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The Perturbation Semigroup of C^*-algebras

A thesis for the master Mathematical Physics

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Abstract

The Perturbation Semigroup that is defined by Chamseddine, Connes and Van Suijlekom for *-algebras is extended in this thesis to C^*-algebras. For this perturbation semigroup several properties are proven, including functoriality, (non)-exactness and direct limits. For the direct limit of nuclear C^*-algebras we prove that the corresponding direct limit of the perturbation semigroups can be interchanged with the perturbation semigroup. We conclude by giving examples of perturbation semigroups of C^*-algebras.

Contents

Introduction

In 2013 the perturbation semigroup was introduced by Chamseddine, Connes and Van Suijlekom [3]. The motivation for this perturbation semigroup lies in physical research into the elementary particles, the building blocks of matter.

The latter research has resulted in a well-established theory that describes the elementary particles and the three relevant fundamental forces between them: electromagnetic, weak and strong force. It was developed in the second part of the 20th century and reached its definite form around 1975. Nowadays it is called the "Standard Model".

In mathematics, the field of non-commutative geometry was developed by Alain Connes in 1980. Non-commutative geometry makes use of a spectral triple consisting of a *-algebra $\mathcal{A}$ acting on a Hilbert space H and an unbounded self-adjoint operator D acting on H satisfying suitable properties such as bounded commutative with $a \in \mathcal{A}$ which is summarized as $(\mathcal{A}, H, D)$. This non-commutative geometry can be used to express the Standard Model of particle physics.

In this thesis we will only look at the mathematical aspects. We start with a spectral triple. Now the perturbation semigroup comes from the map ω. This map gives perturbations of $D \mapsto D + \sum_j a_j[D, b_j]$. Elements $\sum_j a_j \otimes b_j^\circ$ give rise to the perturbation semigroup of a *-algebra $\mathcal{A}$.

The goal of this thesis is to extend this perturbation semigroup of *-algebras to the perturbation semigroup of C^*-algebras.

With this extended definition, we will discover some properties of the perturbation semigroup, where we will focus mainly on homomorphisms between C^*-algebras and the corresponding homomorphisms between their perturbation semigroups. This theory of homomorphisms can be extended by looking at the direct limit of these homomorphisms. This gives the main result of this thesis, namely that taking the direct limit can be interchanged with the perturbation semigroup, i.e.

$$\mathrm{Pert}(\varinjlim A_\alpha) \cong \varinjlim(\mathrm{Pert}(A_\alpha)).$$

Other properties where we will look at is the functoriality of Pert and the (non)-exactness. We will end with examples of the perturbation semigroup. One of these examples is about commutative AF-algebras. We will visualize this algebra in a nice way using Bratteli-diagrams.

The research on the topics above is a part of what I have done this year to finish my study mathematics. These five years of studying mathematics where hard working, but they have given me a good insight into the beautiful world of mathematics.

Last but not least I thank Walter van Suijlekom for supervising me always in a very kind and patient way. I learned much from the discussions and it really helped me to finish this thesis. Also I thank Klaas Landsman for being the second reader. I thank my fellow students Wouter Hetebrij, Niels Neumann and Jop Schouten with who I discussed mathematical problems, how to write a thesis and LaTeX struggles.

1

Preliminaries

In this chapter we will introduce the direct limit and nuclear C^*-algebras. Therefore we first need to recall some definitions and properties from *Functional analysis*. We will also introduce the Bratteli diagrams.

1.1 C^*-algebras

Definition 1.1. *An* algebra over a field $\mathbb{K}$ *is a vector space* A *over* $\mathbb{K}$ *with a* $\mathbb{K}$*-bilinear operation* $\cdot : A \times A \to A,\ (a,b) \mapsto a \cdot b$ *such that*

i. $(a+b) \cdot c = a \cdot b + b \cdot c$

ii. $a \cdot (b+c) = a \cdot b + a \cdot c$

iii. $a \cdot \lambda b = \lambda(a \cdot b) = (\lambda a) \cdot b$

for all $a, b, c \in A$ *and for all* $\lambda \in \mathbb{K}$.

We will work with $\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.2. *A* Banach space *is a vector space* A *over* $\mathbb{R}$ *or* $\mathbb{C}$ *with a norm* $\|\cdot\|$ *such that* A *is complete with respect to this norm.*

An example of a Banach space is a Hilbert space. A Hilbert space is an inner product space which is complete with respect to the norm induced by the inner product.

Definition 1.3. *Let A be a Banach space. If A is an algebra over $\mathbb{C}$ where the multiplication satisfies*

$$\|ab\| \leq \|a\|\|b\|,$$

then A is called a Banach algebra*.*

Definition 1.4. *An* involution *$* : A \to A,\ a \mapsto a^*$ of a Banach algebra is a map that satisfies the properties*

i. $(a^*)^* = a$

ii. $(a+b)^* = a^* + b^*$

iii. $(\lambda a)^* = \bar{\lambda} a^*$

iv. $(ab)^* = b^* a^*$

v. $\|a^*\| = \|a\|$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

An algebra with an involution $*$ is called a *-algebra.

Definition 1.5. *Let A be a Banach algebra with involution $*$. If this involution satisfies*

$$\|a^* a\| = \|a\|^2$$

then A is called a C^*-algebra.

The space of bounded operators $\mathcal{L}(H)$ over a Hilbert space H is an example of a C^*-algebra. In particular, if $H = \mathbb{C}^n$, we see that $A = M_n(\mathbb{C})$ is a C^*-algebra.

Another example of a C^*-algebra is the the set of compact operators on a Hilbert space H, $\mathcal{K}(H)$, where the compact operators are defined as follows.

Definition 1.6. *A bounded operator $K : H \longrightarrow H$ is* compact *if it is the limit in the operator norm of a sequence of finite rank operators.* Finite rank operators *are bounded operators with a finite-dimensional image.*

This algebra is a closed subset of $\mathcal{L}(H)$ and it is closed under the involution so it is a C^*-algebra.

Proposition 1.7. *The norm of a C^*-algebra A is unique.*

Proof. We use the spectral radius theorem, that states

$$\rho(x) = \sup_{\lambda} \sigma(x) := \sup_{\lambda}\{|\lambda| \mid x - \lambda \mathbb{1} \text{ not invertible}\} = \lim_{n\to\infty} \|x^n\|^{1/n},$$

for $x \in A$ where $\sigma(x)$ is called the spectrum of x. The proof can be read in [18, Chapter I.2].

For $a \in A$, it follows from the spectral radius theorem that $\rho(a^*a) := \lim_{n\to\infty} \|(a^*a)^n\|^{1/n} = \|a\|^2$. Namely, for $y \in A$ normal, i.e. $y^*y = yy^*$, we have

$$\|y^*y\|^2 = \|y^*y\|\|y^*y\| = \|y^*y\|\|(y^*y)^*\| = \|y^*yyy^*\| = \|y^*yy^*y\| = \|(y^*y)^2\|. \quad (1.1)$$

Here the third equality follows from the C^*-algebra property.

Now let $a \in A$, then a^*a is normal since $(a^*a)(a^*a)^* = (a^*a)^*(a^*a)$. Write $x = a^*a$, then $\|x\| = \|x\|^{2^n \cdot 2^{-n}} = \|x\|^{2 \cdot 2^{n-1} \cdot 2^{-n}} = \|x^2\|^{2^{n-1} \cdot 2^{-n}}$ by (1.1). If we do this n times, we can move the power 2^n inside the norm, so $\|x\| = \|x\|^{2^n \cdot 2^{-n}} = \|x^{2^n}\|^{2^{-n}}$. Now we apply the spectral radius theorem which results that $\|x\| = \lim_{n\to\infty} \|x\| = \lim_{n\to\infty} \|x^{2^n}\|^{2^{-n}} = \rho(x)$. If we fill in $x = a^*a$, then $\rho(a^*a) = \|a^*a\| = \|a\|^2$.

Hence $\|a\| = (\rho(a^*a))^{1/2}$, so the norm is given by the algebraic structure of A, hence unique. □

A consequence of the uniqueness of the C^*-norm is that ***-homomorphisms**, homomorphisms φ between *-algebras such that $\varphi(a^*) = \varphi(a)^*$, are norm decreasing:

Corollary 1.8. *For a *-homomorphism $\varphi : A \to B$ we have $\|a\| \geq \|\varphi(a)\|$ for all $a \in A$.*

Proof. Suppose $a - \lambda \mathbb{1}$ is invertible, then there is a b such that $\mathbb{1} = (a - \lambda\mathbb{1})b$, so $\varphi(a - \lambda\mathbb{1})\varphi(b) = \varphi((a - \lambda\mathbb{1})b) = \varphi(\mathbb{1}) = \mathbb{1}$ hence $\varphi(a - \lambda\mathbb{1})$ is invertible. Then $\sigma(a) \supset \sigma(\varphi(a))$ for $a \in A$. Therefore

$$\|a\|^2 = \rho(a^*a) \geq \rho(\varphi(a^*a)) = \|\varphi(a^*a)\| = \|\varphi(a)\|^2.$$

Hence $\|a\| \geq \|\varphi(a)\|$. □

Corollary 1.9. *Suppose $\varphi : A \to B$ is a *-homomorphism. Then φ is injective if and only if $\|a\| = \|\varphi(a)\|$, i.e. φ is an* **isometry**.

The proof can be found in [12, Theorem 3.1.5].

We will now mention two useful properties of C^*-algebras.

Proposition 1.10. *Let A be a C^*-algebra, then for all finite sets $\{a_1, ..., a_n\}$ of elements in A and for all $\varepsilon > 0$ there is a $B \subset A$ with $b_1, ..., b_n \in B$ such that $\|a_i - b_i\| \leq \varepsilon$ for $i = 1, ..., k$.*

Proposition 1.11. *Suppose A is a finite-dimensional C^*-algebra. Then A can be written as the direct sum of matrix algebras, so*

$$A = \bigoplus_{i=1}^{n} M_i(\mathbb{C}).$$

Sketch of proof. For a finite-dimensional C^*-algebra A every ideal I of A is of the form Ap with p a central projection of A. We can decompose A as the direct sum of Ap_k with p_k a projection for every element in the spectrum of the center of A. Hence $A = \bigoplus_i Ap_i$. Every Ap_i is simple, so the statement should only be proven for a simple algebra. Let (π, H) be an irreducible representation (see below for the definition) of a simple A then we have $\pi(A) = \mathcal{K}(H) = \mathcal{L}(H) = M_n(\mathbb{C})$ for A finite-dimensional and $\dim H = n$, so the statement follows. □

The exact details of the proof can be found in [12, Chapter 6.2] and in [18, Chapter I.11].

If A is a C^*-algebra, then $M_n(A)$ is also a C^*-algebra. Here $M_n(A)$ denotes the algebra of $n \times n$ matrices with entries $a_{ij} \in A$. To prove this, we recall first some definitions and a theorem.

Definition 1.12. *A representation of an algebra A is a vector space V with a homomorphism $\pi : A \to \mathrm{End}(V)$. The notation for this representation is (V, π). Here $\mathrm{End}(V)$ denotes the endomorphisms from V to itself.*

Most of the time a representation is only denoted by the vector space and the homomorphism π is omitted.

A state f on a C^*-algebra A is a positive linear functional on A with $\|f\| = 1$. For every state we can define a representation $\pi_f : A \to \mathcal{L}(H_f)$. The universal representation (π, H) is the direct sum of the representations of all states, i.e. $(\pi, H) = \bigoplus_f (\pi_f, H_f)$.

A representation (V, π) is called faithful if π is injective.

Theorem 1.13 (Gelfand-Naimark). *Every C^*-algebra A has a faithful representation and especially the universal representation is faithful.*

Corollary 1.14. *If A is a C^*-algebra, then $M_n(A)$ is also a C^*-algebra.*

Proof. Let (π, H) be the universal representation of A, so by the Gelfand-Naimark theorem the map $\pi : A \to \mathcal{L}(H)$ is injective. As a consequence, the map $\pi : M_n(A) \to M_n(\mathcal{L}(H))$ is also injective.

As mentioned before, the bounded operators $\mathcal{L}(H)$ form a C^*-algebra. The map $\varphi : M_n(\mathcal{L}(H)) \to \mathcal{L}(H^{(n)})$ with $\varphi(a)(\xi_1, \ldots, \xi_n) = (\sum_{i=1}^n a_{1i}(\xi_i), \ldots, \sum_{i=1}^n a_{ni}(\xi_i))$ is an isomorphism where $a \in M_n(\mathcal{L}(H))$, so $M_n(\mathcal{L}(H)$ is a C^*-algebra and hence it has a unique norm.

If we combine these two results, we can define the norm of $x \in M_n(A)$ as the norm of $\pi(x)$. Because $M_n(\mathcal{L}(H)$ is a C^*-algebra, the norm $\|\pi(x)\|$ exists, so $\|x\| = \|\pi(x)\|$ is a C^*-norm. Hence $M_n(A)$ has a C^*-norm, so it is a C^*-algebra. □

1.2 Direct Limit

Definition 1.15. *Let I be a directed set and suppose A_α are objects of a category $\mathcal{C}$ for every $\alpha \in I$. If there is for every $\alpha \leq \beta$ a homomorphism $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ with*

$\varphi_{\alpha\alpha} = \mathrm{Id}_{A_\alpha}$ and $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta}$ for $\alpha \leq \beta \leq \gamma$, then $\{A_\alpha, \varphi_{\alpha\beta}\}_{\alpha,\beta\in I}$ is called a direct system.

The basic definitions about categories can be found in Appendix A.1.

Definition 1.16. *If we have a direct system $\{A_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in I}$, then the algebraic direct limit or algebraic inductive limit of this system is*

$$\bigsqcup_\alpha A_\alpha \Big/ \sim$$

where $a_\alpha \sim a_\beta$ for $a_\alpha \in A_\alpha$, $a_\beta \in A_\beta$, if there is $\gamma \in I$ such that $\varphi_{\alpha\gamma}(a_\alpha) = \varphi_{\beta\gamma}(a_\beta)$. We denote this limit with $\operatorname{alg}\lim_\to A_\alpha$.

The disjoint union is defined as $\bigsqcup_\alpha A_\alpha = \bigcup_\alpha \{(a, \alpha) \mid a \in A_\alpha\}$. Another way of writing elements of $\bigsqcup_\alpha A_\alpha$ is as a sequence $(a_\alpha)_{\alpha\in I}$. Then we can write the disjoint union modulo the equivalence relation as

$$\bigcup_\alpha A_\alpha = \{a = (a_\alpha)_{\alpha\in I},\ a_\alpha \in A_\alpha \mid \exists\ N_0,\ \varphi_{\alpha,\alpha+1}(a_\alpha) = a_\alpha \in A_{\alpha+1}\ \forall\ \alpha > N_0\}.$$

With φ_α we denote the homomorphism from A_α to $\operatorname{alg}\lim_\to A_\alpha$.

If we have an element $a_\alpha \in A_\alpha$, then this element can be found in $\operatorname{alg}\lim_\to A_\alpha$ in the sequence $(0, 0, \dots, a_\alpha, \varphi_{\alpha,\alpha+1}(a_\alpha), \dots)$.

Every A_α has a norm $\|\cdot\|_\alpha$. With these norms we define the norm on $\operatorname{alg}\lim_\to A_\alpha$. Because a homomorphism $\varphi_{\alpha\beta}$ between C^*-algebras is norm-decreasing by Corollary 1.8, the norm on a in $\operatorname{alg}\lim_\to A_\alpha$ is defined as

$$\|a\| = \|(0, 0, \dots, a_\alpha, \varphi_{\alpha,\alpha+1}(a_\alpha), \dots)\| = \lim_{\beta>\alpha} \|\varphi_{\alpha\beta}(a_\alpha)\|_\beta = \inf_{\beta>\alpha} \|\varphi_{\alpha\beta}(a_\alpha)\|_\beta.$$

This is a well-defined norm, namely it satisfies

i. $\|a\| = 0 \Leftrightarrow a = 0$. Suppose $\|a\| = 0$ then $\lim_{\beta>\alpha} \|\varphi_{\alpha\beta}(a)\|_\beta = 0$. Because $\|\cdot\|_\beta$ is a norm, we have that $\varphi_{\alpha\beta}(a) = 0$ for β big enough. Suppose that $a = (0, 0, \dots, a_\alpha, \varphi_{\alpha,\alpha+1}, \dots, 0, 0, \dots)$. Then this element is equivalent to $(0, 0, \dots)$ so $a = 0$. The other direction follows because $\|\cdot\|_\beta$ is a norm.

ii. $\|\lambda a\| = |\lambda|\|a\|$ for $\lambda \in \mathbb{C}$. This follows because $\varphi(\lambda a) = \lambda\varphi(a)$.

iii. $\|a + b\| \leq \|a\| + \|b\|$ follows from the triangle inequality of the norm $\|\cdot\|_\beta$.

In this norm we can complete the algebraic inductive limit, which leads to the following definition.

Definition 1.17. *The direct limit or inductive limit $\lim_\to A_\alpha$ is the completion of the algebraic inductive limit, i.e.*

$$\lim_{\longrightarrow} A_\alpha = \overline{\bigsqcup_\alpha A_\alpha \Big/ \sim}.$$

We can write this direct limit also as the norm closure of $\bigcup_\alpha A_\alpha$, so $A = \overline{\bigcup_\alpha A_\alpha}$, but we will use the notation with the disjoint union from now on.

Definition 1.18. *A C^*-algebra A is called approximately finite-dimensional or AF if there exists a sequence of finite-dimensional C^*-algebras A_α*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{\alpha-1}} A_\alpha \xrightarrow{\varphi_\alpha} \ldots$$

*with *-homomorphisms between them such that A is the direct limit of this sequence.*

We will look at an example of an AF-algebra.

Suppose we have $A = \mathcal{K}(H) + \mathbb{C}\mathbb{I}_H$ where $\mathcal{K}(H)$ denotes the set of compact operators. We show that this algebra is an AF-algebra.

We start with defining the sequence

$$\mathbb{C} \oplus \mathbb{C} \xrightarrow{\varphi_1} M_2(\mathbb{C}) \oplus \mathbb{C} \xrightarrow{\varphi_2} M_3(\mathbb{C}) \oplus \mathbb{C} \xrightarrow{\varphi_3} \ldots \tag{1.2}$$

with $\varphi_i(\Lambda, \lambda) = \left(\begin{pmatrix} \Lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \right)$ for $\Lambda \in M_i$ and $\lambda \in \mathbb{C}$.

Furthermore every matrix of dimension n is a bounded operator on a Hilbert space H_n and vice versa, so $M_n(\mathbb{C}) \oplus \mathbb{C} \simeq \mathcal{L}(H_n) \oplus \mathbb{C}$.

Next we define $A_n = \{T \in \mathcal{L}(H) \mid T(\mathbb{1} - P_n) = (\mathbb{1} - P_n)T \in \mathbb{C}(\mathbb{1} - P_n)\}$ with P_n the orthonormal projection onto H_n and $(\mathbb{1} - P_n)$ the projection onto $H_n^\perp$, the complement of H_n. Then for $T \in A_n$ we have

$$\begin{aligned} T(H) &= T(H_n \oplus H_n^\perp) \\ &= T(P_n(H) \oplus (\mathbb{1} - P_n)(H)) \\ &= (TP_n \oplus T(\mathbb{1} - P_n))(H) \\ &= (TP_n \oplus \mathbb{C}(\mathbb{1} - P_n))(H) \\ &= T(H_n) \oplus \mathbb{C}H_n^\perp. \end{aligned}$$

So that $A_n \simeq \mathcal{L}(H_n) \oplus \mathbb{C}$. The sequence in (1.2) is now isomorphic to the sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$$

All the operators in $\mathcal{L}(H_n)$ are finite rank, hence compact, so for every $T \in A_n$, T is the sum of a finite rank operator and multiple of the identity. So $T \in A = \mathcal{K}(H) + \mathbb{C}\mathbb{I}_H$.

Also $T \in \mathcal{K}(H) + \mathbb{C}\mathbb{I}_H$ is the sum of the limit of finite rank operators and a linear combination of element in H. Because the finite rank operators are dense in $\mathcal{K}(H)$ and finite linear combinations of elements of the basis of H are dense in H, we have $T \in \overline{\bigcup_n A_n}$. Hence $A = \lim_\rightarrow A_n \simeq \lim_\rightarrow M_n(\mathbb{C}) \oplus \mathbb{C}$ and therefore A is an AF-algebra.

1.3 Bratteli diagram

There is a way to visualise homomorphisms between matrix algebras due to Bratteli [14]. This will be discussed in this section.

For a matrix algebra, the following proposition holds:

Proposition 1.19. *Suppose* $A = \bigoplus_{i=1}^{r} M_{d_i}(\mathbb{K})$ *then* $V_1 = \mathbb{K}^{d_1}, \dots, V_r = \mathbb{K}^{d_r}$ *are the irreducible representations of* A *and all the finite-dimensional representations of* A *are of the form* $\bigoplus_{i\in R} V_i$ *with* $R \subset \{1, \dots, r\}$.

This proposition implies the next result. More details can be found in [6].

Theorem 1.20. *Let* $A = \bigoplus_{i=1}^{n} M_{d_i}(\mathbb{K})$ *and* $B = \bigoplus_{j=1}^{m} M_{c_j}(\mathbb{K})$ *be matrix algebras and* $\varphi : A \to B$ *a homomorphism. Then* $c_j = \sum_{i=1}^{n} N_{ji} d_i$ *for all* $j = 1, \dots, m$ *with* $N_{ji} \in \mathbb{N}$. *The homomorphism* φ *is of the form*

$$\varphi\left(\bigoplus_{i=1}^{n} a_i\right) = \bigoplus_{j=1}^{m}\left(\bigoplus_{i=1}^{n} N_{ji} a_i\right), \qquad a_i \in M_{d_i}(\mathbb{K}).$$

Remark 1.21. If we have algebras $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$ then the theorem implies that there is a homomorphism $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ only if n divides m.

