Noncommutative Geometry & SU(5) Grand Unification

Matthias Sars

August 2011

revised version, October 2011

Master's Thesis in Physics & Astronomy

Supervisor: Dr. Walter D. van Suijlekom

Department of Theoretical High Energy Physics
Institute for Mathematics, Astrophysics and Particle Physics
Radboud University Nijmegen

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Preface & Acknowledgments

This thesis is the result of my research project for the physics & astronomy research master programme at the Radboud University Nijmegen. The project is supervised by dr. Walter van Suijlekom and done at the department of theoretical high energy physics.

I could not have done this without the help and support from a lot of people. Therefore I would like to thank:

First of all, Walter van Suijlekom, my supervisor, for all his help, advice and patience.

Ronald Kleiss, professor of the department of theoretical high energy physics, for giving me the opportunity to do my project at this department, and to visit the École the Physique in Les Houches.

Everyone who helped me with the questions and problems I had. Especially I want to mention Thijs van den Broek and Koen van den Dungen, my colleague noncommutative geometers, who were always willing to help, and Wim Beenakker for his help with the physical context of my project.

And everyone from the theoretical and experimental high energy physics departments for the great atmosphere. It was a pleasure working here!

Chapter 1

Introduction

In present-day physics, it is believed that there are four fundamental interactions (or forces) in nature: the *electromagnetic force*, the *weak* and *strong nuclear forces* and *gravity*.

To describe gravity, we have a beautiful geometric theory: Einstein's *General Relativity Theory*. Its predictions fit extremely well with observations. An important problem of GRT is that it has not (yet) been quantized; until now it has only a classical approach.

The electromagnetic and the two nuclear interactions are described by the *Standard Model of elementary particle physics*, which also describes all known particles. Just like GRT, its predictions agree extremely well with experiments. However, there are some problems with the Standard Model. To name a few:

i. The *Higgs boson* that is needed in the Standard Model has not been observed experimentally (yet).

ii. The Standard Model does not describe gravity, since the latter has no quantum version yet, as we said before.

iii. Astronomical observations suggest the existence of forms of matter that cannot be described with the Standard Model, such as *dark matter*.

iv. The Standard Model does not explain the asymmetry between matter and antimatter.

v. The structure of the Standard Model seems quite ad hoc and arbitrary (see for example appendix B). This does not need to be a problem, maybe this is just the way nature works, but it would be nice to have an explanation.

vi. In figure 1.1 the dependence on the energy scale of the three coupling strengths of the Standard Model is plotted. You can see that the three lines almost, but not entirely, intersect at an energy scale of about 10^{13} to 10^{16} GeV. If they did intersect, one could assume unification of the three forces of the Standard Model at that energy scale. For this the same thing holds as the previous 'problem': it is not necessarily a problem if this unification is not a property of nature, but it would be nice if it is, because it suggests a deeper underlying structure.

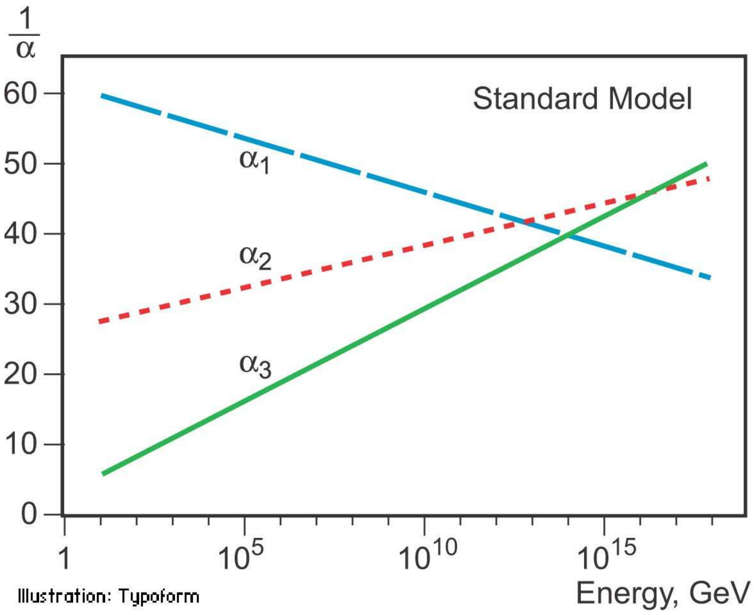

Figure 1.1: The running of the three coupling strengths of the Standard Model.[2]

There are many theories that try to deal with one or more of these problems. Two of them are relevant for this thesis: *SU(5) Grand Unification* and *Noncommutative Geometry*:

SU(5) Grand Unification In a *Grand Unified Theory*, the Standard Model interactions are unified at a high energy scale (*the GUT scale*). At lower energy scales this theory should be the same as the Standard Model. We will study the simplest GUT: the *SU(5) model* or *Georgi–Glashow model* (see chapter 7).

The nice thing is that from this theory the representations of the Standard Model can be reconstructed with little input (§ 7.3); one could say it solves 'problem' v above. Furthermore, the model can explain charge quantization and predicts an acceptable (but not exactly correct) value for the weak mixing angle.

However, this model is not problem-free either:

i. It does not agree with experiments. It predicts proton decay, but the predicted proton lifetime is excluded experimentally.

ii. As we said in problem vi, the Standard Model does not allow a *GUT relation* for the couplings (i.e. equality of the three couplings). This can be solved with certain extensions of the Standard Model, for example with *supersymmetry*, such that the three lines of figure 1.1 do intersect.

[2]The figure is taken from: *The Nobel Prize in Physics 2004, Popular Information*, `http://nobelprize.org/nobel_prizes/physics/laureates/2004/public.html`.

iii. GUTs need extra gauge bosons with a mass of about the GUT scale, so they predict a *big desert* where nothing happens between the electroweak and the GUT scale. This seems very unnatural.

Despite these problems, it is still interesting to study this model, because it is considered as a prototype for all GUTs.

Noncommutative Geometry *Noncommutative geometry* (NCG) is a mathematical theory which generalizes Riemannian geometry. (Riemannian spin geometry, to be more precise.) An introduction is given in chapters 2 to 4.

By considering equivalent ways to describe the same noncommutative space, we see a structure appearing that is similar to gauge theories (chapter 5). An action[3] functional can be assigned to this (the *spectral action*, § 5.2).

For a particular choice for a noncommutative space, this action is the action of the Standard Model, with a coupling to gravity (§ 6.2). One can say that with this model, Einstein's geometrical description of gravity is extended with the Standard Model.

This is not the only thing: the right hypercharges and the Higgs mechanism are obtained without putting them in. Furthermore, certain relations between the parameters of the Standard Model are obtained, such as the GUT relation for the couplings. These relations give an expression for the top quark and the Higgs mass at the GUT scale in terms of other parameters. If one assumes the big desert, these GUT scale mass parameters can be evolved down to the electroweak energy scale. This gives a prediction for the Higgs mass.

So far the good news. This model also has problems:

i. The action is classical; the full model has not been quantized, because of the gravitational terms.

ii. The spacetime manifold is taken to be Riemannian, i.e. locally Euclidean. In reality it is locally Minkowskian. Furthermore, it is taken to be compact.

iii. As we said before: the GUT relation for the couplings is not allowed by the Standard Model.

iv. Most of the calculated range of Higgs masses is excluded experimentally. As said before: the calculation is done with the assumption of a big desert. This means that if one would introduce new particles between the electroweak and the GUT energy scale, this prediction of possible Higgs masses can come out differently.

The aim of this project is to describe the $SU(5)$ model in terms of noncommutative geometry. This is discussed in the last chapter. We will see that it is possible to construct a spectral triple that gives the right gauge theory, but we have to introduce new fermions.

This brings us to some speculations about the implications of this model: Maybe the new fermions change the running of the couplings in such a way

[3]The word 'action' is meant in the physics sense: a functional of the particle fields, that is gauge invariant and leads to the equations of motion.

that Grand Unification is allowed. And maybe, if the GUT-scale parameters are evolved down, an acceptable Higgs mass is obtained.

However, it turns out that this model does not have the right symmetry breaking mechanism: the SU(5) theory is not broken to the Standard Model. From this, draw the conclusion that noncommutative geomerty does not allow the SU(5) model. This is in concordance with a similar conclusion which is drawn in [12] (see § 8.4).

Part I

From Noncommutative Geometry to the Standard Model

Chapter 2

C*-algebras, Gelfand Duality, Noncommutative Topology

In this chapter we introduce the idea of *noncommutative topology*. In chapter 4, it will be refined to noncommutative geometry. The notion of noncommutative topology is based on the theory of *C*-algebras*, which will be introduced here. This is not meant as a rigorous treatment, the results are given without proof. For the proofs an other background information, see [10], §§ 1.A and 1.1–3 and [14], § 1.

2.1 C*-algebras

Definition 2.1 (Banach algebra)**.** A *Banach algebra* $\mathscr{A}$ is an associative $\mathbb{C}$-algebra that is also a complete normed space that satisfies

$$\forall a, b \in \mathscr{A} : \quad \|ab\| \leq \|a\| \|b\|. \tag{2.1}$$

If $\mathscr{A}$ is unital, i.e. $\mathscr{A} \ni \mathbb{1}$, without loss of generality one can assume $\|\mathbb{1}\| = 1$, by rescaling the norm.

Definition 2.2 (*-algebra)**.** A **-algebra* or *involutive algebra* $\mathscr{A}$ is an associative algebra equipped with an *involution*: an antilinear map $(\cdot)^* : \mathscr{A} \to \mathscr{A}$ satisfying

$$\forall a, b \in \mathscr{A} : \quad a^{**} = a \quad \text{and} \quad (ab)^* = b^* a^*. \tag{2.2}$$

Definition 2.3 (C*-algebra)**.** A *C*-algebra* $\mathscr{A}$ is a Banach *-algebra that satisfies

$$\forall a \in \mathscr{A} : \quad \|a^* a\| = \|a\|^2. \tag{2.3}$$

This is called the *C*-identity*.

An algebra that satisfies all conditions to be a C*-algebra, except for completeness, is called a *pre-C*-algebra*.

Due to the C*-identity, the norm of a C*-algebra is unique.

Example 2.4. $\mathrm{M}_N(\mathbb{C})$, the algebra of $N \times N$ matrices, equipped with Hermitian conjugation and the operator norm,

$$\|a\|_{\mathrm{op}} = \max_{\substack{v \in \mathbb{C}^N \\ \|v\|=1}} \|av\|, \tag{2.4}$$

is a C*-algebra. ($\|\cdot\|$ denotes the standard norm on $\mathbb{C}^N$.)

Example 2.5. Let X be a compact Hausdorff topological space. Then $\mathrm{C}(X)$, the algebra of continuous $\mathbb{C}$-valued functions on X, with pointwise multiplication

$$(fg)(x) = f(x)g(x) \tag{2.5}$$

as algebra multiplication, equipped with pointwise complex conjugation

$$\overline{f}(x) = \overline{f(x)} \tag{2.6}$$

as the involution and the supremum norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|, \tag{2.7}$$

is a C*-algebra.

Example 2.6. Let M be a compact smooth manifold. Then the subalgebra $\mathrm{C}^\infty(M)$ of $\mathrm{C}(M)$, consisting of smooth $\mathbb{C}$-valued functions on M, is a pre-C*-algebra. It is dense in in $\mathrm{C}(M)$.

Example 2.7. Let $\mathscr{H}$ be a $\mathbb{C}$-Hilbert space. Then every closed subalgebra of $\mathrm{B}(\mathscr{H})$, the algebra of bounded operators on $\mathscr{H}$, equipped with the operator adjoint as the involution and the operator norm,

$$\|a\|_{\mathrm{op}} = \sup_{\substack{v \in \mathscr{H} \\ \|v\|=1}} \|av\|, \tag{2.8}$$

is a C*-algebra. ($\|\cdot\|$ denotes the norm induced by the inner product on $\mathscr{H}$.)

Note that this is a generalisation of Examples 2.4 (take $\mathscr{H} = \mathbb{C}^N$) and 2.5 (take $\mathscr{H} = \mathrm{L}^2(X)$). In fact, all C*-algebras are of the form of Example 2.7:

Theorem 2.8 (Gelfand–Naĭmark (2^{nd}))**.** *Any C*-algebra has a representation as a closed subalgebra of* $\mathrm{B}(\mathscr{H})$*, for some Hilbert-space* $\mathscr{H}$*.*

2.2 Gelfand Duality, Noncommutative Topology

Definition 2.9 (Character)**.** A *character* of a C*-algebra $\mathscr{A}$ is a nonzero homomorphism $\mathscr{A} \to \mathbb{C}$. We denote the set of all characters as $\Sigma(\mathscr{A})$.

For a commutative C*-algebra $\mathscr{A}$, $\Sigma(\mathscr{A})$ is called the *Gelfand spectrum*, which can be regarded as a topological space.

Definition 2.10 (Gelfand transform)**.** Let $\mathscr{A}$ be a commutative unital C*-algebra (or Banach algebra). The *Gelfand transformation* is the map

$$\widehat{\cdot}\colon \mathscr{A} \to \mathrm{C}\big(\Sigma(\mathscr{A})\big), \quad \widehat{a}(\mu) = \mu(a). \tag{2.9}$$

Theorem 2.11 (Gelfand–Naĭmark (1^{st}, unital case))**.** *For every commutative unital C*-algebra $\mathscr{A}$:*

$$\mathscr{A} \simeq \mathrm{C}\big(\Sigma(\mathscr{A})\big) \tag{2.10}$$

as C-algebras.*

This isomorphism is given by the Gelfand transformation. There is also a way to go back: every compact Hausdorff topological space X is homeomorphic to $\Sigma\big(\mathrm{C}(X)\big)$:

$$X \simeq \Sigma\big(\mathrm{C}(X)\big). \tag{2.11}$$

This homeomorphism given by $x \mapsto \mathrm{ev}_x$, where

$$\mathrm{ev}_x : \mathrm{C}(X) \to \mathbb{C}, \quad \mathrm{ev}_x(f) = f(x) \tag{2.12}$$

is the evaluation map. We have now a duality between unital commutative C*-algebras and compact Hausdorff spaces: *Gelfand duality*.

The same thing can be done for noncompact Haussdorf topological spaces. Let X be one. Then $\mathrm{C}_0(X)$, the algebra of continuous functions vanishing at infinity, is a nonunital C*-algebra. And for a nonunital commutative C*-algebra $\mathscr{A}$, $\Sigma(\mathscr{A})$ is a noncompact Hausdorff topological space. So Gelfand duality, as introduced in the previous paragraph, also holds for noncompact spaces and nonunital algebras.

Now get to the main point of this chapter, defining *noncommutative topology*. Just like a (commutative) topological space is described by a commutative unital C*-algebra, we declare that a *noncommutative topological space* is described by a noncommutative C*-algebra. To express it schematically:

$$\begin{tikzcd}
\begin{array}{c}\text{commutative C*-algebras}\\ \text{(e.g. } \mathrm{C}(X)\text{)}\end{array} \arrow[r, leftrightarrow, "\text{Gelfand duality}"] \arrow[d, "\text{generalize}"'] & \begin{array}{c}\text{topological spaces}\\ \text{(e.g. } X\text{)}\end{array} \\
\text{noncommutative C*-algebras} \arrow[r, leftrightarrow, dotted] & \begin{array}{c}\text{noncommutative}\\ \text{topological spaces}\end{array}
\end{tikzcd}$$

Heuristically speaking, we can say that a Cartesian product of topological spaces corresponds to the tensor product of the corresponding commutative C*-algebras and vice versa. This idea can be extended to noncommutative spaces / algebras.

The idea of noncommutative geometry is to extend this from topology to geometry. It turns out that this is not possible for any Riemannian geometry, but according to Alain Connes, it is possible for Riemannian *spin geometry*, which is a refinement of Riemannian geometry.[1] Spin geometry will be discussed in the next chapter.

[1] We will get back on this in footnote 1 on page 25.

Chapter 3

Spin Geometry

This chapter is an introduction to spin geometry. It is based on [10], §§ 5.1–2 and 9.1–5, [7], §§ 2.1–2 and [14], chapter 2–4. Most results are given without proof, like the previous chapter. For proofs and more background information, see the given references.

3.1 Clifford Algebras

3.1.1 Real Clifford Algebras

Definition 3.1 (Real Clifford algebra)**.** Let $\mathscr{V}$ be a finite-dimensional $\mathbb{R}$-vector space and g a quadratic form on $\mathscr{V}$. The *real Clifford algebra* $\mathrm{Cl}(\mathscr{V}, g)$ is the algebra generated by elements of $\mathscr{V}$ under multiplication $\cdot$ (Clifford multiplication), with the condition

$$v \cdot w + w \cdot v = 2g(v, w), \tag{3.1}$$

for all $v, w \in \mathscr{V}$.

Note that $\mathrm{Cl}(\mathscr{V}, g) \simeq \bigwedge \mathscr{V}$ as vector spaces, which implies $\dim \mathrm{Cl}(\mathscr{V}, g) = 2^{\dim \mathscr{V}}$, and that $\mathrm{Cl}(\mathscr{V}, 0) \simeq \bigwedge \mathscr{V}$ as algebras (see Def. A.3i). We will always assume that g is non-degenerate.

The structure of a real Clifford algebra is completely determined by the dimension of $\mathscr{V}$ and the signature of g. Therefore we define

$$\begin{gathered} \mathrm{Cl}_{p,q} = \mathrm{Cl}(\mathbb{R}^{p+q}, g), \\ \text{where} \quad g(v, w) = v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q}. \end{gathered} \tag{3.2}$$

We will work with these real Clifford algebras.

In physics, Clifford algebras are known as the algebra generated by the Dirac-γ-matrices. For example: for Minkowski space, one has $\mathrm{Cl}_{3,1}$ (or $\mathrm{Cl}_{1,3}$, depending on the convention for the signature of the metric).

Theorem 3.2. *For every real Clifford algebra:*

$$\mathrm{Cl}_{p,q} \simeq \mathrm{M}_N(B_m), \tag{3.3}$$

as algebras, where

$$m \equiv (p-q) \bmod 8, \tag{3.4}$$

the algebra B_m is given by

$$\begin{aligned} B_0 &= \mathbb{R}, & B_4 &= \mathbb{H},^{1} \\ B_1 &= \mathbb{R} \oplus \mathbb{R}, & B_5 &= \mathbb{H} \oplus \mathbb{H}, \\ B_2 &= \mathbb{R}, & B_6 &= \mathbb{H}, \\ B_3 &= \mathbb{C}, & B_7 &= \mathbb{C}. \end{aligned} \tag{3.5}$$

(this is the so-called spinorial clock) and N is such that

$$N^2 \dim_{\mathbb{R}} B_m = \dim_{\mathbb{R}} \mathrm{M}_N(B_m) = \dim \mathrm{Cl}_{p,q} = 2^{p+q}. \tag{3.6}$$

Every Clifford algebra $\mathrm{Cl}_{p,q}$ can be split in two linear subspaces:

$$\mathrm{Cl}_{p,q} \simeq \mathrm{Cl}^+_{p,q} \oplus \mathrm{Cl}^-_{p,q}\,. \tag{3.7}$$

Here $\mathrm{Cl}^+_{p,q}$ and $\mathrm{Cl}^-_{p,q}$ denote the vector spaces spanned by the resp. even and odd products of elements of $\mathbb{R}^{p+q}$. $\mathrm{Cl}^+_{p,q}$ is a subalgebra of $\mathrm{Cl}_{p,q}$.

3.1.2 Complex Clifford Algebras

We can also consider complex Clifford algebras:

Definition 3.3 (Complex Clifford algebra)**.** Let $\mathscr{V}$ be a $\mathbb{C}$-vector space and g a non-degenerate quadratic form on $\mathscr{V}$. We define the *complex Clifford algebra* $\mathbb{C}\mathrm{l}(\mathscr{V}, g)$ analogous to Definition 3.1.

Complex Clifford algebras are completely determined by $\mathscr{V}$'s dimension. Unlike the real case, g's signature is irrelevant. Therefore we define

$$\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}(\mathbb{C}^n, g), \quad \text{where} \quad g(v,w) = v_1 w_1 + \cdots + v_n w_n. \tag{3.8}$$

Complex Clifford algebras can be obtained by complexifying a real ones:

$$\mathbb{C}\mathrm{l}_{p+q} \simeq \mathrm{Cl}_{p,q} \otimes_{\mathbb{R}} \mathbb{C}. \tag{3.9}$$

The complex Clifford algebras can be represented as matrices:

Lemma 3.4. *For every n:*

$$\mathbb{C}\mathrm{l}_n \simeq \begin{cases} \mathrm{M}_{2^{n/2}}(\mathbb{C}) & \text{if } n \text{ is even} \\ \mathrm{M}_{2^{(n-1)/2}}(\mathbb{C}) \oplus \mathrm{M}_{2^{(n-1)/2}}(\mathbb{C}) & \text{if } n \text{ is odd} \end{cases} \tag{3.10}$$

as algebras.

[1] $\mathbb{H}$ denotes Hamilton's algebra of quaternions. We use the following definition:

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C} \right\}.$$

One can check that this is correct by complexifying the results of Thm. 3.2. Note that the structure of complex Clifford (a 2-fold periodicity) is much simpler than the structure of the complex ones (an 8-fold periodicity).

We define the vector spaces $\mathbb{S}_n$ as

$$\mathbb{S}_n = \mathbb{C}^{2^{\lfloor n/2 \rfloor}}. \tag{3.11}$$

$\mathbb{C}\mathrm{l}_n$ has a irreducible representation on $\mathbb{S}_n$. For even n, this representation is faithful; for odd n, it is not.

Given an isomorphism $c : \mathbb{C}\mathrm{l}_{2k} \to \mathrm{M}_{2^k}(\mathbb{C})$ or $\mathbb{C}\mathrm{l}_{2k+1} \to \mathrm{M}_{2^k}(\mathbb{C}) \oplus \mathrm{M}_{2^k}(\mathbb{C})$ (called a *Clifford representation*), we will use the notation

$$c(e_j) = \gamma^j, \tag{3.12}$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^{p+q}$. Obviously, the matrices γ^1, $\ldots, \gamma^{p+q}$ satisfy

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g(e^i, e^j) = 2g^{ij} \tag{3.13}$$

and generate the algebra. We demand that c is such that these matrices are Hermitian.

Like in the real case, we define for a complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ its odd subspace $\mathbb{C}\mathrm{l}_n^-$ and even subalgebra $\mathbb{C}\mathrm{l}_n^+$. For these algebras

$$\mathbb{C}\mathrm{l}_n^+ \simeq \mathbb{C}\mathrm{l}_{n-1} \simeq \begin{cases} \mathrm{M}_{2^{n/2-1}}(\mathbb{C}) \oplus \mathrm{M}_{2^{n/2-1}}(\mathbb{C}) & \text{if } n \text{ is even} \\ \mathrm{M}_{2^{(n-1)/2}}(\mathbb{C}) & \text{if } n \text{ is odd.} \end{cases} \tag{3.14}$$

From this follows: For even n, $\mathbb{C}\mathrm{l}_n^+$ has a representation on $\mathbb{C}^{2^{n/2-1}} \oplus \mathbb{C}^{2^{n/2-1}} = \mathbb{C}^{2^{n/2}} = \mathbb{S}_n$, which is reducible and faithful. For odd n, $\mathbb{C}\mathrm{l}_n^-$ has an irreducible and faithful representation on $\mathbb{C}^{2^{(n-1)/2}} \simeq \mathbb{S}_n$.

On a Clifford algebra, an involution can be defined by

$$(v_1 \cdot \cdots \cdot v_p)^* = \overline{v_p} \cdot \cdots \cdot \overline{v_1}, \tag{3.15}$$

where the bar denotes complex conjugation. Because we chose the γ^i to be Hermitian, a Clifford representation respects the *-structure.

3.1.3 Chirality

Definition 3.5 (Grading)**.** On a Clifford algebra $\mathbb{C}\mathrm{l}_n$ a $\mathbb{Z}_2$-*grading* χ can be defined as the automorphism given by

$$\forall v \in \mathbb{C}^n : \quad \chi(v) = -v, \tag{3.16}$$

or equivalently:

$$\forall a \in \mathbb{C}\mathrm{l}_n^\pm : \quad \chi(a) = \pm a. \tag{3.17}$$

Definition 3.6 (Chirality element)**.** We define the *chirality element* of a Clifford algebra $\mathbb{C}\mathrm{l}_n$ as

$$\Gamma = (-i)^{\lfloor n/2 \rfloor} e_1 \cdot \cdots \cdot e_n. \tag{3.18}$$

It is independent of the choice of the basis $e_1, \dots, e_n$, as long as it is oriented and orthogonal: Let $h \in \mathrm{SO}(n)$, so $he_1, \dots, he_n$ is also an oriented orthogonal basis. Then

$$\begin{aligned}
(-i)^{\lfloor n/2 \rfloor} he_1 \cdot \dots \cdot he_n &= (-i)^{\lfloor n/2 \rfloor} \sum_{i_1,\dots,i_n} h_{i_1 1} \cdots h_{i_n n}\, e_{i_1} \cdot \dots \cdot e_{i_n} \\
&= (-i)^{\lfloor n/2 \rfloor} \sum_{i_1,\dots,i_n} h_{i_1 1} \cdots h_{i_n n}\, \varepsilon_{i_1 \dots i_n}\, e_1 \cdot \dots \cdot e_n \\
&= \det h\ \Gamma = \Gamma,
\end{aligned} \tag{3.19}$$

where we used in the second step that

$$\begin{aligned}
&e_{i_1} \cdot \dots \cdot e_{i_n} \\
&= \left\{ \begin{array}{ll} e_1 \cdot \dots \cdot e_n & \text{if } (i_1 \dots i_n) \text{ is an even permutation of } (i \dots n) \\ -e_1 \cdot \dots \cdot e_n & \text{if } (i_1 \dots i_n) \text{ is an odd permutation of } (i \dots n) \\ 0 & \text{otherwise} \end{array} \right\} \\
&= \varepsilon_{i_1 \dots i_n} e_1 \cdot \dots \cdot e_n .
\end{aligned} \tag{3.20}$$

Γ is self-adjoint:

$$\begin{aligned}
\Gamma^* &= i^{\lfloor n/2 \rfloor} e_n \cdot \dots \cdot e_1 = (-)^{(n-1)+(n-2)+\dots+2+1} i^{\lfloor n/2 \rfloor} e_1 \cdot \dots \cdot e_n \\
&= (-)^{n(n-1)/2} i^{\lfloor n/2 \rfloor} e_1 \cdot \dots \cdot e_n = (-)^{\lfloor n/2 \rfloor} i^{\lfloor n/2 \rfloor} e_1 \cdot \dots \cdot e_n = \Gamma
\end{aligned} \tag{3.21}$$

and unitary:

$$\Gamma^* \cdot \Gamma = e_n \cdot \dots \cdot e_1 \cdot e_1 \cdot \dots \cdot e_n = 1. \tag{3.22}$$

So

$$\Gamma \cdot \Gamma = 1. \tag{3.23}$$

Another property of Γ is:

$$\Gamma \cdot a \cdot \Gamma = \begin{cases} \chi(a) & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd,} \end{cases} \tag{3.24}$$

because

$$\begin{aligned}
\Gamma \cdot e_i &= (-i)^{\lfloor n/2 \rfloor} e_1 \cdot \dots \cdot e_n \cdot e_i \\
&= (-)^{n-1} (-i)^{\lfloor n/2 \rfloor} e_i \cdot e_1 \cdot \dots \cdot e_n = (-)^{n-1} e_i \cdot \Gamma.
\end{aligned} \tag{3.25}$$

Because Γ, and hence $c(\Gamma)$, is unitary and self-adjoint, $c(\Gamma)$'s eigenvalues are 1 and -1. The corresponding eigenspaces we call $\mathbb{S}_n^+$ and $\mathbb{S}_n^-$. In physics, this is interpreted as the two (left- an right-handed) chiralities.

