COHERENT STATES IN QUANTUM MECHANICS

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Abstract

In this thesis we construct coherent states in the $L^2(\mathbb{R})$ Hilbert space, these states minimize the quantum mechanical uncertainty between x and p and obey the classical equations of motion for the harmonic oscillator. We confront this approach, which has an overcomplete basis, with an alternative description using a basis of wavefunctions and their Fourier transforms. However, since a function and its Fourier transform cannot be both supported on arbitrarily small sets, these states do not minimize the quantum mechanical uncertainty between x and p. Lastly, the concept of Planck cells will be introduced. These cells can be mathematically characterized by the above coherent states.

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Coherent states in Quantum Mechanics

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1 Introduction

In classical physics the properties of a certain system can be described using its position x and mass m. With these variables it is possible to determine the velocity v(=dx/dt), the momentum p(=mv) and any other dynamical variable of interest. Quantum mechanics describes the time evolution of physical systems in a different way, through the systems' wavefunction: $\psi(x,t)$. The variables x and p will be replaced by the operators x and p, which are functions. The position operator x multiplies by x and the momentum operator p takes the derivative to x and multiplies by $\hbar/i$, where $\hbar$ is the Planck constant h divided by 2π. The wavefunction of a system does not tell us everything about the corresponding system, one can only use it to predict the state of the system. There is a certain indeterminacy in quantum mechanics.

In the 17th century Christiaan Huygens had a theory of light, which stated that light would consist of waves. Around that same time Isaac Newton proposed a different theory of light, where it would consist of particles. In quantum mechanics it is assumed that all matter has a wave-particle duality, which means that all matter has both wave and particle properties. Einstein was the first scientist who mentioned this duality, in the 20th century his theory about the wave-particle duality of light became scientifically accepted which led to the beginning of quantum mechanics. However, it can not be said that Einstein invented the quantum mechanics. Max Planck, Louis de Broglie and many others all have contributed to the theory of quantum mechanics. In 1925 Werner Heisenberg, Max Born and Pascual Jordan developed the matrix mechanics formulation of quantum mechanics, which is now commonly used. Quantum mechanics differs significantly from classical mechanics in its predictions when looking at the atomic or sub-atomic scale. Many physical phenomena have currently been re-evaluated as the classical limit of quantum mechanics and there are even phenomena such as superconductivity and semiconductors which cannot be explained using classical mechanics. Although successful, quantum theory is still not fully understood, as it might never be. Like Richard Feynman once said: 'I think I can safely say that nobody understands quantum mechanics'.

Unlike classical mechanics, where the variables x and p can be interchanged, the operators x and p do not commute. This results in the Heisenberg uncertainty principle, discovered by Werner Heisenberg in 1927, it states that the uncertainty in position times the uncertainty in momentum is always equal or greater than one half of the reduced Planck constant $\hbar$, so:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Here, σ is the standard deviation. This uncertainty means that the position and momentum cannot be both measured with arbitrarily high precision. The more precise the position is measured, the less precise the momentum can be determined. Vice versa, the more precise the momentum is known, the less precise the position can be determined. The Heisenberg uncertainty principle is not saying anything about the researchers ability to measure the position or momentum, the principle is a law of physics.

Since, in quantum mechanics the position and momentum of a particle cannot be both precisely known at the same time, it is impossible to locate a point in the two dimensional x, p-plane. The most accurate way to localize a point in this plane is to draw a rectangle with area $\hbar/2$. Then the uncertainty in x times the uncertainty in p satisfies the Heisenberg uncertainty principle with equality, and therefore the uncertainty is minimal. The x, p-plane can be partitioned with these rectangles. In this thesis we will try to determine the wavefunctions which mathematically describe this partitioning.

In chapter two some basic knowledge about vector spaces will be mentioned. Thereafter, the concept of Hilbert spaces will be introduced. In quantum mechanics all wavefunctions are elements of a Hilbert space. In this thesis we will use the Hilbert space $L^2(\mathbb{R})$ which will also be introduced in the first chapter.

The main focus in this thesis will be the concept of coherent states, which will be introduced in the third chapter. A coherent state is a specific kind of quantum state of the harmonic oscillator. Its wavefunction will satisfy the Heisenberg uncertainty principle with equality and the expectation values of position and momentum satisfy the classical equations of motion of a harmonic oscillator. To obtain this result we shall study the lecture notes in relativistic quantum mechanics from L. Bergstrom and H. Hansson ([1]). The harmonic oscillator is important in physics since any oscillatory motion is harmonic by approximation as long as the amplitude is small.

In the fourth chapter another approach will be followed to construct states with minimal uncertainty. First, Fourier transformation on the Hilbert space $L^2(\mathbb{R})$ will be defined, from which we will obtain several delta function identities. Following the reasoning of U. Gerlach ([2]), a partitioning of the x, p-plane will be made with the help of one specific wavefunction and its Fourier transform. It turns out that Fourier transformation can not be used to construct wavefunctions with minimal uncertainty, since a function and its Fourier transform can not be both supported on arbitrarily small sets.

2 Preliminaries

In this chapter some basic definitions about vector spaces and the Lebesgue measure will be mentioned. With these definitions the $L^2(\mathbb{R})$ Hilbert space can be defined. Lastly, the position operator x and momentum operator p will be introduced along with some of their properties.

2.1 Matrix operations

Definition 2.1. *The* **transpose** *of a matrix A is another matrix A^T, created by interchanging rows and columns. The $(i,j)^{th}$ element of A^T is the $(j,i)^{th}$ element of A.*

If A is an $m \times n$ matrix then A^T is an $n \times m$ matrix. The transpose of a scalar is the same scalar, the transpose of a column matrix is a row matrix.

Definition 2.2. *The* **complex conjugate** *of A is denoted by $\overline{A}$. If A is a complex number, then $\overline{A}$ is also a complex number with the same real part but with imaginary parts of equal magnitude and opposite sign. The complex conjugate of a matrix consists of the complex conjugate of every element.*

Definition 2.3. *The* **hermitian conjugate** *(or adjoint) of a matrix A, which is denoted by $A^\dagger$, is the transpose conjugate of A.*

2.2 Vector spaces

Definition 2.4. *A* **seminorm** $\|.\|$ *on a vector space E is a function $\|.\| : E \to [0,\infty)$ satisfying:*

1. $\|x+y\| \leq \|x\| + \|y\| \quad (x, y \in E)$,
2. $\|\lambda x\| = |\lambda| \cdot \|x\| \quad (x \in E, \lambda \in \mathbb{C})$.

Definition 2.5. *A* **norm** $\|.\|$ *on a vector space E is a function $\|.\| : E \to [0,\infty)$ satisfying:*

1. $\|x+y\| \leq \|x\| + \|y\| \quad (x, y \in E)$,
2. $\|\lambda x\| = |\lambda| \cdot \|x\| \quad (x \in E, \lambda \in \mathbb{C})$,
3. $\|x\| = 0 \implies x = 0 \quad (x \in E)$.

Definition 2.6. *The* **Cauchy-Schwarz** *inequality states that for all vectors x and y of an inner product space, with inner product $\langle .|. \rangle$:*

$$|\langle x|y\rangle|^2 \leq \langle x|x\rangle\langle y|y\rangle.$$

Equivalently, by taking the square root of both sides and referring to the norms of the vectors,

$$|\langle x|y\rangle| \leq \|x\|\|y\|.$$

Definition 2.7. *Let E and F be normed vector spaces. The maps $E \to F$ that are both linear and continuous form a vector space denoted by $L(E,F)$. For $F = \mathbb{C}$ there is a special notation: $E' = L(E,\mathbb{C})$. The space E' is called the* **dual space** *of E.*

Definition 2.8. *An* **isomorphism** *is a bijective map f such that both f and its inverse f^{-1} are homomorphisms, which means that they are structure-preserving mappings. If there exists an isomorphism between two structures, we call the two structures* **isomorphic**.

Definition 2.9. *Let E_1, E_2 be normed vectorspaces with norm $\|.\|$ and $T : E_1 \to E_2$ a function such that: $\|Tx\| = \|x\|$ for all $x \in E_1$. Then T is called an* **isometry**.

Definition 2.10. *A sequence $v_1, v_2, \dots$ in a vector space V over a field F is a* **basis** *if it satisfies the following properties:*

1. **Linear independence** *property: for all $a_1, a_2, \dots \in F$ and $v_1, v_2 \dots$, if $a_1v_1 + a_2v_2 + \dots = 0$, then necessarily $a_1 = a_2 = \dots = 0$.*
2. **Spanning** *property: for every x in V it is possible to choose $a_1, a_2, \dots \in F$ such that $x = a_1v_1 + a_2v_2 + \dots$.*

Definition 2.11. *A finite or infinite sequence $u_1, u_2, \dots$ in a vector space E is* **orthonormal** *if:*

$$\langle u_n | u_m \rangle = \delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Lemma 2.1. *An orthonormal set is linear independent.*

Proof. If $(u_n)_n$ is an orthonormal set and there are $\lambda_1, \lambda_2, \dots$ so that $\lambda_1 u_1 + \lambda_2 u_2 + \dots = 0$, then $0 = \langle \lambda_1 u_1 + \lambda_2 u_2 + \dots | u_n \rangle = \lambda_n$ for each n, since $\langle \lambda_m u_m | u_n \rangle = \lambda_m \langle u_m | u_n \rangle = \lambda_m \delta_{nm}$. So $\lambda_1 = \lambda_2 = \dots = 0$, and the set $(u_n)_n$ is linear independent. □

Definition 2.12. *A set D is* **dense** *in X if the X-closure of D is equal to X.*

Definition 2.13. *A linear operator T from one topological vector space X, to another one, Y, is said to be* **densely defined** *if the domain of T is a dense subset of X and the range of T is contained within Y.*

2.3 Lebesgue Measure

Definition 2.14. *Let S be a subset of $\mathbb{R}$, let $L(I)$ be the length of an interval $I \subseteq \mathbb{R}$: if $I = (a, b)$, then $L(I) = b - a$. Finally, let M be the set consisting of the values $\sum_{A \in C} L(A)$ for all possible countable collections of open intervals C that cover S, that is, $S \subseteq \cup C$.*
Then the **Lebesgue outer measure** *of S is defined by: $m^*(S) = inf(M)$.*

Since $\sum_{A \in C} L(A) \geq 0$, it follows that $m^*(S) \geq 0$. However, it is possible that $m^*(S)$ could equal $+\infty$. Note that the outer measure can be defined for every single set, because we can take the infimum of any non-empty set.

Definition 2.15. *Let $E \subseteq \mathbb{R}$. It is said that E is* **Lebesgue measurable** *if we have the identity: $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$ for every subset A of $\mathbb{R}$. If E is measurable, the Lebesgue measure of E is defined as $m(E) = m^*(E)$. If E is not Lebesgue measurable, then $m(E)$ is left undefined.*

Definition 2.16. *Let E be a Lebesgue measurable subset of $\mathbb{R}$, and let $f : E \to \mathbb{R}$ be a function. The function f is a* **Lebesgue measurable function** *if $f^{-1}(V)$ is Lebesgue measurable for every open set $V \subseteq \mathbb{R}$.*

2.4 Hilbert Space

Definition 2.17. *Let U, V be two vector spaces. The mapping from U to V is called an* **operator**.

Definition 2.18. *A* **Hilbert space** *is a vector space provided with an inner product such that the space is complete relative to the inner product norm.*

Definition 2.19. *Let E_1, E_2 be Hilbert spaces. A linear map $T : E_1 \to E_2$ is called* **unitary** *if it is bijective and $\langle Tx|Ty\rangle = \langle x|y\rangle$ for all $x, y \in E_1$.*

Definition 2.20. *Let E_1, E_2 be Hilbert spaces and $T \in L(E_1, E_2)$. For $y \in E_2$ the function $x \mapsto \langle Tx|y\rangle$ $(x \in E_1)$ is an element of E_1', hence is of the form $x \mapsto \langle x|\overline{T}y\rangle$ $(x \in E_1)$ for a unique $\overline{T}y$ in E_1. Thus, we obtain a map $\overline{T} : E_2 \to E_1$, determined by:*

$$\langle Tx|y\rangle = \langle x|\overline{T}y\rangle \quad (x \in E_1, y \in E_2).$$

We call $\overline{T}$ the adjoint of T. If $\overline{T} = T$, then T is Hermitian or also called self-adjoint. This can only occur when $E_1 = E_2$.

Definition 2.21. *The* **commutator** *of two operators A, B defined on a Hilbert space is given by: $[A, B] = AB - BA$.*

2.5 L^p-Spaces

Using the previously mentioned definitions, the L^p-space can be constructed.

Definition 2.22. *Let $1 \leq p < \infty$. By $\mathcal{L}^p[\mu]$ we mean the set of all Lebesgue measurable functions f on $X \subseteq \mathbb{R}$ for which:*

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p} < \infty.$$

The set of such functions forms a vector space, with the following natural operations:

$$(f + g)(x) = f(x) + g(x) \text{ and } (\lambda f)(x) = \lambda f(x),$$

for every scalar λ.

It can be shown that the $\mathcal{L}^p$ spaces are normed vector spaces with the help of Minkowski's inequality:

Proposition 2.1. (Minkowski's inequality). *Let $1 \leq p < \infty$ and $f, g \in \mathcal{L}^p[\mu]$ with μ the Lebesgue measure, then: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. This is Theorem 3.5 from [6]. □

Note that Minkowski's inequality is the triangle inequality in $\mathcal{L}^p[\mu]$.

Let $1 \leq p < \infty$ and $f, g \in \mathcal{L}^p[\mu]$. Then $f + g$ and $|f + g|^p$ are also Lebesgue measurable (Corollary 18.5.7 from [7]). With Minkowski's inequality it follows that $f + g \in \mathcal{L}^p[\mu]$. The vectorspace $\mathcal{L}^p[\mu]$ together with the function $\|.\|_p$ is a seminorm.
This seminormed vector space can be made into a normed vector space by taking

the quotient space with respect to the kernel of $\|.\|_p$. The kernel of $\|.\|_p$ is defined by:

$$\mathcal{N} := \{f \in \mathcal{L}^p[\mu] : \|f\|_p = 0\}.$$

The kernel of $\|.\|_p$ does not depend on p since any measurable function f satisfies: $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere.
Now we can define a L^p-space:

Definition 2.23. *Let $1 \leq p < \infty$, then define the L^p-**space** by:*

$$L^p(\mu) := \mathcal{L}^p[\mu]/\mathcal{N}.$$

On $L^p(\mu)$ we have the norm $\|.\|_p$, defined by: $\|f + \mathcal{N}\|_p = \|f\|_p$.