We can use the theorem to visualise the embeddings of finite-dimensional C^*-algebras in each other. To do this, we use Bratteli diagrams. If we have two matrix algebras $A = \bigoplus_{i=1}^{n} M_{d_i}(\mathbb{C})$ and $B = \bigoplus_{j=1}^{m} M_{c_j}(\mathbb{C})$ with a homomorphism $\varphi : A \to B$, between them we can draw a Bratteli diagram of this homomorphism. We draw two lines of vertices with on the first line n vertices with labels $d_1, \dots, d_n$ and on the second line m vertices with labels $c_1, \dots, c_m$. Between every node d_i and c_j we draw N_{ji} lines, with $c_j = \sum_{i=1}^{n} N_{ji} d_i$ as the theorem states. This results in the diagram showed in Figure 1.1.

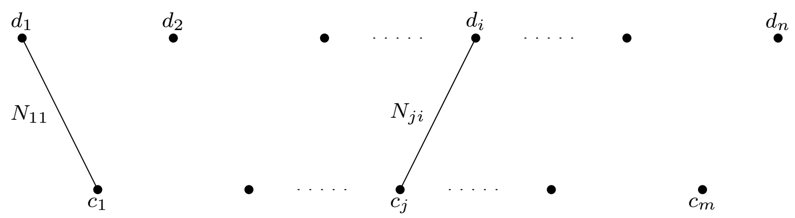

Figure 1.1: Bratteli diagram of $\varphi : \bigoplus_{i=1}^{n} M_{d_i}(\mathbb{C}) \to \bigoplus_{j=1}^{m} M_{c_j}(\mathbb{C})$. The first line of nodes are labelled with the dimensions d_i and the second line with c_j. The N_{ji} satisfy the relation $c_j = \sum_{i=1}^{n} N_{ji} d_i$.

Sometimes there are different homomorphisms between the algebras possible, so this gives different diagrams.

If there is again a homomorphism of the second matrix algebra to another matrix algebra, we can add the dimensions of these matrices in a new line of vertices and connect them to the second line with respect to the N_{ji} of this homomorphism. In this way we can make the Bratteli diagrams of AF-algebras. An example is the sequence

$$M_2(\mathbb{C}) \xrightarrow{\varphi_1} M_4(\mathbb{C}) \xrightarrow{\varphi_2} M_8(\mathbb{C}) \xrightarrow{\varphi_3} M_{16}(\mathbb{C}) \xrightarrow{\varphi_4} \dots$$

The inductive limit of this sequence is called the UHF algebras of type 2^∞. The matrix algebras can be embedded in different ways which results in different Bratteli diagrams, as showed in Figure 1.2.

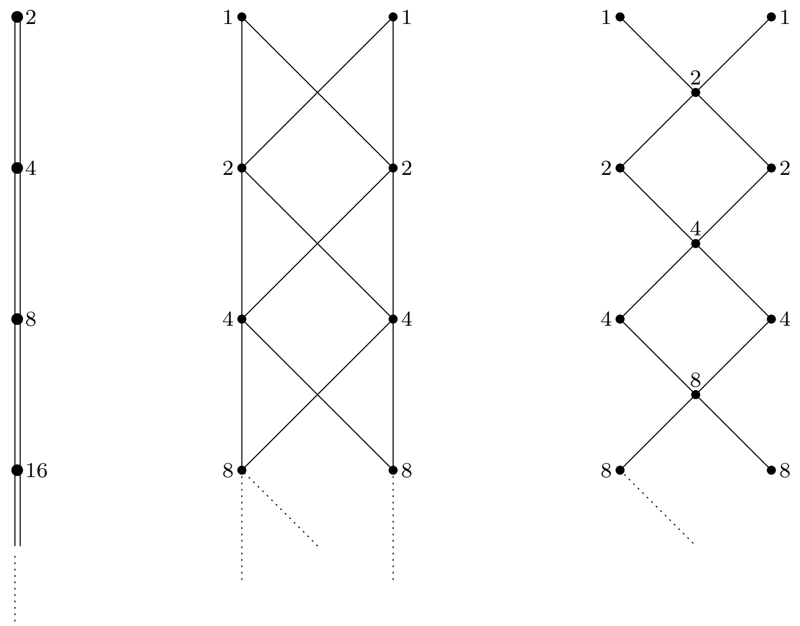

Figure 1.2: The Bratteli diagrams for the UHF algebra $M_{2^\infty}(\mathbb{C})$

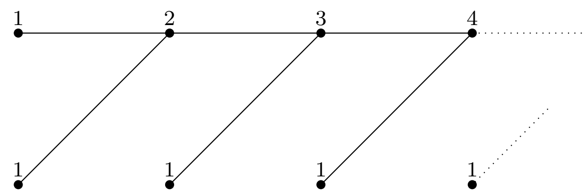

Figure 1.3: Bratteli diagram of $\mathcal{K}(H) + \mathbb{C}\mathbb{I}_H$

Another example is the AF algebra $\mathcal{K}(H) + \mathbb{C}\mathbb{I}_H$ which is the direct limit of the sequence given in the previous section. With this sequence we can draw the Bratteli diagram of $\mathcal{K}(H) + \mathbb{C}\mathbb{I}_H$, as showed in Figure 1.3.

1.4 Tensor products and nuclear C^*-algebras

1.4.1 Tensor products

Let H_1 and H_2 be Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. Then $H_1 \otimes H_2$ has inner product $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle$.
For $S_1 \in \mathcal{L}(H_1)$ and $S_2 \in \mathcal{L}(H_2)$ we define $(S_1 \otimes S_2) \in \mathcal{L}(H_1 \otimes H_2)$ by

$$(S_1 \otimes S_2)(\xi \otimes \eta) = S_1 \xi \otimes S_2 \eta.$$

The norm on $S_1 \otimes S_2$ is a cross norm, that is a norm $\|\cdot\|$ on $A \otimes B$ such that $\|a \otimes b\| = \|a\|\|b\|$ for $a \in A, b \in B$, namely for $\xi \in H_1$ and $\eta \in H_2$ $\|(S_1 \otimes S_2)(\xi \otimes \eta)\| = \|S_1 \xi \otimes S_2 \eta\| = \langle S_1 \xi \otimes S_2 \eta, S_1 \xi \otimes S_2 \eta \rangle = \langle S_1 \xi, S_1 \xi \rangle \langle S_2 \eta, S_2 \eta \rangle = \|S_1 \xi\| \|S_2 \eta\|$ so we can conclude that

$$\|S_1 \otimes S_2\| = \|S_1\| \|S_2\|$$

holds.

Let A and B be C^*-algebras. Then the algebraic tensor product $A \otimes_{alg} B$ is a *-algebra over $\mathbb{C}$ with multiplication $(a_1 \otimes_{alg} b_1)(a_2 \otimes_{alg} b_2) = a_1 a_2 \otimes_{alg} b_1 b_2$ and involution $(a \otimes_{alg} b)^* = a^* \otimes_{alg} b^*$. Most of the time we will denote $\otimes_{\text{alg}}$ with $\otimes$. To make a C^*-algebra of $A \otimes_{alg} B$ this algebra needs to be completed so we need a norm. Also this norm should be a C^*-norm, so $\|ab\| \leq \|a\|\|b\|$ and $\|a^* a\| = \|a\|^2$ for $a, b \in A, B$. The completion $A \overline{\otimes} B$ of $A \otimes_{alg} B$ under any C^*-norm $\| \cdot \|$ is then a C^*-algebra. The only problem is that this norm is not always a cross norm. To construct C^*-norms that satisfy this property, we use the norm of the tensor product of Hilbert spaces.

Suppose π_A and π_B are faithful representations of A and B on the Hilbert spaces H_1 and H_2. The existence of these representations is showed in detail in [1]. Next we can define the representation $\pi = \pi_A \otimes \pi_B$ of $A \otimes_{alg} B$ on $H_1 \otimes H_2$ by $\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b) \in \mathcal{L}(H_1 \otimes H_2)$. If π_A and π_B are injective, then π is also injective. Now we can define the following norm:

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\min} = \sup_{\pi_A, \pi_B} \left\| (\pi_A \otimes \pi_B) \left(\sum_{i=1}^n a_i \otimes b_i \right) \right\|.$$

The completion in this norm is $A \overline{\otimes}_{\min} B$ which is called the minimal or spatial tensor product of A and B.

This is not the only norm that is possible. Another norm that complies is

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\max} = \sup_{\pi} \left\| \pi \left(\sum_{i=1}^n a_i \otimes b_i \right) \right\|.$$

The supremum is taken over all possible representations π of $A \otimes B$. The completion in this norm is called the maximal or projective tensor product of A and B and is denoted as $A\overline{\otimes}_{\max}B$.

For this norms the inequality

$$\|a \otimes b\|_{\min} \leq \|a \otimes b\|_{\max} \qquad \text{with } a \otimes b \in A\overline{\otimes}B$$

holds.

1.4.2 Nuclear C^*-algebras

Definition 1.22. *A C^*-algebra A is called* nuclear *if for every C^*-algebra B there is a unique C^*-norm on $A \otimes_{alg} B$.*

So the completion in different norms gives the same C^*-algebras, hence $A\overline{\otimes}_\gamma B = A\overline{\otimes}_\delta B$ for all well defined norms γ and δ. For example we have $A\overline{\otimes}B = A\overline{\otimes}_{\min}B = A\overline{\otimes}_{\max}B$.

In [17] it is proven that every commutative C^*-algebra is nuclear. Using this, we get the following example of a nuclear C^*-algebra.

Example 1.23. The algebra of continuous functions on a compact hausdorff space X, denoted by $C(X)$, is a nuclear C^*-algebra. This C^*-algebra is namely commutative, hence nuclear.

Another example of a nuclear C^*-algebra follows from the results in the first section.

Corollary 1.24. *Every finite-dimensional C^*-algebra is nuclear.*

Proof. If we have a finite-dimensional C^*-algebra A, then A is the direct sum of matrix algebras by Proposition 1.11. Also for a C^*-algebra B we have $M_n \otimes B \simeq M_n(B)$ because the map defined by

$$\left(\sum_{i,j=1}^{n} e_{ij} \otimes b_{ij}\right) \mapsto (b_{ij})_{ij}$$

is an isomorphism. Here e_{ij} are the basis elements of M_n. By Corollary 1.14 we have that $M_n(B)$ is a C^*-algebra, hence it has a unique norm, so $M_n \otimes B$ has a unique norm, so M_n is nuclear. Hence A is nuclear. □

Nuclearity is closed under most operations and we will prove two of them.

Lemma 1.25. *Nuclearity of C^*-algebras is closed under the direct sum.*

Proof. Suppose A, B are nuclear C^*-algebras and C a C^*-algebra. Then

$$(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$$

has a unique C^*-norm, so $A \oplus B$ is nuclear. □

The next lemma follows from Takesaki [17].

Lemma 1.26. *Nuclearity of C^*-algebras is closed under taking the direct limit.*

Proof. Suppose we have the nuclear C^*-algebras $A_1, A_2, \dots$ and let $A = \lim_{\to} A_\alpha$ be the direct limit. Suppose we have norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ on A. Let B be an arbitrary C^*-algebra. Because A_α is nuclear for all α, we have $\|\sum_{j=1}^n a_j^\alpha \otimes b_j\|_{\min} = \|\sum_{j=1}^n a_j^\alpha \otimes b_j\|_{\max}$. Now take $\sum_{i=1}^n a_i \otimes b_i \in A \otimes B$ arbitrary. For $\varepsilon > 0$ we have nuclear A_β and $a_1^\beta, \dots, a_n^\beta \in A_\beta$ such that $\|a_i^\beta - a_i\| < \varepsilon$ for all i by Proposition 1.10.
This gives

$$\begin{aligned}
\|\sum_{i=1}^n a_i \otimes b_i\|_{\min} &\le \|\sum_{i=1}^n a_i \otimes b_i\|_{\max} \\
&= \|\sum_{i=1}^n a_i \otimes b_i\|_{\max} + \|\sum_{i=1}^n a_i^\beta \otimes b_i\|_{\max} - \|\sum_{i=1}^n a_i^\beta \otimes b_i\|_{\max} \\
&\le \|\sum_{i=1}^n a_i^\beta \otimes b_i\|_{\max} + \|\sum_{i=1}^n (a_i - a_i^\beta) \otimes b_i\|_{\max} \\
&\le \|\sum_{i=1}^n a_i^\beta \otimes b_i\|_{\min} + \sum_{i=1}^n \|a_i - a_i^\beta\|\|b_i\| \\
&\le \|\sum_{i=1}^n a_i^\beta \otimes b_i\|_{\min} + \varepsilon \sum_{i=1}^n \|b_i\| \\
&\le \|\sum_{i=1}^n a_i \otimes b_i\|_{\min} + \|\sum_{i=1}^n (a_i^\beta - a_i) \otimes b_i\|_{\min} + \varepsilon \sum_{i=1}^n \|b_i\| \\
&\le \|\sum_{i=1}^n a_i \otimes b_i\|_{\min} + 2\varepsilon \sum_{i=1}^n \|b_i\|.
\end{aligned}$$

Notice that ε is independent of $\sum_{i=1}^n \|b_i\|$. If there is a B such that this sum becomes bigger, we just choose a bigger set A'_β such that $\|a_i^\beta - a_i\|$ becomes smaller and hence ε can be chosen smaller. Hence we can make $\varepsilon \sum_{i=1}^n \|b_i\|$ arbitrary small.

This results that $\|\sum_{i=1}^n a_i \otimes b_i\|_{\min} = \|\sum_{i=1}^n a_i \otimes b_i\|_{\max}$. For any other C^*-norm $\|\cdot\|_\gamma$ we can do the same calculations as above, so every norm on $A \otimes B$ is the same. Hence the norm on A is unique, so A is nuclear. □

Corollary 1.27. *Every AF-algebra is nuclear.*

We will end this chapter with proving that the direct limit can be interchanged with the direct sum and the tensor product.

Lemma 1.28. *Let $A = \lim_{\to} A_\alpha$ and $B = \lim_{\to} B_\beta$. Then*

$$A \oplus B \cong \varinjlim_{(\alpha,\beta)} (A_\alpha \oplus B_\beta).$$

Proof. The result holds for the algebraic inductive limit. The completion of the direct sum does not change anything, so the completion of the algebraic inductive limit gives us our result. □

Lemma 1.29. *Let* $\operatorname{alg}\lim_{\to} A_\alpha$ *and* $\operatorname{alg}\lim_{\to} B_\beta$ *be the algebraic direct limit of* $\{A_\alpha, \varphi_{\alpha\gamma}\}_{\alpha\in I}$ *resp.* $\{B_\beta, \varphi_{\beta,\delta}\}_{\beta\in J}$. *Then we have*

$$\operatorname{alg}\varinjlim(A_\alpha \otimes B_\beta) = \operatorname{alg}\varinjlim A_\alpha \otimes \operatorname{alg}\varinjlim B_\beta.$$

Proof. Elements in A_α are of the form $\sum_{j=1}^n a_j^\alpha$. Then the algebraic direct limit consists of sequences with elements from the A_α. So

$$(0,0,\dots,\sum_{j=1}^n a_j^\alpha, \varphi_{\alpha,\alpha+1}(\sum_{j=1}^n a_j^\alpha),\dots) \in \operatorname{alg}\varinjlim A_\alpha.$$

For $\operatorname{alg}\lim_{\to} B_\beta$ we have the same, so

$$(0,0,\dots,\sum_{j=1}^n b_j^\beta, \varphi_{\beta,\beta+1}(\sum_{j=1}^n b_j^\beta),\dots) \in \operatorname{alg}\lim_{\to} B_\beta.$$

The tensor product of these elements gives a sequence with zeros until $\max\{\alpha,\beta\}$. Without loss of generality we suppose $\alpha > \beta$. This then results in the sequence

$$(0,0,\dots,\sum_{j=1}^n a_j^\alpha \otimes \varphi_{\beta,\alpha}(b_j^\beta), \sum_{j=1}^n \varphi_{\alpha,\alpha+1}(a_j^\alpha)\otimes\varphi_{\beta,\alpha+1}(b_j^\beta),\dots) \in \operatorname{alg}\varinjlim(A_\alpha\otimes B_\beta).$$

In the same way elements in $\operatorname{alg}\lim_{\to}(A_\alpha \otimes B_\beta)$ can be written in the form of elements in $\operatorname{alg}\lim_{\to} A_\alpha \otimes \operatorname{alg}\lim_{\to} B_\beta$. Hence the elements in both sets are of the same form, so the lemma follows. □

From [16] we have the following theorem.

Theorem 1.30. *Let* $A = \lim_{\to} A_\alpha$ *and* $B = \lim_{\to} B_\beta$. *Suppose* A_α *and* B_β *are nuclear for all* α *and* β. *Then*

$$A\overline{\otimes}B \cong \varinjlim_{(\alpha,\beta)} (A_\alpha\overline{\otimes}B_\beta).$$

Proof. We will first show that $\operatorname{alg}\lim_{\to} A_\alpha \otimes \operatorname{alg}\lim_{\to} B_\beta$ is dense in $A\overline{\otimes}B$. By definition we have that $\operatorname{alg}\lim_{\to} A_\alpha$ is dense in A and $\operatorname{alg}\lim_{\to} B_\beta$ in B. Then for $\sum_{i=1}^n a_i\otimes b_i \in A\otimes B$ and $\varepsilon > 0$ there are $a_i^0 \in \operatorname{alg}\lim_{\to} A_\alpha$, $b_i^0 \in \operatorname{alg}\lim_{\to} B_\beta$ such that

$$\|a_i - a_i^0\| < \varepsilon, \qquad \|b_i - b_i^0\| < \varepsilon \quad \forall i.$$

This shows that $\mathrm{alg}\lim_\to A_\alpha \otimes \mathrm{alg}\lim_\to B_\beta$ is dense in $A \otimes B$, namely

$$\begin{aligned}\|\sum_{i=1}^n a_i \otimes b_i - \sum_{i=1}^n a_i^0 \otimes b_i^0\| &\leq \sum_{i=1}^n \|a_i \otimes b_i - a_i^0 \otimes b_i^0\| \\ &= \sum_{i=1}^n \|a_i \otimes b_i - a_i^0 \otimes b_i + a_i^0 \otimes b_i - a_i^0 \otimes b_i^0\| \\ &= \sum_{i=1}^n \|(a_i - a_i^0) \otimes b_i + a_i^0 \otimes (b_i - b_i^0)\| \\ &\leq \sum_{i=1}^n \|a_i - a_i^0\|\|b_i\| + \|a_i^0\|\|b_i - b_i^0\| \\ &\leq \varepsilon \sum_{i=1}^n \|b_i\| + \varepsilon \sum_{i=1}^n \|a_i^0\|.\end{aligned}$$

Hence the set $\mathrm{alg}\lim_\to A_\alpha \otimes \mathrm{alg}\lim_\to B_\beta$ is dense in $A \otimes B$. We have also that $A \otimes B$ is dense in $A \overline{\otimes} B$. Because density is transitive, we can conclude that $\mathrm{alg}\lim_\to A_\alpha \otimes \mathrm{alg}\lim_\to B_\beta$ is dense in $A \overline{\otimes} B$.

On the other hand we have that $\mathrm{alg}\lim_\to A_\alpha \otimes \mathrm{alg}\lim_\to B_\beta$ is dense in $\lim_\to A_\alpha \overline{\otimes} B_\beta$. Namely

$$\mathrm{alg}\lim_{\to} A_\alpha \otimes \mathrm{alg}\lim_{\to} B_\beta \cong \mathrm{alg}\lim_{\to}(A_\alpha \otimes B_\beta) \subset \lim_{\longrightarrow} A_\alpha \overline{\otimes} B_\beta$$

dense. The isomorphism follows from Lemma 1.29. Now we have that $\mathrm{alg}\lim_\to A_\alpha \otimes \mathrm{alg}\lim_\to B_\beta$ is dense in $A \overline{\otimes} B$ and dense in $\lim_\to A_\alpha \overline{\otimes} B_\beta$. Completion of a metric space is unique up to isometric isomorphisms [11, Proposition 3.2.2]. Because A_α and B_β are nuclear, the norm we use to compete $\mathrm{alg}\lim_\to A_\alpha \otimes B_\beta$ in, is unique, so this results in $A \overline{\otimes} B \cong \lim_\to (A_\alpha \overline{\otimes} B_\beta)$. □

Remark 1.31. For directed sets I, I' we have (I, I') a directed set ordered by $(\alpha, \beta) < (\alpha', \beta')$ if and only if $\alpha < \alpha'$ and $\beta < \beta'$. So the direct limit of $A_\alpha \overline{\otimes} B_\beta$ has been taken over (I, I').
For $A = \lim_{\overrightarrow{\alpha}} A_\alpha$ and $B = \lim_{\overrightarrow{\alpha}} B_\alpha$ we have

$$A \overline{\otimes} B \cong \lim_{\substack{\longrightarrow \\ (\alpha,\alpha)}} A_\alpha \overline{\otimes} B_\alpha = \lim_{\overrightarrow{\alpha}} A_\alpha \overline{\otimes} B_\alpha.$$

2

Perturbation semigroup for C^*-algebras

In this chapter we will recall the definition of the perturbation semigroup for *-algebras $\mathcal{A}$ introduced by A. Chamseddine, A. Connes and W. van Suijlekom in [3] and prepare for the generalization we present in this thesis to C^*-algebras.

After introducing the perturbation semigroup of *-algebras, we will compute $\mathrm{Pert}(M_n(\mathbb{C}))$ as is done in [13]. Thereafter we introduce topological semigroups. This leads us to the definition of the perturbation semigroup of C^*-algebras. The last section will be about the set of unitaries.