3.1.4 Charge Conjugation

As said in § 3.1.2, if one complexifies a real Clifford algebra (eq. (3.9)), one looses certain structure: the signature of the bilinear form. This structure can be recovered by introducing an antiunitary[2] operator C on $\mathbb{S}_n$ with

$$C^2 = \varepsilon \in \{1, -1\}. \tag{3.26}$$

[2] i.e. anti-linear and unitary

In physics, this operator is interpreted as *charge conjugation*.[3]

The sign ε is 1 if the underlying real Clifford consists of matrices over $\mathbb{R}$ ($m \in \{0,1,2\}$, where m is given by eq. (3.4)) and -1 if it consists of matrices over $\mathbb{H}$ ($m \in \{4,5,6\}$). In the complex case ($m \in \{3,7\}$) both things could occur.

The following relation exists between χ and C:

$$\forall a \in \begin{Bmatrix} \mathbb{C}\mathrm{l}_n & \text{if } n \text{ is even} \\ \mathbb{C}\mathrm{l}_n^+ & \text{if } n \text{ is odd} \end{Bmatrix}: \quad Cc(a)C^* = c\big(\chi(\overline{a})\big). \tag{3.27}$$

This implies

$$C\gamma^i\gamma^j C^* = \gamma^i\gamma^j \tag{3.28}$$

for all n, and for even n:

$$C\gamma^i C^* = -\gamma^i. \tag{3.29}$$

In the field of noncommutative geometry, it is customary to choose C such that is reconstructs $\mathrm{Cl}_{0,n}$ from $\mathbb{C}\mathrm{l}_n$.[4] This gives the following values for ε as funcion of n:

$$\begin{array}{c||cccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline m \equiv -n \bmod 8 & 0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \varepsilon & 1 & 1^* & -1 & -1 & -1 & -1^* & 1 & 1 \end{array}. \tag{3.30}$$

Because of the periodicity of the spinorial clock, we only consider the first 8 dimensions.

The two εs marked with a * need some explanation. Consider $\mathbb{C}\mathrm{l}_5$. Recall from (3.14) that $\mathbb{C}\mathrm{l}_5^+ \simeq \mathbb{C}\mathrm{l}_4$, so we can define C on $\mathbb{C}\mathrm{l}_5^+$ to be given by the charge conjugation on $\mathbb{C}\mathrm{l}_4$ and extend this to the whole $\mathbb{C}\mathrm{l}_5$. This operator commutes with the elements of $\mathrm{Cl}_{0,5}^+$ (see eq. (3.27)). It turns out that it also commutes with the elements of $\mathrm{Cl}_{0,5}^-$.[5] The same thing can be done for $\mathbb{C}\mathrm{l}_1$.

We introduce the sign ε' as

$$\forall a \in \mathrm{Cl}_{0,n}^- : \quad Cc(a) = -\varepsilon' c(a)C, \tag{3.31}$$

or equivalently:

$$C\gamma^i = -\varepsilon'\gamma^i C. \tag{3.32}$$

We already saw that for even n $\varepsilon' = 1$ (eq. (3.27)) and for $n = 1$ and 5 $\varepsilon' = -1$. It turns out that $\varepsilon' = 1$ also for $n = 3$ and 7.

For even n, the third sign ε'' is defined to be such that

$$Cc(\Gamma) = \varepsilon'' c(\Gamma)C. \tag{3.33}$$

It can be calculated as

$$\varepsilon'' = (-)^{n/2}, \tag{3.34}$$

because

$$Cc(\Gamma) = C(-i)^{n/2}\gamma^1\cdots\gamma^n = i^{n/2}\gamma^1\cdots\gamma^n C = (-)^{n/2}c(\Gamma)C. \tag{3.35}$$

[3] In [10], the operator $C(\cdot)C^*$ is called the charge conjugation operator instead of C.

[4] According to [14], § 3.6.

[5] In [14], § 3.6 it is stated that it anticommutes. This is related to the extra $-$-sign we have in eq. (3.31), in comparison with [14]. This sign convention is chosen to obtain eq. (3.75).

These ideas are expressed in Connes' sign table:

$$\begin{array}{c||cccccccc} n \bmod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \varepsilon & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ \varepsilon' & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ \varepsilon'' & 1 & & -1 & & 1 & & -1 & \end{array} \tag{3.36}$$

3.2 Spin Groups

Definition 3.7 (Spin groups)**.** We define the groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ as follows:

$$\mathrm{Spin}(n) = \{\zeta v_1 \cdot \cdots \cdot v_p \mid p \text{ even},\ \zeta \in \mu_2,\ \forall i : v_i \in \mathbb{R}^n,\ v_i \cdot v_i = 1\},^{6} \tag{3.37}$$

where Clifford multiplication $\cdot$ is taken from the algebra $\mathrm{Cl}_{n,0}$, or equivalently $\mathrm{Cl}_{0,n}$, and

$$\begin{aligned} \mathrm{Spin}^{\mathbb{C}}(n) = \{\zeta v_1 \cdot \cdots \cdot v_p \mid p \text{ even},\ \zeta \in \mathrm{U}(1), \\ \forall i : v_i \in \mathbb{R}^n,\ v_i \cdot v_i = 1\}, \end{aligned} \tag{3.38}$$

where $\cdot$ is taken from the algebra $\mathbb{C}\mathrm{l}_n$.

They are a groups under Clifford multiplication and the inverse is given by

$$(\zeta v_1 \cdot \cdots \cdot v_p)^{-1} = \bar{\zeta} v_p \cdot \cdots \cdot v_1. \tag{3.39}$$

$\mathrm{Spin}(n)$ is contained is $\mathrm{Cl}^+_{n,0}$, which is contained in $\mathbb{C}\mathrm{l}^+_n \simeq \mathbb{C}\mathrm{l}_{n-1}$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ is also contained in $\mathbb{C}\mathrm{l}^+_n \simeq \mathbb{C}\mathrm{l}_{n-1}$.

The map

$$\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n), \quad \lambda(x)v = x \cdot v \cdot x^{-1} \tag{3.40}$$

(one can show that it indeed maps to $\mathrm{SO}(n)$) is a surjective group homomorphism with $\ker \lambda \simeq \mu_2$. This means that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$, because it can be shown that $\mathrm{Spin}(n)$ is connected (for $n \geq 3$). A similar thing is going on in $\mathrm{Spin}^{\mathbb{C}}(n)$: λ extends to the surjective homomorphism

$$\lambda^{\mathbb{C}} : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \tag{3.41}$$

with $\ker \lambda^{\mathbb{C}} \simeq \mathrm{U}(1)$.

3.3 Spin Manifolds

Before we can introduce the notion of spin geometry, we need a different viewpoint of Riemannian geometry. Mostly, a Riemannian manifold is regarded as a manifold M with a inner product (which is positive definite) on the tangent spaces. An alternative definition is:

[6] $\mu_2 = \{1, -1\}$, see footnote 1 on page 73.

Definition 3.8 (Oriented Riemannian manifold)**.** An oriented Riemannian manifold is an oriented manifold M ($\dim M = n$) together with a principal $\mathrm{SO}(n)$-bundle over M, $\mathrm{SO}(M)$ (called the *orthonormal frame bundle*), and an explicit isomorphism

$$\mathrm{SO}(M) \times_{\mathrm{SO}(n)} \mathbb{R}^n \simeq \mathrm{T}M. \tag{3.42}$$

$\mathrm{SO}(M) \times_{\mathrm{SO}(n)} \mathbb{R}^n$ is a vector bundle over M with typical fibre $\mathbb{R}^n$. What this definition means exactly is explained in [14], § 3.3, but what this expresses is the $\mathrm{SO}(n)$ freedom for choosing local bases. Orthogonality is needed to preserve the Riemannian structure, positive determinants are needed to preserve the orientation.

Analogously, we define Riemannian spin and spin$^{\mathbb{C}}$ manifolds as:

Definition 3.9 (Riemannian spin and spin$^{\mathbb{C}}$ manifold)**.**

i. A *Riemannian spin manifold* is an oriented manifold M ($\dim M = n$) together with a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(M)$ and an explicit isomorphism

$$\mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \simeq \mathrm{T}M. \tag{3.43}$$

$\mathrm{Spin}(M)$ acts on $\mathbb{R}^n$ via the representation (3.40).

ii. A *spin$^{\mathbb{C}}$ manifold* is an oriented manifold M ($\dim M = n$) together with a principal $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle $\mathrm{Spin}^{\mathbb{C}}(M)$ and an explicit isomorphism

$$\mathrm{Spin}^{\mathbb{C}}(M) \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \mathbb{R}^n \simeq \mathrm{T}M. \tag{3.44}$$

$\mathrm{Spin}^{\mathbb{C}}(M)$ acts on $\mathbb{R}^n$ via the representation (3.41).

Every Riemannian spin or spin$^{\mathbb{C}}$ manifold is an oriented Riemannian manifold, but the other way around is not necessarily so.

In above definitions, $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ act on $\mathbb{R}^n$, but recall there is also the spinor representation. We use this to define the spinor bundles:

Definition 3.10 (Spinor bundle)**.** Let M be a n-dimensional Riemannian spin manifold. We define the *spinor bundle* as the complex vector bundle over M

$$S = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{S}_n. \tag{3.45}$$

Its fibres are $S_x \simeq \mathbb{S}_n$. Sections of the spinor bundle (this set is denoted by $\Gamma(S)$) we call *spinors*.

We can define an inner product on $\Gamma(S)$:

$$(\psi_1, \psi_2)_{\Gamma(S)} = \int_M \mathrm{d}^n x \sqrt{g}\, \langle \psi_1(x), \psi_2(x) \rangle_{S_x}, \tag{3.46}$$[7]

where $\langle \cdot, \cdot \rangle_{S_x}$ is the standard inner product on $S_x \simeq \mathbb{S}_n$. We define $\mathrm{L}^2(M, S)$ as the Hilbert space obtained by completion of $\Gamma(S)$ in the norm induced by (3.46).

Definition 3.11 (Clifford bundle)**.** Let M be an n-dimensional spin manifold. We define its *Clifford bundle* $\mathbb{C}\mathrm{l}(M)$ as the algebra bundle over M

$$\mathbb{C}\mathrm{l}(M) = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{C}\mathrm{l}_n\,. \tag{3.47}$$

[7] $\sqrt{g}$ is the usual short-hand notation for $\sqrt{\det g}$.

Its fibres are $\mathbb{C}l_x(M) \simeq \mathbb{C}l_n$. Because $\mathbb{C}l_n$ has a representation c on $\mathbb{S}_n$, $\mathbb{C}l(M)$ has a representation on S, which we also call c.

Locally, $\Gamma\big(\mathbb{C}l(M)\big)$, the algebra of sections of $\mathbb{C}l(M)$, has generators

$$\gamma^\mu = \gamma^a e_a^\mu, \tag{3.48}$$

where the γ^a are given by (3.12) and the e_μ^a are *vielbeine*:

$$\delta^{ab} e_a^\mu e_b^\nu = g^{\mu\nu}, \text{[8]} \tag{3.49}$$

where μ, ν denote local indices on M and a, b indices on the tangent space. The γ^μ satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{3.50}$$

analogous to (3.13).

3.4 Chirality & Charge Conjugation on Spin Manifolds

The chirality element of a Clifford algebra $\mathit{\Gamma} : \mathbb{S}_n \to \mathbb{S}_n$[9] induces a chirality element on a spin manifold, which we also call $\mathit{\Gamma}$:

$$\mathit{\Gamma} : \Gamma(S) \to \Gamma(S), \quad (\mathit{\Gamma}\psi)(x) = \mathit{\Gamma}\big(\psi(x)\big), \tag{3.51}$$

for $\psi \in \Gamma(S)$. This new $\mathit{\Gamma}$ is self-adjoint and unitary (with respect to the inner product (3.46)) and $\mathit{\Gamma}^2 = 1$. The analogue of eq. (3.25) is

$$\mathit{\Gamma}\gamma^\mu = (-)^{n-1}\gamma^\mu\mathit{\Gamma}. \tag{3.52}$$

The same thing can be done with the charge conjugation operator C:

$$C : \Gamma(S) \to \Gamma(S), \quad (C\psi)(x) = C\big(\psi(x)\big). \tag{3.53}$$

The new C is also unitary, it is $C^\infty(M)$-antilinear and $C^2 \in \{1, -1\}$.

If we assume that the charge conjugation operator on $\mathbb{S}_n$ satisfies Connes' sign table (3.36), our new C does too, where the signs are given by

$$C^2 = \varepsilon, \tag{3.54a}$$

$$C\gamma^\mu = -\varepsilon'\gamma^\mu C, \tag{3.54b}$$

$$C\mathit{\Gamma} = \varepsilon''\mathit{\Gamma}C, \tag{3.54c}$$

analogous to equations (3.26), (3.31) and (3.33).

Recall that a $\mathrm{Cl}_{p,q}$ can be reconstructed from $\mathbb{C}l_{p+q}$ with a charge conjugation operator. With this new notion of charge conjugation, the same thing can be done for spin manifolds from spin$^{\mathbb{C}}$ manifolds.

[8] Einstein's summation convention is used.

[9] Actually, we should write $c(\mathit{\Gamma})$ instead of $\mathit{\Gamma}$, but we drop the symbol c.

3.5 The Spin Connection

Recall that every Riemannian manifold (M, g) admits a unique *Levi-Civita connection* ∇^g. If M is also spin, the Levi–Civita connection induces a connection on the spinor bundle:

Definition 3.12 (Spin connection)**.** On a compact Riemannian spin manifold M, the *spin connection* ∇^S is the unique connection acting on spinors that commutes with C and satisfies the following Leibniz rule:

$$\nabla^S\big(c(\alpha)\psi\big) = c(\nabla^g\alpha)\psi + c(\alpha)\nabla^S\psi, \tag{3.55}$$

where $\alpha \in \Gamma\big(\mathbb{C}l(M)\big)$ and $\psi \in \Gamma(S)$. This can be written in a more compact way as

$$[\nabla^S, c(\alpha)] = c(\nabla^g\alpha). \tag{3.56}$$

We write the spin connection locally as follows:

$$\nabla^S_\mu = \partial_\mu + \omega_\mu, \tag{3.57}$$

where $\omega_\mu(x) \in \mathrm{End}(S_x)$ is analogous to the Christoffel symbol.

The curvature of the spin connection turns out to be

$$\Omega^S_{\mu\nu} = [\nabla^S_\mu, \nabla^S_\nu] = \partial_\mu\omega_\nu - \partial_\nu\omega_\nu + [\omega_\mu\omega_\nu] = \tfrac{1}{4}R_{\mu\nu\xi\pi}\gamma^\xi\gamma^\pi, \tag{3.58}$$

where $R_{\mu\nu\xi\pi}$ is the Riemann–Christoffel curvature tensor. (See [10]: Prop. 9.9 and the proof of Thm. 9.16.)

3.6 The Dirac Operator

Recall that in the flat case the Dirac operator is defined as

$$\not{D} = i\gamma^i\partial_i. \tag{3.59}$$

We include a factor i, to make it Hermitian. The square of the Dirac operator gives the Laplacian:

$$(i\gamma^i\partial_i)^2 = -\delta^{ij}\partial_i\partial_j. \tag{3.60}$$

This inspires us to define:

Definition 3.13 (Dirac operator)**.** On a compact Riemannian spin manifold, the *Dirac operator* $\not{D}$ acting on $\Gamma(S)$ is (locally) defined as follows:

$$\not{D} = i\gamma^\mu\nabla^S_\mu. \tag{3.61}$$

For a global definition, see [10], Def. 9.11.

Like in (3.59), we included a factor i, to make $\not{D}$ Hermitian. For a proof, see [10], Prop. 9.13. Actually, it can be shown that $\not{D}$ is self-adjoint for the right choice for its domain ([10], Thm. 9.15).

Let us see what happens if we square $\not{D}$:

Theorem 3.14 (Lichnerowitz formula). *On a compact spin manifold, the square of the Dirac operator is*

$$\not{D}^2 = \Delta^S + \tfrac{1}{4}R, \tag{3.62}$$

where Δ^S is the spinor Laplacian:

$$\Delta^S = -g^{\mu\nu}\nabla^S_\mu \nabla^S_\nu + \Gamma^\mu \nabla^S_\mu. \tag{3.63}$$

Γ^μ is a short-hand notation for

$$\Gamma^\mu = g^{\nu\xi}\Gamma^\mu{}_{\nu\xi}, \tag{3.64}$$

where $\Gamma^\mu{}_{\nu\xi}$ denotes the Christoffel symbols of the second kind of the manifold. R is the scalar curvature.

Proof. First, we calculate the commutator of ∇^S_μ and γ^ν, using (3.48), (3.56) and $\nabla^g_\mu e^\nu_a = -\Gamma^\nu{}_{\mu\xi} e^\xi_a$ (see [10], just below eq. (7.13)):

$$[\nabla^S_\mu, \gamma^\nu] = [\nabla^S_\mu, \gamma^a e^\nu_a] = \gamma^a \nabla^g_\mu e^\nu_a = -\gamma^a \Gamma^\nu{}_{\mu\xi} e^\xi_a = -\Gamma^\nu{}_{\mu\xi}\gamma^\xi \tag{3.65}$$

Using this, we get for $\not{D}^2$:

$$\not{D}^2 = -\gamma^\mu \nabla^S_\mu \gamma^\nu \nabla^S_\nu = -\gamma^\mu\gamma^\nu \nabla^S_\mu \nabla^S_\nu + \gamma^\mu\gamma^\xi \Gamma^\nu{}_{\mu\xi}\nabla^S_\nu. \tag{3.66}$$

The first term can be rewritten as

$$-\gamma^\mu\gamma^\nu\nabla^S_\mu\nabla^S_\nu = -g^{\mu\nu}\nabla^S_\mu\nabla^S_\nu - \tfrac{1}{2}\gamma^\mu\gamma^\nu[\nabla^S_\mu, \nabla^S_\nu] = -g^{\mu\nu}\nabla^S_\mu\nabla^S_\nu + \tfrac{1}{4}R, \tag{3.67}$$

where we use that

$$\gamma^\mu\gamma^\nu[\nabla^S_\mu, \nabla^S_\nu] = \tfrac{1}{4}R_{\mu\nu\xi\pi}\gamma^\mu\gamma^\nu\gamma^\xi\gamma^\pi = -\tfrac{1}{2}R, \tag{3.68}$$

which follows from (3.58) and the symmetries of the Riemann–Christoffel tensor (see [10], proof of Thm. 9.16). The second term can be rewritten, using the Clifford identity (3.50) and the symmetry of Christoffel symbols, as

$$\gamma^\mu\gamma^\xi\Gamma^\nu{}_{\mu\xi}\nabla^S_\nu = \Gamma^\nu\nabla^S_\nu. \tag{3.69}$$

Adding these two terms gives

$$\not{D}^2 = -g^{\mu\nu}\nabla^S_\mu\nabla^S_\nu + \Gamma^\nu\nabla^S_\nu + \tfrac{1}{4}R = \Delta^S + \tfrac{1}{4}R. \qquad \square \tag{3.70}$$

As we expect, the Lichnerowitz formula reduces in the flat case to (3.60).

Lemma 3.15. *i. For all $a \in C^\infty(M)$, the operator $[\not{D}, a]$ equals:*

$$[\not{D}, a] = i\gamma^\mu\partial_\mu a. \tag{3.71}$$

ii. This operator is bounded.

Proof. i. Using Def. 3.13 and (3.56), we get:

$$[\not{D}, a] = i\gamma^\mu[\nabla^S_\mu, a] = i\gamma^\mu\nabla^g_\mu a = i\gamma^\mu\partial_\mu a. \tag{3.72}$$

(This can also be proved by using the local form of the spin connection (eq. (3.57)).

ii. It turns out that (see [10], proof of Prop. 9.12)

$$\|[\not{D}, a]\|_{\text{op}}^2 = \|\gamma^\mu \partial_\mu a\|_{\text{op}}^2 = \|g^{\mu\nu}(\overline{\partial_\mu a})(\partial_\nu a)\|_\infty, \tag{3.73}$$

which is finite, since a is smooth and M is compact. □

To conclude, note that

$$\Gamma \not{D} = i\Gamma \gamma^\mu \nabla_\mu^S = (-)^{n-1} i\gamma^\mu \nabla_\mu^S \Gamma = (-)^{n-1} \not{D}\Gamma, \tag{3.74}$$

where we used that Γ and ∇^S commute[10] and eq. (3.52), and

$$C\not{D} = -iC\gamma^\mu \nabla_\mu^S = i\varepsilon' \gamma^\mu \nabla_\mu^S C = \varepsilon' \not{D} C, \tag{3.75}$$

where we used eq. (3.54b) and $[C, \nabla^S] = 0$ (see Def. 3.12).

[10]See [10], the remark with equations (9.33a and b)..

Chapter 4

Spectral Triples, Noncommutative Geometry

In this chapter, the notion of noncommutative geometry in introduced, using the ideas of the previous two chapters. See also: [10], part III, [4], ch. 1, § 10 and [14], § 5.

4.1 Spectral Triples, Noncommutative Geometry

Recall from § 2.2 the Gelfand duality between topological spaces and commutative C*-algebras and from this the notion of noncommutative topology. According to Alain Connes, this idea can be extended from topology to spin geometry. The role of the *-algebras is replaced by the following structure:

Definition 4.1 (Spectral triple)**.** A *spectral triple* is a triple $(\mathscr{A}, \mathscr{H}, D)$, where:

i. $\mathscr{A}$ is a pre-C*-algebra,

ii. $\mathscr{H}$ is a Hilbert space,

iii. there is a faithful unital *-algebra representation $\pi : \mathscr{A} \to \mathrm{B}(\mathscr{H})$ of $\mathscr{A}$ as bounded operators on $\mathscr{H}$ (which we will not always write explicitly),

iv. D is a self-adjoint operator on (a dense subspace of) $\mathscr{H}$ (called *Dirac operator*),

v. and for all $a \in \mathscr{A}$, $[D, a]$ is a bounded operator.

There are more axioms, see for example [10], § 10.5.

Example 4.2 (Canonical spectral triple)**.** Let M be a Riemannian spin manifold with spinor bundle S and Dirac operator $\not{D}_M$. Then

$$\big(\mathrm{C}^\infty(M),\, \mathrm{L}^2(M,S),\, \not{D}_M\big), \tag{4.1}$$

where $\mathrm{C}^\infty(M)$ acts on $\mathrm{L}^2(M,S)$ by pointwise multiplication, is a commutative spectral triple (i.e. its algebra is commutative). This is called the *canonical spectral triple* for the manifold M.

Example 4.3. The triple

$$\big(\mathrm{M}_N(\mathbb{C}), \mathrm{M}_N(\mathbb{C}), 0\big), \tag{4.2}$$

where $\mathrm{M}_N(\mathbb{C})$ acts on itself by matrix multiplication, and the inner product on $\mathrm{M}_N(\mathbb{C})$ is given by the Hilbert–Schmidt inner product:

$$\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^* B), \tag{4.3}$$

is a spectral triple.

One can regard these two examples as analogies of respectively Examples 2.5 and 2.4 (with the difference that the algebra $\mathrm{M}_N(\mathbb{C})$ is represented on $\mathbb{C}^N$ in 2.4 and on $\mathrm{M}_N(\mathbb{C})$ here). As we will see, Example 4.2 is the motivation to define spectral triples.

Spectral triples can be dressed up with some extra structure:

Definition 4.4. (Even spectral triple) An *even spectral triple* $(\mathscr{A}, \mathscr{H}, D, \Gamma)$ is a spectral triple $(\mathscr{A}, \mathscr{H}, D)$ with a self-adjoint unitary operator Γ on $\mathscr{H}$ which is a $\mathbb{Z}_2$-grading with the properties:

$$\Gamma^2 = 1, \tag{4.4a}$$

$$\Gamma D = -D\Gamma, \tag{4.4b}$$

$$\forall a \in \mathscr{A}: \quad \Gamma a = a\Gamma. \tag{4.4c}$$

Definition 4.5. (Real spectral triple) A *real spectral triple* $(\mathscr{A}, \mathscr{H}, D, (\Gamma,)J)$ of *KO-dimension* $n \in \mathbb{Z}_8$ is a (possibly even) spectral triple $(\mathscr{A}, \mathscr{H}, D\,(,\Gamma))$ with an anti-unitary map J on $\mathscr{H}$ with the properties

$$J^2 = \varepsilon, \tag{4.5a}$$

$$JD = \varepsilon' DJ, \tag{4.5b}$$

$$J\Gamma = \varepsilon'' \Gamma J \quad \text{(in the even case)}, \tag{4.5c}$$

$$\forall a, b \in \mathscr{A}: \quad [a, b^\circ] = 0, \tag{4.5d}$$

$$\text{and} \quad \big[[D, a], b^\circ\big] = 0, \tag{4.5e}$$

where $\varepsilon, \varepsilon', \varepsilon'' \in \{1, -1\}$ depend on the KO-dimension n according to Connes' sign table:

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

, (4.6)

and

$$b^\circ = J b^* J^*. \tag{4.7}$$

Eq. (4.5e) is called the *order-one condition*.

