The Perturbation Semigroup of the Noncommutative Torus

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Contents

1 Introduction

This thesis studies the perturbation semigroup of the noncommutative torus, and may be read in two ways.

You may take this thesis as an example implementing the machinery introduced in [11] by Ali Chamseddine, Alain Connes and Walter van Suijlekom. In this article the authors introduce the perturbation semigroup, and show that it gives rise to an action, or perturbation. They do not use the name "perturbation semigroup", but rather refer to its elements as "inner fluctuations without the first order condition". These elements are obtained when looking at the conditions for a well known group, called the unitaries, and dropping one of these conditions. They show that the action that can be obtained for the unitaries can be extended to the perturbation semigroup, and that this action gives rise to some interesting results. When one considers the perturbation semigroup of different algebras, this action yields different results (sometimes of great interest to physics, like in [12]). Just as [13] investigates the perturbation semigroup of matrix algebras, we investigate the perturbation semigroup of the noncommutative torus.

The second way to percieve this thesis is as a tribute to the noncommutative torus. The noncommutative torus is an algebra, (coming in multiple versions, as we will see,) introduced to the field of noncommutative geometry by Rieffel in [1] and Connes in [2]. Noncommutative geometry is a branch of mathematics that for one part is concerned with generalising the study of algebras of functions to noncommutative algebras. For instance, the noncommutative torus is a generalisation of the algebra of functions on the normal, commutative torus. Just as the normal torus is a good example in topology (the torus is homeomorphic to the surface of a coffee mug) and calculus (the surface of the torus is $4\pi^2 rR$), the noncommutative torus is a hands-on example of an algebra in noncommutative geometry. In this thesis we will make the concept of the noncommutative torus more precise, and prove and apply rules with which this algebra is manageable without deep knowledge of noncommutative geometry. In this view, the fact that we end up calculating a semigroup of perturbations of this algebra is a mere application of the calculating rules which lie at the heart of this text.

Our approach is meant to explain the noncommutative torus without the need of much prior knowledge. However, the reader is assumed to have some very basic knowledge of functional analysis. We will start by sketching the results of Chamseddine, Connes and Van Suijlekom, for these results have motivated our research. We then introduce an algebra called the polynomial noncommutative torus, for which we need analysis nor topology. Later on, we will introduce an extended algebra called the smooth noncommutative torus, which will at the same time introduce the reader to Fréchet spaces. We subsequently consider the noncommutative torus as an algebras of operators, and this will include some elementary functional analysis. We will conclude this thesis with spectral triples, a complicated subject, yet useful for understanding noncommutative geometry. The new concepts are aimed to fall into place in the end.

In good consideration, this thesis is a case study for noncommutative geometry, in which we aim to make some basic concepts accessible to undergraduate students.

1.1 Notational Remarks

We work over the field $\mathbb{C}$ of complex numbers. By a sequence we can simply mean a function $\mathbb{N} \to \mathbb{C}$, or (interpreting the word sequence more broadly) a function $\mathbb{Z}^n \to \mathbb{C}$. In this thesis n will only take the values 2 and 4. When there is no chance for confusion we will leave out this specification. For instance, a sequence could be denoted by $(a_{kl})_{k,l\in\mathbb{Z}}$, and equivalently by (a_{kl}). By 'basis' of a vector space we mean a linearly independent subset which spans this space in terms of finite linear combinations. Such a basis is also called a 'Hamel basis'. By a 'Schauder Basis' we mean[1] a sequence $(e_n)_{n\in\mathbb{N}}$ such that every x has a unique expansion $x = \sum_{n=0}^{\infty} x_n e_n$. All limits are to infinity, so $\lim_m$ must be understood as $\lim_{m\to\infty}$.

[1] We follow the notation in [3] and differ from [4].

2 Preliminaries

2.1 Spectral Triples

We start this thesis with the same concept as we will end with, the concept of a spectral triple. We state here the definition to be used for later purposes, not to fully explore. Knowing at least of the existence of these triples, allows to understand the reason why the perturbation semigroup was introduced. The reader is advised not to pay attention to the details, but rather see this as a glimpse of the destination far up ahead.

Spectral triples where introduced by Alain Connes and play a crucial rôle in noncommutative geometry. There are numerous definitions throughout different articles, all more or less equivalent. The definition below is mainly inspired by that of Connes in [5], but adapted slightly to fit our context. See also [6].

Definition 2.1. *A* ***spectral triple****, also called a* ***(compact) noncommutative geometry*** *is a triple*

$$(A, H, D)$$

consisting of a unital involutive algebra A, a Hilbert space H and an unbouded operator D in H, such that the following hold:

(i) The algebra A is faithfully represented on $B(H)$.

(ii) We have $D = D^*$ and $(D + \mu)^{-1}$ is a compact operator for all $\mu \notin \mathbb{R}$.

(iii) The commutator $[D, a] = Da - aD$ is bounded for any $a \in A$.

We need to explain this definition further before we can talk about its interpretation. For instance, we have not yet specified what we mean by aD or Da. Intended to clear this up we state three preliminary definitions, which are also needed for future reference.

Definition 2.2. *Let H be a Hilbert space, and let $B(H)$ denote the algebra of bounded operators on H, as explained in Section A.2.*

(i) An ***unbounded operator*** *D* ***in*** *H is a subset $\mathrm{dom}(D)$ of H – called the domain of D – together with an operator $D : \mathrm{dom}(D) \to H$.*

(ii) A function $\rho : A \to B(H)$ is called a ***representation of algebras*** *if $\rho(A)$ is an algebra and ρ is an algebra homomorphism.*

*(iii) For A to be **faithfully represented** on $B(H)$ there must exist an injective representation of algebras $\rho : A \to B(H)$.*

More details on unbounded operators can be found in [9]. The reader who is unfamiliar with algebras or algebra homomorphisms is advised to read Section A.1 at this point.

In this thesis we usually treat the representation $\rho : A \to B(H)$ as an identification, meaning that we make no notational distinction between a and $\rho(a)$. This also means that we write $A \subseteq B(H)$, or in other words, A is an algebra of bounded operators on a Hilbert space. Since a and D both operate on the same space (ignoring for now that D is not defined at every point) we can define their compositions, which are written like aD and Da. This is why, in part (iii) of Definition 2.1 we may write $Da - aD$.

The interpretation of spectral triples becomes more clear when the algebra A is commutative, as explained in [7]. In that case we can find a manifold M (a geometric space which locally looks like Euclidean space) such that A becomes the algebra of smooth functions on M. This is called the spectral reconstruction theorem, proven by Connes in [8]. Importantly, the geometry of M is encoded in the spectrum of the operator D, hence the name *spectral* triple. When A is no longer commutative, we cannot find such a space M. The approach of noncommutative geometry is that we still treat a spectral triple as if it encodes a space. For instance, we use formalisms designed for function algebras on A, even though functions commute and the elements of A do not. This technique has helped to explain particle physics with quantum phenomena in a geometric model, with recent progress made in [12].

