Graph C^*-algebras

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1 Introduction & acknowledgements

This paper is an introduction to the theory of a specific class of C^*-algebras: those which can be associated with a graph. We are primarily interested in linking the structure of the graph to that of the associated C^*-algebra. Specifically the occurrence of ideals and the K-theory of these C^*-algebras is discussed in relationship to their graphs. It is often very hard to find these for arbitrary C^*-algebras and it will come as a relieve that both the ideals and the K-theory are solely defined in terms of the graph for graph algebras in an easy manner.

Het voorstel van Walter van Suijlekom om graaf algebras te bestuderen kwam op het juiste moment: ik was toenertijd bezig met een introductie in categorie theorie, C^*-algebras en K-theorie. Ik keek er naar uit om delen uit deze drie disciplines met elkaar te verenigen. Triviale en gemakkelijk te doorgronden voorbeelden van C^*-algebras zijn schaars: in het bijzonder is het vinden van idealen geen sinecure. Grafen zijn een onuitputtelijke bron van interessante algebras en ik was erg blij door middel van deze grafen 'hands on' ervaring te krijgen met C^*-algebras.
Tot slot zou ik zowel Walter als Susanne van Suijlekom willen bedanken. Walter voor zijn inzet, inzicht en de uren die we samen doorgebracht hebben, filosoferend en nadenkend over de materie. En Susanne voor haar aanmoediging mezelf te vermannen en de scriptie eindelijk eens af te schrijven.

2 Preliminaries

2.1 C^*-algebras

Definition 1 (C^*-algebra). *A* ***C^*-algebra*** *A is a complex linear vector space endowed with a norm $\|\cdot\|$, a multiplication $\cdot : A \times A \to A$ and function $* : A \to A$ (called involution) such that for all $a, b \in A$:*

- *A is (Cauchy) complete;*
- *$\|a \cdot b\| \leq \|a\| \, \|b\|$ (A is a Banach-space);*
- *$(za + wb)^* = \overline{z}a^* + \overline{w}b^*$ for each $z, w \in \mathbb{C}$;*
- *$(a \cdot b)^* = b^* \cdot a^*$ and $(a^*)^* = a$;*
- *$\|a^* \cdot a\| = \|a\|^2$.*

We will also come across the little brother of the C^*-algebra, the ***-algebra**.

Definition 2 (*-algebra). *A *-algebra R is a C^*-algebra without a norm, therefore only adhering to points (3) and (4) of the above definition.*

Morphisms between C^*-algebras are complex linear, respect multiplication and involution. $\varphi : A \to B$ is a morphism between C^*-algebras A, B if $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$.
An **isomorphism** between C^*-algebras is a bijective morphism.
A C^*-algebra with a unit element is called **unital**.
A ***-homomorphism** is a complex linear map respecting multiplication and involution.

A very important example is the space of continuous function on a compact Hausdorff-space X, $C_0(X)$, using the definitions: $\|f\| = \sup\{|f(x)|; x \in X\}$, $(f \cdot g)(x) = f(x)g(x)$ and $(f^*)(x) = \overline{f(x)}$ on X.
Secondly, the set of bounded operators on a Hilbert-space $\mathscr{H}$ is a C^*-algebra. As norm we use the operator norm $\|a\| = \sup\{\|ax\| \,; \|x\| = 1\}$ and the involution is defined by $(x, ay) = (a^*x, y)$ for all $x, y \in \mathscr{H}$.

As all proper algebras, C^*-algebras boast **ideals**. An ideal in a C^*-algebra A is a subset $I \subset A$ such that I is closed under multiplication and summation (meaning it is a **sub-C^*-algebra** of a C^*-algebra), I is both an algebraic left- and right-ideal and it is closed in the norm of A. It is a consequence of this definition that an ideal I is closed under involution.
As one would expect any ideal in a C^*-algebra A is the kernel of a morphism $\varphi : A \to B$ (however, this is surprisingly hard to prove: see [10], chapter 10, for a discussion).

The Gelfand-Naimark-Segal representation theorem maybe is one of the most important theorem about C^*-algebras (see [10] for a complete proof):

Theorem 3 (Gelfand-Naimark-Segal)**.** *Every C^*-algebra A is isomorphic to a norm closed *-subalgebra of $B(\mathscr{H})$ for some Hilbert-space $\mathscr{H}$. Or: every C^*-algebra admits an injective morphism $\pi : A \to B(\mathscr{H})$.*

Definition 4 (Projections and partial isometries)**.** *An element $p \in A$ with A a C^*-algebra is a* ***projection*** *if $p^2 = p = p^*$. A* ***partial isometry*** *is an element $s \in A$ such that ss^* and s^*s are both (not necessarily equal) projections. A set of projections $\{p_n\}$ is called mutually orthogonal if $p_i p_j = p_j p_i = 0$ for all $i \neq j$. If two projections are mutually orthogonal their sum is a projection too.*

2.2 C^*-Algebraic K-theory

The roots of C^*-algebraic K-theory lie with the topological K-theory. In topological K-theory a series of maps $\{K^n; n \in \mathbb{N}\}$ are used to encode information about certain equivalence classes of vector bundles over a topological space by assigning an Abelian group to the topological space. Much can be learned about the topology by studying the associated K-groups. For an introduction into the subject, see [13].

A theorem of Gelfand and Naimark states that every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally-compact Hausdorff-space, which makes commutative C^*-algebras intrinsically topological spaces. Due to the fact that certain non-Abelian C^*-algebras can be seen as operators on quantum system we want to study topological properties of these algebras, too. By extending topological K-theory to C^*-algebraic K-theory we see that we can study (through the associated K-groups) some reminiscent topological properties of non-commutative C^*-algebras.

It turns out that in C^*-algebraic K-theory, $K_n(A) \cong K_{n+2}(A)$ for every C^*-algebra A and $n \in \mathbb{N}$. This is called Bott-periodicity.

Before we move to the definition of these K_0 and K_1 maps, we first cover certain equivalence relations used in that definition.

Definition 5 (Equivalence of embedding)**.** *Let A be a unital C^*-algebra. Form $M_n(A)$, the $n \times n$ matrix algebra with entries in A. We call two elements $a, b \in M_n(A)$* ***equivalent up to embedding*** *if $a = b \oplus 0_{m-n}$. Denote this relation by R.*

Definition 6 (Path equivalence)**.** *Using the last definition, define $M_\infty(A) = \coprod M_n(A)/R$, the union of all these matrix algebras. $P_\infty(A)$ is the set of projections in $M_\infty(A)$. For $p, q \in P_\infty(A)$, we call them* ***path equivalent*** *if there is a continuous path of projections $\{e(t)\} \subset P_\infty(A)$ such that $e(0) = p$ and $e(1) = q$.*

Definition 7 (K_0)**.** *For every equivalence class of paths in $P_\infty(A)$, K_0 assigns one copy of $\mathbb{Z}$ to the algebra A.*

Definition 8 (K_1)**.** *Under the same conditions as before, define $S(A)$ as $C_0((0,1), A)$, the space of continuous functions $f : (0,1) \to A$. Then $K_1(A) := K_0(S(A))$.*

2.3 Category theory

Category theory is a unification of all mathematical algebraic notions. Some would even go so far as saying that the whole of mathematics can be defined in terms of category theory. It makes use of the abstract notions of a **category**.

Definition 9 (Categories)**.** *A category consists of two classes C^0 (objects) and C^1 (arrows, or morphisms) with the following properties:*

1. *there are maps $r : C^1 \to C^0$ and $s : C^1 \to C^0$, called the range and source map;*

2. *there is map $\circ : \{(f, g) \in C^1 \times C^1; s(f) = r(g)\} \to C^1$, called composition, such that $r(g \circ f) = r(g)$ and $s(g \circ f) = s(f)$;*

3. *there is a map $i : C^0 \to C^1$, called identity, such that for all $c \in C^0$, $r(i(c)) = c = s(i(c))$.*

There is a natural map between categories called a **functor**:

Definition 10 (Functor)**.** *Let C, D be categories. Then a functor $F : C \to D$ are two maps $F^0 : C^0 \to D^0$ and $F^1 : C^1 \to D^1$ such that:*

1. *For all $c \in C^0$, $F^1 \circ i(c) = i(F^0(c))$;*

2. *If $f : c \to c'$ is an arrow in C^1, then $F^1(f) : F^0(c) \to F^0(c')$ is an arrow in D^1;*

3. *$F^1(f \circ g) = F^1(f) \circ F^1(g)$.*

To show how widespread the use of categories is, we give two examples:

- $\mathbb{N}$. Let $C^0 = \mathbb{N}$ and let C^1 be the collection of arrows pointing from n to m when $m - n = 1$ plus an identity arrow for every number. Composition is then defined by addition.

- Form **Top**, the category of topological spaces, by setting
  $\mathbf{Top}^0 = \{(X, \mathcal{T}); (X, \mathcal{T}) \text{ is a topological space}\}$ and
  $\mathbf{Top}^1 = \{f : X \to Y; f \text{ is a continuous map of topological spaces}\}$.