Lemma 4.6. *Consider a real spectral triple. The operation* $(\cdot)^\circ$ *of eq. (4.7) is a linear map on the algebra* $\mathscr{A}$*, which satisfies*

$$a^{\circ\circ} = a, \quad (ab)^\circ = b^\circ a^\circ \quad \textit{and} \quad a^{*\circ} = a^{\circ *}. \tag{4.8}$$

for all $a, b \in \mathscr{A}$.

Proof. It is linear, because

$$(\lambda a)^\circ = J\overline{\lambda} a^* J^* = \lambda J a^* J^* = \lambda a^\circ \tag{4.9}$$

for $\lambda \in \mathbb{C}$. The other three properties are fulfilled, because:

$$\begin{aligned} a^{\circ\circ} &= J(Ja^*J^*)^*J^* = JJaJ^*J^* = \varepsilon^2 a = a, \\ (ab)^\circ &= J(ab^*)J^* = Jb^*a^*J^* = Jb^*J^*Ja^*J^* = b^\circ a^\circ, \\ a^{*\circ} &= JaJ^* = (Ja^*J^*)^* = a^{\circ *}. \end{aligned} \qquad \square \tag{4.10}$$

Remark 4.7. Because of this Lemma, together with (4.5d), we can say that $(\cdot)^\circ$ implements an *opposite *-representation* of $\mathscr{A}$ on $\mathscr{H}$.

The motivation for these even and real structures arises from the notions of chirality and charge conjugation of a spin manifold (see § 3.4); they are the even and real structures for the canonical spectral triple:

Example 4.8. If M is an n-dimensional (n is even) Riemannian spin manifold with chirality element Γ_M and charge conjugation C_M,

$$\big(\mathrm{C}^\infty(M),\, \mathrm{L}^2(M,S),\, \not{D}_M,\, \Gamma_M,\, C_M\big) \tag{4.11}$$

is an even and real spectral triple of KO-dimension n mod 8.

Example 4.9. For the spectral triple of Example 4.3, $J : \mathrm{M}_N(\mathbb{C}) \to \mathrm{M}_N(\mathbb{C})$, $JT = T^*$ is a real structure.

Proof. $J^2 = 1$, so $J^* = J$. Let $a, b, T \in \mathrm{M}_N(\mathbb{C})$. Then

$$ab^\circ T = aJb^*J^*T = a(b^*T^*)^* = aTb = \big(b^*(aT)^*\big)^* = Jb^*J^*aT = b^\circ aT, \tag{4.12}$$

so eq. (4.5d) is satisfied. $\square$

According to Connes, from any commutative, real and possibly even spectral triple (with some extra technical conditions), a spin geometry can be reconstructed.[1] This means that there is a duality between commutative spectral triples and spin manifolds. In analogy to chapter 2, we have now a notion of *noncommutative geometry*: a *noncommutative spin manifold* is the object described by a noncommutative spectral triple.

4.2 Almost Commutative Geometry

In § 2.2 we saw that the Cartesian product of noncommutative topological spaces corresponds to the tensor product of C*-algebras. In noncommutative geometry a similar thing can be done:

[1]See [10], Thm. 11.2 and the remarks on page 513. Here is also a nice relation to the '*Can one hear the shape of a drum?*'-problem made: "*Connes' spin manifold theorem gives, then, a spectral reformulation of the classical geometry of spin manifolds. It is by now well known that in ordinary Riemannian geometry, the spectrum of the Laplacian does not fully determine the metric,* [. . .] *so the shape of a drum cannot be heard. For spin manifolds, the situation is better: the spectrum of the Dirac operator together with the volume form* [. . .] *fully determines the metric and the spin structure. In that sense, one can hear the shape of a spinorial drum.*"

Definition 4.10. (Tensor product of spectral triples) Let $\mathscr{S}_1 = (\mathscr{A}_1, \mathscr{H}_1, D_1, \Gamma_1)$ and $\mathscr{S}_2 = (\mathscr{A}_2, \mathscr{H}_2, D_2)$ spectral triples, where the $\mathscr{S}_1$ is even. Then we define their tensor product as

$$\mathscr{S}_1 \otimes \mathscr{S}_2 = (\mathscr{A}_1 \otimes \mathscr{A}_2,\ \mathscr{H}_1 \otimes \mathscr{H}_2,\ D_1 \otimes \mathbb{1} + \Gamma_1 \otimes D_2), \tag{4.13}$$

which is also a spectral triple.

If $\mathscr{S}_2$ is even with even structure Γ_2, Γ_2, the even structure for $\mathscr{S}_1 \otimes \mathscr{S}_2$ is $\Gamma_1 \otimes \Gamma_2$. And if $\mathscr{S}_1$ and $\mathscr{S}_2$ are real, with real structures J_1 and J_2 and KO-dimensions n_1 and n_2 respectively, in most cases the real structure for $\mathscr{S}_1 \otimes \mathscr{S}_2$ is $J_1 \otimes J_2$[2] and its KO-dimension is $n_1 + n_2 \bmod 8$.

Definition 4.11. (Almost commutative spectral triple) A spectral triple is called *almost commutative* if it is the product of the canonical spectral triple and a finite-dimensional spectral triple $(\mathscr{A}_F, \mathscr{H}_F, D_F)$ (i.e. $\mathscr{H}_F$ is finite dimensional and hence $\mathscr{A}_F$ is). So, an almost-commutative spectral triple is of the form

$$\big(\mathrm{C}^\infty(M) \otimes \mathscr{A}_F,\ \mathrm{L}^2(M,S) \otimes \mathscr{H}_F,\ \not{D}_M \otimes \mathbb{1} + \Gamma_M \otimes D_F\big). \tag{4.14}$$

Call the real and even structure of the finite spectral triple J_F and Γ_F respectively (if they exist), so for the spectral triple (4.14), we have the real structure $C_M \otimes J_F$ and the even structure $\Gamma_M \otimes \Gamma_F$.[3]

These spectral triples are called almost-commutative, since the noncommutative structure is only finite-dimensional. They are interpreted to describe a (commutative) geometry, together with a finite noncommutative structure at each point.

In [12] a method is introduced to classify finite-dimensional even and real spectral triples (and hence almost-commutative ones) using diagrams, *Krajewski diagrams*. This is amongst other things based on the notion of the opposite representation (see Remark 4.7). On the level of the algebra-representations of spectral triples, it works as follows: Given a finite-dimensional algebra $\mathscr{A}_F$, one looks for algebra-representations of the algebra $\mathscr{A}_F \otimes \mathscr{A}_F^\circ$. In other words, one regards $\mathscr{H}_F$ as a *$\mathscr{A}_F$-bimodule*.

[2] The exceptions are:

- If $n_1 \equiv 6$ and $n_2 \equiv 2 \bmod 8$, on should take $J_1 \otimes J_2\Gamma_2$.
- If $n_1 + n_2 \equiv 1$ or $5 \bmod 8$, on should take $J_1\Gamma_1 \otimes J_2$.

(See [10], page 486.)

[3] Since we will take $\dim M = 4$ at some point, we ignore the exceptions of footnote 2.

Chapter 5

From Noncommutative Geometry to Gauge Theories

In this chapter, the step from mathematics to physics is made. We will see how gauge theories emerge from spectral triples and how we can assign an action to this.

5.1 Equivalence of Spectral Triples, Inner Fluctuations

From a mathematical point of view, it is interesting to look for equivalent spectral triples. There is a straightforward notion of equivalence:[1]

Definition 5.1 (Unitary equivalence of spectral triples)**.** Let $(\mathscr{A}_1, \pi_1, \mathscr{H}_1, D_1)$ and $(\mathscr{A}_2, \pi_2, \mathscr{H}_2, D_2)$ be spectral triples (where we explicitly write the representations $\pi_i : \mathscr{A}_i \to \mathrm{B}(\mathscr{H}_i)$). These spectral triples are said to be *unitary equivalent* if:

i. There is a unital *-algebra isomorphism $\alpha : \mathscr{A}_1 \to \mathscr{A}_2$,

ii. and a unitary map $U : \mathscr{H}_1 \to \mathscr{H}_2$ (which is called the *intertwiner*),

iii. such that $U\pi_1(a)U^* = \pi_2\big(\alpha(a)\big)$ (for all $a \in \mathscr{A}_1$)

iv. and $UD_1U^* = D_2$.

v. If the spectral triples are even (with even structures Γ_1 and Γ_2 respectively), we also we also require that $U\Gamma_1U^* = \Gamma_2$.

vi. If the spectral triples are real (with real structures J_1 and J_2 respectively), we require that $UJ_1U^* = J_2$.

[1]The following definition is based on [13], § 6.9, with the modification as proposed in [5], Def. 5.25.

To summarize iii to vi: The diagrams

$$\begin{tikzcd} \mathscr{H}_1 \arrow[r, "{\pi_1(a), D_1, \Gamma_1, J_1}"] \arrow[d, "U"'] & \mathscr{H}_1 \arrow[d, "U"'] \\ \mathscr{H}_2 \arrow[r, "{\pi_2(\alpha(a)), D_2, \Gamma_2, J_2}"] & \mathscr{H}_2 \end{tikzcd} \tag{5.1}$$

should commute.

Let us have a look at a special class of equivalences: the ones where only the Dirac operator differs:

Proposition 5.2. *Let $(\mathscr{A}, \pi, \mathscr{H}, D, J)$ be a real spectral triple. Then for any $u \in \mathrm{U}(\mathscr{A})$[2], it is unitary equivalent to $(\mathscr{A}, \pi \circ \alpha_u, \mathscr{H}, D_u, J)$, where α_u is the inner automorphism*

$$\alpha_u : \mathscr{A} \to \mathscr{A}, \quad \alpha_u(a) = uau^* \tag{5.2}$$

and

$$D_u = D + u[D, u^*] + \varepsilon' J u[D, u^*] J^*. \tag{5.3}$$

α_u plays the role of the algebra isomorphism and the intertwiner is

$$U = uu^{*\circ} = uJuJ^*. \tag{5.4}$$

(For convenience, we dropped the symbol π.)

Proof.
- First, we show that $(\mathscr{A}, \pi \circ \alpha_u, \mathscr{H}, D_u, J)$ is indeed a real spectral triple:
  - D_u is self-adjoint: because

$$\begin{aligned} D_u^* &= D - [D, u]u^* - \varepsilon' J[D, u]u^* J^* \\ &= D + u[D, u^*] + \varepsilon' Ju[D, u^*]J^* = D_u, \end{aligned} \tag{5.5}$$

    where we used that $uu^* = 1$.
  -

$$\begin{aligned} JD_u &= JD + Ju[D, u^*] + \varepsilon\varepsilon' u[D, u^*]J^* \\ &= \varepsilon' DJ + Ju[D, u^*]J^* J + \varepsilon' u[D, u^*]J = \varepsilon' D_u J, \end{aligned} \tag{5.6}$$

    so eq. (4.5a) is fulfilled.
  -

$$\begin{aligned} \big[[D_u, a], b^\circ\big] = \big[[D, a], b^\circ\big] &+ \Big[\big[u[D, u^*], a\big], b^\circ\Big] \\ &+ \varepsilon' \Big[\big[Ju[D, u^*]J^*, a\big], b^\circ\Big] \end{aligned} \tag{5.7}$$

[2] For a *-algebra $\mathscr{A}$, we define its group of unitary elements as

$$\mathrm{U}(\mathscr{A}) = \{u \in \mathscr{A} \mid uu^* = u^*u = 1\}.$$

The first term is 0, because of eq. (4.5e). For the next two terms we will use that (for any $a \in \mathscr{A}$)

$$\big[u[D,u^*],a^\circ\big] = u\big[[D,u^*],a^\circ\big] + [u,a^\circ][D,u^*] = 0. \tag{5.8}$$

Applying de Jacobi identity to the second term gives

$$\begin{aligned}&\Big[\big[u[D,u^*],a\big],b^\circ\Big]\\ &= \big[[b^\circ,a],u[D,u^*]\big] + \Big[\big[u[D,u^*],b^\circ\big],a\Big] = 0.\end{aligned} \tag{5.9}$$

The third term is 0 because

$$\begin{aligned}\big[Ju[D,u^*]J^*,a\big] &= \varepsilon^2\big[Ju[D,u^*]J^*,JJaJ^*J^*\big]\\ &= J\big[u[D,u^*],a^{*\circ}\big]J^* = 0.\end{aligned} \tag{5.10}$$

So

$$\big[[D_u,a],b^\circ\big] = 0, \tag{5.11}$$

and eq. (4.5e) is fulfilled.

- Now we prove unitary equivalence of the two spectral triples:

ii.
$$UU^* = uu^{*\circ}u^\circ u^* = 1, \quad U^*U = u^\circ u^* uu^{*\circ} = 1. \tag{5.12}$$

iii.
$$UaU^* = uu^{*\circ}au^\circ u^* = uu^{*\circ}u^\circ au^* = uau^* = \alpha_u(a) \tag{5.13}$$

iv.
$$\begin{aligned}UDU^* &= uJuJ^*DJu^*J^*u^* = \varepsilon' uJuDu^*J^*u^*\\ &= \varepsilon' uJu(u^*D + [D,u^*])J^*u^*.\end{aligned} \tag{5.14}$$

The first term can be rewritten as

$$\varepsilon' uJDJ^*u^* = uDu^* = u(u^*D + [D,u^*]) = D + u[D,u^*] \tag{5.15}$$

and the second one as

$$\begin{aligned}\varepsilon' uJu[D,u^*]J^*u^* &= \varepsilon' JJ^*uJu[D,u^*]J^*u^*\\ &= \varepsilon' Ju^{*\circ}u[D,u^*]J^*u^* = \varepsilon' Juu^{*\circ}[D,u^*]J^*u^*\\ &= \varepsilon' Ju[D,u^*]u^{*\circ}J^*u^*\\ &= \varepsilon' Ju[D,u^*]J^*uJJ^*u^* = \varepsilon' Ju[D,u^*]J^*.\end{aligned} \tag{5.16}$$

So

$$UDU^* = D + u[D,u^*] + \varepsilon' Ju[D,u^*]J^* = D_u. \tag{5.17}$$

v.
$$U\Gamma U^* = uJuJ^*\Gamma Ju^*J^*u^* = \varepsilon''^2 uJuJ^*Ju^*J^*u^*\Gamma = \Gamma. \tag{5.18}$$

vi.
$$\begin{aligned}UJU^* &= uJuJ^*JJu^*J^*u^* = \varepsilon uu^{*\circ}u^*J^*u^* = \varepsilon u^{*\circ}uu^*J^*u^*\\ &= \varepsilon u^{*\circ}J^*u^* = \varepsilon JuJ^*J^*u^* = \varepsilon^2 J = J.\end{aligned} \qquad \square \tag{5.19}$$

So, given a real spectral triple $(\mathscr{A}, \mathscr{H}, D, J)$, there is a whole family of equivalent real spectral triples $(\mathscr{A}, \mathscr{H}, D_u, J)$, where $u \in \mathrm{U}(\mathscr{A})$. We see some kind of gauge theory appearing, where $u[D,u^*] + \varepsilon' Ju[D,u^*]J^*$ plays the role of a *pure gauge field*.

There is a weaker notion of equivalence of C*-algebras than isomorphy: *Morita equivalence*. This induces[3] a bigger family of real spectral triples: $(\mathscr{A},$ $\mathscr{H}$, D_A, $J)$, where

$$D_A = D + A + \varepsilon' JAJ^*, \tag{5.20}$$

with self-adjoint

$$A \in \Omega^1_D(\mathscr{A}) = \left\{ \sum_j a_j[D,b_j] \;\middle|\; a_j, b_j \in \mathscr{A} \right\} \tag{5.21}$$

(where the sums are finite). These A are called *inner fluctuations of* D. $\Omega^1_D(\mathscr{A})$ is the noncommutative generalisation of the space of 1-forms. In fact, it is the space of 1-forms in the Clifford representation; see Lemma 3.15i. These $(\mathscr{A}, \mathscr{H}, D_A, J)$ are indeed spectral triples:

Proposition 5.3. *Let $(\mathscr{A}, \mathscr{H}, D, J)$ be a real spectral triple. Then for self-adjoint $A \in \Omega^1_D(\mathscr{A})$, $(\mathscr{A}, \mathscr{H}, D_A, J)$ is also a spectral triple.*

Proof. D_A is self-adjoint because D and A are. For the rest, the arguments are similar to the first half of the proof of Prop. 5.2, where one should read $\sum_j a_j[D,b_j]$ instead of $u[D,u^*]$. ☐

Applying Proposition 5.2 to these fluctuated spectral triples, we see that

$$\begin{aligned}
(D_A)_u &= D_A + u[D_A,u^*] + \varepsilon' Ju[D_A,u^*]J^* \\
&= D + A + \varepsilon' JAJ^* + u[D,u^*] + u[A,u^*] + \varepsilon' u[JAJ^*,u^*] \\
&\quad + \varepsilon' Ju[D,u^*]J^* + \varepsilon' Ju[A,u^*]J^* + Ju[JAJ^*,u^*]J^* \\
&= D + uAu^* + \varepsilon' JuAu^*J^* + u[D,u^*] + \varepsilon' Ju[D,u^*]J^* \\
&= D + A^u + \varepsilon' JA^uJ^* = D_{A^u}.
\end{aligned} \tag{5.22}$$

In the third step we used that $A + u[A,u^*] = uAu^*$ (because $uu^* = 1$) and

$$\begin{aligned}
u[JAJ^*,u^*] &= u\sum_j \left[Ja_j[D,b_j]J^*,u^*\right] = \varepsilon^2 u \sum_j \left[Ja_j[D,b_j]J^*, JJu^*J^*J^*\right] \\
&= uJ\sum_j \left[a_j[D,b_j],u^\circ\right]J^* \\
&= uJ\sum_j \left(a_j\left[[D,b_j],u^\circ\right] + [a_j,u^\circ][D,b_j]\right)J^* = 0.
\end{aligned} \tag{5.23}$$

In the fourth step

$$A^u = uAu^* + u[D,u^*] \tag{5.24}$$

is introduced. This is the transformation property of the inner fluctuation A under transformations in the sense of Prop. 5.2.

[3] See [4], ch. 1, § 10.8.

5.1.1 Gauge and Scalar Bosons

For almost-commutative spectral triples (eq. (4.14)), the inner fluctuations are of the form (for $\alpha_j, \beta_j \in C^\infty(M)$ and $A_j, B_j \in \mathscr{A}_F$):

$$\begin{aligned} A &= \sum_j \alpha_j \otimes A_j [\not{D}_M \otimes \mathbb{1}, \beta_j \otimes B_j] + \sum_j \alpha_j \otimes A_j [\Gamma_M \otimes D_F, \beta_j \otimes B_j] \\ &= i \sum_j \alpha_j (\partial_\mu \beta_j) \gamma^\mu \otimes A_j B_j + \sum_j \alpha_j \beta_j \Gamma_M \otimes A_j [D_F, B_j] \\ &= (\gamma^\mu \otimes \mathbb{1}) A_\mu + (\Gamma_M \otimes \mathbb{1}) \Phi, \end{aligned} \tag{5.25}$$

where we used Lemma 3.15i and rewrote:

$$A_\mu = i \sum_j \alpha_j (\partial_\mu \beta_j) \otimes A_j B_j = \sum_j a_{j\mu} \otimes \widetilde{A}_j, \tag{5.26a}$$

$$\Phi = \sum_j \alpha_j \beta_j \otimes A_j [D_F, B_j] = \sum_j f_j \otimes A_j [D_F, B_j], \tag{5.26b}$$

for $a_{j\mu}, f_j \in C^\infty(M)$ and $\widetilde{A}_j \in \mathscr{A}_F$. Recall that the inner fluctuations are demanded to be self-adjoint. This means that $a_{j\mu}$ and f_j are $\mathbb{R}$-valued, $\widetilde{A}_j$ is selj-adjoint and $(A_j[D_F, B_j])^* = A_j[D_F, B_j]$.

A_μ is interpreted to describe the gauge bosons and Φ the scalar particles of the theory.

5.1.2 The Gauge Group

The group $\mathrm{U}(\mathscr{A})$ is represented on $\mathscr{H}$ by the representation

$$\rho : \mathrm{U}(\mathscr{A}) \to \mathrm{GL}(\mathscr{H}), \quad \rho(u) = uJuJ^*. \tag{5.27}$$

ρ is indeed a group representation because

$$\rho(uu') = uu'Juu'J^* = u\,u'\,JuJ^*\,Ju'J^* = u\,JuJ^*\,u'\,Ju'J^* = \rho(u)\rho(u') \tag{5.28}$$

for $u, u' \in \mathrm{U}(\mathscr{A})$.

We define the *gauge group* $\mathscr{G}(\mathscr{A})$ as the quotient of $\mathrm{U}(\mathscr{A})$ by the kernel of ρ: the elements of $\mathrm{U}(\mathscr{A})$ that act trivially on $\mathscr{H}$:

$$\mathscr{G}(\mathscr{A}) \simeq \mathrm{U}(\mathscr{A}) / \ker \rho. \tag{5.29}$$

This kernel is

$$\begin{aligned} \ker \rho &= \{u \in \mathrm{U}(\mathscr{A}) \mid uJuJ^* = \mathrm{id}_{\mathscr{H}}\} \\ &= \{u \in \mathrm{U}(\mathscr{A}) \mid uJ = Ju^*\} = \mathrm{U}(\widetilde{\mathscr{A}_J}), \end{aligned} \tag{5.30}$$

where $\widetilde{\mathscr{A}_J}$ is the commutative subalgebra

$$\widetilde{\mathscr{A}_J} = \{a \in \mathscr{A} \mid aJ = Ja^*\} = \{a \in \mathscr{A} \mid a = a^\circ\} \tag{5.31}$$

of $\mathscr{A}$. So we can write

$$\mathscr{G}(\mathscr{A}) \simeq \mathrm{U}(\mathscr{A}) / \mathrm{U}(\widetilde{\mathscr{A}_J}). \tag{5.32}$$

Note that

$$\mathscr{G}(\mathscr{A}) \simeq \{uJuJ^* \mid u \in \mathrm{U}(\mathscr{A})\}. \tag{5.33}$$

The spectral triples we will consider are almost-commutative (eq. (4.14)). We have then

$$\mathscr{G}(\mathscr{A}) \simeq \mathrm{C}^\infty\big(M, \mathscr{G}(\mathscr{A}_F)\big). \tag{5.34}$$

We will drop the $\mathrm{C}^\infty(M)$-part, as is customary in physics, and call

$$\mathscr{G}(\mathscr{A}_F) \simeq \mathrm{U}(\mathscr{A}_F)/\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F}) \tag{5.35}$$

the gauge group, which is represented on $\mathscr{H}_F$ as

$$\rho_F : \mathrm{U}(\mathscr{A}_F) \to \mathrm{GL}(\mathscr{H}_F), \quad \rho_F(u) = uJ_F uJ_F^*. \tag{5.36}$$

$\mathbb{C}$ is a subalgebra of any unital $\mathbb{C}$-algebra (take the multiples of the unit element), so it is also for $\mathscr{A}_F$ and $\widetilde{\mathscr{A}_F}_{J_F}$. This means that $\mathrm{U}(1)$ is a subgroup of $\mathrm{U}(\mathscr{A}_F)$ and $\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F})$. This subgroup is normal, which implies that we can write the quotient (5.35) as

$$\mathscr{G}(\mathscr{A}_F) \simeq {}^{\mathrm{U}(\mathscr{A}_F)/\mathrm{U}(1)}\Big/_{\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F})/\mathrm{U}(1)} \tag{5.37}$$

In the examples we will consider (see for example § 6.1), the lower part of the quotient is trivial, which means that

$$\mathscr{G}(\mathscr{A}_F) = \mathrm{U}(\mathscr{A}_F)/\mathrm{U}(1). \tag{5.38}$$

This is related to the *unimodularity condition* (the gauge field is traceless).

Until now, we assumed that $\mathscr{A}$ is a $\mathbb{C}$-algebra and that its representation π is $\mathbb{C}$-linear. In §§ 6.2 and 8.1, we will see examples of spectral triples where this is not the case. The unimodularity condition is then not necessarily satisfied. In these cases, we impose this condition 'by hand' by restricting $\mathrm{U}(\mathscr{A}_F)$ to

$$\mathrm{S}_{\pi_F}\mathrm{U}(\mathscr{A}_F) = \{u \in \mathrm{U}(\mathscr{A}_F) \mid \det \pi_F(u) = 1\}. \tag{5.39}$$

(The representation π_F is shown explicitly, to make clear that the determinant of the us has to be taken in their representation on $\mathscr{H}_F$.)

The ideas of this subsection are dicussed in greater detail in [5], §§ 5.5.2 and 6.2, and [6], § 2.4.3.

5.2 The Spectral Action Principle

We would like to assign an action to our gauge theory. This can be done via Chamseddine and Connes' *spectral action principle*, which asserts that the action only depends on the spectrum of D_A.[4] A natural choice would be $\operatorname{Tr} D_A$. However, this is ill-defined, since the Dirac operator is unbounded, so the following definition is used:

[4] See [2], eq. (1.8).

Definition 5.4 (Spectral action)**.** For an even and real spectral triple of the form (4.14), we define the *spectral action* as

$$S_{\mathrm{sp}}(A,g) = \mathrm{Tr}_{\mathscr{H}}\, f(D_A/\Lambda), \tag{5.40}$$

where $\Lambda \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$. $f(D_A/\Lambda)$ can be defined via functional calculus. f's domain is $\mathbb{R}$, because D_A is self-adjoint. We demand that f is such that $f(D_A/\Lambda)$ is trace class.

f can be interpreted as a cutoff function and Λ as the cutoff scale. Λ could be absorbed into f, but later on we will need it to make an expansion in powers of Λ. The spectral action is invariant under gauge transformations.

Without loss of generality, we can assume that f is even, because of the following lemma:

Lemma 5.5. *For even spectral triples $(\mathscr{A}, \mathscr{H}, D, \Gamma)$, the spectrum of the fluctuated Dirac operator D_A (where $A \in \Omega^1_D(\mathscr{A})$ self-adjoint) is symmetric around* 0.

Proof. Γ and A anticommute, because of (4.4b & c), so Γ and D_A anticommute too. Let $\lambda \in \mathrm{spec}(D_A)$. This means that $\lambda - D_A$ is not invertible, which implies that $\Gamma(\lambda - D_A) = (\lambda + D_A)\Gamma$ is also not invertible. So $-\lambda \in \mathrm{spec}(D_A)$. □

It turns out that the spectral action only describes the bosons of the model, not the fermions. Therefore one defines a second action:

Definition 5.6 (Fermionic action)**.** For a real and even spectral triple, the *fermionic action* is defined as

$$S_{\mathrm{f}}(\widetilde{\Psi}, A) = \tfrac{1}{2}\langle J\widetilde{\Psi}, D_A\widetilde{\Psi}\rangle_{\mathscr{H}}, \tag{5.41}$$

for self-adjoint $A \in \Omega^1_D(\mathscr{A})$ and $\widetilde{\Psi} \in \mathscr{H}^+_{\mathrm{cl}}$. With $\mathscr{H}^+$ we denote the positive eigenspace of Γ. The tilde and the subscript 'cl' denote that the components of $\widetilde{\Psi}$ anticommute (i.e. they are *Graßmann numbers*).