Now that we know a little bit about spectral triples, we set the stage for the perturbation semigroup.

2.2 Background on Perturbations

In [10] the authors Chamseddine and Connes introduced a new principle called the 'Spectral Action Principle' associated to a spectral triple. The setting for this principle is as follows. We know that a commutative spectral triple (where A is commutative) has all the data needed to construct the underlying geometric space M. Of the operator D only its spectrum is needed, and we will see later that the spectrum of D is just the set of eigenvalues of D. Noncommutative spectral triples (A, H, D) hold the data needed for geometric 'spaces' that behave according to the rules of Quantum Mechanics.

As before, the data derived from D to construct this quantum-like space is uniquely determined by its eigenvalues. Here enters the algebra A, which is conceived to be an algebra of operators on the same Hilbert space as D. The algebra has a subgroup of unitaries defined as

$$\mathcal{U}(A) := \{u \in A \mid u^*u = uu^* = 1\}\,.$$

An important fact used by Chamseddine and Connes is one we know from linear algebra, namely that for every unitary $u \in \mathcal{U}(A)$ the operator uDu^* has the same eigenvalues as D. So without changing the underlying geometric space, we have freedom in choosing D up to this fluctuation:

$$D \mapsto uDu^*\,. \tag{1}$$

This means we have a gauge theory, a physical model in which we can fluctuate a parameter without any physical difference. Think about the physical parameter 'energy' which can only be determined up to a constant.

Chamseddine and Connes generalised the fluctuation (1) to something called an internal fluctuation, which can be written as

$$D \mapsto \sum_j a_j D b_j\,, \tag{2}$$

when $a_j, b_j \in A$ are constrained to some conditions. In 2013, in [11], Ali Chamseddine, Alain Connes together with Walter van Suijlekom narrowed down these conditions. This gave rise to a generalised version of the internal fluctuation, called the inner perturbation, which can still be written like (2). Moreover, the set of all inner perturbations forms a semigroup, $\mathrm{Pert}(A)$, which generalises the group $\mathcal{U}(A)$. This semigroup we will call the 'perturbation semigroup'.

2.3 Perturbation Semigroup

Definition 2.3. *For an involutive algebra A, the* ***opposite algebra*** *A^{op} is given by the same vector space A endowed with the same involution but a different product $*$ such that $a * b = ba$.*

When talking about $a \in A$ as being an element of A^{op} we will denote it as a^{op}. As a consequence, the map $A \to A^{op}$, $a \mapsto a^{op}$ is an involutive linear bijection that satisfies

$$(ab)^{op} = b^{op} * a^{op}\,, \qquad\qquad (a, b \in A).$$

It must be stressed that for noncommutative algebras the map $a \mapsto a^{op}$ is not an isomorphism, so in general $A \not\cong A^{op}$. From now on we will restrict ourselves to unital *-algebras A. In this case A^{op} is also unital, with unit 1^{op}.

We now introduce the perturbation semigroup. This is a subset of $A \otimes A^{op}$, the algebraic tensor product of A with its opposite algebra. For more details on tensor products, see [24] or [25].

Definition 2.4. *The **perturbation semigroup** of A is given by* $\mathrm{Pert}(A) :=$

$$\left\{ \sum_j a_j \otimes b_j^{op} \in A \otimes A^{op} \;\middle|\; \sum_j a_j b_j = 1, \sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{*op} \right\},$$

where 1 is the unit in A, and the sums over j are finite. The first condition, $\sum a_j b_j = 1$, is called the normalisation condition; the second condition, $\sum a_j \otimes b_j^{op} = \sum b_j^* \otimes a_j^{*op}$, is called the self-adjointness condition.

This definition was given in [11], and there the authors proved the following: $\mathrm{Pert}(A)$ is a monoid (a semigroup with a unit); for all unitaries $u \in \mathcal{U}(A)$ we have $u \otimes u^{*op} \in \mathrm{Pert}(A)$. In his bachelor thesis, [13], Niels Neumann investigated the perturbation semigroup of several algebras. Among other results he calculated the perturbation semigroup of the algebra of square $n \times n$-matrices, $\mathrm{Pert}(M_n(\mathbb{C}))$. We will do more or less the same for the algebra of the noncommutative torus, which will be introduced shortly.

The perturbation semigroup is a generalisation of the group of unitaries in the following sense. We have an injective multiplicative function $\gamma : \mathcal{U}(A) \to \mathrm{Pert}(A)$ defined by $\gamma(u) := u \otimes u^{*op}$, as was proven in [11]. The reader is encouraged to check this as an exercise. Moreover, when we define the perturbation of D for an element $c = \sum_j a_j \otimes b_j^{op}$ as

$$D'(c) := \sum_j a_j D b_j \,,$$

then we get for all $u \in \mathcal{U}(A)$ that

$$D'(\gamma(u)) = u D u^* \,.$$

Therefore we can rightfully say that (2) is a generalisation of (1).

2.4 Two General Results

Before we introduce the noncommutative torus, and restrict ourselves to a specific algebra, we prove two new results that are valid for any unital *-algebra A. The first gives us insight in how the unitaries are embedded in the perturbation semigroup.

Proposition 2.1. *The embedding* $\gamma : \mathcal{U}(A) \to \mathrm{Pert}(A)$ *defined earlier has the image*

$$\gamma(\mathcal{U}(A)) = \{a \otimes b^{op} \in \mathrm{Pert}(A) \mid ba = 1\} \,. \tag{3}$$

Proof. Let $u \in \mathcal{U}(A)$ be arbitrary, then $\gamma(u) = a \otimes b^{op} \in \mathrm{Pert}(A)$ when we define $a := u$ and $b := u^*$. Because u is unitary, we have in particular that $ba = u^*u = 1$.

Let $a \otimes b^{op} \in \mathrm{Pert}(A)$ be such that $ba = 1$. The normalisation condition gives $ab = 1$ and the self-adjointness condition gives that $a \otimes b^{op} = b^* \otimes a^{*op}$, which in particular implies $a^* = b$. Define $u := a$, this gives

$$u^*u = a^*a = ba = 1 \qquad \text{and} \qquad uu^* = ab = 1\,.$$

Therefore $u \in \mathcal{U}(A)$ and so $a \otimes b^{op} = u \otimes u^{*op} \in \gamma(\mathcal{U}(A))$. □

As we can see from (3) the elements of $\mathrm{Pert}(A)$ which correspond to unitaries are homogeneous elements (elements that can be written without a sum) satisfying an additional condition ($ba = 1$) which looks like the normalisation condition ($ab = 1$).