The strong point of an approach via category theory is that we can make far-fetching generalizations of known algebraic structures. One of these is the a so-called **universal construction**. See [11], chapter 3, for a profound introduction into this matter. We will only quote here two examples to see how this universality works and work with this understanding.

- Direct product of topological spaces. Let X, Y be two topological spaces. Then $X \times Y$, the topological direct product, goes with two projections $p : X \times Y \to X$ and $q : X \times Y \to Y$ such that $p(x, y) = x$ and $q(x, y) = y$. Now let U be a topological space such that there exist $f : X \to U$ and $g : Y \to U$ with the property $f \circ p = g \circ q$. The construction of the direct product is universal in the sense that there is a unique morphism (= a continuous map) $h : X \times Y \to U$. You guessed it, the morphism we are looking for is $h = (f, g)$.

- Cokernels. We work with a category which has a zero-object 0 such that for every pair of objects u, v we have unique arrows f, g with $f : u \to 0$ and $g : 0 \to v$. Let $f : u \to v$ be a morphism. Then coker$f$ is defined by an object coker$f$ and an arrow $p : v \to \text{coker} f$ such that $pf = 0$ and when there is an object b with arrow $q : v \to b$ such that $qf = 0$, there is a unique arrow $h : \text{coker} f \to b$ with $p \circ h = q$.

The word universal comes from the fact that these construction are the most general way of constructing an object of that particular type. It is therefore no surprise that they are so ubiquitous. The direct product, for instance, is available for many types of mathematical structures.

3 Graph algebras

3.1 Definitions

Definition 11 (Row-finite directed graph)**.** *A **directed graph** is a class consisting of a set of vertices E^0, a set of edges E^1 and maps $s : E^1 \to E^0$ and $r : E^1 \to E^0$ called respectively the source and the range of an edge. A row-finite directed graph is all of the above with the extra condition that $r^{-1}(v) = \{e \in E^1; r(e) = v\}$ is finite for all $v \in E^0$.*

The reason for demanding row-finiteness will become apparent in the definition of a graph algebra. Since every directed graph we will encounter is row-finite we will call 'row-finite directed graphs' simply 'graphs'.

A clear example of a graph is the following one.

$$\begin{tikzcd} w \arrow[r, "g"'] \arrow[rr, bend left, "h"] & v \arrow[r, "f"'] & u \arrow[loop right, "e"] \end{tikzcd} \tag{1}$$

We see that $E^0 = \{u, v, w\}$ and $E^1 = \{e, f, g, h\}$. $r(e) = r(h) = r(f) = u$ and $r(g) = v$. $s(e) = u$, $s(h) = s(g) = w$ and $s(f) = v$.

Definition 12 (Graph algebras)**.** *Let E be a graph, $\{S_e; e \in E^1\}$ a collection of partial isometries and $\{P_v; v \in E^0\}$ a collection of mutual orthogonal projections in a C^*-algebra B (also called a Cuntz-Krieger family $\{S, P\}$, or just simply CK-family) if they satisfy:*

1. $S_e^* S_e = P_{s(e)}$,
2. $P_v = \sum\limits_{e \in E^1; r(e) = v} S_e S_e^*$ *iff* $r(v)^{-1} \neq \emptyset$

*then the C^*-algebra generated (over $\mathbb{C}$) by all S_e and P_v is a **graph algebra**. Let us denote this algebra with $C^*(S, P)$.*

Note that our requirement $\#(r^{-1}(v)) < \infty$ now makes sense; in general $\sum\limits_{e \in E^1; r(e) = v}^{\infty} S_e S_e^*$ is *not* a projection. However, it is possible to define a CK-family on any directed graph and the interested reader should look into [14] chapter 5.
Let us make two important observations. At this stage, we have to embed $C^*(S, P)$ into an ambient C^*-algebra B since $C^*(S, P)$ is not naturally endowed with a norm: we have to borrow one from B[1]. Secondly, the definition does not exclude the association of several CK-families to the same graph. As we shall

[1] The reader familiar with the theory of C^*-algebras notices that the norm of a C^*-algebra is entirely defined by the algebraic structure, hence the reference to B is in fact redundant. In section 4.1 we will construct the graph algebra from scratch, omitting any reference to an ambient algebra.

see in section 4.4, these different graph algebras associated to a graph are in general *not* isomorphic.

Definition 13 (Paths). *A path is a string of edges $\mu := \mu_1 \cdots \mu_n$ with $\mu_i \in E^1$ such that $s(\mu_i) = r(\mu_{i+1})$. $|\mu| = n$, the length of the path, $s(\mu) = s(\mu_{|\mu|})$ and $r(\mu) = r(\mu_1)$. For every $n \in \mathbb{N}^*$ E^n is defined as the collection of paths of length n. We extend this to include E^0 and we simply call paths of length zero vertices for which $r(v) = s(v) = v$. Denote $E^* = \bigcup_{n \in \mathbb{N}^*} E^n$. For every $\mu \in \prod_{i=1}^{n} E^1$ (not just paths) we define $S_\mu = S_{\mu_1} \cdots S_{\mu_n}$.*

A **simple** graph is a graph without any cycles, that is, for every vertex v there is no path in the graph which begins and ends at v.

3.2 Structure of graph algebras: the multiplication rules

The astute reader has probably already noticed that the way we can chase arrows in the graph tells us what the structure of multiplication in $C^*(S,P)$ will be. In fact, there is a very elegant way of expressing arbitrary elements of $C^*(S,P)$:

Theorem 14. $C^*(S,P) = \overline{\text{span}}\{S_\mu S_\nu^*; \mu,\nu \in E^*, s(\mu) = s(\nu)\}$.

Before we start with the actual proof, we will prove a set of lemmas first which will be needed to prove that $\text{span}\{S_\mu S_\nu^*; \mu,\nu \in E^*, s(\mu) = s(\nu)\}$ is a subalgebra of $C^*(S,P)$.

In the following calculations we make the assumption that all projections associated to a graph are non-zero. An extension to the general case is trivial, but it would ruin the elegant formulation of the multiplication rules.

The first one conveniently elucidates the nature of S_μ.

Lemma 15 ($\mathbf{S_\mu S_\nu}$ & $\mathbf{S_\mu^* S_\nu^*}$)**.**

1. *If $\mu \notin E^*$ then $S_\mu = 0$.*
2. *S_μ is a partial isometry.*
3. *$S_\mu S_\nu = S_{\mu\nu}$.*
4. *$S_\mu^* S_\nu^* = S_{\nu\mu}^*$.*

Proof. *1.* Choose an i such that $r(\mu_{i+1}) \neq s(\mu_i)$. Then $S_{\mu_{i+1}} S_{\mu_i} = S_{\mu_{i+1}} P_{r(\mu_{i+1})} P_{s(\mu_i)} S_{\mu_i} = 0$ because the projections are mutually orthogonal. Thus $S_\mu = 0$.
2. If $\mu \notin E^*$ then $S_\mu^* S_\mu = 0$ is definitely a projection. Otherwise let us take a look at $S_\mu^* S_\mu$:

$$\begin{aligned}
&S_\mu^* S_\mu = S_{\mu_n}^* \cdots S_{\mu_1}^* S_{\mu_1} \cdots S_{\mu_n} = S_{\mu_n}^* \cdots S_{\mu_2}^* P_{s(\mu_1)} S_{\mu_2} \cdots S_{\mu_n} = \\
&S_{\mu_n}^* \cdots S_{\mu_2}^* P_{r(\mu_2)} S_{\mu_2} \cdots S_{\mu_n} = S_{\mu_n}^* \cdots S_{\mu_2}^* S_{\mu_2} \cdots S_{\mu_n} = \quad \ldots \\
&= S_{\mu_n}^* S_{\mu_n} = P_{s(\mu_n)} = P_{s(\mu)}.
\end{aligned}$$

3: Whether or not $\mu\nu \in E^*$ this statement is checked by writing it out fully.
4: Apply the adjoint to *3*.

□

To see how this works on a diagrammatical level we take a look at (1). Then we can infer that to construct a non-zero composition of S_μ's composition must be tail-to-head and against the direction of the arrows. For example, consider $S_h S_f$. Since $w = s(h) \neq r(f) = v$ this is zero. Also $S_{gfe} = 0$ while S_{efg} is not.