This object describes behaviour of the fermions of the model and their coupling to the bosons. The restriction to the positive eigenspace is done to solve the *fermion doubling problem*: In almost commutative geometries one has $\mathscr{H} = \mathrm{L}^2(M,S) \otimes \mathscr{H}_F$. Both $\mathrm{L}^2(M,S)$ and $\mathscr{H}_F$ are even, so they both contain left- and right-handed modes. For this double counting is compensated by the restriction to $\mathscr{H}^+$.

5.3 Heat Kernel Expansion of the Spectral Action

In this section a method for computing the spectral action is discussed. For this, the *heat kernel expansion* is used, which is explained in the following subsection. In § 5.3.2 it will be applied to the spectral action.

5.3.1 Heat Kernel Expansion, Gilkey's Theorem

Let V be a vector bundle on M and P a differential operator acting on $\Gamma(V)$ with the metric tensor as the leading symbol. That is: P is locally of the form:

$$P = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B), \tag{5.42}$$

where $A^\mu, B \in \mathrm{End}(V)$.

Let us rewrite P in a convenient form. Let ∇ be a connection on V. We write locally

$$\nabla_\mu = \partial_\mu + \omega'_\mu. \tag{5.43}$$

∇'s Laplacian is (like in eq. (3.63))

$$\begin{aligned} \Delta &= -g^{\mu\nu}\nabla_\mu\nabla_\nu + \Gamma^\mu\nabla_\mu \\ &= -g^{\mu\nu}\partial_\mu\partial_\nu + (-2g^{\mu\nu}\omega'_\nu + \Gamma^\mu)\partial_\mu - g^{\mu\nu}(\partial_\mu\omega'_\nu + \omega'_\mu\omega'_\nu) + \Gamma^\mu\omega'_\mu. \end{aligned} \tag{5.44}$$

Subtract P, and call this difference E:

$$\begin{aligned} E &= \Delta - P \\ &= (-2g^{\mu\nu}\omega'_\nu + \Gamma^\mu + A^\mu)\partial_\mu - g^{\mu\nu}(\partial_\mu\omega'_\nu + \omega'_\mu\omega'_\nu) + \Gamma^\mu\omega'_\mu + B \\ &= -g^{\mu\nu}(2\partial_\mu\omega'_\nu + \omega'_\mu\omega'_\nu) + \Gamma^\mu\omega'_\mu + B, \end{aligned} \tag{5.45}$$

where we took ω' to be

$$\omega'_\mu = \tfrac{1}{2}(\Gamma_\mu + A_\mu). \tag{5.46}$$

Now we can write P as

$$P = \Delta - E. \tag{5.47}$$

According to Gilkey,[5] the trace of the operator e^{-tP} (the *heat kernel*) has the following expansion in t:

$$\mathrm{Tr}_{\mathrm{L}^2(M,V)}\, e^{-tP} \sim \sum_{k=0}^{\infty} t^{(k-n)/2} a_n(P) \quad \text{as } t \to 0,^{6} \tag{5.48}$$

where the *Seeley–DeWitt coefficients* are

$$a_k(P) = \int_M \mathrm{d}^n x \sqrt{g}\, a_k(x, P), \tag{5.49}$$

with

$$a_k(x,P) = 0 \quad \text{for odd } k \tag{5.50a}$$

$$a_0(x,P) = \frac{1}{(4\pi)^{n/2}} \mathrm{Tr}_{V_x}(1), \tag{5.50b}$$

$$a_2(x,P) = \frac{1}{(4\pi)^{n/2}} \mathrm{Tr}_{V_x}\big(-\tfrac{1}{6}R + E\big), \tag{5.50c}$$

$$\begin{aligned} a_4(x,P) = \frac{1}{(4\pi)^{n/2}} \frac{1}{360} \mathrm{Tr}_{V_x}(&-12R_{;\mu}{}^{\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\ &+ 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RE + 180E^2 + 60E_{;\mu}{}^{\mu} + 30\Omega_{\mu\nu}\Omega^{\mu\nu}). \end{aligned} \tag{5.50d}$$

[5] See [9], §4.8.

[6] '$\sim$' denotes an expansion: for all large enough N,

$$\mathrm{Tr}_{\mathrm{L}^2(M,V)}\, e^{-tP} = \sum_{k=0}^{N} t^{(k-n)/2} a_k(P) + \mathscr{O}(t^{(N-n)/2+1}) \quad \text{as } t \to 0.$$

The sum does not need to converge for $N \to \infty$.

Ω is the curvature of ∇:

$$\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu \omega'_\nu - \partial_\mu \omega'_\nu + [\omega'_\mu, \omega'_\nu]. \tag{5.51}$$

and the short-hand notation for double covariant derivatives

$$R_{;\mu}{}^{\mu} = \nabla_\mu \nabla^\mu R \quad \text{and} \quad E_{;\mu}{}^{\mu} = \nabla_\mu \nabla^\mu E \tag{5.52}$$

is used. The coefficient $a_6(x, P)$ is also known, but its expression is quite complicated and we will not need it.

5.3.2 Heat Kernel Expansion of the Spectral Action

Let us return to the spectral action (Defn. 5.4). We said that the function f is even, so there exists a function g such that

$$f(u) = g(u^2). \tag{5.53}$$

Write this function as the Laplace transform of a function h:

$$g(u) = \int_0^\infty \mathrm{d}s\, e^{-su} h(s). \tag{5.54}$$

With functional calculus, we can write

$$f(D_A/\Lambda) = g(D_A^2/\Lambda^2) = \int_0^\infty \mathrm{d}s\, e^{-sD_A^2/\Lambda^2} h(s). \tag{5.55}$$

D_A^2 is of the form (5.42), so we can apply the heat kernel expansion to the operator $e^{-sD_A^2/\Lambda^2}$, where we take Λ^{-2} to be the expansion variable:

$$\mathrm{Tr}_{\mathscr{H}}\, e^{-sD_A^2/\Lambda^2} \sim \sum_{k=0}^{\infty} s^{(k-n)/2} \Lambda^{n-k} a_k(D_A^2) \quad \text{as } \Lambda \to \infty. \tag{5.56}$$

Using this, (5.50a) and formally interchanging the trace, the integral and the infinite sum, the spectral action can be expanded as:

$$\begin{aligned} \mathrm{Tr}_{\mathscr{H}}\, f(D_A/\Lambda) &= \int_0^\infty \mathrm{d}s\ \mathrm{Tr}_{\mathscr{H}}\, e^{-sD_A^2/\Lambda^2} h(s) \\ &\sim \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \Lambda^{n-k} a_k(D_A^2) \int_0^\infty \mathrm{d}s\, s^{(k-n)/2} h(s). \end{aligned} \tag{5.57}$$

In Def. 5.4, it is assumed that the spectral triple is even and almost commutative. Therefore we take n, the dimension of the background manifold M, to be even.

Let us calculate the integral for two cases:

- For even $k < n$:
  Using the identity

$$s^{-\alpha} = \frac{1}{(\alpha-1)!} \int_0^\infty \mathrm{d}v \, e^{-sv} v^{\alpha-1} \tag{5.58}$$

  (for $\alpha \in \mathbb{N}_{>0}$), we get

$$\begin{aligned}
\int_0^\infty \mathrm{d}s \, s^{(k-n)/2} h(s) &= \frac{1}{(\frac{n-k}{2}-1)!} \int_0^\infty \mathrm{d}v \, v^{(n-k)/2-1} \int_0^\infty ds \, s^{-sv} h(s) \\
&= \frac{1}{(\frac{n-k}{2}-1)!} \int_0^\infty \mathrm{d}v \, v^{(n-k)/2-1} g(v) \\
&= \frac{2}{(\frac{n-k}{2}-1)!} \int_0^\infty \mathrm{d}u \, u^{n-k-1} f(u) \\
&= \frac{2}{(\frac{n-k}{2}-1)!} f_{n-k}.
\end{aligned} \tag{5.59}$$

  We recognized equation (5.54), substituted $v = u^2$ and introduced

$$f_\alpha = \int_0^\infty \mathrm{d}u \, f(u) u^{\alpha-1}, \tag{5.60}$$

  the *moments* of the function f.

- And for even $k \geq n$:

$$\begin{aligned}
\int_0^\infty \mathrm{d}s \, s^{(k-n)/2} h(s) &= (-)^{(k-n)/2} \int_0^\infty \mathrm{d}s \, (-s)^{(k-n)/2} h(s) \\
&= (-)^{(k-n)/2} g^{((k-n)/2)}(0) \\
&= (-)^{(k-n)/2} \frac{\frac{k-n}{2}!}{(k-n)!} f^{(k-n)}(0).
\end{aligned} \tag{5.61}$$

  We used that for even α

$$f^{(\alpha)}(0) = \frac{\alpha!}{(\alpha/2)!} g^{(\alpha/2)}(0). \tag{5.62}$$

So we have:

$$\begin{aligned}
\mathrm{Tr}_{\mathscr{H}} f(D_A/\Lambda) \sim & \sum_{\substack{k=0 \\ k \text{ even}}}^{n-2} \frac{2}{(\frac{n-k}{2}-1)!} f_{n-k} \Lambda^{n-k} a_k(D_A^2) \\
& + \sum_{\substack{k=n \\ k \text{ even}}}^{\infty} (-)^{(k-n)/2} \frac{\frac{k-n}{2}!}{(k-n)!} f^{(k-n)}(0) \Lambda^{n-k} a_k(D_A^2).
\end{aligned} \tag{5.63}$$

The $a_k(D_A^2)$ can be calculated using (5.50). In the latter, the traces must be taken over $S_x \otimes \mathscr{H}_F = \mathbb{C}^{2^{n/2}} \otimes \mathscr{H}_F$.

In our problems, the background manifold M plays the role of space-time (although it is Riemannian), so we take $n = 4$. In this case we have, neglecting the $\mathscr{O}(\Lambda^{-2})$-terms:

$$\begin{aligned} \mathrm{Tr}_{\mathscr{H}}\, f(D_A/\Lambda) = 2f_4\Lambda^4 a_0(D_A^2) + 2f_2\Lambda^2 a_2(D_A^2) \\ + f(0)a_4(D_A^2) + \mathscr{O}(\Lambda^{-2}). \end{aligned} \tag{5.64}$$

Chapter 6

The Einstein–Yang–Mills and the Standard Model

In this chapter, two important examples of gauge theories from noncommutative geometry are studied. First, the so-called *Einstein–Yang–Mills system* is studied, including a derivation of the spectral action. Also the NCG approach to the Standard Model will be introduced to a certain extend. These discussions are based on [4], ch. 1, § 11.4 and §§ 13.0–4 respectively.

6.1 The Einstein–Yang–Mills Model

6.1.1 Spectral Triple and Gauge Theory

Consider the spectral triple of the form (4.14) with

$$(\mathscr{A}_F, \mathscr{H}_F, D_F) = \big(\mathrm{M}_N(\mathbb{C}), \mathrm{M}_N(\mathbb{C}), 0\big) \tag{6.1}$$

(see Example 4.3). This spectral triple is even, although the finite part is not: $\Gamma_M \otimes \mathbb{1}$ is a grading element, because $D_F = 0$. In Example 4.9 a real structure is defined: $J_F T = T^*$.

The algebra $\widetilde{\mathscr{A}_F}_{J_F}$ is (eq. (5.31))

$$\widetilde{\mathscr{A}_F}_{J_F} = \{a \in \mathrm{M}_N(\mathbb{C}) \mid \forall T \in \mathrm{M}_N(\mathbb{C}) : aT = Ta\} = \mathrm{span}\{1_N\} \simeq \mathbb{C}, \tag{6.2}$$

so

$$\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F}) \simeq \mathrm{U}(1) \tag{6.3}$$

According to eq. (5.35), this spectral triple gives a gauge theory with the gauge group

$$\mathscr{G}\big(\mathrm{M}_N(\mathbb{C})\big) \simeq \mathrm{U}(N)/\mathrm{U}(1) \simeq \mathrm{PSU}(N) \sim \mathrm{SU}(N). \tag{6.4}$$

'$\sim$' stands for isomorphic groups quotiented finite abelian groups. We forget about these finite groups, since we do not see them in the corresponding Lie algebras and hence in the gauge fields.

According to eq. (5.36), the representation of $\mathrm{SU}(N)$ on $\mathrm{M}_N(\mathbb{C})$ is

$$\rho_F : \mathrm{SU}(N) \to \mathrm{GL}\big(\mathrm{M}_N(\mathbb{C})\big), \quad \rho_F(u)T = uJ_F u J_F^* T = u(uT^*)^* = uTu^*. \tag{6.5}$$

In the language of § A.1, this is the representation $\mathrm{St} \otimes \mathrm{St}^*$, which decomposes in the adjoint plus the trivial representation (Proposition A.9ii):

$$\rho_F \sim \mathrm{St} \otimes \mathrm{St}^* \sim \mathrm{Ad} \oplus \mathrm{Triv}. \tag{6.6}$$

6.1.2 The Spectral Action

In this subsection, we compute the spectral action for this model.

Let A be a gauge field: a Hermitian operator on $\mathscr{H}$ of the form (5.21):

$$A = \sum_j a_j[D, b_j] = \sum_j \alpha_j[\not{D}_M, \beta_j] \otimes A_j B_j = i \sum_j \gamma^\mu \alpha_j(\partial_\mu \beta_j) \otimes A_j B_j, \tag{6.7}$$

where $a_j = \alpha_j \otimes A_j, b_j = \beta_j \otimes B_j \in \mathscr{A}$ (see eq. (5.21)). In the last step we used Lemma 3.15i. Let us rewrite A as

$$A = \sum_j \gamma^\mu a_{j\mu} \otimes \widetilde{A}_j = (\gamma^\mu \otimes \mathbb{1}) A_\mu, \tag{6.8}$$

where $a_{j\mu} \in C^\infty(M)$, $\widetilde{A}_j \in \mathrm{M}_N(\mathbb{C})$ and

$$A_\mu = \sum_j a_{j\mu} \otimes \widetilde{A}_j. \tag{6.9}$$

Next, we calculate JAJ^* by applying it on a $\psi \otimes T \in \mathscr{H}$, using the self-adjointness of A and (3.54b) (with $\varepsilon' = 1$, because $\dim M = 4$):

$$JAJ^*(\psi \otimes T) = \sum_j C_M \gamma^\mu a_{j\mu} C_M^* \psi \otimes (\widetilde{A}_j T^*)^* = -\sum_j \gamma^\mu a_{j\mu} \psi \otimes T \widetilde{A}_j. \tag{6.10}$$

Now we have an expression for the fluctuated Dirac operator:

$$\begin{aligned} D_A = D + A + JAJ^* &= \not{D}_M \otimes \mathbb{1} + \sum_j \gamma^\mu a_{j\mu} \otimes [\widetilde{A}_j, \cdot] \\ &= \not{D}_M \otimes \mathbb{1} + i(\gamma^\mu \otimes \mathbb{1}) \boldsymbol{A}_\mu, \end{aligned} \tag{6.11}$$

where we introduced the notation

$$\boldsymbol{A}_\mu = -i \operatorname{ad}(A_\mu) = -i[A_\mu, \cdot] = -i \sum_j a_{j\mu} \otimes [\widetilde{A}_j, \cdot]. \tag{6.12}$$

We want to calculate D_A^2. Applying (6.11) twice to a $\psi \otimes T \in \mathscr{H}$ gives:

$$\begin{aligned} D_A^2(\psi \otimes T) = \not{D}_M^2 \psi \otimes T &+ \sum_j (\not{D}_M \gamma^\mu a_{j\mu} \psi + \gamma^\mu a_{j\mu} \not{D}_M \psi) \otimes [\widetilde{A}_j, T] \\ &+ \sum_{j,k} \gamma^\mu \gamma^\nu a_{j\mu} a_{k\nu} \psi \otimes \big[\widetilde{A}_j, [\widetilde{A}_k, T]\big]. \end{aligned} \tag{6.13}$$

We will write this out term by term:

- Using the Lichnerowitz formula (Thm. 3.14) and the local expression of the spin connection (eq. (3.57)), the first term can be written out as

$$\begin{aligned}
\not{D}_M^2\psi &= \Delta^S\psi + \tfrac{1}{4}R\psi \\
&= -\, g^{\mu\nu}\nabla^S_\mu\nabla^S_\nu\psi + \Gamma^\mu\nabla^S_\mu\psi + \tfrac{1}{4}R\psi \\
&= -\, g^{\mu\nu}(\partial_\mu\partial_\nu + 2\omega_\nu\partial_\mu + \partial_\mu\omega_\nu + \omega_\mu\omega_\nu)\psi \\
&\quad + \Gamma^\mu(\partial_\mu + \omega_\mu)\psi + \tfrac{1}{4}R\psi.
\end{aligned} \tag{6.14}$$

- Writing out the expression between the brackets of the second term gives (leaving out a factor i):

$$\begin{aligned}
-i(\not{D}_M\gamma^\mu a_{j\mu}\psi + \gamma^\mu a_{j\mu}\not{D}_M\psi) &= \gamma^\mu\nabla^S_\mu\gamma^\nu a_{j\nu}\psi + \gamma^\mu\gamma^\nu a_{j\mu}\nabla^S_\nu\psi \\
&= \gamma^\mu\gamma^\nu(\nabla^g_\mu a_{j\nu})\psi + \gamma^\mu\gamma^\nu a_{j\nu}\nabla^S_\mu\psi \\
&\quad + \gamma^\mu\gamma^\nu a_{j\mu}\nabla^S_\nu\psi \\
&= \gamma^\mu\gamma^\nu(\nabla^g_\mu a_{j\nu})\psi + 2g^{\mu\nu}a_{j\nu}\nabla^S_\mu\psi \\
&= \tfrac{1}{2}\gamma^\mu\gamma^\nu(\nabla^g_\mu a_{j\nu} - \nabla^g_\nu a_{j\mu})\psi \\
&\quad + g^{\mu\nu}(\nabla^g_\mu a_{j\nu})\psi + 2g^{\mu\nu}a_{j\nu}\nabla^S_\mu\psi,
\end{aligned} \tag{6.15}$$

  where we used the Clifford identity (eq. (3.50)) and

$$\nabla^S_\mu\gamma^\nu a_{j\nu}\psi = \gamma^\nu(\nabla^g_\mu a_{j\nu})\psi + \gamma^\nu a_{j\nu}\nabla^S_\mu\psi, \tag{6.16}$$

  which follows from eq. (3.55). Using the local expression of both connections and $\Gamma^\xi_{\ \mu\nu} = \Gamma^\xi_{\ \nu\mu}$:

$$\begin{aligned}
&-i(\not{D}_M\gamma^\mu a_{j\mu}\psi + \gamma^\mu a_{j\mu}\not{D}_M\psi) \\
&= \tfrac{1}{2}\gamma^\mu\gamma^\nu(\partial_\mu a_{j\nu} - \partial_\nu a_{j\mu} - \Gamma^\xi_{\ \mu\nu}a_{j\xi} + \Gamma^\xi_{\ \nu\mu}a_{j\xi})\psi \\
&\quad + g^{\mu\nu}(\partial_\mu a_{j\nu} - \Gamma^\xi_{\ \mu\nu}a_{j\xi})\psi + 2g^{\mu\nu}a_{j\nu}(\partial_\mu + \omega_\mu)\psi \\
&= \tfrac{1}{2}\gamma^\mu\gamma^\nu(\partial_\mu a_{j\nu} - \partial_\nu a_{j\mu})\psi + g^{\mu\nu}(\partial_\mu a_{j\nu})\psi \\
&\quad - \Gamma^\mu a_{j\mu}\psi + 2g^{\mu\nu}a_{j\nu}(\partial_\mu + \omega_\mu)\psi.
\end{aligned} \tag{6.17}$$

  Now we can write the second term of (6.13) as

$$\begin{aligned}
&\sum_j(\not{D}_M\gamma^\mu a_{j\mu}\psi + \gamma^\mu a_{j\mu}\not{D}_M\psi)\otimes[\widetilde{A}_j, T] \\
&= \sum_j\big(\tfrac{1}{2}i\gamma^\mu\gamma^\nu(\partial_\mu a_{j\nu} - \partial_\nu a_{j\mu})\psi + ig^{\mu\nu}(\partial_\mu a_{j\nu})\psi \\
&\qquad - i\Gamma^\mu a_{j\mu}\psi + 2ig^{\mu\nu}a_{j\nu}(\partial_\mu + \omega_\mu)\psi\big)\otimes[\widetilde{A}_j, T] \\
&= -\tfrac{1}{2}(\gamma^\mu\gamma^\nu\otimes\mathbb{1})(\partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu)(\psi\otimes T) - g^{\mu\nu}(\partial_\mu\boldsymbol{A}_\nu)(\psi\otimes T) \\
&\quad + \Gamma^\mu\boldsymbol{A}_\mu(\psi\otimes T) - 2g^{\mu\nu}\boldsymbol{A}_\nu\big((\partial_\mu + \omega_\mu)\psi\big)\otimes T,
\end{aligned} \tag{6.18}$$

  using (6.12).

- Using the Jacobi and Clifford identities and (6.12) and relabelling some indices, the third term of (6.13) can be written as:

$$\begin{aligned}
&\sum_{j,k} \gamma^\mu \gamma^\nu a_{j\mu} a_{k\nu} \psi \otimes [\widetilde{A}_j, [\widetilde{A}_k, T]] \\
&= \tfrac{1}{2} \sum_{j,k} \big(\gamma^\mu \gamma^\nu a_{j\mu} a_{k\nu} \psi \otimes [\widetilde{A}_j, [\widetilde{A}_k, T]] + \gamma^\nu \gamma^\mu a_{j\nu} a_{k\mu} \psi \otimes [\widetilde{A}_k, [\widetilde{A}_j, T]] \big) \\
&= \sum_{j,k} \Big(\tfrac{1}{2} \gamma^\mu \gamma^\nu a_{j\mu} a_{k\nu} \psi \otimes \big([\widetilde{A}_j, [\widetilde{A}_k, T]] - [\widetilde{A}_k, [\widetilde{A}_j, T]] \big) \\
&\qquad + g^{\mu\nu} a_{j\nu} a_{k\mu} \psi \otimes [\widetilde{A}_k, [\widetilde{A}_j, T]] \Big) \\
&= \sum_{j,k} \big(\tfrac{1}{2} \gamma^\mu \gamma^\nu a_{j\mu} a_{k\nu} \psi \otimes [[\widetilde{A}_j, \widetilde{A}_k], T] + g^{\mu\nu} a_{j\mu} a_{k\nu} \psi \otimes [\widetilde{A}_j, [\widetilde{A}_k, T]] \big) \\
&= -\tfrac{1}{2} (\gamma^\mu \gamma^\nu \otimes \mathbb{1}) [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] (\psi \otimes T) - g^{\mu\nu} \boldsymbol{A}_\mu \boldsymbol{A}_\nu (\psi \otimes T)
\end{aligned} \tag{6.19}$$

Putting everything together gives:

$$\begin{aligned}
D_A^2(\psi \otimes T) = &- g^{\mu\nu} \partial_\mu \partial_\nu \psi \otimes T + \big(-2 g^{\mu\nu} (\omega_\mu + \boldsymbol{A}_\nu) + \Gamma^\mu \big) (\partial_\mu \psi \otimes T) \\
&+ \big(- g^{\mu\nu} \partial_\mu (\omega_\nu + \boldsymbol{A}_\nu) - g^{\mu\nu} (\omega_\mu + \boldsymbol{A}_\mu)(\omega_\nu + \boldsymbol{A}_\nu) \\
&\quad + \Gamma^\mu (\omega_\mu + \boldsymbol{A}_\mu) + \tfrac{1}{4} R - \tfrac{1}{2} (\gamma^\mu \gamma^\nu \otimes \mathbb{1}) \boldsymbol{F}_{\mu\nu} \big) (\psi \otimes T),
\end{aligned} \tag{6.20}$$

where we introduced

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu], \tag{6.21}$$

the curvature corresponding to $\boldsymbol{A}_\mu$.

Now we have D_A^2 in the form (5.42) with

$$A^\mu = 2 g^{\mu\nu} (\omega_\nu + \boldsymbol{A}_\nu) - \Gamma^\mu {}^{1} \tag{6.22}$$

and

$$\begin{aligned}
B = &\, g^{\mu\nu} \big(\partial_\mu (\omega_\nu + \boldsymbol{A}_\nu) + (\omega_\mu + \boldsymbol{A}_\mu)(\omega_\nu + \boldsymbol{A}_\nu) \big) \\
&- \Gamma^\mu (\omega_\mu + \boldsymbol{A}_\mu) - \tfrac{1}{4} R + \tfrac{1}{2} \gamma^\mu \gamma^\nu \boldsymbol{F}_{\mu\nu}
\end{aligned} \tag{6.23}$$

D_A^2 can be written in the form (5.47) using equations (5.45), (5.46) and (5.43):

$$\omega'_\mu = \tfrac{1}{2} g_{\mu\nu} (A^\mu + \Gamma^\mu) = \omega_\mu + \boldsymbol{A}_\mu, \tag{6.24}$$

$$\nabla_\mu = \partial_\mu + \omega'_\mu = \partial_\mu + \omega_\mu + \boldsymbol{A}_\mu, \tag{6.25}$$

$$E = B - g^{\mu\nu} (\partial_\mu \omega'_\nu + \omega'_\mu \omega'_\nu) + \Gamma^\mu \omega'_\mu = -\tfrac{1}{4} R + \tfrac{1}{2} \gamma^\mu \gamma^\nu \boldsymbol{F}_{\mu\nu}.^{2} \tag{6.26}$$

[1] We have a clash of notations here: this A^μ does not denote a gauge field, but the A^μ of eq. (5.42).

[2] The second term has a different sign, compared with [4], eq. (1.593). This does not change the final result.