The second result of this section will be of great importance throughout this thesis, because it allows us to write the perturbation semigroup in a concrete way. The inspiration for this came from [13], where multiple algebras are introduced for which the perturbation semigroup is calculated. All treated algebras A satisfy $A^{op} \cong A$, and this allows the authors to write down new expressions for $\mathrm{Pert}(A)$. We will prove a generalisation of this technique of rewriting $\mathrm{Pert}(A)$ when the isomorphism between A^{op} and A is very similar to its own inverse.

Proposition 2.2. *Let* $\phi : A^{op} \to A$ *be an algebra isomorphism such that* $\phi((\phi(a^{op})^{op}) = a$*. Then the semigroup* $\mathrm{Pert}(A)$ *is isomorphic to*

$$\left\{ \sum_j a_j \otimes b_j \in A \otimes A \;\middle|\; \sum_j a_j \phi(b_j^{op}) = 1, \sum_j a_j \otimes b_j = \sum_j \phi(b_j^{*op}) \otimes \phi(a_j^{*op}) \right\}.$$

Proof.
Using Definition 2.4 we find that $\text{Pert}(A)$ is equal to

$$\left\{ \sum_j a_j \otimes b_j^{op} \in A \otimes A^{op} \;\middle|\; \sum_j a_j b_j = 1, \sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{*op} \right\} .$$

It follows easily that $(id \otimes \phi) : A_\lambda \otimes A_\lambda^{op} \to A_\lambda \otimes A_\lambda$ is an isomorphism. Therefore $\text{Pert}(A)$ is isomorphic to its image under $(id \otimes \phi)$, which is

$$\left\{ \sum_j a_j \otimes \phi(b_j^{op}) \in A \otimes A \;\middle|\; \sum_j a_j b_j = 1, \sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{*op} \right\} .$$

In particular, the map $(id \otimes \phi)$ is injective. Therefore

$$\sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{*op} \iff \sum_j a_j \otimes \phi(b_j^{op}) = \sum_j b_j^* \otimes \phi(a_j^{*op}) .$$

When we combine the last two results, it follows that $\text{Pert}(A) \cong$

$$\left\{ \sum_j a_j \otimes \phi(b_j^{op}) \in A \otimes A \;\middle|\; \sum_j a_j b_j = 1, \sum_j a_j \otimes \phi(b_j^{op}) = \sum_j b_j^* \otimes \phi(a_j^{*op}) \right\} .$$

Replacing b_j everywhere with $\phi(b_j^{op})$, and using that the function $\phi(\cdot^{op})$ is its own inverse, gives the claim immediately. □

This result will be applied to the algebra of the noncommutative torus.

3 Polynomial Noncommutative Torus

We have talked about unital *-algebras in a general way, without giving any examples. Throughout this thesis we will see two main examples of unital *-algebras, both can be called (the algebra of) the noncommutative torus. In this section we introduce the polynomial noncommutative torus, which can be related to a space of polynomials. In next section, Section 4, we introduce the smooth commutative torus, which contains the polynomial version and can be related to a space of smooth functions. Just like smooth functions on a compact set can be approximated by polynomials, the elements of the smooth noncommutative torus can be approximated by elements of the polynomial noncommutative torus. This is used much in Sections 4 and 5, and is the reason we introduce the polynomial noncommutative torus first.

A third algebra which is called the noncommutative torus is a so-called C*-algebra which contains the smooth noncommutative torus (and therefore also the polynomial noncommutative torus), and was the first version of the noncommutative torus ever studied. This was done by Marc Rieffel in [1], in 1981. However, this C*-algebra proved unfit for our specific applications. We therefore restrict ourselves to the two versions of the noncommutative torus stated earlier.

We will now introduce the polynomial noncommutative torus and describe its structure. This will lead us, at the end of this section, to write an expression for the perturbation semigroup of the polynomial noncommutative torus.

3.1 Definitions

Definition 3.1. *Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. The* ***polynomial noncommutative 2-torus*** *A_λ is given as the involutive algebra generated by unitaries u, v, with*

$$vu = \lambda uv\,. \tag{4}$$

Since A_λ is generated by u and v, all linear combinations of finite products of u,v,u^* and v^* are in A_λ. Since u and v are defined to be unitary, we have $u^{-1} = u^*$ and $v^{-1} = v^*$. Together with the commutation relation (4) this ensures that a generic element $a \in A_\lambda$ can be written as

$$a = \sum_{k,l\in\mathbb{Z}} a_{kl} u^k v^l\,, \tag{5}$$

where $(a_{kl}) \in C_c(\mathbb{Z}^2)$. Here $C_c(\mathbb{Z}^2)$ is the space of sequences $\mathbb{Z}^2 \to \mathbb{C}$ with a finite number of nonzero terms, for convenience called **finite sequences**. As proven in Proposition A.2 of the appendix, $\{u^k v^l | k, l \in \mathbb{Z}\}$ is in fact a basis for A_λ. Therefore the finite sequence (a_{kl}) such that (5) holds is unique.[2] We can thus describe every element of A_λ by its coefficients. This description plays a sufficiently big rôle in this text, that we introduce an explicit notation based on (5).

Notation 1. *For every $a \in A_\lambda$ we denote by a_{kl} the unique coefficient in front of $u^k v^l$.*

Since we have a basis for A_λ, and $a \mapsto a^{op}$ is a linear bijection, we immediately have a basis for A_λ^{op}, namely:

$$\{(u^k v^l)^{op} | k, l \in \mathbb{Z}\} = \{(v^{op})^l (u^{op})^k | k, l \in \mathbb{Z}\}\,.$$

[2] We will sometimes write $a_{k,l}$ for clarity.

This means that every element of A_λ^{op} is generated by u^{op} and v^{op}. Keeping this in mind we define a linear map from A_λ^{op} back towards A_λ, which interchanges u and v. We only need to define this map on the two elements u^{op} and v^{op}.

Definition 3.2. *Define the function $\phi : A_\lambda^{op} \to A_\lambda$ determined by $\phi(u^{op}) := v$, $\phi(v^{op}) := u$, and extended to all $a^{op} \in A_\lambda^{op}$ as algebra homomorphism.*

Since we let ϕ extend as algebra homomorphism we get $\phi((v^{op})^l(u^{op})^k) := \phi(u^{op})^l\phi(v^{op})^k = u^lv^k$, using multiplicativity. Using linearity as well gives an expression for $\phi(a^{op})$ for all a^{op}. It will turn out that this specific ϕ satisfies the conditions of Proposition 2.2. This fact will help us write down $\mathrm{Pert}(A_\lambda)$ in a simpler way. First, we will prove a few calculating rules that will be applied throughout this thesis.