Theorem 2.1. *Let $1 \leq p < \infty$, then $C(X)$ is dense in $L^p(\mu)$.*

Proof. This is Theorem 3.14 from [6]. □

It turns out that a L^2-space is complete, to prove this we need the following theorem:

Theorem 2.2. *Let E a normed vector space. The space E is complete if every sequence $(x_n)_n$ in E with $\sum \|x_n\| < \infty$, is summable. This means that the sequence $(x_1 + \cdots + x_N)_N$ converges.*

Proof. Suppose every sequence $(x_n)_n$ in E with $\sum \|x_n\| < \infty$, is summable. Let $(u_n)_n$ be a Cauchy sequence in E, we have to show that this Cauchy sequence has a limit u and that u is also in E.
Because $(u_n)_n$ is a Cauchy sequence we can choose $N(1), N(2), \ldots$ in $\mathbb{N}$ so that

$$\|u_n - u_{N(i)}\| \leq 2^{-i} \text{ as soon as } n \geq N(i) \quad (*),$$

and $N(2) > N(1), N(3) > N(2)$, etc. Take $i \in \mathbb{N}$ arbitrary and $n \geq N(i+1)$ then:

$$\|u_{N(i+1)} - u_{N(i)}\| \leq \|u_{N(i+1)} - u_n\| + \|u_n - u_{N(i)}\| \leq 2 \cdot 2^{-(i+1)} = 2^{-i}.$$

Consider the sequence $(v_i)_{i=0}^{\infty}$ in E where $v_0 = u_{N(1)}$ and $v_i = u_{N(i+1)} - u_{N(i)}$ for $i \geq 1$. We have $\sum_i v_i \leq \sum_i 2^{-i} \leq \infty$. So the sequence v_i is summable. The partial sums of this sequence are $\sum_{i=0}^{M} v_i = u_{N(M+1)}$, so $u := \lim_{i\to\infty} u_{N(i)}$ exists. From the inequality (*) we get $u_n \to u$, so the Cauchy sequence $(u_n)_n$ has a limit u in E. □

Theorem 2.3. *The space $L^p(\mu)$ is complete for $1 \leq p < \infty$ with respect to the Lebesgue measure μ.*

Proof. Let $1 \leq p < \infty$. Because $L^p(\mu)$ is the quotient space of $\mathcal{L}^p[\mu]$ with respect to the kernel of $\|.\|_p$, we can consider sequences in $\mathcal{L}^p[\mu]$ to prove that $L^p(\mu)$ is complete. We are going to use Theorem 2.2. So, take a sequence $(f_n)_n$ in $\mathcal{L}^p[\mu]$ such that $s := \sum_n \|f_n\|_p$ is finite. We have to show that the sequence $(f_1 + f_2 + \cdots + f_N)_N$ converges. As every function can be written as a linear combination of non negative real functions, we may assume $f_n \geq 0$ for all n.

Now, for each n, set $g_n := f_1 + \cdots + f_n$ in $\mathcal{L}^p[\mu]$. The sequence $(g_n^p)_n$ is an increasing sequence in $\mathcal{L}^1[\mu]$. Here,

$$\int g_n^p d\mu = \|f_1 + \cdots + f_n\|_p^p \leq (\|f_1\|_p + \ldots \|f_n\|_p)^p \leq s^p \leq \infty,$$

for every n. Note that $g_1^p \leq g_2^p \leq \ldots$ and $\sup_n \int g_n^p d\mu \leq \infty$, so we can use Levi's Theorem (Theorem C.16 from [5]) which states that: There is a g in $\mathcal{L}^p[\mu]^+$ with $g_n^p \to g^p$ almost everywhere. Then $(g_n - g)^p \to 0$ almost everywhere, and $\|g_n - g\|_p \to 0$ by Lebesgue's Theorem (Theorem C.18 from [5]). So we have proven that $\|g - (f_1 + \cdots + f_n)\|_p \to 0$. Therefore, the sequence $(f_1 + \cdots + f_n)_n$ converges and from Theorem 2.2 it follows that $L^p(\mu)$ is complete. □

For now, let $p = 2$.

Definition 2.24. *Let $f(x), g(x) \in \mathbf{L^2}(\mathbb{R})$, then the inner product $\langle f|g\rangle$ is defined by:*

$$\langle f|g\rangle := \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx.$$

Since $f(x)$ and $g(x)$ are both in $L^2(\mathbb{R})$, the expression $\langle f|g\rangle$ is guaranteed to exist i.e. it converges to a finite number. This follows from the Cauchy-Schwarz inequality (Definition 2.6):

$$\left| \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx \right| \leq \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx \int_{-\infty}^{\infty} |g(x)|^2 dx} \leq \infty.$$

The inner product has the following properties:

$$\begin{aligned}
\langle y|x\rangle &= \overline{\langle x|y\rangle} \quad \text{for all } x, y \in L^2(\mathbb{R}), \\
\langle x + y|z\rangle &= \langle x|z\rangle + \langle y|z\rangle \quad \text{for all } x, y, z \in L^2(\mathbb{R}), \\
\langle x|\alpha y\rangle &= \alpha\langle x|y\rangle \quad \text{for all } x, y \in L^2(\mathbb{R}) \text{ and } \alpha \in \mathbb{C}, \\
\langle x|x\rangle &\geq 0 \quad \text{for all } x \in L^2(\mathbb{R}), \\
\langle x|x\rangle &= 0 \implies x = 0.
\end{aligned}$$

We can define the norm $\|x\|_2$ of $x \in L^2(\mathbb{R})$ by $\|x\|_2^2 = \langle x|x\rangle$.
If a set of functions $\{f_n\}$ with $f_n \in L^2(\mathbb{R})$ is orthonormal, then $\langle f_n|f_m\rangle = \delta_{nm}$.
Functions in $L^2(\mathbb{R})$ are called **square-integrable functions**.

Corollary 2.1. *The space $L^2(\mathbb{R})$ is a Hilbert space.*

Proof. The space $L^2(\mathbb{R})$ is a complete vector space (Theorem 2.3), provided with an inner product. □

2.6 Position and momentum operator

Let x be the position operator and $p := \frac{\hbar}{i}\frac{d}{dx}$ the momentum operator. Their commutator can be calculated with the help of a so called 'test function' $f(x)$ with the following property: $f(x) \in \mathcal{D} := \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n, m \in \mathbb{N}\}$. Let $f \in \mathcal{D}$ then: $xf \in \mathcal{D}$ and $pf = \frac{\hbar}{i}\frac{d}{dx}f = \frac{\hbar}{i}f^{(1)}(x) \in \mathcal{D}$. So, $x\mathcal{D} \subset \mathcal{D}$ and

$p\mathcal{D} \subset \mathcal{D}$. Therefore, $x : \mathcal{D} \to \mathcal{D}$ and $p : \mathcal{D} \to \mathcal{D}$.
Now use an arbitrary $f \in \mathcal{D}$ as a test function to calculate the commutator of x and p, so:

$$[x,p]f = x\frac{\hbar}{i}\frac{d}{dx}(f) - \frac{\hbar}{i}\frac{d}{dx}(xf) = \frac{\hbar}{i}\left(x\frac{df}{dx} - x\frac{df}{dx} - f\right) = i\hbar f.$$

Dropping the test function, we obtain:

$$[x,p] = i\hbar. \tag{1}$$

In this thesis the *bra-ket notation* will be used, this is a standard notation for describing quantum states in quantum mechanics. Let H a Hilbert space. The inner product of two states is denoted by a *bracket*, $\langle\alpha|\beta\rangle$. The left part is called the *bra* and the right part the *ket*. The ket $|\beta\rangle$ is a column vector and is an element of the Hilbert space H. The bra $\langle\alpha|$ is a row vector and is an element of the dual space H' of H. Further, $\langle\alpha| = |\alpha\rangle^\dagger$.

Definition 2.25. *Let H a Hilbert space, with inner product $\langle .|.\rangle$. A continuous linear operator $Q : H \to H$ is* **hermitian** *if:*

$$\langle f|Qg\rangle = \langle Qf|g\rangle \quad \textit{for all } f,g \in H.$$

Definition 2.26. *Let H a Hilbert space, with inner product $\langle .|.\rangle$. A linear operator $Q : H \to H$ is* **symmetric** *on the domain $\mathcal{D} \subset H$ if:*

$$\langle f|Qg\rangle = \langle Qf|g\rangle \quad \textit{for all } f,g \in \mathcal{D}.$$

Corollary 2.2. *Let H a Hilbert space, with inner product $\langle .|.\rangle$ and Q a hermitian operator. Then the operator Q is also symmetric.*

Proof. Since Q is hermitian, it is linear. Furthermore, for all $f,g \in H$ it satisfies the equation: $\langle fQ|g\rangle = \langle f|Qg\rangle$, and therefore also for all $f,g \in D \subset H$. □

Theorem 2.4. *The position operator x is symmetric on the domain $\mathcal{D} = \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n,m \in \mathbb{N}\}$.*

Proof. Let $f(x), g(x) \in \mathcal{D}$. Using that $x = \overline{x}$ we find:

$$\begin{aligned}\langle f|xg\rangle &= \int_{-\infty}^{\infty} \overline{f(x)}[xg(x)]dx = \int_{-\infty}^{\infty} \overline{[xf(x)]}g(x)dx \\ &= \langle xf|g\rangle.\end{aligned}$$

So indeed, x is symmetric on $\mathcal{D}$. □

Theorem 2.5. *The momentum operator $p = \frac{\hbar}{i}\frac{d}{dx}$ is symmetric on $\mathcal{D} = \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n,m \in \mathbb{N}\}$.*

Proof. Let $f(x), g(x) \in \mathcal{D}$, then:

$$\begin{aligned}\langle f|pg\rangle &= \int_{-\infty}^{\infty} \overline{f(x)}\left(\frac{\hbar}{i}\frac{dg(x)}{dx}\right)dx = \left[\frac{\hbar}{i}\overline{f(x)}g(x)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \overline{\left(\frac{\hbar}{i}\frac{df(x)}{dx}\right)}g(x)dx \\ &= \langle pf|g\rangle.\end{aligned}$$

We applied integration by parts where the boundary term was equal to zero because f and g are square integrable functions which go to zero at $\pm\infty$. □

Lemma 2.2. *Define* $\mathcal{D} := \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n, m \in \mathbb{N}\}$*, and let* Q *a symmetric operator on* $\mathcal{D}$*. If* $Qf(x) = qf(x)$*, with* $f(x) \in \mathcal{D}$*, then* $q \in \mathbb{R}$*.*

Proof. Since Q is symmetric, it follows that:

$$q\langle f|f\rangle = \langle f|Qf\rangle = \langle Qf|f\rangle = \overline{q}\langle f|f\rangle.$$

Here, $\langle f|f\rangle$ cannot be zero because $f = 0$ is not a legal eigenfunction, so $q = \overline{q}$ and hence q is real. □

Corollary 2.3. *Let* x *the position operator and* $f(x) \in \mathcal{D} = \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n, m \in \mathbb{N}\}$*. If* $xf(x) = qf(x)$*, then* $q \in \mathbb{R}$*.*

Proof. From Theorem 2.4 we know that the position operator x is symmetric on $\mathcal{D}$. Therefore, applying Lemma 2.2, we find that if $f(x) \in \mathcal{D}$ and if $xf(x) = qf(x)$, then $q \in \mathbb{R}$. □

3 Coherent states of the harmonic oscillator

In this chapter the concept of coherent states will be introduced, inspired on section 4.3 of [1]. First, we will investigate the harmonic oscillator in quantum mechanics. It will turn out that coherent states represent the equations of motion of the classical harmonic oscillator.

3.1 The harmonic oscillator

In classical mechanics we can talk about the position of a particle at any given time: $x(t)$. The quantum mechanics analog to this is a particle's wavefunction: $|\psi(x,t)\rangle$. This wavefunction has a statistical interpretation, $|\psi(x,t)|^2$ gives the probability of finding the particle at position x at time t. More precisely we could say that $\int_a^b |\psi(x,t)|^2 dx$ is the probability of finding the particle between a and b, at time t. The wavefunction can be obtained by solving the Schrödinger equation:

Definition 3.1. *The following equation is called the one dimensional* **Schrödinger equation***:*

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2|\psi\rangle}{\partial x^2} + V|\psi\rangle.$$

Here, i is the square root of -1 and $\hbar = \frac{h}{2\pi}$, with h the Planck constant.

The paradigm for a classical harmonic oscillator is a mass m attached to a spring of force constant k. Ignoring friction, the potential energy is given by: $V(x) = \frac{1}{2}kx^2$. The frequence of oscillation, denoted by ω, is given by: $\omega = \sqrt{\frac{k}{m}}$. The quantum problem is to solve the one dimensional Schrödinger equation for the potential $V(x) = \frac{1}{2}m\omega^2x^2$. Because this potential is not time dependent we can solve the Schrödinger equation by the method of separation of variables. For more detail about this method see section 2.1 of [3]. It suffices to solve the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2|\psi\rangle}{dx^2} + V(x)|\psi\rangle = E|\psi\rangle.$$

We can rewrite this equation with the help of the momentum operator $p = \frac{\hbar}{i}\frac{d}{dx}$, which results in:

$$\underbrace{\left(\frac{p^2}{2m} + V(x)\right)}_{=H}|\psi\rangle = E|\psi\rangle,$$

where H is called the Hamiltonian. The Hamiltonian of the harmonic oscillator is given by:

$$H = \frac{1}{2m}[p^2 + (m\omega x)^2].$$

The wavefunctions of the harmonic oscillator can be determined using *ladder operators*:

Definition 3.2. *The following quantities are called* **ladder operators***:*

$$a = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x),$$
$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x).$$

Here a is called the lowering operator, and $a^\dagger$ is called the raising operator.

The commutator of a and $a^\dagger$ can be calculated directly from their definition:

$$[a, a^\dagger] = \frac{i}{\hbar}[p, x] = 1.$$

Here, we used that the commutator of p and x is equal to $-i\hbar$, which follows from equation (1).
The operators x and p expressed in terms of these ladder operators are:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger),$$
$$p = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger).$$

So,

$$x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2,$$
$$p^2 = -\frac{m\omega\hbar}{2}(a - a^\dagger)^2.$$

We can express the Hamiltonian of the harmonic oscillator in terms of the ladder operators using that:

$$aa^\dagger = \frac{1}{2\hbar m\omega}(p^2 + (m\omega x)^2) - \frac{i}{2\hbar}[x, p] = \frac{1}{\hbar\omega}H + \frac{1}{2}. \tag{2}$$

Here we recognized the Hamiltonian of the harmonic oscillator and the commutator of x and p, which is equal to $i\hbar$.
Now we can express the Hamiltonian H in terms of the ladder operators:

$$H = \hbar\omega(aa^\dagger - \frac{1}{2}).$$

And it follows that:

$$[H, a] = \hbar\omega a[a^\dagger, a] = -\hbar\omega a,$$
$$[H, a^\dagger] = \hbar\omega a^\dagger[a, a^\dagger] = \hbar\omega a^\dagger.$$

The lowering operator will always reduce the energy of the state, since:

$$Ha|\psi\rangle = (aH - \hbar\omega a)|\psi\rangle = (E - \hbar\omega)a|\psi\rangle.$$

Similarly, the raising operator will always raise the energy of the state, hence the name ladder operator:

$$Ha^\dagger|\psi\rangle = (a^\dagger H + \hbar\omega a^\dagger)|\psi\rangle = (E + \hbar\omega)a^\dagger|\psi\rangle.$$

The ground state of a system is the state with the lowest energy. Since, the lowering operator will always reduce the energy of the state, the ground state wavefunction of the harmonic oscillator $|0\rangle$ must satisfy the equation $a|0\rangle = 0$. Consequently, the ground state wavefunction can be determined (see section 2.3 of [3] for more detail).