2.1 The perturbation semigroup for *-algebras

For a start we define the map $\omega : D \mapsto uDu^*$. Here D is a self-adjoint operator on a Hilbert space H and $u \in \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = 1 = u^*u\}$. These set $\mathcal{U}(\mathcal{A})$ of unitaries will be treated also in Section 2.5. With the commutator $[a, b] = ab - ba$ we can rewrite the map ω as

$$\omega : D \mapsto uDu^* = D + u[D, u^*].$$

We see this map as a perturbation of the operator D by u where $u[D, u^*]$ is the perturbation part. This map can be extended by defining $Z^* = Z = \sum_j a_j[D, b_j]$ for $a_j, b_j \in \mathcal{A}$, so $D \mapsto D + Z$. We make a set of the elements a, b that act on D as $\sum a_j[D, b_j]$. This set is called the perturbation semigroup and $\sum a_j \otimes b_j^\circ \in \mathrm{Pert}(\mathcal{A})$. To do this, Z needs to satisfy two conditions, which we will see later in the definition of the perturbation semigroup. Now this map gives a perturbation of D. This perturbation semigroup is the main topic of this thesis, so we will continue by taking a closer look at some definitions before we formulate the perturbation semigroup.

Definition 2.1. *Let $\mathcal{A}$ be an algebra. The opposite algebra $\mathcal{A}^\circ$ of $\mathcal{A}$ is defined as the algebra given by $\mathcal{A}$ as vector space, but with opposite product. So $a^\circ b^\circ = (ba)^\circ$ for $a^\circ, b^\circ \in \mathcal{A}^\circ$ and $a, b \in \mathcal{A}$.*

The multiplication of the opposite algebra $\mathcal{A}^\circ$ is different from the algebra $\mathcal{A}$ but the linear structure, involution and norm are the same as in the original algebra $\mathcal{A}$. For a commutative algebra we have naturally $\mathcal{A}^\circ = \mathcal{A}$.

Definition 2.2. *A semigroup is a set S with a bilinear operation $\cdot : S \times S \to S$, $(s,t) \mapsto s \cdot t$ that is associative, i.e. $(s \cdot t) \cdot r = s \cdot (t \cdot r)$. If S has a unit, S is called a monoid.*

Now we have all the ingredients to formulate the perturbation semigroup of a *-algebra.

Definition 2.3. *The perturbation semigroup of $\mathcal{A}$ is given by*

$$\mathrm{Pert}(\mathcal{A}) = \left\{ \sum_{j=1}^n a_j \otimes b_j^\circ \in \mathcal{A} \otimes \mathcal{A}^\circ \;\middle|\; \sum_{j=1}^n a_j b_j = 1 \text{ and } \sum_{j=1}^n a_j \otimes b_j^\circ = \sum_{j=1}^n b_j^* \otimes a_j^{\circ *} \right\}$$

with 1 the unit of $\mathcal{A}$. Remark that n is arbitrary but finite.

We call the condition $\sum_{j=1}^n a_j b_j = 1$ the normalisation condition and $\sum_{j=1}^n a_j \otimes b_j^\circ = \sum_{j=1}^n b_j \otimes a_j^{\circ *}$ the self-adjointness condition.

In [13] is showed that the perturbation semigroup is indeed a semigroup, which can be proven in a similar way as Lemma 2.17 below is done.

We will give an easy example of a perturbation semigroup.

Example 2.4. Let $\mathcal{A} = \mathbb{C}$. Because $\mathbb{C}$ is commutative, we have $\mathbb{C} \cong \mathbb{C}^\circ$. There is an isomorphism between $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C}$ so $a_j \otimes b_j \cong a_j b_j$. Then $\sum_{j=1}^n a_j \otimes b_j^\circ \cong \sum_{j=1}^n a_j \otimes b_j \cong \sum_{j=1}^n a_j b_j$. By definition of the perturbation semigroup, $\sum_{j=1}^n a_j b_j$ needs to be 1, so the elements $\sum_{j=1}^n a_j \otimes b_j^\circ \in \mathrm{Pert}(\mathbb{C})$ are equal to 1. Hence $\mathrm{Pert}(\mathbb{C}) = 1$.

Remark 2.5. The perturbation semigroup is not a *-algebra anymore. If we have $\mathcal{A}$ a *-algebra, then $\mathcal{A} \otimes \mathcal{A}^\circ$ is a *-algebra. If $\mathrm{Pert}(\mathcal{A})$ is a *-algebra, it needs to be at least a linear subspace, so for $\sum_{j=1}^n a_j \otimes b_j^\circ \in \mathrm{Pert}(\mathcal{A})$ with $\sum_{j=1}^n a_j b_j = 1$ we need $c \sum_{j=1}^n a_j \otimes b_j^\circ$ to be in $\mathrm{Pert}(\mathcal{A})$ for any scalar c. But $c \sum_{j=1}^n a_j b_j = c \neq 1$ for $c \neq 1$, hence it is not in $\mathrm{Pert}(\mathcal{A})$.

2.2 Perturbation semigroup for matrix algebras

The structure of the perturbation semigroup of matrix algebras has already been treated in [13] by N. Neumann and W. van Suijlekom. In this section, we will recall some results from that paper to give more intuition about the perturbation semigroup. We will come back to this section in Chapter 4 when we give some examples.

Suppose $\mathcal{A} = M_n(\mathbb{C})$. Then every element of the perturbation semigroup $\mathrm{Pert}(M_n(\mathbb{C}))$ can be written as an element of the form $\sum_{ij,kl=1}^n C_{ij,kl} e_{ij} \otimes e_{kl}^\circ$ where $(e_{ij})_{i,j=1}^n$ is a basis of $M_n(\mathbb{C})$. We can make the identification

$$\begin{aligned} M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^\circ &\to M_{n^2}(\mathbb{C}) \\ C_{ij,kl} e_{ij} \otimes e_{kl}^\circ &\mapsto C_{ij,kl} e_{ij} \otimes e_{lk} = C_{ij,kl} e_{n(i-1)+l,n(j-1)+k} \end{aligned}$$

where the last equality follows from the Kronecker-product of matrix algebras.

The normalisation and self-adjointness condition for $\mathrm{Pert}(M_n(\mathbb{C}))$ can be given explicitly.

Lemma 2.6. *The normalisation condition of* $\mathrm{Pert}(M_n(\mathbb{C}))$ *is equivalent to*

$$\begin{cases} \sum_{j=1}^n C_{ij,ji} = 1 & i \in \{1, \dots, n\} \\ \sum_{j=1}^n C_{ij,jl} = 0 & i \neq l \in \{1, \dots, n\}. \end{cases}$$

The self-adjointness condition is equivalent to $C_{ij,kl} = \overline{C_{lk,ji}}$.

If we calculate the perturbation matrix explicit, the effect of the normalisation condition is that every element in the last column depends on $n-1$ elements in the same row of n^2 elements.

Remark 2.7. The coefficient $C_{ij,kl}$ belongs to the element $e_{ij} \otimes e_{kl}^\circ$ and $\overline{C_{lk,ji}}$ to the element $e_{lk}^* \otimes e_{ji}^{\circ *}$. Because $e_{ij} \otimes e_{kl}^\circ = e_{ij} \otimes e_{lk}$ and $e_{lk}^* \otimes e_{ji}^{\circ *} = e_{lk}^* \otimes e_{ij}^*$ we get that the elements on position ij, kl and kl, ij in the matrix $M_{n^2}(\mathbb{C})$ are each other complex conjugates.

In order to descripe $\mathrm{Pert}(M_n(\mathbb{C}))$, we introduce the matrix $\hat{\Omega} = \sum e_{ij} \otimes e_{ji}^\circ$. Indeed, this matrix has the property

$$C_{ij,kl} = \overline{C_{lk,ji}} \iff \hat{\Omega}\overline{A} = A\hat{\Omega} \tag{2.1}$$

with $A = \sum_{ij,kl} C_{ij,kl} e_{ij} \otimes e_{lk}$.

A property of $\hat{\Omega}$ is that it is a hermitian matrix, namely

$$(\sum e_{ij} \otimes e_{ji}^\circ)^T = \sum e_{ij}^T \otimes e_{ji}^{\circ T} = \sum e_{ji} \otimes e_{ij}^\circ.$$

After relabelling, this is again $\sum e_{ij} \otimes e_{ji}^\circ$, hence $\hat{\Omega}$ is hermitian. A hermitian matrix has always a basis consisting of eigenvectors. The eigenvectors of $\hat{\Omega}$ are given by

$$\begin{cases} f_{kl}^+ = e_k \otimes e_l + e_l \otimes e_k & k \neq l \\ f_{kl}^- = e_k \otimes e_l - e_l \otimes e_k & k \neq l \\ f_k = e_k \otimes e_k \end{cases}$$

where the eigenvalue of $e_k \otimes e_l + e_l \otimes e_k$ and $e_k \otimes e_k$ are 1 and that of $e_k \otimes e_l - e_l \otimes e_k$ are -1. In this basis we replace one of the eigenvectors $e_k \otimes e_k$ by $\sum_k e_k \otimes e_k$ which has also eigenvalue 1. This eigenvector will be the first eigenvector ω in our basis of eigenvalues. Now we can diagonalize $\hat{\Omega}$ and get

$$\Omega = \begin{pmatrix} I_{n(n+1)/2} & 0 \\ 0 & -I_{n(n-1)/2} \end{pmatrix}.$$

If we now combine all these results, the perturbation semigroup for $M_n(\mathbb{C})$ is given by

$$\mathrm{Pert}(M_n(\mathbb{C})) \cong \{A \in M_{n^2}(\mathbb{C}) \mid A\omega = \omega, \Omega\bar{A} = A\Omega\}.$$

Explicitly, these are the matrices of the form:

$$\begin{pmatrix} 1 & v & iw \\ 0 & E & iF \\ 0 & iG & H \end{pmatrix}$$

with row vectors

$$v \in \mathbb{R}^{(n-1)(n+2)/2}, \; w \in \mathbb{R}^{n(n-1)/2}$$

and matrices

$$E \in M_{((n-1)(n+2)/2}(\mathbb{R}), \; F \in M_{(n-1)(n+2)/2, n(n-1)/2}(\mathbb{R}),$$
$$G \in M_{n(n-1)/2,(n-1)(n+2)/2}(\mathbb{R}), \; H \in M_{n(n-1)/2}(\mathbb{R}).$$

Note that E and H are square matrices, but F and G are not.

We will call this form the standard form.

If we multiply two matrices of the standard form, we get a matrix that is again from this form, so this matrix is then still in $\mathrm{Pert}(M_n(\mathbb{C}))$, hence the semigroup structure is fulfilled.

The perturbation semigroup of $\mathbb{C}^n$ is given by $\mathbb{C}^{n(n-1)/2}$. Remark that $n(n-1)/2$ is exactly the amount of the positive eigenvalues of $\hat{\Omega}$.

Another way of representing $\mathrm{Pert}(M_n(\mathbb{C}))$ is as semidirect product.

Definition 2.8. *Let S and T be semigroups and suppose that T works on S. Then $S \rtimes T$ is the semidirect product which can be seen as the Cartesian product $S \times T$ with multiplication $* : (S \times T) \times (S \times T) \to S \rtimes T$ defined as*

$$(s_1, t_1) * (s_2, t_2) = (s_1\phi_{t_1}(s_2), t_1 t_2)$$

for a homomorphism $\phi : T \to \mathrm{Aut}(S)$.

Now we write $\mathrm{Pert}(M_n(\mathbb{C}))$ as

$$\mathrm{Pert}(M_n(\mathbb{C})) \cong V \rtimes S$$

where

$$V = \{v \in \mathbb{C}^{n^2-1} \mid \overline{v} = v\Omega'\},$$
$$S = \{B \in M_{n^2-1}(\mathbb{C}) \mid \Omega'\overline{B} = B\Omega'\},$$

and

$$\Omega' = \begin{pmatrix} I_{n(n+1)/2-1} & 0 \\ 0 & -I_{n(n-1)/2} \end{pmatrix}.$$

The multiplication of $V \rtimes S$ is given by

$$(v, B) * (v', B') = (v' + vB', BB').$$

We end this section with an important proposition (Proposition 2.6 in [13]) and give a corollary of it.

Proposition 2.9. *Let $\mathcal{A}, \mathcal{B}$ be *-algebras, then*

$$\mathrm{Pert}(\mathcal{A} \oplus \mathcal{B}) \cong \mathrm{Pert}(\mathcal{A}) \times \mathrm{Pert}(\mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B}^\circ \oplus \mathcal{B} \otimes \mathcal{A}^\circ)^{(s.a.)}$$

where (s.a.) stands for the selfadjoint elements $\sum a_i \otimes b_i^\circ + b_i^* \otimes a_i^{\circ *}$.

Proof. The following isomorphism holds

$$(\mathcal{A} \oplus \mathcal{B}) \otimes (\mathcal{A} \oplus \mathcal{B})^\circ \cong \mathcal{A} \otimes \mathcal{A}^\circ \oplus \mathcal{B} \otimes \mathcal{B}^\circ \oplus \mathcal{A} \otimes \mathcal{B}^\circ \oplus \mathcal{B} \otimes \mathcal{A}^\circ.$$

If we now impose on both sides the normalisation and self-adjointness condition, then the left side become $\mathrm{Pert}(\mathcal{A} \oplus \mathcal{B})$, whereof on $\mathcal{A} \otimes \mathcal{A}^\circ$ and $\mathcal{B} \otimes \mathcal{B}^\circ$ they rise exactly to $\mathrm{Pert}(\mathcal{A})$ and $\mathrm{Pert}(\mathcal{B})$. The normalisation condition does not affect $(\mathcal{A} \otimes \mathcal{B}^\circ \oplus \mathcal{B} \otimes \mathcal{A}^\circ)$ so only the self-adjointness condition is imposed here, which results in $(\mathcal{A} \otimes \mathcal{B}^\circ \oplus \mathcal{B} \otimes \mathcal{A}^\circ)^{(s.a.)}$. □

Corollary 2.10. *Let $\mathcal{A}_i$ be *-algebras for every i. Then we have*

$$\mathrm{Pert}(\bigoplus_{i=1}^n \mathcal{A}_i) = \prod_{i=1}^n \mathrm{Pert}(\mathcal{A}_i) \times \prod_{i<j} (\mathcal{A}_i \otimes \mathcal{A}_j^\circ \oplus \mathcal{A}_j \otimes \mathcal{A}_i^\circ)^{(s.a.)}$$

Proof. We prove this with the following calculation:

$$\begin{aligned}\mathrm{Pert}(\bigoplus_{i=1}^n \mathcal{A}_i) &= \mathrm{Pert}(\mathcal{A}_1 \oplus \bigoplus_{i=2}^n \mathcal{A}_i) \\ &= \mathrm{Pert}(\mathcal{A}_1) \times \mathrm{Pert}(\bigoplus_{i=2}^n \mathcal{A}_i) \times (\mathcal{A}_1 \otimes (\bigoplus_{i=2}^n \mathcal{A}_i)^\circ \oplus \bigoplus_{i=2}^n \mathcal{A}_i \otimes \mathcal{A}_1^\circ)^{(s.a.)}.\end{aligned}$$

Here the second equality follows from Proposition 2.9. In the same way we will rewrite $\mathrm{Pert}(\bigoplus_{i=2}^n \mathcal{A}_i)$. This results in

$$\mathrm{Pert}(\mathcal{A}_1) \times \mathrm{Pert}(\mathcal{A}_2) \times \mathrm{Pert}(\bigoplus_{i=3}^n \mathcal{A}_i) \times (\mathcal{A}_1 \otimes (\bigoplus_{i=2}^n \mathcal{A}_i)^\circ \oplus \bigoplus_{i=2}^n \mathcal{A}_i \otimes \mathcal{A}_1^\circ)^{(s.a.)} \times (\mathcal{A}_2 \otimes (\bigoplus_{i=3}^n \mathcal{A}_i)^\circ \oplus \bigoplus_{i=3}^n A_i \otimes A_2^\circ)^{(s.a.)}.$$

Now we calculate $\mathrm{Pert}(\bigoplus_{i=k}^n \mathcal{A}_i)$ for every k and substitute this in the formula above:

$$\mathrm{Pert}(\mathcal{A}_1) \times \cdots \times \mathrm{Pert}(\mathcal{A}_n) \times (\mathcal{A}_1 \otimes (\bigoplus_{i=2}^n \mathcal{A}_i)^\circ \oplus \bigoplus_{i=2}^n \mathcal{A}_i \otimes \mathcal{A}_1^\circ)^{(s.a.)} \times (\mathcal{A}_2 \otimes (\bigoplus_{i=3}^n \mathcal{A}_i)^\circ \oplus \bigoplus_{i=3}^n A_i \otimes A_2^\circ)^{(s.a.)} \times \cdots \times (\mathcal{A}_{n-1} \otimes \mathcal{A}_n^\circ \oplus \mathcal{A}_n \otimes \mathcal{A}_{n-1}^\circ)^{(s.a.)}.$$

A shorter way to write this is

$$\prod_{i=1}^n \mathrm{Pert}(\mathcal{A}_i) \times \prod_{i=1}^n \left(\mathcal{A}_i \otimes (\bigoplus_{k=i+1}^n \mathcal{A}_k)^\circ \oplus \bigoplus_{k=i+1}^n \mathcal{A}_k \otimes \mathcal{A}_i^\circ \right)^{(s.a.)}.$$

For a finite direct sum we have $\bigoplus_{i=1}^n \mathcal{A}_i = \prod_{i=1}^n \mathcal{A}_i$, so if we write out the self-adjoint part, this results in

$$\prod_{i=1}^n \mathrm{Pert}(\mathcal{A}_i) \times \prod_{i<j} (\mathcal{A}_i \otimes \mathcal{A}_j^\circ \oplus \mathcal{A}_j \otimes \mathcal{A}_i^\circ)^{(s.a.)},$$

which completes the proof. □

Suppose $\mathcal{A}_i = M_{n_i}(\mathbb{C})$, then we get

$$\mathrm{Pert}(\bigoplus_{i=1}^n M_{n_i}(\mathbb{C})) = \prod_{i=1}^n \mathrm{Pert}(M_{n_i}(\mathbb{C})) \times \prod_{i<j} (M_{n_i}(\mathbb{C}) \otimes M_{n_j}(\mathbb{C})^\circ \oplus M_{n_j}(\mathbb{C}) \otimes M_{n_i}(\mathbb{C})^\circ)^{(s.a.)}.$$

Every finite-dimensional C^*-algebra can be written as a finite direct sum of matrix algebras, so this gives the explicit form of the perturbation semigroup of a finite-dimensional C^*-algebra.

2.2.1 Number of variables

The perturbation semigroup of $M_n(\mathbb{C})$ has n^4 entries. The amount of variables in the matrix is a smaller number because of the normalisation and self-adjointness condition. The normalisation condition gives for every row an equation. We have n^2 rows hence also n^2 equations, so $n^4 - n^2$ variables remain.

By the self-adjointness we have that the elements on position ij, ij are real, so n^2 elements are real. The n real elements in the last column are already fixed by the normalisation condition, so $n(n-1)$ real variables remain. The other variables are complex. Half of them are determined by the self-adjointness condition. This gives $\frac{1}{2}(n^4 - n^2 - n(n-1))$ complex variables.

The $n(n-1)$ real variables can be expressed in $\frac{1}{2}n(n-1)$ complex variables. Hence the total amount of complex variables is $\frac{1}{2}(n^4 - n^2 - n(n-1)) + \frac{1}{2}n(n-1) = \frac{1}{2}n^2(n^2-1)$.

2.3 Topological semigroups

Before we saw that the perturbation semigroup is a semigroup, we will look in some more detail to semigroups. More about this subject can be found in [5].

For a direct system of semigroups, the algebraic inductive limit can be taken.

Proposition 2.11. *Suppose we have a direct system $\{S_\alpha, \varphi_{\alpha\beta}\}_{\alpha \in I}$ of semigroups. Then the algebraic inductive limit*

$$\bigsqcup_\alpha S_\alpha \Big/ \sim$$

is a semigroup.

Proof. Because S_α is a semigroup for every α, we have that the multiplication is associative. Also for elements $s, t \in S_\alpha$ we have $st \in S_\alpha$. Hence the multiplication of elements in the algebraic direct limit is also in the algebraic inductive limit. □

If we want to extent this algebraic inductive limit to the inductive limit, we need a topology on the algebraic case. This brings us to topological semigroups.

Definition 2.12. *A* topological semigroup *is a semigroup where the multiplication operation is continuous. A* topological group *is a group where in addition of continuity of the multiplication operation also taking the inverse is continuous.*

The next proposition is a short intermezzo about the relation between topological semigroups and topological groups.

Proposition 2.13. *Let S be a topological semigroup. Then the set*

$$S^\times = \{s \in S \mid \exists\ g \in S\ sg = gs = e\}$$

is a topological group if the multiplication in S is differentiable.

Proof. Because the multiplication in S is continuous, the multiplication in $S^\times$ is continuous. Taking the inverse is continuous by the following argument. Multiplication is continuous and differentiable, so the map $f : S^\times \times S^\times \to S^\times \times S^\times$, $f(a,b) = (a, ab)$ is continuous and differential. By the inverse function theorem, $f^{-1} : S^\times \times S^\times \to S^\times \times S^\times$, $f^{-1}(a, c) = (a, a^{-1}c)$, is continuous (and differentiable). But then the composition map

$$\begin{array}{ccccccc} S^\times & \longrightarrow & S^\times \times S^\times & \xrightarrow{f^{-1}} & S^\times \times S^\times & \longrightarrow & S^\times \\ s & \longmapsto & (s, e) & \longmapsto & (s, s^{-1}) & \longmapsto & s^{-1} \end{array}$$

is continuous, hence taking the inverse in $S^\times$ is continuous. □

Because the operations of taking the inverse and differentiate are continuous for Lie Groups, we have that $S^\times$ is a topological group if it is a Lie Group.

2.4 The perturbation semigroup for C^*-algebras

We are now ready to extend the definition of the perturbation semigroup of *-algebras to that of C^*-algebras. So from now on, we work with nuclear C^*-algebras which we denote as in Chapter 1 with A.

Because A is nuclear, there is for every C^*-algebra B a unique C^*-norm on $A \otimes B$. So we have a unique norm on $A \otimes A^\circ$.

First we give a proposition before we define the perturbation semigroup of nuclear C^*-algebras.

Proposition 2.14. *Let A be finite-dimensional, then the operator*

$$\mu : A \overline{\otimes} A^\circ \to A \qquad a \otimes b^\circ \mapsto ab$$

is continuous.