Lemma 16 ($\mathrm{S_\mu S_\nu^*}$)**.** *If $S_\mu S_\nu^* \neq 0$ then $s(\mu) = s(\nu)$.*

Proof.

$$S_\mu S_\nu^* = S_{\mu_1} \cdots S_{\mu_n} S_{\nu_k}^* \cdots S_{\nu_1}^* = S_{\mu_1} \cdots S_{\mu_n} P_{s(\mu_n)} P_{s(\nu_k)} S_{\nu_k}^* \cdots S_{\nu_1}^* \neq 0$$

Which means $P_{s(\mu_n)} P_{s(\nu_k)} \neq 0$. Both of these projections are associated with the sources of some arrow, which, by definition, implies that they are mutually orthogonal. The product being non-zero then shows $P_{s(\mu_n)} = P_{s(\nu_k)}$ so that $s(\mu) = s(\mu_n) = s(\nu_k) = s(\nu)$. □

Lemma 17 ($\mathrm{S}_\mu^*\mathrm{S}_\nu$)**.**

$$S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \textit{if } \mu = \nu\mu' \textit{ for some } \mu' \in E^* \\ S_{\nu'} & \textit{if } \nu = \mu\nu' \textit{ for some } \nu' \in E^* \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let us look a the case $S_\mu^* S_\nu \neq 0$ and assume $|\nu| \leq |\mu|$. Choose an α such that $\mu = \alpha\mu'$ with $|\alpha| = |\nu|$. Then $S_\mu^* S_\nu = S_{\alpha\mu'}^* S_\nu = S_{\mu'}^*(S_\alpha^* S_\nu)$. If $\alpha = \nu$ then $S_\mu^* S_\nu = S_{\mu'}^*(S_\nu^* S_\nu) = S_{\mu'}^* P_{s(\nu)} = S_{\mu'}^* P_{r(\mu')} = S_{\mu'}^*$. If $\alpha \neq \nu$ choose the smallest integer such that $\alpha_i \neq \nu_i$. Then

$$S_\alpha^* S_\nu = (S_{\alpha_1} \cdots S_{\alpha_n})^* S_{\nu_1} \cdots S_{\nu_n} = S_{\alpha_n}^* \cdots S_{\alpha_i}^* (S_{\alpha_{i-1}}^* \cdots S_{\alpha_1}^* S_{\nu_1} \cdots S_{\nu_{i-1}}) S_{\nu_i} \cdots S_{\nu_n} =$$
$$S_{\alpha_n}^* \cdots S_{\alpha_i}^* P_{r(\alpha_i)} S_{\nu_i} \cdots S_\nu = S_{\alpha_n}^* \cdots S_{\alpha_i}^* S_{\nu_i} \cdots S_\nu.$$

Which vanishes, as we have seen before. Applying the same line of reasoning to the case $|\mu| \leq |\nu|$ yields the second expression. □

Apply the last lemma to obtain the important relation:

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^* & \text{if } \alpha = \nu\alpha' \text{ for some } \alpha' \in E^* \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Now for the proof of theorem 2.

Proof. Equation (2) shows that any word formed in $S_\mu S_\nu^*$ is of the form $S_\alpha S_\beta^*$ for some paths α and β. $\mathrm{span}\{S_\mu S_\nu^*; \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ is closed under multiplication and closed under involution and hence is a proper subalgebra of $C^*(S, P)$. It contains all the generators of $C^*(S, P)$, so its norm closure must be the whole of $C^*(S, P)$. □

3.3 Examples of graph algebras: simple graphs

In this section we will present all examples of simple graphs and provide a complete classification of graph algebras they generate.

Example 18 (Matrix algebras)**.**

Let us start with a description of a **matrix algebra** as a C^*-algebra. Define $M_n(\mathbb{C}) = \{A : \mathbb{C}^n \to \mathbb{C}^n; A \text{ linear}\}$. $M_n(\mathbb{C})$ is a vector space over the complex numbers and multiplication is given by the usual matrix multiplication. We endow this space with the operator norm $\|A\| = \{\|A\mathbf{x}\| \,;\, \|\mathbf{x}\| = 1\}$ with respect to the inner product on $\mathbb{C}^n$. Involution is given by $A^* = \overline{A}^T$, complex conjugation and transposition. The projections in $M_n(\mathbb{C})$ are the matrices with ones on the diagonal and zeros everywhere else.
We can choose $\{e_{ij}, i, j \in \{1, \ldots, n\}\}$ as a base for $M_n(\mathbb{C})$ where e_{ij} is a $n \times n$-matrix with a 1 on (i, j) and zeros everywhere else. Note that e_{ij} is a partial isometry for all pairs i, j: $e_{ij}e_{ij}^* = e_{ij}e_{ji} = e_{ii}$ and $e_{ij}^*e_{ij} = e_{jj}$. We therefore can form the *-algebra A_n generated by $\{e_{1j}; j \in \{1, \ldots, n\}\}$ of partial isometries. It follows that $M_n(\mathbb{C}) = \operatorname{span}(A_n)$. Note that there are multiple ways to construct A_n.
So if we were to find a graph E, a set of (non-zero) partial isometries Q and a set of projections P which generate $C^*(S, P) \cong M_n(\mathbb{C})$ then by definition E should have a finite number of edges. Futhermore, for any morphism between $C^*(S, P) \cong M_n(\mathbb{C})$ to respect composition, we need property (2) to boil down to the relation $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \delta_{\nu,\alpha} S_\mu S_\beta^*$. This implies that α' and ν' are always vertices, meaning whenever $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) \neq 0$, $s(\mu) = s(\beta)$. So in general, a graph corresponding to $M_n(\mathbb{C})$ has n different partial isometries emanating from the same (single) source. Since every projection in $M_n(\mathbb{C})$ is of the form $e_{ij}e_{ij}^*$ every vertex in the graph receives at least one edge, which makes the graph connected. Surely, the graph has no cycles because of the finite number of partial isometries involved.

Returning to the discussion in the introduction of this section we can show that the members of $(C^*)^{-1}(M_n(\mathbb{C}))$ are in general not graph-isomorphic. For instance

$$
\begin{tikzcd}
w \arrow[r, "f"] & u_1 \arrow[r, "g"] & u_2 \arrow[r, "h"] & u_3
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
w \arrow[rr, "i"] \arrow[dr, "j"'] & & u_1 \\
& u_2 \arrow[ur, "k"'] &
\end{tikzcd}
\tag{3}
$$

both generate $M_4(\mathbb{C})$. Namely, we for both graph algebras we have the set $\{S_w, S_f, S_{gf}, S_{hgf}\}$ and $\{S_w, S_j, S_{kj}, S_i\}$ satisfying the matrix multiplication rules of base vectors. These sets are therefore isomorphic to a set of generators of $M_4(\mathbb{C})$, which proves that both graphs generate $M_4(\mathbb{C})$.
They differ in which base vectors of $M_4(\mathbb{C})$ the isomorphism maps to. We can use the graph relations to determine the image of the isomorphism on the edges when

we have fixed the edges. For the first graph, map $w \to e_{11}, u_1 \to e_{22}, u_2 \to e_{33}$ and $u_3 \to e_{44}$. Then $\{S_w, S_f, S_g, S_h\} \cong \{e_{11}, e_{21}, e_{32}, e_{43}\}$, which gives us the generators $\{e_{11}, e_{21}, e_{31}, e_{41}\}$. For the second graph, $w \to e_{11}, u_1 \to e_{33} + e_{44}$ and $u_2 \to e_{22}$. Then $\{S_w, S_i, S_j, S_k\} \cong \{e_{11}, e_{31}, e_{21}, e_{42}\}$ which also gives the generators $\{e_{11}, e_{31}, e_{21}, e_{41}\}$.

Example 19 (Multimatrix algebra)**.**

A **multimatrix algebra** $M(\vec{m})$ for $m \in \mathbb{N}^n$ is defined by $M(\vec{m}) = \bigoplus_{i=1}^{n} M_{m(i)}(\mathbb{C})$. This is also known as a Jordan normal form. They are a direct extension of the matrix algebras. Namely, Let E be a finite graph without cycles and let $\{w_1, \ldots, w_n\}$ the collection of sources. Then $C^*(S, P) \cong \bigoplus_{i=1}^{n} M_{\#\{s^{-1}(w_i)\}}(\mathbb{C})$.

Proof. Let $A_i = \text{span}\{S_\mu S_\nu^*; s(\mu) = s(\nu) = w_i\}$. Then following the line of reasoning in example 18 we see that $A_i \cong M_{\#\{s^{-1}(w_i)\}}$. Let $i \neq j$ and $S_\mu S_\nu^*$ with $s(\mu) = s(\nu) = w_i$ and $S_\alpha S_\beta^*$ with $s(\alpha) = s(\beta) = w_j$. Then $S_\mu S_\nu^* S_\alpha S_\beta^* = 0$ unless ν extends to α or α extends to ν. But since they do not share a source, this is not possible. Therefore $A_i A_j = 0$. It is now easy to see that $\text{span}\{A_i \cup A_j\} \cong A_i \oplus A_j$. □

Example 20 (Compact operators on a separable Hilbert space)**.**

The correct generalization of the matrix algebras (which act on a finite dimensional space) to spaces of infinite (but countable) dimension are the **compact operators**. As illustrated in [2] chapter II they exhibit a great degree of similarity with matrix operators. The algebra of compact operators, $B_0(H)$, is a closed algebraic ideal of $B(H)$ and thus a C^*-algebra in its own right. As a graph algebra, $B_0(H)$'s ancestry is quite evident. Namely,

$$\cdots \longrightarrow u_2 \longrightarrow u_1 \longrightarrow w \tag{4}$$

generates $B_0(H)$. Let $\{E_n; n \in \mathbb{N}\}$ be the basis of H and define the linear operator $E_i \otimes E_j : H \to H$ as $E_i \otimes E_j(h) := E_{ij}(h) = (h, E_j)E_i$. It is obviously a finite rank operator, so $A_n = \text{span}\{E_{ij}, i, j \in \{1, \ldots, n\}\} \subset B_0(H)$ for every $n \in \mathbb{N}$.
We can derive two useful identities.