3.2 Coefficient Sequences

Lemma 3.1 (Rules for coefficients)**.** *For any $a, b \in A_\lambda$ and $k, l \in \mathbb{Z}$ we have:*

(i) $(ab)_{kl} = \sum_{m,n} a_{mn}b_{k-m,l-n}\lambda^{n(k-m)}$

(ii) $(a^*)_{kl} = \overline{a_{-k,-l}}\lambda^{kl}$

(iii) $\phi(a^{op})_{kl} = a_{lk}$

(iv) $1_{00} = 1$ *and* $1_{kl} = 0$ *for* $(k, l) \neq (0, 0)$

Here we used Notation 1 with subsequently ab, a^*, $\phi(a^{op})$ and 1 instead of a.

Proof. When one needs to prove $x_{kl} = f(k, l)$, it is sufficient to show $x = \sum f(k, l)u^kv^l$. Also note that $v^lu^k = \lambda^{kl}u^kv^l$.

(i)

$$\begin{aligned}
ab &= \left(\sum_{m,n} a_{mn}u^mv^n\right)\left(\sum_{k,l} b_{kl}u^kv^l\right) \\
&= \sum_{m,n,k,l} a_{mn}b_{kl}u^mv^nu^kv^l \\
&= \sum_{k,l,m,n} a_{mn}b_{kl}\lambda^{nk}u^{m+k}v^{n+l} \\
&= \sum_{k,l,m,n} a_{mn}b_{k-m,l-n}\lambda^{n(k-m)}u^kv^l \\
&= \sum_{k,l}\left(\sum_{m,n} a_{mn}b_{k-m,l-n}\lambda^{n(k-m)}\right)u^kv^l
\end{aligned}$$

(ii)

$$\begin{aligned} a^* &= \left(\sum a_{kl} u^k v^l\right)^* = \sum \overline{a_{kl}} (u^k v^l)^* = \sum \overline{a_{kl}} (v^*)^l (u^*)^k \\ &= \sum \overline{a_{kl}} v^{-l} u^{-k} \ = \sum \overline{a_{-k,-l}} v^l u^k = \sum \overline{a_{-k,-l}} \lambda^{kl} u^k v^l \end{aligned}$$

(iii)

$$\begin{aligned} \phi(a^{op}) &= \phi\left(\left(\sum a_{kl} u^k v^l\right)^{op}\right) = \phi\left(\sum a_{kl} (v^{op})^l (u^{op})^k\right) \\ &= \sum a_{kl} \phi(v^{op})^l \phi(u^{op})^k \ = \sum a_{kl} u^l v^k \\ &= \sum a_{lk} u^k v^l \end{aligned}$$

(iv)

$$\begin{aligned} 1 &= u^0 v^0 + 0 \\ &= 1\, u^0 v^0 + \sum_{(k,l)\neq 0} 0\, u^k v^l \end{aligned}$$

Here 1 and 0 on the first line are elements of A_λ and 1 and 0 on the second line are (complex) numbers.

□

The multiplication rule for coefficients in (i) looks very much like a discrete convolution product. The difference is the phase factor $\lambda^{n(k-m)}$, which can not be taken out of the sum. Convolution products with this kind of factor are called twisted convolution products, and are encountered in discrete Fourier analysis. See [18] for comparison. The reason for this link to Fourier analysis can be found in viewing the transformation $a \mapsto (a_{kl})$ as a Fourier transformation, see [14].

Writing elements of the noncommutative torus in terms of the basis $\{u^k v^l\}$ will be done throughout the rest of this thesis, and therefore the coefficients in this basis will be of great importance. It is for this reason that the four rules for coefficients of Lemma 3.1 form the foundation on which we build the rest of this thesis. As a first application of our rules for coefficients we prove that the function ϕ satisfies the criteria in Proposition 2.2.

Proposition 3.2. *The map ϕ is an isomorphism of unital *-algebras, and therefore $A_\lambda^{op} \cong A_\lambda$. Moreover, $\phi(\phi(a^{op})^{op}) = a$.*

Proof. We have linearity and multiplicativity by definition. Moreover, ϕ maps basis vectors uniquely to basis vectors, since

$$\phi((u^k v^l)^{op}) = u^l v^k ,$$

and therefore it is bijective. Furthermore, using Lemma 3.1, we have:

- $(\phi(a^{op})^*)_{kl} = \overline{\phi(a^{op})_{-k,-l}}\lambda^{kl} = \overline{a_{-l,-k}}\lambda^{lk} = (a^*)_{lk} = \phi((a^*)^{op})_{kl}$
  $= \phi((a^{op})^*)_{kl} ,$
- $\phi(1^{op})_{kl} = 1_{lk} = 1_{kl} .$

Therefore ϕ also conserves involution and the unit, so it is an isomorphism.

The last claim follows from the definition of ϕ, since we have $\phi(\phi(u^{op})^{op}) = \phi(v^{op}) = u$ and $\phi(\phi(v^{op})^{op}) = \phi(u^{op}) = v$, so the claim holds for the generators u and v. If $\mu \in \mathbb{C}$ and the claim holds for a and b, it also holds for $a+\mu b$ by linearity of ϕ. By multiplicativity also $\phi(\phi((ab)^{op})^{op}) = \phi((\phi(b^{op})\phi(a^{op}))^{op}) = \phi(\phi(a^{op})^{op})\phi(\phi(b^{op})^{op})$. Therefore, by extension of ϕ as algebra homomorphism, the claim holds for all $a \in A_\lambda$. □

3.3 Rewriting Pert(A_λ)

The perturbation semigroup Pert(A_λ) is a subset of the algebraic tensor product $A_\lambda \otimes A_\lambda^{op}$. Hinting that we will later on use $A_\lambda^{op} \cong A_\lambda$ and Propostion 2.2, we are going to look at elements of $A_\lambda \otimes A_\lambda$. Every $c \in A_\lambda \otimes A_\lambda$ can be written as a finite sum $c = \sum_j a_j \otimes b_j$ where $a_j, b_j \in A_\lambda$. Using the decomposition into basis vectors of (5) we get

$$\begin{aligned}\sum_j a_j \otimes b_j &= \sum_j \sum_{k,l} (a_j)_{kl} \sum_{m,n} (b_j)_{mn} (u^k v^l) \otimes (u^m v^n) \\ &= \sum_{k,l,m,n} \left(\sum_j (a_j)_{kl} (b_j)_{mn} \right) (u^k v^l) \otimes (u^m v^n) . \end{aligned} \tag{6}$$

Trusting that this will not confuse the reader, we from now on drop the brackets around $u^k v^l$ and $u^m v^n$, as well as the commas in k, l, m, n under the summation sign. One can check that the elements $u^k v^l \otimes u^m v^n$ form a basis of $A_\lambda \otimes A_\lambda$. For the coefficients of an element c in terms of this basis we introduce a special name.