Proof. A linear operator between two finite-dimensional normed spaces is always bounded, so μ is continuous. □

Remark 2.15. For A infinite-dimensional, this operator is not continuous in general with respect to a C^*-norm. This result follows from [2].

Definition 2.16. *The perturbation semigroup for nuclear C^*-algebras A is defined by*

$$\mathrm{Pert}(A) = \left\{ \sum_{j=1}^{\infty} a_j \otimes b_j^\circ \in A \overline{\otimes} A^\circ \;\middle|\; \sum_{j=1}^{\infty} a_j b_j = 1 \text{ and } \sum_{j=1}^{\infty} a_j \otimes b_j^\circ = \sum_{j=1}^{\infty} b_j^* \otimes a_j^{\circ *} \right\}$$

with 1 the unit of A.

To avoid confusion with the previously defined perturbation semigroup for *-algebras, we will denote that one with $\mathrm{Pert}^{\mathrm{alg}}(A)$.

Lemma 2.17. *The perturbation semigroup for a nuclear C^*-algebra A is a semigroup with unit.*

Proof. Because A is associative, $A \otimes A^\circ$ is associative. Hence multiplication in $\mathrm{Pert}(A)$ is associative. The element $1 \otimes 1$ is the unit of $\mathrm{Pert}(A)$.
Suppose $\sum_{j=1}^\infty a_j \otimes b_j^\circ$, $\sum_{k=1}^\infty \tilde{a}_k \otimes \tilde{b}_k^\circ \in \mathrm{Pert}(A)$. Then their product $\sum_{j,k=1}^\infty a_j\tilde{a}_k \otimes (\tilde{b}_k b_j)^\circ$ is also in $\mathrm{Pert}(A)$, namely

$$\sum_{j,k=1}^\infty a_j\tilde{a}_k\tilde{b}_k b_j = \sum_{j=1}^\infty a_j(\sum_{k=1}^\infty \tilde{a}_k\tilde{b}_k)b_j = \sum_{j=1}^\infty a_j b_j = 1$$

and

$$\begin{aligned}\textstyle\sum_{j,k=1}^\infty a_j\tilde{a}_k \otimes (\tilde{b}_k b_j)^\circ &= \left(\textstyle\sum_{j=1}^\infty a_j \otimes b_j^\circ\right)\left(\textstyle\sum_{k=1}^\infty \tilde{a}_k \otimes \tilde{b}_k^\circ\right)\\ &= \left(\textstyle\sum_{j=1}^\infty b_j^* \otimes a_j^{\circ *}\right)\left(\textstyle\sum_{k=1}^\infty \tilde{b}_k^* \otimes \tilde{a}_k^{*\circ}\right)\\ &= \textstyle\sum_{j,k=1}^\infty b_j^*\tilde{b}_k^* \otimes (\tilde{a}_k^* a_j^*)^\circ\\ &= \textstyle\sum_{j,k=1}^\infty (\tilde{b}_k b_j)^* \otimes (a_j\tilde{a}_k)^{\circ *}.\end{aligned}$$

□

The difference with the previous definition of the perturbation semigroup is that for a C^*-algebra A, $A \otimes A^\circ$ is typically not complete, so it is not a C^*-algebra. The completion of $A \otimes A^\circ$ can be taken. This gives the C^*-algebra $A \overline{\otimes} A^\circ$ which is unique because A is nuclear. The consequence of this is that this perturbation semigroup is complete now. Namely if we have a sequence $S_N = \sum_{j=1}^N a_j \otimes b_j^\circ$ in $\mathrm{Pert}(A)$ that converge to $S = \sum_{j=1}^\infty a_j \otimes b_j^\circ$ in $A \overline{\otimes} A^\circ$, then we have $S \in \mathrm{Pert}(A)$. This follows from the next propositions.

Proposition 2.18. *The subset*

$$B := \{\sum_{j=1}^\infty a_j \otimes b_j^\circ \mid \sum_{j=1}^\infty a_j b_j = 1\} \subset A \overline{\otimes} A^\circ$$

is closed in $A \overline{\otimes} A^\circ$.

Proof. On the subset $D = \{\sum_{j=1}^\infty a_j \otimes b_j^\circ \mid \sum_{j=1}^\infty a_j b_j \text{ is convergent }\} \subset A \overline{\otimes} A^\circ$ the operator $\mu_{|D}$ is bounded, so continuous. Because $\sum_{j=1}^\infty a_j b_j$ is convergent, the set D is closed in $A \overline{\otimes} A^\circ$.
The set

$$D \setminus B = \{\sum_{j=1}^\infty a_j \otimes b_j^\circ \mid \sum_{j=1}^\infty a_j b_j \neq 1\ , \sum_{j=1}^\infty a_j b_j \text{ is convergent }\}$$

is open in D. Namely $\tilde{A} = \{\sum_{j=1}^\infty a_j b_j \mid \sum_{j=1}^\infty a_j b_j \neq 1, \sum_{j=1}^\infty a_j b_j \text{ is convergent }\} \subset \tilde{D} = \mu(D)$ is open in $\tilde{D}$, because for every $\sum_{j=1}^\infty a_j b_j \in \tilde{A}$ there is an open ball $\mathcal{B}$ around this point with radius $\frac{\|\sum_{j=1}^\infty a_j b_j - 1\|}{2}$ in $\tilde{A}$. By continuity of $\mu_{|D}$ follows that $\mu_{|D}^{-1}(\tilde{A}) = D \setminus B \subset D$ is open in D. But then B is closed in D. Because D is closed in $A \overline{\otimes} A^\circ$, B is also closed in $A \overline{\otimes} A^\circ$. □

Proposition 2.19. *The subset*

$$C\overline{\otimes}C^\circ := \left\{ \sum_{j=1}^{\infty} a_j \otimes b_j^\circ \mid \sum_{j=1}^{\infty} a_j \otimes b_j^\circ = \sum_{j=1}^{\infty} b_j^* \otimes a_j^{\circ *} \right\} \subset A\overline{\otimes}A^\circ$$

is closed in $A\overline{\otimes}A^\circ$.

Proof. Suppose there is a sequence $S_N = \sum_{j=1}^N a_j \otimes b_j^\circ$ that converges to $S = \sum_{j=1}^\infty a_j \otimes b_j^\circ$. Then S is in $C\overline{\otimes}C^\circ$ if $\sum_{j=1}^\infty a_j \otimes b_j^\circ = \sum_{j=1}^\infty b_j^* \otimes a_j^{\circ *}$. But $S_N = \sum_{j=1}^N b_j^* \otimes a_j^{\circ *}$ for all N and this sequence converges to $\sum_{j=1}^\infty b_j^* \otimes a_j^{\circ *}$. Because $C\overline{\otimes}C^\circ$ is a metric space, the limit is unique and hence $\sum_{j=1}^\infty a_j \otimes b_j^\circ = \sum_{j=1}^\infty b_j^* \otimes a_j^{\circ *}$, so $\sum_{j=1}^\infty a_j \otimes b_j^\circ$ is in $C\overline{\otimes}C^\circ$ and hence the set is closed in $A\overline{\otimes}A^\circ$. □

Hence we have two sets that are closed in the complete set $A\overline{\otimes}A^\circ$ so the intersection of these two sets is also closed in $A\overline{\otimes}A^\circ$. Because a closed set of a complete metric space is itself complete the following theorem results.

Theorem 2.20. *Suppose A is a nuclear C^*-algebra. Then* $\mathrm{Pert}(A)$ *is complete. In that case* $\mathrm{Pert}(A) = \overline{\mathrm{Pert}^{\mathrm{alg}}(A)}$ *completed in the topology of* $A\overline{\otimes}A^\circ$.

In Remark 2.5 we noted that the perturbation semigroup of a *-algebra is no longer a *-algebra. With the same argument, the perturbation semigroup of C^*-algebras are no more C^*-algebras. The normalisation and self-adjointness conditions make $\mathrm{Pert}(A)$ only a closed subset of $A\overline{\otimes}A^\circ$.

We will continue by looking at the topology of $\mathrm{Pert}(A)$.

We will define the topology on the semigroup $\mathrm{Pert}(A)$ by a metric $d(\cdot,\cdot)$. Because $\mathrm{Pert}(A) \subset A\overline{\otimes}A^\circ$, the metric $d(\cdot,\cdot)$ in $\mathrm{Pert}(A)$ is induced by the norm of $A\overline{\otimes}A^\circ$, so for $s,t \in \mathrm{Pert}(A)$, $d(s,t) = \|s-t\|$. With this metric, the multiplication in $\mathrm{Pert}(A)$ is continuous, namely let $s_n, t_n, s, t \in \mathrm{Pert}(A)$ for all n and suppose we have $s_n \to s$ and $t_n \to t$ if $n \to \infty$, then $d(s_n t_n, st) = \|s_n t_n - st\| = \|s_n t_n - s_n t + s_n t - st\| \le \|s_n t_n - s_n t\| + \|s_n t - st\| \le \|s_n\|\|t_n - t\| + \|t\|\|s_n - s\| \le \varepsilon\|s_n\| + \varepsilon'\|t\| \le \varepsilon''$. Hence the multiplication on $\mathrm{Pert}(A)$ is continuous. This results in the next proposition:

Proposition 2.21. *Let A be a nuclear C^*-algebra. Then the perturbation semigroup of A,* $\mathrm{Pert}(A)$*, with metric $d(\cdot,\cdot)$ is a topological semigroup.*

From this proposition we have that $\mathrm{Pert}(A)$ is a topological semigroup. Because $\mathrm{Pert}(A)^\times$ is a group and from Proposition 2.13 above, it follows that $\mathrm{Pert}(A)^\times$ becomes a topological group if we add the condition that the multiplication is differentiable.

2.5 Unitaries

In this section we will look at an example of a topological group and its relation with the perturbation semigroup, namely the group of unitary elements. This group is defined as follows.

Definition 2.22. *The group of unitary elements $\mathcal{U}(\mathcal{A})$ of a *-algebra $\mathcal{A}$ is defined as*

$$\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = 1 = u^*u\}.$$

We will look at the unitaries of a C^*-algebra A. By the properties of a C^*-algebra, this gives a norm on A. Because $\mathcal{U}(A) \subset A$, we can again define the metric $d(\cdot,\cdot)$ on $\mathcal{U}(A)$ by the norm of A so $d(a,b) = \|a-b\|$. Also we have that the norm of every $u \in \mathcal{U}(A)$ is 1, namely $1 = \|1\| = \|u^*u\| = \|u\|^2$.

Proposition 2.23. *Suppose A is a C^*-algebra, then $\mathcal{U}(A)$ is a topological group.*

Proof. $\mathcal{U}(A)$ is by definition a group. The topology on $\mathcal{U}(A)$ comes from the norm of A. The proof that the multiplication is continuous in this topology is similar to the proof of Proposition 2.21, namely suppose for $u_n, u, v_n, v \in \mathcal{U}(A)$ that $u_n \to u$ and $v_n \to v$ if $n \to \infty$, then $d(u_nv_n, uv) = \|u_nv_n - uv\| = \|u_nv_n - u_nv + u_nv - uv\| \leq \|u_nv_n - u_nv\| + \|u_nv - uv\| \leq \|u_n\|\|v_n - v\| + \|v\|\|u_n - u\| \leq \varepsilon\|u_n\| + \varepsilon'\|v\| \leq \varepsilon''$.

The inverse operation is continuous by the following argument. Let $T : \mathcal{U}(A) \to \mathcal{U}(A)$, $T(u) = u^{-1}$ be the inverse operator. Then $\|T(u)\| = \|u^{-1}\| = \|u^*\| = \|u\|$, so T is a bounded operator, hence continuous. Thus $\mathcal{U}(A)$ is a topological group. □

The unitaries can be embedded in the perturbation semigroup, as stated in the following proposition.

Proposition 2.24. *If $u \in \mathcal{U}(A)$ then $u \otimes u^{*\circ} \in \mathrm{Pert}(A)$.*

Proof. Because $uu^* = 1$ the normalisation condition is satisfied. Also we have $u \otimes u^{*\circ} = u^{**} \otimes u^{*\circ}$ so the self-adjointness condition is satisfied. □

The resulting operator $\eta : \mathcal{U}(A) \to \mathrm{Pert}(A)$, $u \mapsto u \otimes u^{*\circ}$ is continuous, namely $\|\eta(u)\| = \|u \otimes u^{*\circ}\| = \|u\|\|u^{*\circ}\| = 2\|u\|$, so the operator is bounded, hence continuous.

For $u \otimes u^{*\circ} \in \mathrm{Pert}(\mathcal{U}(A))$ we have $u^* \otimes u^\circ \in \mathrm{Pert}(\mathcal{U}(A))$. Because $(u \otimes u^{*\circ})(u^* \otimes u^\circ) = (1 \otimes 1)$, it follows that $u^* \otimes u^\circ$ the inverse is of $u \otimes u^{*\circ}$. This gives a group

$$\eta(\mathcal{U}(A)) := \{u \otimes u^{*\circ} \mid u \in \mathcal{U}(A)\} \subset \mathrm{Pert}(\mathcal{U}(A)). \tag{2.2}$$

Proposition 2.25. *The set $\eta(\mathcal{U}(A))$ as defined in (2.2) is complete.*

Proof. The set $\mathcal{U}(A)$ is complete. Now suppose $u_n \otimes u_n^{*\circ}$ is a sequence that converge to $u \otimes u^{*\circ}$. By continuity of η we have that $\eta^{-1}(u_n \otimes u_n^{*\circ}) = u_n$ converge to $\eta^{-1}(u \otimes u^{*\circ}) = u$. Because $\mathcal{U}(A)$ is complete we have $u \in \mathcal{U}(A)$. Hence $u \otimes u^{*\circ} \in \eta(\mathcal{U}(A))$. □

Proposition 2.26. *The subgroup $\eta(\mathcal{U}(A)) \subset \mathrm{Pert}(A)$ is a topological group.*

Proof. The multiplication in $\eta(\mathcal{U}(A))$ is defined as $(u \otimes u^{*\circ})(v \otimes v^{*\circ}) = (uv \otimes (uv)^{*\circ})$. Because $\mathcal{U}(A)$ is a topological group, the multiplication uv is continuous, hence multiplication in $\eta(\mathcal{U}(A))$ is continuous.

To show that taking the inverse is continuous, we define the operator T that gives the inverse, so $T(u \otimes u^{*\circ}) = (u \otimes u^{*\circ})^{-1}$. This operator is bounded: $\|T(u \otimes u^{*\circ})\| = \|(u \otimes u^{*\circ})^{-1}\| = \|u^* \otimes u^\circ\| = \|u^*\|\|u^\circ\| = \|u\|\|u^{*\circ}\| = \|u \otimes u^{*\circ}\|$. Hence taking the inverse is continuous, so the proposition follows. □

3

Properties of Pert(A)

This chapter will be about properties of the perturbation semigroup. We start with computing the perturbation semigroup of a direct sum of C^*-algebras. Thereafter we will look at the functoriality of Pert; in specific we will look at morphisms, ideals and exact sequences. We will end with interchanging the perturbation semigroup with the direct limit, wherefore the preparation was done in the previous chapters.

3.1 The perturbation semigroup of direct sums

In Proposition 2.9 the perturbation semigroup of the direct sum was calculated. With our new definition of the perturbation semigroup we will do the same for nuclear C^*-algebras in the next proposition.

Proposition 3.1. *Let A, B be nuclear C^*-algebras, then*

$$\mathrm{Pert}(A \oplus B) \cong \mathrm{Pert}(A) \times \mathrm{Pert}(B) \times (A \overline{\otimes} B^\circ \oplus B \overline{\otimes} A^\circ)^{(s.a.)}$$

with (s.a.) the self-adjoint elements $\sum a_i \otimes b_i^\circ + b_i^* \otimes a_i^{\circ *}$.

Proof. A and B are nuclear, so $A \oplus B$ and $(A \oplus B)^\circ$ are nuclear. If we have $(A \oplus B) \overline{\otimes} (A \oplus B)^\circ$, then this is the unique completion of $(A \oplus B) \otimes (A \oplus B)^\circ$ by nuclearity. With the same argument, it follows that $A \overline{\otimes} A^\circ \oplus B \overline{\otimes} B^\circ \oplus A \overline{\otimes} B^\circ \oplus B \overline{\otimes} A^\circ$ is the unique completion of $A \otimes A^\circ \oplus B \otimes B^\circ \oplus A \otimes B^\circ \oplus B \otimes A^\circ$. We have

$$(A \oplus B) \otimes (A \oplus B)^\circ \cong A \otimes A^\circ \oplus B \otimes B^\circ \oplus A \otimes B^\circ \oplus B \otimes A^\circ.$$

Hence we also have

$$(A \oplus B)\overline{\otimes}(A \oplus B)^\circ \cong A\overline{\otimes}A^\circ \oplus B\overline{\otimes}B^\circ \oplus A\overline{\otimes}B^\circ \oplus B\overline{\otimes}A^\circ.$$

In the same way as in the proof of Proposition 2.9 above, it follows that the left-hand side is equal to $\mathrm{Pert}(A \oplus B)$ and the right-hand side equal to $\mathrm{Pert}(A) \times \mathrm{Pert}(B) \times (A\overline{\otimes}B^\circ \oplus B\overline{\otimes}A^\circ)^{(s.a.)}$ if we add the normalisation and self-adjointness conditions to it. □

3.2 Functoriality of the perturbation semigroup

In this section we will discover some categorical characteristics of the perturbation semigroup.

We work only with small categories, that is a category where the class of objects and class of morphisms are sets. In our case we have two small categories. The first category is of C^*-algebras $A, B, \ldots$ with *-homomorphisms φ and the second category is the categories of topological semigroups with $\mathrm{Pert}(A), \mathrm{Pert}(B), \ldots$ and homomorphism φ^*. Between these categories we can make a functor Pert that sends A to $\mathrm{Pert}(A)$ and φ to φ^*. The diagram below illustrate this.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \Big\downarrow \mathrm{Pert} & \\ \mathrm{Pert}(A) & \xrightarrow{\varphi^*} & \mathrm{Pert}(B) \end{array}$$

The existence of this homomorphism φ^* follows from the next Proposition.

Proposition 3.2. *Let A and B be C^*-algebras and $\varphi : A \to B$ a homomorphism. Then there exists a homomorphism between* $\mathrm{Pert}(A)$ *and* $\mathrm{Pert}(B)$.

Proof. We define a homomorphism $\varphi^\circ : A^\circ \to B^\circ$ by $\varphi^\circ(a^\circ) = (\varphi(a))^\circ$. This is indeed a homomorphism. The unit in A° is the same unit as in A, so φ° sends the unit of A° to the unit of B°. Also the linear structure is the same of the opposite algebra, so it remains to check the multiplication property:

$$\varphi^\circ(a^\circ b^\circ) = \varphi^\circ((ba)^\circ) = (\varphi(ba))^\circ = (\varphi(b)\varphi(a))^\circ = \varphi(a)^\circ\varphi(b)^\circ = \varphi^\circ(a^\circ)\varphi^\circ(b^\circ)$$

These two homomorphisms φ and φ° define together the homomorphism $\varphi \otimes \varphi^\circ : A \otimes A^\circ \to B \otimes B^\circ$, with $(\varphi \otimes \varphi^\circ)(\sum_j a_j \otimes \tilde{a}_j^\circ) = \sum_j \varphi(a_j) \otimes \varphi^\circ(\tilde{a}_j^\circ)$.

Let $\sum a_j \otimes b_j^\circ \in \mathrm{Pert}(A)$, then

$$\sum \varphi(a_j)\varphi(b_j) = \varphi(\sum a_j b_j) = \varphi(1) = 1$$

and

$$\begin{aligned}\sum \varphi(a_j) \otimes \varphi^\circ(b_j^\circ) &= (\varphi \otimes \varphi^\circ)(\sum a_j \otimes b_j^\circ) \\ &= (\varphi \otimes \varphi^\circ)(\sum b_j^* \otimes a_j^{\circ *}) \\ &= \sum \varphi(b_j^*) \otimes \varphi^\circ(a_j^{\circ *}) = \sum \varphi(b_j)^* \otimes \varphi^\circ(a_j^\circ)^*.\end{aligned}$$

So for $\sum_{j=1}^n a_j \otimes b_j^\circ \in \mathrm{Pert}(A)$ we have $\sum_{j=1}^n \varphi(a_j) \otimes \varphi^\circ(b_j^\circ) \in \mathrm{Pert}(B)$.
Hence $\varphi \otimes \varphi^\circ$ is the homomorphism between $\mathrm{Pert}(A)$ and $\mathrm{Pert}(B)$. □

We will denote this homomorphism with $\varphi^* = \varphi \otimes \varphi^\circ$.

Now we are ready to look in detail at the characteristics of these categories.

