. A good treatment of such a spontaneous breaking in the case of conformal gravity with a conformally coupled scalar field can be found in [29].

8.13 Remark. Note that, because of the freedom we have in choosing $\Omega(x)$, we are free to take $v_0 = 1$ (provided that $v(x)^2 > 0$) through a *global* conformal transformation. However, for clarity we will simply leave v_0 as it is, without specifying its value.

Furthermore, an interesting situation arises when the scalar curvature s becomes large. When $\frac{1}{12}af(0)s(x) > 2af_2\Lambda^2 - ef(0)$, the total coefficient in front of $|H|^2$ becomes positive, and hence the minimum of the Higgs potential is obtained for $H = 0$. In this case, there will be no spontaneous symmetry breaking.

8.14 Remark. Consider the effect of a conformal transformation given by $\Omega(x)$ on the Higgs field Φ, which transforms as

$$\Phi = \begin{pmatrix} S+\phi & T^* \\ T & \overline{(S+\phi)} \end{pmatrix} \longrightarrow \Omega^{-1}(x)\Phi = \begin{pmatrix} \Omega^{-1}(x)(S+\phi) & \Omega^{-1}(x)T^* \\ \Omega^{-1}(x)T & \Omega^{-1}(x)\overline{(S+\phi)} \end{pmatrix}. \tag{8.51}$$

The rescaling of $S+\phi$ is given by the rescaling of the doublet $H \to \Omega(x)^{-1}H$. However, the conformal transformation also affects the off-diagonal part T (which gives the Majorana mass for the right-handed neutrinos) and hence it affects the constants c, d and e of Eq. (8.14). So, when performing a conformal transformation, these constants must be transformed accordingly.

8.15 Theorem. *The (gauge and conformal) transformation of the Higgs field, given by*

$$H = \Omega^{-1}(x)u(x)\begin{pmatrix} v_0 + h(x) \\ 0 \end{pmatrix}, \qquad h(x) := \Omega^{-1}(x)|H(x)| - v_0, \tag{8.52}$$

breaks both the gauge symmetry and the conformal symmetry. The resulting spontaneously broken bosonic action (ignoring topological and boundary terms) is given by

$$\begin{aligned} S_B = \int_M \Bigg(& \frac{4f_4\Lambda^4}{\pi^2} - \frac{cf_2\Lambda^2}{\pi^2} + \frac{df(0)}{4\pi^2} - \frac{b\pi^2}{2a^2f(0)}{v_0}^4 \\ & + \frac{cf(0)s}{24\pi^2} - \frac{1}{3\pi^2}f_2\Lambda^2 s - \frac{f(0)}{40\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \\ & + \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu,a} + \frac{f_2\Lambda^2}{3\pi^2}(\partial^\beta\eta)(\partial_\beta\eta) \\ & + \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{b\pi^2}{2a^2f(0)}\left(h^4 + 4v_0h^3 + 4{v_0}^2h^2\right) \\ & + \frac{1}{4}{g_2}^2(v_0+h)^2W^{\mu*}W_\mu + \frac{1}{8}\frac{{g_2}^2}{{c_w}^2}(v_0+h)^2Z^\mu Z_\mu\Bigg)\sqrt{|g|}d^4x. \end{aligned} \tag{8.53}$$

Here we have introduced the dilaton field η by setting $\Omega(x) = e^{\eta(x)}$, as in Remark 6.32.

Proof. As has been shown in Proposition 6.31, the full Lagrangian $\mathcal{L}_H$ (when integrated over M) is conformally invariant. Thus, we can simply replace H by the doublet $(v_0 + h(x), 0)$ in the formula for $\mathcal{L}_H$ given in Proposition 8.12. Using the rescaling of the Higgs field of Eq. (8.43), the Higgs potential and the Higgs kinetic term are then rewritten as in Eqs. (8.45) and (8.46). For the gauge kinetic terms, we use Eq. (8.25) and impose the relations Eq. (8.27). The gravitational Lagrangian $\mathcal{L}_M$ of Proposition 8.12 obtains a kinetic term for a dilaton field η by Remark 6.32. Combining all terms then proves the proposition. □

8.16 Remark. In the Higgs Lagrangian $\mathcal{L}_H$, as given in Proposition 8.12, there is one term proportional to $s|H|^2$. One might expect that after the spontaneous symmetry breaking this would yield a contribution to the Einstein-Hilbert Lagrangian in the form of $s{v_0}^2$. However, this is not the case, since this term combines with the other terms proportional to ${v_0}^2$, and then their coefficient is also seen to be proportional to ${v_0}^2$. In this way, we are only left with the constant term $-\frac{b\pi^2}{2a^2f(0)}{v_0}^4$, which contributes to the cosmological constant.

Furthermore, after the conformal transformation there also remains no coupling between the Higgs field h and the scalar curvature s, since this coupling has been absorbed into the mass term ${v_0}^2h^2$. Hence, the term $s|H|^2$ has completely disappeared from the action.

9 THE STANDARD MODEL

One of the major applications of noncommutative geometry to physics has been the derivation of the Standard Model of high energy physics from a suitably chosen almost commutative geometry [8, 13]. In Chapter 8 we have already discussed the electroweak sector (for one generation) of the Standard Model. In this chapter we will also incorporate the quark sector with the strong interactions, and show that we obtain the full Standard Model.

9.1 THE FINITE TRIPLE

In [8], the starting point is a left-right symmetric algebra $\mathcal{A}_{LR}$. One then obtains a subalgebra $\mathcal{A}_F \subset \mathcal{A}_{LR}$ by requiring that $\mathcal{A}_F$ should admit the Dirac operator D_F to contain an off-diagonal part. A discussion of how the algebra $\mathcal{A}_{LR}$ occurs naturally is given in [7]. For the purpose of this chapter we will not go into these details. Instead, we simply state the spectral triple that will be used. Keeping in mind the previous chapters and the fact that we now wish to obtain the Standard Model, the choices below should not be too mysterious.

We take the finite spectral triple of Section 8.1 as our starting point. In order to incorporate the strong interactions, we add the 3×3 complex matrices $M_3(\mathbb{C})$ to the algebra, and define

$$\mathcal{A}_F := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \tag{9.1}$$

We keep the Hilbert spaces $\mathcal{H}_l = \mathbb{C}^4$ and $\mathcal{H}_{\bar{l}} = \mathbb{C}^4$ for the description of the leptons and antileptons. For the quarks, we define $\mathcal{H}_q = \mathbb{C}^4 \otimes \mathbb{C}^3$, where the basis of $\mathbb{C}^4$ is given by $\{u_R, d_R, u_L, d_L\}$ and the three colors of the quarks are given by the factor $\mathbb{C}^3$. Similarly, we also have the antiquarks in $\mathcal{H}_{\bar{q}}$. Combined, we obtain the 96-dimensional Hilbert space for three generations of fermions and antifermions:

$$\mathcal{H}_F := \left(\mathcal{H}_l \oplus \mathcal{H}_{\bar{l}} \oplus \mathcal{H}_q \oplus \mathcal{H}_{\bar{q}}\right)^{\oplus 3}. \tag{9.2}$$

An element of the algebra $\mathcal{A}_F$ is given by $a = (\lambda, q, m)$, where the quaternion q can be written as $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. The representation $\pi\colon \mathcal{A}_F \to B(\mathcal{H}_F)$ on the leptons $\mathcal{H}_l$ and quarks $\mathcal{H}_q$ is given just as in Eq. (8.1) by

$$\pi(a)|_{\mathcal{H}_l} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \pi(a)|_{\mathcal{H}_q} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \otimes \mathbb{1}_3. \tag{9.3}$$

The representation is given by $\pi(a) = \lambda \mathbb{1}_4$ on the antileptons $\mathcal{H}_{\bar{l}}$ and by $\pi(a) = \mathbb{1}_4 \otimes m$ on the antiquarks $\mathcal{H}_{\bar{q}}$.

The grading and the conjugation operator are also chosen in the same way as in Section 8.1. The grading γ_F is such that all left-handed fermions have eigenvalue $+1$, and all right-handed fermions have eigenvalue -1. The conjugation operator

J_F interchanges a fermion with its antifermion. The Dirac operator D_F is again of the form $\begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$. The operator S is now given by

$$S_l := S|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \quad S_q \otimes \mathbb{1}_3 := S|_{\mathcal{H}_q} = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_3, \quad (9.4)$$

where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations. The symmetric operator T only acts on the right-handed (anti)neutrinos, so it is given by $T\nu_R = Y_R \overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

9.1 Proposition. *The data $(\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ given above define a real even spectral triple of KO-dimension 6.*

Proof. The representation of $\mathcal{A}_F$ on $\mathcal{H}_{\overline{q}}$, given by $\mathbb{1}_4 \otimes m$, commutes with all other operators, and hence it has no effect on the commutation relations. The proof is then the same as in Proposition 8.1. □

9.2 THE GAUGE THEORY

We shall now consider the product of the canonical spectral triple, describing a 4-dimensional spin manifold M, with the finite triple described in the previous section.

9.2 Lemma. *The subalgebra $(\widetilde{\mathcal{A}}_F)_{J_F}$ of the algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with the representation on $B(\mathcal{H}_F)$ as above, is given by $(\widetilde{\mathcal{A}}_F)_{J_F} = \mathbb{R}$. The Lie algebra $\mathfrak{h}_F$ of Definition 6.12 is then given by the trivial subset $\mathfrak{h}_F = \{0\}$.*

Proof. The proof is similar to Lemma 8.2. For an element $a = (\lambda, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, the relation $aJ_F = J_F a^*$ now yields $\lambda = \overline{\lambda} = \alpha = \overline{\alpha}$ and $\beta = 0$, as well as $m = \lambda \mathbb{1}_3$. So, $a \in (\widetilde{\mathcal{A}}_F)_{J_F}$ if and only if $a = (x, x, x)$ for $x \in \mathbb{R}$. Since $\mathfrak{u}(\mathcal{A}_F)$ consists of the anti-hermitian elements of $\mathcal{A}_F$, we again obtain that the cross-section $\mathfrak{h}_F = \mathfrak{u}(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F}$ is given by the trivial subalgebra $\{0\}$. □

9.3 Proposition. *The gauge group $\mathcal{G}(\mathcal{A}_F)$ is given by $\big(U(1) \times SU(2) \times U(3)\big)/\{1, -1\}$.*

Proof. As in Proposition 8.3, we find that $U(\mathbb{H}) = SU(2)$, so the unitary group $U(\mathcal{A}_F)$ is given by $U(1) \times SU(2) \times U(3)$. The subgroup $H_F = U(\mathcal{A}_F) \cap (\widetilde{\mathcal{A}}_F)_{J_F}$ is again given by $H_F = \{1, -1\}$. By Proposition 5.28, the gauge group is given by the quotient of the unitary group with this subgroup. □

9.4 Remark. The gauge group that we have obtained is not the gauge group of the Standard Model, because (even ignoring the quotient with the finite group $\{1, -1\}$) we have a factor $U(3)$ instead of $SU(3)$. As mentioned in Proposition 6.18, the unimodularity condition is only satisfied for complex algebras. In our case, we have a real algebra representation $\pi \colon \mathcal{A}_F \to B(\mathcal{H}_F)$, so the unimodularity condition is not satisfied. As in [13, Ch. 1, §15.5], we shall now *demand* that the unimodularity condition is satisfied, so for $u = (\lambda, q, m) \in U(1) \times SU(2) \times U(3)$ we require

$$\det|_{\mathcal{H}_F}(u) = 1 \quad \Longrightarrow \quad \big(\lambda \det m\big)^{12} = 1. \quad (9.5)$$

For $u \in U(1) \times SU(2) \times U(3)$, we denote $[u]$ for the equivalence class of u in the quotient $\mathcal{G}(\mathcal{A}_F)$. We shall then consider the gauge group

$$S\mathcal{G}(\mathcal{A}_F) = \Big\{ [u] = [(\lambda, q, m)] \in \mathcal{G}(\mathcal{A}_F) \mid \big(\lambda \det m\big)^{12} = 1 \Big\}. \quad (9.6)$$

The effect of the unimodularity condition is that the determinant of $m \in U(3)$ is identified (modulo the finite group μ_{12} of 12th-roots of unity) to $\overline{\lambda}$. In other words, imposing the unimodularity condition provides us, modulo some finite abelian group, with the gauge group

$$\mathcal{G}_{SM} := U(1) \times SU(2) \times SU(3). \tag{9.7}$$

Let us go into a little more detail, following (but slightly modifying) [13, Prop. 1.185], and prove the following proposition.