Definition 3.3. *For an arbitrary element $c = \sum_j a_j \otimes b_j \in A_\lambda \otimes A_\lambda$, the corresponding **coefficient sequence** (c_{klmn}) is defined by*

$$c_{klmn} := \sum_j (a_j)_{kl} (b_j)_{mn} . \tag{7}$$

Since we are dealing with the *polynomial* noncommutative torus and an *algebraic* tensor product, it follows that the coefficient sequence is always a finite sequence. To put it more concretely: $(c_{klmn}) \in C_c(\mathbb{Z}^4)$. Combining everything we have so far, we can write c as follows:

$$c = \sum_{klmn} c_{klmn} u^k v^l \otimes u^m v^n . \tag{8}$$

We can give some structure to the space $C_c(\mathbb{Z}^4)$ of finite sequences, by defining a twisted convolution product $*_\lambda$ as:[3]

$$(c_{klmn}) *_\lambda (d_{klmn}) := \left(\sum_{pqrs} c_{pqrs} d_{k-p,l-q,m-r,n-s} \lambda^{q(k-p)+s(m-r)} \right)_{k,l,m,n \in \mathbb{Z}} .$$

The definition of $*_\lambda$ may seem to appear from thin air, but it is chosen such that the function $c \mapsto (c_{klmn})$ is multiplicative, which we will now prove.

Proposition 3.3. *The function $A_\lambda \otimes A_\lambda \to C_c(\mathbb{Z}^4)$ defined by $c \mapsto (c_{klmn})$ is a unital algebra isomorphism, when we equip $C_c(\mathbb{Z}^4)$ with the product $*_\lambda$.*

Proof. We need to show that $c \mapsto (c_{klmn})$ is well defined. We already discussed why coefficient sequences always are in $C_c(\mathbb{Z}^4)$. Assume $c = \sum a_j \otimes b_j = \sum a'_j \otimes b'_j$. We want to show that the definition of c_{klmn} gives the same result for these different sums. Using (6) and assembling the sums over k, l, m and n gives

$$\sum_{klmn} \left(\sum_j (a_j)_{kl} (b_j)_{mn} - \sum_j (a'_j)_{kl} (b'_j)_{mn} \right) u^k v^l \otimes u^m v^n = 0 .$$

By independence of the basis vectors $u^k v^l \otimes u^m v^n$ this results in

$$\sum_j (a_j)_{kl} (b_j)_{mn} = \sum_j (a'_j)_{kl} (b'_j)_{mn} .$$

This means $c \mapsto (c_{klmn})$ is unambiguously defined.

Linearity follows directly from the observation that $a \mapsto a_{kl}$ is a linear map for all $k, l \in \mathbb{Z}$.

We show that $c \mapsto (c_{klmn})$ is injective and surjective. If the coefficient sequence of c is zero, then we get $c = \sum 0\, u^k v^l \otimes u^m v^n = 0$. If (c_{klmn}) is an arbitrary sequence in $C_c(\mathbb{Z}^4)$, then there are finitely many pairs $(k, l) \in \mathbb{Z}^2$ such that $c_{klmn} \neq 0$ for certain m, n. Interpret the indices j, for now, as

[3] Again, see [18] for more theory and background of twisted convolutions.

elements of $\{(k,l) \mid c_{klmn} \neq 0\}$. Define $(a_j)_{kl} := \delta_{j=(k,l)}$ and $(b_{(k,l)})_{mn} := c_{klmn}$. One can check that these definitions give $a_j, b_j \in A_\lambda$ such that (7) holds. Therefore $c \mapsto (c_{klmn})$ is bijective.

To see how multiplication of $c, d \in A_\lambda \otimes A_\lambda$ can be written in terms of coefficient sequences, we use the multiplication rule for $a, b \in A_\lambda$ from Lemma 3.1. Write $c = \sum_i a_i \otimes b_i$ and $d = \sum_j a'_j \otimes b'_j$. By definition of the tensor product of algebras we have $cd = \sum_{i,j} a_i a'_j \otimes b_i b'_j$. If $((cd)_{klmn})$ denotes the coefficient sequence of cd then

$$
\begin{aligned}
(cd)_{klmn} &= \sum_{i,j} (a_i a'_j)_{kl} (b_i b'_j)_{mn} \\
&= \sum_{i,j} \sum_{p,q} (a_i)_{pq} (a'_j)_{k-p,l-q} \lambda^{q(k-p)} \sum_{r,s} (b_i)_{rs} (b'_j)_{m-r,n-s} \lambda^{s(m-r)} \\
&= \sum_{pqrs} \sum_{i} (a_i)_{pq} (b_i)_{rs} \sum_{j} (a'_j)_{k-p,l-q} (b'_j)_{m-r,n-s} \lambda^{q(k-p)+s(m-r)} \\
&= \sum_{pqrs} c_{pqrs} d_{k-p,l-q,m-r,n-s} \lambda^{q(k-p)+s(m-r)} .
\end{aligned}
$$

This agrees with the definition of $*_\lambda$. Therefore $((cd)_{klmn}) = (c_{klmn}) *_\lambda (d_{klmn})$ as sequences in $C_c(\mathbb{Z}^4)$, and that settles multiplicativity.

The unit in $A_\lambda \otimes A_\lambda$ is $1 \otimes 1$. Its coefficient sequence is given by $(1_{kl} 1_{mn})$, which is 1 if $k, l, m, n = 0$ and 0 otherwise. A quick calculation shows that $(c_{klmn}) *_\lambda (1_{kl} 1_{mn}) = (1_{kl} 1_{mn}) *_\lambda (c_{klmn}) = (c_{klmn})$, which implies that $(1_{kl} 1_{mn})$ is the unit in $C_c(\mathbb{Z}^4)$.

Since A_λ is a unital algebra and we have a map $A_\lambda \to C_c(\mathbb{Z}^4)$ which is linear, bijective, multiplicative and unital, the space $C_c(\mathbb{Z}^4)$ must be a unital algebra as well. We conclude that $c \mapsto (c_{klmn})$ is a unital algebra isomorphism. □

By the above result we can view $A_\lambda \otimes A_\lambda$ as a space of sequences, $C_c(\mathbb{Z}^4)$. We already had an isomorphism between A_λ^{op} and A_λ, and therefore we may view $\mathrm{Pert}(A_\lambda)$ as a subset of $A_\lambda \otimes A_\lambda$. By these statements it is obvious that we can see the perturbation semigroup as a set of sequences. We will now make this more precise, and find a useful expression for the perturbation semigroup.