3.2.1 Morphisms

Lemma 3.3. *Suppose $\varphi : A \to B$ is a surjective homomorphism with A, B C^*-algebras. Then $\varphi^* : \mathrm{Pert}(A) \to \mathrm{Pert}(B)$ is a surjection.*

Proof. First remark that the elements in B° are the same as the elements in B. This applies also for A and A°. Hence φ° is a surjection. Let $\sum_j a_j \otimes b_j^\circ \in \mathrm{Pert}(B)$. Then a_j, b_j are in B for all j. Because φ is surjective, there are $\tilde{a}_j$ and $\tilde{b}_j$ in A with $\varphi(\tilde{a}_j) = a_j$ and $\varphi(\tilde{b}_j) = b_j$, so $\sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ \in A \otimes A^\circ$. This does not imply that the normalisation and self-adjointness condition are satisfied, so suppose not, i.e. $\sum_j \tilde{a}_j \tilde{b}_j = R \neq 1$. Then take

$$\frac{1}{2}(1 \otimes (1-R)^\circ + (1-R) \otimes 1) + \sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ \in A \otimes A^\circ.$$

The normalisation condition holds for this element, namely

$$\frac{1}{2}(1 \cdot (1-R) + (1-R) \cdot 1) + \sum_j \tilde{a}_j \tilde{b}_j = 1 - R + R = 1.$$

If we apply φ to R this gives 1, namely $\varphi(R) = \varphi(\sum_j \tilde{a}_j \tilde{b}_j) = \sum_j \varphi(\tilde{a}_j)\varphi(\tilde{b}_j) = \sum_j a_j b_j = 1$. For $\varphi^\circ(R)$ holds the same. Then

$$\begin{aligned}&\varphi^*(\frac{1}{2}(1 \otimes (1-R)^\circ + (1-R) \otimes 1) + \sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ) \\ &\qquad = \frac{1}{2}(\varphi(1) \otimes (\varphi^\circ(1) - \varphi^\circ(R)) + (\varphi(1) - \varphi(R)) \otimes \varphi(1)) + \varphi^*(\sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ) \\ &\qquad = \frac{1}{2}(1 \otimes 0 - 0 \otimes 1) + \varphi^*(\sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ) \\ &\qquad = \sum_j a_j \otimes b_j^\circ\end{aligned}$$

so this element is the pre-image of $\sum_j a_j \otimes b_j^\circ$.
Now suppose the normalisation condition is satisfied for $\sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ$, but that is does not satisfy the self-adjointness condition. Then take $\frac{1}{2}(\sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ + \tilde{b}_j^* \otimes \tilde{a}_j^{\circ *})$. As we saw before, this element is self-adjoint. Its image is

$$\begin{aligned}\varphi^*(\frac{1}{2}(\sum_j \tilde{a}_j \otimes \tilde{b}_j^\circ + \tilde{b}_j^* \otimes \tilde{a}_j^{\circ *})) &= \frac{1}{2}(\sum_j \varphi^*(\tilde{a}_j \otimes \tilde{b}_j^\circ) + \varphi^*(\tilde{b}_j^* \otimes \tilde{a}_j^{\circ *})) \\ &= \frac{1}{2}(\sum_j a_j \otimes b_j^\circ + b_j^* \otimes a_j^{\circ *}) \\ &= \sum_j a_j \otimes b_j^\circ.\end{aligned}$$

Hence for every $\sum_j a_j \otimes b_j^\circ \in \mathrm{Pert}(B)$ there is an element in $\mathrm{Pert}(A)$ such that its image is equal to $\sum_j a_j \otimes b_j^\circ$. Therefore φ^* is surjective. □

Lemma 3.4. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an injective homomorphism and $\mathcal{A}, \mathcal{B}$ *-algebras. Then $\varphi^* : \mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{B})$ is injective.*

Proof. We start with proving that $\tilde{\varphi}^* : \mathcal{A} \otimes \mathcal{A}^\circ \to \mathcal{B} \otimes \mathcal{B}^\circ$ is injective. Suppose $\tilde{\varphi}^*(\sum_{j=1}^n a_j \otimes b_j^\circ) = 0$. Then we have to show that $\sum_{j=1}^n a_j \otimes b_j^\circ = 0$. We choose linear independent $c_1^\circ, \ldots, c_m^\circ \in B^\circ$ such that $\mathrm{span}\{b_1^\circ, \ldots, b_n^\circ\} = \mathrm{span}\{c_1^\circ, \ldots, c_m^\circ\}$. Because φ° is injective, the elements $\varphi^\circ(c_1^\circ), \ldots, \varphi^\circ(c_m^\circ)$ are also linear independent and $\mathrm{span}\{\varphi^\circ(b_1^\circ), \ldots, \varphi^\circ(b_n^\circ)\} = \mathrm{span}\{\varphi^\circ(c_1^\circ), \ldots, \varphi^\circ(c_m^\circ)\}$.
Every b_j° can be written as $b_j^\circ = \sum_{i=1}^m \lambda_{ij} c_i^\circ$. Then also $\varphi^\circ(b_j^\circ) = \varphi^\circ(\sum_{i=1}^m \lambda_{ij} c_i^\circ) = \sum_{i=1}^m \lambda_{ij} \varphi^\circ(c_i^\circ)$. Now

$$\begin{aligned}0 &= \tilde{\varphi}^*(\sum_{j=1}^n a_j \otimes b_j^\circ) \\ &= \sum_{j=1}^n \varphi(a_j) \otimes \varphi^\circ(b_j^\circ) \\ &= \sum_{j=1}^n \varphi(a_j) \otimes \sum_{i=1}^m \lambda_{ij} \varphi^\circ(c_i^\circ) \\ &= \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} \varphi(a_j) \otimes \varphi^\circ(c_i^\circ) = \sum_{i=1}^m \varphi(\sum_{j=1}^n \lambda_{ij} a_j) \otimes \varphi^\circ(c_i^\circ).\end{aligned}$$

Because of the linear independence of the $\varphi^\circ(c_i^\circ)$, it follows that $\varphi(\sum_{j=1}^n \lambda_{ij} a_j) = 0$ for all i. Now again by injectivity of φ, we have $\sum_{j=1}^n \lambda_{ij} a_j = 0$ for all i. Then

$$\sum_{j=1}^n a_j \otimes b_j^\circ = \sum_{j=1}^n a_j \otimes \sum_{i=1}^m \lambda_{ij} c_i^\circ = \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} a_j \otimes c_i^\circ = \sum_{i=1}^m 0 \otimes c_i^\circ = 0.$$

Hence $\tilde{\varphi}^*$ is injective. Because $\mathrm{Pert}(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{A}^\circ$, the map φ^* is a submap of $\tilde{\varphi}^*$, hence φ^* is also injective. □

This lemma holds also for C^*-algebras instead of *-algebras.

Lemma 3.5. *Let $\varphi : A \to B$ be an injective homomorphism and A, B nuclear C^*-algebras. Then $\varphi^* : \mathrm{Pert}(A) \to \mathrm{Pert}(B)$ is injective.*

Proof. Again we proof that $\tilde{\varphi}^* : A \otimes A^\circ \to B \otimes B^\circ$ is injective. By Corollary 1.9 we have that $\|\varphi(a)\| = \|a\|$ and $\|\varphi^\circ(b^\circ)\| = \|b^\circ\|$. Then

$$\|\tilde{\varphi}^*(a \otimes b^\circ)\| = \|\varphi(a) \otimes \varphi^\circ(b^\circ)\| = \|\varphi(a)\|\|\varphi^\circ(b^\circ)\| = \|a\|\|b^\circ\| = \|a \otimes b^\circ\|.$$

With the same corollary it follows that $\tilde{\varphi}$ is injective. Hence φ^* is injective. □

3.2.2 Ideals

Let A be a subalgebra of B, then $\mathrm{Pert}(A) \subset \mathrm{Pert}(B)$ is a subsemigroup. $\mathrm{Pert}(A)$ is namely a subset of $\mathrm{Pert}(B)$ and also a semigroup, so it is a subsemigroup.

Lemma 3.6. *Let A be an C^*-algebra and $I \subset A$ an ideal. Then*

i. $\mathrm{Pert}(I)$ *is an ideal of* $\mathrm{Pert}(A)$*,*

ii. $\mathrm{Pert}(I)^\times$ *is a normal subgroup of* $\mathrm{Pert}(A)^\times$*.*

Proof. i. Let $\sum_k i_k \otimes j_k^\circ \in \mathrm{Pert}(I)$ and $\sum_l a_l \otimes b_l^\circ \in \mathrm{Pert}(A)$. Then $(\sum_k i_k \otimes j_k^\circ)(\sum_l a_l \otimes b_l^\circ) = \sum_{k,l}(i_k \otimes j_k^\circ)(a_l \otimes b_l^\circ) = \sum_{k,l}(i_k a_l) \otimes (j_k^\circ b_l^\circ) = \sum_{k,l}(i_k a_l) \otimes (b_l j_k)^\circ$. Because I is an ideal, $i_k a_l$ and $b_l j_k$ are both in I. Because $\mathrm{Pert}(A)$ is a semigroup, the product of our elements satisfies the perturbation conditions, hence $(\sum_k i_k \otimes j_k^\circ)(\sum_l a_l \otimes b_l^\circ)$ is in $\mathrm{Pert}(I)$. For $(\sum_l a_l \otimes b_l^\circ)(\sum_k i_k \otimes j_k^\circ)$ the same argument holds. This gives that $\mathrm{Pert}(I)$ an ideal of $\mathrm{Pert}(A)$ is.

ii. We already saw that $\mathrm{Pert}(I)$ a subsemigroup of $\mathrm{Pert}(A)$ is. Because $\mathrm{Pert}(I)^\times$ consists by definition an inverse for every element, it is a subgroup of $\mathrm{Pert}(A)^\times$.

Suppose $\sum_l a_l \otimes b_l^\circ \in \mathrm{Pert}(A)^\times$ with inverse $(\sum_l a_l \otimes b_l^\circ)^{-1} = \sum_k c_k \otimes d_k^\circ$ and $\sum_n i_n \otimes j_n^\circ \in \mathrm{Pert}(I)^\times$. Then

$$\begin{aligned}(\sum_k c_k \otimes d_k^\circ)(\sum_n i_n \otimes j_n^\circ)(\sum_l a_l \otimes b_l^\circ) &= \sum_{k,n,l}(c_k \otimes d_k^\circ)(i_n \otimes j_n^\circ)(a_l \otimes b_l^\circ) \\ &= \sum_{k,n,l}(c_k i_n a_l) \otimes (d_k^\circ j_n^\circ b_l^\circ) \\ &= \sum_{k,n,l}(c_k i_n a_l) \otimes (b_k j_n d_k)^\circ.\end{aligned}$$

Again because I is an ideal, $c_k i_n a_l$ and $b_k j_n d_k$ are in I. The perturbation semigroup conditions are also satisfied, so this element is in $\mathrm{Pert}(I)^\times$, hence $\mathrm{Pert}(I)^\times$ is a normal subgroup of $\mathrm{Pert}(A)^\times$.

□

3.2.3 Non-exactness of the perturbation semigroup

We will end this section about the functoriality of the perturbation semigroup by determine whether the functor Pert is exact or not.

Definition 3.7. *A sequence*

$$\cdots \xrightarrow{\varphi_{i-2}} A_{i-2} \xrightarrow{\varphi_{i-1}} A_{i-1} \xrightarrow{\varphi_i} A_i \xrightarrow{\varphi_{i+1}} A_{i+1} \ldots$$

is called an **exact sequence** *if* $\mathrm{Im}(\varphi_i) = \mathrm{Ker}(\varphi_{i+1})$*. A* **short exact sequence** *is an exact sequence of the form*

$$0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \xrightarrow{i} 0.$$

Proposition 3.8. *For a short exact sequence*

$$0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \xrightarrow{i} 0$$

the map g is injective and h surjective.

Proof. By definition of the functions f and i we have $\mathrm{Im}(f) = 0$ and $\mathrm{Ker}(i) = C$. Injectivity of g holds if and only if $\mathrm{Ker}(g) = 0$. Because $0 = \mathrm{Im}(f) = \mathrm{Ker}(g)$, injectivity follows. On the other hand it follows that h is surjective because $\mathrm{Im}(h) = \mathrm{Ker}(i) = C$. □

One might expect that if we have an exact sequence

$$A \xrightarrow{\varphi_1} B \xrightarrow{\varphi_2} C$$

that

$$\mathrm{Pert}(A) \xrightarrow{\varphi_1^*} \mathrm{Pert}(B) \xrightarrow{\varphi_2^*} \mathrm{Pert}(C)$$

is also exact. This would mean that the functor Pert is exact. But unfortunately this is not true. The sequence $\mathbb{C} \xrightarrow{\alpha} \mathbb{C}^2 \xrightarrow{\beta} \mathbb{C}$ with $\alpha : \lambda \mapsto (\lambda, 0)$ and $\beta : (\cdot, \mu) \mapsto \mu$ is exact. However, $\mathrm{Pert}(\mathbb{C}) \xrightarrow{\alpha^*} \mathrm{Pert}(\mathbb{C}^2) \xrightarrow{\beta^*} \mathrm{Pert}(\mathbb{C})$, which is equal to $1 \xrightarrow{\alpha^*} \mathbb{C} \xrightarrow{\beta^*} 1$, cannot be exact.

3.3 The direct limit of the perturbation semigroup

We are ready to combine the theory about the perturbation semigroup of C^*-algebras and that of the direct limit. In this section we will work with nuclear C^*-algebras.

Our goal is to combine taking the direct limit of the perturbation semigroup. If we could prove that

$$\mathrm{Pert}(\varinjlim A_\alpha) = \varinjlim(\mathrm{Pert}(A_\alpha)), \tag{3.1}$$

then this gives a way to compute the direct limit of the perturbation semigroup also in another way.

Below is a diagram of the homomorphisms that we are looking at.

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} A_4 \longrightarrow \cdots$$

$$\mathrm{Pert}(A_1) \xrightarrow{\varphi_1^*} \mathrm{Pert}(A_2) \xrightarrow{\varphi_2^*} \mathrm{Pert}(A_3) \xrightarrow{\varphi_3^*} \mathrm{Pert}(A_4) \longrightarrow \cdots$$

In the previous chapter we saw that the perturbation semigroup of a C^*-algebra A is a complete subset of $A\overline{\otimes}A^\circ$. Therefore the left side of (3.1) is complete. So to prove the equality of (3.1), the right side needs to be complete too. But here we do not take any more the limit of C^*-algebras because $\mathrm{Pert}(A_\alpha)$ is only a semigroup. In a semigroup we do not have a norm as in a C^*-algebra to take the completion, so we need a new definition of the direct limit for semigroups. We have already introduced the algebraic direct limit of a semigroup. We will extend this to the direct limit.

The notation that we will use is as follows:

- $A = \lim_{\rightarrow} A_\alpha$ the direct limit of C^*-algebras A_α
- $S_\alpha = \mathrm{Pert}(A_\alpha)$ with elements $\sum_{j=1}^{\infty} a_j^\alpha \otimes b_j^{\alpha\circ} \in S_\alpha$ denoted by s_α and t_α
- Norm $\|\cdot\|_\alpha$ on A_α and metric $d(\cdot,\cdot)_\alpha$ on S_α

In the previous section we saw that for every α, $S_\alpha = \mathrm{Pert}(A_\alpha)$ is a topological semigroup for metric $d_\alpha(\cdot,\cdot)$. Suppose $s_\alpha \to s$ and $t_\alpha \to t$ if $\alpha \to \infty$. On the algebraic limit of these S_α we will define a metric too, namely

$$d(s,t) = \lim_{\gamma\to\infty} d_\gamma(\varphi_{\alpha\gamma}(s_\alpha), \varphi_{\beta\gamma}(t_\beta)). \tag{3.2}$$

Proposition 3.9. *Suppose $S_\alpha = \mathrm{Pert}(A_\alpha)$ and $\{S_\alpha, \varphi_{\alpha\beta}\}$ is a direct system of semigroups then*

$$S = \bigsqcup_\alpha S_\alpha \Big/ \sim$$

is a topological semigroup with the metric from (3.2).

Proof. In Proposition 2.21 we saw that $\mathrm{Pert}(A_\alpha)$ is a topological semigroup for every α. So the multiplication is continuous in $d_\alpha(\cdot,\cdot)$ for every α. If $\alpha\to\infty$, the multiplication then is again continuous. Namely suppose $s_n \to s$ and $t_n \to t$ if $n\to\infty$ for $s_n, t_n, s, t \in S$. Because φ is a norm decreasing function we have

$$\begin{aligned} d_{\gamma+1}(\varphi_{\alpha,\gamma+1}(s_\alpha), \varphi_{\alpha,\gamma+1}(t_\alpha)) &= \|\varphi_{\alpha,\gamma+1}(s_\alpha - t_\alpha)\|_{\gamma+1} \\ &\le \|\varphi_{\alpha,\gamma}(s_\alpha - t_\beta)\|_\gamma = d_\gamma(\varphi_{\alpha,\gamma}(s_\alpha), \varphi_{\alpha,\gamma}(t_\alpha)). \end{aligned}$$

Then $d(s,t) = \lim_{\gamma\to\infty} d_\gamma(\varphi_{\alpha,\gamma}(s_\alpha), \varphi_{\alpha,\gamma}(t_\alpha)) \le d_\gamma(\varphi_{\alpha,\gamma}(s_\alpha), \varphi_{\alpha,\gamma}(t_\alpha))$. So $d(s,t) \le d_\alpha(s_\alpha, t_\alpha)$ for every α. Now $d(s_n t_n, st) \le d_\alpha(s_{\alpha n} t_{\alpha n}, s_\alpha t_\alpha) = \|s_{\alpha n} t_{\alpha n} - s_\alpha t_\alpha\|_\alpha \le \|s_{\alpha n}\|_\alpha \|t_{\alpha n} - t_\alpha\|_\alpha + \|s_{\alpha n} - s_\alpha\|_\alpha \|t_\alpha\|_\alpha \le \|s_{\alpha n}\|_\alpha d_\alpha(t_{\alpha n}, t_\alpha) + \|t\|_\alpha d_\alpha(s_{\alpha n}, s_\alpha)$ for all α. Because this holds for all α, we have $d(s_n t_n, st) \le \|s_n\| d(t_n, t) + \|t\| d(s_n, s) \le \varepsilon$. Hence S is a topological semigroup. □

Because we have now a topology on the semigroup given by the norm $d(\cdot,\cdot)$ we can take the completion of the algebraic direct limit and hence we can define the direct limit for semigroups.

Definition 3.10. *Let $\{S_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in I}$ be a direct system of semigroups. Then the direct limit of this system is*

$$\overline{\bigsqcup_\alpha S_\alpha \Big/ \sim}$$

with the completion in the metric from (3.2). We denote this limit with $S = \lim_\to S_\alpha$.

To get the direct limit from the algebraic direct limit, the algebraic inductive limit is completed. So there are elements added. If a sequence x_n converge to a certain x in the metric $d(\cdot,\cdot)$, then this x is added in the direct limit. Because this metric comes from the norm of $A_\alpha \overline{\otimes} A_\alpha^\circ$, this sequence converge also in this norm, so is also added to the direct limit of $A_\alpha \overline{\otimes} A_\alpha^\circ$. Hence

$$\varinjlim \operatorname{Pert}(A_\alpha) \subset \varinjlim (A_\alpha \overline{\otimes} A_\alpha^\circ).$$

First we will prove the algebraic version of our main theorem.

Lemma 3.11. *Suppose we have a direct system $\{A_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in I}$.*
Then

$$\operatorname{alg} \varinjlim (\operatorname{Pert}^{\mathrm{alg}}(A_\alpha)) = \operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg} \varinjlim A_\alpha).$$

Proof. We proof this by inclusion of both sets in each other.

$\supseteq$) Suppose we have $\sum_{j=1}^n a_j \otimes b_j^\circ \in \operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_\to A_\alpha)$ then this element is of the form

$$\sum_{j=1}^n (0,0,\ldots,a_j^{\alpha_j}, \varphi_{\alpha_j,\alpha_j+1}(a_j^{\alpha_j}),\ldots) \otimes (0,0,\ldots,b_j^{\circ\beta_j}, \varphi^\circ_{\beta_j,\beta_j+1}(b_i^{\circ\beta_j}),\ldots).$$

Suppose without loss of generality that $\alpha > \beta$. Because we have

$$\operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\varinjlim A_\alpha) \subset \operatorname{alg}\varinjlim A_\alpha \otimes \operatorname{alg}\varinjlim A_\alpha \cong \operatorname{alg}\varinjlim (A_\alpha \otimes A_\alpha^\circ) \subset \varinjlim (A_\alpha \overline{\otimes} A_\alpha^\circ),$$

we can rewrite the equation from above to

$$(0,0,\ldots,\sum_{j=1} a_j^{\alpha_j} \otimes \varphi^\circ_{\beta_j,\alpha_j}(b_j^{\circ\beta_j}), \sum_{j=1}^n \varphi_{\alpha_j,\alpha_j+1}(a_j^{\alpha_j}) \otimes \varphi^\circ_{\beta_j,\alpha_j+1}(b_j^{\circ\beta_j}),\ldots).$$

Because this element is in the perturbation semigroup, we have that

$$\sum_{j=1}^n (a_j^1 b_j^1, a_j^2 b_j^2,\ldots) = (\sum_{j=1}^n a_j^1 b_j^1, \sum_{j=1}^n a_j^2 b_j^2,\ldots) = (1,1,\ldots),$$

so for every element of the sequence the normalisation condition should hold. The other demand is that $\sum_{j=1}^{n} a_j \otimes b_j^\circ = \sum_{j=1}^{n} b_j^* \otimes a_j^{\circ *}$. So

$$\sum_{j=1}^{n}(0,0,\ldots,a_j^{\alpha_j},\varphi_{\alpha_j,\alpha_j+1}(a_j^{\alpha_j}),\ldots)\otimes(0,0,\ldots,b_j^{\circ\beta_j},\varphi^\circ_{\beta_j,\beta_j+1}(b_i^{\circ\beta_j}),\ldots)$$
$$=\sum_{j=1}^{n}(0,0,\ldots,b_j^{\beta_j *},\varphi_{\beta_j,\beta_j+1}(b_i^{\beta_j *}),\ldots)\otimes(0,0,\ldots,a_j^{\circ\alpha_j *},\varphi^\circ_{\alpha_j,\alpha_j+1}(a_j^{\circ\alpha_j *}),\ldots).$$

In the same way it can be shown that this condition should hold for every element of the sequence.

Hence the α^{th} element in the sequence is in $\mathrm{Pert}^{\mathrm{alg}}(A_\alpha)$. This holds for every α so the inclusion follows.

$\subseteq$) Elements in $\mathrm{alg}\lim_{\rightarrow}\mathrm{Pert}^{\mathrm{alg}}(A_\alpha)$ are of the form

$$(0,0,\ldots,\sum_{j=1}^{n} x_j^\alpha \otimes y_j^{\alpha\circ},\sum_{j=1}^{n}\varphi(x_j^\alpha)\otimes\varphi^\circ(y_i^{\circ\alpha}),\ldots).$$

Because the elements in the sequence come from $\mathrm{Pert}^{\mathrm{alg}}(A_\alpha)$, every component satisfies the conditions of the perturbation semigroup.
Now we can rewrite the sequence in the same way as we did above and hence this element is in $\mathrm{Pert}^{\mathrm{alg}}(\mathrm{alg}\lim_{\rightarrow} A_\alpha)$.