9.5 Proposition. *There is an exact sequence of group homomorphisms*

$$1 \to \mu_6 \to \mathcal{G}_{SM} \xrightarrow{\varphi} S\mathcal{G}(\mathcal{A}_F) \xrightarrow{\rho} \mu_{12} \to 1.$$

Proof. We define the homomorphism $\rho\colon [(\lambda, q, m)] \mapsto \lambda \det m$, and by definition of $S\mathcal{G}(\mathcal{A}_F)$ this maps onto μ_{12}. The kernel of ρ is given by

$$\operatorname{Ker}(\rho) = \{[(\lambda, q, m)] \in \mathcal{G}(\mathcal{A}_F) \mid \lambda \det m = 1\}.$$

The homomorphism $\varphi\colon U(1) \times SU(2) \times SU(3) \to S\mathcal{G}(\mathcal{A}_F)$ is given by $\varphi(\lambda, q, m) = [(\lambda^3, g, \lambda^{-1}m)]$. We see that $\lambda^3 \det(\lambda^{-1}m) = \det m = 1$, so that φ maps onto $S\mathcal{G}(\mathcal{A}_F)$, and we obtain that $\operatorname{Im}(\varphi) = \operatorname{Ker}(\rho)$. The kernel of φ is given by all (λ, q, m) for which $(\lambda^3, q, \lambda^{-1}m) = \pm 1$. This implies that $\lambda^3 = \pm 1$, and thus $q = \lambda^3 \mathbb{1}_2$ and $m = \lambda^4 \mathbb{1}_3$. The requirement $\lambda^3 = \pm 1$ implies $\lambda \in \mu_6$, so we obtain that $\operatorname{Ker}(\varphi) \cong \mu_6$. □

9.6 Proposition. *The gauge field A_μ is given by*

$$A_\mu|_{\mathcal{H}_l} = \begin{pmatrix} \Lambda_\mu & 0 & \\ 0 & -\Lambda_\mu & \\ & & Q_\mu \end{pmatrix}, \qquad A_\mu|_{\mathcal{H}_q} = \begin{pmatrix} \Lambda_\mu & 0 & \\ 0 & -\Lambda_\mu & \\ & & Q_\mu \end{pmatrix} \otimes \mathbb{1}_3,$$
$$A_\mu|_{\mathcal{H}_{\bar{l}}} = \Lambda_\mu \mathbb{1}_4, \qquad A_\mu|_{\mathcal{H}_{\bar{q}}} = -\mathbb{1}_4 \otimes (V_\mu + \frac{1}{3}\Lambda_\mu), \tag{9.8}$$

for a $U(1)$ gauge field Λ_μ, an $SU(2)$ gauge field Q_μ and an $SU(3)$ gauge field V_μ. The Higgs field ϕ is given by

$$\phi|_{\mathcal{H}_l} = \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix}, \quad \phi|_{\mathcal{H}_q} = \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \otimes \mathbb{1}_3, \quad \phi|_{\mathcal{H}_{\bar{l}}} = 0, \quad \phi|_{\mathcal{H}_{\bar{q}}} = 0, \tag{9.9}$$

where, for $\phi_1, \phi_2 \in \mathbb{C}$, we have

$$Y = \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}, \qquad X = \begin{pmatrix} Y_u \phi_1 & -Y_d \overline{\phi}_2 \\ Y_u \phi_2 & Y_d \overline{\phi}_1 \end{pmatrix}. \tag{9.10}$$

Proof. The proof is similar to Proposition 8.5, and the formulas for Λ_μ, Q_μ and ϕ follow immediately. The only difference is the occurrence of the field $V'_\mu := im\partial_\mu m'$, acting on $\mathcal{H}_{\bar{q}}$, for $m, m' \in M_3(\mathbb{C})$. Demanding selfadjointness yields $V'_\mu \in i\,\mathfrak{u}(3)$, so V'_μ is a $U(3)$ gauge field instead of an $SU(3)$ gauge field. As mentioned in Remark 9.4, we need to impose the unimodularity condition to obtain an $SU(3)$ gauge field. Hence, we require that the trace of the gauge field A_μ over $\mathcal{H}_F$ vanishes, and we obtain

$$\operatorname{Tr}|_{\mathcal{H}_{\bar{l}}} (\Lambda_\mu \mathbb{1}_4) + \operatorname{Tr}|_{\mathcal{H}_{\bar{q}}} (\mathbb{1}_4 \otimes V'_\mu) = 0 \quad \Longrightarrow \quad \operatorname{Tr}(V'_\mu) = -\Lambda_\mu.$$

So, we can define a traceless $SU(3)$ gauge field by $V_\mu := -V'_\mu - \frac{1}{3}\Lambda_\mu$. □

9.7 Corollary. *The gauge field* $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$ *is given on the fermions by*

$$B_\mu\big|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & \\ 0 & -2\Lambda_\mu & \\ & & Q_\mu - \Lambda_\mu \mathbb{1}_2 \end{pmatrix}, \tag{9.11}$$

$$B_\mu\big|_{\mathcal{H}_q} = \begin{pmatrix} \frac{4}{3}\Lambda_\mu \mathbb{1}_3 + V_\mu & 0 & \\ 0 & -\frac{2}{3}\Lambda_\mu \mathbb{1}_3 + V_\mu & \\ & & (Q_\mu + \frac{1}{3}\Lambda_\mu \mathbb{1}_2) \otimes \mathbb{1}_3 + \mathbb{1}_2 \otimes V_\mu \end{pmatrix}. \tag{9.12}$$

From the coefficients in front of Λ_μ in the above formulas, we recognize the correct hypercharges of the leptons and fermions.

9.3 THE SPECTRAL ACTION

We are now ready to determine the bosonic part of the Lagrangian of the Standard Model from the spectral action. As in Section 8.3, we start with a few lemmas to capture most of the calculations.

9.8 Lemma. *The trace of the square of the curvature of* B_μ *is given by*

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 24\Big(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu})\Big). \tag{9.13}$$

PROOF. The lepton sector yields the same result as in Lemma 8.9, only multiplied by a factor 3 for the number of generations. For the quark sector, we obtain on $\mathcal{H}_q$ the curvature

$$F_{\mu\nu} = \begin{pmatrix} \frac{4}{3}\Lambda_{\mu\nu} \mathbb{1}_3 + V_{\mu\nu} & 0 & \\ 0 & -\frac{2}{3}\Lambda_{\mu\nu} \mathbb{1}_3 + V_{\mu\nu} & \\ & & (Q_{\mu\nu} + \frac{1}{3}\Lambda_{\mu\nu} \mathbb{1}_2) \otimes \mathbb{1}_3 + \mathbb{1}_2 \otimes V_{\mu\nu} \end{pmatrix}, \tag{9.14}$$

where we have now defined the curvature of the $SU(3)$ gauge field by

$$V_{\mu\nu} := \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]. \tag{9.15}$$

If we calculate the trace of the square of the curvature $F_{\mu\nu}$, the cross-terms again vanish, so we obtain

$$\mathrm{Tr}\big|_{\mathcal{H}_q}(F_{\mu\nu}F^{\mu\nu}) = \left(\frac{16}{3} + \frac{4}{3} + \frac{1}{3} + \frac{1}{3}\right)\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 3\mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + 4\mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}).$$

We multiply this by a factor 2 to include the trace over the antiquarks, and by a factor 3 for the number of generations. Adding the result to the trace over the lepton sector, we finally obtain

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 80\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 24\mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + 24\mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}). \qquad \square$$

9.9 Lemma. *The traces of* Φ^2 *and* Φ^4 *are given by*

$$\mathrm{Tr}\big(\Phi^2\big) = 4a|H|^2 + 2c, \tag{9.16}$$

$$\mathrm{Tr}\big(\Phi^4\big) = 4b|H|^4 + 8e|H|^2 + 2d, \tag{9.17}$$

where H *denotes the complex doublet* $(\phi_1 + 1, \phi_2)$ *and (following [13, Ch. 1, §15.3])*

$$\begin{aligned} a &= \mathrm{Tr}\big(Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + 3Y_d^* Y_d\big), \\ b &= \mathrm{Tr}\big((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + 3(Y_d^* Y_d)^2\big), \\ c &= \mathrm{Tr}\big(Y_R^* Y_R\big), \\ d &= \mathrm{Tr}\big((Y_R^* Y_R)^2\big), \\ e &= \mathrm{Tr}\big(Y_R^* Y_R Y_\nu^* Y_\nu\big). \end{aligned} \tag{9.18}$$

PROOF. The proof is analoguous to Lemma 8.10, where the coefficients a, b, c, d, e have now been redefined to incorporate the quark sector, and the trace is taken over the generation space. □

9.10 Lemma. *The trace of $(D_\mu\Phi)(D^\mu\Phi)$ is given by*

$$\mathrm{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) = 4a|\widetilde{D}_\mu H|^2, \tag{9.19}$$

where H denotes the complex doublet $(\phi_1 + 1, \phi_2)$, and the covariant derivative $\widetilde{D}_\mu$ on H is defined as

$$\widetilde{D}_\mu H = \partial_\mu H - iQ_\mu^a\sigma^a H + i\Lambda_\mu H. \tag{9.20}$$

PROOF. The proof is as in Lemma 8.11. Since Φ commutes with the gauge field V_μ, this gauge field does not contribute to the covariant derivative $\widetilde{D}_\mu$. □

9.11 Proposition. *The spectral action of the spectral triple*

$$\Big(C^\infty(M, \mathcal{A}_F), L^2(M, S\otimes\mathcal{H}_F), \not{D}\otimes 1 + \gamma_5\otimes D_F, \gamma_5\otimes\gamma_F, J_M\otimes J_F\Big)$$

defined in this chapter is given by

$$\mathrm{Tr}\left(f\Big(\frac{D_A}{\Lambda}\Big)\right) \sim \frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu}, \Lambda_\mu, Q_\mu, V_\mu, H)\sqrt{|g|}d^4x + O(\Lambda^{-1}), \tag{9.21}$$

for

$$\mathcal{L}(g_{\mu\nu}, \Lambda_\mu, Q_\mu, V_\mu, H) := 96\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_A(\Lambda_\mu, Q_\mu, V_\mu) + \mathcal{L}_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H). \tag{9.22}$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 6.27. $\mathcal{L}_A$ gives the kinetic terms of the gauge fields and equals

$$\mathcal{L}_A(\Lambda_\mu, Q_\mu, V_\mu) := 4f(0)\Big(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu})\Big). \tag{9.23}$$

The Higgs potential (ignoring the boundary term) gives

$$\begin{aligned}\mathcal{L}_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H) := {}& 2bf(0)|H|^4 + \big(-8af_2\Lambda^2 + 4ef(0)\big)|H|^2 \\ & - 4cf_2\Lambda^2 + df(0) + \frac{1}{3}f(0)as|H|^2 + \frac{1}{6}f(0)cs + 2af(0)|\widetilde{D}_\mu H|^2.\end{aligned} \tag{9.24}$$

PROOF. We will use the general form of the spectral action of an almost commutative spectral triple as calculated in Proposition 6.29. The gravitational Lagrangian $\mathcal{L}_M$ now obtains a factor 96 from the trace over $\mathcal{H}_F$. From Lemma 9.8 we immediately find the term $\mathcal{L}_A$. For the newly defined coefficients a, b, c, d, e of Eq. (9.18), the Higgs potential has exactly the same form as in Proposition 8.12. □

9.3.1 *The coupling constants and unification*

The $SU(3)$ gauge field V_μ can be written as $V_\mu = V_\mu^i\lambda^i$, for the Gell-Mann matrices λ^i and real coefficients V_μ^i. As in Section 8.4, we will introduce coupling constants into the model by rescaling the gauge fields as

$$\Lambda_\mu = \frac{1}{2}g_1 B_\mu, \qquad Q_\mu^a = \frac{1}{2}g_2 W_\mu^a, \qquad V_\mu^i = \frac{1}{2}g_3 G_\mu^i. \tag{9.25}$$

By using the relations $\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}$ and $\mathrm{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}$, we now find that the Lagrangian $\mathcal{L}_A$ of Proposition 9.11 can be written as

$$\frac{1}{4\pi^2}\mathcal{L}_A(B_\mu, W_\mu, G_\mu) := \frac{f(0)}{2\pi^2}\Big(\frac{5}{3}{g_1}^2 B_{\mu\nu}B^{\mu\nu} + {g_2}^2 W_{\mu\nu}W^{\mu\nu} + {g_3}^2 G_{\mu\nu}G^{\mu\nu}\Big). \tag{9.26}$$

It is natural to require that these kinetic terms are properly normalized, and this imposes the relations

$$\frac{f(0)}{2\pi^2}{g_3}^2 = \frac{f(0)}{2\pi^2}{g_2}^2 = \frac{5f(0)}{6\pi^2}{g_1}^2 = \frac{1}{4}. \tag{9.27}$$

The coupling constants are then related by

$${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2, \tag{9.28}$$

which is precisely the relation between the coupling constants at unification, common to grand unified theories (GUT). Thus, it is now natural to assume that our model is defined at the scale Λ_{GUT}. As discussed in [13, Ch. 1, §17.10], one can then use the renormalization group equations to obtain a prediction for the Higgs mass.

9.3.2 *The Higgs mechanism*

In Section 8.5 we have evaluated the Higgs mechanism for the Glashow-Weinberg-Salam model. We have shown that by performing a gauge transformation as well as a conformal transformation of the Higgs field, given by Eq. (8.52), we obtain a spontaneous breaking of both the gauge symmetry and the conformal symmetry. In the Standard Model, this Higgs mechanism can be applied in exactly the same way. We will not redo the calculations, but simply state the result below.

First, we shall rescale the Higgs field as in Eq. (8.43), in order to normalize its kinetic term. The transformation of Eq. (8.52) gives a real parameter v_0 (which corresponds to the vacuum expectation value of the Higgs field) and a real field $h(x)$ (which describes the physical Higgs particle). From the conformal transformation we obtain a dilaton field $\eta(x)$, given by $\Omega(x) = e^\eta$. The gauge bosons W_μ, W_μ^*, Z_μ and A_μ are defined as in Eq. (8.40). In exactly the same way as in Theorem 8.15, we can now rewrite the spectral action of Proposition 9.11, and obtain the following result.

9.12 Theorem. *The spontaneously broken bosonic action (ignoring topological and boundary terms) is given by*

$$\begin{aligned} S_B = \int_M \Bigg(& \frac{48 f_4\Lambda^4}{\pi^2} - \frac{c f_2\Lambda^2}{\pi^2} + \frac{d f(0)}{4\pi^2} - \frac{b\pi^2}{2a^2 f(0)}{v_0}^4 \\ & + \frac{c f(0) s}{24\pi^2} - \frac{4 f_2\Lambda^2 s}{\pi^2} - \frac{3f(0)}{10\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \\ & + \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu,a} + \frac{1}{4}G^i_{\mu\nu}G^{\mu\nu,i} + \frac{4f_2\Lambda^2}{\pi^2}(\partial^\beta\eta)(\partial_\beta\eta) \\ & + \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{b\pi^2}{2a^2 f(0)}\Big(h^4 + 4v_0 h^3 + 4{v_0}^2 h^2\Big) \\ & + \frac{1}{4}{g_2}^2(v_0+h)^2 W^{\mu *}W_\mu + \frac{1}{8}\frac{{g_2}^2}{{c_w}^2}(v_0+h)^2 Z^\mu Z_\mu \Bigg)\sqrt{|g|}d^4x. \end{aligned} \tag{9.29}$$

9.4 THE FERMIONIC ACTION

By an abuse of notation, let us take a set of independent anticommuting Dirac spinors $\nu^\lambda, \overline{\nu}^\lambda, e^\lambda, \overline{e}^\lambda, u^{\lambda c}, \overline{u}^{\lambda c}, d^{\lambda c}, \overline{d}^{\lambda c}$. We then write a generic Grassmann vector $\widetilde{\xi} \in \mathcal{H}^+_{\mathrm{cl}}$ as follows:

$$\begin{aligned}\widetilde{\xi} = {} & \nu_L^\lambda \otimes \nu_L^\lambda + \nu_R^\lambda \otimes \nu_R^\lambda + \overline{\nu}_R^\lambda \otimes \overline{\nu_L^\lambda} + \overline{\nu}_L^\lambda \otimes \overline{\nu_R^\lambda} \\ & + e_L^\lambda \otimes e_L^\lambda + e_R^\lambda \otimes e_R^\lambda + \overline{e}_R^\lambda \otimes \overline{e_L^\lambda} + \overline{e}_L^\lambda \otimes \overline{e_R^\lambda} \\ & + u_L^{\lambda c} \otimes u_L^{\lambda c} + u_R^{\lambda c} \otimes u_R^{\lambda c} + \overline{u}_R^{\lambda c} \otimes \overline{u_L^{\lambda c}} + \overline{u}_L^{\lambda c} \otimes \overline{u_R^{\lambda c}} \\ & + d_L^{\lambda c} \otimes d_L^{\lambda c} + d_R^{\lambda c} \otimes d_R^{\lambda c} + \overline{d}_R^{\lambda c} \otimes \overline{d_L^{\lambda c}} + \overline{d}_L^{\lambda c} \otimes \overline{d_R^{\lambda c}},\end{aligned} \tag{9.30}$$

where in each tensor product it should be clear that the first component is a Weyl spinor, and the second component is a basis element of $\mathcal{H}_F$. Here $\lambda = 1, 2, 3$ labels the generation of the fermions, and $c = r, g, b$ labels the color index of the quarks.