$$E_{ij}(E_{kl}(h)) = E_{ij}(h, E_l)E_k = (h, E_l)(E_k, E_j)E_i = \delta_{j,k}(h, E_l)E_i = \delta_{j,k}E_{il}(h)$$
$$(g, E_{ij}h) = (g, (h, E_j)E_i) = (E_j, h)\overline{(E_i, g)} = ((E_i, g)E_j, h) = (E_{ji}g, h)$$

This means we can construct an isomorphism $\varphi : A_n \to M_n(\mathbb{C})$ by simply identifying $\varphi(E_{ij}) = e_{ij}$.
By using the results from 18 we can conclude that the algebra generated by (4) equals $\overline{\text{span}}\{e_{ij}; i, j \in \mathbb{N}\} = \overline{\text{span}}\{E_{ij}; i, j \in \mathbb{N}\}$. Since every element in $B_0(H)$ is

the norm limit of some series of finite rank operator, and $\text{span}\{E_{ij}; i, j \in \mathbb{N}\}$ contains every finite rank operator, we can conclude $\overline{\text{span}}\{E_{ij}; i, j \in \mathbb{N}\} = B_0(H)$.

The algebra of compact operators is an example of an approximately finite algebra.

Example 21 (AF-algebra)**.**

An AF-algebra A is defined by $A = \text{clo} \bigcup\limits_{n=1}^{\infty} A_n$ with $\{A_n\}$ a sequence of multimatrix algebras ordered by inclusion. For instance, if we include $M_n(\mathbb{C}) \subset M_{n+1}(\mathbb{C})$ by $M_n(\mathbb{C}) \oplus 0 \in M_{n+1}(\mathbb{C})$ then we see that $\text{clo} \bigcup\limits_{n=1}^{\infty} M_n(\mathbb{C}) = B_0(H)$. In general, AF-algebras will be generated by graphs of multimatrix algebras with an infinite path added to each sink.

These possibilities exhaust all the graph algebras generated by simple graphs and we have obtained a complete classification.

4 Categorical aspects of graph algebras

4.1 The universal construction

To study K-theory properly we have to delve into the functorial nature of the construction of a graph algebra from a graph. Heuristically speaking, we want to examine the maps $\{\text{graphs}\} \xrightarrow{C^*} \{C^* - \text{algebras}\} \xrightarrow{K^i} \{\text{Abelian groups}\}$ for $i \in \{0, 1\}$. As we have seen there is an ambiguity in our choice for a graph algebra which first needs to be resolved. Only if we can associate a unique C^*-algebra to a given graph, a functor C^* can be formulated. Reference [14] proposes a universal construction:

Theorem 22 ($C^*(E)$)**.** *For any graph E there is a C^*-algebra $C^*(E)$ with a formal Cuntz-Krieger family $\{s, p\}$ such that if $\{T, Q\}$ is a Cuntz-Krieger family generating an algebra B, there is a unique morphism $\pi_{T,Q} : C^*(E) \to B$ such that $\pi_{T,Q}(s_e) = T_e$ and $\pi_{T,Q}(p_v) = Q_v$ for every vertex and edge.*
Moreover, this construction is universal. Let $\{w, r\}$ be a CK-family generating C such that for every CK-family $\{T, Q\}$ generating B there is a homomorphism $\rho_{T,Q} : C \to B$ such that $\rho_{T,Q}(w_e) = T_e$ for all edges and $\rho_{T,Q}(r_v) = Q_v$ for all vertices. Then there is an isomorphism $\varphi : C^*(E) \to C$ such that $\varphi(s_e) = w_e$ and $\varphi(p_v) = r_v$ for all edges respectively vertices.
This implies the following diagram is commutative:

$$\begin{tikzcd} C \arrow[r, "\rho_{T,Q}"] & B \\ C^*(E) \arrow[u, "!\varphi"] \arrow[ur, "\pi_{T,Q}"'] & \end{tikzcd}$$

Proof. Let $z_{\mu,\nu} \in \mathbb{C}$ and $d_{\mu,\nu}$ formal symbols for all paths $\mu, \nu \in E^*$. Then $V = \{\sum z_{\mu,\nu} d_{\mu,\nu}; \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ is a $*$-algebra under the following definitions:

- $a \sum z_{\mu,\nu} d_{\mu,\nu} + b \sum w_{\mu,\nu} d_{\mu,\nu} = \sum (a z_{\mu,\nu} + b w_{\mu,\nu}) d_{\mu,\nu}$;
- $d^*_{\mu,\nu} = d_{\nu,\mu}$;
- $d_{\mu,\nu} d_{\alpha,\beta}$ satisfies the same relations as (2).

We need to define a norm on V to construct a C^*-algebra. We will borrow the norm from the other C^*-algebras generated by E.
For every Cuntz-Krieger family $\{S, P\}$ (generating C^*-algebra A) we can define a $*$-homomorphism with the property $\pi_{S,P}(d_{\mu,\nu}) = S_\mu S^*_\nu$. Then $\|v\|_1 = \sup\{\|\pi_{S,P}(v)\|; \{S, P\}$ is a Cuntz-Krieger family$\}$ is a semi-norm on V. This follows from:

$$\left\| \pi_{S,P} \left(\sum z_{\mu,\nu} d_{\mu,\nu} \right) \right\| \leq \sum |z_{\mu,\nu}| \, \|S_\mu S^*_\nu\| \leq \sum |z_{\mu,\nu}|,$$

since A is a Banach space and every partial isometry has norm 1.
It is easily checked that $\|\cdot\|_1$ satisfies $\|v^* v\|_1 = \|v\|_1^2$, since $\pi_{S,P}$ is a *-homomorphism.

Taking V_0 as the quotient algebra of V and the kernel of the norm gives us a new norm $\|\cdot\|_0$ which is constant on equivalence classes. Forming the closure of V_0 then gives us a C^*-algebra, which we call $C^*(E)$.
Now take $s_e := d_{e,s(e)}$ and $p_v := d_{v,v}$. It is easily verified this a Cuntz-Krieger family which generates V_0. Using the Gelfand-Naimark theorem to find a faithful representation $\rho : B \to B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, $\pi_{T,Q} = \rho^{-1} \circ \pi_{\rho(T),\rho(Q)}$ then does the trick.
Lastly, the universality property. Since $\{w, r\}$ is a Cuntz-Krieger family, there is a map $\pi_{w,r} : C^*(E) \to C$. We will prove this is an isomorphism.
The range of $\pi_{w,r}$ contains all the generators of C, namely all $\{w_e, r_v\}$. Hence it is surjective. Vice versa, there is a $\rho_{s,p} : C \to C^*(E)$. This makes $\rho_{s,p} \circ \pi_{w,r}$ the identity on $\{s, p\}$, thus all of $C^*(E)$. Injectivity then follows from $\pi_{w,r}(a) = 0 \Rightarrow a = \rho_{s,p} \circ \pi_{w,r}(a) = 0$. $\square$

Now a central result in the theory of graph algebra's is the uniqueness criterion of graph algebras, which is the subject of chapter 3 in [14]. It can be formulated in terms of this universal construction.

Corollary 23. *If E is a graph for which every cycle has an entry, then every Cuntz-Krieger family $\{T, Q\}$ defined on E has the universal property described above.*

To conclude: if in a graph E every cycle has an entry all graph algebras it generates are isomorphic. If the graph does not exhibit this property there is still a unique graph algebra we can associate it with.

4.2 The functor C^*

Now we are ready to deploy the machinery of category theory to our graph algebras. First of all, we need to define proper categories which we will be working with.