Together these inclusions give that the sets are equal. □

In Remark 2.15 we saw that the operator μ is in general not continuous. The next lemma shows that μ is continuous if it is continuous for every A_α.

Lemma 3.12. *Let $\{A_\alpha,\varphi_{\alpha,\beta}\}_{\alpha\in I}$ be a direct system and $A=\lim_{\rightarrow} A_\alpha$. If $\mu_\alpha : A_\alpha \otimes A_\alpha^\circ \to A_\alpha$, $\mu_\alpha(\sum_j a_j^\alpha \otimes b_j^{\alpha\circ}) = \sum_j a_j^\alpha b_j^\alpha$, is continuous for every α, then $\mu : A\overline{\otimes}A^\circ \to A$, $\mu(\sum_j a_j \otimes b_j^\circ) = \sum_j a_j b_j$, is continuous.*

Proof. Suppose $\sum_j a_{j,k}^\alpha \otimes b_{j,k}^{\circ\alpha}$ converges to $\sum_j a_j^\alpha \otimes b_j^{\circ\alpha}$ for all α if $k\to\infty$. Then by continuity of μ_α we have that $\sum_j a_{j,k}^\alpha b_{j,k}^\alpha$ converges to $\sum_j a_j^\alpha b_j^\alpha$. Suppose

$$(0,0,\ldots,\sum_j a_{j,k}^\alpha \otimes b_{j,k}^{\circ\alpha},\varphi^*_{\alpha,\alpha+1}(\sum_j a_{j,k}^\alpha \otimes b_{j,k}^{\circ\alpha}),\ldots)\in \mathrm{alg}\varinjlim(A_\alpha\otimes A_\alpha^\circ)$$

converges to

$$(0,0,\ldots,\sum_j a_j^\alpha \otimes b_j^{\circ\alpha},\varphi^*_{\alpha,\alpha+1}(\sum_j a_j^\alpha \otimes b_j^{\circ\alpha}),\ldots)\in \mathrm{alg}\varinjlim(A_\alpha\otimes A_\alpha^\circ)$$

Then

$$\sum_j a_{j,k}^\alpha \otimes b_{j,k}^{\circ\alpha} \longrightarrow \sum_j a_j^\alpha \otimes b_j^{\circ\alpha} \text{ for all } \alpha$$

and by continuity of μ_α it follows that

$$\sum_j a^\alpha_{j,k} b^\alpha_{j,k} \longrightarrow \sum_j a^\alpha_j b^\alpha_j.$$

We let μ work on $(0,0,\dots,\sum_j a^\alpha_{j,k} \otimes b^{\circ\alpha}_{j,k}, \varphi^*_{\alpha,\alpha+1}(\sum_j a^\alpha_{j,k} \otimes b^{\circ\alpha}_{j,k}),\dots)$ which results in

$$(0,0,\dots,\sum_j a^\alpha_{j,k} b^\alpha_{j,k}, \varphi_{\alpha,\alpha+1}(\sum_j a^\alpha_{j,k} b^\alpha_{j,k}),\dots) \in \operatorname{alg} \varinjlim A_\alpha.$$

By continuity of the μ_α this converges to

$$(0,0,\dots,\sum_j a^\alpha_j b^\alpha_j, \varphi_{\alpha,\alpha+1}(\sum_j a^\alpha_j b^\alpha_j),\dots) \in \operatorname{alg} \varinjlim A_\alpha.$$

Hence μ is continuous on $\operatorname{alg} \lim_\rightarrow (A_\alpha \otimes A^\circ_\alpha)$. This is a dense subset of $A \overline{\otimes} A^\circ$, so μ is also continuous on $A \overline{\otimes} A^\circ$. □

It only remains to show that if the complete tensor and the direct limit instead of the algebraic direct limit are used, the statement of Lemma 3.11 still holds. This will be the main theorem of this thesis.

Theorem 3.13. *Suppose we have a direct system $\{A_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in I}$ of nuclear C^*-algebras with direct limit A and continuous operators μ_α as defined in Lemma 3.12. Then*

$$\operatorname{Pert}(\varinjlim A_\alpha) \cong \varinjlim(\operatorname{Pert}(A_\alpha)). \tag{3.3}$$

Proof. We have

$$\operatorname{Pert}(\varinjlim A_\alpha) \subset A \overline{\otimes} A^\circ \cong \varinjlim(A_\alpha \overline{\otimes} A^\circ_\alpha) \supset \varinjlim(\operatorname{Pert}(A_\alpha)),$$

where the isomorphism follows from Theorem 1.30.

In Theorem 2.20 we saw that the perturbation semigroup of a C^*-algebra is complete, so $\operatorname{Pert}(\lim_\rightarrow A_\alpha)$ is complete.

The direct limit of semigroups is defined as the completion of the algebraic direct limit. Hence the right side of (3.3) is also a complete set.

The set $\operatorname{alg}\lim_\rightarrow A_\alpha \otimes \operatorname{alg}\lim_\rightarrow A^\circ_\alpha$ is dense in $A \overline{\otimes} A^\circ$. Recall that in Lemma 1.29 we proved that $\operatorname{alg}\lim_\rightarrow A_\alpha \otimes \operatorname{alg}\lim_\rightarrow A^\circ_\alpha = \operatorname{alg}\lim_\rightarrow(A_\alpha \otimes A^\circ_\alpha)$. We will now define a set $X \subset \operatorname{alg}\lim_\longrightarrow(A_\alpha \otimes A^\circ_\alpha)$ as

$$X = \operatorname{Pert}(\varinjlim A_\alpha) \cap \operatorname{alg} \varinjlim(A_\alpha \otimes A^\circ_\alpha).$$

Claim 1.

$$\operatorname{Pert}(\varinjlim A_\alpha) = \overline{X}$$

Proof of Claim 1. We prove this by inclusion of the sets in each other. Suppose we have $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ \in \mathrm{Pert}(\lim_\rightarrow A_\alpha)$. Then by definition of the perturbation semigroup we have $\sum_{j=1}^{\infty} a_j b_j = 1$ and $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ = \sum_{j=1}^{\infty} b_j^* \otimes a_j^{\circ *}$. We are looking for a sequence in X that converge to this element. Suppose $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ} \in \mathrm{alg}\lim_\rightarrow (A_\alpha \otimes A_\alpha^\circ)$ converges to $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ$ as $k \longrightarrow \infty$. If this limit sequence is in X we are done, so suppose not.

First suppose we have a sequence $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ}$ which converge to $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ$ but does not satisfy the normalisation condition, so $\sum_{j=1}^{\infty} a_j^k b_j^k = c_k \neq 1$ for some k. By continuity of the operator μ from Lemma 3.12 we have that $c_k = \sum_{j=1}^{\infty} a_j^k b_j^k \longrightarrow \sum_{j=1}^{\infty} a_j b_j = 1$. Then we define the sequence

$$\sum_{j=0}^{\infty} \tilde{a}_j^k \otimes \tilde{b}_j^{k\circ} = (1 - c_k) \otimes 1 + \sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ}.$$

Here $(1 - c_k) \otimes 1 \in \mathrm{alg}\lim_\rightarrow (A_\alpha \otimes A_\alpha^\circ)$ because $1 - c_k$ is just a finite linear combination of elements in A. This sequence converges to $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ$ in the topology of $A \overline{\otimes} A^\circ$ because $c_k \longrightarrow 1$.

Moreover, this sequence now satisfies the normalisation condition, namely

$$\sum_{j=0}^{\infty} \tilde{a}_j^k \tilde{b}_j^k = (1 - c_k) \cdot 1 + \sum_{j=1}^{\infty} a_j^k b_j^k = 1 - c_k + c_k = 1.$$

Now we have a sequence that satisfies the normalisation condition, so we proceed to check the self-adjointness condition.

So suppose we have a sequence $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ}$ that converges to $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ$ that satisfies the normalisation condition but not the self-adjointness condition. So we have $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ} \longrightarrow \sum_{j=1}^{\infty} a_j \otimes b_j^\circ$ and $\sum_{j=1}^{\infty} b_j^{k*} \otimes a_j^{k\circ *} \longrightarrow \sum_{j=1}^{\infty} b_j^* \otimes a_j^{\circ *}$ but $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ} \neq \sum_{j=1}^{\infty} b_j^{k*} \otimes a_j^{k\circ *}$. Then we define the sequence $\frac{1}{2}(\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ} + b_j^{k*} \otimes a_j^{k\circ *})$. This sequence is self-adjoint. Also it does converge to our element, namely $\frac{1}{2}(\sum_{j=1}^{\infty} a_j \otimes b_j^\circ + b_j^* \otimes a_j^{\circ *}) = \frac{1}{2} \cdot 2 \sum_{j=1}^{\infty} a_j \otimes b_j^\circ$. The normalisation condition for this sequence holds too; $\frac{1}{2}(\sum_{j=1}^{\infty} a_j^k b_j^k + b_j^{k*} a_j^{k*}) = \frac{1}{2}(\sum_{j=1}^{\infty} a_j^k b_j^k + (\sum_{j=1}^{\infty} a_j^k b_j^k)^*) = \frac{1}{2}(1 + 1^*) = 1$. Hence we have constructed for every element in $\mathrm{Pert}(\lim_\rightarrow A_\alpha)$ a sequence that converges to it and satisfies the normalisation and self-adjointness conditions, so this is a sequence in X. This shows $\mathrm{Pert}(\lim_\rightarrow A_\alpha) \subset \overline{X}$.

On the other hand suppose we have a sequence $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ} \in X$ which converge to $\sum_{j=1}^{\infty} a_j \otimes b_j^\circ \in A \overline{\otimes} A^\circ$. Then for all k we have $\sum_{j=1}^{\infty} a_j^k b_j^k = 1$ and $\sum_{j=1}^{\infty} a_j^k \otimes b_j^{k\circ} = \sum_{j=1}^{n} b_j^{k*} \otimes a_j^{k\circ *}$. If we take the limit these conditions also hold, so the limit is in $\mathrm{Pert}(\lim A_\alpha)$. Hence Claim 1 follows.

Claim 2.

$$\overline{X} \cong \varinjlim (\mathrm{Pert}\, A_\alpha)$$

Proof of Claim 2. First we show that $\operatorname{alg}\lim_{\rightarrow} \operatorname{Pert}^{\mathrm{alg}} A_\alpha = X$. In Lemma 3.11 we saw that $\operatorname{alg}\lim_{\rightarrow}(\operatorname{Pert}^{\mathrm{alg}} A_\alpha) = \operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_{\rightarrow} A_\alpha)$. Also $\operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_{\rightarrow} A_\alpha) \subset \operatorname{alg}\lim_{\rightarrow}(A_\alpha \otimes A_\alpha^\circ)$ and $\operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_{\rightarrow} A_\alpha) \subset \operatorname{Pert}\lim_{\rightarrow} A_\alpha$. Hence it follows that $\operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_{\rightarrow} A_\alpha) \subset X$. For the other inclusion we look at the elements that are in $\operatorname{Pert}\lim_{\rightarrow} A_\alpha$ but not in $\operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_{\rightarrow} A_\alpha)$. These elements come as limit points in the completion of the direct limit or from the completion of the tensor product. Because we take the intersection with $\operatorname{alg}\lim_{\rightarrow}(A_\alpha \otimes A_\alpha^\circ)$ these elements are not in X. So $X \subset \operatorname{Pert}^{\mathrm{alg}}(\operatorname{alg}\lim_{\rightarrow} A_\alpha)$. Hence $\operatorname{alg}\lim_{\rightarrow} \operatorname{Pert}^{\mathrm{alg}} A_\alpha = X$.

Also we have $\operatorname{alg}\lim_{\rightarrow}(A_\alpha \otimes A_\alpha^\circ) \subset \operatorname{alg}\lim_{\rightarrow}(A_\alpha \overline{\otimes} A_\alpha^\circ)$ densely. Namely suppose we have an element $(0, 0, \ldots, \sum_{j=1}^n a_j^\alpha \otimes b_j^\alpha, \varphi(\sum_{j=1}^n a_j^\alpha \otimes b_j^\alpha), \ldots) \in \operatorname{alg}\lim A_\alpha \overline{\otimes} A_\alpha^\circ$ then because $A_\alpha \otimes A_\alpha^\circ$ is dense in $A_\alpha \overline{\otimes} A_\alpha^\circ$ for all α, we have a sequence in $\operatorname{alg}\lim_{\rightarrow} A_\alpha \otimes A_\alpha^\circ$ that converges to our point. From this we conclude that $\operatorname{alg}\lim_{\rightarrow} \operatorname{Pert}^{\mathrm{alg}} A_\alpha \subset \operatorname{alg}\lim_{\rightarrow} \operatorname{Pert} A_\alpha$ dense.

By definition we have that $\operatorname{alg}\lim_{\rightarrow} \operatorname{Pert} A_\alpha$ is dense in $\lim_{\rightarrow} \operatorname{Pert} A_\alpha$. Now we combine these results. So we start with $\operatorname{alg}\lim_{\rightarrow} \operatorname{Pert}^{\mathrm{alg}} A_\alpha = X$ and we take the completion of both sides and get $\lim_{\rightarrow} \operatorname{Pert} A_\alpha = \overline{X}$ which completes the proof of Claim 2.

Taking the completion is unique up to isomorphism, so if we combine Claim 1 and Claim 2, we get exactly the statement of the theorem. Hence the theorem is proved. □

From Proposition 2.14 followed that μ was continuous for finite-dimensional C^*-algebras. Because an AF-algebra is the direct limit of finite-dimensional C^*-algebras, we have with Lemma 3.12 that μ is also continuous for AF-algebras. This results in the corollary given below.

Corollary 3.14. *Let $A = \lim_{\rightarrow} A_\alpha$ be an AF-algebra. Then*

$$\operatorname{Pert}(A) \cong \varinjlim(\operatorname{Pert}(A_\alpha)).$$

4

Examples of Pert(A)

In this chapter we will look at some examples of the perturbation semigroup where the focus will be on examples of homomorphisms between the perturbation semigroup. This results in the following subjects:

- Commutative AF-algebras
- Homomorphisms between $\mathbb{C}^n$ and $M_n(\mathbb{C})$
- Homomorphism $\varphi : M_2(\mathbb{C}) \to M_4(\mathbb{C})$
- Continuous functions with values in A

4.1 Commutative AF-algebras

In this section we will visualise the perturbation semigroup a bit more and combine this with the Bratteli diagrams.

We will do this for $\mathbb{C}^n$. We could see elements in $\mathbb{C}^n$ as vectors $(v_1, \ldots, v_n)$ but also as a matrices

$$\begin{pmatrix} v_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_n \end{pmatrix}.$$

The multiplication on $\mathbb{C}^n$ is then defined as matrix multiplication.
Remark that $\mathbb{C}^n$ is commutative, so the opposite algebra is the same. The following lemma gives the perturbation conditions of $\mathbb{C}^n$.

Lemma 4.1. *$\sum_{i,j} \lambda_{ij} e_i \otimes e_j$ is in $\mathrm{Pert}(\mathbb{C}^n)$ if $\lambda_{ii} = 1$ and $\lambda_{ij} = \overline{\lambda}_{ji}$ for $1 \leq i, j \leq n$.*

We can represent an element in $\mathrm{Pert}(\mathbb{C}^n)$ as a matrix (a_{ij}). The entry a_{ij} is equal to the coefficient λ_{ij}. Because of the conditions from Lemma 4.1, the elements on the diagonal are 1 and the triangles below and above the diagonal are each others complex conjugates.

An example is $\mathrm{Pert}(\mathbb{C}^3)$ represented by the matrix

$$\begin{pmatrix} 1 & \lambda_{12} & \lambda_{13} \\ \overline{\lambda}_{12} & 1 & \lambda_{23} \\ \overline{\lambda}_{13} & \overline{\lambda}_{23} & 1 \end{pmatrix}.$$

Another way of representing $\mathrm{Pert}(\mathbb{C}^n)$ is by a graph. We draw n nodes and connect nodes i and j if $\lambda_{ij} \neq 0$. The result is a complete graph, as showed in Figure 4.1.

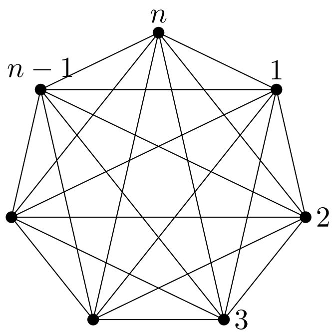

Figure 4.1: The graph of $\mathrm{Pert}(\mathbb{C}^n)$

We will continue with homomorphism between the perturbation semigroups. Suppose $\varphi : \mathbb{C}^n \to \mathbb{C}^m$ is a homomorphism. Then there exists a homomorphism $\varphi^* : \mathrm{Pert}(\mathbb{C}^n) \to \mathrm{Pert}(\mathbb{C}^m)$. We will look in more detail to this homomorphism.

We have two interesting cases, namely $m < n$ or $m > n$.

$\boldsymbol{m < n}$: We start with an example. Let

$$\varphi : \mathbb{C}^3 \to \mathbb{C}^2, \qquad (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_2)$$

be a homomorphism with corresponding Bratteli diagram given in Figure 4.2. This defines a homomorphism $\varphi^* : \mathrm{Pert}(\mathbb{C}^3) \to \mathrm{Pert}(\mathbb{C}^2)$. In the matrix notation this results in

$$\begin{pmatrix} 1 & \lambda_{12} & \lambda_{13} \\ \overline{\lambda}_{12} & 1 & \lambda_{23} \\ \overline{\lambda}_{13} & \overline{\lambda}_{23} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \lambda_{12} \\ \overline{\lambda}_{12} & 1 \end{pmatrix}$$

and for the basis

$$\lambda_{12} e_1 \otimes e_2 + \lambda_{13} e_1 \otimes e_3 + \lambda_{23} e_2 \otimes e_3 \mapsto \lambda_{12} e_1 \otimes e_2.$$

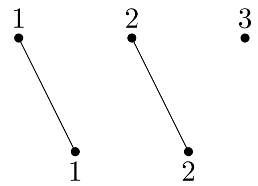

Figure 4.2: Bratteli diagram of $\varphi : \mathbb{C}^3 \to \mathbb{C}^2$

The complex conjugated terms are omitted for simplicity. This results in the graphs of Figure 4.3.

Figure 4.3: Perturbation graphs of $\varphi^* : \mathrm{Pert}(\mathbb{C}^3) \to \mathrm{Pert}(\mathbb{C}^2)$

Now we go to the general case. The homomorphism $\mathbb{C}^n$ to $\mathbb{C}^m$ is defined by omitting $n - m$ terms. The complete graph of $\mathrm{Pert}(\mathbb{C}^n)$ as showed above is send to the subgraph in Figure 4.4.

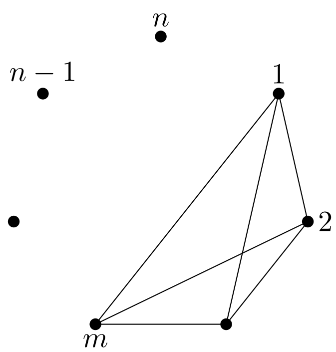

Figure 4.4: Perturbation graph from homomorphism $\varphi : \mathbb{C}^n \to \mathbb{C}^m$

$\boldsymbol{m > n}$: In this case elements needs to be recycled. There are different ways to do this, which is showed in the next example.

$$\varphi : \mathbb{C}^3 \to \mathbb{C}^5, \qquad (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_2, \lambda_3, \lambda_1, \lambda_2)$$
$$\tilde{\varphi} : \mathbb{C}^3 \to \mathbb{C}^5, \qquad (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_3)$$

The perturbation semigroup is in the case of φ of the form

$$\lambda_{12}e_1\otimes e_2+\lambda_{13}e_1\otimes e_3+\lambda_{12}e_1\otimes e_5+\lambda_{23}e_2\otimes e_3+\lambda_{12}e_4\otimes e_2+\lambda_{12}e_4\otimes e_3+\lambda_{23}e_5\otimes e_3+\lambda_{12}e_4\otimes e_5$$

with graph as showed in Figure 4.5.

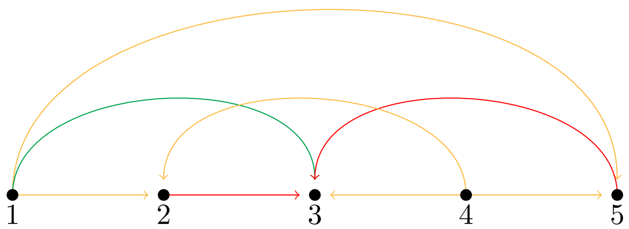

Figure 4.5: Perturbation graph from homomorphism φ

For $\tilde{\varphi}$ it looks as

$$\lambda_{12}e_1 \otimes e_2 + \lambda_{13}e_1 \otimes e_3 + \lambda_{13}e_1 \otimes e_4 + \lambda_{13}e_1 \otimes e_5 + \lambda_{23}e_2 \otimes e_3 + \lambda_{23}e_2 \otimes e_4 + \lambda_{23}e_2 \otimes e_5$$

with the graph from Figure 4.6 below.

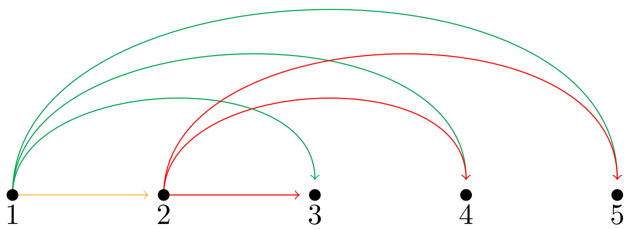

Figure 4.6: Perturbation graph from homomorphism $\tilde{\varphi}$

The Bratteli diagrams these homomorphisms φ and $\tilde{\varphi}$ are given in Figure 4.7

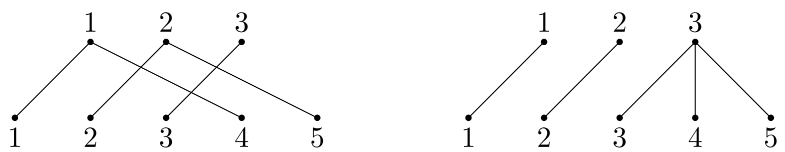

Figure 4.7: Bratteli diagrams of φ and $\tilde{\varphi}$

If we compare these Bratteli diagrams with the perturbation graph, we see that all the nodes on the second line that are not connected with each other in the Bratteli diagram are connected in the perturbation graph.