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \in L^2(\mathbb{R}). \tag{3}$$

Using the raising operator, the excited states $|n\rangle$ can be calculated (see section 2.3 of [3] for more detail). This gives:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \in L^2(\mathbb{R}). \tag{4}$$

With,

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{5}$$
$$a|n\rangle = \sqrt{n}|n-1\rangle. \tag{6}$$

Furthermore, the wavefunctions are orthonormal. So, $\langle n|m\rangle = \delta_{nm}$.
We also know from [3] that the energy of the harmonic oscillator is quantized:

$$H|n\rangle = E_n|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle. \tag{7}$$

Using the above two relations, for the ladder operators acting on the wavefunction $|n\rangle$, and *Hermite polynomials*, we can prove that the collection of wavefunctions of the harmonic oscillator forms a basis for $L^2(\mathbb{R})$.

Definition 3.3. *For $n = 1, 2, \ldots$ we define the* **Hermite polynomial** H_n *by:*

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

The Hermite polynomials obey the following recursion formula:

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x). \tag{8}$$

Further, their derivative satisfies:

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x). \tag{9}$$

Now we can prove the following theorem:

Theorem 3.1. *The collection of wavefunctions $\{|n\rangle\}$ of the harmonic oscillator forms a basis for $L^2(\mathbb{R})$. This basis is called the $|n\rangle$ basis.*

Proof. To prove this theorem we will consider the functions:

$$|\psi_n\rangle := \underbrace{(2^n n! \pi^{1/2} x_0)^{-1/2}}_{:=\alpha_n} e^{-y^2/2} H_n(y).$$

Here, $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. We will show that these functions are eigenfunction of the Hamiltonian of the harmonic oscillator and that they are equal to the wavefunctions $|n\rangle$. First we are going to rewrite the Schrödinger equation for the harmonic oscillator, which is given by:

$$H|\psi\rangle = \left(\frac{-\hbar}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right)|\psi\rangle = E|\psi\rangle.$$

Using $y = x/x_0$ we obtain:

$$\frac{d^2|\psi\rangle}{dy^2} - y^2|\psi\rangle = -\frac{2E}{\hbar\omega}|\psi\rangle.$$

For $|\psi_n\rangle = \alpha_n e^{-y^2/2} H_n(y) = \alpha_n \phi(y)$ we have:

$$\frac{d\phi(y)}{dy} = \left(\frac{dH_n(y)}{dy} - yH_n(y)\right)e^{-y^2/2}.$$

And,

$$\frac{d^2\phi(y)}{dy^2} = \left(\frac{d^2H_n(y)}{dy^2} - 2y\frac{dH_n(y)}{dy} + (y^2-1)H_n(y)\right)e^{-y^2/2}.$$

Here, we can use equation (9) for the derivative of H_n, then we obtain:

$$= (4n(n-1)H_{n-2}(y) - 4ynH_{n-1}(y) + (y^2-1)H_n(y))e^{-y^2/2}.$$

Now use the recursion relation (8) for the Hermite polynomials:

$$= H_n(y)(-2n-1+y^2)e^{-y^2/2}.$$

So the Schrödinger equation becomes:

$$-(2n+1)\alpha_n e^{-y^2/2} H_n(y) = -\frac{2E}{\hbar\omega}\alpha_n e^{-y^2/2} H_n(y).$$

Therefore, $|\psi_n\rangle = \alpha_n e^{-y^2/2} H_n(y)$ is an eigenfunction of H with eigenvalue $E_n = \hbar\omega(n+\frac{1}{2})$. Rewriting $|\psi_n\rangle$ we see that this function is equal to $|n\rangle$, the wavefunction of the harmonic oscillator:

$$\begin{aligned}|\psi_n\rangle &= \alpha_n e^{-y^2/2} H_n(y) = \alpha_n (-1)^n e^{y^2/2}\frac{d^n}{dy^n}e^{-y^2} \\ &= (2^n n!\pi^{1/2}x_0)^{-1/2}\left(y - \frac{d^n}{dy^n}\right)^n e^{-y^2/2}.\end{aligned}$$

Substituting $x = yx_0$ and $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, we obtain:

$$= \frac{1}{\sqrt{n!}}\left(\frac{1}{2\hbar m\omega}\right)^{n/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(m\omega x - \hbar\frac{d}{dx}\right)^{n} e^{-m\omega x^2/(2\hbar)}.$$

Using Definition 3.2 for $a^\dagger$ we can recognize expression (4) for $|n\rangle$, so that:

$$= \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = |n\rangle.$$

So, $|\psi_n\rangle = \alpha_n e^{-y^2/2}H_n(y)$ is equal to $|n\rangle$. Using Theorem 14.18 from [5] we see that the functions $\alpha_n e^{-y^2/2}H_n(y)$ form a basis for $L^2(\mathbb{R})$. Consequently, the collection $\{|n\rangle\}$ forms a basis for $L^2(\mathbb{R})$. □

Corollary 3.1. *The closure relation for the wavefunctions of the harmonic oscillator is:*

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{1}.$$

Where $\hat{1}$ *is the identity operator.*

Proof. Let $f \in L^2(\mathbb{R})$, then since the collection $\{|n\rangle\}$ forms a basis for $L^2(\mathbb{R})$ we could write:

$$f = \sum_{m=0}^{\infty} a_m|m\rangle.$$

The wavefunctions of the harmonic oscillator are orthonormal so, $\langle m|m\rangle = 1$ and $\langle n|m\rangle = 0$ if $n \neq m$. Using this, we obtain:

$$\left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right)f = \left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right)\sum_{m=0}^{\infty}a_m|m\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_m|n\rangle\langle n|m\rangle = \sum_{n=0}^{\infty}a_n|n\rangle = f.$$

So we can conclude that:

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{1}.$$

□

3.2 Coherent states

Before the coherent states will be defined, we introduce the uncertainty principle:

Theorem 3.2. (Uncertainty principle) *Consider a system with suitable normalized wavefunction* $|\psi\rangle$*, e.g.* $|\psi\rangle \in \mathcal{D} = \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n, m \in \mathbb{N}\}$*, and two symmetric operators* A *and* B*. Then:*

$$(\sigma_A)(\sigma_B) \geq \frac{1}{2}|\langle[A, B]\rangle|.$$

This is called the uncertainty principle.

The *uncertainty* in operators A and B is defined by:

$$\sigma_A = (\langle A^2\rangle - \langle A\rangle^2)^{1/2} \geq 0,$$
$$\sigma_B = (\langle B^2\rangle - \langle B\rangle^2)^{1/2} \geq 0.$$

And the *expectation values* by:

$$\langle A\rangle = \langle\psi|A|\psi\rangle,$$
$$\langle B\rangle = \langle\psi|B|\psi\rangle.$$

Proof. If $\sigma_B \neq 0$, define the following operator:

$$C := A - \langle A\rangle + i\lambda(B - \langle B\rangle) \quad (\lambda \in \mathbb{R}).$$

Using this operator, the fact that A and B are symmetric operators and the fact that $|\psi\rangle$ is normalized, we obtain the following inequality for every real λ:

$$0 \leq \langle C\psi|C\psi\rangle = \langle\psi|C^\dagger C|\psi\rangle = (\sigma_A)^2 + \lambda^2(\sigma_B)^2 + \lambda\langle i[A,B]\rangle. \tag{10}$$

The right side of this equation has a minimum for $\lambda = -\frac{1}{2}\langle i[A,B]\rangle/(\sigma_B)^2$. This minimum is equal to:

$$(\sigma_A)^2 - \frac{\langle i[A,B]\rangle^2}{4(\sigma_B)^2} \geq 0.$$

Rearranging gives us the uncertainty principle,

$$(\sigma_A)(\sigma_B) \geq \frac{1}{2}|\langle[A,B]\rangle|.$$

When $\sigma_B = 0$ and $\sigma_A \neq 0$, we can obtain the uncertainty principle in the same way but with the roles of A and B interchanged. If $\sigma_A = \sigma_B = 0$, then from equation (10) follows that $\langle[A,B]\rangle = 0$, because λ can be negative. This result is in accordance with the uncertainty principle. □

Heisenberg discovered this uncertainty relation in 1926. He realized that every pair of physical properties that do not commute results in an uncertainty relation. This implication led the foundation of the contemporary quantum mechanics. In this thesis we will use the uncertainty relation with the physical properties position and momentum.

Theorem 3.3. *The* **Heisenberg uncertainty principle** *states that:*

$$(\sigma_x)(\sigma_p) \geq \frac{\hbar}{2}.$$

Proof. It is known from equation (1) that $[x,p] = i\hbar$. Since x and p are symmetric operators (Theorem 2.4 and 2.5), we can apply the uncertainty principle (Theorem 3.2) to x and p, then:

$$(\sigma_x)(\sigma_p) \geq \frac{1}{2}|\langle i\hbar\rangle| = \frac{\hbar}{2}.$$

□

Our main interest in this thesis is to find states that satisfy this uncertainty with equality.

Definition 3.4. *Wavefunctions that satisfy the Heisenberg uncertainty principle with equality are called* **minimum uncertainty wavefunctions**.

Theorem 3.4. *The ground state wavefunction $|0\rangle$ of the harmonic oscillator is a minimum uncertainty wavefunction.*

Proof. We need the following expectation values to prove the theorem:

$$\begin{aligned}\langle 0|(a+a^\dagger)(a+a^\dagger)|0\rangle &= \langle 0|aa + a^\dagger a + aa^\dagger + a^\dagger a^\dagger|0\rangle \\ &= \langle 0|aa|0\rangle + \langle 0|a^\dagger a|0\rangle + \langle 0|aa^\dagger|0\rangle + \langle 0|a^\dagger a^\dagger|0\rangle.\end{aligned}$$

Because $a|0\rangle = 0$ and $\langle 0|2\rangle = 0$, there is only one term nonzero:

$$= \langle 0|aa^\dagger|0\rangle = \langle 0|\left(\frac{1}{\hbar\omega}H + \frac{1}{2}\right)|0\rangle = \frac{1}{\hbar\omega}E_0 + \frac{1}{2} = 1. \tag{11}$$

In a similar way it follows that:

$$\langle 0|(a-a^\dagger)(a-a^\dagger)|0\rangle = \langle 0| - aa^\dagger|0\rangle = -1. \tag{12}$$

Now the expectation values for position and momentum can be calculated. We obtain:

$$\begin{aligned}\langle x\rangle_0 &:= \langle 0|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a+a^\dagger)|0\rangle = 0, \\ \langle p\rangle_0 &:= \langle 0|p|0\rangle = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\langle 0|(a-a^\dagger)|0\rangle = 0.\end{aligned}$$

Where we used that $a|0\rangle = 0$ and that the wavefunctions are orthonormal, so $\langle 0|1\rangle = 0$. Further, using equations (11) and (12) it follows that:

$$\begin{aligned}\langle x^2\rangle_0 &:= \langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(a+a^\dagger)^2|0\rangle = \frac{\hbar}{2m\omega}, \\ \langle p^2\rangle_0 &:= \langle 0|p^2|0\rangle = -\frac{\hbar m\omega}{2}\langle 0|(a-a^\dagger)^2|0\rangle = \frac{\hbar m\omega}{2}.\end{aligned}$$

Now we can calculate the uncertainty in x and p, which results in:

$$\begin{aligned}(\sigma_x)_0^2 &:= \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}, \\ (\sigma_p)_0^2 &:= \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar m\omega}{2}.\end{aligned}$$

We obtain the Heisenberg uncertainty principle with equality:

$$(\sigma_x)_0^2(\sigma_p)_0^2 = \frac{\hbar^2}{4}.$$

So, the ground state wavefunction $|0\rangle$ of the harmonic oscillator is a minimum uncertainty wavefunction. □

Theorem 3.5. *The wavefunctions $|n\rangle$ $(n \neq 0)$, are not minimum uncertainty wavefunctions.*

Proof. Since the wavefunctions are orthonormal, we obtain that:

$$\langle x\rangle_n := \langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a + a^\dagger|n\rangle = 0,$$

and that:

$$\langle p\rangle_n := \langle n|p|n\rangle = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\langle n|a - a^\dagger|n\rangle = 0.$$

Furthermore,

$$\langle x^2\rangle_n = \frac{\hbar}{2m\omega}\langle n|(a + a^\dagger)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|[a, a^\dagger] + 2a^\dagger a|n\rangle = \frac{\hbar}{2m\omega}(1 + 2n),$$
$$\langle p^2\rangle_n = -\frac{\hbar m\omega}{2}\langle n|(a - a^\dagger)^2|n\rangle = \frac{\hbar m\omega}{2}\langle n|[a, a^\dagger] + 2a^\dagger a|n\rangle = \frac{\hbar m\omega}{2}(1 + 2n).$$

Where relations (5) and (6) were used combined with the orthonormality of the wavefunctions of the harmonic oscillator.
Now we can calculate the uncertainty in x and p, which results in:

$$(\sigma_x)_n^2(\sigma_p)_n^2 = \frac{\hbar^2}{4}(1 + 2n)^2.$$

This is not a minimum uncertainty, as the uncertainty increases with n. □

The crucial point that causes the wavefunction $|n\rangle$ $(n \neq 0)$ to have no minimal uncertainty is that $a|n\rangle \neq 0$ so that: $\langle n|a^\dagger a|n\rangle \neq 0$. Further, $|n\rangle$ is not an eigenfunction of the operator a since $a|n\rangle = \sqrt{n}|n - 1\rangle$.
The minimum uncertainty wavefunction $|0\rangle$ did satisfy the relation $a|0\rangle = 0$, so that: $\langle 0|a^\dagger a|0\rangle = 0$. We would expect other minimum uncertainty wavefunctions $|\alpha\rangle$ if they satisfy the relation: $a|\alpha\rangle = \alpha|\alpha\rangle$.

Definition 3.5. *The states $|\alpha\rangle$ defined by: $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\langle\alpha|\alpha\rangle = 1$, are called* **coherent states**.

Later in this chapter we will see that a coherent state is a specific kind of quantum state of the quantum harmonic oscillator whose dynamics closely resemble the oscillating behavior of a classical harmonic oscillator system.