Definition 24. *1. **DGrph** is the category of directed graphs. Its objects are directed graphs and its arrows are defined as follows. $\varphi : E \to F$ is a graph morphism if $\varphi(v)$ is a vertex in F for every $v \in E^0$ and $\varphi(e)$ is an edge in F for every $e \in E^1$ such that $r(\varphi(e)) = \varphi(r(e))$ and $s(\varphi(e)) = \varphi(s(e))$. Composition of $\varphi : E \to F$ and $\psi : F \to G$ is defined as $\psi \circ \varphi : E \to G$. Lastly, if $\mu \in E^*$ with $\mu = \mu_1 \cdots \mu_n$, $\varphi(\mu) = \prod \varphi(\mu_i) \in F^*$.*
*2. **CAlg** is the category of C^*-algebras with as objects C^*-algebras. An arrow $\varphi : A \to B$ with A, B C^*-algebras is a complex linear map satisfying $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$.*

Corollary 25. *There is a functor $C^* :$ **DGrph** $\to$ **Calg** adding to every graph the graph algebra by universal construction.*

Proof. Let $\varphi : E \to F$ be a graph morphism such that $\{s, p\}$ and $\{t, p\}$ are the Cuntz-Krieger families generating $C^*(E)$ and $C^*(F)$.
Define $C^*(\varphi)(\alpha s_\mu + \beta s_\nu) := \alpha t_{\varphi(\mu)} + \beta t_{\varphi(\nu)}$ for any $\mu, \nu \in E^*$ and $\alpha, \beta \in \mathbb{C}$. Then $C^*(\varphi)$ is a morphism of C^*-algebras: $C^*(\varphi)(s_\mu s_\nu) = C^*(\varphi)(s_{\mu\nu}) = t_{\varphi(\mu\nu)} = t_{\varphi(\mu)} t_{\varphi(\nu)} = C^*(\varphi)(s_\mu) C^*(\varphi)(s_\nu)$ and $C^*(\varphi)(s_\mu^*) = t^*_{\varphi(\mu)} = (C^*(\varphi)(s_\mu))^*$. Now using the fact that $C^*(F)$ is norm-closed, we can extend $C^*(\varphi)$ to any element $a \in C^*(E)$ by continuity.
Let 1_E be the identity on graph E. $C^*(1_E)(s_\mu) = s_{1_E(\mu)} = s_\mu = 1_{C^*(E)}(s_\mu)$.
For $\psi : F \to G$ with $C^*(G)$ generated by $\{s, r\}$ we have $C^*(\psi \circ \varphi)(s_\mu) = r_{\psi \circ \varphi(\mu)} = C^*(\psi)(s_{\varphi(\mu)}) = C^*(\psi) \circ C^*(\varphi)(s_\mu)$.
□

It is desirable to have a functor U : **Calg** $\to$ **DGrph** such that it is the left-adjoint of C^* and such that C^* is the right-adjoint to U. This would allow us to prove the result in [7] with less effort: it's a known result that when U and C^* relate in such a manner, C^* is both stable under take limits and colimits. Which would imply $C^*(E \oplus F) = C^*(E) \oplus C^*(F)$ and $C^*(E \times F) = C^*(E) \otimes C^*(F)$. Calculating for instance the K-theory of tensor products of C^*-algebras is elaborate and these results would prove to be useful tool. See for instance [11] for more information on this subject. Here, we will only quote a result of this conjecture, which can also be proved independently:

Proposition 26 (Direct sums of graphs)**.** *Let E, F be graphs then $E \oplus F$ is constructed by laying E and F in the same plane without touching each other. Then $C^*(E \oplus F) = C^*(E) \oplus C^*(F)$.*

We foresee two major technical difficulties in this endeavor. Firstly, how to restrict the category **Calg**. The two most important reasons for this is that not every C^*-algebra is the norm-limit span of a set of partial isometries. Secondly, the multiplication rules (2) are in general not valid for C^*-algebras. Lastly, it would be a challenge to retrieve partial isometries in a given C^*-algebra. Via the representation on a Hilbert-space we can always dissect the bounded operators into sums of one-dimensional projections. An abstract procedure to create partial isometries is available for us: let $p, q \in A$ be projections in a C^*-algebra A. Form the representations $p = (\cdot, e_i)e_i$ and $q = (\cdot, e_j)e_j$ on the basis $\{e_i\}_{i \in I}$ of the Hilbert-space $\mathscr{H}$ on which we represent the algebra. Then $e_i \otimes e_j$ as defined in 20 does the trick. However, it is not obvious that there is an element $a \in A$ such that $\pi(a) = e_i \otimes e_j$ and how we would construct such an a.

I am grateful for a discussion with W. Szymański who pointed out that one should look at the $\mathbb{Z}$-graded C^*-algebras with graded morphisms as subcategory. The functor then should take us to the category of Hilbert-bimodules, which has to be decreased sufficiently to narrow down to a directed graph, which is in particular a special case of a Hilbert-bimodule.

4.3 Graphs of rank 2

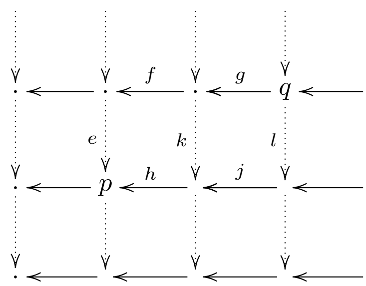
We now go into the theory of higher-rank graphs. The following is based on the material in chapter 10 of [14].

Definition 27 (Graph of rank 2)**.** *Let $\mathbb{N}^2$ be the category with one object, two morphisms given by two generators of $\mathbb{N}$ and addition as multiplication. Then a* ***graph of rank 2*** *is a countable category Λ and a functor $d : \Lambda \to \mathbb{N}^2$, called the degree map. d has the following property: if $d(\lambda) = m + n$ for some morphism λ and $m, n \in \mathbb{N}^2$, then there are two morphisms μ, ν such that $d(\mu) = m$ and $d(\nu) = n$ with $\lambda = \mu\nu$.*

This abstract definition basically means that d *colors* the arrows of a graph, painting them blue and red, and that decomposition in $\mathbb{N}^2$ is guaranteed to be reflected in the way we can take paths of different colors on the graph (see also example 28).
A graph of rank 2 has no sources if every vertex receives at least one arrow of every color.

Example 28. *[Ω_2] A good example is Ω_2; let $\Omega_2^0 = \mathbb{N}^2$ and define the collection of paths on Ω_2 as: $\Omega_2^* = \{(p,q) \in \mathbb{N}^2 \times \mathbb{N}^2; p_1 \leq q_1, p_2 \leq q_2\}$. Define $d : \Omega_2 \to \mathbb{N}^2$ as $d(p,q) = q - p$, $r(p,q) = p$, $s(p,q) = q$ and composition $(p,q)(q,r) = (p,r)$. This is a graph of rank 2.*

Proof. Ω_2 is a category with objects pairs of positive integers. The morphisms are the pairs (p,q) such that $q - p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (blue arrows) or $q - p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (red arrows).
Then d is a functor; if (p,q) is the identity morphism then $p = q$, so that $d(p,q) = 0$. Let (p,q), (q,r) be a pair of morphisms then $d((p,q)(q,r)) = d(p,r) = r - p = r - q + (q - p) = d(p,q) + d(q,r)$. So d is a functor.
Now let $d(p,q) = m + n$. To prove we can find two paths $(p,r), (r,q)$ such that $d(p,r) = m$ and $d(r,q) = n$ we take a look at the visualization of the 2 graph in terms of its **1-skeleton**.



Then $d(p,q) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and depending on the values of m and n we can find paths efg, hkg and hjl for the composition $\mu\nu$ to satisfy $d(p,q) = d(\mu) + d(\nu)$. $\square$

Definition 29 (2-rank graph algebra)**.** *Let Λ be a graph of rank 2. Then a **Cuntz-Krieger-Λ-family** $S = \{S_\lambda; \lambda \in \Lambda^*\}$ is a collection of partial isometries satisfying:*

1. $\{S_v; v \in \Lambda^0\}$ *are mutually orthogonal projections;*
2. $S_\lambda S_\mu = S_{\lambda\mu}$ *when* $s(\lambda) = r(\mu)$.
3. $S_\lambda^* S_\lambda = S_{s(\lambda)}$.
4. $S_v = \sum_{\lambda \in \Lambda^n; r(\lambda)=v} S_\lambda S_\lambda^*$ *for all* $v \in \Lambda^0$ *and all* $n \in \mathbb{N}^2$.

Important to note is that condition (4) implies that paths of different color share the same projection. This implies that Λ is a so-called *2-category*. See [11], chapter 2.5, for more information on these special type of categories.

We will use the following important theorem about two-graphs, which is the following result by [7].

Theorem 30. *Let E and F be two ordinary graphs. Then $E \times F$ is a 2-graph and $C^*(E \times F) = C^*(E) \otimes C^*(F)$ with $\otimes$ the tensor product for C^*-algebras.*

4.4 Examples of graph algebras: graphs with cycles

We now illustrate the theory of the preceding chapter with examples of graphs with cycles.

Example 31 (C($\mathbb{T}$))**.** *The graph*

$$\begin{tikzcd} v \arrow[loop left, "e"] \end{tikzcd} \tag{5}$$

does not have an entry into its single cycle, so it might as well generate multiple non-isomorphic C^*-algebras. In fact we can immediately find two distinct algebras.
This graph allows only one projection. If we conveniently assume $P_v = 1$ then every possible partial isometry S_e is unitary. These are the only restrictions we face so if we choose $B = B(\mathbb{C}) \cong \mathbb{C}$ then we can choose $S_e = \exp(it)$ for any $t \in \mathbb{R}$. Since $C^*(S, P)$ is the linear span of $\exp(it)$, $C^*(S, P) \cong \mathbb{C}$ too.
$B = C(\mathbb{T})$, the algebra of continuous functions over the circle, is also generated by this graph. By fixing $S_e = e_1 \in C(\mathbb{T})$ with $e_k(x) = \exp(ikx)$ for all $x \in [0, 2\pi)$ we see that $S_e^k = e_k$ for all $k \in \mathbb{Z}$. Therefore $C^*(S, P) = \text{closure}\left\{\sum_{k=-N}^{M} a_k e_k ; a_k \in \ell^1(\mathbb{Z})\right\} = C(\mathbb{T})$, which is obviously not isomorphic to $\mathbb{C}$.
We also can examine the universal graph algebra, $C^*(E)$. When represented on a suitable Hilbert-space, S_e forms an isometry $S_e : P_v\mathscr{H} \to P_v\mathscr{H}$ so for $C^*(E)$, P_V is the unit element. This means $C^*(E)$ is universal for all C^*-algebras generated by a single unitary.
Now, $C(\mathbb{T})$ has the same universal property. Construct $i : \mathbb{T} \to \mathbb{C}$ the inclusion function. Then the spectrum of i coincides with that of any unitary element U in a C^*-algebra B. Hence we have a morphism $\pi : C(\mathbb{T}) \to B$ such that $\pi(i) = U$. All of this implies $C(\mathbb{T}) \cong C^*(E)$.