We can do this also for the general case. If we have a homomorphism given by the Bratteli diagram, we can derive the perturbation graph. Namely we connect the nodes that are not connected in the Bratelli diagram. If we contemplate it, this can be quickly understood. The points that are connected have the same coefficient λ_i. Then in the tensor product $e_p \otimes e_q$ gets coefficient λ_{ii}, but this is 1 and these lines are not drawn in the perturbation graph. Lines from a node to itself are omitted, so they are also not in the graph. Nodes that are not connected have different coefficients so they are connected in the graph.
This is also the case for $m < n$. None of the nodes on the second line of the Bratteli diagrams are connected with another node. The result is that the perturbation graph is a complete graph, which can be seen in the graphs above.

We can also combine the Bratteli diagrams and the perturbation graph. Between the perturbation graphs we draw the lines of the Bratteli diagram. Suppose we have the perturbation graph of just two points i and j and the perturbationgraph of $i_1, \dots, i_k, j_1, \dots j_l$. The homomorphism sends i to $i_1, \dots, i_k$ and j to $j_1, \dots, j_l$. The Bratteli diagram follows. From this Bratelli diagram follows then the perturbation graph on the next level, namely only the points that do not come from the same node in the Bratteli diagram are connected. This gives the diagram as showed in Figure 4.8.

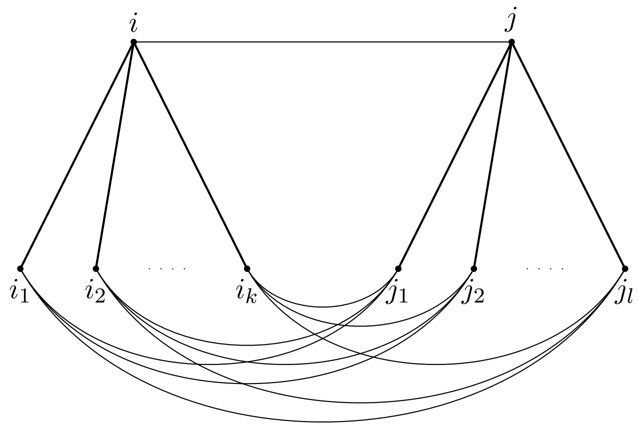

Figure 4.8: Bratteli diagram combined with the perturbation graph

We can extend this diagram by adding a homomorphism, so for example we have the following homomorphism.

$$\mathbb{C}^3 \xrightarrow{\varphi_1} \mathbb{C}^4 \xrightarrow{\varphi_2} \mathbb{C}^5$$

$$\varphi_1(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_2, \lambda_3, \lambda_3) \qquad \varphi_2(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_2)$$

The result is the diagram in Figure 4.9. In the same way all other homomorphisms between the algebras $\mathbb{C}^i$ can be visualized. The direct limit of these homomorphisms

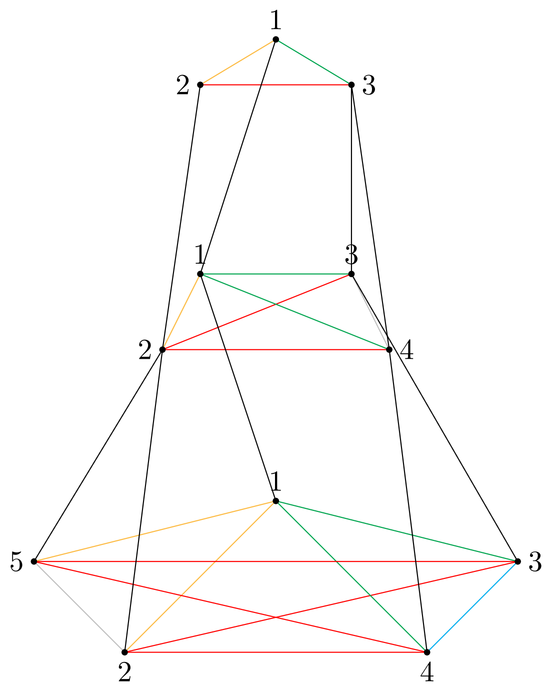

Figure 4.9: Example of the first steps of a diagram of the perturbation graphs of an AF-algebra

is a commutative AF-algebra. Hence we have a way to represent the direct limit of Pert($\mathbb{C}^n$) if the direct limit of the commutative algebras $\mathbb{C}^n$ is known.

Conjecture 4.2. *Such a diagram also exists for the perturbation semigroup of non-commutative AF-algebras.*

4.2 Homomorphism between $\mathbb{C}^n$ and $M_n(\mathbb{C})$

We will continue our examples with homomorphisms from $\mathbb{C}^n$ to matrices $\mathbb{M}_n(\mathbb{C})$. We start with the easy homomorphism $\mathbb{C}^2 \to M_2(\mathbb{C})$ that is given by

$$(\lambda, \mu) \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

If $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \otimes \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{pmatrix}$ is in Pert($M_2(\mathbb{C})$), then it is of the form

$$\begin{pmatrix} \lambda_1\lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1\mu_2 & 0 & 0 \\ 0 & 0 & \mu_1\lambda_2 & 0 \\ 0 & 0 & 0 & \mu_1\lambda_2 \end{pmatrix}.$$

Because this matrix needs to satisfy the conditions of the perturbation semigroup, the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1\mu_2 & 0 & 0 \\ 0 & 0 & \overline{\lambda_1\mu_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

remains. If we write this matrix on the basis of eigenvectors, we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Re(\lambda\delta) & i\Im(\lambda\delta) \\ 0 & 0 & i\Im(\lambda\delta) & \Re(\lambda\delta) \end{pmatrix}.$$

This matrix is multiplicative.

We extend this example to a homomorphism between $\varphi : \mathbb{C}^3 \to M_3(\mathbb{C})$. Calculating the perturbation semigroup of $\varphi(\lambda_1, \lambda_2, \lambda_3) = \left(\begin{smallmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{smallmatrix}\right)$ results after basis transformation in

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Re(\lambda_1\mu_2) & 0 & 0 & i\Im(\lambda_1\mu_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Re(\lambda_1\mu_3) & 0 & 0 & i\Im(\lambda_1\mu_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Re(\lambda_2\mu_3) & 0 & 0 & i\Im(\lambda_2\mu_3) \\ 0 & 0 & 0 & i\Im(\lambda_1\mu_2) & 0 & 0 & \Re(\lambda_1\mu_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & i\Im(\lambda_1\mu_3) & 0 & 0 & \Re(\lambda_1\mu_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & i\Im(\lambda_2\mu_3) & 0 & 0 & \Re(\lambda_2\mu_3) \end{pmatrix}.$$

From these examples can the general form be deduced. Suppose we have a homomorphism $\mathbb{C}^n \to M_n(\mathbb{C})$, with the corresponding Bratteli diagram as given in Figure 4.10.

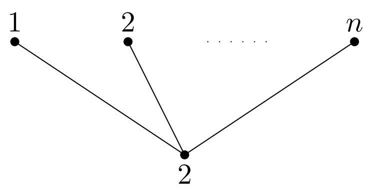

Figure 4.10: Bratteli diagram of $\mathbb{C}^n \to M_n(\mathbb{C})$

Then the general form of this homomorphism is given in the next proposition.

Proposition 4.3. *Let* $\mathbb{C}^n \to M_n(\mathbb{C})$ *be a homomorphism defined as* $(\lambda_1, \ldots, \lambda_n) \mapsto$

diag$(\lambda_1, \ldots, \lambda_n)$. *Then* Pert$(M_n(\mathbb{C}))$ *is of the form*

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \Re(\Lambda) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & J_n \\ 0 & J_n & 0 \end{pmatrix} \Im(\Lambda)$$

where $\Lambda = \operatorname{diag}(\lambda_1\mu_2, \ldots, \lambda_1\mu_n, \lambda_2\mu_3, \ldots, \lambda_2\mu_n, \ldots, \lambda_{n-1}\mu_n)$ *and* $J_n = iI_n$.

Proof. Suppose Pert$(M_n(\mathbb{C})) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \otimes \operatorname{diag}(\mu_1, \ldots, \mu_n)$. Then the normalisation condition gives the relation $\lambda_i\mu_i = 1$. The elements $\lambda_i\mu_i$ are on position $(i-1)n+i, (i-n)+i$ in the matrices in Pert$(M_n(\mathbb{C}))$. The eigenvectors $e_i \otimes e_i = e_{(i-1)n+i}$ are send by these points in the matrix to itself. We have n such eigenvectors, so this gives

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the self-adjointness condition follows that the relation $\lambda_i\mu_j = \overline{\lambda_j\mu_i}$ should hold. So on position $(i-1)n+j, (i-1)n+j$ is $\lambda_i\mu_j$ and on position $(j-1)n+i, (j-1)n+i$ there is $\overline{\lambda_j\mu_i}$ which are equal. The eigenvector $f_{ij}^+ = e_i \otimes e_j + e_j \otimes e_i = e_{(i-1)n+j} + e_{(j-1)n+i}$ is send by the original matrix to $\lambda_i\mu_j e_{(i-1)n+j} + \overline{\lambda_j\mu_i} e_{(j-1)n+i} = (\Re(\lambda_i\mu_j) + i\Im(\lambda_i\mu_j))e_{(i-1)n+j} + (\Re(\lambda_i\mu_j) - i\Im(\lambda_i\mu_j))e_{(j-1)n+i} = \Re(\lambda_i\mu_j)f_{ij}^+ + i\Im(\lambda_i\mu_j)f_{ij}^-$. The same calculation can be done for f_{ij}^-. If we put these results together, we get exactly the matrix that is asked for. □

Remark 4.4. Remark that here J_n gives a complex structure where a complex structure on a real vector space V is defined as a linear endomorphism J of V such that $J^2 = -\operatorname{Id}$.

4.3 Homomorphism between Pert$(M_2(\mathbb{C}))$ and Pert$(M_4(\mathbb{C}))$

In Section 2.2 we showed the results of calculating the perturbation semigroup of matrix algebras. We will take a closer look on the matrix $M_2(\mathbb{C})$ and combine it with a homomorphism between these matrix algebra and $M_4(\mathbb{C})$.

4.3.1 Explicit calculation of $M_2(\mathbb{C})$

From the rules in Section 2.2 we see that the perturbation semigroup of the algebra $M_2(\mathbb{C})$ looks in the canonical basis as

$$\hat{C} = \begin{pmatrix} x_1 & z_3 & \overline{z_3} & 1-x_1 \\ z_1 & z_2 & \overline{z_5} & -z_1 \\ \overline{z_1} & z_5 & \overline{z_2} & -\overline{z_1} \\ x_2 & z_4 & \overline{z_4} & 1-x_2 \end{pmatrix} \qquad \text{with } x_1, x_2 \in \mathbb{R}, z_1, \ldots, z_5 \in \mathbb{C}.$$

We write $\hat{\Omega}$ on the basis of eigenvalues

$$f_1 = e_1 \otimes e_1 + e_2 \otimes e_2 \qquad f_2 = e_2 \otimes e_2$$
$$f_3 = e_1 \otimes e_2 + e_2 \otimes e_1 \qquad f_4 = e_1 \otimes e_2 - e_2 \otimes e_1$$

to diagonalize it. This results in $\Omega = \text{diag}(1, 1, 1, -1)$ where diag is the diagonal matrix with $1, 1, 1$ and -1 on the diagonal. Calculating $C = M^{-1}\hat{C}M$ with

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

gives A written on this basis of eigenvalues. This results in

$$C = \begin{pmatrix} 1 & 1 - x_1 & 2\Re(z_3) & 2i\Im(z_3) \\ 0 & x_1 - x_2 & 2\Re(-z_3 + z_4) & 2i\Im(-z_3 + z_4) \\ 0 & -\Re(z_1) & \Re(z_2 + z_5) & i\Im(z_2 + z_5) \\ 0 & -i\Im(z_1) & i\Im(z_2 - z_5) & \Re(z_2 - z_5) \end{pmatrix}.$$

We could also do this the other way around, so suppose we have a matrix that is written on the basis of eigenvectors of $\hat{\Omega}$, so it is of the form

$$\begin{pmatrix} 1 & v_1 & v_2 & iw \\ 0 & e_1 & e_2 & if_1 \\ 0 & e_3 & e_4 & if_2 \\ 0 & ig_1 & ig_2 & h \end{pmatrix}. \tag{4.1}$$

Then the original matrix on the canonical basis is given by

$$\begin{pmatrix} 1 - v_1 & \frac{1}{2}(v_2 + iw) & \frac{1}{2}(v_2 - iw) & v_1 \\ -(e_3 + ig_1) & \frac{1}{2}(e_4 + h + i(f_2 + g_2)) & \frac{1}{2}(e_4 - h - i(f_2 - g_2)) & e_3 + ig_1 \\ -(e_3 - ig_1) & \frac{1}{2}(e_4 - h + i(f_2 - g_2)) & \frac{1}{2}(e_4 + h - i(f_2 + g_2)) & e_3 - ig_1 \\ 1 - v_1 - e_1 & \frac{1}{2}(v_2 + e_2 + i(w + f_1)) & \frac{1}{2}(v_2 + e_2 - i(w + f_1) & v_1 + e_1 \end{pmatrix}.$$

4.3.2 Homomorphism between Pert($M_2(\mathbb{C})$) and Pert($M_4(\mathbb{C})$)

We know now precise how the perturbation semigroup of $M_2(\mathbb{C})$ looks, so we are ready to determine how the perturbation semigroup looks when we first apply a homomorphisms on $M_2(\mathbb{C})$. So we will give an explicit calculation of the perturbation semigroup of $\varphi(M_2(\mathbb{C}))$.

Suppose we have $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B^\circ = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$ with $C = A \otimes B^\circ \in \text{Pert}(M_2(\mathbb{C}))$. Then this matrix is of the form

$$C = A \otimes B^\circ = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{21} & \overline{a_{11}b_{21}} & 1 - a_{11}b_{11} \\ a_{11}b_{12} & a_{11}b_{22} & \overline{a_{21}b_{21}} & -\overline{a_{11}b_{12}} \\ \overline{a_{11}b_{12}} & a_{21}b_{21} & \overline{a_{11}b_{22}} & -\overline{a_{11}b_{12}} \\ a_{21}b_{12} & a_{21}b_{22} & \overline{a_{21}b_{22}} & 1 - a_{21b_12} \end{pmatrix} = \begin{pmatrix} x_1 & z_3 & \overline{z_3} & 1 - x_1 \\ z_1 & z_2 & \overline{z_5} & -z_1 \\ \overline{z_1} & z_5 & \overline{z_2} & -\overline{z_1} \\ x_2 & z_4 & \overline{z_4} & 1 - x_2 \end{pmatrix}. \tag{4.2}$$

If we first apply the homomorphism φ^* on A and B°, this gives

$$\varphi^*(A\otimes B^\circ)=\varphi(A)\otimes\varphi^\circ(B^\circ)=\begin{pmatrix}a_{11}&a_{12}&0&0\\a_{21}&a_{22}&0&0\\0&0&a_{11}&a_{12}\\0&0&a_{21}&a_{22}\end{pmatrix}\otimes\begin{pmatrix}b_{11}&b_{21}&0&0\\b_{12}&b_{22}&0&0\\0&0&b_{11}&b_{21}\\0&0&b_{12}&b_{22}\end{pmatrix}.$$

We calculate the tensor product and apply the the perturbation conditions on it. The result is the matrix on the next page. We use equation (4.2) to rewrite the matrix with entries $x_1, x_2 \in \mathbb{R}$ and $z_1, \dots, z_5 \in \mathbb{C}$ as showed below.

$$\begin{pmatrix}
x_1 & z_3 & 0 & 0 & \overline{z_3} & 1-x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
z_1 & z_2 & 0 & 0 & \overline{z_5} & -z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & x_1 & z_3 & 0 & 0 & \overline{z_3} & 1-x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & z_1 & z_2 & 0 & 0 & \overline{z_5} & -z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\overline{z_1} & z_5 & 0 & 0 & \overline{z_2} & -\overline{z_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
x_2 & z_4 & 0 & 0 & \overline{z_4} & 1-x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \overline{z_1} & z_5 & 0 & 0 & \overline{z_2} & -\overline{z_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & x_2 & z_4 & 0 & 0 & \overline{z_4} & 1-x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & z_3 & 0 & 0 & \overline{z_3} & 1-x_1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_1 & z_2 & 0 & 0 & \overline{z_5} & -z_1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & z_3 & 0 & 0 & \overline{z_3} & 1-x_1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_1 & z_2 & 0 & 0 & \overline{z_5} & -z_1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{z_1} & z_5 & 0 & 0 & \overline{z_2} & -\overline{z_1} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & z_4 & 0 & 0 & \overline{z_4} & 1-x_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{z_1} & z_5 & 0 & 0 & \overline{z_2} & -\overline{z_1}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & z_4 & 0 & 0 & \overline{z_4} & 1-x_2
\end{pmatrix}$$

$$\begin{pmatrix}
a_{11}b_{11} & a_{11}b_{21} & 0 & 0 & \overline{a_{11}b_{21}} & 1-a_{11}b_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{11}b_{12} & a_{11}b_{22} & 0 & 0 & \overline{a_{21}b_{21}} & -a_{11}b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{11}b_{11} & a_{11}b_{21} & 0 & 0 & \overline{a_{11}b_{21}} & 1-a_{11}b_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{11}b_{12} & a_{11}b_{22} & 0 & 0 & \overline{a_{21}b_{21}} & -a_{11}b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{a_{11}b_{12}} & a_{21}b_{21} & 0 & 0 & \overline{a_{11}b_{22}} & -\overline{a_{11}b_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21}b_{12} & a_{21}b_{22} & 0 & 0 & \overline{a_{21}b_{22}} & 1-a_{21}b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \overline{a_{11}b_{12}} & a_{21}b_{21} & 0 & 0 & \overline{a_{11}b_{22}} & -\overline{a_{11}b_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{21}b_{12} & a_{21}b_{22} & 0 & 0 & \overline{a_{21}b_{22}} & 1-a_{21}b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11}b_{11} & a_{11}b_{21} & 0 & 0 & \overline{a_{11}b_{21}} & 1-a_{11}b_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11}b_{12} & a_{11}b_{22} & 0 & 0 & \overline{a_{21}b_{21}} & -a_{11}b_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11}b_{11} & a_{11}b_{21} & 0 & 0 & \overline{a_{11}b_{21}} & 1-a_{11}b_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11}b_{12} & a_{11}b_{22} & 0 & 0 & \overline{a_{21}b_{21}} & -a_{11}b_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{a_{11}b_{12}} & a_{21}b_{21} & 0 & 0 & \overline{a_{11}b_{22}} & -\overline{a_{11}b_{12}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{21}b_{12} & a_{21}b_{22} & 0 & 0 & \overline{a_{21}b_{22}} & 1-a_{21}b_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{a_{11}b_{12}} & a_{21}b_{21} & 0 & 0 & \overline{a_{11}b_{22}} & -\overline{a_{11}b_{12}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{21}b_{12} & a_{21}b_{22} & 0 & 0 & \overline{a_{21}b_{22}} & 1-a_{21}b_{12}
\end{pmatrix}$$

We see that the matrix $A \otimes B^\circ$ is stretched a bit and that the two matrices $A \otimes B^\circ$ are mingled with each other, but the order of elements in $A \otimes B^\circ$ is not changed.

We can also give the homomorphism explicitly. The matrix $A \otimes B^\circ$ can be written as $\sum_{i,j,k,l=1}^{n} a_{ij}b_{kl}e_{ij} \otimes e_{kl}^\circ$. If we choose one entry of the matrix, this entry is given by a certain $a_{ij}b_{kl}e_{ij} \otimes e_{kl}^\circ$. The homomorphism sends this element to $a_{ij,kl}(e_{ij} + e_{i+n,j+n}) \otimes (e_{ij} + e_{i+n,j+n})^\circ$, which looks like a diagonal with four times $a_{ij}b_{ij}$ on this diagonal and zero's between them where $e_{ij} \otimes e_{ij}^\circ$ is the starting point of the diagonal.

Next we transform the matrix to the standard form

$$\begin{pmatrix} 1 & v & iw \\ 0 & E & iF \\ 0 & iG & H \end{pmatrix}.$$

To do this, we change the canonical basis to the basis of eigenvalues of $\hat{\Omega}$. We use the matrix M, given below, to calculate $M^{-1}\varphi^*(A \otimes B^\circ)M$.