Let us have a closer look at the gauge fields of the electroweak sector. For the physical gauge fields of Eq. (8.40) we can write

$$\begin{aligned} & Q_\mu^1 + iQ_\mu^2 = \frac{1}{\sqrt{2}} g_2 W_\mu, \qquad Q_\mu^1 - iQ_\mu^2 = \frac{1}{\sqrt{2}} g_2 W_\mu^*, \\ & Q_\mu^3 - \Lambda_\mu = \frac{g_2}{2c_w} Z_\mu, \qquad \Lambda_\mu = \frac{1}{2} s_w g_2 A_\mu - \frac{1}{2}\frac{{s_w}^2 g_2}{c_w} Z_\mu, \\ & -Q_\mu^3 - \Lambda_\mu = -s_w g_2 A_\mu + \frac{g_2}{2c_w}(1 - 2{c_w}^2) Z_\mu, \\ & Q_\mu^3 + \frac{1}{3}\Lambda_\mu = \frac{2}{3} s_w g_2 A_\mu - \frac{g_2}{6c_w}(1 - 4{c_w}^2) Z_\mu, \\ & -Q_\mu^3 + \frac{1}{3}\Lambda_\mu = -\frac{1}{3} s_w g_2 A_\mu - \frac{g_2}{6c_w}(1 + 2{c_w}^2) Z_\mu. \end{aligned} \tag{9.31}$$

We have rescaled the Higgs field in Eq. (8.43), so we can write $H = \frac{\sqrt{af(0)}}{\pi}(\phi_1 + 1, \phi_2)$. We shall parametrize the Higgs field as $H = (v_0 + h + i\phi^0, i\sqrt{2}\phi^-)$, where ϕ^0 is real and ϕ^- is complex. We write ϕ^+ for the complex conjugate of ϕ^-. Thus, we can write

$$(\phi_1 + 1, \phi_2) = \frac{\pi}{\sqrt{af(0)}}(v_0 + h + i\phi^0, i\sqrt{2}\phi^-). \tag{9.32}$$

9.13 Remark. As in Remark 7.11, we will need to impose a further restriction on the mass matrices in D_F, in order to obtain physical mass terms in the fermionic action. From here on, we will require that the matrices Y_x are antihermitian, for $x = \nu, e, u, d$. In other words, the hermitian operator S in D_F is required to be antisymmetric (as opposed to T, which was already required to be symmetric). We shall then define the hermitian mass matrices m_x by writing

$$Y_x =: i\frac{\sqrt{af(0)}}{\pi v_0} m_x. \tag{9.33}$$

9.14 Theorem. *The fermionic action is given by*

$$S_F = \int_M (\mathcal{L}_{kin} + \mathcal{L}_{gf} + \mathcal{L}_{Hf}) \sqrt{|g|} d^4x. \tag{9.34}$$

We suppress all generation and color indices. The kinetic terms of the fermions are given by

$$\mathcal{L}_{kin} := (J_M \overline{\nu}, \not{D}\nu) + (J_M \overline{e}, \not{D}e) + (J_M \overline{u}, \not{D}u) + (J_M \overline{d}, \not{D}d). \tag{9.35}$$

The minimal coupling of the gauge fields to the fermions is given by

$$\begin{aligned}
\mathcal{L}_{gf} := s_w g_2 A_\mu \Big(&- (J_M\overline{e}, \gamma^\mu e) + \tfrac{2}{3}(J_M\overline{u}, \gamma^\mu u) - \tfrac{1}{3}(J_M\overline{d}, \gamma^\mu d)\Big) \\
&+ \frac{g_2}{4c_w} Z_\mu \Big((J_M\overline{\nu}, \gamma^\mu(1+\gamma_5)\nu) + (J_M\overline{e}, \gamma^\mu(4{s_w}^2 - 1 - \gamma_5)e) \\
&\qquad + (J_M\overline{u}, \gamma^\mu(-\tfrac{8}{3}{s_w}^2 + 1 + \gamma_5)u) + (J_M\overline{d}, \gamma^\mu(\tfrac{4}{3}{s_w}^2 - 1 - \gamma_5)d)\Big) \\
&+ \frac{g_2}{2\sqrt{2}} W_\mu \Big((J_M\overline{e}, \gamma^\mu(1+\gamma_5)\nu) + (J_M\overline{d}, \gamma^\mu(1+\gamma_5)u)\Big) \\
&+ \frac{g_2}{2\sqrt{2}} W_\mu^* \Big((J_M\overline{\nu}, \gamma^\mu(1+\gamma_5)e) + (J_M\overline{u}, \gamma^\mu(1+\gamma_5)d)\Big) \\
&+ \frac{g_3}{2} G_\mu^i \Big((J_M\overline{u}, \gamma^\mu\lambda_i u) + (J_M\overline{d}, \gamma^\mu\lambda_i d)\Big). \qquad (9.36)
\end{aligned}$$

The Yukawa couplings of the Higgs field to the fermions are given by

$$\begin{aligned}
\mathcal{L}_{Hf} := -i(1 + \frac{h}{v_0})\Big(&(J_M\overline{\nu}, m_\nu \nu) + (J_M\overline{e}, m_e e) + (J_M\overline{u}, m_u u) + (J_M\overline{d}, m_d d)\Big) \\
&+ \frac{\phi^0}{v_0}\Big(- (J_M\overline{\nu}, \gamma_5 m_\nu \nu) + (J_M\overline{e}, \gamma_5 m_e e) - (J_M\overline{u}, \gamma_5 m_u u) + (J_M\overline{d}, \gamma_5 m_d d)\Big) \\
&+ \frac{\phi^-}{\sqrt{2}v_0}\Big((J_M\overline{e}, m_\nu(1-\gamma_5)\nu) - (J_M\overline{e}, m_e(1+\gamma_5)\nu)\Big) \\
&+ \frac{\phi^+}{\sqrt{2}v_0}\Big((J_M\overline{\nu}, m_e(1-\gamma_5)e) - (J_M\overline{\nu}, m_\nu(1+\gamma_5)e)\Big) \\
&+ \frac{\phi^-}{\sqrt{2}v_0}\Big((J_M\overline{d}, m_u(1-\gamma_5)u) - (J_M\overline{d}, m_d(1+\gamma_5)u)\Big) \\
&+ \frac{\phi^+}{\sqrt{2}v_0}\Big((J_M\overline{u}, m_d(1-\gamma_5)d) - (J_M\overline{u}, m_u(1+\gamma_5)d)\Big). \qquad (9.37)
\end{aligned}$$

Proof. The proof is similar to Proposition 7.10, though the calculations are now a little more complicated. From Definition 5.21 we know that the fermionic action is given by $S_F = \frac{1}{2}\langle J\widetilde{\xi}, D_A\widetilde{\xi}\rangle$, where the fluctuated Dirac operator is given by

$$D_A = \not{D} \otimes \mathbb{1} + \gamma^\mu \otimes B_\mu + \gamma_5 \otimes \Phi.$$

We rewrite the inner product on $\mathcal{H}$ as $\langle \xi, \psi\rangle = \int_M (\xi, \psi)\sqrt{|g|}d^4x$. As in Proposition 7.10, the expressions for $J\widetilde{\xi} = (J_M \otimes J_F)\widetilde{\xi}$ and $(\not{D} \otimes \mathbb{1})\widetilde{\xi}$ are obtained straightforwardly. We will use the symmetry of the form $(J_M\widetilde{\chi}, \not{D}\widetilde{\psi})$, and then we obtain the kinetic terms as

$$\frac{1}{2}(J\widetilde{\xi}, (\not{D} \otimes \mathbb{1})\widetilde{\xi}) = (J_M\overline{\nu}^\lambda, \not{D}\nu^\lambda) + (J_M\overline{e}^\lambda, \not{D}e^\lambda) + (J_M\overline{u}^{\lambda c}, \not{D}u^{\lambda c}) + (J_M\overline{d}^{\lambda c}, \not{D}d^{\lambda c}).$$

The other two terms in the fluctuated Dirac operator yield more complicated expressions. For the calculation of $(\gamma^\mu \otimes B_\mu)\widetilde{\xi}$, we use Corollary 9.7 for the gauge field B_μ, and we can insert the expressions of Eq. (9.31). As in Proposition 7.10, we now use the antisymmetry of the form $(J_M\widetilde{\chi}, \gamma^\mu\widetilde{\psi})$. For the coupling of the fermions to the gauge fields, a direct calculation then yields

$$\begin{aligned}
&\frac{1}{2}(J\widetilde{\xi}, (\gamma^\mu \otimes B_\mu)\widetilde{\xi}) = \\
&\quad s_w g_2 A_\mu \Big(- (J_M\overline{e}^\lambda, \gamma^\mu e^\lambda) + \tfrac{2}{3}(J_M\overline{u}^{\lambda c}, \gamma^\mu u^{\lambda c}) - \tfrac{1}{3}(J_M\overline{d}^{\lambda c}, \gamma^\mu d^{\lambda c})\Big) \\
&\quad + \frac{g_2}{4c_w} Z_\mu \Big((J_M\overline{\nu}^\lambda, \gamma^\mu(1+\gamma_5)\nu^\lambda) + (J_M\overline{e}^\lambda, \gamma^\mu(4{s_w}^2 - 1 - \gamma_5)e^\lambda) \\
&\qquad + (J_M\overline{u}^{\lambda c}, \gamma^\mu(-\tfrac{8}{3}{s_w}^2 + 1 + \gamma_5)u^{\lambda c}) + (J_M\overline{d}^{\lambda c}, \gamma^\mu(\tfrac{4}{3}{s_w}^2 - 1 - \gamma_5)d^{\lambda c})\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{g_2}{2\sqrt{2}} W_\mu \Big((J_M \overline{e}^\lambda, \gamma^\mu(1+\gamma_5)\nu^\lambda) + (J_M \overline{d}^{\lambda c}, \gamma^\mu(1+\gamma_5)u^{\lambda c}) \Big) \\
&+ \frac{g_2}{2\sqrt{2}} W_\mu^* \Big((J_M \overline{\nu}^\lambda, \gamma^\mu(1+\gamma_5)e^\lambda) + (J_M \overline{u}^{\lambda c}, \gamma^\mu(1+\gamma_5)d^{\lambda c}) \Big) \\
&+ \frac{g_3}{2} G_\mu^i \lambda_i^{dc} \Big((J_M \overline{u}^{\lambda d}, \gamma^\mu u^{\lambda c}) + (J_M \overline{d}^{\lambda d}, \gamma^\mu d^{\lambda c}) \Big),
\end{aligned}
$$

where in the weak interactions the projection operator $\frac{1}{2}(1+\gamma_5)$ is used to select only the left-handed spinors.

Next, we need to calculate $\frac{1}{2}(J\widetilde{\xi}, (\gamma_5 \otimes \Phi)\widetilde{\xi})$. The Higgs field is given by $\Phi = D_F + \phi + J_F \phi J_F^*$, where ϕ is given by Proposition 9.6. Let us first focus on the four terms involving only the neutrinos. Using the symmetry of the form $(J_M\widetilde{\chi}, \gamma_5\widetilde{\psi})$, we obtain

$$
\begin{aligned}
&\frac{1}{2}(J_M \overline{\nu}_R^\kappa, \gamma_5 Y_\nu^{\kappa\lambda}(\phi_1+1)\nu_R^\lambda) + \frac{1}{2}(J_M \nu_R^\kappa, \gamma_5 Y_\nu^{\lambda\kappa}(\phi_1+1)\overline{\nu}_R^\lambda) \\
&\qquad + \frac{1}{2}(J_M \overline{\nu}_L^\kappa, \gamma_5 \overline{Y}_\nu^{\lambda\kappa}(\overline{\phi}_1+1)\nu_L^\lambda) + \frac{1}{2}(J_M \nu_L^\kappa, \gamma_5 \overline{Y}_\nu^{\kappa\lambda}(\overline{\phi}_1+1)\overline{\nu}_L^\lambda) \\
&\qquad\qquad = (J_M \overline{\nu}_R^\kappa, \gamma_5 Y_\nu^{\kappa\lambda}(\phi_1+1)\nu_R^\lambda) + (J_M \overline{\nu}_L^\kappa, \gamma_5 \overline{Y}_\nu^{\lambda\kappa}(\overline{\phi}_1+1)\nu_L^\lambda).
\end{aligned}
$$

Using Eqs. (9.32) and (9.33), and dropping the generation labels, we can now rewrite

$$
\begin{aligned}
&(J_M \overline{\nu}_R, \gamma_5 Y_\nu(\phi_1+1)\nu_R) + (J_M \overline{\nu}_L, \gamma_5 \overline{Y}_\nu(\overline{\phi}_1+1)\nu_L) \\
&\qquad = -i(1+\frac{h}{v_0})(J_M\overline{\nu}, m_\nu \nu) - \frac{\phi^0}{v_0}(J_M \overline{\nu}, \gamma_5 m_\nu \nu).
\end{aligned}
$$