Example 32 ($\mathcal{T}$)**.** *Consider the graph*

$$\begin{tikzcd} v \arrow[loop left, "e"] & w \arrow[l, "f"'] \end{tikzcd} \tag{6}$$

Since every cycle has an entry we only have to select any graph algebra generated by the graph to prove that algebra is *the* graph algebra.
Let us represent $C^*(S, P)$ on $B(\ell^2(\mathbb{N}))$. Choose $P_v(\{e_n\}) = (e_1, 0, 0, \ldots)$ then $P_w(\{e_n\}) = (0, e_2, e_3, \ldots)$. The CK-relations imply $S_f(\{e_n\}) = (0, e_1, 0, \ldots)$ and $S_f^*(\{e_n\}) = (e_2, 0, 0, \ldots)$ while $S_e(\{e_n\}) = (0, 0, e_2, e_3, \ldots)$ and $S_e^*(\{e_n\}) = (0, e_3, e_4, \ldots)$.
We conclude this graph generates the Töplitz-algebra.

Example 33 ($SU_q(2)$)**.** *For $q \in [0, 1)$ Let $SU_q(2)$ be the C^*-algebra generated by a and b subject to the following relations:*

$$\begin{aligned} a^*a + b^*b &= 1 & ab &= qba \\ aa^* + q^2b^*b &= 1 & ab^* &= qb^*a \\ b^*b &= bb^* \end{aligned}$$

Theorem A2.2 in [20] proves that for all q the C^*-algebras they generate are isomorphic. So let us focus on the case $q = 0$. Now take a look at the following graph:

$$\begin{tikzcd} v \arrow[loop left, "e"] \arrow[r, "f"'] & w \arrow[loop right, "g"] \end{tikzcd} \tag{7}$$

We have the following relations: $P_v = S_f^* S_f = S_e^* S_e = S_e S_e^*$ and $P_w = S_g^* S_g = S_g S_g^* + S_f S_f^*$. We call the universal C^*-algebra generated by these projections and partial isometries $C^*(L_3)$.
We now claim $\varphi : C(SU_0(2)) \to C^*(L_3)$ given on generators by $\varphi(a) = S_f^* + S_g^*$ and $\varphi(b) = S_e$ is an isomorphism.

Proof. We claim that the image of the generators also adheres to the relations of $SU_0(2)$:
$\varphi(b^*b) = S_e^* S_e = S_e S_e^* = \varphi(bb^*)$.
$\varphi(ab) = (S_f^* + S_g^*)S_e = S_f^* S_e + S_g^* S_e$. There is no path μ' such that $f = e\mu'$, neither is there any ν' such that $e = f\nu'$. Same goes for $S_g^* S_e$, so this relation is zero.
$\varphi(a^*b) = (S_f + S_g)S_e = S_{fe} + S_{ge} = 0$.
$\varphi(aa^*) = (S_f^* + S_g^*)(S_f + S_g) = 1 + S_f^* S_g + S_g^* S_f = 0$ for the same reason as the one above.
$\varphi(a^*a + b^*b) = (S_f + S_g)(S_f^* + S_g^*) + S_e^* S_e = 1 + S_g S_f^* + S_f S_g^* = 0$ since f and g do not have the same source.
Now construct φ^{-1} as follows: $\varphi^{-1}(P_v) = a^*a$, $\varphi^{-1}(P_w) = bb^*$, $\varphi^{-1}(S_e) = a^*(1 - bb^*)$, $\varphi^{-1}(S_g) = b$ and $\varphi^{-1}(S_f) = a^*bb^*$. Again, it is an easy exercise to check if these images obey the relations of a graph algebra and is the inverse of φ.
By extending φ to linearity we now see that $C(SU_q(2)) \cong C^*(L_3)$. □

Example 34 (A_θ)**.** *The irrational rotation algebra, or A_θ, is the C^*-algebra generated by u, v on which we define the relationship $uv = \exp(i\theta)vu$ with θ irrational.*

An important result achieved by Elliot and Evans is the description of A_θ as a direct limit in [4]. Let $\{q_n\}$ be the coefficients of θ in a continuous fraction expansion. The rotational algebra takes the following form $A_\theta = \varinjlim \left(M_{q_{n+1}}(C(\mathbb{T}) \oplus M_{q_n}(C(\mathbb{T}))\right)$ where we take the direct limit in the category of C^*-algebras. See [8], proposition 5.2, for more information. For our algebra this narrows down to $A_\theta = \overline{\bigcup_n M_{q_{n+1}}(\mathbb{C}) \otimes C(\mathbb{T})} \oplus \overline{\bigcup_n M_{q_n}(\mathbb{C}) \otimes C(\mathbb{T})}$[2]. Using the identities we have found and the result in the discussion of chapter 4.2 two copies of the following 2-rank graph generate the irrational rotational algebra:

$$\tag{8}$$

[2]With the added technical detail that we have to take care of a construction of a new norm on this limit. But since it is not relevant for our discussion, we omit this here.

5 The ideal structure of graph algebras

The purposes of this section is to link the structure of a graph to the (possible) ideals of the generated C^*-algebra. First of all, we derive a condition for simplicity of the algebra. For graph algebras with ideals we will show that every ideal determines a unique subgraph. Using both lemma 39 and theorem 40 we derive the structure of this subgraph. At the end of the section we can formulate a 1-to-1 correspondence of certain subgraphs and ideals for graphs without cycles.

First two definitions. The first one is for notational purposes, used to compare vertices:

Definition 35 (Preorder on vertices)**.** *For $w, v \in E^0$, $v \leq w$ if there is a $\mu \in E^*$ such that $s(\mu) = w$ and $r(\mu) = v$.*

The second one is the criterion for simplicity:

Definition 36 (Cofinality)**.** *Let $E^{\leq\infty}$ be the collection of infinite paths together with the finite paths starting at a source. A graph E is cofinal if for every $\mu \in E^{\leq\infty}$ and every $v \in E^0$ there is a vertex w on μ such that $v \leq w$.*

Loosely speaking, from every infinite path and from every source we can reach the rest of the graph by following the arrows.

Theorem 37 (Simplicity)**.** *Let E be graph such that E is cofinal and every cycle has an entry. Then the graph algebra is a simple algebra.*

Proof. Take a graph algebra $C^*(S, P)$. Every ideal is the kernel of a representation $\pi : C^*(S, P) \to A \subseteq B(\mathscr{H})$. So it suffices to show injectivity for every representation π and to prove that $C^*(S, P)$ is unique. To do this we first prove $P_v \neq 0$ for all $v \in E^0$.
Take a non-zero P_v and assume for a moment P_v is not a source. Since $v = r(e)$ for some edge e we have $S_e S_e^* \neq 0$, implying $P_{s(e)} = S_e^* S_e \neq 0$. We can follow this procedure until we arrive at a source or form an infinite path. Either way, we form a path $\mu \in E^{\leq\infty}$. Take a $w \in E^0$. There is a path $\mu' \in E^*$ with $r(\mu') = w$ and $s(\mu')$ a vertex in μ. By construction, $P_{s(\mu')} = S_{\mu'}^* S_{\mu'} \neq 0$, which implies $S_{\mu'} S_{\mu'}^* \neq 0$. And since $P_w S_{\mu'} S_{\mu'}^* = S_{\mu'} S_{\mu'}^* \neq 0$. Hence all projections are non-zero (implying all partial isometries are non-zero too) and we can apply theorem 23 to see that $C^*(S, P) \cong C^*(E)$. Since every representation is linear and the multiplication rule, injectivity now follows. $\square$

Corollary 38. *Matrix algebras and B_0 are simple.*

Proof. The graphs of these C^*-algebras have no cycle. So we have to show they are cofinal. For B_0, $E^{\leq\infty}$ consists of all paths extending from $-\infty$ to an arbitrary vertex. For reference, use (4).
Let us choose μ the path finishing at v and an arbitrary $u_i \in E^0$. If u_i is on μ, then we are finished. If u_i is to the right of v, there is a path starting at v and finishing at u_i just be following the arrows connecting these two vertices. So B_0

is simple.
The matrix algebras do not have infinite paths, so $E^{\leq\infty}$ are all the paths starting at the source w. For an arbitrary vertex v, there is a path starting at w and finishing at v, which immediately shows that the graph is cofinal. $\square$

On the other hand, multimatrix algebra's or not simple; they have multiple sources which directly implies there is no path way to connect paths from one source to an other source.