Theorem 3.6. *Coherent states satisfy the minimum uncertainty relation*

Proof. From $a|\alpha\rangle = \alpha|\alpha\rangle$ it follows that $\langle\alpha|a^\dagger = \langle\alpha|\overline{\alpha}$. So, $\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$. Furthermore,

$$\langle\alpha|a + a^\dagger|\alpha\rangle = \alpha + \overline{\alpha}, \tag{13}$$
$$\langle\alpha|a - a^\dagger|\alpha\rangle = \alpha - \overline{\alpha}, \tag{14}$$
$$\begin{aligned}\langle\alpha|(a + a^\dagger)^2|\alpha\rangle &= \langle\alpha|aa|\alpha\rangle + \langle\alpha|a^\dagger a^\dagger|\alpha\rangle + \langle\alpha|[a, a^\dagger]|\alpha\rangle + 2\langle\alpha|a^\dagger a|\alpha\rangle \\ &= \alpha^2 + (\overline{\alpha})^2 + 1 + 2\alpha\overline{\alpha} = (\alpha + \overline{\alpha})^2 + 1.\end{aligned}$$

In a similar way it follows that:

$$\langle\alpha|(a-a^\dagger)^2|\alpha\rangle = \alpha^2 + (\overline{\alpha})^2 - 1 - 2\alpha\overline{\alpha} = (\alpha-\overline{\alpha})^2 - 1.$$

Now we can calculate the expectation values for position and momentum:

$$\begin{aligned}(\sigma_x)^2_\alpha &= \langle x^2\rangle_\alpha - \langle x\rangle^2_\alpha = \frac{\hbar}{2m\omega}(\langle\alpha|(a+a^\dagger)^2|\alpha\rangle - \langle\alpha|a+a^\dagger|\alpha\rangle)\\ &= \frac{\hbar}{2m\omega}[(\alpha+\overline{\alpha})^2 + 1 - (\alpha+\overline{\alpha})^2] = \frac{\hbar}{2m\omega},\\ (\sigma_p)^2_\alpha &= \langle p^2\rangle_\alpha - \langle p\rangle^2_\alpha = -\frac{\hbar m\omega}{2}(\langle\alpha|(a-a^\dagger)^2|\alpha\rangle - \langle\alpha|a-a^\dagger|\alpha\rangle)\\ &= -\frac{\hbar m\omega}{2}[(\alpha-\overline{\alpha})^2 - 1 - (\alpha-\overline{\alpha})^2] = \frac{\hbar m\omega}{2}.\end{aligned}$$

Which implies:

$$(\sigma_x)^2_\alpha(\sigma_p)^2_\alpha = \frac{\hbar^2}{4}.$$

So, the coherent states satisfy the minimum uncertainty relation. □

3.3 Coherent states in the n-representation

The wavefunctions of the harmonic oscillator form a basis for $L^2(\mathbb{R})$, this is Theorem 3.1. Therefore, we can express the coherent states in these wavefunctions. In the $|n\rangle$ basis the coherent state $|\alpha\rangle$ is written as:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \quad (c_n \in \mathbb{R}).$$

Multiplying this expression from the left with the bra $\langle m|$ gives a expression for the coefficients c_m, so:

$$\langle m|\alpha\rangle = \sum_{n=0}^{\infty} c_n\langle m|n\rangle = c_m.$$

Here, we used the fact that the wavefunctions of the harmonic oscillator are orthonormal. As a result we obtain the following expression:

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\alpha\rangle. \tag{15}$$

Now use expression (4) for the wavefunction $|n\rangle$, then:

$$\langle n|\alpha\rangle = \frac{1}{\sqrt{n!}}\langle 0|a^n\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle.$$

Combining this with expression (15) we obtain:

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{16}$$

The constant factor $\langle 0|\alpha\rangle$ must still be determined, which can be done using normalisation since the coherent state $|\alpha\rangle$ has to be normalized. So,

$$1 = \langle\alpha|\alpha\rangle = \Big(\sum_{m=0}^{\infty}\langle\alpha|m\rangle\langle m|\Big)\Big(\sum_{n=0}^{\infty}|n\rangle\langle n|\alpha\rangle\Big).$$

Here, expression (15) was used, after rearranging the obtained expression, orthonormality can be used resulting in:

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\alpha|m\rangle\langle m|n\rangle\langle n|\alpha\rangle = \sum_{n=0}^{\infty}\langle\alpha|n\rangle\langle n|\alpha\rangle.$$

Now we can use expression (4) for the functions $|n\rangle$, this results in:

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(\overline{\alpha})^n\alpha^n\langle\alpha|0\rangle\langle 0|\alpha\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}|\alpha|^{2n}|\langle 0|\alpha\rangle|^2.$$

Here, the exponential function of $|\alpha|^2$ can be recognized:

$$= |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}.$$

Solving for $|\langle\alpha|0\rangle|$ we get:

$$|\langle 0|\alpha\rangle| = e^{-\frac{1}{2}|\alpha|^2}.$$

Now we know $\langle 0|\alpha\rangle$ up to a constant phase factor:

$$\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{i\beta}.$$

Substituting this into equation (16) we obtain the final form:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{17}$$

The constant phase factor $e^{i\beta}$ is left out because it does not contribute to the expectation value of the wavefunction, since $|e^{i\beta}|^2 = 1$, and because every multiple of a coherent state by a nonzero constant factor is still a coherent state. We can check that the coherent state $|\alpha\rangle$ is indeed orthonormal:

$$\langle\alpha|\alpha\rangle = e^{-|\alpha|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^n(\overline{\alpha})^m}{\sqrt{n!m!}}\langle m|n\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2}e^{|\alpha|^2} = 1.$$

The coherent state $|\alpha\rangle$ is not an eigenfunction of the harmonic oscillator, which can be seen from:

$$H|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}H|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}E_n|n\rangle \neq \lambda|\alpha\rangle,$$

with $\lambda \in \mathbb{C}$.

The coherent state $|\alpha\rangle$ can be expressed in terms of the *displacement operator* $D(\alpha)$, which is given by $D(\alpha) = e^{\alpha a^\dagger - \overline{\alpha} a}$. From expression (17) we know that:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

Now use expression (4) for $|n\rangle$, this gives:

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (a^\dagger)^n |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle, \tag{18}$$

where we recognized the exponential function of $\alpha a^\dagger$.
To rewrite this expression we need the *Baker−Campbell−Hausdorff formula* which states that if X and Y are Hilbert space operators that both commute with $[X, Y]$ then:

$$e^{X+Y} = e^{-\frac{1}{2}[X,Y]} e^X e^Y. \tag{19}$$

We apply this formula on the displacement operator with $X = \alpha a^\dagger$ and $Y = -\overline{\alpha} a$. Their commutator is: $[\alpha a^\dagger, -\overline{\alpha} a] = -|\alpha|^2 [a^\dagger, a] = |\alpha|^2$, since $[a, a^\dagger] = 1$. So this commutator commutes with both $\alpha a^\dagger$ and $\overline{\alpha} a$. Now, applying the Baker−Campbell−Hausdorff formula results in:

$$e^{\alpha a^\dagger - \overline{\alpha} a} = e^{\alpha a^\dagger} e^{-\overline{\alpha} a} e^{-\frac{1}{2}|\alpha|^2}.$$

Since $a|\alpha\rangle = 0$, it follows that $e^{-\overline{\alpha} a}|0\rangle = e^0 = 1$. Therefore,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle = e^{\alpha a^\dagger} e^{-\overline{\alpha} a} e^{-\frac{1}{2}|\alpha|^2} |0\rangle = e^{\alpha a^\dagger - \overline{\alpha} a} |0\rangle = D(\alpha)|0\rangle.$$

So, the coherent state $|\alpha\rangle$ is equal to the displacement operator $D(\alpha)$ operating on the ground state of the harmonic oscillator.

3.4 Orthogonality and completeness relations

We can calculate the overlap between two coherent states. Let $|\alpha\rangle$ and $|\beta\rangle$ be two coherent states, so $a|\alpha\rangle = \alpha|\alpha\rangle$ and $a|\beta\rangle = \beta|\beta\rangle$. Using expression (17) these two states can be expressed by:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad |\beta\rangle = e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} |m\rangle.$$

Then the overlap is calculated using:

$$\begin{aligned} \langle\alpha|\beta\rangle &= \left(e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\overline{\alpha})^n}{\sqrt{n!}} \langle n| \right) \left(e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} |m\rangle \right) \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{(\overline{\alpha})^n \beta^n}{n!} = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{\overline{\alpha}\beta}, \end{aligned}$$

and similarly,

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{\overline{\beta}\alpha}.$$

So that the overlap is given by:

$$|\langle\alpha|\beta\rangle|^2 = \langle\beta|\alpha\rangle\langle\alpha|\beta\rangle = e^{-|\alpha|^2-|\beta|^2+\overline{\alpha}\beta+\alpha\overline{\beta}} = e^{-|\alpha-\beta|^2}.$$

Suppose a system is in quantum state $|\alpha\rangle$, then there is a nonzero chance that the system is in quantum state $|\beta\rangle$ because $|\langle\alpha|\beta\rangle|^2 \neq 0$ if $\alpha \neq \beta$. Consequently, since $\langle\alpha|\beta\rangle \neq 0$ if $\alpha \neq \beta$, the collection of coherent states forms an overcomplete set. The number of coherent states is greater than the needed number for a basis.
Nevertheless, there is a closure relation:

$$\int_{\mathbb{C}} d^2\alpha |\alpha\rangle\langle\alpha| = \int_{\mathbb{C}} d^2\alpha e^{-|\alpha|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\overline{\alpha})^n \alpha^m}{\sqrt{n!m!}} |m\rangle\langle n|.$$

Now, writing α in polar form: $\alpha = re^{i\phi}$ and $d^2\alpha = rdrd\phi$, gives:

$$\begin{aligned} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \underbrace{\int_0^{2\pi} e^{i\phi(m-n)} d\phi}_{=2\pi\delta_{n,m}} \int_0^{\infty} re^{-r^2} r^{(n+m)} dr \\ &= \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} 2\pi \int_0^{\infty} re^{-r^2} r^{2n} dr. \end{aligned}$$

Change variable from r to $x = r^2$, then $dx = d(r^2) = 2rdr$ and we obtain:

$$= \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} 2\pi \underbrace{\int_0^{\infty} \frac{1}{2} e^{-x} x^n dx}_{=\frac{1}{2}\Gamma(n+1)} = \pi \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1}\pi.$$

Here, $(n+1) \in \mathbb{N} - \{0\}$, so $\Gamma(n+1) = n!$.
We used the closure relation from Corollary 3.1 to obtain this result.
We can conclude that the closure relation for coherent states is given by:

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1}.$$

3.5 Coherent states in the x-representation

In this section the wavefunction of a coherent state in the x-representation will be determined. We shall introduce some constants to simplify the calculation. Using equation (13) and (14) we find that:

$$\begin{aligned} \mathcal{R}(\alpha) &:= \frac{1}{2}(\alpha + \overline{\alpha}) = \langle\alpha| \frac{a + a^\dagger}{2} |\alpha\rangle = \sqrt{\frac{m\omega}{2\hbar}} \langle x\rangle_\alpha, \\ \mathcal{I}(\alpha) &:= \frac{1}{2i}(\alpha - \overline{\alpha}) = \langle\alpha| \frac{a - a^\dagger}{2i} |\alpha\rangle = \frac{1}{\sqrt{2m\omega\hbar}} \langle p\rangle_\alpha. \end{aligned}$$

We could also write,

$$\langle x\rangle_\alpha = \sqrt{\frac{2\hbar}{m\omega}}\mathcal{R}(\alpha), \tag{20}$$

$$\langle p\rangle_\alpha = \sqrt{2m\omega\hbar}\mathcal{I}(\alpha). \tag{21}$$

Here $\mathcal{R}(\alpha)$ is the real part of α and $\mathcal{I}(\alpha)$ is the imaginary part of α.

Now, we are going to write the coherent state $|\alpha\rangle$ in the x-representation. Let $|x'\rangle$ be an eigenstate of the operator x, so $x|x'\rangle = x'|x'\rangle$.
Using the previously derived form of the coherent states, expressed in terms of the ground state of the harmonic oscillator (equation (18)), we get:

$$\langle x'|\alpha\rangle = \langle x'|e^{-\frac{1}{2}|\alpha|^2+\alpha a^\dagger}|0\rangle.$$

Now use Definition 3.2 for $a^\dagger$, then:

$$= e^{-\frac{1}{2}|\alpha|^2}\langle x'|e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}(x-\frac{ip}{m\omega})}|0\rangle.$$

Multiplying from the left with $\langle x'|$, remembering that the eigenvalues of x' are real (Corollary 2.3), so $\langle x'|x = \langle x'|x'$ and $\langle x'|p = \frac{\hbar}{i}\frac{d}{dx'}$, we obtain:

$$= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}(x'-\frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx'})}\langle x'|0\rangle.$$

Substitute the constants $N = (\frac{m\omega}{\pi\hbar})^{1/4}$ and $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. Further, use the explicit expression for $|0\rangle$ which is given by equation (3), then:

$$\begin{aligned}&= e^{-\frac{1}{2}|\alpha|^2}e^{\frac{\alpha}{x_0\sqrt{2}}(x'-x_0^2\frac{d}{dx'})}\langle x'|Ne^{-\frac{1}{2}(\frac{x}{x_0})^2}\\ &= e^{-\frac{1}{2}|\alpha|^2}e^{\frac{\alpha}{x_0\sqrt{2}}(x'-x_0^2\frac{d}{dx'})}Ne^{-\frac{1}{2}(\frac{x'}{x_0})^2}.\end{aligned}$$

For notational simplicity use $y' = x'/x_0$, which results in:

$$= e^{-\frac{1}{2}|\alpha|^2}e^{\frac{\alpha}{\sqrt{2}}(y'-\frac{d}{dy'})}Ne^{-\frac{1}{2}y'^2}.$$

Now we are going to use the Baker−Campbell−Hausdorff formula, given by formula (19). Apply this formula with $X = \frac{\alpha}{\sqrt{2}}y'$ and $Y = -\frac{\alpha}{\sqrt{2}}\frac{d}{dy'}$. The commutator of X and Y can be calculated using a test function $f \in L^2(\mathbb{R})$ with the following property: $f \in \mathcal{D} = \{f(x) : x^n f^{(m)}(x) \in L^2(\mathbb{R}) \quad \forall n, m \in \mathbb{N}\}$. Then:

$$[X,Y]f = -\frac{\alpha^2 y'}{2}\frac{d}{dy'}(f) + \frac{\alpha^2}{2}\frac{d}{dy'}(y'f) = -\frac{\alpha^2 y'}{2}\frac{df}{dy'} + \frac{\alpha^2}{2}(f + y'\frac{df}{dy'}) = \frac{\alpha^2 f}{2}.$$

Dropping the test function f we obtain:

$$[X,Y] = \frac{\alpha^2}{2}.$$

Operators X and Y both commute with this commutator, so we can apply the Baker−Campbell−Hausdorff formula:

$$e^{\frac{\alpha}{\sqrt{2}}(y'-\frac{d}{dy'})} = e^{-\frac{\alpha^2}{4}} e^{\frac{\alpha}{\sqrt{2}}y'} e^{-\frac{\alpha}{\sqrt{2}}\frac{d}{dy'}}.$$

To rewrite this expression we need the following definition:

Definition 3.6. *Define a* **translation operator** $T(a)$*, which acts on a wavefunction* $|\psi(x)\rangle$*, as an operator which translates the wavefunction over* a*, so:* $T(a)|\psi(x)\rangle = |\psi(x+a)\rangle$*.*

Consider the function $\psi(y'-\alpha)$, using a Taylor series we can rewrite this as:

$$\psi(y'-\alpha) = \sum_{n=0}^{\infty} \frac{\psi^n(y')}{n!}(-\alpha)^n = e^{-\alpha\frac{d}{dy'}}\psi(y').$$

So, $e^{-\alpha\frac{d}{dy'}}$ is a translation operator.