Theorem 3.4. *The perturbation semigroup is isomorphic to*

$$
\mathrm{Pert}(A_\lambda) \cong \left\{ (c_{klmn}) \in C_c(\mathbb{Z}^4) \;\middle|\; \begin{array}{l} \sum_{m,n} c_{m,l-n,n,k-m} \lambda^{(k-m)(l-n)} = 1_{kl} \\ c_{klmn} = \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn} \end{array} \right\}
$$

when we equip $C_c(\mathbb{Z}^4)$ with the product $*_\lambda$.

Proof. Notice that the conditions for Proposition 2.2 are satisfied when we take $A = A_\lambda$ and ϕ as the isomorphism defined in Definition 3.2. We thus have $\mathrm{Pert}(A_\lambda) \cong$

$$\left\{ \sum_j a_j \otimes b_j \in A_\lambda \otimes A_\lambda \;\middle|\; \begin{array}{l} \sum_j a_j \phi(b_j^{op}) = 1\,, \\ \sum_j a_j \otimes b_j = \sum_j \phi(b_j^{*op}) \otimes \phi(a_j^{*op}) \end{array} \right\} . \tag{9}$$

The function $c \mapsto (c_{klmn})$ is a unital algebra isomorphism by Proposition 3.3, so in particular preserves the structure of the unital semigroup (9). We are done when we show that $c \mapsto (c_{klmn})$ maps (9) precisely to the right-hand side of Theorem 3.4. For this purpose let $\sum_j a_j \otimes b_j \in A_\lambda \otimes A_\lambda$ be arbitrary, and let (c_{klmn}) be its coefficient sequence.

We will rewrite the first condition of (9) – also called the normalisation condition – using Definition 3.3. The following are equivalent:

$$\begin{aligned} \sum_j a_j \phi(b_j^{op}) &= 1 \\ \sum_{k,l} \left(\sum_j (a_j \phi(b_j^{op}))_{kl} \right) (u^k v^l) &= 1 \\ \sum_j (a_j \phi(b_j^{op}))_{kl} &= 1_{kl} \\ \sum_{m,n} \left(\sum_j (a_j)_{mn} \phi(b_j^{op})_{k-m,l-n} \right) \lambda^{n(k-m)} &= 1_{kl} \\ \sum_{m,n} c_{m,l-n,n,k-m} \lambda^{(k-m)(l-n)} = \sum_{m,n} c_{m,n,l-n,k-m} \lambda^{n(k-m)} &= 1_{kl} \end{aligned} \tag{10}$$

The same applies to the self-adjointness condition. The following are equivalent:

$$\begin{aligned} \sum_j a_j \otimes b_j &= \sum_j \phi(b_j^{*op}) \otimes \phi(a_j^{*op}) \\ \sum_{klmn} \left(\sum_j (a_j)_{kl} (b_j)_{mn} \right) u^k v^l \otimes u^m v^n &= \sum_{klmn} \left(\sum_j (b_j^*)_{lk} (a_j^*)_{nm} \right) u^k v^l \otimes u^m v^n \\ \sum_j (a_j)_{kl} (b_j)_{mn} &= \sum_j (b_j^*)_{lk} (a_j^*)_{nm} \\ \sum_j (a_j)_{kl} (b_j)_{mn} &= \sum_j \overline{(a_j)_{-n,-m} (b_j)_{-l,-k}} \lambda^{kl+nm} \\ c_{klmn} &= \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn} \end{aligned} \tag{11}$$

Therefore the element $\sum_j a_j \otimes b_j$ is in (9) if and only if its coefficient sequence (c_{klmn}) satisfies (10) and (11). □

We have just seen that the normalisation condition is equivalent to $\sum_{m,n} c_{m,l-n,n,k-m} \lambda^{(k-m)(l-n)} = 1_{kl}$, when (c_{klmn}) is the corresponding coefficient sequence. We shall from now on refer to this formula as the normalisation condition as well. In the same way we will refer to $c_{klmn} = \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn}$ as the self-adjointness condition.

This covers the second section, in which we have introduced the polynomial noncommutative torus. We described the structure of this algebra in terms of sequences of basis coefficients. Finally, we found a new expression for the perturbation semigroup in terms of similar sequences called 'coefficient sequences'. All of these concepts will return in the next section, in which we talk about the smooth noncommutative torus.

4 Smooth Noncommutative Torus

4.1 A Fréchet Space

We introduce the smooth version of the noncommutative torus, which will turn out to be a rich extension of the polynomial version. Almost every result we had so far on the polynomial noncommutative torus can be generalized to the so-called *smooth noncommutative torus*. However, proving these generalisations can be very subtle, and requires more machinery than before. As a start, we introduce the concepts of Fréchet spaces.

Definition 4.1. *A* ***Fréchet space*** *is a topological vector space with a non-decreasing family of seminorms* $\|\cdot\|_{(0)} \leqslant \|\cdot\|_{(1)} \leqslant \|\cdot\|_{(2)} \leqslant \dots$ *which induce a topology in the following way:*

$$a_i \to a \iff \|a_i - a\|_{(p)} \to 0 \text{ for all } p\,,$$

and the space is complete in this topology.

This definition is formulated differently to what one may find in the literature, but this formulation is better suited for our purposes. To see that it is equivalent we refer to [15].

Definition 4.2. *The topology of a Fréchet space is called the* ***Fréchet topology****.*

Definition 4.3. *A* ***Fréchet algebra*** *is a Fréchet space which is also an algebra, in which multiplication is jointly continuous in the Fréchet topology.*

We will define the smooth noncommutative torus as an extension of A_λ. For this we will need a topology on A_λ. Recall the Schwarz space,

$$\mathcal{S}(\mathbb{Z}^2) = \left\{ (a_{kl}) \;\middle|\; \forall p \in \mathbb{N} : \sup_{k,l} \{|a_{kl}|(|k|+1)^p(|l|+1)^p\} < \infty \right\} ,$$

of rapidly decreasing sequences. Loosely speaking, we want to look at torus-elements where the coefficient sequence is no longer finite, but instead rapidly decreasing. For us to rigorize this concept, we will define a family of norms on A_λ, inspired by the definition of $\mathcal{S}$. For every $p \in \mathbb{N}$ and $a \in A_\lambda$, recall that (a_{kl}) is the sequence of basis-coefficients of a. Define the p-th norm of a, and at the same time the p-th norm of (a_{kl}), by

$$\|a\|_{(p)} := \|(a_{kl})\|_{(p)} := \sup_{k,l} |a_{kl}|(|k|+1)^p(|l|+1)^p \,. \tag{12}$$

It is straightforward to check that this is a norm. Moreover, the norms increase in size every step, because $(|k|+1)^p \leqslant (|k|+1)^{p+1}$, and therefore

$$\sup_{kl} |a_{kl}|(|k|+1)^p(|l|+1)^p \leqslant \sup_{kl} |a_{kl}|(|k|+1)^{p+1}(|l|+1)^{p+1} ,$$

and thus $\|a\|_{(p)} \leqslant \|a\|_{(p+1)}$. Using these norms, we will now define the Fréchet topology on A_λ. Its name will be justified in a while.