The graph of the Töplitz-algebra is a good example of a graph which does satisfy the condition that every cycle has and entry but does not exhibit cofinality.

$$\begin{tikzcd} u \arrow[loop left, "e"] & v \arrow[l, "f"'] \end{tikzcd} \tag{9}$$

Since $I = \overline{\text{span}}\{P_v, S_e\}$ is an ideal of $\mathcal{T}$. The reader can verify this by examining the multiplication of the generators of I with the generators of $\mathcal{T}$ and then extending this result by linearity and continuity.

The converse of 37 is also true: a graph algebra is simple if and only if every cycle has an entry and the graph is cofinal. For a proof, see chapter 4 of [14].

In general, given a graph algebra $C^*(E)$ with a non-trivial ideal I, I determines a subgraph of E in the following way:

Lemma 39. *If I is an ideal of the graph algebra $C^*(S,P)$ on graph E the set $E\backslash H_I$ generates $C^*(S,P)/I$.*

Proof. The quotient-algebra $C^*(S,P)/I$ defines q, the quotient map , satisfying $q(I) = 0$. This proves that $\{q(P_v); v \notin H_I\}$ is a set of non-zero projections. This implies for $s(e) \notin H_I$ that $0 \neq q(P_{s(e)}) = q(S_e)^*q(S_e)$. Since $q(P_{r(e)}) = q(S_e)q(S_e)^* + \dots$ (some others projections) we can write down the intuitive statement $q(P_{r(e)}) \geq q(S_e)q(S_e)^* > 0$[3]. Other way around, if $r(e) \notin H_I$ then $q(S_e)q(S_e)^* = 0$ so $q(P_{s(e)}) = 0$.
So $E\backslash H_I = \{E^0\backslash H_I; s^{-1}(E^0\backslash H_I), r, s\}$ is a graph. One verifies easily that $\{q(P_v), q(S_e)\}$ generates the graph algebras of $E\backslash H_I$, of which some (uniqueness is not guaranteed!) are isomorphic to $C^*(S,P)/I$. $\square$

The defining property of H_I representing an ideal is that H_I is an a sense an isolated part of the graph. To make this concretely:

Theorem 40. *If I is a non-zero ideal in the graph algebra $C^*(S,P)$ of graph E then:*

- *if $w \in H_I$ and $w \leq v$, then $v \in H_I$ (H_I is hereditary);*

[3] This may seem like complete nonsense since we seem to pretend $q(S_e)q(S_e)^*$ and $q(P_{r(e)})$ are real numbers which they are not. However, in the theory of C^*-algebras '$\leq$' has a very precise meaning having to do with 'positivity' of operators. See also [10] section 9. Using this theory one easily sees $s(e) \notin H_I \Rightarrow q(P_{r(e)}) \neq 0$.

- $r^{-1}(v) \neq \emptyset$ *and* $\{s(e); r(e) = v\} \subset H_I$, *then* $v \in H_I$ *(H_I is saturated).*

Proof.

- Let $w \in H_I$ and μ a path with $s(\mu) = v$ and $r(\mu) = w$. We will proof that if $r(\mu_1) = w$ then $s(\mu_1) \in H_I$ from which follows that $v \in H_I$ (we can repeat the argument to show $s(\mu_2) \in H_I$, $s(\mu_3) \in H_I$ etc up to v).
  $P_w \in I$ which means $P_w S_{\mu_1} \in I$ because of the fact that P_w is an ideal. This means $P_w S_{\mu_1} = P_{r(\mu_1)} S_{\mu_1} = S_{\mu_1} \in I$.

- Let $v \in E^0$ such that $r^{-1}(v) \neq \emptyset$ and $\{s(e); r(e) = v\} \subset H_I$. Since for every e with $r(e) = v$ we know $S_e = S_e P_{s(e)} \in I$, $P_v = \sum\limits_{e;r(e)=v} S_e S_e^* \in I$.

$\square$

In fact, for graphs without cycles we can formulate the opposite of theorem 40.

Theorem 41. *For a graph E, if $H \subseteq E^0$ is both hereditary and saturated and E has no cycles, then H determines an ideal I.*

We start with defining an equivalence relation on $C^*(S, P)$. Let I_H be the $\mathbb{C}$-linear combination of $\{P_v; v \in H\}$ then $a \sim b$ iff there is an $i \in I_H$ such that $a + i = b$. The map $q : C^*(S, P) \to C^*(S, P)/\sim$ is the canonical projection map with $\ker(q) = I_H$.
Our goal will to prove that if H_I is hereditary and saturated $C^*(S, P)/\sim$ is a C^*-algebra in its own right. This means that the map q is a proper morphism of which I_H is the kernel, from which we will infer that I_H is an ideal in $C^*(S, P)$.

Proof. (of theorem 41)
From the construction of the quotient map we see that q is surjective. So if we can prove that q is a morphism then according to theorem 10.4 in [10] we can conclude $C^*(S, P)/\sim$ is a C^*-algebra. $q(a^*) = \{b \in C^*(S, P); \exists i \in I_H \quad b = a^* + i\}$. I_H is closed under involution, so $q(a^*) = \{b^* \in C^*(S, P); \exists i \in I_H \quad b = a + i\} = q(a)^*$.

$$q(a_1)q(a_2) = \{b_1 \cdot b_2 \in C^*(S, P); \exists i_1, i_2 \in I_H; a_1 = b_1 + i_1, a_2 = b_2 + i_2\}$$
$$q(a_1 a_2) = \{b \in C^*(S, P); \exists i \in I_H; a_1 \cdot a_2 = i + b\}$$

If $b_1 b_2 \in q(a_1)q(a_2)$ then $b_1 b_2 \in q(a_1 a_2)$ only if $a_1 a_2 = b_1 b_2 + i_1 b_2 + b_1 i_2 + i_1 i_2 = b_1 b_2 + i$ for some $i \in I_H$.
Assume $P_v \in H_I$. Since multiplication on a C^*-algebra is continuous in every argument we have:

$$P_v b_2 = P_v \cdot \left(\sum_{n=1}^{\infty} S_{\mu_n} S_{\nu_n}^* \right) = \sum_{n=1}^{\infty} P_v \cdot S_{\mu_n} S_{\nu_n}^*.$$

Assume S_{μ_n} is a projection. Then either $P_v S_{\mu_n} = P_v$ or zero. In the first case, examine $P_v S^*_{\nu_n}$. If again $S^*_{\nu_n}$ is a projection then $P_v S_{\mu_n} S^*_{\nu_n} \in I_H$. Otherwise, $P_v S^*_{\nu_n} = P_{r(e)} S_e S^*_{\nu_n}$ for every e with $r(e) = v$. If this is non-zero, then $s(e) = s(\nu_n)$.

We will show first that $I_H = \text{span}\{S_\alpha S^*_\beta; s(\alpha) = s(\beta) \in H\}$ is an algebraic ideal in $C^*(S, P)$.
Let $S_\alpha \in I_H$. The multiplication in any C^*-algebra is continuous in both arguments. So if $a \in C^*(S, P)$ is of the form as derived in theorem **??** then:

$$S_\alpha S^*_\beta \cdot a = S_\alpha S^*_\beta \cdot \left(\sum_{n=1}^{\infty} S_{\mu_n} S^*_{\nu_n} \right) = \sum_{n=1}^{\infty} S_\alpha S^*_\beta S_{\mu_n} S^*_{\nu_n}$$

Which motivates us to study multiplication of generators of I_H with those of $C^*(S, P)$.
Assume $S_\alpha S_\mu \neq 0$ for $S_\alpha \in I_H$. Then $s(\alpha) = r(\mu)$. Which means $v \leq s(\mu)$ implying $S_\mu \in I_H$ because H is hereditary.
If $S_\alpha S^*_\nu \neq 0$ then $s(\alpha) = s(\mu)$. If ν lies in H then we are finished. Otherwise, take the largest i such that either $r(\nu_i)$ or $s(\nu_i) \notin H$. Saturation now implies that if $r(\nu_i) \notin H$ then $s(\nu_i) \notin H$. If $S_\alpha S^*_\beta S_\mu S^*_\nu \neq 0$ then either there is a path β' with $\beta = \mu\beta'$ or μ' with $\mu = \beta\mu'$.
In the first case we see that $\beta \leq \beta'$. Because H is hereditary, β' is a path in H and $S_\alpha S^*_{\beta'} \in I_H$. We now answer the question is whether or not $S_\nu S_{\beta'} \in I_H$. If ν is a path in H we are done. Otherwise, choose the smallest i such that either $s(\nu_i) \notin H$ or $r(\nu_i) \notin H$. Because H is saturated, if $s(\nu_i) \notin H$ then $r(\nu_i) \notin H$. Using this result:

$$S_\nu S_{\beta'} = S_{\nu_1} \cdots S_{\nu_{i-1}} S_{\nu_i} \cdots S_{\beta'} = S_{\nu_1} \cdots S_{\nu_{i-1}} P_{s(\nu_{i-1})} P_{r(\nu_i)} S_{\nu_i} \cdots S_{\beta'}$$

$P_{s(\nu_{i-1})} \in I_H$ and definitely $P_{r(\nu_i)} \notin I_H$. So this expression is zero, which yields a contradiction. Following the reasoning all back from where we started we conclude:
$S_\alpha S^*_\beta S_\mu S^*_\nu \in I_H$ if there is some β' with $\beta = \mu\beta'$.
The case in which there is some μ' with $\mu = \beta\mu'$ yields the same result, which proves I_H is an algebraic right ideal. The proof that I_H is an algebraic left ideal is similar.