For e, u, d we obtain similar terms, with the only difference that for e and d the sign for ϕ^0 is changed. We also find terms that mix the neutrinos and electrons, and by the symmetry of the form $(J_M\widetilde{\chi}, \gamma_5\widetilde{\psi})$, these are given by the four terms

$$
\frac{\sqrt{2}}{v_0}\Big(\phi^-(J_M\overline{e}_R, m_\nu \nu_R) + \phi^+(J_M \overline{\nu}_R, m_e e_R) - \phi^-(J_M \overline{e}_L, m_e \nu_L) - \phi^+(J_M \overline{\nu}_L, m_\nu e_L) \Big).
$$

There are four similar terms with ν and e replaced by u and d, respectively. We can use the projection operators $\frac{1}{2}(1 \pm \gamma_5)$ to select left- or right-handed spinors. We thus obtain that the mass terms of the fermions and their couplings to the Higgs field are given by

$$
\begin{aligned}
&\frac{1}{2}(J\widetilde{\xi}, (\gamma_5\otimes\Phi)\widetilde{\xi}) = \\
&\quad - i(1+\frac{h}{v_0})\Big((J_M\overline{\nu}, m_\nu \nu) + (J_M\overline{e}, m_e e) + (J_M \overline{u}, m_u u) + (J_M \overline{d}, m_d d) \Big) \\
&\quad + \frac{\phi^0}{v_0}\Big(- (J_M\overline{\nu}, \gamma_5 m_\nu \nu) + (J_M \overline{e}, \gamma_5 m_e e) - (J_M \overline{u}, \gamma_5 m_u u) + (J_M \overline{d}, \gamma_5 m_d d) \Big) \\
&\quad + \frac{\phi^-}{\sqrt{2}v_0}\Big((J_M \overline{e}, m_\nu(1-\gamma_5)\nu) - (J_M\overline{e}, m_e(1+\gamma_5)\nu) \Big) \\
&\quad + \frac{\phi^+}{\sqrt{2}v_0}\Big((J_M \overline{\nu}, m_e(1-\gamma_5)e) - (J_M\overline{\nu}, m_\nu(1+\gamma_5)e) \Big) \\
&\quad + \frac{\phi^-}{\sqrt{2}v_0}\Big((J_M \overline{d}, m_u(1-\gamma_5)u) - (J_M\overline{d}, m_d(1+\gamma_5)u) \Big) \\
&\quad + \frac{\phi^+}{\sqrt{2}v_0}\Big((J_M \overline{u}, m_d(1-\gamma_5)d) - (J_M\overline{u}, m_u(1+\gamma_5)d) \Big),
\end{aligned}
$$

where we have suppressed all indices. □

9.15 Remark. In Theorems 9.12 and 9.14 we have calculated the action functional of Definition 5.21 for the almost commutative geometry defined in this chapter. However, we should still check whether this action coincides with the action of the Standard Model. This comparison has been worked out in detail in [13, Ch. 1, §17], and it confirms that our almost commutative geometry indeed yields the action of the Standard Model.

Part IV

CONCLUSION & OUTLOOK

I never see what has been done;
I only see what remains to be done.

— Buddha

10 CONCLUSION

The main subject of this thesis is the concept of an almost commutative geometry, which is given by the product of the canonical spectral triple (describing a 4-dimensional Riemannian space-time M) and some finite (non)commutative spectral triple (describing the internal degrees of freedom). A real even almost commutative geometry is described by a spectral triple

$$(\mathcal{A}, \mathcal{H}, D) = \big(C^\infty(M, \mathcal{A}_F), L^2(M, S) \otimes \mathcal{H}_F, D\!\!\!/ \otimes \mathbb{1} + \gamma_5 \otimes D_F\big),$$

along with a grading $\gamma_5 \otimes \gamma_F$ and a conjugation operator $J_M \otimes J_F$. We have studied the properties of such almost commutative geometries in general, i.e. without any assumptions on the finite spectral triple. One of the main results is Theorem 6.22, where we have shown that any almost commutative geometry describes a gauge theory, with the gauge group given by $\mathcal{G}(\mathcal{A}) = C^\infty(M, \mathcal{G}(\mathcal{A}_F))$. In particular:

- the gauge group $\mathcal{G}(\mathcal{A})$ is completely determined by the algebra $\mathcal{A}$ and the conjugation operator J;
- we can construct a (trivial) principal fibre bundle $P = M \times \mathcal{G}(\mathcal{A}_F)$, for which $M \times \mathcal{H}_F$ is an associated vector bundle;
- the inner fluctuations of the form $\sum_j a_j[D\!\!\!/, b_j]$ determine a connection A_μ on P;
- the gauge group $\mathcal{G}(\mathcal{A})$ of the almost commutative geometry is equal to the gauge group $\mathcal{G}(P)$ of the principal fibre bundle P;
- both the spectral action and the fermionic action are invariant under the gauge group.

Besides a gauge field, the inner fluctuations of the form $\sum_j a_j[D_F, b_j]$ determine a field $\Phi \in \Gamma\big(\mathrm{End}(M \times \mathcal{H}_F)\big)$, which we call the Higgs field. We have calculated in Proposition 6.29 the general form of the bosonic part of the Lagrangian, which arises from the heat expansion of the spectral action. We have written this Lagrangian in terms of the Higgs field Φ and the gauge field $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$. In this Lagrangian, we recognize (besides gravitational terms) the familiar terms we would expect for gauge fields and Higgs fields:

- a kinetic term for the gauge field B_μ, given by its curvature $F_{\mu\nu}$;
- a Higgs potential for the field Φ, such that the coefficient of the quadratic term is negative and the coefficient of the quartic term is positive;
- a kinetic term for the Higgs field, including its minimal coupling to the gauge field.

In addition, we also obtain a term which couples Φ^2 to the scalar curvature s of the manifold M.

In the gravitational part of the Lagrangian, we recognize the Einstein-Hilbert action including a cosmological constant. In addition, we have obtained terms that depend quadratically on the Riemannian curvature tensor. These terms can be rewritten such that (ignoring the topological term R^*R^*) we are only left with the Weyl gravity term $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, which is invariant under conformal transformations. It is interesting to note that this is not the most general form a gravitational Lagrangian could have (see Proposition 4.22), since the term s^2 is absent. The Weyl gravity is not the only part of the Lagrangian that is conformally invariant. We have shown in Proposition 6.31 that most terms are invariant under conformal transformations, except for one; the conformal transformation of the Einstein-Hilbert action yields a kinetic term of a dilaton field.

The justification for using the name *Higgs field* for the field Φ is not just that it is a scalar field, but more so the occurrence of the Higgs potential. This potential allows us to apply the Higgs mechanism. For the example of the Glashow-Weinberg-Salam model in Chapter 8, we have seen that this Higgs mechanism yields a spontaneous breaking of both the gauge symmetry and the conformal symmetry. Through the breaking of the conformal symmetry, the physical Higgs field is no longer coupled to the scalar curvature.

The general framework of Part II can easily be applied to yield specific examples, by simply making an explicit choice for the finite spectral triple. In this thesis we have given three examples, in which we have worked our way towards the Standard Model. The first example is the simplest one, but nonetheless an important one. It has long been thought impossible to describe abelian gauge theories within the framework of noncommutative geometry. In Chapter 7 we have shown that it is not impossible, and we have provided a description of electrodynamics as an almost commutative geometry. We have calculated both the spectral action and the fermionic action, and have seen that these indeed yield the Euclidean action of electrodynamics.

For the second example in Chapter 8, we have expanded the model of electrodynamics to also incorporate the weak interactions. This provides a good example of the application of the Higgs mechanism in an almost commutative geometry. We have shown that, because of the coupling of the Higgs field to the scalar curvature, the Higgs mechanism yields a spontaneous breaking of not only the gauge symmetry, but also of the conformal symmetry. By breaking this conformal symmetry, the physical Higgs field is no longer coupled to the scalar curvature.

These two examples have provided a good preparation for Chapter 9, in which we finally discussed the full Standard Model. A new feature in this model (compared to the previous two examples) was that we needed to impose the unimodularity condition on the gauge group, in order to obtain the correct gauge group for the Standard Model (see Remark 9.4). We have explicitly calculated the full Lagrangian of the Standard Model from the spectral action and the fermionic action.

11 OUTLOOK

In this thesis we have studied the concept of almost commutative geometries in general. There is of course a lot more work to be done, and this thesis provides a good starting point for further research. Below we will mention several open questions or ideas that could be pursued.

GENERAL DIMENSIONS

We have only considered almost commutative geometries for which the spin manifold M has dimension 4. One could generalize our results to spin manifolds of other dimensions. One complication that needs to be properly handled is that the precise form of the real structure of the almost commutative geometry depends on the KO-dimensions of both the canonical triple and the finite triple, as has been worked out in [33].

NON-TRIVIAL BUNDLES

The principal fibre bundle $P = M \times \mathcal{G}(\mathcal{A}_F)$ of Theorem 6.22, which describes the gauge theory of an almost commutative geometry, is a *trivial* fibre bundle. This is related to the fact that the finite triple has no bundle structure over M. Thus, it would be interesting to generalize our work to describe gauge theories on non-trivial fibre bundles. For the case of $SU(N)$ Yang-Mills theory, this has already been accomplished [3, 4]. The approach would be to replace the finite triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ by a triple

$$\Big(\Gamma(M, B), L^2(M, E), D\Big),$$

where B is an *algebra bundle* with fibre $\mathcal{A}_F$, E is a vector bundle with fibre $\mathcal{H}_F$, and D is a self-adjoint operator in $\Gamma(\text{End}(E))$. Note that now the Dirac operator might vary on M. When taking the product with the canonical spectral triple, one would then take as the total Dirac operator $\not{D} \otimes \mathbb{1} + \gamma_5 \otimes D + D_E$, where the third term D_E arises from combining the Clifford representation of the canonical triple with a connection ∇^E on E, so locally we would have $D_E = i\gamma^\mu \otimes \nabla^E_\mu$. The main task is then to show that we can construct a principal fibre bundle P from the spectral triple given above, such that the gauge group $\mathcal{G}(\mathcal{A})$ of this spectral triple corresponds to the group of gauge transformations of the principal bundle P.

THE SPIN BUNDLE

In the case of the trivial principal bundle $P = M \times \mathcal{G}(\mathcal{A}_F)$, the vector bundle $E = M \times \mathcal{H}_F$ is an associated vector bundle. However, the total Hilbert space is given by $L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes E)$. Thus the actual fermion fields are sections of $S \otimes E$, and the vector bundle $S \otimes E$ is *not* an associated vector bundle (as mentioned in Remark 6.23). The question arises whether or not we should find a larger principal fibre bundle, which incorporates the bundle structure of S and has $S \otimes E$ as an associated vector bundle.

In other words, can we expand the canonical triple in such a way that we can apply the same procedure as for gauge theory to obtain a principal fibre bundle, for which the spinor bundle S is an associated vector bundle? For this purpose, we would need to replace the algebra $C^\infty(M)$ in the canonical spectral triple by a larger algebra $\mathcal{A}$, such that the group $\mathcal{G}(\mathcal{A})$ corresponds to the group of gauge transformations of the principal fibre bundle $P = \text{Spin}(M)$ with fibre $\text{Spin}(n)$. The logical starting point would be to take

$\mathcal{A} = \Gamma\big(\mathbb{C}l^+(M)\big)$, the sections of the even Clifford bundle. Once a correct description of a spectral triple has been given, one will obtain an alternative model for the description of gravity on a spin manifold.

CLASSIFICATION

We have considered almost commutative geometries in their most general form. As mentioned in Remark 5.11, we have not yet considered the implications of the commutation rule $J_F D_F = J_F D_F$ or the order one condition $\big[[D,a],b^0\big] = 0$. These conditions impose further restrictions on the possible almost commutative geometries. A lot of work has already been done on the classification of finite spectral triples (and thus of almost commutative geometries). This classification started with papers by Paschke and Sitarz [31] and Krajewski [25]. Though these papers only focused on finite spectral triples of KO-dimension 0, their work has by now been generalized to allow for other KO-dimensions [5, 23]. It would be interesting to study the implications of this classification for the gauge group and the precise form of the Lagrangian.

REAL MASS PARAMETERS

In order to obtain physical mass terms in the Lagrangian, we needed to impose a restriction on the finite Dirac operator D_F (cf. Remarks 7.11 and 9.13), by demanding that the mass matrices are antihermitian. One might wonder whether this can be explained differently, without the need for such an ad hoc restriction of the Dirac operator.

ELECTROWEAK UNIFICATION

In the Glashow-Weinberg-Salam model, we have obtained in Section 8.4.1 the relation ${g_2}^2 = 3{g_1}^2$ for the coupling constants. It thus remains to determine the scale Λ_{EW} at which this relation holds from the running of g_1 and g_2. One can then obtain physical predictions from this model by using the renormalization group equations to 'run down' the model from the scale Λ_{EW} to lower energies.

CONFORMAL COSMOLOGY

The bosonic Lagrangian of an almost commutative geometry provides a new perspective for describing conformal cosmology. Traditionally, a conformally invariant Lagrangian may not contain a term proportional to the square of the Higgs field Φ. A new feature in noncommutative geometry is the presence of the scale Λ, which is affected by a conformal transformation. We now have a term $\Lambda^2\Phi^2$ in the Lagrangian which is conformally invariant (see Proposition 6.31).

One could study the effect of this extra term on cosmological models. Subsequently, one can study the spontaneous breaking of the conformal symmetry by the Higgs mechanism, and the resulting decoupling of the physical Higgs field from the scalar curvature, within such models.

LARGE SCALAR CURVATURE

Application of the Higgs mechanism relies on the fact that the quadratic term in the Higgs potential has a negative coefficient, so that the minimum of the potential is acquired for a non-zero Higgs field. However, taking into account the coupling of the Higgs field to the scalar curvature, we see that this coefficient becomes positive for a sufficiently large scalar curvature. As mentioned in Remark 8.13, this implies that there will be no spontaneous symmetry breaking. If the scalar curvature varies from point to point, there could at some point occur a transition from a broken to an unbroken theory.

Part V

APPENDIX

The scholar who cherishes the love of comfort is not fit to be deemed a scholar.