Because of this form of ∇, its curvature is of a neat form: the sum of the spin curvature (3.58) and the curvature of $\boldsymbol{A}_\mu$ (6.21):

$$\begin{aligned}\Omega_{\mu\nu} &= \partial_\mu \omega'_\nu - \partial_\mu \omega'_\nu + [\omega'_\mu, \omega'_\nu] \\ &= \partial_\mu \omega_\nu - \partial_\mu \omega_\nu + [\omega_\mu, \omega_\nu] + \partial_\mu \boldsymbol{A}_\nu - \partial_\mu \boldsymbol{A}_\nu + [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \\ &= \Omega^S_{\mu\nu} + \boldsymbol{F}_{\mu\nu} = \tfrac{1}{4} R_{\mu\nu\xi\pi} \gamma^\xi \gamma^\pi + \boldsymbol{F}_{\mu\nu}.\end{aligned} \tag{6.27}$$

Now we can calculate the Seeley–DeWitt coefficients (5.50):

- For $k = 0$ we simply have:

$$a_0(x, D_A^2) = \frac{1}{(4\pi)^2} \operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \mathbb{1} = \frac{N^2}{4\pi^2}, \tag{6.28}$$

$$a_0(D_A^2) = \frac{N^2}{4\pi^2} \int_M \mathrm{d}^4 x \sqrt{g}. \tag{6.29}$$

- For $k = 2$ we get

$$\begin{aligned} a_2(x, D_A^2) &= \frac{1}{(4\pi)^2} \operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \big(-\tfrac{1}{6} R + E \big) \\ &= \frac{1}{(4\pi)^2} \operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \big(\tfrac{1}{12} R + \tfrac{1}{2} \gamma^\mu \gamma^\nu \boldsymbol{F}_{\mu\nu} \big) \\ &= \frac{1}{48\pi^2} N^2 R. \end{aligned} \tag{6.30}$$

We used that

$$\operatorname{Tr}_{\mathbb{C}^4} (\gamma^\mu \gamma^\nu) \boldsymbol{F}_{\mu\nu} = 0, \tag{6.31}$$

since $\operatorname{Tr}_{\mathbb{C}^4} \gamma^\mu \gamma^\nu$ is symmetric and $\boldsymbol{F}_{\mu\nu}$ is antisymmetric in μ and ν. So:

$$a_2(D_A^2) = \frac{N^2}{48\pi^2} \int_M \mathrm{d}^4 x \sqrt{g}\, R. \tag{6.32}$$

- To calculate the coefficient for $k = 4$, we first compute

$$\begin{aligned} &\operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \big(\tfrac{1}{6} R + E \big)^2 \\ &= \operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \big(\tfrac{1}{12} R + \tfrac{1}{2} \gamma^\mu \gamma^\nu \boldsymbol{F}_{\mu\nu} \big)^2 \\ &= \operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \big(\tfrac{1}{144} R^2 + \tfrac{1}{4} \gamma^\mu \gamma^\nu \gamma^\xi \gamma^\pi \boldsymbol{F}_{\mu\nu} \boldsymbol{F}_{\xi\pi} + \tfrac{1}{12} \gamma^\mu \gamma^\nu \boldsymbol{F}_{\mu\nu} R \big) \\ &= \tfrac{1}{36} N^2 R^2 + (g^{\mu\nu} g^{\xi\pi} - g^{\mu\xi} g^{\nu\pi} + g^{\mu\pi} g^{\nu\xi}) \operatorname{Tr}_{\mathrm{M}_N(\mathbb{C})} \boldsymbol{F}_{\mu\nu} \boldsymbol{F}_{\xi\pi} \\ &= \tfrac{1}{36} N^2 R^2 - 2 \operatorname{Tr}_{\mathrm{M}_N(\mathbb{C})} \boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu} \end{aligned} \tag{6.33}$$

and

$$\begin{aligned} &\operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \Omega_{\mu\nu} \Omega^{\mu\nu} \\ &= \operatorname{Tr}_{\mathbb{C}^4 \otimes \mathrm{M}_N(\mathbb{C})} \big(\tfrac{1}{16} R_{\mu\nu\xi\pi} R^{\mu\nu}{}_{\rho\sigma} \gamma^\xi \gamma^\pi \gamma^\rho \gamma^\sigma + \boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu} \\ &\qquad + \tfrac{1}{2} R_{\mu\nu\xi\pi} \gamma^\xi \gamma^\pi \boldsymbol{F}^{\mu\nu} \big) \\ &= \tfrac{1}{4} N^2 R_{\mu\nu\xi\pi} R^{\mu\nu}{}_{\rho\sigma} (g^{\xi\pi} g^{\rho\sigma} - g^{\xi\rho} g^{\pi\sigma} + g^{\xi\sigma} g^{\pi\rho}) \\ &\quad + 4 \operatorname{Tr}_{\mathrm{M}_N(\mathbb{C})} \boldsymbol{F}^{\mu\nu} \boldsymbol{F}_{\mu\nu} \\ &= -\tfrac{1}{2} N^2 R_{\mu\nu\xi\pi} R^{\mu\nu\xi\pi} + 4 \operatorname{Tr}_{\mathrm{M}_N(\mathbb{C})} \boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu}, \end{aligned} \tag{6.34}$$

using eq. (6.31),

$$\mathrm{Tr}_{\mathbb{C}^4}\,\gamma^\mu\gamma^\nu\gamma^\xi\gamma^\pi = g^{\mu\nu}g^{\xi\pi} - g^{\mu\xi}g^{\nu\pi} + g^{\mu\pi}g^{\nu\xi} \tag{6.35}$$

and $R_{\mu\nu\xi\pi}\,\mathrm{Tr}_{\mathbb{C}^4}\,\gamma^\xi\gamma^\pi = 0$ (because $R_{\mu\nu\xi\pi} = -R_{\mu\nu\pi\xi}$). So:

$$\begin{aligned}
&a_4(x, D_A^2)\\
&= \frac{1}{(4\pi)^2}\,\mathrm{Tr}_{\mathbb{C}^4\otimes \mathrm{M}_N(\mathbb{C})}\Big(\tfrac{1}{2}\big(-\tfrac{1}{6}R + E\big)^2 - \tfrac{1}{180}R_{\mu\nu}R^{\mu\nu}\\
&\qquad\qquad + \tfrac{1}{180}R_{\mu\nu\xi\pi}R^{\mu\nu\xi\pi} + \tfrac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu}\Big) + \text{b.t.}\\
&= \frac{1}{(4\pi)^2}\big(\tfrac{1}{72}N^2R^2 - \tfrac{1}{45}N^2R_{\mu\nu}R^{\mu\nu} - \tfrac{7}{360}N^2R_{\mu\nu\xi\pi}R^{\mu\nu\xi\pi}\\
&\qquad - \tfrac{2}{3}\,\mathrm{Tr}_{\mathrm{M}_N(\mathbb{C})}\,\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\big) + \text{b.t.}\\
&= \frac{1}{(4\pi)^2}\big(-\tfrac{1}{20}N^2C_{\mu\nu\xi\pi}C^{\mu\nu\xi\pi} + N^2\tfrac{11}{360}R^*R^*\\
&\qquad - \tfrac{2}{3}\,\mathrm{Tr}_{\mathrm{M}_N(\mathbb{C})}\,\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\big) + \text{b.t.}
\end{aligned} \tag{6.36}$$

'b.t.' stands for 'boundary terms', the terms proportional to ${R_{;\mu}}^{\mu}$ and ${E_{;\mu}}^{\mu}$. We assume M to be boundaryless, so by Stokes' Theorem these terms vanish after integration. In the last line, $C_{\mu\nu\xi\pi}$ stands for the *Weyl tensor* (the traceless part of the Riemann tensor) and R^*R^* for the *Pontrjagin class*, for which

$$\chi(M) = \frac{1}{32\pi^2}\int_M \mathrm{d}^4x\sqrt{g}\,R^*R^*, \tag{6.37}$$

where $\chi(M)$ is M's *Euler characteristic*.[3] So the Seeley–DeWitt coefficient is

$$\begin{aligned}
a_4(D_A^2) = \int_M \mathrm{d}^4x\sqrt{g}\,\frac{1}{(4\pi)^2}\big(&-\tfrac{1}{20}N^2C_{\mu\nu\xi\pi}C^{\mu\nu\xi\pi}\\
&- \tfrac{2}{3}N^2\,\mathrm{Tr}_{\mathrm{M}_N(\mathbb{C})}\,\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\big) + \tfrac{44}{45}\pi^2N^2\chi(M).
\end{aligned} \tag{6.38}$$

Inserting these coefficients in eq. (5.64) gives the following expression for the spectral action:

$$\mathrm{Tr}_{\mathscr{H}}\big(f(D_A/\Lambda)\big) = \frac{1}{4\pi^2}\int_M \mathrm{d}^4x\sqrt{g}\,\mathscr{L}(g,A) + \mathscr{O}(\Lambda^{-2}) + \text{t.t.}, \tag{6.39}$$

where $\mathscr{L}$ is the Lagrangian

$$\begin{aligned}
\mathscr{L}(g,A) = 2N^2f_4\Lambda^4 &+ \tfrac{1}{6}N^2f_2\Lambda^2R\\
&+ N^2f(0)\big(-\tfrac{1}{80}C_{\mu\nu\xi\pi}C^{\mu\nu\xi\pi} - \tfrac{1}{6}\,\mathrm{Tr}_{\mathrm{M}_N(\mathbb{C})}\,\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\big)
\end{aligned} \tag{6.40}$$

and 't.t.' stands for 'topological term': something proportional to $\chi(M)$. This is just a constant number, which we will not see in the equations of motion.

[3] For more details: see [4], the proof of Thm. 1.158.

This Lagrangian is interpreted as follows: the first term is a cosmological term, the second one the Einstein–Hilbert Lagrangian,[4] the third term corresponds to higher-oder gravity (which is suppressed with a factor Λ^{-2} with respect to Einsteinian gravity) and the fourth is the gauge-boson part of the Yang–Mills Lagrangian.[5]

6.2 The Standard Model

6.2.1 Algebra, Representation & Gauge Theory

We take the spectral triple of the form (4.14), with the finite-dimensional algebra

$$\mathscr{A}_F = \mathrm{M}_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C}.^6 \tag{6.41}$$

Note that this is not a complex algebra, but only a real one.[7]

We take the 96-dimensional Hilbert space

$$\mathscr{H}_F = (\mathscr{H}' \oplus \mathscr{H}')^{\oplus 3}, \tag{6.42}$$

where

$$\mathscr{H}' = (\mathbb{C}^3 \oplus \mathbb{C}^3) \oplus (\mathbb{C}^3 \oplus \mathbb{C}^3) \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus (\mathbb{C} \oplus \mathbb{C}). \tag{6.43}$$

The action of $\mathscr{A}_F$ on $\mathscr{H}_F$ is given by

$$\begin{aligned}&\pi_F(m, q, \lambda)\\ &= \mathbb{1}_3 \otimes \left(\begin{array}{cccc|cccc} \mathbb{1}_2 \otimes m & & & & & & & \\ & \mathbb{1}_2 \otimes m & & & & & & \\ & & \lambda \mathbb{1}_2 & & & & & \\ & & & \lambda \mathbb{1}_2 & & & & \\ \hline & & & & q \otimes \mathbb{1}_3 & & & \\ & & & & & \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix} \otimes \mathbb{1}_3 & & \\ & & & & & & q & \\ & & & & & & & \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix} \end{array}\right).\end{aligned} \tag{6.44}$$

Note that π_F is only real linear and not complex linear.

For the real structure, we take

$$J_F = \mathbb{1}_3 \otimes \begin{pmatrix} & \mathbb{1}_{16} \\ \mathbb{1}_{16} & \end{pmatrix} \circ \overline{(\cdot)}, \tag{6.45}$$

which gives

$$\begin{aligned}&J_F \pi_F(m, q, \lambda) J_F^*\\ &= \mathbb{1}_3 \otimes \left(\begin{array}{cccc|cccc} \bar{q} \otimes \mathbb{1}_3 & & & & & & & \\ & \begin{pmatrix} \bar{\lambda} & \\ & \lambda \end{pmatrix} \otimes \mathbb{1}_3 & & & & & & \\ & & \bar{q} & & & & & \\ & & & \begin{pmatrix} \bar{\lambda} & \\ & \lambda \end{pmatrix} & & & & \\ \hline & & & & \mathbb{1}_2 \otimes \overline{m} & & & \\ & & & & & \mathbb{1}_2 \otimes \overline{m} & & \\ & & & & & & \bar{\lambda} \mathbb{1}_2 & \\ & & & & & & & \bar{\lambda} \mathbb{1}_2 \end{array}\right).\end{aligned} \tag{6.46}$$

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[4] See [20], eq. (E.1.12).

[5] See [21], eq. (15.2.3).

[6] $\mathbb{H}$ stands for the algebra of quaternions, see footnote 1 on page 13.

[7] The algebra of a spectral triple can be complexified by taking $\mathscr{A}_F + i\mathscr{A}_F$.

For this spectral triple we have $\varepsilon_F = J_F^2 = 1$.

The group of unitary elements of $\mathscr{A}_F$ is

$$\mathrm{U}(\mathscr{A}_F) = \mathrm{U}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1). \tag{6.47}$$

The algebra $\widetilde{\mathscr{A}_F}_{J_F}$ is

$$\begin{aligned} \widetilde{\mathscr{A}_F}_{J_F} &= \{a \in \mathscr{A}_F \mid \pi_F(a) = \pi_F^\circ(a)\} \\ &= \{(m, q, \lambda) \in \mathscr{A}_F \mid \lambda = \overline{\lambda},\ q = \lambda \mathbb{1}_2,\ m = \lambda \mathbb{1}_3\} \\ &= \operatorname{span}_{\mathbb{R}}\{(\mathbb{1}_3, \mathbb{1}_2, 1)\} \simeq \mathbb{R}, \end{aligned} \tag{6.48}$$

so we have

$$\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F}) \simeq \mu_2,^8 \tag{6.49}$$

which is only a finite abelian group. In § 6.1.1 we argued that finite groups can be ignored. This means that we have (eq. (6.47)) for the gauge group.

Because $\mathscr{A}_F$ is not a complex algebra, and π_F is not complex linear, the unimodularity condition is not satisfied naturally. Therefore we impose it 'by hand': we restrict the gauge group to

$$\begin{aligned} \mathrm{S}_{\pi_F}\mathscr{G}(\mathscr{A}_F) &= \{(u_3, u_2, \zeta) \in \mathrm{U}(\mathscr{A}_F) \mid \det \pi_F(u_3, u_2, \zeta) = 1\} \\ &= \{(u_3, u_2, \zeta) \in \mathrm{U}(\mathscr{A}_F) \mid (\zeta \det u_3)^{12} = 1\} \\ &\sim \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1). \end{aligned} \tag{6.50}$$

In the last step we used that kernel of the surjective homomorphism

$$\phi : \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{S}_{\pi_F}\mathrm{U}(\mathscr{A}_F), \quad \phi(u_3, u_2, \zeta) = (\zeta u_3, u_2, \zeta^{-3}) \tag{6.51}$$

is finite and abelian:

$$\ker \phi = \{(\zeta^{-1}\mathbb{1}_3, \mathbb{1}_2, \zeta) \mid \zeta \in \mathrm{U}(1), \zeta^3 = 1\} \simeq \mu_3. \tag{6.52}$$

The gauge group elements $(u_3, u_2, \zeta) \in \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ are represented on $\mathscr{H}_F$ as

$$\begin{aligned} &\rho(\phi(u_3, u_2, \zeta)) = \rho(\zeta u_3, u_2, \zeta^{-3}) = \pi_F(\zeta u_3, u_2, \zeta^{-3}) J_F \pi_F(\zeta u_3, u_2, \zeta^{-3}) J_F^* \\ &= \mathbb{1}_3 \otimes \left(\begin{array}{cccc|cccc} \zeta\overline{u}_2 \otimes u_3 &&&&&&& \\ & \begin{pmatrix} \zeta^4 & \\ & \zeta^{-2} \end{pmatrix} \otimes u_3 &&&&&& \\ && \zeta^{-3}\overline{u}_2 &&&&& \\ &&& \begin{pmatrix} 1 & \\ & \zeta^{-6} \end{pmatrix} &&&& \\ \hline &&&& \zeta^{-1}u_2 \otimes \overline{u}_3 &&& \\ &&&&& \begin{pmatrix} \zeta^{-4} & \\ & \zeta^2 \end{pmatrix} \otimes \overline{u}_3 && \\ &&&&&& \zeta^3 u_2 & \\ &&&&&&& \begin{pmatrix} 1 & \\ & \zeta^6 \end{pmatrix} \end{array}\right). \end{aligned} \tag{6.53}$$

In the language of appendix B, this representation is written as (3 copies of)

$$\left(\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\right) \oplus \left(\mathbf{3}, \mathbf{1}, \tfrac{2}{3}\right) \oplus \left(\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\right) \oplus \left(\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}\right) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{1}, 1) \tag{6.54}$$

---

[8] $\mu_2 = \{1, -1\}$, see footnote 1 on page 73.

plus the conjugates. Recall that the standard representation of $SU(2)$ is real: $\overline{\mathbf{2}} = \mathbf{2}$ (eq. (A.39)). This is exactly the particle content of the Standard Model! The right-handed neutrinos are included. $\mathscr{H}'$ is interpreted with the following particle content (for the first generation):

$$\mathscr{H}' = (\underbrace{\mathbb{C}^3 \oplus \mathbb{C}^3}_{\begin{pmatrix} u_L \\ d_L \end{pmatrix}}) \oplus (\underbrace{\mathbb{C}^3}_{u_R} \oplus \underbrace{\mathbb{C}^3}_{d_R}) \oplus (\underbrace{\mathbb{C} \oplus \mathbb{C}}_{\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}}) \oplus (\underbrace{\mathbb{C}}_{\nu_{eR}} \oplus \underbrace{\mathbb{C}}_{e_R^-}). \tag{6.55}$$

The operator J_F interchanges particles and antiparticles.

Γ_F is defined as

$$\Gamma_F = \mathbb{1}_3 \otimes \begin{pmatrix} \mathbb{1}_6 & & & \\ & -\mathbb{1}_6 & & \\ & & \mathbb{1}_2 & \\ & & & -\mathbb{1}_2 \end{pmatrix}. \tag{6.56}$$

The positive eigenspace of Γ_F consists of all left-handed fermions and antifermions, and the negative eigenspace the right-handed ones. This Γ_F gives $\varepsilon''_F = -1$.

6.2.2 Higgs Mechanism, Predictions

The Dirac operator for this model is discussed in [4], ch. 1, § 13.4 and expressed in the mass matrices of the Standard Model. Computing the inner fluctuations and the spectral and fermionic action, analogous to § 6.1.2, gives the full action of the Standard Model. This includes a Higgs sector and a coupling to gravitation.[9] Recall from § 5.1.1 that the inner fluctuations give rise to the gauge bosons and scalars. The latter is the Higgs field. The nice thing of this NCG-approach is that the Higgs sector is not added to the theory 'by hand', but appears in the same way as the gauge fields.

Certain parameters in the spectral action are related. As said in the introduction, the GUT relation for the couplings is obtained, as well as the top quark and Higgs mass at the GUT energy scale. Assuming the big desert, a Higgs mass can be computed. This is done in [4], ch. 1, § 17.10 and [6], § 8. According to the latter, this model implies a Higgs mass (at the scale of the Z mass) between 167 and 176 GeV. This is not in conflict with the experimental lower bond of 114 GeV, as measured by the LEP experiments ([17]). The Tevatron experiments have excluded Higgs masses between 158 and 173 GeV ([19]). This largely overlaps with the predicted range. One has to keep in mind that this prediction is done under the big desert hypothesis; changing this can change the prediction of this Higgs mass.

[9]See [4], ch. 1, §§ 16.1 and 17 or [22], § 10.3.

Part II

Noncommutative Geometry & SU(5) Grand Unification

Chapter 7

The SU(5) Grand Unification Model

7.1 Introduction

The idea of Grand Unified Theories (GUTs) is to extend the gauge group of the Standard Model (G_{SM}) to a larger group G, where the representations of G are such that if one restricts them to G_{SM}, they reproduce the representations of the Standard Model. Most physicists call such a model only a GUT if G is simple (like SU(5)), because in that case there is only one coupling constant.

We will study the simplest GUT: the *SU(5) model*, also known as the *Georgi–Glashow model*. It is first described by Howard Georgi and Sheldon Glashow in 1974 ([8]): "*We present a series of hypotheses and speculations leading inescapably to the conclusion that* $\mathrm{SU}(5)$ *is the gauge group of the world — that all elementary particle forces (strong, weak, electromagnetic) are different manifestations of the same fundamental interaction involving a single coupling strength, the fine-structure constant. Our hypotheses may be wrong and our speculations idle, but the uniqueness and simplicity of our scheme are reasons enough that it be taken seriously.*"

The most important motivation to study GUTs, is the apparent arbitrariness of the Standard Model, as you see appendix B. As we will see, with little input, the $\mathrm{SU}(5)$ model gives back all fermions representations of the Standard Model.

Discussions about this model and GUTs in general can be found in [1], [3], ch. 14, [11], ch. 18, [15], and [18], ch. 4.

7.2 The Gauge Group

As the name suggests, this model is based on the group $\mathrm{SU}(5)$. How does the Standard Model gauge group extend to $\mathrm{SU}(5)$? Clearly, $\mathrm{U}(3) \times \mathrm{U}(2)$ is a subgroup of $\mathrm{U}(5)$ and hence

$$\mathrm{S}\big(\mathrm{U}(3) \times \mathrm{U}(2)\big) = \left\{ \begin{pmatrix} u_3 & \\ & u_2 \end{pmatrix} \in \mathrm{U}(3) \times \mathrm{U}(2) \;\middle|\; \det u_3 \, \det u_2 = 1 \right\} \tag{7.1}$$

is a subgroup of $\mathrm{SU}(5)$.

Let ϕ be the surjective group homomorphism

$$\begin{aligned}\phi : \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) &\to \mathrm{S}\big(\mathrm{U}(3) \times \mathrm{U}(2)\big),\\ \phi(u_3, u_2, \zeta) &= \begin{pmatrix} \zeta^{-2}u_3 & \\ & \zeta^3 u_2 \end{pmatrix}.\end{aligned} \tag{7.2}$$

It has a nontrivial kernel:

$$\begin{aligned}\ker\phi &= \{(u_3, u_2, \zeta) \in \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \mid \zeta^{-2}u_3 = \mathbb{1}_3,\ \zeta^3 u_2 = \mathbb{1}_2\}\\ &= \{(\zeta^2 \mathbb{1}_3, \zeta^{-3}\mathbb{1}_2, \zeta) \in \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\}\\ &= \{(\zeta^2 \mathbb{1}_3, \zeta^{-3}\mathbb{1}_2, \zeta) \mid \zeta \in \mathrm{U}(1),\ \zeta^6 = 1\}\\ &= \{(\zeta^2 \mathbb{1}_3, \zeta^{-3}\mathbb{1}_2, \zeta) \mid \zeta \in \mu_6\} \simeq \mu_6.\end{aligned} \tag{7.3}$$

We can conclude that

$$\mathrm{S}\big(\mathrm{U}(3) \times \mathrm{U}(2)\big) \simeq \big(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\big)/\mu_6, \tag{7.4}$$

which is exactly the 'true' gauge group of the Standard Model (eq. (B.7)). This makes $\mathrm{SU}(5)$ a very interesting candidate for the gauge group of our unifying theory, because this subgroup respects the pattern of the hypercharges. To use the words of [1]: "$\mathrm{SU}(5)$ *passes the test, not despite the bizarre pattern followed by hypercharges, but because of it!*"

The next thing we have to do is finding a representation of $\mathrm{SU}(5)$ that extends the Standard Model.

7.3 The Fermion Representations

To simplify things, we will assume in this section only one generation of particles. This means also that we forget about Cabibbo–Kobayashi–Maskawa mixing of families.

Recall that the fermion representation of the Standard Model is 30-dimensional: 15 for the particles and 15 for the antiparticles, if we do not include the right-handed neutrino (see appendix B). For $\mathrm{SU}(5)$, we can make a 15 dimensional representation by taking $\mathbf{5} \oplus \mathbf{10}$ (see § A.1.1). Adding the conjugate to this representation gives the 30-dimensional representation

$$\mathbf{5} \oplus \overline{\mathbf{5}} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}. \tag{7.5}$$

Let us see what happens if we restrict these representations to G_{SM}. We use the embedding of G_{SM} in $\mathrm{SU}(5)$ given by (7.2):

$$\begin{pmatrix} \zeta^{-2}u_3 & \\ & \zeta^3 u_2 \end{pmatrix} \in \mathrm{SU}(5), \tag{7.6}$$

where $u_3 \in \mathrm{SU}(3)$, $u_2 \in \mathrm{SU}(2)$ and $\zeta \in \mathrm{U}(1)$. The action of the $\mathrm{SU}(5)$ matrix on $\mathbb{C}^5$ is now split in the action of $\zeta^{-2}u_3$ on $\mathbb{C}^3$ and $\zeta^3 u_2$ on $\mathbb{C}^2$.

- First, we look at the $\mathbf{5}$:

– The matrix (7.6) acts as follows on $\mathbb{C}^3$:

$$v \mapsto \zeta^{-2} u_3 v. \tag{7.7}$$

In the language of appendix B, this is the $\left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right)$ and corresponds to the $\mathrm{d_R}$.[1]

– On a vector in $\mathbb{C}^2$ it acts as

$$v \mapsto \zeta^3 u_2 v, \tag{7.8}$$

which is the $\left(\mathbf{1}, \mathbf{2}, \frac{1}{2}\right)$ and corresponds to $\begin{pmatrix} \overline{\nu}_{\mathrm{eR}} \\ \mathrm{e_R^+} \end{pmatrix}$.

So we interpret the particle content of the **5** as

$$\mathbf{5} : \begin{pmatrix} \mathrm{d_R^r} \\ \mathrm{d_R^g} \\ \mathrm{d_R^b} \\ \overline{\nu}_{\mathrm{eR}} \\ \mathrm{e_R^+} \end{pmatrix}. \tag{7.9}$$

With this notation we mean that e_1 corresponds to $\mathrm{d_R^r}$ etc. (The e_i denote the 5 standard basis vectors of $\mathbb{C}^5$.) Taking the conjugate gives us the content of the $\overline{\mathbf{5}}$:

$$\overline{\mathbf{5}} : \begin{pmatrix} \overline{\mathrm{d}}_{\mathrm{L}}^{\mathrm{r}} \\ \overline{\mathrm{d}}_{\mathrm{L}}^{\mathrm{g}} \\ \overline{\mathrm{d}}_{\mathrm{L}}^{\mathrm{b}} \\ \nu_{\mathrm{eL}} \\ \mathrm{e_L^-} \end{pmatrix}. \tag{7.10}$$

- Now we look at the **10**. First, note that

$$\textstyle\bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C}^2) \simeq \bigwedge^2 \mathbb{C}^3 \oplus \bigwedge^2 \mathbb{C}^2 \oplus \mathbb{C}^3 \otimes \mathbb{C}^2 \tag{7.11}$$

as vector spaces. The matrix (7.6) acts on this as follows:

– On $\bigwedge^2 \mathbb{C}^3$:

$$v \wedge w \mapsto \zeta^{-4}\, u_3 v \wedge u_3 w \tag{7.12}$$

which is the $\left(\bigwedge^2 \mathbf{3}, \mathbf{1}, -\frac{2}{3}\right) = \left(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3}\right)$, where we used eq. (A.34). This corresponds to the $\overline{\mathrm{u}}_{\mathrm{L}}$.