Definition 4.4. *The* ***Fréchet topology*** *on A_λ is the topology defined by the above family of seminorms $\|\cdot\|_{(p)}$. That is, a sequence (a_i) in A_λ converges to $a \in A_\lambda$ iff*

$$\|a_i - a\|_{(p)} \to 0 \text{ for every } p \in \mathbb{N} .$$

Remark 1. *A sequence (a_i) in A_λ is Cauchy in the Fréchet topology iff it is Cauchy in the p-th norm for every p.*

Definition 4.5. *The* ***smooth noncommutative 2-torus*** *$\mathcal{A}_\lambda$ is the completion of A_λ in the Fréchet topology.*

The word 'completion' should be interpreted in terms of Fréchet spaces. So $\mathcal{A}_\lambda$ is not only a topological space but in fact a Fréchet space, carrying all extended structure of the Fréchet space A_λ. For instance, for a sequence (a_i) in A_λ converging to $a \in \mathcal{A}_\lambda$, we have by definition $\|a\|_{(p)} = \lim_i \|a_i\|_{(p)}$.

Remark 2. *The polynomial noncommutatitive torus A_λ is a dense subalgebra of $\mathcal{A}_\lambda$.*

By the results we have gathered $\mathcal{A}_\lambda$ is a complete topological vector space with a topology induced by a nondecreasing family of norms, called the Fréchet norms. Therefore it is by Definition 4.1 a *Fréchet space*. In general, a topology on A is called a *Fréchet topology* if it makes A into a Fréchet space. This justifies the name of Definition 4.4.

The following lemma generalises the formula in (12).

Lemma 4.1. *For all elements $a = \sum_{k,l} a_{kl} u^k v^l$ of the smooth noncommutative torus $\mathcal{A}_\lambda$, we have*

$$\|a\|_{(p)} = \sup_{k,l} |a_{kl}|(|k|+1)^p(|l|+1)^p \,.$$

Proof. Since $\mathcal{A}_\lambda$ is the completion of A_λ, extending the Fréchet norms by continuity to $\mathcal{A}_\lambda$ gives

$$\|a\|_{(p)} = \left\| \lim_m \sum_{|k|,|l|\leqslant m} a_{kl} u^k v^l \right\|_{(p)} = \lim_m \left\| \sum_{|k|,|l|\leqslant m} a_{kl} u^k v^l \right\|_{(p)} .$$

Now we use (12) on the partial sums, and obtain

$$\|a\|_{(p)} = \lim_m \sup_{|k|,|l|\leqslant m} |a_{kl}|(|k|+1)^p(|l|+1)^p \,,$$

which implies the result. □

We can now give a more elegant description of $\mathcal{A}_\lambda$ at which we hinted at the start of this section. Other authors (see for instance [14]) often take this description as the definition of the smooth noncommutative torus.

Theorem 4.2. *We can write the smooth noncommutative 2-torus as*

$$\mathcal{A}_\lambda = \left\{ \sum_{k,l} a_{kl} u^k v^l \;\middle|\; (a_{kl}) \in \mathcal{S}(\mathbb{Z}^2) \right\} .$$

Proof. We need to do two things:

(i) For all $(a_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$ we prove that $(\sum_{|k|,|l|\leqslant m} a_{kl} u^k v^l)_{m\in\mathbb{N}}$ is a Cauchy sequence, so $\sum_{k,l} a_{kl} u^k v^l \in \mathcal{A}_\lambda$.

(ii) For every $a \in \mathcal{A}_\lambda$ we construct a sequence $(a_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$ such that $a = \sum_{k,l} a_{kl} u^k v^l$.

For (i), let $(a_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$ and define for $m \in \mathbb{N}$ the partial sum up to m:

$$a^{(m)} := \sum_{|k|,|l| \leqslant m} a_{kl} u^k v^l \in A_\lambda \, .$$

Using the p^{th} Fréchet norm gives

$$\begin{aligned} \left\| a^{(m)} - a^{(n)} \right\|_{(p)} &= \left\| \sum_{n < |k|,|l| \leqslant m} a_{kl} u^k v^l \right\|_{(p)} \\ &= \sup_{n < |k|,|l| \leqslant m} |a_{kl}| (|k|+1)^p (|l|+1)^p \\ &\leqslant \sup_{n < |k|,|l| \leqslant m} |a_{kl}| \frac{(|k|+1)^{p+1}}{n+1} \frac{(|l|+1)^{p+1}}{n+1} \\ &\leqslant \frac{1}{(n+1)^2} \sup_{k,l} |a_{kl}| (|k|+1)^{p+1} (|l|+1)^{p+1} \, . \end{aligned}$$

Since $(a_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$ and $1/(n+1)^2 \to 0$ for $n \to \infty$, we find that $\left\| a^{(m)} - a^{(n)} \right\|_{(p)} \to 0$ for $m, n \to \infty$. So $(a^{(m)})$ is Cauchy in every p-norm, hence Cauchy in the Fréchet topology, and therefore convergent by completeness of the smooth torus. We thus have $\sum_{k,l} a_{kl} u^k v^l = \lim_m a^{(m)} \in \mathcal{A}_\lambda$, which settles (i).