— Confucius

A CALCULATIONS IN CLIFFORD ALGEBRAS

On a Riemannian manifold M, with the metric locally given by $g^{\mu\nu}$, a Clifford algebra is generated by a set of *gamma matrices* $\{\gamma^\mu\}$ which obey the Clifford relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{A.1}$$

Throughout the thesis, we have $\gamma^\mu = c(dx^\mu)$. Here we will derive some useful *trace identities*. By using the periodicity of the trace and the above relation, we see that

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}\mathrm{Tr}(\mathrm{Id}) = 4g^{\mu\nu}. \tag{A.2}$$

For a product of four gamma matrices, we can use the Clifford relation three times to pull the first gamma matrix to the end, and so obtain

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu + 2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho. \tag{A.3}$$

We use this and the previous result to calculate its trace:

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)\\ &= \mathrm{Tr}(g^{\mu\nu}\gamma^\rho\gamma^\sigma - g^{\mu\rho}\gamma^\nu\gamma^\sigma + g^{\mu\sigma}\gamma^\nu\gamma^\rho)\\ &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \end{aligned}\tag{A.4}$$

The chirality element γ_5 anticommutes with γ^μ, so periodicity of the trace implies that

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu + \gamma^\mu\gamma_5\gamma_5) &= \mathrm{Tr}(\gamma^\mu + \gamma_5\gamma^\mu\gamma_5)\\ &= \mathrm{Tr}(\gamma^\mu - \gamma^\mu\gamma_5\gamma_5) = \mathrm{Tr}(\gamma^\mu - \gamma^\mu) = 0. \end{aligned}\tag{A.5}$$

The same method shows that the trace of a product of any odd number of gamma matrices also vanishes. In particular, we also have

$$\mathrm{Tr}(\gamma^\mu\gamma_5) = 0. \tag{A.6}$$

B

CALCULATIONS IN RIEMANNIAN GEOMETRY

In this chapter, we gather many straightforward but sometimes tedious calculations concerning local expressions of the metric, the Christoffel symbols and mostly the Riemannian tensor and its derived tensors.

B.1 LOCAL FORMULAE FOR COVARIANT DERIVATIVES

Recall from Definition 2.36 that the Christoffel symbols *of the second kind* of the Levi-Civita connection ∇ on the tangent bundle TM are given by

$$\nabla_\mu \partial_\nu = \Gamma^\rho{}_{\mu\nu}\partial_\rho, \tag{B.1}$$

where we have written $\nabla_\mu := \nabla_{\partial_\mu}$. The Levi-Civita connection is torsion-free, so Eq. (2.20) implies that

$$\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}. \tag{B.2}$$

If we define the *Christoffel symbols of the first kind* $\Gamma_{\lambda\mu\nu} = g_{\lambda\rho}\Gamma^\rho{}_{\mu\nu}$, the metric compatibility of Eq. (2.21) states that

$$\partial_\rho g_{\mu\nu} = \Gamma_{\nu\rho\mu} + \Gamma_{\mu\rho\nu}, \tag{B.3}$$

i.e. $\nabla_\rho g_{\mu\nu} = 0$. We then see that

$$\frac{1}{2}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}) = \frac{1}{2}(\Gamma_{\mu\nu\rho} + \Gamma_{\rho\nu\mu} + \Gamma_{\nu\mu\rho} + \Gamma_{\rho\mu\nu} - \Gamma_{\nu\rho\mu} - \Gamma_{\mu\rho\nu}) = \Gamma_{\rho\mu\nu}, \tag{B.4}$$

where we have used the symmetry of Eq. (B.2). We have thus obtained an explicit formula for the Christoffel symbols in terms of derivatives of the metric.

For a vector field X given locally by $X = X^\nu\partial_\nu$, the Leibniz rule gives us

$$\begin{aligned}\nabla_\mu X = \nabla_\mu(X^\nu\partial_\nu) &= \partial_\mu X^\nu \partial_\nu + \Gamma^\rho{}_{\mu\nu}X^\nu\partial_\rho \\ &= \left(\partial_\mu X^\rho + \Gamma^\rho{}_{\mu\nu}X^\nu\right)\partial_\rho.\end{aligned} \tag{B.5}$$

We can use this formula to define the *covariant derivative* of the *component functions* X^ρ of a vector field X. These functions $\nabla_\mu X^\rho$ are defined by

$$\nabla_\mu X = \left(\nabla_\mu X^\rho\right)\partial_\rho, \tag{B.6}$$

which then yields the familiar formula

$$\nabla_\mu X^\rho = \partial_\mu X^\rho + \Gamma^\rho{}_{\mu\nu}X^\nu. \tag{B.7}$$

Similarly, we can also define the covariant derivative of the component functions of a 1-form $\omega = \omega_\rho dx^\rho$ by

$$\nabla_\mu \omega = \left(\nabla_\mu \omega_\rho\right) dx^\rho, \tag{B.8}$$

and we now obtain the formula

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho{}_{\mu\nu}\omega_\rho. \tag{B.9}$$

We shall sometimes use the compact notation $X^{\rho}{}_{;\mu} := \nabla_\mu X^\rho$. A tensor field T of type (k,l) can locally be written as

$$T = T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_l}. \tag{B.10}$$

In a similar manner as for vector fields, the covariant derivatives of the components $T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}$ are given by

$$\begin{aligned}\nabla_\lambda T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} = \partial_\lambda T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} &+ \Gamma^{\mu_1}{}_{\lambda\rho}T^{\rho\cdots\mu_k}_{\nu_1\cdots\nu_l} + \ldots + \Gamma^{\mu_k}{}_{\lambda\rho}T^{\mu_1\cdots\rho}_{\nu_1\cdots\nu_l} \\ &- \Gamma^{\rho}{}_{\lambda\nu_1}T^{\mu_1\cdots\mu_k}_{\rho\cdots\nu_l} - \ldots - \Gamma^{\rho}{}_{\lambda\nu_l}T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\rho}.\end{aligned} \tag{B.11}$$

B.1.1 *Local formulae of the Riemann tensor*

Recall from Definition 2.38 that the Riemannian curvature $R \in \mathcal{A}^2(M, \mathrm{End}(TM))$ is defined by

$$R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \tag{B.12}$$

where ∇ is the Levi-Civita connection on the tangent bundle TM. The local components of R are defined by

$$R_{\mu\nu\rho\sigma} := g(\partial_\mu, R(\partial_\rho,\partial_\sigma)\partial_\nu). \tag{B.13}$$

Equivalently, we can define $R^{\lambda}{}_{\nu\rho\sigma}$ by

$$R(\partial_\rho,\partial_\sigma)\partial_\nu = R^{\lambda}{}_{\nu\rho\sigma}\partial_\lambda, \tag{B.14}$$

so that $R^{\lambda}{}_{\nu\rho\sigma} = g^{\mu\lambda}R_{\mu\nu\rho\sigma}$.

The curvature is defined as an endomorphism. Let us evaluate what this means locally. For a vector field $X = X^\nu\partial_\nu$, we can write

$$R(\partial_\rho,\partial_\sigma)X^\nu\partial_\nu = X^\nu R^{\lambda}{}_{\nu\rho\sigma}\partial_\lambda = X^\lambda R^{\nu}{}_{\lambda\rho\sigma}\partial_\nu, \tag{B.15}$$

so we obtain in the compact notation $X^{\nu}{}_{;\rho;\sigma} = \nabla_\sigma\nabla_\rho X^\nu$ that

$$X^{\nu}{}_{;\rho;\sigma} - X^{\nu}{}_{;\sigma;\rho} = -R^{\nu}{}_{\lambda\rho\sigma}X^\lambda. \tag{B.16}$$

Similarly we have for arbitrary tensor fields that

$$\begin{aligned}T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l;\lambda;\kappa} - T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l;\kappa;\lambda} = &- R^{\mu_1}{}_{\rho\lambda\kappa}T^{\rho\cdots\mu_k}_{\nu_1\cdots\nu_l} - \ldots - R^{\mu_k}{}_{\rho\lambda\kappa}T^{\mu_1\cdots\rho}_{\nu_1\cdots\nu_l} \\ &+ R^{\rho}{}_{\nu_1\lambda\kappa}T^{\mu_1\cdots\mu_k}_{\rho\cdots\nu_l} + \ldots + R^{\rho}{}_{\nu_l\lambda\kappa}T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\rho}.\end{aligned} \tag{B.17}$$

Using Eq. (B.12) we can obtain an explicit formula for the components $R_{\mu\nu\rho\sigma}$. We use that $[\partial_\rho,\partial_\sigma] = 0$ and see that

$$\begin{aligned}R_{\mu\nu\rho\sigma} &= g\big(\partial_\mu, (\nabla_{\partial_\rho}\nabla_{\partial_\sigma} - \nabla_{\partial_\sigma}\nabla_{\partial_\rho})\partial_\nu\big) \\ &= g\big(\partial_\mu, \nabla_{\partial_\rho}(\Gamma^{\lambda}{}_{\sigma\nu}\partial_\lambda) - \nabla_{\partial_\sigma}(\Gamma^{\lambda}{}_{\rho\nu}\partial_\lambda)\big) \\ &= g\big(\partial_\mu, \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\kappa}{}_{\rho\lambda}\partial_\kappa + (\partial_\rho\Gamma^{\lambda}{}_{\sigma\nu})\partial_\lambda - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\kappa}{}_{\sigma\lambda}\partial_\kappa - (\partial_\sigma\Gamma^{\lambda}{}_{\rho\nu})\partial_\lambda\big) \\ &= \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\kappa}{}_{\rho\lambda}g_{\mu\kappa} + (\partial_\rho\Gamma^{\lambda}{}_{\sigma\nu})g_{\mu\lambda} - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\kappa}{}_{\sigma\lambda}g_{\mu\kappa} - (\partial_\sigma\Gamma^{\lambda}{}_{\rho\nu})g_{\mu\lambda} \\ &= g_{\mu\lambda}\big(\Gamma^{\kappa}{}_{\sigma\nu}\Gamma^{\lambda}{}_{\rho\kappa} + \partial_\rho\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\kappa}{}_{\rho\nu}\Gamma^{\lambda}{}_{\sigma\kappa} - \partial_\sigma\Gamma^{\lambda}{}_{\rho\nu}\big),\end{aligned} \tag{B.18}$$

and thus

$$R^{\lambda}{}_{\nu\rho\sigma} = \Gamma^{\kappa}{}_{\sigma\nu}\Gamma^{\lambda}{}_{\rho\kappa} + \partial_\rho\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\kappa}{}_{\rho\nu}\Gamma^{\lambda}{}_{\sigma\kappa} - \partial_\sigma\Gamma^{\lambda}{}_{\rho\nu}. \tag{B.19}$$

The Ricci tensor is defined as $R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma}$. By using Eq. (2.25), we can write

$$\begin{aligned} R_{\nu\sigma} &= \delta^{\rho}{}_{\lambda} \left(\Gamma^{\kappa}{}_{\sigma\nu} \Gamma^{\lambda}{}_{\rho\kappa} + \partial_\rho \Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\kappa}{}_{\rho\nu} \Gamma^{\lambda}{}_{\sigma\kappa} - \partial_\sigma \Gamma^{\lambda}{}_{\rho\nu} \right) \\ &= \Gamma^{\kappa}{}_{\sigma\nu} \Gamma^{\rho}{}_{\rho\kappa} + \partial_\rho \Gamma^{\rho}{}_{\sigma\nu} - \Gamma^{\kappa}{}_{\rho\nu} \Gamma^{\rho}{}_{\sigma\kappa} - \partial_\sigma \Gamma^{\rho}{}_{\rho\nu}. \end{aligned} \tag{B.20}$$

The scalar curvature $s = g^{\nu\sigma} R_{\nu\sigma}$ then equals

$$s = g^{\nu\sigma} \left(\Gamma^{\kappa}{}_{\sigma\nu} \Gamma^{\rho}{}_{\rho\kappa} + \partial_\rho \Gamma^{\rho}{}_{\sigma\nu} - \Gamma^{\kappa}{}_{\rho\nu} \Gamma^{\rho}{}_{\sigma\kappa} - \partial_\sigma \Gamma^{\rho}{}_{\rho\nu} \right). \tag{B.21}$$

B.1.2 *Rewriting derivatives*

First, for the derivative of $g^{\mu\nu}$ we have

$$\begin{aligned} \partial_\rho g^{\mu\nu} &= \partial_\rho (g^{\mu\lambda} g^{\nu\sigma} g_{\lambda\sigma}) = \delta^{\nu}{}_{\lambda} \partial_\rho g^{\mu\lambda} + \delta^{\mu}{}_{\sigma} \partial_\rho g^{\nu\sigma} + g^{\mu\lambda} g^{\nu\sigma} \partial_\rho g_{\lambda\sigma} \\ &= 2\partial_\rho g^{\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \partial_\rho g_{\lambda\sigma}, \end{aligned}$$

so

$$\partial_\rho g^{\mu\nu} = -g^{\mu\lambda} g^{\nu\sigma} \partial_\rho g_{\lambda\sigma}. \tag{B.22}$$

Second, for the determinant of g, we have the formula

$$\partial_\rho |g| = |g| g^{\mu\nu} \partial_\rho g_{\mu\nu}, \tag{B.23}$$

which then yields

$$\partial_\rho \sqrt{|g|} = \frac{1}{2\sqrt{|g|}} \partial_\rho |g| = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \partial_\rho g_{\mu\nu}. \tag{B.24}$$

By using Eq. (B.22) we note that

$$g^{\mu\nu} \partial_\rho g_{\mu\nu} = -g^{\mu\nu} g_{\mu\lambda} g_{\nu\sigma} \partial_\rho g^{\lambda\sigma} = -\delta^{\nu}{}_{\lambda} g_{\nu\sigma} \partial_\rho g^{\lambda\sigma} = -g_{\lambda\sigma} \partial_\rho g^{\lambda\sigma}, \tag{B.25}$$

so we can rewrite

$$\partial_\rho \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\mu\nu} \partial_\rho g^{\mu\nu}. \tag{B.26}$$

B.2 THE DUAL TENSOR

We will regard the particular case that $\dim(M) = 4$. We define the permutation tensor by

$$\delta_{\mu\nu\rho\sigma} := \sqrt{|g|} \epsilon_{\mu\nu\rho\sigma}, \tag{B.27}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita symbol for which $\epsilon_{0123} = 1$ and $|g|$ is the determinant $\det(g_{\mu\nu})$ of the matrix of the covariant metric, given by

$$|g| = \epsilon^{\mu\nu\rho\sigma} g_{1\mu} g_{2\nu} g_{3\rho} g_{4\sigma}. \tag{B.28}$$

Note that $\epsilon^{\mu\nu\rho\sigma} := \epsilon_{\mu\nu\rho\sigma}$, so the position of the indices is irrelevant for the symbol ϵ. The contravariant metric $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$ and hence we have that

$$|g|^{-1} = \epsilon_{\mu\nu\rho\sigma} g^{1\mu} g^{2\nu} g^{3\rho} g^{4\sigma}. \tag{B.29}$$

Note that

$$\delta_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \sqrt{|g|} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} d^4x = 24\sqrt{|g|} d^4x = 24 v_g, \tag{B.30}$$

where ν_g is the Riemannian volume form of Definition 2.52, which is invariant under coordinate transformations. Hence the $\delta_{\mu\nu\rho\sigma}$ form the components of a tensor.