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Remark that the order of putting the eigenvectors in the columns of the matrix M has an effect on how $M^{-1}\varphi^*(A \otimes B^\circ)M$ will look. Also the choice of which vector will be replaced by $\sum e_i \otimes e_i$ effects the result. For now we have the order

$$f_1 + f_2 + f_3 + f_4, f_2, f_3, f_4, f_{12}^+, f_{13}^+, f_{14}^+, f_{23}^+, f_{24}^+, f_{34}^+, f_{12}^-, f_{13}^-, f_{14}^-, f_{23}^-, f_{24}^-, f_{34}^-$$

and f_1 is replaced by $f_1 + f_2 + f_3 + f_4$. The result is the following matrix on the next page.

$$
\begin{pmatrix}
1 & 1-x_1 & 0 & 0 & 2\Re(z_3) & 0 & 0 & 0 & 0 & 0 & 2i\Im(z_3) & 0 & 0 & 0 & 0 & 0 \\
0 & x_1-x_2 & 0 & 0 & -2\Re(z_3-z_4) & 0 & 0 & 0 & 0 & 0 & 2i\Im(-z_3+z_4) & 0 & 0 & 0 & 0 & 0 \\
0 & x_1-1 & x_1 & 1-x_1 & -2\Re(z_3) & 0 & 0 & 0 & 0 & 2\Re(z_3) & -2i\Im(z_3) & 0 & 0 & 0 & 0 & 2i\Im(z_3) \\
0 & x_1-1 & x_2 & 1-x_2 & -2\Re(z_3) & 0 & 0 & 0 & 0 & 2\Re(z_4) & -2i\Im(z_3) & 0 & 0 & 0 & 0 & 2i\Im(z_4) \\
0 & -\Re(z_1) & 0 & 0 & \Re(z_2+z_5) & 0 & 0 & 0 & 0 & 0 & i\Im(z_2+z_5) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_1 & \Re(z_3) & \Re(z_3) & 1-x_1 & 0 & 0 & 0 & i\Im(z_3) & -i\Im(z_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Re(z_1) & \Re(z_2) & \Re(z_5) & -\Re(z_1) & 0 & 0 & i\Im(z_1) & i\Im(z_2) & -i\Im(z_5) & -i\Im(z_1) & 0 \\
0 & 0 & 0 & 0 & 0 & \Re(z_1) & \Re(z_5) & \Re(z_2) & -\Re(z_1) & 0 & 0 & -i\Im(z_1) & i\Im(z_5) & -i\Im(z_2) & i\Im(z_1) & 0 \\
0 & 0 & 0 & 0 & 0 & x_2 & \Re(z_4) & \Re(z_4) & 1-x_2 & 0 & 0 & 0 & i\Im(z_4) & -i\Im(z_4) & 0 & 0 \\
0 & 0 & \Re(z_1) & -\Re(z_1) & 0 & 0 & 0 & 0 & 0 & \Re(z_2+z_5) & 0 & 0 & 0 & 0 & 0 & i\Im(z_2+z_5) \\
0 & -i\Im(z_1) & 0 & 0 & i\Im(z_2-z_5) & 0 & 0 & 0 & 0 & 0 & \Re(z_2-z_5) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i\Im(z_3) & -i\Im(z_3) & 0 & 0 & 0 & x_1 & \Re(z_3) & \Re(z_3) & 1-x_1 & 0 \\
0 & 0 & 0 & 0 & 0 & i\Im(z_1) & i\Im(z_2) & -i\Im(z_5) & -i\Im(z_1) & 0 & 0 & \Re(z_1) & \Re(z_2) & \Re(z_5) & -\Re(z_1) & 0 \\
0 & 0 & 0 & 0 & 0 & -i\Im(z_1) & i\Im(z_5) & -i\Im(z_2) & i\Im(z_1) & 0 & 0 & \Re(z_1) & \Re(z_5) & \Re(z_2) & -\Re(z_2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i\Im(z_4) & -i\Im(z_4) & 0 & 0 & 0 & x_2 & \Re(z_4) & \Re(z_4) & 1-x_2 & 0 \\
0 & 0 & i\Im(z_1) & -i\Im(z_1) & 0 & 0 & 0 & 0 & 0 & i\Im(z_2-z_5) & 0 & 0 & 0 & 0 & 0 & \Re(z_2-z_5)
\end{pmatrix}
$$

In (4.1) was the common form for a matrix in the standard form given. This gives

$$
\begin{pmatrix}
1 & v_1 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 & 0 & iw & 0 & 0 & 0 & 0 & 0 \\
0 & e_1 & 0 & 0 & e_2 & 0 & 0 & 0 & 0 & 0 & if_1 & 0 & 0 & 0 & 0 & 0 \\
0 & -v_1 & 1-v_1 & v_1 & -v_2 & 0 & 0 & 0 & 0 & v_2 & -iw & 0 & 0 & 0 & 0 & iw \\
0 & -v_1 & 1+e_1-v_1 & v_1-e_1 & -v_2 & 0 & 0 & 0 & 0 & v_2+e_2 & -iw & 0 & 0 & 0 & 0 & i(f_1+w) \\
0 & e_3 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & if_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-v_1 & \frac{1}{2}v_2 & \frac{1}{2}v_2 & v_1 & 0 & 0 & 0 & \frac{1}{2}iw & -\frac{1}{2}iw & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -e_3 & \frac{1}{2}(e_4+h) & \frac{1}{2}(e_4-h) & e_3 & 0 & 0 & -ig_1 & \frac{1}{2}i(f_2+g_2) & -\frac{1}{2}i(f_2-g_2) & ig_1 & 0 \\
0 & 0 & 0 & 0 & 0 & -e_3 & \frac{1}{2}(e_4-h) & \frac{1}{2}(e_4+h) & e_3 & 0 & 0 & -ig_1 & \frac{1}{2}i(f_2-g_2) & -\frac{1}{2}i(f_2+g_2) & -ig_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-v_1-e_1 & \frac{1}{2}(v_2+e_2) & \frac{1}{2}(v_2+e_2) & v_1+e_1 & 0 & 0 & 0 & \frac{1}{2}i(w+f_1) & -\frac{1}{2}i(w+f_1) & 0 & 0 \\
0 & 0 & -e_3 & e_3 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & if_2 \\
0 & ig_1 & 0 & 0 & ig_2 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}iw & -\frac{1}{2}iw & 0 & 0 & 0 & 1-v_1 & \frac{1}{2}v_2 & \frac{1}{2}v_2 & v_1 & 0 \\
0 & 0 & 0 & 0 & 0 & -ig_1 & \frac{1}{2}i(f_2+g_2) & -\frac{1}{2}i(f_2-g_2) & ig_1 & 0 & 0 & -e_3 & \frac{1}{2}(e_4+h) & \frac{1}{2}(e_4-h) & e_3 & 0 \\
0 & 0 & 0 & 0 & 0 & -ig_1 & \frac{1}{2}i(f_2-g_2) & -\frac{1}{2}i(f_2+g_2) & -ig_1 & 0 & 0 & -e_3 & \frac{1}{2}(e_4-h) & \frac{1}{2}(e_4+h) & e_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i(w+f_1) & -\frac{1}{2}i(w+f_1) & 0 & 0 & 0 & 1-v_1-e_1 & \frac{1}{2}(v_2+e_2) & \frac{1}{2}(v_2+e_2) & v_1+e_1 & 0 \\
0 & 0 & -ig_1 & ig_1 & 0 & 0 & 0 & 0 & 0 & ig_2 & 0 & 0 & 0 & 0 & 0 & h
\end{pmatrix}
$$

If we take a closer look at this matrix, we can recognise a stretched standard form of the matrix $\mathrm{Pert}(A \otimes B^\circ) = \mathrm{Pert}(C)$ at places i, j with $i, j = 1, 2, 5, 11$. This entries are shown in the matrix below in blue. The original matrix $\mathrm{Pert}(C)$ can also be found. At place $6, 6$ starts the matrix $\Re\,\mathrm{Pert}(C)$. Next to this matrix at starting point $6, 12$ we find the matrix $i\Im\,\mathrm{Pert}(C)$. At starting point $12, 6$ we recognize $i\Im\,\mathrm{Pert}(C)$ and at $12, 12$ the matrix $\Re\,\mathrm{Pert}(C)$. Hence we can rewrite the matrix in the more compact form:

$$\begin{pmatrix}
1 & v_1 & 0 & 0 & v_2 & 0_4 & 0 & iw & 0_4 & 0 \\
0 & e_1 & 0 & 0 & e_2 & 0_4 & 0 & if_1 & 0_4 & 0 \\
0 & -v_1 & 1-v_1 & v_1 & -v_2 & 0_4 & v_2 & -iw & 0_4 & iw \\
0 & -v_1 & 1+e_1-v_1 & v_1-e_1 & -v_2 & 0_4 & v_2+e_2 & -iw & 0_4 & i(f_1+w) \\
0 & e_3 & 0 & 0 & e_4 & 0_4 & 0 & if_2 & 0_4 & 0 \\
0^4 & 0^4 & 0^4 & 0^4 & 0^4 & \Re\,\mathrm{Pert}(C) & 0^4 & 0^4 & \Im\,\mathrm{Pert}(C) & 0^4 \\
0 & 0 & -e_3 & e_3 & 0 & 0^4 & e_4 & 0 & 0^4 & if_2 \\
0 & ig_1 & 0 & 0 & ig_2 & 0^4 & 0 & h & 0^4 & 0 \\
0^4 & 0^4 & 0^4 & 0^4 & 0^4 & \Im\,\mathrm{Pert}(C) & 0^4 & 0^4 & \Re\,\mathrm{Pert}(C) & 0^4 \\
0 & 0 & -ig_1 & ig_1 & 0 & 0_4 & ig_2 & 0 & 0_4 & h
\end{pmatrix}.$$

Here 0_4 denotes $(0, 0, 0, 0)$ and $0^4 = (0, 0, 0, 0)^T$.

If we compute this also for the next homomorphisms, $M_4(\mathbb{C}) \to M_8(\mathbb{C})$ and so on, we can compute the perturbation semigroup of the UHF algebra. But as the matrix above is already quite tough, we have to omit to do this.

4.4 Continuous functions with values in A

In this section we will look at the C^*-algebra $C(X)$ of continuous functions on a compact hausdorff space X. We will compute the perturbation semigroup of $C(X)$ and that of $C(X, A)$ where A is a C^*-algebra.

We already saw in Example 1.23 that $C(X)$ is commutative and nuclear. Because $C(X)$ is commutative, we have $C(X) \cong C(X)^\circ$.

The C^*-algebras $C(X) \otimes A$ and $C(X, A)$ are isomorphic and this isomorphism is induced by the map Φ defined as $\Phi(\varphi, a) = \varphi(x)a = f(x)$. This results in $\Phi(\sum_{i=1}^n \varphi_i \otimes a_i) = \sum \varphi_i(x)a_i$. For the complete tensor, there is still an isomorphism which we will show in the next lemma.

Lemma 4.5. *Let $C(X)$ be the set of continuous functions on a compact hausdorff space and A a C^*-algebra. Then $C(X)\overline{\otimes}A \cong C(X, A)$.*

Proof. We show that $C(X) \otimes A$ is dense in $C(X, A)$. Take $f \in C(X, A)$ and $\varepsilon > 0$. X

is compact so there are m open sets U_i such that $\bigcup_{i=1}^m U_i = X$ and

$$\sup_{x_k, x_l \in U_j} \|f(x_k) - f(x_l)\| \leq \varepsilon.$$

Then by partition of unity we have $\varphi_1, \ldots, \varphi_m \in C(X, [0,1])$ such that $\sum_{i=1}^m \varphi_i(x) = 1$ for all $x \in X$ and $\text{supp}(\varphi_i) = \overline{\{x \in X \mid \varphi_i(x) \neq 0\}} \subset U_i$ for all i. For every i we take a $x_i \in U_i$ and define $g(x) = \sum_{i=1}^m \varphi_i \otimes f(x_i) \in C(X) \otimes A$. Then for all $x \in X$ we have

$$\begin{aligned}
\|f(x) - g(x)\| &= \|f(x) - \sum_{i=1}^m \varphi_i \otimes f(x_i)\| \\
&= \|1 \cdot f(x) - \sum_{i=1}^m \varphi_i(x) f(x_i)\| \\
&= \|\sum_{i=1}^m \varphi_i(x)(f(x) - f(x_i))\| \\
&\leq \sum_{i=1}^m \varphi_i(x) \sup_{x \in U_i} \|f(x) - f(x_i)\| \leq 1 \cdot \varepsilon.
\end{aligned}$$

Then $\|f - g\| = \sup_{x \in X} \|f(x) - g(x)\| \leq \varepsilon$, so density follows. Now we take the completion and the isomorphism results. □

Corollary 4.6. *Suppose X and Y are compact hausdorff spaces, then $C(X)\overline{\otimes}C(Y) \cong C(X \times Y)$.*

Proof. We take A from the previous lemma equal to $C(Y)$. Then $C(X)\overline{\otimes}C(Y) \cong C(X, C(Y))$. The map $F : C(X, C(Y)) \to C(X \times Y)$ defined as $F(f)(x, y) = (f(x))(y)$ for $f \in C(X, C(Y))$ gives an isomorphism, so the statement follows. □

Proposition 4.7. *The operator $\mu : C(X)\overline{\otimes}C(X) \longrightarrow C(X)$, $\mu(f \otimes g) = fg$, is continuous.*

Proof. We have

$$\|\mu(f \otimes g)\| = \|fg\| \leq \sup_{x \in X} \|fg(x)\|.$$

Here fg is a function in one variable, so we could take the supremum only over the compact set X. Hence μ is bounded, thus continuous. □

For the perturbation semigroup of $C(X)$ should the normalisation and self-adjointness condition hold. So

$$\text{Pert}(C(X)) = \left\{ \sum g_j \otimes h_j^\circ \in C(X)\overline{\otimes}C(X)^\circ \;\middle|\; \begin{array}{l} \sum g_j h_j = 1 \\ \sum g_j \otimes h_j^\circ = \sum h_j^* \otimes g_j^{\circ *} \end{array} \right\}.$$

By Corollary 4.6 we can identify a function $\sum g_j \otimes h_j^\circ \in C(X)\overline{\otimes}C(Y)$ with a function $f \in C(X \times Y)$. Explicitly, $f(x, y) = \sum(g_j(x), h_j(y))$.

We will take a look at the normalisation and self-adjointness conditions for this set $C(X \times Y)$.

- The normalisation condition states that $\sum g_j(x)h_j(x) = 1$, so only $f(x,x)$ needs to be normalized. Hence the normalisation condition can be formulated as the condition $f(x,x) = 1$ for all $x \in X$.
- We can formulate the self-adjointness condition for f as $f(x,y) = \overline{f(y.x)}$. Namely

$$f(x,y) = \sum g_j \otimes h_j^\circ = \sum h_j^* \otimes g_j^{\circ *} = f(y,x)^* = \overline{f(y,x)}.$$

The next theorem follows.

Theorem 4.8. *The perturbation semigroup of the continuous functions on compact hausdorff space X is given by*

$$\mathrm{Pert}(C(X)) = \left\{ f \in C(X \times X) \;\middle|\; f(x,x) = 1, f(x,y) = \overline{f(y,x)} \right\}.$$

There is also another way of writing $\mathrm{Pert}(C(X))$, namely

Corollary 4.9. *If X is a compact hausdorff space, then* $\mathrm{Pert}(C(X)) \cong (C_0(X \times X - \Delta))^{(s.a.)}$ *where $\Delta = \{(x,y) \in X \times Y \mid x = y\}$ is the diagonal of $X \times X$.*

Here $C_0(X)$ is the set of continuous functions with the property that for every $\varepsilon > 0$ there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. This means intuitive that f vanish at infinity. This set is isomorphic to the set $C_1(X)$, the set of functions that are 1 outside X, because we can add the function $f \equiv 1$ to every function in $C_0(X)$ to become in $C_1(X)$. $C_0(X)$ is a more common notation, so we will use this set.

Example 4.10. Let X be the interval $[-1,1]$. Then on the diagonal the functions are 1 and below and above the diagonal are each others complex conjugates. The result is drawn in Figure 4.11.

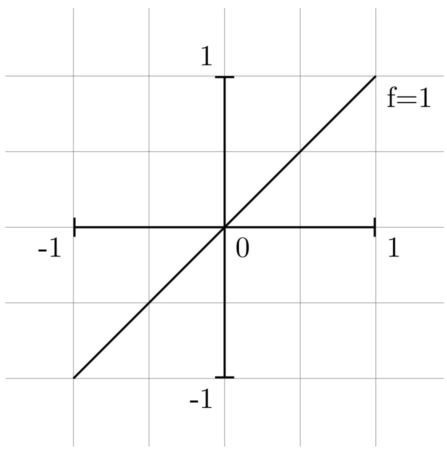

Figure 4.11: $\mathrm{Pert}(C(X))$ for $X = [-1,1]$

Next we will go to a more general case, namely the perturbation semigroup of $C(X, A)$. We start with a lemma.

Lemma 4.11. *The following isomorphism holds:*

$$C(X \times X) \cong C_0(X \times X - \Delta) \oplus C(\Delta)$$

Proof. We define an isomorphism $\varphi : C(X \times X) \to C_0(X \times X - \Delta) \oplus C(\Delta)$ by

$$\varphi(f(x,y)) = \begin{cases} f(x,y) - f(x,x) & x \neq y \\ f(x,x) & x = y. \end{cases}$$

We show that φ is multiplicative if $x \neq y$. The other isomorphism conditions are straightforward.

$$\begin{aligned} \varphi(f(x,y))\varphi(g(x,y)) &= (f(x,y) - f(x,x))(g(x,y) - g(x,x)) \\ &= f(x,y)g(x,y) - f(x,y)g(x,x) - f(x,x)g(x,y) - f(x,x)g(x,x) \\ &= f(x,y)g(x,y) - f(x,x)g(x,x) \\ &= \varphi(f(x,y)g(x,y)) \end{aligned}$$

The third equality follows because $f(x,x)$ and $g(x,x)$ are 0 on $C_0(X \times X - \Delta)$. □

Theorem 4.12. *Let X be a compact hausdorff space and A a C^*-algebra. Then*

$$\mathrm{Pert}(C(X, A)) \cong C(X, \mathrm{Pert}(A)) \times C_0(X \times X - \Delta, A \otimes A^\circ)^{(s.a.)}.$$

Proof. We start with rewriting $C(X, A) \otimes C(X, A)^\circ$ with use of Lemma 4.5 and Corollary 4.6:

$$\begin{aligned} C(X, A) \otimes C(X, A)^\circ &\cong (C(X) \otimes A) \otimes (C(X) \otimes A)^\circ \\ &\cong (C(X) \otimes C(X)) \otimes (A \otimes A^\circ) \\ &\cong C(X \times X) \otimes (A \otimes A^\circ) \\ &\cong C(X \times X, A \otimes A^\circ) \end{aligned}$$

We have $\mathrm{Pert}(C(X, A)) \subset C(X \times X, A \otimes A^\circ)$ so we are looking at functions of the form $f : X \times X \to A \otimes A^\circ$. For the normalisation condition we split $X \times X$ in two sets, namely in Δ and $X \times X - \Delta$. By Lemma 4.11 this results in

$$\mathrm{Pert}(C(X, A)) \subset C(\Delta, A \otimes A^\circ) \times C_0(X \times X - \Delta, A \otimes A^\circ).$$

Now we add the normalisation and self-adjointness condition. For the first part does this result in $C(\Delta, \mathrm{Pert}(A))$. Because $\Delta \cong X$, we get $C(X, \mathrm{Pert}(A))$. For the other part we use Theorem 4.8. There we saw that the normalisation condition only works on the diagonal Δ. Hence this part only needs to be self-adjoint, so the theorem follows. □

The self-adjointness condition gives $f(x,y) = f(y,x)^*$ because $f(x,y) \in A \otimes A^\circ$ which has an involution map $*$.

We use this theorem to calculate $\mathrm{Pert}(C(X) \otimes C(Y))$. If we take $A = C(Y)$ we get

$$\mathrm{Pert}(C(X, C(Y)) \cong C(X, \mathrm{Pert}(C(Y)) \times C_0(X \times X - \Delta, C(Y \times Y))^{(s.a.)}.$$

We can write this formula in a symmetric form, which is done below.

$$\begin{aligned}\mathrm{Pert}(C(X, C(Y)) &\cong C(X, \mathrm{Pert}(C(Y)) \times C_0(X \times X - \Delta_X, C(Y \times Y))^{(s.a.)} \\ &\cong (C(X, C(Y \times Y))^{\substack{(s.a.)\\(norm.)}} \times (C_0(X \times X - \Delta_X, C(Y \times Y)))^{(s.a.)}\end{aligned}$$

The first part gives the functions that are 1 on the diagonal, so we can omit this part. The elements on the diagonal of Y need to be 1 too, so this part can also be omitted. The result is $C_0(X \times X - \Delta_X) \otimes C_0(Y \times Y - \Delta_Y)^{(s.a.)}$, which can be written as

$$C_0(X \times Y) \times (X \times Y) - \Delta_{X \times Y})^{(s.a.)}.$$

This gives a symmetric formula for $\mathrm{Pert}(C(X) \otimes C(Y))$. Remark also that if we apply Corollary 4.9 on $\mathrm{Pert}(C(X) \otimes C(Y)$, we get the same result.

A

Appendix

A.1 Categories

Definition A.1. *A Category $\mathcal{C}$ consists of*

i. *a class of objects $A, B, C, \dots,$ denoted with* $\text{Obj}\,\mathcal{C}$*,*

ii. *a class of morphisms $f, g, h, \dots$ between all $X, Y \in \text{Obj}\ \mathcal{C}$ denoted with* $\hom_{\mathcal{C}}(X, Y)$*,*

iii. *a binary operation $\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$, the composition of morphisms. For $f \in \hom_{\mathcal{C}}(X, Y)$ and $g \in \hom_{\mathcal{C}}(Y, Z)$ we write $g \circ f = h \in \hom_{\mathcal{C}}(X, Z)$,*

such that the next two axioms hold

i. *For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.*

ii. *For every $X \in \text{Obj}\ \mathcal{C}$ there is a map $\text{Id}_X : X \to X$, the identity. For $f : X \to Y$ we have $f \circ \text{Id}_X = f = \text{Id}_Y \circ f$.*

Definition A.2. *Let $\mathcal{C}$ and $\mathcal{D}$ categories. Then a functor from $\mathcal{C}$ to $\mathcal{D}$, $F : \mathcal{C} \to \mathcal{D}$ is a mapping that*

- *sends every $X \in \text{Obj}\ \mathcal{C}$ to an $F(X) \in \text{Obj}\ \mathcal{D}$*
- *sends every $f \in \hom_{\mathcal{C}}(X, Y)$ to $F(f) \in \hom_{\mathcal{D}}(F(X), F(Y))$*

such that the following axioms hold:

i. for every $X \in \mathcal{C}$, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$

ii. For $X \xrightarrow{f} Y \xrightarrow{g} Z$ *we have* $F(g \circ f) = F(f) \circ F(g)$.

Definition A.3. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is called*

- *faithful if the maps* $F_{X,Y} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ *are injective for all* $X, Y \in \mathcal{C}$.
- *full if the maps* $F_{X,Y} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ *are surjective* $X, Y \in \mathcal{C}$.

Definition A.4. *A category is called a small category if the class of objects and the class of morphisms are sets.*

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