– On $\bigwedge^2 \mathbb{C}^2$:

$$v \wedge w \mapsto \zeta^6\, u_2 v \wedge u_2 w, \tag{7.13}$$

which is the $(\mathbf{1}, \bigwedge^2 \mathbf{2}, 1) = (\mathbf{1}, \mathbf{1}, 1)$ (because of (A.41)) and corresponds to the $\mathrm{e_L^+}$.

– On $\mathbb{C}^3 \otimes \mathbb{C}^2$:

$$v_3 \otimes v_2 \mapsto \zeta\, u_3 v_3 \otimes u_2 v_2, \tag{7.14}$$

which is the $\left(\mathbf{3}, \mathbf{2}, \frac{1}{6}\right)$ and corresponds to $\begin{pmatrix} \mathrm{u_L} \\ \mathrm{d_L} \end{pmatrix}$.

[1]Recall that in this notation, the third number is $\frac{1}{6}q$, where q is the power of the element in $\mathrm{U}(1)$; so in this case we have $\frac{1}{6} \cdot -2 = -\frac{1}{3}$.

So we interpret the content of the **10** as follows:

$$\mathbf{10}: \begin{pmatrix} 0 & \overline{\mathrm{u}}_{\mathrm{L}}^{\mathrm{b}} & -\overline{\mathrm{u}}_{\mathrm{L}}^{\mathrm{g}} & \mathrm{u}_{\mathrm{L}}^{\mathrm{r}} & \mathrm{d}_{\mathrm{L}}^{\mathrm{r}} \\ & 0 & \overline{\mathrm{u}}_{\mathrm{L}}^{\mathrm{r}} & \mathrm{u}_{\mathrm{L}}^{\mathrm{g}} & \mathrm{d}_{\mathrm{L}}^{\mathrm{g}} \\ & & 0 & \mathrm{u}_{\mathrm{L}}^{\mathrm{b}} & \mathrm{d}_{\mathrm{L}}^{\mathrm{b}} \\ & & & 0 & \mathrm{e}_{\mathrm{L}}^{+} \\ & & & & 0 \end{pmatrix}. \tag{7.15}$$

With this notation we mean that $e_1 \wedge e_2$ corresponds to $\overline{\mathrm{u}}_{\mathrm{L}}^{\mathrm{b}}$ etc. The meaning of the $-$ sign before $\overline{\mathrm{u}}_{\mathrm{R}}^{\mathrm{b}}$ will be explained in § 7.3.1. The content of the $\overline{\mathbf{10}}$ is

$$\overline{\mathbf{10}}: \begin{pmatrix} 0 & \mathrm{u}_{\mathrm{R}}^{\mathrm{b}} & -\mathrm{u}_{\mathrm{R}}^{\mathrm{g}} & \overline{\mathrm{u}}_{\mathrm{R}}^{\mathrm{r}} & \overline{\mathrm{d}}_{\mathrm{R}}^{\mathrm{r}} \\ & 0 & \mathrm{u}_{\mathrm{R}}^{\mathrm{r}} & \overline{\mathrm{u}}_{\mathrm{R}}^{\mathrm{g}} & \overline{\mathrm{d}}_{\mathrm{R}}^{\mathrm{g}} \\ & & 0 & \overline{\mathrm{u}}_{\mathrm{R}}^{\mathrm{b}} & \overline{\mathrm{d}}_{\mathrm{R}}^{\mathrm{b}} \\ & & & 0 & \mathrm{e}_{\mathrm{R}}^{-} \\ & & & & 0 \end{pmatrix}. \tag{7.16}$$

To summarize this, we denote the symmetry breaking of the fermion representation as:

$$\mathbf{5} \rightsquigarrow \left(\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\right) \oplus \left(\mathbf{1}, \mathbf{2}, \tfrac{1}{2}\right), \tag{7.17a}$$

$$\mathbf{10} \rightsquigarrow \left(\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}\right) \oplus \left(\mathbf{1}, \mathbf{1}, 1\right) \oplus \left(\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\right). \tag{7.17b}$$

Note that the $\overline{\mathbf{5}}$ and the **10** contain the left-handed particles and antiparticles and the **5** and $\overline{\mathbf{10}}$ the right-handed ones.

We can conclude that the representation (7.5) gives the fermions of the Standard Model by breaking SU(5) to the Standard Model gauge group.

7.3.1 Fermion Representations & Exterior Products

In [1],[2] the SU(5) model is described from a nice point of view. In this approach, it is nice to include right-handed neutrinos, so we need a 32-dimensional representation. The full exterior algebra of $\mathbb{C}^5$,

$$\bigwedge \mathbb{C}^5 = \mathbb{C} \oplus \mathbb{C}^5 \oplus \bigwedge^2 \mathbb{C}^5 \oplus \bigwedge^3 \mathbb{C}^5 \oplus \bigwedge^4 \mathbb{C}^5 \oplus \bigwedge^5 \mathbb{C}^5 \tag{7.18}$$

is 32-dimensional.[3]

We let an $u \in \mathrm{SU}(5)$ act on this space by the representation

$$\rho : \mathrm{SU}(5) \to \mathrm{GL}(\bigwedge \mathbb{C}^5), \quad \rho(u)(v_1 \wedge \cdots \wedge v_k) = uv_1 \wedge \cdots \wedge uv_k, \tag{7.19}$$

or in the language of § A.1:

$$\begin{aligned} \rho &= \mathrm{Triv} \oplus \mathrm{St} \oplus \textstyle\bigwedge^2 \mathrm{St} \oplus \bigwedge^3 \mathrm{St} \oplus \bigwedge^4 \mathrm{St} \oplus \bigwedge^5 \mathrm{St} \\ &\sim \mathrm{Triv} \oplus \mathrm{St} \oplus \textstyle\bigwedge^2 \mathrm{St} \oplus \bigwedge^2 \overline{\mathrm{St}} \oplus \overline{\mathrm{St}} \oplus \mathrm{Triv}, \end{aligned} \tag{7.20}$$

where we used Prop. A.11. In the language of § A.1.1, this can be written as

$$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \overline{\mathbf{5}} \oplus \mathbf{1}. \tag{7.21}$$

[2]Some $-$-signs are chosen here differently as in [1].

[3]See Def. A.3i.

Except for the two singlets, this is exactly (7.5)!

In order of interpret the basis of $\bigwedge \mathbb{C}^5$ as particles, we will call the basis vectors of $\mathbb{C}^5$ r, g, b, u and d. r, g and b stand for the thee quark colours and u and d for weak isospin up and down.

Given (7.9), we have the following correspondence between particles and rank-1 tensors:

$$\begin{aligned} \mathrm{d}_\mathrm{R}^\mathrm{r} &: r \\ \mathrm{d}_\mathrm{R}^\mathrm{g} &: g \\ \mathrm{d}_\mathrm{R}^\mathrm{b} &: b \end{aligned} \qquad \begin{aligned} \overline{\nu}_\mathrm{eR} &: u \\ \mathrm{e}_\mathrm{R}^+ &: d. \end{aligned} \tag{7.22}$$

The $\wedge$-structure of our representation gives a nice implementation of charge conjugation: we let it correspond with taking the Hodge dual, since this corresponds to taking the conjugate representation (see Prop. A.11). So for the rank-4 antisymmetric tensors we have the correspondences:

$$\begin{aligned} \overline{\mathrm{d}}_\mathrm{L}^\mathrm{r} &: *r = g \wedge b \wedge u \wedge d \\ \overline{\mathrm{d}}_\mathrm{L}^\mathrm{g} &: *g = b \wedge r \wedge u \wedge d \\ \overline{\mathrm{d}}_\mathrm{L}^\mathrm{b} &: *b = r \wedge g \wedge u \wedge d \end{aligned} \qquad \begin{aligned} \nu_\mathrm{eL} &: *u = g \wedge r \wedge b \wedge d \\ \mathrm{e}_\mathrm{L}^- &: *d = r \wedge g \wedge b \wedge u. \end{aligned} \tag{7.23}$$

Given (7.15), we have for the rank-2 tensors:

$$\begin{aligned} \overline{\mathrm{u}}_\mathrm{L}^\mathrm{b} &: r \wedge g \\ \overline{\mathrm{u}}_\mathrm{L}^\mathrm{g} &: b \wedge r \\ \overline{\mathrm{u}}_\mathrm{L}^\mathrm{g} &: g \wedge b \\ \\ \mathrm{e}_\mathrm{L}^+ &: u \wedge d \end{aligned} \qquad \begin{aligned} \mathrm{u}_\mathrm{L}^\mathrm{r} &: r \wedge u \\ \mathrm{u}_\mathrm{L}^\mathrm{g} &: g \wedge u \\ \mathrm{u}_\mathrm{L}^\mathrm{b} &: b \wedge u \\ \mathrm{d}_\mathrm{L}^\mathrm{r} &: r \wedge d \\ \mathrm{d}_\mathrm{L}^\mathrm{g} &: g \wedge d \\ \mathrm{d}_\mathrm{L}^\mathrm{b} &: b \wedge d \end{aligned} \tag{7.24}$$

Taking the conjugates/Hodge duals, we get for the rank-3 tensors:

$$\begin{aligned} \mathrm{u}_\mathrm{R}^\mathrm{b} &: *r \wedge g = b \wedge u \wedge d \\ \mathrm{u}_\mathrm{R}^\mathrm{g} &: *b \wedge r = g \wedge u \wedge d \\ \mathrm{u}_\mathrm{R}^\mathrm{g} &: *g \wedge b = r \wedge u \wedge d \\ \\ \mathrm{e}_\mathrm{R}^- &: *u \wedge d = r \wedge g \wedge b \end{aligned} \qquad \begin{aligned} \overline{\mathrm{u}}_\mathrm{R}^\mathrm{r} &: *r \wedge u = g \wedge b \wedge d \\ \overline{\mathrm{u}}_\mathrm{R}^\mathrm{g} &: *g \wedge u = b \wedge r \wedge d \\ \overline{\mathrm{u}}_\mathrm{R}^\mathrm{b} &: *b \wedge u = r \wedge g \wedge d \\ \overline{\mathrm{d}}_\mathrm{R}^\mathrm{r} &: *r \wedge d = b \wedge g \wedge u \\ \overline{\mathrm{d}}_\mathrm{R}^\mathrm{g} &: *g \wedge d = r \wedge b \wedge u \\ \overline{\mathrm{d}}_\mathrm{R}^\mathrm{b} &: *b \wedge d = g \wedge r \wedge u \end{aligned} \tag{7.25}$$

Now it is clear why it is nice to choose the minus sign in front of the $\overline{\mathrm{u}}_\mathrm{L}^\mathrm{g}$ in (7.15): the correspondence $\overline{\mathrm{u}}_\mathrm{L}^\mathrm{b} : r \wedge g$ holds for all cyclic permutations of r, g and b and consequently, $\mathrm{u}_\mathrm{R}^c : c \wedge u \wedge d$ holds for all $c \in \{r, g, b\}$.

Now only the rank-0 and rank-5 tensors have no interpretation yet. It does not matter which one corresponds to the right-handed neutrino and the left-handed anti-neutrino, they correspond both to the trivial representation. It is nice to choose

$$\overline{\nu}_\mathrm{eL} : 1, \qquad \nu_\mathrm{eR} : r \wedge g \wedge b \wedge u \wedge d, \tag{7.26}$$

such that all left-handed particles and antiparticles correspond to even-rank tensors and all right-handed ones to odd-rank tensors.

7.4 New Gauge and Higgs Bosons

The new symmetry which is introduced in the SU(5) model correspond to new gauge bosons. In the Standard Model we have $(3^2-1)+(2^2-1)+1=12$ gauge bosons and in the SU(5) model we have $5^2-1=24$ of them. The 12 new ones are sometimes called the *X* and *Y bosons* or *lepto-quarks* and *diquarks*.

The breaking of the SU(5)-symmetry to the Standard Model gauge group via the embedding (7.6) is achieved with a Higgs field in the adjoint representation (the **24**). The corresponding Higgs potential is such that the vacuum state is a multiple of

$$\begin{pmatrix} 2\mathbb{1}_3 & \\ & -3\mathbb{1}_2 \end{pmatrix} \in \mathrm{M}_5(\mathbb{C})_{\mathrm{Tr}=0}. \tag{7.27}$$

The subgroup of SU(5) that leaves the vacuum state invariant is

$$\begin{aligned} &\left\{ u = \mathrm{SU}(5) \;\middle|\; u \begin{pmatrix} 2\mathbb{1}_3 & \\ & -3\mathbb{1}_2 \end{pmatrix} u^* = \begin{pmatrix} 2\mathbb{1}_3 & \\ & -3\mathbb{1}_2 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} u_3 & \\ & u_2 \end{pmatrix} \in \mathrm{SU}(5) \;\middle|\; u_3 \in \mathrm{U}(3),\ u_2 \in \mathrm{U}(2) \right\} = \mathrm{S}\big(\mathrm{U}(3)\times\mathrm{U}(2)\big), \end{aligned} \tag{7.28}$$

which is indeed exactly what we need (see § 7.2).

To achieve the electroweak symmetry breaking (see § B.2), a Higgs field in the **5** is introduced. Recall from eq. (7.17a) that this representation breaks to

$$\mathbf{5} \rightsquigarrow \big(\mathbf{3},\mathbf{1},-\tfrac{1}{3}\big) \oplus \big(\mathbf{1},\mathbf{2},\tfrac{1}{2}\big). \tag{7.29}$$

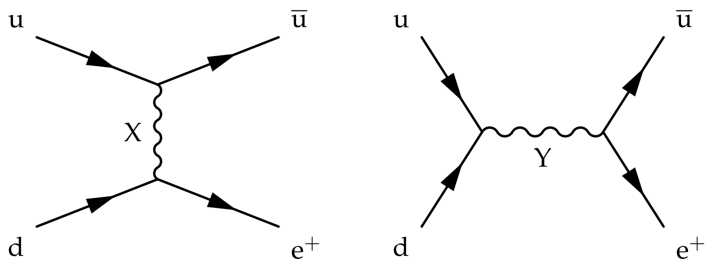
This is what we need, since the Standard Model Higgs transforms as a $\big(\mathbf{1},\mathbf{2},\tfrac{1}{2}\big)$ (eq. (B.8)). The Higgs potential has to be such that the vacuum state is a multiple of

$$\begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} \in \mathbb{C}^5. \tag{7.30}$$

Note that we do not only obtain the Standard Model Higgs, but also a color triplet scalar. The latter has vacuum expectation value 0 and causes no spontaneous symmetry breaking.

7.5 Phenomenology

What does the SU(5) model imply phenomenologically? That is, what does it predict, which can be verified experimentally? There is no way to produce X and Y bosons in a collider experiment, because of their enormous mass of about $4 \cdot 10^{14}$ GeV.[4] But there is another method: the SU(5) model predicts proton decay, for example in a e^{+} plus a $\pi^{0} = u\overline{u}$ via the following channels:[5]



(7.31)

In the SU(5) model, the decay rate of the proton into a positron and a pion is calculated to be $4.5 \times 10^{29 \pm 1.7}$ yr.[6] However, the experimental lower bound is $8.2 \cdot 10^{33}$ yr ([17]).

Furthermore, the model predicts a weak mixing angle (θ_W) with $(\sin\theta_W)^2 \sim 0.206$. This is not far from the experimental observed value of about 0.231, but this is measured precise enough to rule this value out.[7]

[4] See [11], eq. (18.35).
[5] See [15], page 264 and Fig. 3.3 and [18], § 4.3.
[6] See [11], eq. (18.51).
[7] See [11], eq. (18.35) and [17].

Chapter 8

The SU(5) Model and Noncommutative Geometry

In this chapter, a spectral triple is studied (only the algebra and its representation) that gives the same gauge theory as the SU(5) model, including the right fermion representations. However, the symmetry breaking mechanism turns out to be incorrect.

8.1 Algebra, Representation & Gauge Theory

We take a real spectral triple of the form (4.14) with

$$\mathscr{A}_F = \mathrm{M}_5(\mathbb{C}) \oplus \mathbb{R}, \tag{8.1}$$

$$\mathscr{H}_F = \mathbb{C}^3 \otimes \mathscr{H}' \quad \text{where} \quad \mathscr{H}' = \mathbb{C}^5 \oplus \mathbb{C}^5 \oplus \mathbb{C}^5 \otimes \mathbb{C}^5 \oplus \mathbb{C}^5 \otimes \mathbb{C}^5, \tag{8.2}$$

and $\mathscr{A}_F$ is represented on $\mathscr{H}_F$ as

$$\pi_F(m, \lambda) = \mathbb{1}_3 \otimes \pi'(m, \lambda) \tag{8.3}$$

where

$$\pi' : \mathscr{A}_F \to \mathrm{End}(\mathscr{H}'), \quad \pi'(m, \lambda) = \begin{pmatrix} m & & & \\ & \lambda \mathbb{1}_5 & & \\ & & m \otimes \mathbb{1}_5 & \\ & & & \mathbb{1}_5 \otimes \overline{m}. \end{pmatrix}. \tag{8.4}$$

Note that $\mathscr{A}_F$ is not complex but real and π_F is not complex linear, but real linear, like the spectral triple of the Standard Model (equations (6.41) and (6.44)).

We take for the real structure

$$J_F = \mathbb{1}_3 \otimes J', \quad \text{where} \quad J' = \begin{pmatrix} & \mathbb{1}_5 & & \\ \mathbb{1}_5 & & & \\ & & & \mathbb{1}_{25} \\ & & \mathbb{1}_{25} & \end{pmatrix} \circ \overline{(\cdot)}, \tag{8.5}$$

for which $\varepsilon_F = J_F^2 = 1$. This gives

$$J'\pi'(m,\lambda)J'^* = \begin{pmatrix} \lambda\mathbb{1}_5 & & & \\ & \overline{m} & & \\ & & \mathbb{1}_5 \otimes m & \\ & & & \overline{m} \otimes \mathbb{1}_5 \end{pmatrix}. \tag{8.6}$$

The group of unitary elements in $\mathscr{A}_F$ is

$$\mathrm{U}(\mathscr{A}_F) = \mathrm{U}(5) \times \mu_2, \tag{8.7}$$

the algebra $\widetilde{\mathscr{A}_F}_{J_F}$ is

$$\begin{aligned} \widetilde{\mathscr{A}_F}_{J_F} &= \{(m,\lambda) \in \mathscr{A}_F \mid m = \lambda\mathbb{1}_5, m \otimes \mathbb{1}_5 = \mathbb{1}_5 \otimes m\} \\ &= \mathrm{span}_{\mathbb{R}}\{(\mathbb{1}_5, 1)\} \simeq \mathbb{R}, \end{aligned} \tag{8.8}$$

so the group $\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F})$ is just

$$\mathrm{U}(\widetilde{\mathscr{A}_F}_{J_F}) \simeq \mu_2. \tag{8.9}$$

This implies that the gauge group is

$$\mathscr{G}(\mathscr{A}_F) = \mathrm{U}(5) \times \{1\} \simeq \mathrm{U}(5). \tag{8.10}$$

Like in the Standard Model, we impose the unimodularity condition, which gives the group

$$\begin{aligned} \mathrm{S}_{\pi_F}\mathscr{G}(\mathscr{A}_F) &= \big\{(u,1) \in \mathrm{U}(5) \times \{1\} \;\big|\; \det \pi_F(u,1) = 1\big\} \\ &\simeq \{u \in \mathrm{U}(5) \mid \det u^3 = 1\} \sim \mathrm{SU}(5). \end{aligned} \tag{8.11}$$

The fermion representation for this gauge group is given by

$$\rho_F(u,1) = \pi_F(u,1)J_F\pi_F(u,1)J_F^* = \mathbb{1}_3 \otimes \rho'(u), \tag{8.12}$$

where

$$\rho'(u) = \pi'(u,1)J'\pi'(u,1)J' = \begin{pmatrix} u & & & \\ & \overline{u} & & \\ & & u \otimes u & \\ & & & \overline{u} \otimes \overline{u} \end{pmatrix}. \tag{8.13}$$

In the language of § A.1.1, this is three copies of

$$\mathbf{5} \oplus \overline{\mathbf{5}} \oplus \mathbf{5} \otimes \mathbf{5} \oplus \overline{\mathbf{5}} \otimes \overline{\mathbf{5}} = \mathbf{5} \oplus \overline{\mathbf{5}} \oplus \mathbf{10} \oplus \mathbf{15} \oplus \overline{\mathbf{10}} \oplus \overline{\mathbf{15}}, \tag{8.14}$$

using eq. (A.30). Except for the **15** and its conjugate, these are exactly the fermions of the SU(5) model (without right-handed neutrinos). J_F interchanges indeed particles and antiparticles.

Note that this model does not contain the right-handed neutrinos. To include these particles, one can simply extend the algebra representation with two extra representations of $\mathbb{R}$, on order to obtain two singlets.

Recall from § 7.3 that the $\overline{\mathbf{5}}$ and the $\mathbf{10}$ (i.e. the even-rank antisymmetric tensors, in the language of § 7.3.1) contain the lefthanded particles and antiparticles and the $\mathbf{5}$ and the $\overline{\mathbf{10}}$ (the odd-rank tensors) contain the righthanded ones. So it makes sense to take the grading on our spectral triple to be

$$\Gamma_F = \mathbb{1}_3 \otimes \Gamma' \quad \text{where} \quad \Gamma' = \begin{pmatrix} -\mathbb{1}_5 & & & \\ & \mathbb{1}_5 & & \\ & & \mathbb{1}_5 \otimes \mathbb{1}_5 & \\ & & & -\mathbb{1}_5 \otimes \mathbb{1}_5 \end{pmatrix}. \tag{8.15}$$

So $\varepsilon''_F = -1$.

8.2 The Dirac operator

The next step is to construct the Dirac operator D_F for our spectral triple. We make the assumption that it is of the form

$$D_F = \begin{pmatrix} D^{(1)} & & \\ & D^{(2)} & \\ & & D^{(3)} \end{pmatrix}, \tag{8.16}$$

where the $D^{(i)}$ are hermitian operators on $\mathscr{H}'$. This assumption corresponds to ignoring Cabibbo–Kobayashi–Maskawa-like mixing of fermion masses between the families.

Because D_F and Γ_F anticommute (eq. (4.4b)), $D^{(i)}$ and Γ' do, which implies that $D^{(i)}$ is of the form

$$D^{(i)} = \begin{pmatrix} & D_{\mathrm{I}}^{(i)} & D_{\mathrm{II}}^{(i)} & \\ D_{\mathrm{I}}^{(i)*} & & & D_{\mathrm{III}}^{(i)} \\ D_{\mathrm{II}}^{(i)*} & & & D_{\mathrm{IV}}^{(i)} \\ & D_{\mathrm{III}}^{(i)*} & D_{\mathrm{IV}}^{(i)*} & \end{pmatrix}, \tag{8.17}$$

where

$$\begin{gathered} D_{\mathrm{I}}^{(i)} \in \mathrm{End}(\mathbb{C}^5) = \mathrm{M}_5(\mathbb{C}), \\ D_{\mathrm{II}}^{(i)}, D_{\mathrm{III}}^{(i)} \in \mathrm{Hom}(\mathbb{C}^5 \otimes \mathbb{C}^5, \mathbb{C}^5) = \mathrm{M}_{5\times 25}(\mathbb{C}), \\ D_{\mathrm{IV}}^{(i)} \in \mathrm{End}(\mathbb{C}^5 \otimes \mathbb{C}^5) = \mathrm{M}_{25}(\mathbb{C}). \end{gathered} \tag{8.18}$$

We take $\varepsilon'_F = 1$, in other words: D_F ($D^{(i)}$) and J_F (J') commute (4.5b). Writing this out gives

$$D_{\mathrm{I}}^{(i)} = D_{\mathrm{I}}^{(i)\top}, \quad D_{\mathrm{IV}}^{(i)} = D_{\mathrm{IV}}^{(i)\top}, \quad D_{\mathrm{II}}^{(i)} = \overline{D_{\mathrm{III}}^{(i)}}. \tag{8.19}$$

Writing out the order-one condition (eq. (4.5e)) gives

$$\begin{gathered} \lambda D_{\mathrm{I}}^{(i)} \overline{n} - m D_{\mathrm{I}}^{(i)} \overline{n} = \mu \lambda D_{\mathrm{I}}^{(i)} - \mu m D_{\mathrm{I}}^{(i)}, \\ D_{\mathrm{II}}^{(i)} (m \otimes n) - m D_{\mathrm{II}}^{(i)} (\mathbb{1}_5 \otimes n) = \mu D_{\mathrm{II}}^{(i)} (m \otimes \mathbb{1}_5) - \mu m D_{\mathrm{II}}^{(i)}, \\ D_{\mathrm{IV}}^{(i)*} (m \otimes n) - (\mathbb{1}_5 \otimes \overline{m}) D_{\mathrm{IV}}^{(i)*} n = (\overline{n} \otimes 1) D_{\mathrm{IV}}^{(i)*} (m \otimes \mathbb{1}_5) - (\overline{n} \otimes \overline{m}) D_{\mathrm{IV}}^{(i)*} \end{gathered} \tag{8.20}$$

for all $m, n \in M_5(\mathbb{C})$ and $\lambda, \mu \in \mathbb{R}$. From this it follows directly that $D_{\mathrm{I}}^{(i)} = 0$ (take $m, n = 0$ and $\mu, \lambda = 1$), $D_{\mathrm{IV}}^{(i)} = 0$ (take $m, n = i\mathbb{1}_5$) and

$$D_{\mathrm{II}}^{(i)}(m \otimes \mathbb{1}_5) = mD_{\mathrm{II}}^{(i)} \tag{8.21}$$

(take $\mu = 0$ and $n = \mathbb{1}_5$).