From this definition we find that raising the indices of this tensor δ yields

$$\delta^{\alpha\beta\gamma\delta} = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} \delta_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{|g|}} \epsilon^{\alpha\beta\gamma\delta}. \tag{B.31}$$

We can now define a *dual tensor* of the Riemannian curvature tensor by [26]

$$*R_{\gamma\delta\rho\sigma} := \frac{1}{4} \delta_{\alpha\beta\gamma\delta} \delta_{\mu\nu\rho\sigma} R^{\alpha\beta\mu\nu}, \qquad *R^{\gamma\delta\rho\sigma} = \frac{1}{4} \delta^{\alpha\beta\gamma\delta} \delta^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu}. \tag{B.32}$$

We will derive an explicit formula for this dual tensor in terms of the Riemann tensor. First, Eq. (B.29) can be rewritten into the identity

$$\delta^{\mu\nu\rho\sigma} \delta^{\alpha\beta\gamma\delta} = \sum_{\pi} (-1)^{|\pi|} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma}, \tag{B.33}$$

where the sum is over all 24 permutations of $\alpha, \beta, \gamma, \delta$. We multiply this identity by $R_{\alpha\beta\mu\nu}$, and explicitly write out the sum. The first six terms containing $g^{\alpha\mu}$ yield

$$s(g^{\gamma\rho} g^{\delta\sigma} - g^{\delta\rho} g^{\gamma\sigma}) - R^{\gamma\rho} g^{\delta\sigma} - R^{\delta\sigma} g^{\gamma\rho} + R^{\delta\rho} g^{\gamma\sigma} + R^{\gamma\sigma} g^{\delta\rho}. \tag{B.34}$$

The six terms containing $g^{\beta\mu}$ yield the same result. The six terms containing $g^{\gamma\mu}$ yield

$$2R^{\gamma\sigma} g^{\delta\rho} - 2R^{\gamma\rho} g^{\delta\sigma} + 2R^{\gamma\delta\rho\sigma}, \tag{B.35}$$

and the last six terms containing $g^{\delta\mu}$ yield

$$2R^{\delta\rho} g^{\gamma\sigma} - 2R^{\delta\sigma} g^{\gamma\rho} + 2R^{\gamma\delta\rho\sigma}. \tag{B.36}$$

Collecting all terms and dividing by four then gives us

$$*R^{\gamma\delta\rho\sigma} = R^{\gamma\delta\rho\sigma} - R^{\gamma\rho} g^{\delta\sigma} - R^{\delta\sigma} g^{\gamma\rho} + R^{\delta\rho} g^{\gamma\sigma} + R^{\gamma\sigma} g^{\delta\rho} + \frac{1}{2} s(g^{\gamma\rho} g^{\delta\sigma} - g^{\delta\rho} g^{\gamma\sigma}). \tag{B.37}$$

Multiplying by $R_{\gamma\delta\rho\sigma}$ yields the identity

$$R^* R^* := *R^{\gamma\delta\rho\sigma} R_{\gamma\delta\rho\sigma} = R^{\gamma\delta\rho\sigma} R_{\gamma\delta\rho\sigma} - 4R^{\gamma\rho} R_{\gamma\rho} + s^2, \tag{B.38}$$

where we introduce the common notation $R^* R^*$, which is often referred to as the Pontryagin class (see also Section B.4.4).

B.3 THE WEYL TENSOR

Again we regard the case of $\dim(M) = 4$. The Ricci decomposition of the Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = S_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma} + C_{\mu\nu\rho\sigma}. \tag{B.39}$$

where the *scalar part* is given by $S_{\mu\nu\rho\sigma} := \frac{1}{(n-1)(n-2)} s(g_{\mu\sigma} g_{\rho\nu} - g_{\mu\rho} g_{\sigma\nu})$ and the *semi-traceless part* is given by $E_{\mu\nu\rho\sigma} := \frac{1}{n-2}(g_{\mu\rho} R_{\sigma\nu} - g_{\mu\sigma} R_{\rho\nu} - g_{\nu\rho} R_{\sigma\mu} + g_{\nu\sigma} R_{\rho\mu})$. Note that these definitions imply that

$$g^{\mu\rho}(S_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma}) = -\frac{1}{n-2} s g_{\nu\sigma} + R_{\nu\sigma} + \frac{1}{n-2} s g_{\nu\sigma} = R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma}, \tag{B.40}$$

so that $g^{\mu\rho}C_{\mu\nu\rho\sigma} = 0$. Hence this Ricci decomposition defines the *fully traceless part* $C_{\mu\nu\rho\sigma}$, which is called the *Weyl tensor*, by

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - S_{\mu\nu\rho\sigma} - E_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{(n-1)(n-2)} s(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\rho}g_{\sigma\nu}) - \frac{1}{n-2}(g_{\mu\rho}R_{\sigma\nu} - g_{\mu\sigma}R_{\rho\nu} - g_{\nu\rho}R_{\sigma\mu} + g_{\nu\sigma}R_{\rho\mu}). \quad \text{(B.41)}$$

From here on we will take $n = \dim(M) = 4$, and we then have

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{6}s(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\rho}g_{\sigma\nu}) - \frac{1}{2}(g_{\mu\rho}R_{\sigma\nu} - g_{\mu\sigma}R_{\rho\nu} - g_{\nu\rho}R_{\sigma\mu} + g_{\nu\sigma}R_{\rho\mu}). \quad \text{(B.42)}$$

We would like to calculate its square $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$. We use the Ricci decomposition and can write $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ as the sum of three squares and three cross products:

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, & -2R_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma} &= \frac{2}{3}s^2, \\ S_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma} &= \frac{2}{3}s^2, & -2R_{\mu\nu\rho\sigma}E^{\mu\nu\rho\sigma} &= -4R_{\nu\sigma}R^{\nu\sigma}, \\ E_{\mu\nu\rho\sigma}E^{\mu\nu\rho\sigma} &= 2R_{\nu\sigma}R^{\nu\sigma} + s^2, & 2E_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma} &= -2s^2. \end{aligned}$$

Summing these six terms together we obtain

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\nu\sigma}R^{\nu\sigma} + \frac{1}{3}s^2. \quad \text{(B.43)}$$

Using the Pontryagin class of Eq. (B.38), we can rewrite

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R^*R^* + 2R_{\nu\sigma}R^{\nu\sigma} - \frac{2}{3}s^2. \quad \text{(B.44)}$$

In a similar manner, we can calculate that

$$C_{\lambda\nu\rho\sigma}C_\kappa{}^{\nu\rho\sigma} = R_{\lambda\nu\rho\sigma}R_\kappa{}^{\nu\rho\sigma} - \frac{1}{6}s^2 g_{\lambda\kappa} + \frac{1}{2}g_{\lambda\kappa}R_{\nu\sigma}R^{\nu\sigma} + sR_{\lambda\kappa} - 2R_{\lambda\nu\kappa\sigma}R^{\nu\sigma} - 2R_{\lambda\sigma}R_\kappa{}^\sigma, \quad \text{(B.45)}$$

by summing the terms

$$\begin{aligned} R_{\lambda\nu\rho\sigma}R_\kappa{}^{\nu\rho\sigma} &= R_{\lambda\nu\rho\sigma}R_\kappa{}^{\nu\rho\sigma}, \\ S_{\lambda\nu\rho\sigma}S_\kappa{}^{\nu\rho\sigma} &= \frac{1}{6}s^2 g_{\lambda\kappa}, \\ E_{\lambda\nu\rho\sigma}E_\kappa{}^{\nu\rho\sigma} &= \frac{1}{2}g_{\lambda\kappa}R_{\nu\sigma}R^{\nu\sigma} + sR_{\lambda\kappa}, \\ -R_{\lambda\nu\rho\sigma}S_\kappa{}^{\nu\rho\sigma} - S_{\lambda\nu\rho\sigma}R_\kappa{}^{\nu\rho\sigma} &= \frac{2}{3}sR_{\lambda\kappa}, \\ -R_{\lambda\nu\rho\sigma}E_\kappa{}^{\nu\rho\sigma} - E_{\lambda\nu\rho\sigma}R_\kappa{}^{\nu\rho\sigma} &= -2R_{\lambda\nu\kappa\sigma}R^{\nu\sigma} - 2R_{\lambda\sigma}R_\kappa{}^\sigma, \\ E_{\lambda\nu\rho\sigma}S_\kappa{}^{\nu\rho\sigma} + S_{\lambda\nu\rho\sigma}E_\kappa{}^{\nu\rho\sigma} &= -\frac{2}{3}sR_{\lambda\kappa} - \frac{1}{3}s^2 g_{\lambda\kappa}. \end{aligned}$$

In the case $\dim(M) = 4$, the Weyl tensor obeys an interesting identity, which we shall now derive. We calculate

$$C_{\lambda\nu\rho\sigma}C_\kappa{}^{\nu\rho\sigma} - \frac{1}{4}g_{\lambda\kappa}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\lambda\nu\rho\sigma}R_\kappa{}^{\nu\rho\sigma} - \frac{1}{4}s^2 g_{\lambda\kappa} + g_{\lambda\kappa}R_{\nu\sigma}R^{\nu\sigma} + sR_{\lambda\kappa} - 2R_{\lambda\nu\kappa\sigma}R^{\nu\sigma} - 2R_{\lambda\sigma}R_\kappa{}^\sigma - \frac{1}{4}g_{\lambda\kappa}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad \text{(B.46)}$$

Since we are in the case where $\dim M = 4$, we have the identity

$$\sum(\pm) g^{\mu\nu} g^{\sigma\iota} g^{\tau\kappa} g^{\rho\eta} g^{\lambda\zeta} = 0, \tag{B.47}$$

where the sum is over the 120 permutations of ν, ι, κ, η and ζ with the sign chosen according to the eveness or oddness of each permutation. If we multiply this identity by $R_{\sigma\tau\eta\zeta} R_{\iota\kappa\rho\lambda}$, a tedious calculation will show (according to [14, p.720]) that we obtain the *Bach-Lanczos identity*

$$C_{\mu\rho\sigma\lambda} C_{\nu}{}^{\rho\sigma\lambda} - \frac{1}{4} g_{\mu\nu} C_{\rho\sigma\lambda\tau} C^{\rho\sigma\lambda\tau} = 0. \tag{B.48}$$

B.4 VARIATIONS OF THE METRIC

Suppose we take an infinitesimal variation of the metric:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}. \tag{B.49}$$

We shall now calculate the variations of several quantities depending on this metric. First, as in Eq. (B.22), we note that

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\beta\nu} \delta g_{\alpha\beta}. \tag{B.50}$$

Second, we have as in Eq. (B.26) that

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}\, g_{\alpha\beta} \delta g^{\alpha\beta}. \tag{B.51}$$

The variation of the Christoffel symbols (given by Eq. (B.4)) depends on the derivatives of the variation of the metric, and we have

$$\delta\Gamma_{\mu\nu\rho} = \frac{1}{2}\big(\partial_\rho(\delta g_{\mu\nu}) + \partial_\nu(\delta g_{\mu\rho}) - \partial_\mu(\delta g_{\nu\rho})\big). \tag{B.52}$$

From this we derive that

$$\begin{aligned} \delta\Gamma^{\sigma}{}_{\nu\rho} &= \delta\big(g^{\mu\sigma}\Gamma_{\mu\nu\rho}\big) = \Gamma_{\mu\nu\rho}\delta g^{\mu\sigma} + g^{\mu\sigma}\delta\Gamma_{\mu\nu\rho} \\ &= -\Gamma^{\alpha}{}_{\nu\rho} g^{\beta\sigma} \delta g_{\alpha\beta} + \frac{1}{2} g^{\alpha\sigma}\big(\partial_\rho(\delta g_{\alpha\nu}) + \partial_\nu(\delta g_{\alpha\rho}) - \partial_\alpha(\delta g_{\nu\rho})\big). \end{aligned} \tag{B.53}$$

Replacing the partial derivatives by covariant derivatives adds extra terms containing Christoffel symbols, and then these terms conveniently cancel each other:

$$\begin{aligned} \delta\Gamma^{\sigma}{}_{\nu\rho} &= -\Gamma^{\alpha}{}_{\nu\rho} g^{\beta\sigma}\delta g_{\alpha\beta} + \frac{1}{2} g^{\alpha\sigma}\big(\nabla_\rho(\delta g_{\alpha\nu}) + \nabla_\nu(\delta g_{\alpha\rho}) - \nabla_\alpha(\delta g_{\nu\rho})\big) \\ &\quad + \frac{1}{2} g^{\alpha\sigma}\Big(\Gamma^{\beta}{}_{\rho\alpha}\delta g_{\beta\nu} + \Gamma^{\beta}{}_{\rho\nu}\delta g_{\beta\alpha} + \Gamma^{\beta}{}_{\nu\alpha}\delta g_{\beta\rho} \\ &\qquad + \Gamma^{\beta}{}_{\nu\rho}\delta g_{\beta\alpha} - \Gamma^{\beta}{}_{\nu\alpha}\delta g_{\beta\rho} - \Gamma^{\beta}{}_{\rho\alpha}\delta g_{\beta\nu}\Big) \\ &= \frac{1}{2} g^{\alpha\sigma}\big(\nabla_\rho(\delta g_{\alpha\nu}) + \nabla_\nu(\delta g_{\alpha\rho}) - \nabla_\alpha(\delta g_{\nu\rho})\big). \end{aligned} \tag{B.54}$$