Using the fact that $e^{-\alpha\frac{d}{dy'}}$ is a translation operator we find:

$$e^{-\frac{\alpha}{\sqrt{2}}\frac{d}{dy'}} e^{-\frac{1}{2}y'^2} = e^{-\frac{1}{2}(y'-\frac{\alpha}{\sqrt{2}})^2} = e^{-\frac{1}{2}(y'^2+\frac{\alpha^2}{2}-\sqrt{2}\alpha y')}.$$

Thus,

$$e^{-\frac{\alpha^2}{4}} e^{\frac{\alpha}{\sqrt{2}}y'} e^{-\frac{\alpha}{\sqrt{2}}\frac{d}{dy'}} e^{-\frac{1}{2}y'^2} = e^{-\frac{\alpha^2}{4}} e^{\frac{\alpha}{\sqrt{2}}y'} e^{-\frac{1}{2}(y'^2+\frac{\alpha^2}{2}-\sqrt{2}\alpha y')} = e^{-\frac{1}{2}\alpha^2+\sqrt{2}\alpha y'-\frac{1}{2}y'^2}.$$

For the final expression we used that, since $[y', \alpha] = 0$, every term commutes with each other and the expression could be written as a single exponent. Now, we obtain:

$$\langle x'|\alpha\rangle = Ne^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}\alpha^2+\sqrt{2}\alpha y'-\frac{1}{2}y'^2} = Ne^{-\frac{1}{2}(y'-\sqrt{2}\mathcal{R}(\alpha))^2+i\sqrt{2}\mathcal{I}(\alpha)y'-i\mathcal{I}(\alpha)\mathcal{R}(\alpha)}.$$

Using expressions (20) and (21), and substituting $x' = x_0 y'$ the resulting expression for the wavefunction of the coherent state is:

$$\langle x'|\alpha\rangle = Ne^{-\frac{m\omega}{2\hbar}(x'-\langle x\rangle_\alpha)^2+\frac{i}{\hbar}\langle p\rangle_\alpha x'-\frac{i}{2\hbar}\langle p\rangle_\alpha\langle x\rangle_\alpha}.$$

Since the last term is a constant phase factor it can be ignored and we finally obtain:

$$\psi_\alpha(x') := \langle x'|\alpha\rangle = (\frac{m\omega}{\pi\hbar})^{1/4} e^{\frac{i}{\hbar}\langle p\rangle_\alpha x'-\frac{m\omega}{2\hbar}(x'-\langle x\rangle_\alpha)^2}. \tag{22}$$

This wavefunction is a *Gaussian function*:

Definition 3.7. *A* **Gaussian function** *is a function of the form:*

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}},$$

for some real constants $a, b, c > 0$*.*

The graph of a Gaussian has a symmetric shape that quickly falls off towards $\pm\infty$. The parameter a is the height of the curve's peak, b is the position of the center of the peak and c controls the width.

3.6 Time evolution of coherent states

In this section we will investigate the time evolution of a coherent state, it turns out that a coherent state remains coherent under time evolution.

The time evolution of a state is given by the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{23}$$

Here, $H(t)$ is the Hamilton operator.
The Schrödinger equation is a first order differential equation, when a state $|\psi(t)\rangle$ is known on a time $t = t_0$, then the state can be determined for every t.

Definition 3.8. *The* **time evolution operator** $U(t, t_0)$ *gives the time evolution of a state. It has the following properties:*

$$|\psi(t)\rangle := U(t, t_0)|\psi(t_0)\rangle,$$
$$U(t_0, t_0) = \hat{1}, \quad \forall_{t_1} U(t, t_0) = U(t, t_1)U(t_1, t_0), \quad U^{-1}(t, t_0) = U(t_0, t).$$

According to equation (23), it is required that:

$$i\hbar\frac{\partial}{\partial t}U(t, t_0)|\psi(t_0)\rangle = H(t)U(t, t_0)|\psi(t_0)\rangle.$$

So that,

$$i\hbar\frac{\partial}{\partial t}U(t, t_0) = H(t)U(t, t_0), \tag{24}$$

with precondition $U(t_0, t_0) = \hat{1}$.

The Hamiltonian of the harmonic oscillator, $H = \frac{1}{2m}[p^2 + (m\omega x)^2]$, is time independent, so $\frac{\partial}{\partial t}H(t) = 0$. Therefore, differential equation (24) has a direct solution given by:

$$U(t, t_0) = e^{-i(t-t_0)H/\hbar}.$$

So, the time evolution of a state is:

$$|\psi(t)\rangle = e^{-i(t-t_0)H/\hbar}|\psi(t_0)\rangle. \tag{25}$$

This expression can be used to determine the time evolution of a coherent state. We use expression (17) to define a coherent state at $t_0 = 0$ as:

$$|\alpha(0)\rangle = e^{-\frac{1}{2}|\alpha(0)|^2}\sum_{n=0}^{\infty}\frac{\alpha(0)^n}{\sqrt{n!}}|n\rangle.$$

The coherent state at an arbitrary time t is found by applying equation (25), resulting in:

$$|\alpha(t)\rangle = U(t, 0)|\alpha(0)\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha(0)\rangle = e^{-\frac{1}{2}|\alpha(0)|^2}\sum_{n=0}^{\infty}e^{-\frac{i}{\hbar}Ht}\frac{\alpha(0)^n}{\sqrt{n!}}|n\rangle.$$

Since the wavefunctions $|n\rangle$ are eigenstates of the Hamiltonian with eigenvalue E_n we obtain:

$$= e^{-\frac{1}{2}|\alpha(0)|^2} \sum_{n=0}^{\infty} e^{-\frac{i}{\hbar}E_n t} \frac{\alpha(0)^n}{\sqrt{n!}} |n\rangle.$$

Here, $E_n = \hbar\omega(n + 1/2)$, see equation (7). Substituting $|n\rangle$ from expression (4) results in:

$$\begin{aligned} &= e^{-it\omega/2} e^{-\frac{1}{2}|\alpha(0)|^2} \sum_{n=0}^{\infty} \frac{[\alpha(0)(a^\dagger)e^{-it\omega}]^n}{n!} |0\rangle \\ &= e^{-it\omega/2} e^{-\frac{1}{2}|\alpha(0)|^2} e^{\alpha(0)(a^\dagger)e^{-it\omega}} |0\rangle. \end{aligned}$$

We recognized an exponential function, now rewrite the obtained expression using that $|e^{i\omega t}|^2 = 1$, then:

$$= e^{-it\omega/2} \big(e^{-\frac{1}{2}|\alpha(0)|^2 |e^{i\omega t}|^2 + \alpha(0) e^{-it\omega}(a^\dagger)} |0\rangle \big).$$

Comparing the expression between the parentheses with equation (18), we see that this gives a coherent state with the time dependent eigenvalue $e^{-i\omega t}\alpha(0)$:

$$= e^{-it\omega/2} |e^{-i\omega t}\alpha(0)\rangle.$$

We can conclude that a coherent state remains coherent under time evolution. Furthermore,

$$\alpha(t) = e^{-i\omega t}\alpha(0).$$

Which implies that:

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t). \tag{26}$$

This differential equation can be rewritten using the real part of $\alpha(t)$, denoted by $\mathcal{R}(\alpha(t))$ and the imaginary part, $\mathcal{I}(\alpha(t))$. These are defined by:

$$\begin{aligned} \mathcal{R}(\alpha(t)) &:= \frac{1}{2}(\alpha(t) + \overline{\alpha(t)}) = \langle\alpha(t)| \frac{a + a^\dagger}{2} |\alpha(t)\rangle, \\ \mathcal{I}(\alpha(t)) &:= \frac{1}{2i}(\alpha(t) - \overline{\alpha(t)}) = \langle\alpha(t)| \frac{a - a^\dagger}{2i} |\alpha(t)\rangle. \end{aligned}$$

From equation (26) follows that:

$$\frac{d}{dt}\mathcal{R}(\alpha(t)) = \omega\mathcal{I}(\alpha(t)), \tag{27}$$

$$\frac{d}{dt}\mathcal{I}(\alpha(t)) = -\omega\mathcal{R}(\alpha(t)). \tag{28}$$

Therefore, the expectation values for position and momentum are given by:

$$x_c(t) := \langle\alpha(t)|x|\alpha(t)\rangle = \sqrt{\frac{2\hbar}{m\omega}}\mathcal{R}(\alpha(t)),$$
$$p_c(t) := \langle\alpha(t)|p|\alpha(t)\rangle = \sqrt{2m\omega\hbar}\mathcal{I}(\alpha(t)).$$

Here, the subscript c stands for classical. Combining these expectation values with expressions (27) and (28) results into the following differential equations:

$$\frac{d}{dt}x_c(t) = \sqrt{\frac{\hbar}{2m\omega}}2\frac{d}{dt}\mathcal{R}(\alpha(t)) = \sqrt{\frac{\hbar}{2m\omega}}2\omega\mathcal{I}(\alpha(t)) = \frac{p_c(t)}{m},$$
$$\frac{d}{dt}p_c(t) = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}2i\frac{d}{dt}\mathcal{I}(\alpha(t)) = -\sqrt{\frac{\hbar m\omega}{2}}2\omega\mathcal{R}(\alpha(t)) = -m\omega^2 x_c(t).$$

After rewriting and introducing $v_c(t) = \frac{d}{dt}x_c(t)$ we obtain a more familiar form:

$$p_c(t) = m\frac{d}{dt}x_c(t) = mv_c(t), \tag{29}$$
$$\frac{d}{dt}p_c(t) = -m\omega^2 x_c(t). \tag{30}$$

So, the equations of motion for the classical harmonic oscillator are valid in terms of the quantum mechanical expectation values for x and p. We could have expected this because of the following theorem:

Theorem 3.7. *The* **Ehrenfest Theorem** *relates the time derivative of the expectation value for a quantum mechanical operator to the commutator of that operator with the Hamiltonian of the system in the following way:*

$$\frac{d}{dt}\langle A\rangle = \frac{1}{ih}\langle[A,H]\rangle + \langle\frac{\partial A}{\partial t}\rangle.$$

Where A is some quantum mechanical operator and $\langle A\rangle$ is its expectation value.

We are working with the Hamiltonian of the harmonic oscillator, then with the Ehrenfest Theorem (Theorem 3.7) we find that:

$$\frac{d}{dt}\langle p\rangle = \frac{1}{ih}\langle[p,\frac{1}{2m}(p^2+(m\omega x)^2)]\rangle + \langle\frac{\partial p}{\partial t}\rangle = \frac{1}{i\hbar}\langle[p,\frac{m\omega^2x^2}{2}]\rangle = -m\omega^2\langle x\rangle,$$
$$\frac{d}{dt}\langle x\rangle = \frac{1}{ih}\langle[x,\frac{1}{2m}(p^2+(m\omega x)^2)]\rangle + \langle\frac{\partial x}{\partial t}\rangle = \frac{1}{ih}\langle[x,\frac{1}{2m}p^2]\rangle = \frac{1}{m}\langle p\rangle.$$

So, from the Ehrenfest Theorem we could have expected equations (29) and (30) to be found.

The quantum mechanical expectation values for position and momentum, determined with the coherent states, satisfy the classical equations of motion for a harmonic oscillator. Furthermore, as we already saw, the coherent states remain coherent under time evolution. That is why the coherent states are used to study the classical limit of quantum mechanics.

4 Phase space representation

In this chapter we will try another approach to construct minimal wavefunctions. Hereby, we will follow the reasoning of U. Gerlach ([2]). First we have to define a Fourier transformation on the $L^2(\mathbb{R})$ space. We will use this to construct wavefunctions that partition the x, p-plane into cells with minimal uncertainty.

4.1 The Fourier Transform

In this section we will define Fourier transformation on the L^2 space. First we will define a Fourier transformation on functions $f \in L^1$. We cannot directly use this definition on functions $g \in L^2$, but with the help of an extension from the Fourier transformation $\mathcal{F} : L^1 \cap L^2 \to L^2$ to $\mathcal{F} : L^2 \to L^2$ we can define the Fourier transformation on L^2.

Definition 4.1. *Let $f \in L^1(\mathbb{R})$, the* **Fourier transform** *of f is the function $\hat{f}$ defined by:*

$$\hat{f}(x) = \mathcal{F}(f(p)) = \int_{-\infty}^{\infty} f(p) \frac{e^{-ixp}}{\sqrt{2\pi}} dp.$$

This integral is well defined for every real x, because $f \in L^1(\mathbb{R})$.

Theorem 4.1. ***The Inversion Theorem.*** *If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, and if:*

$$g(p) = \int_{-\infty}^{\infty} \hat{f}(x) \frac{e^{ixp}}{\sqrt{2\pi}} dx \quad (p \in \mathbb{R}),$$

then $g \in C_0(\mathbb{R})$ and $f(p) = g(p)$ almost everywhere.
Here, $C_0(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ which vanish at infinity.

Proof. This is Theorem 9.11 from [6]. □

Corollary 4.1. ***The Uniqueness Theorem.*** *If f_1 and f_2 belong to $L^1(\mathbb{R})$ and $\hat{f}_1(x) = \hat{f}_2(x)$ for all $x \in \mathbb{R}$, then $f_1(p) = f_2(p)$ almost everywhere.*

Proof. Using Theorem 4.1 we obtain that if $\hat{f}(x) = 0$ then $f(p) = 0$ almost everywhere. Applying this to $\hat{f}_1(x) - \hat{f}_2(x) = 0$ we find that $f_1(p) - f_2(p) = 0$ almost everywhere and consequently that $f_1(p) = f_2(p)$ almost everywhere. □

The definition of the Fourier transform by Definition 4.1 is not directly applicable to every $f \in L^2(\mathbb{R})$, because $L^2(\mathbb{R})$ is not a subset of $L^1(\mathbb{R})$. However, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the definition does apply.
It is known from Theorem 2.1 that $C(X)$ is dense in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. As $C(X)$ is contained in the intersection we know that the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. This implies that the Fourier transform map restricted to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ has a unique extension to a linear isometric map $L^2 \longrightarrow L^2$. This extension defines the Fourier transform of every $f \in L^2$, called the *Plancherel transform*. This makes it possible to speak of the Fourier transform of a square integrable function.