Furthermore we claim that if $I \in \overline{I_H}$ is approximated by a number of elements $I_n \in I_H$ then (again due to the continuity of multiplication):

$$\sum_{i=1}^{n} I_i \sum_{k=1}^{\infty} S_{\mu_k} S^*_{\nu_k} = \sum_{k=1}^{\infty} \sum_{i=1}^{n} I_i S_{\mu_k} S^*_{\nu_k} \Rightarrow$$
$$\sum_{i=1}^{\infty} I_i \sum_{k=1}^{\infty} S_{\mu_k} S^*_{\nu_k} = \sum_{i,k=1}^{\infty} I_i S_{\mu_k} S^*_{\nu_k} \in I.$$

Same for left multiplication, so I is ideal. □

Final remark: As elaborated on in chapter 4 of [14], one can formulate a general theorem of which we shall briefly state the results. If every cycle in a graph E has a return path then there is a bijection $I \leftrightarrow H$ of closed, two sided ideals and hereditary and saturated subsets. As in 39, $C^*(E/H) \cong C^*(E)/I$ and $C^*(H) \cong p_H I p_H$ for the formal sum of projections in H which satisfies $p_H S_\mu S_\nu^* = S_\mu S_\nu^*$ if $r(\mu) \in H$ and zero otherwise.

6 The K-theory of graph algebras

The K-groups of a graph algebra have a particular elegant form for they can be calculated solely using the properties of the underlying graph. The classical result of Raeburn in [14] chapter 7 is restricted to the use of graphs without sources, we here quote a more general theorem found in [18], proposition 2, which applies to graphs with sources and a countable amount of vertices:

Theorem 42. *For a directed graph E let V_E be the collection of all the vertices which receive at least one edge but only finitely many. Let $\mathbb{Z}V_E$ and $\mathbb{Z}E^0$ be the free abelian groups on free generators V_E and E^0. Define $\Delta_E : \mathbb{Z}V_E \to \mathbb{Z}E^0$ on generators as follows:*

$$\Delta_E(v) = \sum_{e \in E^1; r(e)=v} s(e) - v.$$

Then $K_0(C^*(E)) \cong \mathit{coker}(\Delta_E)$ and $K_1(C^*(E)) \cong \ker(\Delta_E)$.

Notice that this formula extends to graphs with a countable number of vertices, not necessarily finite.

Example 43 (Matrix algebras)**.**

A familiar example of a graph with a source is $M_2(\mathbb{C})$. V_E consists of one vertex only and E^0 of two. So in terms of the generator (1) of $\mathbb{Z}V_E$ on one side and the generators $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have:

$$\Delta_E((1)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Such that $K_0(M_2(\mathbb{C}) = \text{coker}(\Delta_E) = \mathbb{Z}^2/\mathbb{Z} = \mathbb{Z}$ and $K_1(M_2(\mathbb{C})) = \ker(\Delta_E) = 0$, a familiar result. Using the representation of $M_n(\mathbb{C})$ in (3) we can extend our result to any matrix algebra. We have exactly $n-1$ vertices receiving one edge, from the vertex to the left of it, and one source. So Δ_E is a $n \times n-1$ matrix defined on the generators of $\mathbb{Z}^{n-1}$ as:

$$\Delta_E e_i = e_{i-1} - e_i \quad \Rightarrow \quad \Delta_E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \quad \Rightarrow$$

$$\Delta_E \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 - k_1 \\ k_3 - k_2 \\ \vdots \\ k_{n-1} - k_{n-2} \end{pmatrix}$$

So the range of Δ_E is $\mathbb{Z}^{n-1}$, hence $K_0(M_n(\mathbb{C})) = \mathbb{Z}$ and the kernel of $\Delta_E = 0$.

Example 44 ($SU_q(2)$)**.**

This is now very easy! We do not have to bother with sources or sinks since the graph contains none. Δ_E is defined by the following relations:

$$\Delta_E \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \Delta_E \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow$$

$$\Delta_E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $K_0(SU_q(2)) = K_1(SU_q(2)) = \mathbb{Z}$.

In the paper [3], D. Evans has extended the calculating of K-theory to higher rank graph algebras without sources. We quote the central result:

Theorem 45 (K-theory of higher rank graph algebras)**.** *Let $E \times F$ be the direct product of two 1-graphs forming a 2-graph without sources. They define the C^*-algebra $C^*(\Lambda)$. Let Δ_E and Δ_F be the maps defined by 42 on E, respectively F. Then:*

$$K_0(C^*(\Lambda)) = \mathit{coker}(\Delta_E, \Delta_F) \oplus \ker \begin{pmatrix} -\Delta_F \\ \Delta_E \end{pmatrix} \tag{10}$$

$$K_1(C^*(\Lambda)) = \ker(\Delta_E, \Delta_F)/\mathit{im} \begin{pmatrix} -\Delta_F \\ \Delta_E \end{pmatrix}. \tag{11}$$

Where we view $(\Delta_E, \Delta_F) : \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 \to \mathbb{Z}\Lambda^0$ and $\begin{pmatrix} -\Delta_F \\ \Delta_E \end{pmatrix} : \mathbb{Z}\Lambda^0 \to \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0$ as group morphisms on resp. $\mathbb{Z}\Lambda^0$ and $\mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0$, the 1-skeleton of the graph.

Example 46 (A_θ)**.**

Let us define E as the graph of (4) and F as (5) then $C^*(E \times F \oplus E \times F) = C^*(E \times F) \oplus C^*(E \times F) \cong A_\theta$ (see proposition 26) and:

$$\Delta_E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Delta_F = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We should evaluate these maps on the free generators of $\mathbb{Z}\Lambda^0$, which implies that we should evaluate the formulas for all $k \in \mathbb{Z}^{\mathbb{N}}$ with a finite number of non-zero terms. We have

$$\Delta_E \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 - k_1 \\ k_3 - k_2 \\ \vdots \\ k_n - k_{n-1} \\ -k_n \\ 0 \\ \vdots \end{pmatrix}$$

as element of $\mathbb{Z}^{\mathbb{N}}$ with at the right-hand side $n+1$ non-zero terms. This means:

$$(\Delta_E, \Delta_F) \left[\begin{pmatrix} k_1 \\ \vdots \\ k_n \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} n_1 \\ \vdots \\ n_n \\ 0 \\ \vdots \end{pmatrix} \right] = \begin{pmatrix} k_1 + n_1 \\ k_2 - k_1 + n_2 \\ k_3 - k_2 + n_3 \\ \vdots \\ k_n - k_{n+1} + n_n \\ -k_n \\ 0 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} -\Delta_F \\ \Delta_E \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \\ 0 \\ \vdots \end{pmatrix} = \left[\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} -k_1 \\ -k_2 + k_1 \\ -k_3 + k_2 \\ \vdots \\ -k_n + k_{n-1} \\ k_n \\ 0 \\ \vdots \end{pmatrix} \right].$$

From these expressions we can read that $\operatorname{coker}(\Delta_E, \Delta_F) = \mathbb{Z}^{n+1}/\mathbb{Z}^n = \mathbb{Z}$ and $\ker \begin{pmatrix} -\Delta_F \\ \Delta_E \end{pmatrix} = 0$. Therefore, $K_0(A_\theta) = K_0(C^*(E) \times C^*(F)) \oplus K_0(C^*(E) \times C^*(F)) = \mathbb{Z} \oplus \mathbb{Z}$.
Lastly, $\ker(\Delta_E, \Delta_F) = \mathbb{Z}^{2n}$ while $\operatorname{im} \begin{pmatrix} -\Delta_F \\ \Delta_E \end{pmatrix}$ is only $\mathbb{Z}^{2n-1}$; therefore $K_0(A_\theta) = \mathbb{Z}^{2n}/\mathbb{Z}^{2n-1} \oplus \mathbb{Z}^{2n}/\mathbb{Z}^{2n-1} = \mathbb{Z} \oplus \mathbb{Z}$.

For C^*-algebras there are many equivalences which lead to the same K-groups. For instance homotopy equivalence and the excision property. Also, a very natural one is Morita-equivalence. It would be a nice endeavor to explore what the conditions on a graph would be to create equivalent graph algebras. The interested reader is referred to Bates and Pask for their paper [1] to see some results.

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