Claim 8.1. *All solutions for $D_{\mathrm{II}}^{(i)}$ of eq. (8.21) are of the form*

$$D_{\mathrm{II}}^{(i)} = \mathbb{1}_5 \otimes d^{(i)*}, \tag{8.22}$$

for $d^{(i)} \in \mathbb{C}^5$. In other words: $D_{\mathrm{II}}^{(i)}$ acts on $v \otimes w \in \mathbb{C}^5 \otimes \mathbb{C}^5$ as follows:

$$D_{\mathrm{II}}^{(i)}(v \otimes w) = (d^{(i)*}w)v = \langle d^{(i)}, w \rangle v. \tag{8.23}$$

Proof. We write $\widetilde{D}^{(i)}(v \otimes w) = D_{\mathrm{II}}^{(i)}(w \otimes v)$, because writing out tensor products as Kronecker products is more convenint this way. Eq. (8.21) reads then

$$\widetilde{D}^{(i)}(\mathbb{1}_5 \otimes m) = m\widetilde{D}^{(i)}. \tag{8.24}$$

Writing $\widetilde{D}^{(i)} = \begin{pmatrix} \widetilde{D}_1^{(i)} & \cdots & \widetilde{D}_5^{(i)} \end{pmatrix}$, where $\widetilde{D}_j^{(i)} \in M_5(\mathbb{C})$, gives us

$$\begin{aligned}
\begin{pmatrix} \widetilde{D}_1^{(i)} m & \cdots & \widetilde{D}_5^{(i)} m \end{pmatrix} &= \begin{pmatrix} \widetilde{D}_1^{(i)} & \cdots & \widetilde{D}_5^{(i)} \end{pmatrix} \begin{pmatrix} m & & \\ & \ddots & \\ & & m \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{D}_1^{(i)} & \cdots & \widetilde{D}_5^{(i)} \end{pmatrix} (\mathbb{1}_5 \otimes m) \\
&= m \begin{pmatrix} \widetilde{D}_1^{(i)} & \cdots & \widetilde{D}_5^{(i)} \end{pmatrix} \\
&= \begin{pmatrix} m\widetilde{D}_1^{(i)} & \cdots & m\widetilde{D}_5^{(i)} \end{pmatrix},
\end{aligned} \tag{8.25}$$

where we used (8.24) in the third step. The only solution for the $\widetilde{D}_j^{(i)}$ is then $\widetilde{D}_j^{(i)} = \overline{d_j^{(i)}}\mathbb{1}_5$, where

$$d^{(i)} = \begin{pmatrix} d_1^{(i)} \\ \vdots \\ d_5^{(i)} \end{pmatrix} \in \mathbb{C}^5. \tag{8.26}$$

So we have

$$\widetilde{D}^{(i)} = \begin{pmatrix} \overline{d_1^{(i)}}\mathbb{1}_5 & \cdots & \overline{d_5^{(i)}}\mathbb{1}_5 \end{pmatrix} = d^{(i)*} \otimes \mathbb{1}_5 \tag{8.27}$$

and hence

$$D_{\mathrm{II}}^{(i)} = \mathbb{1}_5 \otimes d^{(i)*} \qquad \square \tag{8.28}$$

For $D^{(i)}$ we have now:

$$D^{(i)} = \begin{pmatrix} & & \mathbb{1}_5 \otimes d^{(i)*} & \\ & & & \mathbb{1}_5 \otimes d^{(i)\top} \\ \mathbb{1}_5 \otimes d^{(i)} & & & \\ & \mathbb{1}_5 \otimes \overline{d^{(i)}} & & \end{pmatrix}. \tag{8.29}$$

For the spectral triple $(\mathscr{A}_F\mathscr{H}_F, D_F)$ we have $\varepsilon_F = 1$, $\varepsilon'_F = 1$ and $\varepsilon''_F = -1$, so its KO-dimension is 6. The KO-dimension of the full product space is $4 + 6 \equiv 2 \pmod 8$.

8.3 The Symmetry Breaking Mechanism

8.3.1 Assuming the Correct Mechanism

Suppose that we get the correct symmetry breaking mechanism from the spectral action, although we will see in the next two subsections that this is not the case. So suppose that we get the Standard Model gauge group for the broken theory, with the embedding given by eq. (7.2). How do the new fermions transform under the Standard Model gauge group?

We follow the same procedure for the **15** as we did for the **10** on page 50: We start with the vector space isomorphism

$$\mathrm{Sym}^2(\mathbb{C}^3 \oplus \mathbb{C}^2) \;\simeq\; \mathrm{Sym}^2\,\mathbb{C}^3 \;\oplus\; \mathrm{Sym}^2\,\mathbb{C}^2 \;\oplus\; \mathbb{C}^3 \otimes \mathbb{C}^2. \tag{8.30}$$

The matrix (7.6) acts on this as follows:

- On $\mathrm{Sym}^2\,\mathbb{C}^3$:
$$v \otimes_S w \mapsto \zeta^{-4}\, u_3 v \otimes_S u_3 w, \tag{8.31}$$
which is the $\left(\mathrm{Sym}^2\,\mathbf{3}, \mathbf{1}, -\tfrac{2}{3}\right) = \left(\mathbf{6}, \mathbf{1}, -\tfrac{2}{3}\right)$, because of eq. (A.36).
- On $\mathrm{Sym}^2\,\mathbb{C}^2$:
$$v \otimes_S w \mapsto \zeta^{6}\, u_2 v \otimes_S u_2 w, \tag{8.32}$$
which is the $(\mathbf{1}, \mathrm{Sym}^2\,\mathbf{2}, 1) = (\mathbf{1}, \mathbf{3}, 1)$, because of eq. (A.44).
- And on $\mathrm{Sym}^2\,\mathbb{C}^2$:
$$v_2 \otimes_S v_3 \mapsto \zeta\, u_3 v_3 \otimes_S u_2 v_2, \tag{8.33}$$
which is the $\left(\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\right)$, which is the same representation as for the $\begin{pmatrix} \mathrm{u_L} \\ \mathrm{d_L} \end{pmatrix}$. This means that these particles cannot be distinguished from left-handed quarks, except for their mass.

Let us see how these particles behave after electroweak symmetry breaking (see § B.2):

- Because the $\left(\mathbf{6}, \mathbf{1}, -\tfrac{2}{3}\right)$ is a singlet under SU(2), it is the $\left(\mathbf{6}, -\tfrac{2}{3}\right)$ under $\mathrm{SU}(3) \times \mathrm{U}(1)$.
- The $(\mathbf{1}, \mathbf{3}, 1)$ is a **3** under SU(2). In this case, I_3 takes the values 1, 0 and -1, so it breaks to $(\mathbf{1}, 2) \oplus (\mathbf{1}, 1) \oplus (\mathbf{1}, 0)$.
- $\left(\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\right)$ breaks of course the same way as the left-handed quarks of the Standard Model: to $\left(\mathbf{3}, \tfrac{2}{3}\right) \oplus \left(\mathbf{3}, -\tfrac{1}{3}\right)$.

To summarize, we write:

$$\begin{aligned} \mathbf{15} \;\rightsquigarrow\; & \left(\mathbf{6}, \mathbf{1}, -\tfrac{2}{3}\right) \;\oplus\; && (\mathbf{1}, \mathbf{3}, 1) && \oplus\; \left(\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\right) \\ \rightsquigarrow\; & \left(\mathbf{6}, -\tfrac{2}{3}\right) \;\oplus\; && (\mathbf{1}, 2) \oplus (\mathbf{1}, 1) \oplus (\mathbf{1}, 0) && \oplus\; \left(\mathbf{3}, \tfrac{2}{3}\right) \oplus \left(\mathbf{3}, -\tfrac{1}{3}\right), \end{aligned} \tag{8.34}$$

where the first arrow denotes the breaking of SU(5) to the Standard Model, and the second one the electroweak symmetry breaking.

8.3.2 The Scalar Fields

In the previous subsection we assumed that our model has the correct symmetry breaking mechanism to break the SU(5)-theory to the Standard Model. In this and the next subsection we will see what mechanism the spectral triple gives us actually.

In § 5.1.1 we have seen that fluctuating the Dirac operator of an almost-commutative spectral triple gives rise to gauge and scalar bosons. The scalar field is given by (eq. (5.26b))

$$\Phi = \sum_j f_j \otimes \pi_F(m_j, \lambda_j)[D_F, \pi_F(n_j, \mu_j)], \tag{8.35}$$

where $f_j \in \mathrm{C}^\infty(M, \mathbb{R})$, $m_j, n_j \in \mathrm{M}_5(\mathbb{C})_{\mathrm{sa}}$ and $\lambda_j, \mu_j \in \mathbb{R}$. Writing this out explicitly gives:

$$\begin{aligned} \Phi &= \sum_j f_j \otimes \bigoplus_{i=1}^{3} \begin{pmatrix} & & 0 & \\ & & & \lambda_j \mathbb{1}_5 \otimes d^{(i)\top}\overline{n}_j - \lambda_j\mu_j\mathbb{1}_5 \otimes d^{(i)\top} \\ 0 & & & \\ & \mu_j\mathbb{1}_5 \otimes \overline{m_j d^{(i)}} - \mathbb{1}_5 \otimes \overline{m_j n_j d^{(i)}} & & \end{pmatrix} \\ &= \bigoplus_{i=1}^{3} \begin{pmatrix} & & 0 & \\ & & & \mathbb{1}_5 \otimes (\phi d^{(i)})^\top \\ 0 & & & \\ & \mathbb{1}_5 \otimes \overline{\phi d^{(i)}} & & \end{pmatrix}, \end{aligned} \tag{8.36}$$

where we have used the self-adjointness of Φ and have rewritten

$$\begin{aligned} \phi &= \sum_j f_j \otimes (\lambda_j n_j^* - \lambda_j \mu_j \mathbb{1}_5) = \sum_j f_j \otimes (\mu_j m_j - m_j n_j) \\ &\in \mathrm{C}^\infty(M) \otimes \mathrm{M}_5(\mathbb{C}) \simeq \mathrm{C}^\infty\big(M, \mathrm{M}_5(\mathbb{C})\big). \end{aligned} \tag{8.37}$$

On Φ, a $u \in \mathrm{C}^\infty\big(M, \mathrm{SU}(5)\big)$ acts as follows:

$$\pi(u,1)\Phi\pi(u^*,1) = \bigoplus_{i=1}^{3} \begin{pmatrix} & & 0 & \\ & & & \mathbb{1}_5 \otimes (u\phi d^{(i)})^\top \\ 0 & & & \\ & \mathbb{1}_5 \otimes \overline{u\phi d^{(i)}} & & \end{pmatrix}, \tag{8.38}$$

so on $\phi = \begin{pmatrix} \phi_1 & \cdots & \phi_5 \end{pmatrix}$ ($\phi_j \in \mathrm{C}^\infty(M, \mathbb{C}^5)$) it acts as $u\phi = \begin{pmatrix} u\phi_1 & \cdots & u\phi_5 \end{pmatrix}$. This representation decomposes into five irreps: five times the standard representation. In the language of § A.1.1 we can write it as

$$\mathbf{5} \oplus \mathbf{5} \oplus \mathbf{5} \oplus \mathbf{5} \oplus \mathbf{5}. \tag{8.39}$$

An equivalent way to see this, is as follows: Φ appears in the fluctuated Dirac operator as (see eq. (5.20))

$$\begin{aligned} (\Gamma_M \otimes \mathbb{1})\Phi + J(\Gamma_M \otimes \mathbb{1})\Phi J^* &= (\Gamma_M \otimes \mathbb{1})\Phi + (\Gamma_M \otimes \mathbb{1})J\Phi J^* \\ &= (\Gamma_M \otimes \mathbb{1})\boldsymbol{\Phi}. \end{aligned} \tag{8.40}$$

In the first step we used that $\varepsilon''_M = 1$, and in the second one we introduced:

$$\begin{aligned}\boldsymbol{\Phi} &= \Phi + J\Phi J^* \\ &= \bigoplus_{i=1}^{3} \begin{pmatrix} & & \mathbb{1}_5 \otimes (\phi d^{(i)})^* & \\ & & & \mathbb{1}_5 \otimes (\phi d^{(i)})^\top \\ \mathbb{1}_5 \otimes \phi d^{(i)} & & & \\ & \mathbb{1}_5 \otimes \overline{\phi d^{(i)}} & & \end{pmatrix},\end{aligned} \tag{8.41}$$

$\boldsymbol{\Phi}$ transforms under conjugation with $\pi(u,1)J\pi(u,1)J^* = \rho(u,1)$ as:

$$\begin{aligned}&\rho(u,1)\boldsymbol{\Phi}\rho(u^*,1) \\ &= \bigoplus_{i=1}^{3} \begin{pmatrix} & & \mathbb{1}_5 \otimes (u\phi d^{(i)})^* & \\ & & & \mathbb{1}_5 \otimes (u\phi d^{(i)})^\top \\ \mathbb{1}_5 \otimes u\phi d^{(i)} & & & \\ & \mathbb{1}_5 \otimes \overline{u\phi d^{(i)}} & & \end{pmatrix}.\end{aligned} \tag{8.42}$$

So u acts indeed on ϕ as $u\phi$.

8.3.3 The Higgs Potential & the Broken Theory

Recall from §7.4 that a Higgs field in the adjoint (**24**) is needed to break the SU(5) model to the Standard Model. Our field ϕ is not able to do this. Depending on the vacuum expectation values, the only broken gauge groups that can be obtained are $\mathrm{SU}(5-l)$, for $l \leq 5$:

Proposition 8.2. *Consider a* $\mathrm{SU}(N)$ *gauge theory with k scalar fields in the standard representation with vacuum expectation values* $v_1, \dots, v_k \in \mathbb{C}^N$. *They break the* $\mathrm{SU}(N)$ *symmetry to an* $\mathrm{SU}(N-l)$ *symmetry, for some* $l \leq N, k$.

In other words:

$$\big\{u \in \mathrm{SU}(N) \;\big|\; \forall i \in \{1,\dots,k\} : uv_i = v_i\big\} \simeq \mathrm{SU}(N-l) \tag{8.43}$$

as groups. The number l is given by $l = \dim \mathscr{V}$, *for* $\mathscr{V} = \mathrm{span}\{v_1, \dots, v_k\}$.

Proof. The condition

$$\forall i \in \{1,\dots,k\} : \quad uv_i = v_i \tag{8.44}$$

is equivalent to

$$\forall v \in \mathscr{V} : \quad uv = v. \tag{8.45}$$

Let $\mathfrak{B} = v'_1, \dots, v'_N$ be a basis for $\mathbb{C}^N$ with the property that $v'_1, \dots, v'_l$ is a basis for $\mathscr{V}$. The condition (8.45) is then equivalent to

$$\forall i \in \{1,\dots,l\} : \quad uv'_i = v'_i. \tag{8.46}$$

This means that in the basis $\mathfrak{B}$, u is of the form

$$u = \begin{pmatrix} \mathbb{1}_l & \\ & u' \end{pmatrix}_{\mathfrak{B}}, \tag{8.47}$$

where $u' \in \mathrm{SU}(N-l)$. ☐

Let us see which broken theory we get. The *Higgs potential* is the sector of the Lagrangian which only contains $\boldsymbol{\Phi}$ (and hence ϕ), without derivatives or couplings to other fiels (such as the gauge or gravitational field). We will derive this potential, without computing the full spectral action as in § 6.1. $\boldsymbol{\Phi}$ appears in D_A as $(\Gamma_M \otimes \mathbb{1})\boldsymbol{\Phi}$, so it appears in the endomorphism $E = \Delta - D_A^2$ of eq. (5.45) as $-\boldsymbol{\Phi}^2$ (forgetting about the cross terms). For the Seeley–DeWitt coefficients (eq. (5.50)) $a_2(D_A^2, x)$ and $a_4(D_A^2, x)$, we have the contributions

$$-\frac{1}{4\pi^2}\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^2 \quad \text{and} \quad \frac{1}{8\pi^2}\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^4 \tag{8.48}$$

respectively. The spectral action (eq. (5.64)) gets thus a contribution

$$\begin{aligned}&\int_M \mathrm{d}^4x\sqrt{g}\,\Big(-\frac{1}{2\pi^2}f_2\Lambda^2\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^2 + \frac{1}{8\pi^2}f(0)\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^4\Big)\\ &= \frac{1}{4\pi^2}\int_M \mathrm{d}^4x\sqrt{g}\,V,\end{aligned} \tag{8.49}$$

where V is the Higgs potential

$$V = -2f_2\Lambda^2\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^2 + \tfrac{1}{2}f(0)\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^4. \tag{8.50}$$

In our case we have

$$\boldsymbol{\Phi}^2 = \bigoplus_{i=1}^{3}\begin{pmatrix}(\phi d^{(i)})^*\phi d^{(i)}\mathbb{1}_5 & & & \\ & (\phi d^{(i)})^\top\overline{\phi d^{(i)}}\mathbb{1}_5 & & \\ & & \mathbb{1}_5\otimes\phi d^{(i)}(\phi d^{(i)})^* & \\ & & & \mathbb{1}_5\otimes\overline{\phi d^{(i)}}(\phi d^{(i)})^\top\end{pmatrix}, \tag{8.51}$$

so

$$\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^2 = 20\sum_{i=1}^{3}(\phi d^{(i)})^*\phi d^{(i)} = 20\sum_{i=1}^{3}\langle\phi d^{(i)}, \phi d^{(i)}\rangle; \tag{8.52}$$

and similarly

$$\operatorname{Tr}_{\mathscr{H}_F}\boldsymbol{\Phi}^4 = 20\sum_{i=1}^{3}\langle\phi d^{(i)}, \phi d^{(i)}\rangle^2. \tag{8.53}$$

The Higgs potential is then

$$V = 20\sum_{i=1}^{3}\big(-2f_2\Lambda\langle\phi d^{(i)}, \phi d^{(i)}\rangle + \tfrac{1}{2}f(0)\langle\phi d^{(i)}, \phi d^{(i)}\rangle^2\big). \tag{8.54}$$

Assume that the vectors $d^{(1)}$, $d^{(2)}$ and $d^{(3)}$ are linearly independent. There exists then a basis transformation $T \in \mathrm{GL}(5, \mathbb{C})$ with

$$Td^{(i)} = e_i. \tag{8.55}$$

We write

$$\phi T^{-1} = \phi' = \begin{pmatrix}\phi'_1 & \cdots & \phi'_5\end{pmatrix}, \tag{8.56}$$

which implies

$$\phi d^{(i)} = \phi T^{-1}e_i = \phi' e_i = \phi'_i, \tag{8.57}$$

so the potential can be written as

$$V = 20 \sum_{i=1}^{3} \big(-2 f_2 \Lambda^2 \langle \phi'_i, \phi'_i \rangle + \tfrac{1}{2} f(0) \langle \phi'_i, \phi'_i \rangle^2 \big). \tag{8.58}$$

We had 5 scalar fields $\phi_1, \dots, \phi_5$ in the **5** of $\mathrm{SU}(5)$, but we just have shown with a basis transformation that only 3 of them, $\phi'_1, \phi'_2, \phi'_3$, appear in the potential; the other two are unphysical.

Using Prop. 8.2, we can conclude that this symmetry breaking mechanism leads to a $\mathrm{SU}(5-m)$ theory, for $m \in \{0,1,2,3\}$, depending on the linear dependence of the vacuum expectation values of the three fields.

We assumed that the $d^{(i)}$ are linearly independent. Relaxing this assumptions and using a similar reasoning as above, one can conclude that the broken theory is $\mathrm{SU}(5-m)$-invariant, where $m \leq \dim \operatorname{span}\{d^{(1)}, d^{(2)}, d^{(3)}\}$. A special case is $d^{(1)}, d^{(2)}, d^{(3)} = 0$ (that is $D_F = 0$): then $m = 0$, so no symmetry breaking occurs.

8.4 Literature

In [16], it is argued that it is impossible to construct a spectral triple that gives the representations (7.5). The argument they use, is that the **10** is not an algebra representation of $M_5(\mathbb{C})$. Recall that the fermion representation is not the representation of the algebra in the spectral triple, but it is given by eq. (5.36). This difference of interpretation can be explained: [16] is based on the Connes–Lott model, an older NCG-approach to the Standard Model.

In § 4.2, it was mentioned that in [12] a diagrammatical method is developed to classify finite-dimensional spectral triples. In that paper, the $\mathrm{SU}(5)$ model is mentioned as an example (§ 5.3). A similar spectral triple is chosen, but for $\mathscr{A}_F$, $M_5(\mathbb{C}) \oplus \mathbb{C}$ is taken. This gives (modulo finite abelian groups) the gauge group $\mathrm{U}(5) \times \mathrm{U}(1)$. [12] suggests to apply the unimodularity condition to the two components of the algebra separately, to get the $\mathrm{SU}(5)$. With $\mathscr{A}_F = M_5(\mathbb{C}) \oplus \mathbb{R}$, this is not necessary.

In § 5.4 of the same paper it is claimed here that it is not possible to have Higgs fields in the adjoint epresentation (in our NCG-framework). From this, the same conclusion is drawn about the symmetry breaking mechanism as we did: this model does allow for the correct one.

Appendices

Appendix A

Some Results from Representation Theory

In this appendix, we give some results of group representation theory, mainly the representation theory of $\mathrm{SU}(N)$. This is not meant as a full treatment of this theory; only results which are relevant for this thesis are given.

A.1 SU(N)

Definition A.1 (Trivial representation)**.** The *trivial representation* is

$$\mathrm{Triv} : \mathrm{SU}(N) \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times, \quad \mathrm{Triv}(u) = 1. \tag{A.1}$$

Definition A.2 (Standard representation)**.** The irrep

$$\mathrm{St} : \mathrm{SU}(N) \to \mathrm{GL}(\mathbb{C}^N) = \mathrm{GL}(N, \mathbb{C}), \quad \mathrm{St}(u) = u \tag{A.2}$$

is called the *defining*, *standard* or *fundamental representation* of $\mathrm{SU}(N)$. This is just the inclusion map of $\mathrm{SU}(N)$ in $\mathrm{GL}(N, \mathbb{C})$.

Definition A.3 ((Anti)symmetric product)**.** Let $\mathscr{V}$ be an n-dimensional vector space.

i. We define the *antisymmetric product* as

$$v_1 \wedge v_2 = \tfrac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \tag{A.3}$$

(for $v_1, v_2 \in \mathscr{V}$) and the $\binom{n}{k}$-dimensional vector spaces $\bigwedge^k \mathscr{V}$ as

$$\textstyle\bigwedge^k \mathscr{V} = \mathrm{span}\{v_1 \wedge \cdots \wedge v_k \mid v_1, \ldots, v_k \in \mathscr{V}\}. \tag{A.4}$$

Equipped with the multiplication $\wedge$, $\bigwedge \mathscr{V} = \bigoplus_{k=0}^{N} \bigwedge^k \mathscr{V}$ is a 2^n-dimensional algebra, called the *exterior algebra* of $\mathscr{V}$.

ii. In the same way, we define the *symmetric product* as

$$v_1 \otimes_S v_2 = \tfrac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) \tag{A.5}$$

and the $\binom{n+k-1}{k}$-dimensional vector spaces $\mathrm{Sym}^k \mathscr{V}$ as

$$\mathrm{Sym}^k \mathscr{V} = \mathrm{span}\{v_1 \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} v_k \mid v_1, \ldots, v_k \in \mathscr{V}\}. \tag{A.6}$$

Definition A.4 ((Anti)symmetric representations)**.** Let $\rho : \mathrm{SU}(N) \to \mathrm{GL}(\mathscr{V})$ be a representation.

i. We define the *antisymmetric representation of rank $k \in \mathbb{N}_{\leq N}$* as

$$\begin{gathered} \textstyle\bigwedge^k \rho : \mathrm{SU}(N) \to \mathrm{GL}(\bigwedge^k \mathscr{V}) \\ (\textstyle\bigwedge^k \rho)(u)(v_1 \wedge \cdots \wedge v_k) = uv_1 \wedge \cdots \wedge uv_k, \end{gathered} \tag{A.7}$$

ii. and the *symmetric representation of rank $k \in \mathbb{N}$* as

$$\begin{gathered} \mathrm{Sym}^k \rho : \mathrm{SU}(N) \to \mathrm{GL}(\mathrm{Sym}^k \mathscr{V}) \\ (\mathrm{Sym}^k \rho)(u)(v_1 \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} v_k) = uv_1 \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} uv_k \end{gathered} \tag{A.8}$$

Proposition A.5. *Let $\rho : \mathrm{SU}(N) \to \mathrm{GL}(\mathscr{V})$ be an irrep. Then:*

i. $\bigwedge^k \rho$ and $\mathrm{Sym}^k \rho$ are irreps too (for any k),

ii. and the rank-2 tensor product of ρ has the following decomposition in irreps:

$$\rho \otimes \rho \sim \mathrm{Sym}^2 \rho \oplus \textstyle\bigwedge^2 \rho. \tag{A.9}$$

Definition A.6 (Conjugate & dual representation)**.** For a representation $\rho : \mathrm{SU}(N) \to \mathrm{GL}(\mathscr{V})$, one defines:

i. the *conjugate representation $\overline{\rho}$* as:

$$\overline{\rho} : \mathrm{SU}(N) \to \mathrm{GL}(\mathscr{V}), \quad \overline{\rho}(u)v = \overline{\rho(u)}v, \tag{A.10}$$

ii. and the *dual representation ρ^** as:

$$\rho^* : \mathrm{SU}(N) \to \mathrm{GL}(\mathscr{V}^*), \quad \rho^*(u)v^* = v^*\rho(u)^*. \tag{A.11}$$

Lemma A.7. *These representations are equivalent*

$$\overline{\rho} \sim \rho^*, \tag{A.12}$$

intertwined by matrix transposition; i.e.: for any $u \in \mathrm{SU}(N)$, the diagram

$$\begin{tikzcd} \mathscr{V} \arrow[r, "\overline{\rho}(u)"] \arrow[d, "(\bullet)^\top"'] & \mathscr{V} \arrow[d, "(\bullet)^\top"'] \\ \mathscr{V}^* \arrow[r, "\rho^*(u)"] & \mathscr{V}^* \end{tikzcd} \tag{A.13}$$

commutes.