Theorem 4.2. *One can associate to each $f \in L^2(\mathbb{R})$ a function $\hat{f} \in L^2(\mathbb{R})$ so that the following properties hold:*

1. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f}$ is the Fourier transform of f as defined in Definition 4.1.*
2. *The mapping $f \to \hat{f}$ is an unitary map with respect to the L^2 norm $\|.\|_2$.*
3. *For every $f \in L^2(\mathbb{R})$, $\|\hat{f}\|_2 = \|f\|_2$.*
4. *The mapping $f \to \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.*
5. *The following symmetric relation exists between f and $\hat{f}$: if $h(x) = \int_{-A}^{A} f(p)\frac{e^{-ixp}}{\sqrt{2\pi}}dp$ and $g(p) = \int_{-A}^{A} \hat{f}(x)\frac{e^{ixp}}{\sqrt{2\pi}}dx$ then $\|h - \hat{f}\|_2 \to 0$ and $\|g - f\|_2 \to 0$ as $A \to \infty$.*

Proof. Properties (1) to (4) will be proven here, for the proof of property (5) see Theorem 9.13 of [6].
If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the Fourier transform $\hat{f}$ is defined as in Definition 4.1, since $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subset L^1(\mathbb{R})$. This proves the first item.
Let $\hat{f}(x), \hat{g}(x) \in L^2(\mathbb{R})$ and $\alpha \in \mathbb{C}$ then:

$$\begin{aligned}
\mathcal{F}^{-1}(\hat{f}(x) + \hat{g}(x)) &= \int_{-\infty}^{\infty} (\hat{f}(x) + \hat{g}(x))\frac{e^{ixp}}{\sqrt{2\pi}}dx \\
&= \int_{-\infty}^{\infty} \hat{f}(x)\frac{e^{ixp}}{\sqrt{2\pi}}dx + \int_{-\infty}^{\infty} \hat{g}(x)\frac{e^{ixp}}{\sqrt{2\pi}}dx \\
&= \mathcal{F}^{-1}(\hat{f}(x)) + \mathcal{F}^{-1}(\hat{g}(x)), \\
\mathcal{F}^{-1}(\alpha\hat{f}(x)) &= \int_{-\infty}^{\infty} \alpha\hat{f}(x)\frac{e^{ixp}}{\sqrt{2\pi}}dx = \alpha\int_{-\infty}^{\infty} \hat{f}(x)\frac{e^{ixp}}{\sqrt{2\pi}}dx \\
&= \alpha\mathcal{F}^{-1}(\hat{f}(x)).
\end{aligned}$$

So, the Fourier transformation is a linear map.
For $f(p), g(p) \in L^2(\mathbb{R})$ we have:

$$\begin{aligned}
\langle\hat{f}(x)|\hat{g}(x)\rangle &= \int_{-\infty}^{\infty} \overline{\hat{f}(x)}\hat{g}(x)dx = \int_{-\infty}^{\infty} \overline{\hat{f}(x)}\left(\int_{-\infty}^{\infty} g(p)\frac{e^{-ixp}}{\sqrt{2\pi}}dp\right)dx \\
&= \int_{-\infty}^{\infty} \left(\overline{\int_{-\infty}^{\infty} \hat{f}(x)\frac{e^{ixp}}{\sqrt{2\pi}}dx}\right)g(p)dp = \int_{-\infty}^{\infty} \overline{f(p)}g(p)dp = \langle f(p)|g(p)\rangle.
\end{aligned}$$

We used the fact that $\hat{g}$ is the Fourier transform of g, and then after reorganizing where we changed the order of integration using Fubini's Theorem, we recognized the Fourier transform of $\hat{f}$, which is f.
We can conclude that the Fourier transformation is a unitary transformation. With this result we can easily verify that for $f \in L^2(\mathbb{R})$: $\|\hat{f}\|_2 = \|f\|_2$. Take $g = f$ in the previous result, then:

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx = \langle\hat{f}(x)|\hat{f}(x)\rangle = \langle f(p)|f(p)\rangle = \int_{-\infty}^{\infty} |f(p)|^2 dp.$$

So indeed, $\|f\|_2 = \|\hat{f}\|_2$.
Now we are going to prove property (4). Let $f \in L^2(\mathbb{R})$ then $\|f\|_2 \leq \infty$, so $\|\hat{f}\|_2 = \|f\|_2 \leq \infty$. From this it is clear that $\hat{f}$ is also in $L^2(\mathbb{R})$ and that the mapping $f \to \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. □

Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, properties (1) and (3) determine the mapping $f \to \hat{f}$ uniquely:
Let $f \in L^2$ and let $X \subseteq L^1 \cap L^2$ be a dense subspace of $L^2(\mathbb{R})$, for instance $L^1 \cap L^2$ itself. Now, choose a sequence $f_n \in X$, such that $\|f_n - f\|_2 \to 0$ if $n \to \infty$. Then the sequence $(f_n)_n$ is a Cauchy sequence in X and $\hat{f}_n$ is well defined because $f_n \in L^1$. Since $\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2$, it follows that $\hat{f}_n$ is a Cauchy sequence in $L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ is complete, therefore there exists a unique limit in $L^2(\mathbb{R})$ and $\hat{f}$ is well defined as $\hat{f} := \lim_{n\to\infty} \hat{f}_n$.

Corollary 4.2. *If $f \in L^2(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then:*

$$f(p) = \int_{-\infty}^{\infty} \hat{f}(x) \frac{e^{ixp}}{\sqrt{2\pi}} dx \quad a.e.$$

Proof. This is a Corollary of Theorem 4.2, property (5). □

If $f \in L^2(\mathbb{R})$, the Plancherel transform defines $\hat{f}$ uniquely as an element of the Hilbert space $L^2(\mathbb{R})$. But in terms of pointwise functions the Fourier transform $\hat{f}(x)$ is determined almost everywhere on $\mathbb{R}$.

4.2 Delta functions

To define some delta identities on the $L^2(\mathbb{R})$ space we will use trigonometric polynomials:

Definition 4.2. *A* **trigonometric polynomial** *is a finite sum of the form:*

$$f(p) = a_0 + \sum_{n=1}^{N} (a_n \cos(np) + b_n \sin(np)) \quad (p \in \mathbb{R}),$$

where $N \in \mathbb{N}$ and $a_0, a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ are complex numbers. We can also write this in the form:

$$f(p) = \sum_{n=-N}^{N} c_n e^{inp},$$

where $c_{-N}, \ldots, c_N$ are complex numbers.

It is clear that every trigonometric polynomial has period 2π. This means that $f(p - \pi) = f(p + \pi)$ for all real p.

Theorem 4.3. *Let $g \in C_{\mathbb{C}}(T)$, which consists of all continuous complex 2π-periodic functions on $[-\pi, \pi]$ with norm $\|g\|_\infty = \sup_p |g(p)|$, and let $\varepsilon > 0$. Here, 2π-periodic means that $g(-\pi) = g(\pi)$. Then, there exists a trigonometric polynomial P such that: $\|g - P\|_\infty < \varepsilon$.*

Proof. This is Theorem 4.25 from [6]. □

Theorem 4.4. *Let $\varepsilon > 0$ and f a complex function in $L^2([-\pi, \pi])$ with $f(-\pi) = f(\pi)$. Then there exists a trigonometric polynomial P such that: $\|f - P\|_2 < \varepsilon$.*

Proof. From Theorem 2.1 we know that since $f \in L^2([-\pi,\pi])$, there exists a continuous function g on $[-\pi,\pi]$ such that: $\|f-g\|_2 < \frac{1}{2}\varepsilon$.
From Theorem 4.3 we know that there is a trigonometric polynomial P such that: $\|g-P\|_\infty < \frac{1}{2\sqrt{2\pi}}\varepsilon$ on $[-\pi,\pi]$.

Now use that for h a continuous function: $\|h\|_2 \leq \left(\int_{-\pi}^{\pi} \sup_p |h(p)|dp\right)^{1/2} \leq \sqrt{2\pi}\|h\|_\infty$. So restricted to $[-\pi,\pi]$, we obtain:

$$\|f-P\|_2 \leq \|f-g\|_2 + \|g-P\|_2 \leq \frac{1}{2}\varepsilon + \sqrt{2\pi}\|g-P\|_\infty \leq \varepsilon.$$

□

By taking the limit of ε to zero it follows that we can write every complex function $f \in L^2([-\pi,\pi])$, with $f(-\pi) = f(\pi)$, as a limit of trigonometric polynomials, so:

$$f(p) = \lim_{M\to\infty} \sum_{m=-M}^{M} a_m e^{imp}. \tag{31}$$

We would like to define a delta distribution on functions $f \in L^2([-\pi,\pi])$. First we will define the delta distribution on continuous functions f on the interval $[-\pi,\pi]$, with $f(-\pi) = f(\pi)$.

Theorem 4.5. *Let $f \in C([-\pi,\pi])$, with $f(-\pi) = f(\pi)$ then:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{in(p-p')} f(p)dp = f(p').$$

As a distribution on $C([-\pi,\pi])$ we could write:

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(p-p')} = \delta(p-p').$$

Proof. To prove this theorem we have to calculate the following integral:

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{in(p-p')} f(p)dp = \lim_{M\to\infty} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-M}^{M} a_m e^{in(p-p')} e^{imp} dp.$$

The order of integration and taking the limit could be changed due to the dominated convergence Theorem.
Suppose $n \neq -m$, then since $(n+m) \in \mathbb{Z}$ it follows that:

$$a_m \int_{-\pi}^{\pi} e^{ip(n+m)}dp = \frac{a_m}{i(n+m)}(e^{i\pi(n+m)} - e^{-i\pi(n+m)}) = 0.$$

Suppose $n = -m$, then:

$$a_m \int_{-\pi}^{\pi} e^{ip(n+m)}dp = a_m \int_{-\pi}^{\pi} dp = a_m 2\pi.$$

So for fixed m:

$$\sum_n a_m e^{-ip'n} \int_{-\pi}^{\pi} e^{ip(n+m)} dp = a_m e^{ip'm} 2\pi.$$

From which it follows that:

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{in(p-p')} f(p) dp = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-M}^{M} a_m e^{in(p-p')} e^{imp} dp.$$

The order of summation can be changed because the summation is absolute convergent. The order of integration and summation can be changed because the summation is uniform convergent and the function $e^{inp}e^{imp}$ is Riemann integrable. We obtain:

$$= \sum_{m=-M}^{M} \sum_{n=-\infty}^{\infty} a_m e^{-ip'n} \int_{-\pi}^{\pi} e^{ip(n+m)} dp = 2\pi \sum_{m=-M}^{M} a_m e^{ip'm}.$$

If we take the limit of M to infinity we recognize the function $f(p')$, so that:

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{in(p-p')} f(p) dp = 2\pi f(p').$$

□

Corollary 4.3. *Let $f \in C([-\pi, \pi])$, then:*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{inp} f(p) dp = f(0).$$

Proof. Let $p' = 0$ in Theorem 4.5. □

The continuous functions are dense in L^2, and the action of the delta distribution on such functions is well defined on the interval $[-\pi, \pi]$. Therefore, the Dirac delta distribution is densely defined on $L^2([-\pi, \pi])$.

4.3 Basis of $L^2(\mathbb{R})$

We are going to define a basis for the Hilbert space $L^2(\mathbb{R})$ with the help of Fourier transformation.

Divide the real line, $-\infty < x < \infty$, of the Fourier domain into equal intervals of length ε and consider a function $P_{jl}(p)$ whose Fourier transform $F_{jl}(x)$ is zero everywhere except in one of these intervals:

$$F_{jl}(x) = \begin{cases} 0 & x \notin [j\varepsilon, (j+1)\varepsilon], l \in \mathbb{Z} \\ \frac{1}{\sqrt{\varepsilon}} e^{\frac{-2\pi i l x}{\varepsilon}} & x \in [j\varepsilon, (j+1)\varepsilon], l \in \mathbb{Z} \end{cases} \tag{32}$$

We demand that $l \in \mathbb{Z}$, so that $F_{jl}(x)$ can be pictured as a finite complex amplitude in the j^{th} position interval: $j\varepsilon \leq x \leq (j+1)\varepsilon$. For $l = 4$, the imaginary part of $F_{jl}(x)$ is pictured in Figure 1.

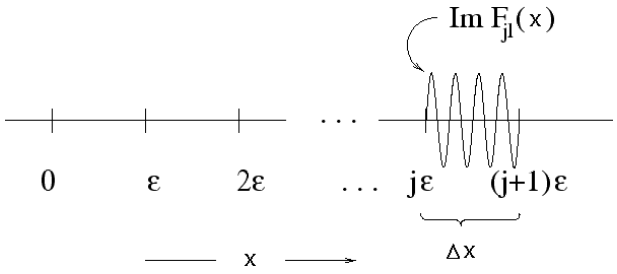

Figure 1: The imaginary part of $F_{jl}(x)$ for $l = 4$.

Theorem 4.6. *The set $\{F_{jl}(x)\}$ forms an orthonormal set and $F_{jl}(x) \in L^2(\mathbb{R})$.*

Proof. Using the fact that $\overline{F_{jl}(x)}F_{j'l'}(x) = 0$ if $j \neq j'$, we compute:

$$\langle F_{jl}|F_{j'l'}\rangle = \int_{-\infty}^{\infty} \overline{F_{jl}(x)}F_{j'l'}(x)dx = \delta_{jj'}\int_{-\infty}^{\infty} \overline{F_{jl}(x)}F_{jl'}(x)dx.$$

Further, since F_{jl} is zero outside the interval $[j\varepsilon, (j+1)\varepsilon]$ we obtain:

$$\begin{aligned}
&= \delta_{jj'}\int_{j\varepsilon}^{(j+1)\varepsilon} \frac{1}{\varepsilon}e^{\frac{2\pi i x(l'-l)}{\varepsilon}}dx = \delta_{jj'}\int_{j}^{(j+1)} e^{2\pi i x(l'-l)}dx \\
&= \begin{cases} \delta_{jj'}\frac{e^{2\pi i j(l'-l)}}{2\pi i(l'-l)}[e^{2\pi i(l'-l)} - 1] = 0 & \text{for } l \neq l', (l'-l) \in \mathbb{Z} \\ \delta_{jj'}\int_{j}^{(j+1)} dx = \delta_{jj'} & \text{for } l = l' \end{cases} \\
&= \delta_{jj'}\delta_{ll'}.
\end{aligned}$$

Also, $F_{jl} \in L^2(\mathbb{R})$, because:

$$\|F_{jl}(x)\|_2 = \langle F_{jl}(x)|F_{jl}(x)\rangle = 1 < \infty.$$

□

Corollary 4.4. *The set $\{F_{jl}(x)\}$ forms a linear independent set.*

Proof. From Theorem 4.6 we know that the set $\{F_{jl}(x)\}$ forms an orthonormal set. Since Lemma 2.1 tells us that an orthonormal set is also linear independent, it follows that the set $\{F_{jl}(x)\}$ is linear independent. □

In order to form a basis for $L^2(\mathbb{R})$, the set $\{F_{jl}(x)\}$ has to span this space. First, we will prove that the set $\{F_{jl}(x)\}$ spans the space $C(\mathbb{R})$.