Using Eq. (B.19), the variation of the Riemann tensor is given by

$$\begin{aligned} \delta R^{\sigma}{}_{\mu\rho\nu} &= \partial_\rho \delta\Gamma^{\sigma}{}_{\mu\nu} - \partial_\nu \delta\Gamma^{\sigma}{}_{\rho\mu} \\ &\quad + \delta\Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\sigma}{}_{\lambda\rho} + \Gamma^{\lambda}{}_{\mu\nu}\delta\Gamma^{\sigma}{}_{\lambda\rho} - \delta\Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\mu\rho}\delta\Gamma^{\sigma}{}_{\nu\lambda}. \end{aligned} \tag{B.55}$$

By Lemma 2.28, the variations of Christoffel symbols form the components of a tensor, so their covariant derivative is given by

$$\nabla_\alpha \delta\Gamma^{\rho}{}_{\mu\nu} = \partial_\alpha \delta\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\alpha\beta}\delta\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\beta}{}_{\alpha\mu}\delta\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\beta}{}_{\alpha\nu}\delta\Gamma^{\rho}{}_{\mu\beta}. \tag{B.56}$$

The difference of two conveniently chosen terms of this form equals

$$\nabla_\rho \delta \Gamma^\sigma{}_{\mu\nu} - \nabla_\nu \delta \Gamma^\sigma{}_{\rho\mu} = \partial_\rho \delta \Gamma^\sigma{}_{\mu\nu} + \Gamma^\sigma{}_{\rho\beta} \delta \Gamma^\beta{}_{\mu\nu} - \Gamma^\beta{}_{\rho\mu} \delta \Gamma^\sigma{}_{\beta\nu} - \Gamma^\beta{}_{\rho\nu} \delta \Gamma^\sigma{}_{\mu\beta}$$
$$- \partial_\nu \delta \Gamma^\sigma{}_{\rho\mu} - \Gamma^\sigma{}_{\nu\beta} \delta \Gamma^\beta{}_{\rho\mu} + \Gamma^\beta{}_{\nu\rho} \delta \Gamma^\sigma{}_{\beta\mu} + \Gamma^\beta{}_{\nu\mu} \delta \Gamma^\sigma{}_{\rho\beta}. \tag{B.57}$$

We see that in the above expression, the fourth and seventh terms cancel each other. The remaining terms in Eq. (B.57) are exactly equal to the terms of $\delta R^\sigma{}_{\mu\rho\nu}$ in Eq. (B.55), so we obtain

$$\delta R^\sigma{}_{\mu\rho\nu} = \nabla_\rho \delta \Gamma^\sigma{}_{\mu\nu} - \nabla_\nu \delta \Gamma^\sigma{}_{\rho\mu}. \tag{B.58}$$

The variation of the Ricci tensor then easily follows:

$$\delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu} = \nabla_\rho \delta \Gamma^\rho{}_{\mu\nu} - \nabla_\nu \delta \Gamma^\rho{}_{\rho\mu}. \tag{B.59}$$

Since we have $\nabla_\rho g^{\mu\nu}$ by the metric compatibility of the connection, this implies that

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\rho \left(g^{\mu\nu} \delta \Gamma^\rho{}_{\mu\nu} \right) - \nabla_\nu \left(g^{\mu\nu} \delta \Gamma^\rho{}_{\rho\mu} \right)$$
$$= \nabla_\rho \left(g^{\mu\nu} \delta \Gamma^\rho{}_{\mu\nu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right). \tag{B.60}$$

By defining the components of a vector field v by

$$v^\rho := g^{\mu\nu} \delta \Gamma^\rho{}_{\mu\nu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu}, \tag{B.61}$$

and using Proposition 2.54, we see that

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\rho v^\rho = \nabla \cdot v. \tag{B.62}$$

B.4.1 *Variations of the scalar curvature squared*

The variation of $s^2 \sqrt{|g|}$ is given by

$$\delta \left(s^2 \sqrt{|g|} \right) = 2s \delta s \sqrt{|g|} + s^2 \delta \sqrt{|g|}$$
$$= 2s \left(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right) \sqrt{|g|} - \frac{1}{2} s^2 g_{\mu\nu} \sqrt{|g|} \delta g^{\mu\nu}. \tag{B.63}$$

Let us have another look at the vector field v, defined in Eq. (B.61). By inserting Eq. (B.54) and then rearranging some indices and using that $\nabla_\rho g^{\alpha\sigma} = 0$, we obtain

$$v^\rho = \frac{1}{2} g^{\mu\nu} g^{\alpha\rho} \left(\nabla_\nu \delta g_{\alpha\mu} + \nabla_\mu \delta g_{\alpha\nu} - \nabla_\alpha \delta g_{\mu\nu} \right)$$
$$- \frac{1}{2} g^{\mu\rho} g^{\nu\alpha} \left(\nabla_\mu \delta g_{\alpha\nu} + \nabla_\nu \delta g_{\alpha\mu} - \nabla_\alpha \delta g_{\mu\nu} \right)$$
$$= \nabla_\mu \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big). \tag{B.64}$$

Inserting this into Eq. (B.62), we see that we can write

$$2s g^{\mu\nu} \delta R_{\mu\nu} = 2s \nabla_\rho \nabla_\mu \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big)$$
$$= 2 \nabla_\rho \Big(s \nabla_\mu \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big) \Big)$$
$$- 2 \nabla_\rho (s) \nabla_\mu \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big)$$
$$= 2 \nabla_\rho \Big(s \nabla_\mu \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big) \Big)$$
$$- 2 \nabla_\mu \Big(\nabla_\rho (s) \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big) \Big)$$
$$+ 2 \nabla_\mu \left(\nabla_\rho (s) \right) \Big(\left(g^{\mu\nu} g^{\alpha\rho} - g^{\mu\rho} g^{\nu\alpha} \right) \delta g_{\alpha\nu} \Big). \tag{B.65}$$

The first two terms are again total divergences, and as in Section 4.3.1 they can be ignored by Stokes' theorem. Using Eq. (B.50), the last term can be rewritten as

$$2\nabla_\mu(\nabla_\rho(s))\Big((g^{\mu\nu}g^{\alpha\rho}-g^{\mu\rho}g^{\nu\alpha})\delta g_{\alpha\nu}\Big) = \\ -2\Big(\nabla_\mu(\nabla_\nu(s))-g_{\mu\nu}g^{\beta\rho}\nabla_\beta(\nabla_\rho(s))\Big)\delta g^{\mu\nu}. \tag{B.66}$$

This finally yields (ignoring the boundary terms)

$$\delta\left(s^2\sqrt{|g|}\right) = \left(2sR_{\mu\nu}-2s_{;\mu;\nu}+2g_{\mu\nu}s^{;\beta}{}_{;\beta}-\frac{1}{2}s^2g_{\mu\nu}\right)\sqrt{|g|}\delta g^{\mu\nu}. \tag{B.67}$$

B.4.2 *Variations of the Ricci curvature squared*

Next we will have a look at the variation of $R_{\mu\nu}R^{\mu\nu}\sqrt{|g|}$, which is given by

$$\delta\left(R_{\mu\nu}R^{\mu\nu}\sqrt{|g|}\right) = \delta\left(R_{\mu\nu}\right)R^{\mu\nu}\sqrt{|g|}+R_{\mu\nu}\delta\left(R^{\mu\nu}\right)\sqrt{|g|}+R_{\mu\nu}R^{\mu\nu}\delta\left(\sqrt{|g|}\right). \tag{B.68}$$

The last term is known by Eq. (B.51). The middle term can be rewritten as

$$\begin{aligned} R_{\mu\nu}\delta(R^{\mu\nu}) &= R_{\mu\nu}\delta(g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta}) = R_{\mu\nu}\left(R^{\nu}{}_{\alpha}\delta g^{\mu\alpha}+R^{\mu}{}_{\beta}\delta g^{\nu\beta}+g^{\mu\alpha}g^{\nu\beta}\delta R_{\alpha\beta}\right) \\ &= 2R_{\mu\nu}R^{\mu}{}_{\alpha}\delta g^{\nu\alpha}+R^{\alpha\beta}\delta R_{\alpha\beta}. \end{aligned} \tag{B.69}$$

All that remains is to determine $R^{\alpha\beta}\delta R_{\alpha\beta}$. Using Eq. (B.59) and Eq. (B.54), we write out

$$\begin{aligned} R^{\alpha\beta}\delta R_{\alpha\beta} &= R^{\alpha\beta}\left(\nabla_\rho\delta\Gamma^{\rho}{}_{\alpha\beta}-\nabla_\beta\delta\Gamma^{\rho}{}_{\rho\alpha}\right) \\ &= R^{\alpha\beta}\Big(\frac{1}{2}g^{\rho\sigma}\nabla_\rho(\nabla_\alpha(\delta g_{\beta\sigma})+\nabla_\beta(\delta g_{\sigma\alpha})-\nabla_\sigma(\delta g_{\alpha\beta})) \\ &\quad -\frac{1}{2}g^{\rho\sigma}\nabla_\beta(\nabla_\rho(\delta g_{\alpha\sigma})+\nabla_\alpha(\delta g_{\sigma\rho})-\nabla_\sigma(\delta g_{\rho\alpha}))\Big) \\ &= g^{\rho\sigma}\Big(\frac{1}{2}\nabla_\alpha(\nabla_\rho(R^{\alpha\beta}))\delta g_{\beta\sigma}+\frac{1}{2}\nabla_\beta(\nabla_\rho(R^{\alpha\beta}))\delta g_{\sigma\alpha} \\ &\quad -\frac{1}{2}\nabla_\sigma(\nabla_\rho(R^{\alpha\beta}))\delta g_{\alpha\beta}-\frac{1}{2}\nabla_\alpha(\nabla_\beta(R^{\alpha\beta}))\delta g_{\sigma\rho}\Big) \\ &\quad +\text{boundary terms} \\ &= \left(-\frac{1}{2}R^{\alpha}{}_{\mu;\nu;\alpha}-\frac{1}{2}R^{\alpha}{}_{\nu;\mu;\alpha}+\frac{1}{2}R_{\mu\nu;\beta}{}^{;\beta}+\frac{1}{2}R^{\alpha\beta}{}_{;\alpha;\beta}g_{\mu\nu}\right)\delta g^{\mu\nu} \\ &\quad +\text{boundary terms}. \end{aligned} \tag{B.70}$$

Using the second Bianchi identity of Eq. (2.35), we can rewrite

$$2R^{\alpha\beta}{}_{;\alpha;\beta} = -R^{\alpha\gamma\beta}{}_{\alpha;\gamma;\beta}-R^{\alpha\gamma}{}_{\gamma}{}^{\beta}{}_{;\alpha;\beta} = R^{\alpha\gamma}{}_{\alpha\gamma}{}^{;\beta}{}_{;\beta} = s^{;\beta}{}_{;\beta}. \tag{B.71}$$

Now, putting everything together yields

$$\begin{aligned} \delta\left(R_{\mu\nu}R^{\mu\nu}\sqrt{|g|}\right) = \Big(&-R^{\alpha}{}_{\mu;\nu;\alpha}-R^{\alpha}{}_{\nu;\mu;\alpha}+R_{\mu\nu;\beta}{}^{;\beta}+\frac{1}{2}s^{;\beta}{}_{;\beta}g_{\mu\nu} \\ &+2R_{\mu\alpha}R^{\alpha}{}_{\nu}-\frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}\Big)\sqrt{|g|}\delta g^{\mu\nu}+\text{boundary terms}. \end{aligned} \tag{B.72}$$

B.4.3 *Variations of the Riemann curvature squared*

The variation of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\sqrt{|g|}$ is given by

$$\delta\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\sqrt{|g|}\right) = \delta\left(R_{\mu\nu\rho\sigma}\right)R^{\mu\nu\rho\sigma}\sqrt{|g|} + R_{\mu\nu\rho\sigma}\delta\left(R^{\mu\nu\rho\sigma}\right)\sqrt{|g|} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\delta\left(\sqrt{|g|}\right). \tag{B.73}$$

In a similar matter as for the Ricci tensor, we can rewrite the middle term as

$$\begin{aligned} R_{\mu\nu\rho\sigma}\delta\left(R^{\mu\nu\rho\sigma}\right) &= R_{\mu\nu\rho\sigma}\delta\left(g^{\mu\alpha}g^{\nu\beta}g^{\rho\lambda}g^{\sigma\kappa}R_{\alpha\beta\lambda\kappa}\right) \\ &= \left(R_{\alpha\nu\rho\sigma}R_{\beta}{}^{\nu\rho\sigma} + R_{\mu\alpha\rho\sigma}R^{\mu}{}_{\beta}{}^{\rho\sigma} + R_{\mu\nu\alpha\sigma}R^{\mu\nu}{}_{\beta}{}^{\sigma} + R_{\mu\nu\rho\alpha}R^{\mu\nu\rho}{}_{\beta}\right)\delta g^{\alpha\beta} \\ &\quad + R^{\alpha\beta\lambda\kappa}\delta\left(R_{\alpha\beta\lambda\kappa}\right). \end{aligned} \tag{B.74}$$

From the (skew)symmetry relations in Eq. (2.33) for the Riemann tensor, we see that the four terms within the brackets are identical, so we have

$$R_{\mu\nu\rho\sigma}\delta\left(R^{\mu\nu\rho\sigma}\right) = 4R_{\alpha\nu\rho\sigma}R_{\beta}{}^{\nu\rho\sigma}\delta g^{\alpha\beta} + R^{\alpha\beta\lambda\kappa}\delta\left(R_{\alpha\beta\lambda\kappa}\right). \tag{B.75}$$

We shall rewrite

$$\begin{aligned} R^{\alpha\beta\lambda\kappa}\delta\left(R_{\alpha\beta\lambda\kappa}\right) &= R^{\alpha\beta\lambda\kappa}\delta\left(g_{\alpha\sigma}R^{\sigma}{}_{\beta\lambda\kappa}\right) \\ &= -R_{\mu}{}^{\beta\lambda\kappa}R_{\nu\beta\lambda\kappa}\delta g^{\mu\nu} + R_{\sigma}{}^{\beta\lambda\kappa}\delta\left(R^{\sigma}{}_{\beta\lambda\kappa}\right). \end{aligned} \tag{B.76}$$