Proof. It commutes, because

$$\big(\overline{\rho}(u)v\big)^\top = \big(\overline{\rho(u)}v\big)^\top = v^\top\overline{\rho(u)}^\top = v^\top\rho(u)^* = \rho^*(u)v^\top. \qquad \square \tag{A.14}$$

Definition A.8 (Adjoint representation)**.** The *adjoint representation* of $\mathrm{SU}(N)$ is defined as the (N^2-1)-dimensional representation

$$\mathrm{Ad} : \mathrm{SU}(N) \to \mathrm{GL}\big(\mathrm{M}_N(\mathbb{C})_{\mathrm{Tr}=0}\big), \quad \mathrm{Ad}(u)m = umu^*. \tag{A.15}$$

It is well-defined, because

$$\mathrm{Tr}\,\big(\,\mathrm{Ad}(u)m\big) = \mathrm{Tr}\,umu^* = \mathrm{Tr}\,m = 0. \tag{A.16}$$

Proposition A.9. *i. The adjoint representation is irreducible.*

ii. The tensor product of the standard representation and its conjugate has the following decomposition in irreps:

$$\mathrm{St} \otimes \overline{\mathrm{St}} \sim \mathrm{Ad} \oplus \mathrm{Triv}\,. \tag{A.17}$$

Let us have a closer look at the representations $\bigwedge^k \mathrm{St}$. Note that

$$\textstyle\bigwedge^k\mathbb{C}^N \simeq \bigwedge^{N-k}\mathbb{C}^N \tag{A.18}$$

as vector spaces, because

$$\dim \textstyle\bigwedge^{N-k}\mathbb{C}^N = \displaystyle\binom{N}{N-k} = \binom{N}{k} = \dim \textstyle\bigwedge^k\mathbb{C}^N. \tag{A.19}$$

The corresponding isomorphism is:

Definition A.10 (Hodge dual)**.** For every $k \in \mathbb{N}_{\leq N}$ we define the linear map:

$$\begin{gathered} * : \textstyle\bigwedge^k\mathbb{C}^N \to \bigwedge^{N-k}\mathbb{C}^N \\ * e_{i_1} \wedge \cdots \wedge e_{i_k} = \displaystyle\frac{1}{(N-k)!} \sum_{i_{k+1},\dots,i_N} \varepsilon_{i_1\dots i_N}\, e_{i_{k+1}} \wedge \cdots \wedge e_N, \end{gathered} \tag{A.20}$$

where $e_1, \dots, e_N$ denotes the standard basis of $\mathbb{C}^N$ and $\varepsilon_{i_1\dots i_N}$ is the Levi–Civita symbol. This map is called the *Hodge dual* or *Hodge $*$-operator*.

We can show that this notion of duality extends to $\mathrm{SU}(N)$ representations:

Proposition A.11. *For any $k \in \mathbb{N}_{\leq 0}$:*

$$\textstyle\bigwedge^k\overline{\mathrm{St}} \sim \bigwedge^{N-k}\mathrm{St}\,, \tag{A.21}$$

intertwined by the Hodge-$*$. In other words, for any $u \in \mathrm{SU}(N)$, the diagram

$$\begin{tikzcd} \bigwedge^k \mathbb{C}^N \arrow[r, "(\bigwedge^k \overline{\mathrm{St}})(u)"] \arrow[d, "*"'] & \bigwedge^k \mathbb{C}^N \arrow[d, "*"'] \\ \bigwedge^{N-k} \mathbb{C}^N \arrow[r, "(\bigwedge^{N-k} \mathrm{St})(u)"] & \bigwedge^{N-k} \mathbb{C}^N \end{tikzcd} \tag{A.22}$$

commutes.

Proof. It commutes if for any $i_1, \ldots, i_k \in \mathbb{N}$

$$*(\textstyle\bigwedge^k \overline{\mathrm{St}})(u)(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (\textstyle\bigwedge^{N-k} \mathrm{St})(u)(* e_{i_1} \wedge \cdots \wedge e_{i_k}). \tag{A.23}$$

Let us write this out. For the left-hand side we get

$$\begin{aligned}
&*(\textstyle\bigwedge^k \overline{\mathrm{St}})(u)(e_{i_1} \wedge \cdots \wedge e_{i_k}) \\
&= *(\overline{u}e_{i_1} \wedge \cdots \overline{u}e_{i_k}) \\
&= \sum_{j_1,\ldots,j_k} \overline{u}_{j_1 i_1} \cdots \overline{u}_{j_k i_k} * e_{j_1} \wedge \cdots \wedge e_{j_k} \\
&= \frac{1}{(N-k)!} \sum_{j_1,\ldots,j_N} \varepsilon_{j_1 \ldots j_N} \overline{u}_{j_1 i_1} \cdots \overline{u}_{j_k i_k}\, e_{j_{k+1}} \wedge \cdots \wedge e_{j_N},
\end{aligned} \tag{A.24}$$

and for the right-hand side

$$\begin{aligned}
&(\textstyle\bigwedge^{N-k} \mathrm{St})(u)(* e_{i_1} \wedge \cdots \wedge e_{i_k}) \\
&= \frac{1}{(N-k)!} \sum_{i_{k+1},\ldots,i_N} \varepsilon_{i_1 \ldots i_N} (\textstyle\bigwedge^{N-k} \mathrm{St})(u)(e_{i_{k+1}} \wedge \cdots \wedge e_{i_N}) \\
&= \frac{1}{(N-k)!} \sum_{i_{k+1},\ldots,i_N} \varepsilon_{i_1 \ldots i_N}\, u e_{i_{k+1}} \wedge \cdots \wedge u e_{i_N} \\
&= \frac{1}{(N-k)!} \sum_{i_{k+1},\ldots,i_N, j_{k+1},\ldots,j_N} \varepsilon_{i_1 \ldots i_N} u_{j_{k+1} i_{k+1}} \cdots u_{j_N i_N}\, e_{j_{k+1}} \wedge \cdots \wedge e_{i_N}.
\end{aligned} \tag{A.25}$$

Both sides are equal if

$$\sum_{j_1,\ldots,j_k} \varepsilon_{j_1 \ldots j_N} \overline{u}_{j_1 i_1} \cdots \overline{u}_{j_k i_k} = \sum_{i_{k+1},\ldots,i_N} \varepsilon_{i_1 \ldots i_N} u_{j_{k+1} i_{k+1}} \cdots u_{j_N i_N}. \tag{A.26}$$

We will prove this by induction:

- Let $k = 0$. Then the left hand side of eq. (A.26) reduces to $\varepsilon_{j_1 \ldots j_N}$. Using the definition of the determinant and $\det u = 1$, the left-hand side is

  $$\sum_{i_1,\ldots,i_N} \varepsilon_{i_1 \ldots i_N} u_{j_1 i_1} \cdots u_{j_N i_N} = \varepsilon_{j_1 \ldots j_N} \det u = \varepsilon_{j_1 \ldots j_N}. \tag{A.27}$$

  So eq. (A.26) is true for $k = 0$.

- Suppose (A.26) holds for some k. Then:

  $$\begin{aligned}
  &\sum_{j_1,\ldots,j_{k+1}} \varepsilon_{j_1 \ldots j_N} \overline{u}_{j_1 i_1} \cdots \overline{u}_{j_{k+1} i_{k+1}} \\
  &= \sum_{j_{k+1}, l, i_{k+2},\ldots,i_N} \varepsilon_{i_1 \ldots i_k\, l\, i_{k+2} \ldots, i_N} \overline{u}_{j_{k+1} i_{k+1}} u_{j_{k+1} l} u_{j_{k+2} i_{k+2}} \cdots u_{j_N i_N} \\
  &= \sum_{i_{k+2},\ldots,i_N} \varepsilon_{i_1 \ldots i_N} u_{j_{k+2} i_{k+2}} \cdots u_{j_N i_N}.
  \end{aligned} \tag{A.28}$$

  In the first step we used the induction hypothesis (A.26), where we renamed the dummy index i_{k+1} to l, to avoid a clash of notation, since the 'open' indices of eq. (A.28) are $i_1, \ldots, i_{k+1}, j_{k+2}, \ldots, j_N$. In the second step we used the unitarity of u:

$$\sum_{j_{k+1}} \overline{u}_{j_{k+1} i_{k+1}} u_{j_{k+1} l} = (u^* u)_{i_{k+1} l} = \delta_{i_{k+1} l}. \tag{A.29}$$

So (A.26) also holds for $k+1$. □

A.1.1 SU(5), SU(3) and SU(2)

In physics, it is customary to denote an irrep —actually: an equivalence class of irreps— by its dimension in boldface, for a given N. For example, for every N, the trivial representation is denoted by $[\mathrm{Triv}] = \mathbf{1}$[1] and the standard representation by $[\mathrm{St}] = \boldsymbol{N}$. To distinguish an irrep from its conjugate, we use a bar, e.g. $[\overline{\mathrm{St}}] = \overline{\boldsymbol{N}}$.

We will apply the results of this appendix to three cases, which are relevant in this thesis: $N = 5$, 3 and 2.

For SU(5): The standard representation is the $[\mathrm{St}] = \mathbf{5}$. Plugging this into Proposition A.5ii gives:

$$\mathbf{5} \otimes \mathbf{5} = \mathrm{Sym}^2\, \mathbf{5} \oplus \textstyle\bigwedge^2 \mathbf{5} = \mathbf{15} \oplus \mathbf{10}, \tag{A.30}$$

where we denoted

$$\mathrm{Sym}^2\, \mathbf{5} = \mathbf{15} \quad \text{and} \quad \textstyle\bigwedge^2 \mathbf{5} = \mathbf{10}. \tag{A.31}$$

Recall from Def. A.3 that the dimensions of these irreps are indeed $\dim \mathrm{Sym}^2 \mathbb{C}^5 = \binom{6}{2} = 15$ and $\dim \bigwedge^2 \mathbb{C}^5 = \binom{5}{2} = 10$.

Proposition A.9ii reads:

$$\mathbf{5} \otimes \overline{\mathbf{5}} = \mathbf{24} \oplus \mathbf{1}, \tag{A.32}$$

where we denoted

$$[\mathrm{Ad}] = \mathbf{24}, \tag{A.33}$$

since $\dim \mathrm{M}_5(\mathbb{C})_{\mathrm{Tr}=0} = 24$.

For SU(3): The standard representation is the $[\mathrm{St}] = \mathbf{3}$. From Proposition A.11 it follows that

$$\overline{\mathbf{3}} = \textstyle\bigwedge^2 \mathbf{3}. \tag{A.34}$$

Plugging the **3** into Proposition A.5ii gives:

$$\mathbf{3} \otimes \mathbf{3} = \textstyle\bigwedge^2 \mathbf{3} \oplus \mathrm{Sym}^2\, \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}, \tag{A.35}$$

where we used eq. (A.34) and denoted

$$\mathrm{Sym}^2\, \mathbf{3} = \mathbf{6}, \tag{A.36}$$

since $\dim \mathrm{Sym}^2 \mathbb{C}^3 = \binom{4}{2} = 6$.

[1] The brackets $[\cdots]$ denote such an equivalence class.

Proposition A.9ii reads in this case:

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{A.37}$$

where we denoted

$$[\mathrm{Ad}] = \mathbf{8}, \tag{A.38}$$

since $\dim \mathrm{M}_3(\mathbb{C})_{\mathrm{Tr}=0} = 8$.

For SU(2): The standard representation is the $[\mathrm{St}] = \mathbf{2}$. From Proposition A.11 it follows that it is real:

$$\overline{\mathbf{2}} = \bigwedge^1 \overline{\mathbf{2}} = \bigwedge^1 \mathbf{2} = \mathbf{2}. \tag{A.39}$$

Plugging the **2** into Proposition A.5ii gives

$$\mathbf{2} \otimes \mathbf{2} = \mathrm{Sym}^2 \mathbf{2} \oplus \bigwedge^2 \mathbf{2} = \mathrm{Sym}^2 \mathbf{2} \oplus \mathbf{1}, \tag{A.40}$$

where we used Proposition A.11 again:

$$\bigwedge^2 \mathbf{2} = \bigwedge^0 \overline{\mathbf{2}} = \mathbf{1}. \tag{A.41}$$

Proposition A.9ii reads:

$$\mathbf{2} \otimes \overline{\mathbf{2}} = [\mathrm{Ad}] \oplus \mathbf{1}. \tag{A.42}$$

Using the reality of the **2** (eq. (A.39)) we see that

$$\mathrm{Sym}^2 \mathbf{2} \oplus \mathbf{1} = \mathbf{2} \otimes \mathbf{2} = \mathbf{2} \otimes \overline{\mathbf{2}} = [\mathrm{Ad}] \oplus \mathbf{1}, \tag{A.43}$$

so we can write:

$$\mathrm{Sym}^2 \mathbf{2} = [\mathrm{Ad}] = \mathbf{3}. \tag{A.44}$$

It is indeed 3-dimensional: $\dim \mathrm{Sym}^2 \mathbb{C}^2 = \binom{3}{2} = 3$ and $\dim \mathrm{M}_2(\mathbb{C})_{\mathrm{Tr}=0} = 2^2 - 1 = 3$.

To conclude:

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{2} \otimes \overline{\mathbf{2}} = \mathbf{3} \oplus \mathbf{1}. \tag{A.45}$$

A.2 U(1)

The representations of $\mathrm{U}(1)$ are

$$\rho_q : \mathrm{U}(1) \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times, \quad \rho_q(\zeta) = \zeta^q, \tag{A.46}$$

where $q \in \mathbb{Z}$. Because $\mathrm{U}(1)$ is commutative, these ρ_q are indeed representations.

Note that ρ_0 is the trivial representation, ρ_1 the standard representation and ρ_{-q} is ρ_q's conjugate.

In physics, q is interpreted as a *charge* (up to some factors), for example electric charge or hypercharge.

A.3 Semisimple Groups

Definition A.12. Let $G_1, \ldots, G_k$ be simple groups and let (for every i) $\rho_i : G_i \to \mathrm{GL}(\mathscr{V}_i)$ be a representation of G_i. Then we define the representation $(\rho_1, \ldots, \rho_k)$ of $G_1 \times \cdots \times G_k$ as

$$\begin{aligned} (\rho_1, \ldots, \rho_k) : G_1 \times \cdots \times G_k \to \mathrm{GL}(\mathscr{V}_1 \otimes \cdots \otimes \mathscr{V}_k), \\ (\rho_1, \ldots, \rho_k)(g_1, \ldots, g_k)(v_1 \otimes \cdots \otimes v_k) = \rho_1(g_1)v_1 \otimes \cdots \otimes \rho_k(g_k)v_k. \end{aligned} \tag{A.47}$$

All irreps of $G_1 \times \cdots \times G_k$ are of this form, for irreducible ρ_i.

Do not confuse this definition with the tensor product of representations of one group.

Appendix B

The Representation Theory of the Standard Model

B.1 The Unbroken Standard Model

The particles in the Standard Model are described as irreps of the group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, the gauge group of the Standard Model. According to § A.3, we can label these irreps as follows: $\big(\boldsymbol{n}_3, \boldsymbol{n}_2, \tfrac{1}{2}Y\big)$. $\boldsymbol{n}_3$ and $\boldsymbol{n}_2$ are irreps of $\mathrm{SU}(3)$ and $\mathrm{SU}(2)$ respectively, where we use the notation of § A.1.1. The number $\frac{1}{2}Y$ is a multiple of $\frac{1}{6}$ and defines the $\mathrm{U}(1)$-representation by the identification $\frac{1}{6}q = \frac{1}{2}Y$, where q is as in § A.2. The factors $\frac{1}{2}$ and $\frac{1}{6}$ are included because they are customary in physics. Y is called the *weak hypercharge*.

The first-generation of (spin-$\frac{1}{2}$) fermions of the Standard Model correspond to the following irreps:

$$\begin{aligned}
\begin{pmatrix} \mathrm{u_L} \\ \mathrm{d_L} \end{pmatrix} &: \big(\mathbf{3}, \mathbf{2}, \tfrac{1}{6}\big), & \begin{pmatrix} \overline{\mathrm{u}}_\mathrm{R} \\ \overline{\mathrm{d}}_\mathrm{R} \end{pmatrix} &: \big(\overline{\mathbf{3}}, \mathbf{2}, -\tfrac{1}{6}\big), \\
\begin{pmatrix} \nu_{\mathrm{eL}} \\ \mathrm{e_L^-} \end{pmatrix} &: \big(\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}\big), & \begin{pmatrix} \overline{\nu}_{\mathrm{eR}} \\ \mathrm{e_R^+} \end{pmatrix} &: \big(\mathbf{1}, \mathbf{2}, \tfrac{1}{2}\big), \\
\mathrm{u_R} &: \big(\mathbf{3}, \mathbf{1}, \tfrac{2}{3}\big), & \overline{\mathrm{u}}_\mathrm{L} &: \big(\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}\big), \\
\mathrm{d_R} &: \big(\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}\big), & \overline{\mathrm{d}}_\mathrm{L} &: \big(\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}\big), \\
\mathrm{e_R^-} &: (\mathbf{1}, \mathbf{1}, -1), & \mathrm{e_L^+} &: (\mathbf{1}, \mathbf{1}, 1).
\end{aligned} \tag{B.1}$$

Charge conjugation corresponds to taking the conjugate representation. One can include the right-handed neutrino and its antiparticle, which do not interact with any gauge boson:

$$\nu_{\mathrm{eR}} : (\mathbf{1}, \mathbf{1}, 0), \qquad \overline{\nu}_{\mathrm{eL}} : (\mathbf{1}, \mathbf{1}, 0). \tag{B.2}$$

The (spin-1) gauge bosons correspond to the adjoint representations of the three simple components of the gauge group:

$$\mathrm{g} : (\mathbf{8}, \mathbf{1}, 0), \qquad \mathrm{W} : (\mathbf{1}, \mathbf{3}, 0), \qquad \mathrm{B} : (\mathbf{1}, \mathbf{1}, 0). \tag{B.3}$$

B.1.1 The 'True' Gauge Group

We said that $SU(3) \times SU(2) \times U(1)$ is the gauge group of the Standard Model. This can be nuanced a bit. As we will see, a finite abelian subgroup acts trivially under the representations (B.1). In other words, the kernel of the representation is non-trivial. The 'true' gauge group is then the quotient of $SU(3) \times SU(2) \times U(1)$ with this kernel.

The representation (B.1) can be written as follows in its full glory:

$$\rho = (\rho' \oplus \overline{\rho'})^{\oplus 3}, \tag{B.4}$$

where ρ' is the representation of one generation of fermions (without the anti-fermions):

$$\begin{aligned} \rho' &: SU(3) \times SU(2) \times U(1) \to GL(\mathbb{C}^3 \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}), \\ \rho'(u_3, u_2, \zeta) &= \begin{pmatrix} \zeta u_3 \otimes u_2 & & & & \\ & \zeta^{-3} u_2 & & & \\ & & \zeta^4 u_3 & & \\ & & & \zeta^{-2} u_3 & \\ & & & & \zeta^{-6} \end{pmatrix}. \end{aligned} \tag{B.5}$$

Its kernel is indeed nontrivial:

$$\ker \rho = \ker \rho' = \{(\zeta^2 \mathbb{1}_3, \zeta^3 \mathbb{1}_2, \zeta) \mid \zeta \in \mu_6\} \simeq \mu_6.^{1} \tag{B.6}$$

So we can say that

$$\big(SU(3) \times SU(2) \times U(1)\big)/\mu_6 \tag{B.7}$$

is the 'true' gauge group of the Standard Model. Note that the gauge boson representations also act trivially under the μ_6-subgroup.

B.2 The Broken Standard Model

Besides the mentioned fermions and gauge bosons, the Standard Model contains the (spin-0) Higgs boson, which transforms as

$$H : \left(\mathbf{1}, \mathbf{2}, \tfrac{1}{2}\right). \tag{B.8}$$

In order for the *electro-weak symmetry breaking* or *Higgs mechanism*[2] to work, the *Higgs potential* (the famous *Mexican hat*) is such that its vacuum states (its minima) are not symmetric under the full Standard Model gauge group. Traditionally, the vacuum state is taken to be a certain multiple of $\binom{0}{1}$. This is invariant (in the representation (B.8)) under the subgroup

$$\begin{aligned} &\left\{(u_3, u_2, \zeta) \in SU(3) \times SU(2) \times U(1) \;\middle|\; \zeta^3 u_2 \tbinom{0}{1} = \tbinom{0}{1}\right\} \\ &= \left\{ \left(u_3, \begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}, \zeta \right) \;\middle|\; u_3 \in SU(3),\ \zeta \in U(1) \right\} \simeq SU(3) \times U(1). \end{aligned} \tag{B.9}$$

[1] μ_N denotes the finite abelian group

$$\mu_N = \{\zeta \in U(1) \mid \zeta^N = 1\} = \{e^{2n\pi i/N} \mid n \in \mathbb{Z}_N\},$$

which is isomorphic to the additive group $\mathbb{Z}_N$.

[2] or *Englert–Brout–Higgs–Guralnik–Hagen–Kibble mechanism*

We write the representations of $\mathrm{SU}(3) \times \mathrm{U}(1)$ in the form $(\boldsymbol{n}_3, Q)$, where Q is interpreted as the *electric charge*. The relation between Q and the q of § A.2 is $Q = \frac{1}{6}q$.

Let us see how a $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$-representation $(\boldsymbol{n}_3, \boldsymbol{n}_2, \frac{1}{2}Y)$ behaves under this symmetry breaking.

- *For $\boldsymbol{n}_2 = \mathbf{1}$:* Since the $\mathrm{SU}(2)$-action is trivial, the symmetry breaking does not really anything. We write this as

$$(\boldsymbol{n}_3, \mathbf{1}, \tfrac{1}{2}Y) \rightsquigarrow (\boldsymbol{n}_3, \tfrac{1}{2}Y). \tag{B.10}$$

- *For $\boldsymbol{n}_2 = \mathbf{2}$:* Explicitly, this representation is

$$\rho(u_3, u_2, \zeta) = \zeta^{3Y} \rho_3(u_3) \otimes u_2 \tag{B.11}$$

(where we write $[\rho_3] = \boldsymbol{n}_3$). Restricting ρ to the group (B.9) gives

$$\rho\left(u_3, \begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}, \zeta\right) = \rho_3(u_3) \otimes \begin{pmatrix} \zeta^{3+3Y} & \\ & \zeta^{-3+3Y} \end{pmatrix}. \tag{B.12}$$

We denote this symmetry breaking as

$$(\boldsymbol{n}_3, \mathbf{2}, \tfrac{1}{2}Y) \rightsquigarrow (\boldsymbol{n}_3, \tfrac{1}{2} + \tfrac{1}{2}Y) \oplus (\boldsymbol{n}_3, -\tfrac{1}{2} + \tfrac{1}{2}Y). \tag{B.13}$$

- *For $\boldsymbol{n}_2 = \mathbf{3}$:* We can write this representation explicitly as

$$\rho(u_3, u_2, \zeta) = \zeta^{3Y} \rho_3(u_3) \otimes \mathrm{Ad}(u_2). \tag{B.14}$$

Restricting this to (B.9) gives:

$$\rho\left(u_3, \begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}, \zeta\right) = \zeta^{3Y} \rho_3(u_3) \otimes \mathrm{Ad}\begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}. \tag{B.15}$$

The operator $\mathrm{Ad}\left(\begin{smallmatrix} \zeta^3 & \\ & \zeta^{-3} \end{smallmatrix}\right)$ has 3 invariant subspaces, which are 1-dimensional: the spans of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$:

$$\begin{aligned} \mathrm{Ad}\begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \zeta^6 \\ 0 & 0 \end{pmatrix}, \\ \mathrm{Ad}\begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \\ \mathrm{Ad}\begin{pmatrix} \zeta^3 & \\ & \zeta^{-3} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ \zeta^{-6} & 0 \end{pmatrix}. \end{aligned} \tag{B.16}$$

This means that we can write our restricted representation as

$$(\boldsymbol{n}_3, \mathbf{3}, \tfrac{1}{2}Y) \rightsquigarrow (\boldsymbol{n}_3, 1 + \tfrac{1}{2}Y) \oplus (\boldsymbol{n}_3, \tfrac{1}{2}Y) \oplus (\boldsymbol{n}_3, -1 + \tfrac{1}{2}Y). \tag{B.17}$$

In fact, we have now derived the *Gell-Mann–Nishijima formula* for these cases:

$$Q = I_3 + \tfrac{1}{2}Y, \tag{B.18}$$

where I_3 is the *(third component of) weak isospin*. $I_3 = 0$ for SU(2)-singlets, $I_3 \in \left\{\tfrac{1}{2}, -\tfrac{1}{2}\right\}$ for doublets and $I_3 \in \{1, 0, -1\}$ for triplets.

Applying this to our fermion representations gives:

$$\begin{aligned}
\mathrm{u_L} &: \left(\mathbf{3}, \tfrac{2}{3}\right), & \overline{\mathrm{u}}_\mathrm{R} &: \left(\overline{\mathbf{3}}, -\tfrac{2}{3}\right),\\
\mathrm{d_L} &: \left(\mathbf{3}, -\tfrac{1}{3}\right), & \overline{\mathrm{d}}_\mathrm{R} &: \left(\overline{\mathbf{3}}, \tfrac{1}{3}\right),\\
\nu_\mathrm{eL} &: (\mathbf{1}, 0), & \overline{\nu}_\mathrm{eR} &: (\mathbf{1}, 0),\\
\mathrm{e_L^-} &: (\mathbf{1}, -1), & \mathrm{e_R^+} &: (\mathbf{1}, 1),\\
\mathrm{u_R} &: \left(\mathbf{3}, \tfrac{2}{3}\right), & \overline{\mathrm{u}}_\mathrm{L} &: \left(\overline{\mathbf{3}}, -\tfrac{2}{3}\right),\\
\mathrm{d_R} &: \left(\mathbf{3}, -\tfrac{1}{3}\right), & \overline{\mathrm{d}}_\mathrm{L} &: \left(\overline{\mathbf{3}}, \tfrac{1}{3}\right),\\
\nu_\mathrm{eR} &: (\mathbf{1}, 0), & \overline{\nu}_\mathrm{eL} &: (\mathbf{1}, 0),\\
\mathrm{e_R^-} &: (\mathbf{1}, -1), & \mathrm{e_L^-} &: (\mathbf{1}, 1),
\end{aligned} \tag{B.19}$$

where we recognize indeed the electric charges. Note that there is chiral (i.e. left-right) symmetry. For the gauge bosons we get:

$$\mathrm{g} : (\mathbf{8}, 0), \qquad \begin{aligned} \mathrm{W^+} &: (\mathbf{1}, 1),\\ \mathrm{Z} &: (\mathbf{1}, 0),\\ \mathrm{W^-} &: (\mathbf{1}, -1), \end{aligned} \qquad \gamma : (\mathbf{1}, 0). \tag{B.20}$$

A very important feature of this Higgs mechanism is that it generates the masses of the massive particles, in particular for the W en Z bosons.

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