Proposition 4.1. *Let $f \in C(\mathbb{R})$, then:*

$$\int_{-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \overline{F_{jl}(x')}F_{jl}(x)f(x')dx' = f(x),$$

with convergence in the L^2 norm. This can also be written as a distribution on $C(\mathbb{R})$,

$$\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \overline{F_{jl}(x')}F_{jl}(x) = \delta(x-x').$$

Proof. Let $x, x' \in [j\varepsilon, (j+1)\varepsilon]$ then for that fixed j we obtain:

$$F_{jl}(x)\overline{F_{jl}(x')} = \frac{1}{\varepsilon} e^{\frac{2\pi i l(x'-x)}{\varepsilon}}.$$

For the other $j's$ in $\mathbb{Z}$ the product of $F_{jl}(x)\overline{F_{jl}(x')}$ is zero. Therefore,

$$\int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \overline{F_{jl}(x')} F_{jl}(x) f(x') dx' = \int_{j\varepsilon}^{(j+1)\varepsilon} \sum_{l=-\infty}^{\infty} \frac{1}{\varepsilon} e^{\frac{2\pi i l(x'-x)}{\varepsilon}} f(x') dx'.$$

Since the summation is absolute convergent, the order of summation could be changed. After rescaling we recognize the delta identity from Theorem 4.5:

$$= \int_{-\pi}^{\pi} \Big(\sum_{l=-\infty}^{\infty} \frac{e^{il(x'-x)}}{2\pi} \Big) f\Big(\frac{\varepsilon(x' + 2\pi j + \pi)}{2\pi} \Big) dx' = f\Big(\frac{\varepsilon(x + 2\pi j + \pi)}{2\pi} \Big)$$
$$= f(x).$$

Suppose $x \in [j\varepsilon, (j+1)\varepsilon]$ and $x' \in [j'\varepsilon, (j'+1)\varepsilon]$ with $j' \neq j$, then $F_{jl}(x)\overline{F_{jl}(x')} = 0$ for every $j \in \mathbb{Z}$. And, because $x' \notin [j\varepsilon, (j+1)\varepsilon]$, we obtain:

$$\int_{j\varepsilon}^{(j+1)\varepsilon} \delta(x - x') f(x') dx' = 0 = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \overline{F_{jl}(x')} F_{jl}(x) f(x') dx'.$$

We can conclude that:

$$\int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \overline{F_{jl}(x')} F_{jl}(x) f(x') dx' = f(x),$$

with convergence in the L^2 norm. □

Corollary 4.5. *Let $f \in C(\mathbb{R})$, then:*

$$f = \sum_{j \leq |J|} \sum_{l \leq |L|} a_{jl} F_{jl}(x),$$

in $L^2(\mathbb{R})$ with $a_{jl} \in \mathbb{R}$.

Proof. It is known from Theorem 4.1 that if $f \in C(\mathbb{R})$, then:

$$\| \int_{\mathbb{R}} \sum_{j \leq |J|} \sum_{l \leq |L|} \overline{F_{jl}(x')} F_{jl}(x) f(x') dx' - f(x) \|_2 \to 0,$$

as $J, L \to \infty$. So if $J, L \to \infty$ then:

$$\| \int_{\mathbb{R}} \sum_{j \leq |J|} \sum_{l \leq |L|} \overline{F_{jl}(x')} F_{jl}(x) f(x') dx' - f(x) \|_2$$
$$= \| \sum_{j \leq |J|} \sum_{l \leq |L|} \Big(\int_{\mathbb{R}} \overline{F_{jl}(x')} f(x') dx' \Big) F_{jl}(x) - f(x) \|_2,$$

We recognize the inner product of $F_{jl}(x')$ and $f(x')$, so:

$$\begin{aligned} &= \| \sum_{j\leq|J|} \sum_{l\leq|L|} \langle F_{jl}(x')|f(x')\rangle F_{jl}(x) - f(x)\|_2 \\ &=: \| \sum_{j\leq|J|} \sum_{l\leq|L|} a_{jl}F_{jl}(x) - f(x)\|_2 \to 0, \end{aligned}$$

where $a_{jl} \in \mathbb{R}$. The order of summation and integration could be changed because the summation is uniform convergent and the function $\overline{F_{jl}(x')}f(x')$ is Riemann integrable. □

We can conclude that:

Theorem 4.7. *The collection $\{F_{jl}\}$, with:*

$$F_{jl}(x) = \begin{cases} 0 & x \notin [j\varepsilon,(j+1)\varepsilon], l \in \mathbb{Z} \\ \frac{1}{\sqrt{\varepsilon}} e^{\frac{-2\pi i l x}{\varepsilon}} & x \in [j\varepsilon,(j+1)\varepsilon], l \in \mathbb{Z} \end{cases}$$

forms a basis for $L^2(\mathbb{R})$.

Proof. Since $C(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$ (Theorem 2.1), every $f \in L^2(\mathbb{R})$ is either in $C(\mathbb{R})$ or is a limit point of $C(\mathbb{R})$. So, for every $f \in L^2(\mathbb{R})$ there is a $g \in C(\mathbb{R})$ such that: $\|f-g\|_2 \to 0$. Using Corollary 4.5 we obtain that $f = \sum_{j\leq|J|} \sum_{l\leq|L|} a_{jl}F_{jl}(x)$ in $L^2(\mathbb{R})$ with $a_{jl} \in \mathbb{R}$. Consequently, the collection $\{F_{jl}(x)\}$ spans $L^2(\mathbb{R})$. Combining this with Corollary 4.4, which tells us that the set $\{F_{jl}(x)\}$ is linear independent, we obtain that the collection $\{F_{jl}(x)\}$ forms a basis for $L^2(\mathbb{R})$. □

With the help of the square integrable functions $F_{jl}(x)$ we can construct the wavefunctions $P_{jl}(p)$ with the momentum domain $-\infty < p < \infty$. Fourier transformation of $F_{jl}(x)$ gives:

$$P_{jl}(p) \quad = \quad \int_{-\infty}^{\infty} F_{jl}(x) \frac{e^{ixp}}{\sqrt{2\pi}}\, dx.$$

The real part of $P_{jl}(p)$ is pictured in Figure 2 for $l = 1$ and $j = 2$ and in Figure 3 for $l = 2$ and $j = 2$.

Theorem 4.8. *The collection $\{P_{jl}(p)\}$ forms a basis for $L^2(\mathbb{R})$.*

Proof. In Theorem 4.2 we proved that a Fourier transformation is a unitary transformation from which it follows that:

$$\langle P_{jl}|P_{j'l'}\rangle = \langle F_{jl}|F_{j'l'}\rangle = \delta_{jj'}\delta_{ll'}.$$

So, the collection $\{P_{jl}(p)\}$ forms an orthonormal set and therefore also a linear independent set due to Lemma 2.1. Theorem 4.2 noted that, since P_{jl} is the Fourier transform of F_{jl} and $F_{jl} \in L^2(\mathbb{R})$, also $P_{jl} \in L^2(\mathbb{R})$.
Let $f(p) \in C(\mathbb{R})$, write $f(p)$ as the Fourier transform of $f(x)$, so:

$$f(p) = \int_{-\infty}^{\infty} \frac{e^{ixp}}{\sqrt{2\pi}} f(x)dx.$$

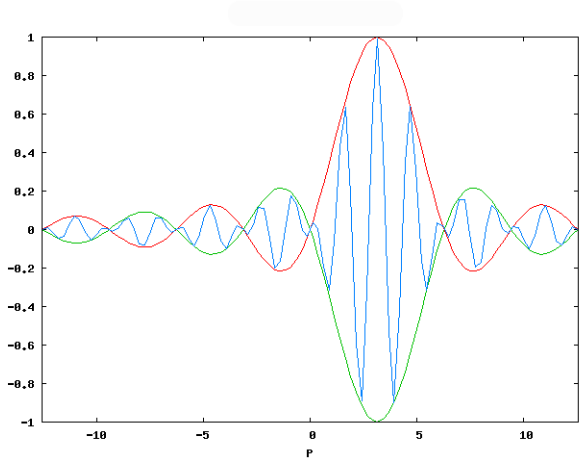

Figure 2: The real part of $P_{jl}(p)$ for $l = 1$ and $j = 2$.

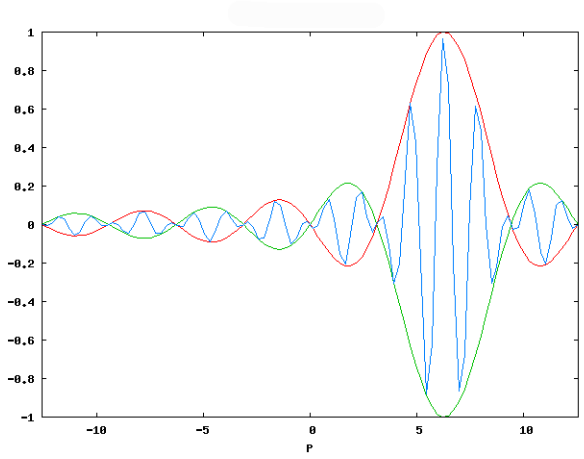

Figure 3: The real part of $P_{jl}(p)$ for $l = 2$ and $j = 2$.

Using the spanning property of the set $\{F_{jl}(x)\}$ we could write:

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} F_{jl}(x).$$

Substituting this in the expression for $f(p)$ gives:

$$f(p) = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} F_{jl}(x) \frac{e^{ixp}}{\sqrt{2\pi}} dx.$$

After reorganizing we can recognize P_{jl} and we obtain:

$$= \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} \int_{-\infty}^{\infty} F_{jl}(x) \frac{e^{ixp}}{\sqrt{2\pi}} dx = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} P_{jl}(p).$$

The order of summation and integration could be changed because the summation is uniform convergent and the functions $F_{jl}(x)\frac{e^{ixp}}{\sqrt{2\pi}}$ are Riemann integrable.

We can conclude that the collection $\{P_{jl}(p)\}$ spans the space $C(\mathbb{R})$. Since $C(\mathbb{R}) \subset L^2(\mathbb{R})$ dense, every $f \in L^2(\mathbb{R})$ is either in $C(\mathbb{R})$ or is a limit point of $C(\mathbb{R})$. So, for every $f \in L^2(\mathbb{R})$ there is a $g \in C(\mathbb{R})$ such that: $\|f - g\|_2 \to 0$. Therefore, we obtain that $f = \sum_{j \leq |J|} \sum_{l \leq |L|} a_{jl} P_{jl}(p)$ in $L^2(\mathbb{R})$ with $a_{jl} \in \mathbb{R}$. Consequently, the collection $\{P_{jl}(p)\}$ spans $L^2(\mathbb{R})$. Since the collection $\{P_{jl}(p)\}$ forms a linear independent set we obtain that the collection $\{P_{jl}(p)\}$ forms a basis for $L^2(\mathbb{R})$.

□

Note that $f(p)$ and $f(x)$ have the same expansion coefficients a_{jl} relative to the Fourier-related bases $\{P_{jl}(p)\}$ and $\{F_{jl}(x)\}$.

Instead of writing $P_{jl}(p)$ as the Fourier transform of $F_{jl}(x)$ we could also calculate $P_{jl}(p)$ explicitly.
For $p = \frac{2\pi l}{\varepsilon}$ we obtain:

$$P_{jl}(2\pi l/\varepsilon) = \int_{-\infty}^{\infty} F_{jl}(x) \frac{e^{ix2\pi l/\varepsilon}}{\sqrt{2\pi}} dx = \int_{j\varepsilon}^{(j+1)\varepsilon} \frac{1}{\sqrt{\varepsilon 2\pi}} dx = \sqrt{\frac{\varepsilon}{2\pi}}.$$

For $p \neq \frac{2\pi l}{\varepsilon}$ we obtain:

$$\begin{aligned} P_{jl}(p) &= \int_{-\infty}^{\infty} F_{jl}(x) \frac{e^{ixp}}{\sqrt{2\pi}} dx = \int_{j\varepsilon}^{(j+1)\varepsilon} \frac{e^{-2\pi i l x/\varepsilon}}{\sqrt{\varepsilon}} \frac{e^{ixp}}{\sqrt{2\pi}}\, dx \\ &= \frac{1}{\sqrt{2\pi\varepsilon}} \left[\frac{e^{i(p-2\pi l/\varepsilon)(j+1)\varepsilon} - e^{i(p-2\pi l/\varepsilon)j\varepsilon}}{i(p - 2\pi l/\varepsilon)} \right] \\ &= \frac{2}{\sqrt{2\pi\varepsilon}} e^{i(p-2\pi l/\varepsilon)(j+1/2)\varepsilon} \frac{\sin[(p - \frac{2\pi l}{\varepsilon})\frac{\varepsilon}{2}]}{p - \frac{2\pi l}{\varepsilon}}. \end{aligned} \tag{33}$$

These two expressions for $P_{jl}(p)$ are equivalent, which can be seen using a Taylor expansion for the exponential functions in expression (33). If y is small then $e^y \approx 1 + y + \ldots$. For $p \to \frac{2\pi l}{\varepsilon}$, a Taylor expansion can be used in expression (33), resulting in:

$$\begin{aligned} P_{jl}(p) &= \frac{1}{\sqrt{2\pi\varepsilon}} \left[\frac{e^{i(p-2\pi l/\varepsilon)(j+1)\varepsilon} - e^{i(p-2\pi l/\varepsilon)j\varepsilon}}{i(x - 2\pi l/\varepsilon)} \right] \\ &\approx \frac{1}{\sqrt{2\pi\varepsilon}} \left[\frac{i(p - 2\pi l/\varepsilon)(j+1)\varepsilon - i(p - 2\pi l/\varepsilon)j\varepsilon}{i(p - 2\pi l/\varepsilon)} \right] = \sqrt{\frac{\varepsilon}{2\pi}}. \end{aligned}$$

We can conclude that:

$$P_{jl}(p) = \frac{2}{\sqrt{2\pi\varepsilon}} e^{i(p-2\pi l/\varepsilon)(j+1/2)\varepsilon} \frac{\sin[(p - \frac{2\pi l}{\varepsilon})\frac{\varepsilon}{2}]}{p - \frac{2\pi l}{\varepsilon}}.$$

Note that this wavefunction consists of a real amplitude, a sinc function[1], multiplied by an exponential phase factor which is oscillating. At $p = \frac{2\pi l}{\varepsilon}$, the function $P_{jl}(p)$ has maximum modulus with value: $\sqrt{\frac{\varepsilon}{2\pi}}$.
For $p = \frac{2\pi k}{\varepsilon}$, where $k \in \mathbb{Z} - \{l\}$, the function $P_{jl}(p)$ equals zero. We define Δp to be the distance between two such zero's, so: $\Delta p = \frac{2\pi}{\varepsilon}$.