Using Eq. (B.54) and Eq. (B.58), and using partial integration twice, we obtain for the second term

$$\begin{aligned} R_{\sigma}{}^{\beta\lambda\kappa}\delta\left(R^{\sigma}{}_{\beta\lambda\kappa}\right) &= R_{\sigma}{}^{\beta\lambda\kappa}\left(\nabla_{\lambda}\delta\Gamma^{\sigma}{}_{\beta\kappa} - \nabla_{\kappa}\delta\Gamma^{\sigma}{}_{\lambda\beta}\right) = 2R_{\sigma}{}^{\beta\lambda\kappa}\nabla_{\lambda}\delta\Gamma^{\sigma}{}_{\beta\kappa} \\ &= R_{\sigma}{}^{\beta\lambda\kappa}\left(g^{\rho\sigma}\nabla_{\lambda}\left(\nabla_{\kappa}(\delta g_{\rho\beta}) + \nabla_{\beta}(\delta g_{\rho\kappa}) - \nabla_{\rho}(\delta g_{\kappa\beta})\right)\right) \\ &= g^{\rho\sigma}\Big(\nabla_{\kappa}\left(\nabla_{\lambda}(R_{\sigma}{}^{\beta\lambda\kappa})\right)\delta g_{\rho\beta} + \nabla_{\beta}\left(\nabla_{\lambda}(R_{\sigma}{}^{\beta\lambda\kappa})\right)\delta g_{\rho\kappa} \\ &\qquad - \nabla_{\rho}\left(\nabla_{\lambda}(R_{\sigma}{}^{\beta\lambda\kappa})\right)\delta g_{\kappa\beta}\Big) + \text{boundary terms} \\ &= \left(-R_{\mu\nu}{}^{\lambda\kappa}{}_{;\lambda;\kappa} + R_{\mu}{}^{\beta}{}_{\nu}{}^{\lambda}{}_{;\lambda;\beta} + R_{\mu}{}^{\lambda}{}_{\nu}{}^{\rho}{}_{;\lambda;\rho}\right)\delta g^{\mu\nu} \\ &\quad + \text{boundary terms.} \end{aligned} \tag{B.77}$$

The first term $R_{\mu\nu}{}^{\lambda\kappa}{}_{;\lambda;\kappa}$ can be rewritten using the skewsymmetries of the Riemann tensor into

$$\begin{aligned} 2R_{\mu\nu}{}^{\lambda\kappa}{}_{;\lambda;\kappa} &= R_{\mu\nu}{}^{\lambda\kappa}{}_{;\lambda;\kappa} - R_{\mu\nu}{}^{\kappa\lambda}{}_{;\lambda;\kappa} = R_{\mu\nu}{}^{\lambda\kappa}{}_{;\lambda;\kappa} - R_{\mu\nu}{}^{\lambda\kappa}{}_{;\kappa;\lambda} \\ &= R^{\alpha}{}_{\mu\lambda\kappa}R_{\alpha\nu}{}^{\lambda\kappa} + R^{\alpha}{}_{\nu\lambda\kappa}R_{\mu\alpha}{}^{\lambda\kappa} - R^{\lambda}{}_{\alpha\lambda\kappa}R_{\mu\nu}{}^{\alpha\kappa} - R^{\kappa}{}_{\alpha\lambda\kappa}R_{\mu\nu}{}^{\lambda\alpha}, \end{aligned} \tag{B.78}$$

where we have used Eq. (B.17). The first two terms cancel each other. The last two terms are equal, and also vanish because of the symmetry of the Ricci tensor and the skewsymmetry of the Riemann tensor:

$$R^{\lambda}{}_{\alpha\lambda\kappa}R_{\mu\nu}{}^{\alpha\kappa} = R_{\alpha\kappa}R_{\mu\nu}{}^{\alpha\kappa} = -R_{\kappa\alpha}R_{\mu\nu}{}^{\kappa\alpha}. \tag{B.79}$$

Hence we see that

$$R_{\mu\nu}{}^{\lambda\kappa}{}_{;\lambda;\kappa} = 0. \tag{B.80}$$

Putting everything together finally yields

$$\delta\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\sqrt{|g|}\right) = \left(2R_{\mu\ \nu\ ;\beta;\alpha}^{\ \alpha\ \beta} + 2R_{\mu\ \nu\ ;\alpha;\beta}^{\ \alpha\ \beta} + 2R_{\mu\alpha\rho\sigma}R_{\nu}^{\ \alpha\rho\sigma} - \frac{1}{2}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}g_{\mu\nu}\right)\sqrt{|g|}\delta g^{\mu\nu} + \text{boundary terms.} \quad \text{(B.81)}$$

B.4.4 *Pontryagin class*

The following derivations are partly based on [14, Chapter 16]. From Corollary 2.67 we have obtained the topological term R^*R^* in the case $\dim(M) = 4$, also called the *Pontryagin class*, given in Eq. (B.38) by

$$R^*R^* = s^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad \text{(B.82)}$$

We obtain its variation by summing the contributions of the three terms obtained in the previous subsections and (ignoring boundary terms) we find

$$\delta\left(R^*R^*\sqrt{|g|}\right) = \left(2sR_{\mu\nu} - 2s_{;\mu;\nu} + 2g_{\mu\nu}s^{;\beta}_{\ \ ;\beta} - \frac{1}{2}g_{\mu\nu}s^2 + 4R^{\alpha}_{\ \mu;\nu;\alpha} + 4R^{\alpha}_{\ \nu;\mu;\alpha} - 4R_{\mu\nu;\beta}^{\ \ \ \ \ ;\beta} - 4g_{\mu\nu}R^{\alpha\beta}_{\ \ \ ;\alpha;\beta} - 8R_{\mu\alpha}R^{\alpha}_{\ \nu} + 2g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2R_{\mu\alpha\nu\beta}^{\ \ \ \ \ \ ;\beta;\alpha} + 2R_{\mu\alpha\nu\beta}^{\ \ \ \ \ \ ;\alpha;\beta} + 2R_{\mu\alpha\rho\sigma}R_{\nu}^{\ \alpha\rho\sigma} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}\right)\sqrt{|g|}\delta g^{\mu\nu}. \quad \text{(B.83)}$$

First we show that all terms containing covariant derivatives will drop out. Using the second Bianchi identity of Eq. (2.36) we easily derive

$$\begin{aligned} R^{\alpha\beta}_{\ \ \ ;\alpha;\beta} &= \frac{1}{2}s^{;\beta}_{\ \ ;\beta}, \\ R_{\mu\alpha\nu\beta}^{\ \ \ \ \ \ ;\alpha;\beta} &= R_{\mu\nu;\beta}^{\ \ \ \ \ ;\beta} - R_{\mu\beta;\nu}^{\ \ \ \ \ ;\beta}, \\ R^{\alpha}_{\ \mu;\alpha;\nu} &= \frac{1}{2}s_{;\mu;\nu}. \end{aligned} \quad \text{(B.84)}$$

Using the third identity and the fact that

$$R^{\alpha}_{\ \mu;\nu;\alpha} - R^{\alpha}_{\ \mu;\alpha;\nu} = -R^{\alpha}_{\ \rho\nu\alpha}R^{\rho}_{\ \mu} + R^{\rho}_{\ \mu\nu\alpha}R^{\alpha}_{\ \rho}, \quad \text{(B.85)}$$

we also find the identity

$$R^{\alpha}_{\ \mu;\nu;\alpha} = \frac{1}{2}s_{;\mu;\nu} + R_{\mu\rho}R^{\rho}_{\ \nu} - R_{\mu\rho\nu\alpha}R^{\rho\alpha}. \quad \text{(B.86)}$$

By substituting these identities into Eq. (B.83), we obtain

$$\delta\left(R^*R^*\sqrt{|g|}\right) = \left(2sR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}s^2 - 4R_{\mu\alpha}R^{\alpha}_{\ \nu} + 2g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2R_{\mu\alpha\rho\sigma}R_{\nu}^{\ \alpha\rho\sigma} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - 4R_{\mu\rho\nu\alpha}R^{\rho\alpha}\right)\sqrt{|g|}\delta g^{\mu\nu}. \quad \text{(B.87)}$$

Comparing this result to Eq. (B.46) and using the Bach-Lanczos identity of Eq. (B.48), we find

$$\frac{1}{2}\frac{1}{\sqrt{|g|}}\frac{\delta(R^*R^*\sqrt{|g|})}{\delta g^{\mu\nu}} = C_{\mu\rho\sigma\lambda}C_{\nu}^{\ \rho\sigma\lambda} - \frac{1}{4}g_{\mu\nu}C_{\rho\sigma\lambda\tau}C^{\rho\sigma\lambda\tau} = 0. \quad \text{(B.88)}$$

Hence we have shown directly that the variation of $R^*R^*\sqrt{|g|}$ vanishes identically.

B.4.5 *Variation of the Weyl curvature squared*

Finally, we will have a look at the variation of $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}$. By Eq. (B.44), this variation is given by

$$\delta\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}\right)=\delta\left(R^*R^*\sqrt{|g|}\right)+2\delta\left(R_{\mu\nu}R^{\mu\nu}\sqrt{|g|}\right)-\frac{2}{3}\delta\left(s^2\sqrt{|g|}\right). \tag{B.89}$$

The first term vanishes by Eq. (B.88). Using Eq. (B.67) and Eq. (B.72), we find

$$\begin{aligned}\delta\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}\right)=\Big(&-\frac{4}{3}sR_{\mu\nu}+\frac{4}{3}s_{;\mu;\nu}-\frac{1}{3}g_{\mu\nu}s^{;\beta}{}_{;\beta}+\frac{1}{3}s^2g_{\mu\nu}\\&-2R^{\alpha}{}_{\mu;\nu;\alpha}-2R^{\alpha}{}_{\nu;\mu;\alpha}+2R_{\mu\nu;\beta}{}^{;\beta}+4R_{\mu\alpha}R^{\alpha}{}_{\nu}-R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}\Big)\sqrt{|g|}\delta g^{\mu\nu}.\end{aligned} \tag{B.90}$$

Using Eq. (B.86), this can be rewritten as

$$\begin{aligned}\delta\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}\right)=\Big(&-\frac{4}{3}sR_{\mu\nu}-\frac{2}{3}s_{;\mu;\nu}-\frac{1}{3}g_{\mu\nu}s^{;\beta}{}_{;\beta}\\&+\frac{1}{3}s^2g_{\mu\nu}+4R_{\mu\rho\nu\alpha}R^{\rho\alpha}+2R_{\mu\nu;\beta}{}^{;\beta}-R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}\Big)\sqrt{|g|}\delta g^{\mu\nu}.\end{aligned} \tag{B.91}$$

We will show that this can be written in a more compact form. Using Eq. (B.41), we calculate

$$\begin{aligned}C_{\mu\rho\nu\sigma}{}^{;\rho;\sigma}=&R_{\mu\rho\nu\sigma}{}^{;\rho;\sigma}-\frac{1}{6}s^{;\rho;\sigma}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma})\\&-\frac{1}{2}(g_{\mu\nu}R_{\rho\sigma}{}^{;\rho;\sigma}-g_{\mu\sigma}R_{\rho\nu}{}^{;\rho;\sigma}-g_{\rho\nu}R_{\mu\sigma}{}^{;\rho;\sigma}+g_{\rho\sigma}R_{\mu\nu}{}^{;\rho;\sigma}).\end{aligned} \tag{B.92}$$

Using the identities of Eq. (B.84) and Eq. (B.86), we obtain

$$\begin{aligned}C_{\mu\rho\nu\sigma}{}^{;\rho;\sigma}&=R_{\mu\nu;\sigma}{}^{;\sigma}-(\frac{1}{2}s_{;\mu;\nu}+R_{\mu\rho}R^{\rho}{}_{\nu}-R_{\mu\rho\nu\alpha}R^{\rho\alpha})-\frac{1}{6}s_{;\nu;\mu}+\frac{1}{6}g_{\mu\nu}s_{;\sigma}{}^{;\sigma}\\&\quad-\frac{1}{2}\left(\frac{1}{2}g_{\mu\nu}s_{;\sigma}{}^{;\sigma}-\frac{1}{2}s_{;\mu;\nu}-(\frac{1}{2}s_{;\mu;\nu}+R_{\mu\rho}R^{\rho}{}_{\nu}-R_{\mu\rho\nu\alpha}R^{\rho\alpha})+R_{\mu\nu;\rho}{}^{;\rho}\right)\\&=\frac{1}{2}R_{\mu\nu;\sigma}{}^{;\sigma}-\frac{1}{6}s_{;\mu;\nu}-\frac{1}{12}g_{\mu\nu}s_{;\sigma}{}^{;\sigma}-\frac{1}{2}R_{\mu\rho}R^{\rho}{}_{\nu}+\frac{1}{2}R_{\mu\rho\nu\alpha}R^{\rho\alpha}.\end{aligned} \tag{B.93}$$

Secondly, we calculate

$$\begin{aligned}C_{\mu\rho\nu\sigma}R^{\rho\sigma}&=R_{\mu\rho\nu\sigma}R^{\rho\sigma}-\frac{1}{6}s(R_{\mu\nu}-g_{\mu\nu}s)-\frac{1}{2}(g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}-2R_{\mu\sigma}R^{\sigma}{}_{\nu}+sR_{\mu\nu})\\&=R_{\mu\rho\nu\sigma}R^{\rho\sigma}-\frac{2}{3}sR_{\mu\nu}+\frac{1}{6}g_{\mu\nu}s^2-\frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}+R_{\mu\sigma}R^{\sigma}{}_{\nu}.\end{aligned} \tag{B.94}$$

From this we find that

$$\begin{aligned}4C_{\mu\rho\nu\sigma}{}^{;\rho;\sigma}+2C_{\mu\rho\nu\sigma}R^{\rho\sigma}=&\,2R_{\mu\nu;\sigma}{}^{;\sigma}-\frac{2}{3}s_{;\mu;\nu}-\frac{1}{3}g_{\mu\nu}s_{;\sigma}{}^{;\sigma}\\&+4R_{\mu\rho\nu\sigma}R^{\rho\sigma}-\frac{4}{3}sR_{\mu\nu}+\frac{1}{3}g_{\mu\nu}s^2-g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}.\end{aligned} \tag{B.95}$$

We conclude that

$$\delta\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{|g|}\right)=\left(4C_{\mu\rho\nu\sigma}{}^{;\rho;\sigma}+2C_{\mu\rho\nu\sigma}R^{\rho\sigma}\right)\sqrt{|g|}\delta g^{\mu\nu}. \tag{B.96}$$

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