4.4 Phase Space

Consider the two-dimensional space spanned by the domain $-\infty < p < \infty$, and the Fourier domain $-\infty < x < \infty$, of the set of square-integrable functions $f(p)$. In this section we will see that the set of wavefunctions $P_{jl}(p)$ determines a partitioning of this space into elements of area whose shape and location are determined by this wavefunctions. This partitioned two-dimensional space is called the *Phase Space*. The partition of the phase space exist of elements of equal area called the *phase space cells*.

The wavefunction representation for a function $f(p) \in L^2(\mathbb{R})$, which is given by:

$$f(p) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} P_{jl}(p),$$

can be represented geometrically as a set of complex amplitudes a_{jl} assigned to their corresponding phase space cells. The location of the phase space cells, as determined by $P_{jl}(p)$ and $F_{jl}(x)$, is given by the mean position $\langle x \rangle$ and the mean momentum $\langle p \rangle$.

In the article of U. Gerlach ([2]), each cell has a surface of size $\Delta x \Delta p$. Here, Δx and Δp both depend on the distance between two zero points of the function $F_{jl}(x)$, respectively $P_{jl}(p)$. The surface of a cell can also be calculated with help of the standard deviations of the position and momentum, given by σ_x and σ_p. Both approaches will be discussed in this section.

For now, let us follow the reasoning of U. Gerlach. We already saw that $\Delta p = \frac{2\pi}{\varepsilon}$. We will define Δx in a similar way, Δx is the length of the interval where $F_{jl}(x) \neq 0$, so $\Delta x = \varepsilon$. In this way we see that the surface of a phase space cell is given by: $S = \Delta x \Delta p = 2\pi$.
For the position of a phase space cell we calculate:

$$\langle x \rangle = \int_{-\infty}^{\infty} F_{jl}(x) x \overline{F_{jl}(x)} dx = \frac{1}{\varepsilon} \int_{j\varepsilon}^{(j+1)\varepsilon} x dx = \varepsilon(\frac{1}{2} + j), \tag{34}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} P_{jl}(p) p \overline{P_{jl}(p)} dp = \int_{-\infty}^{\infty} \frac{4p}{2\pi\varepsilon} \left(\frac{\sin[(p - \frac{2\pi l}{\varepsilon})\frac{\varepsilon}{2}]}{p - \frac{2\pi l}{\varepsilon}} \right)^2 dp.$$

[1] $\mathrm{sinc}(y) = \sin(y)/y$

Change of variable to $t = (p - \frac{2\pi l}{\varepsilon})\frac{\varepsilon}{2}$, and using $dp = \frac{2}{\varepsilon}dt$ results in:

$$= \int_{-\infty}^{\infty} \left(\frac{\sin(t)}{t}\right)^2 \frac{2t - 2\pi l}{\varepsilon} \frac{1}{\pi} dt = \frac{2}{\pi\varepsilon} \underbrace{\int_{-\infty}^{\infty} \frac{\sin(t)^2}{t} dt}_{=0} + \frac{2l}{\varepsilon} \underbrace{\int_{-\infty}^{\infty} \left(\frac{\sin(t)}{t}\right)^2 dt}_{=\pi}$$

$$= \frac{2\pi l}{\varepsilon}. \tag{35}$$

So, the location of the $(j,l)^{th}$ phase space cell is given by its mean position $(j + \frac{1}{2})\varepsilon$, and its mean momentum $\frac{2\pi l}{\varepsilon}$.

Now we can assign the complex amplitude a_{jl} to the $(j,l)^{th}$ phase space cell. Typically, the squared norm:

$$\|f(p)\|^2 = \int_{-\infty}^{\infty} |f(p)|^2 dp,$$

is proportional to the total 'energy' of the signal represented by $f(p)$. Using the spanning property of $\{P_{jl}(p)\}$ we obtain:

$$\|f(p)\|^2 = \int_{-\infty}^{\infty} |f(p)|^2 dp = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |a_{jl}|^2 |P_{jl}(p)|^2 dp.$$

Using $\langle P_{jl} | P_{j'l'} \rangle = \delta_{jj'}\delta_{ll'}$ and changing the order of summation and integration gives:

$$= \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |a_{jl}|^2 \int_{-\infty}^{\infty} |P_{jl}(p)|^2 dp = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |a_{jl}|^2.$$

The order of summation and integration could be changed because the summation is uniform convergent and the function $|a_{jl}|^2|P_{jl}(p)|^2$ is Riemann integrable. This implies that $|a_{jl}|^2$ is proportional to the 'energy' contained in the $(j,l)^{th}$ phase space cell. In other words $\{|a_{jl}|^2 : j,l \in \mathbb{Z}\}$ is a decomposition of the 'energy' of $f(p)$ relative to the chosen wavefunction basis $\{P_{jl}(p)\}$. The wavefunction representation of a signal $f(p)$ assigns to each phase space cell an intensity $|a_{jl}|^2$, which we can represent as a level of grayness. This is illustrated in Figure 4.

We already mentioned that the surface of a phase space cell can be determined with the standard deviations: σ_x and σ_p, where $\sigma_A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$. The surface obtained in this way is given by: $S = \sigma_x \sigma_p$. It turns out that an other surface is found using this approach then with the approach of U. Gerlach, where the distance between two zero points of $P_{jl}(p)$ and $F_{jl}(x)$ was used.
From equation (34) we know that:

$$\langle x \rangle = \varepsilon\left(\frac{1}{2} + j\right).$$

Further,

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} F_{jl}(x) x^2 \overline{F_{jl}(x)} dx = \frac{1}{\varepsilon} \int_{j\varepsilon}^{(j+1)\varepsilon} x^2 dx = \varepsilon^2 (j^2 + j + \frac{1}{3}).$$

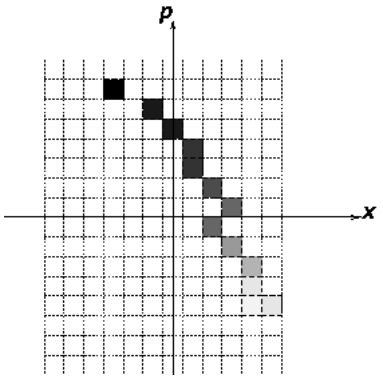

Figure 4: Phase space representation of a wavefunction.

And therefore,

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \varepsilon^2 \Big(\frac{1}{3} - \frac{1}{4} \Big) = \frac{\varepsilon^2}{12},$$
$$\sigma_x = \frac{\varepsilon}{2\sqrt{3}}.$$

For the momentum p, we already know from equation (35) that:

$$\langle p \rangle = \frac{2\pi l}{\varepsilon}.$$

Further,

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} P_{jl}(p) p^2 \overline{P_{jl}(p)} dp = \int_{-\infty}^{\infty} \frac{4}{2\pi\varepsilon} \left(\frac{\sin[(p - \frac{2\pi l}{\varepsilon})\frac{\varepsilon}{2}]}{p - \frac{2\pi l}{\varepsilon}} \right)^2 p^2 dp.$$

Change of variable to $t = (p - \frac{2\pi l}{\varepsilon})\frac{\varepsilon}{2}$, and using $dp = \frac{2}{\varepsilon} dt$ results in:

$$= \int_{-\infty}^{\infty} \left(\frac{\sin(t)}{t} \right)^2 \left(\frac{2t - 2\pi l}{\varepsilon} \right)^2 \frac{1}{\pi} dt$$
$$= \frac{4}{\pi\varepsilon^2} \underbrace{\int_{-\infty}^{\infty} \sin(t)^2 dt}_{\to\infty} + \frac{4\pi l^2}{\varepsilon^2} \underbrace{\int_{-\infty}^{\infty} \left(\frac{\sin(t)}{t} \right)^2 dt}_{=\pi} + \frac{4l}{\varepsilon^2} \underbrace{\int_{-\infty}^{\infty} \frac{\sin(t)^2}{t} dt}_{=0} \to \infty.$$

So,

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 \to \infty.$$

We see that the surface of a cell calculated in this way goes to infinity. This is not unexpected. When looking at the graphs of the functions $F_{jl}(x)$ (Figure

1) and $P_{jl}(p)$ (Figure 2), we see that the graph of $F_{jl}(x)$ is localized on a finite interval, so we expected σ_x to be finite. On the contrary, the graph of $P_{jl}(p)$ is not localized on a finite interval, which intuitively explains the fact that $\sigma_p \to \infty$.

The construction of functions $F_{jl}(x)$ and $P_{jl}(p)$ using Fourier transformation leads to the concept of gray coloring. Since the collection $\{P_{jl}(p)\}$ forms a basis for $L^2(\mathbb{R})$ we can express a function $\psi \in L^2(\mathbb{R})$ as the sum of complex amplitudes a_{jl} and functions $P_{jl}(p)$. With the help of these amplitudes a_{jl}, we can represent the function ψ in the two-dimensional space spanned by x and p, by gray coloring of the phase space cells. However, the functions $P_{jl}(p)$ and $F_{jl}(x)$ do not describe the phase cells mathematically. We could have known this right from the beginning because a function f and its Fourier transform $\hat{f}$ cannot be both supported on arbitrarily small sets (see Theorem 2.1 from [4]). This can also be seen from the definition of the Fourier transform from Plancherel's transformation, Theorem 4.2. Suppose f is only supported on a small interval $I \subset \mathbb{R}$, then:

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(p) \frac{e^{-ixp}}{\sqrt{2\pi}} dp = \int_I f(p) \frac{e^{-ixp}}{\sqrt{2\pi}} dp.$$

So, for every $x \in \mathbb{R}$ we know that $\hat{f}(x)$ is nonzero and therefore can not be supported on a small interval.

5 Conclusion

From the Heisenberg uncertainty principle we know that: $\sigma_x \sigma_p \geq \frac{\hbar}{2}$. This means that the smaller the uncertainty is in position, the larger it is in momentum. And the other way around, the larger the uncertainty is in position, the smaller it is in momentum. Consequently, it is not possible to localize points in the two dimension space spanned by x and p.
Nevertheless, it is possible to determine a partition of this space into cells of size $\hbar/2$, then the uncertainty in x times the uncertainty in p is minimal. Consider a particle whose position x and momentum p we would like to indicate in the x, p-plane. Suppose that the uncertainty in position: σ_x is known, for example $\sigma_x = \hbar$, then σ_p has to be $1/2$. Therefore, we could draw a rectangle in the x, p-plane with sides: $\sigma_x = \hbar$ and $\sigma_p = 1/2$. Since we do not know the precise uncertainty in the position or momentum we obtain a collection of rectangles, for each combination of uncertainties in x and p. All the obtained cells have area $\hbar/2$ and will lie in a star-shaped cell, this is illustrated in Figure 5. The obtained rectangles are called Planck cells. The cell with equal uncertainty in position and momentum is called the central Planck cell, this cell is drawn in red in Figure 5. A central Planck cell has sides: $\sigma_x = \sigma_p = \sqrt{\hbar/2}$. There are infinitely many Planck cells, but there is only one central Planck cell. To create the partitioning of the x, p-plane by Planck cells, minimal uncertainty wavefunctions have to be used. From [4] we know that if $\psi \in L^2(\mathbb{R})$ then: The wavefunction ψ is a minimum uncertainty wavefunction $\Leftrightarrow$ The wavefunction ψ is a Gaussian.

In Chapter 3 we defined coherent states. These coherent states satisfy the Heisenberg uncertainty relation with equality and indeed, their wavefunction is a Gaussian. The coherent states mathematically characterize Planck cells. A disadvantage of coherent states is that they form an overcomplete basis for $L^2(\mathbb{R})$. Nevertheless, since coherent states remain coherent under time evolution and their expectation values for position and momentum satisfy the equations of motion for the classical harmonic oscillator, they can be used to study the classical limit of quantum mechanics.

In Chapter 4 we studied an article by U. Gerlach ([2]), where cells in the x, p-plane were constructed using one specific function $P_{jl}(p)$ and its Fourier transform $F_{jl}(x)$. U. Gerlach defined his cells using Δx and Δp, both derived from the graphs of $F_{jl}(x)$, respectively $P_{jl}(p)$. We confronted this approach with an alternative description of the cells using the standard deviations σ_x and σ_p. This approach led to a surface that goes to infinity, so there is no minimal uncertainty. This is not unexpected since the function $P_{jl}(p)$ is not a Gaussian function and therefore does not satisfy the Heisenberg uncertainty principle with equality. The advantage of using the function $P_{jl}(p)$ and its Fourier transform, is that the collection $\{P_{jl}(p)\}$ forms a basis for $L^2(\mathbb{R})$. Therefore, a wavefunction $\psi(p) \in L^2(\mathbb{R})$ can be written as:

$$\psi(p) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} P_{jl}(p),$$

and the amplitude a_{jl} can be assigned to its corresponding phase space cell constructed in the way U. Gerlach did, with Δx and Δp. This results in a

visual display of the wavefunction in the x, p-plane.

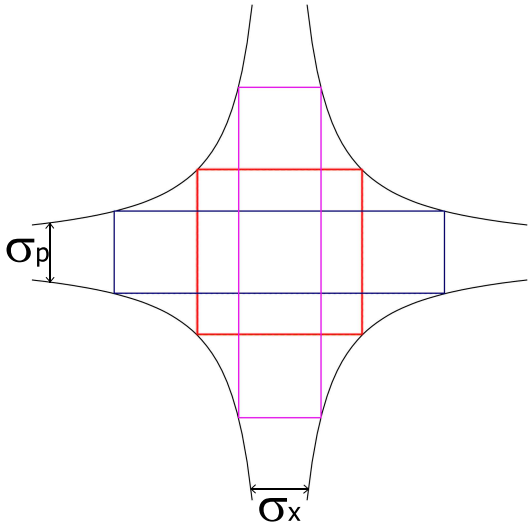

Figure 5: The star-shaped cell which contains all Planck cells, three of them are drawn in the cell. The central Planck cell is drawn in red.

References

[1] Bergstrom, L. & Hansson, H. *Lecture Notes in Relativistic Quantum Mechanics.* Stockholm: Stockholm University, 1999.

[2] Gerlach, U. *Phase Space Representation.* Ohio: Author, 2010.

[3] Griffiths, D. *Introduction to Quantum Mechanics: International edition.* Pearson: Prenctice Hall, 2005.

[4] Gröchenig, K. *Foundations of Time-Frequency Analysis.* Boston: Birkhäuser, 2001

[5] Pagter, B. de, & Rooij, A.C.M. van. *An invitation to functional analysis.* Nijmegen: Radboud University.

[6] Rudin, W. *Real and Complex Analysis: International student edition.* New York: McGraw-Hill Inc, 1987

[7] Tao, T. *Analysis II: Texts and Readings in Mathematics 38.* Los Angeles: Hindustan Book Agency, 2006