For (ii), suppose $a \in \mathcal{A}_\lambda$. We fix a sequence $(b^{(m)})$ in A_λ such that $b^{(m)} \to a$ in the Fréchet topology. It follows that $(b^{(m)})$ is Cauchy with respect to every p-norm. Using this fact for $p = 0$ gives

$$\sup_{k,l} |b_{kl}^{(m)} - b_{kl}^{(n)}| \to 0 \qquad \text{for} \qquad m, n \to \infty \, .$$

Notice that we used Notation 1, which is valid since $b^{(m)}$ is in the polynomial torus. It follows that the sequences $(b_{kl}^{(m)})_{m \in \mathbb{N}}$ are Cauchy in $\mathbb{C}$ for every k and l. Therefore we may define

$$a_{kl} := \lim_m b_{kl}^{(m)} \, . \tag{13}$$

If $c^{(m)}$ is another sequence in A_λ, convergent to a, then $(b^{(m)} - c^{(m)})$ converges to zero. Using $\left\| b^{(m)} - c^{(m)} \right\|_{(0)} \to 0$ and (12) we get for all k and l that $\lim_m b_{kl}^{(m)} = \lim_m c_{kl}^{(m)}$. Therefore a_{kl} in (13) is uniquely defined. Now let

$p \in \mathbb{N}$ and use (13) to find:

$$\begin{aligned}\sup_{k,l} |a_{kl}|(|k|+1)^p(|l|+1)^p &= \sup_{k,l} \lim_m |b_{kl}^{(m)}|(|k|+1)^p(|l|+1)^p \\ &\leqslant \lim_m \sup_{k,l} |b_{kl}^{(m)}|(|k|+1)^p(|l|+1)^p \\ &= \lim_m \left\|b^{(m)}\right\|_{(p)} \\ &= \|a\|_{(p)} \ .\end{aligned}$$

We know that $\|a\|_{(p)}$ is finite since $a \in \mathcal{A}_\lambda$. So $(a_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$ and as we have seen in part (i) this implies $\sum_{k,l} a_{kl}u^k v^l \in \mathcal{A}_\lambda$.

We use Lemma 4.1 together with some basic rules for limits to find

$$\begin{aligned}\left\|b^{(m)} - \sum_{k,l} a_{kl}u^k v^l\right\|_{(p)} &= \sup_{k,l} \left|b_{kl}^{(m)} - a_{kl}\right| (|k|+1)^p(|l|+1)^p \\ &= \sup_{k,l} \left|b_{kl}^{(m)} - \lim_n b_{kl}^{(n)}\right| (|k|+1)^p(|l|+1)^p \\ &= \sup_{k,l} \lim_n \left|b_{kl}^{(m)} - b_{kl}^{(n)}\right| (|k|+1)^p(|l|+1)^p \\ &\leqslant \lim_n \left\|b^{(m)} - b^{(n)}\right\|_{(p)} \ .\end{aligned}$$

Since we have assumed $(b^{(m)})$ is Cauchy, we arrive at the final result that $\lim_m \left\|b^{(m)} - \sum a_{kl}u^k v^l\right\|_{(p)} \leqslant \lim_{m,n} \left\|b^{(m)} - b^{(n)}\right\|_{(p)} = 0$. This concludes part (ii), since we have

$$a = \lim_m b^{(m)} = \sum_{k,l} a_{kl}u^k v^l \ .$$

□

Corollary 1. *The Schwarz space $\mathcal{S}(\mathbb{Z}^2)$ is complete.*

Proof. Suppose (ξ_i) is a Cauchy sequence in $\mathcal{S}(\mathbb{Z}^2)$. Define $a_i := \sum \xi_i(k,l)u^k v^l$ for every $i \in \mathbb{N}$. By the previous theorem $a_i \in \mathcal{A}_\lambda$, and (a_i) is Cauchy since $\|a_i - a_j\|_{(p)} = \|\xi_i - \xi_j\|_{(p)} \to 0$. Since $\mathcal{A}_\lambda$ is complete by definition, there exists a limit $a \in \mathcal{A}_\lambda$, which can be written like $a = \sum a_{kl}u^k v^l$. Now $\|\xi_i - (a_{kl})\|_{(p)} = \|a_i - a\|_{(p)} \to 0$, which means that (ξ_i) converges. □

Corollary 2. *The set $\{u^k v^l | k,l \in \mathbb{Z}\}$ is a Schauder basis for $\mathcal{A}_\lambda$, in the sense that every $a \in \mathcal{A}_\lambda$ has a unique sequence (a_{kl}) such that $a = \sum a_{kl}u^k v^l$.*

Proof. The fact that every element of $\mathcal{A}_\lambda$ has such a sequence follows directly from Theorem 4.2. For the proof of uniqueness, assume $\sum a_{kl}u^k v^l = \sum b_{k,l}u^k v^l$. This implies $\sum(a_{kl} - b_{kl})u^k v^l = 0$. Take the 0^{th} Frechet norm and use (12) (which is allowed by Lemma 4.1) to find $\sup_{k,l} |a_{kl} - b_{kl}| = 0$, which in turn implies that $a_{kl} = b_{kl}$ for all $k, l \in \mathbb{Z}$. □

This result justifies the following notation, which extends Notation 1 to elements of $\mathcal{A}_\lambda$. On A_λ both notations agree.

Notation 2. *Write a_{kl} for the coefficient of $a \in \mathcal{A}_\lambda$ in front of $u^k v^l$.*

In other literature the maps $a \mapsto a_{kl}$ are called the coordinate functionals, and the set of coordinate functionals is called the dual basis. Some authors (like [16]) only talk about a Schauder basis if the coordinate functionals are continuous. For a Frechet space the coordinate functionals are always continuous, (see [3],) but rather than using this general fact we prove it for our special case.

Lemma 4.3. *For a convergent sequence (a_i) in $\mathcal{A}_\lambda$ we have $(\lim_i a_i)_{kl} = \lim_i (a_i)_{kl}$.*

Proof. By definition of the Frechet topology, $\|a_i - \lim a_i\|_{(p)} \to 0$ for every $p \in \mathbb{N}$. We take $p = 0$ and employ (12) to find $\sup_{k,l} |(a_i)_{kl} - (\lim_i a_i)_{kl}| \to 0$. Thus for all k and l we get $|(a_i)_{kl} - (\lim_i a_i)_{kl}| \to 0$, or equivalently: $\lim_i (a_i)_{kl} = (\lim_i a_i)_{kl}$. □

4.2 Continuous Extensions

We want to use the results we have gathered in Section 3, and prove generalizations of these results on the larger algebra $\mathcal{A}_\lambda$. The main idea behind this is that A_λ is a dense subset of $\mathcal{A}_\lambda$. It is well known within the field of functional analysis that a bounded operator on a dense subset can be uniquely extended to the whole space. This claim can be strengthened, for which we need the notion of 'Cauchy continuous'.

Definition 4.6. *Let X, Y be metric spaces.*[4] *A function $f : X \to Y$ is called **Cauchy continuous** iff, for all Cauchy sequences (x_i) in X, the sequence $(f(x_i))$ is Cauchy in Y.*

[4]This includes spaces with a topology defined by a suitable family of seminorms, like A_λ together with Definition 4.4.

It can easily be checked that every Cauchy continuous function on A_λ extends to a unique continuous function on $\mathcal{A}_\lambda$.

Equip A_λ^{op} with the same norms as A_λ: $\|a^{op}\|_{(p)} := \|a\|_{(p)}$. It is then obvious that the completion of A_λ^{op} equals $\mathcal{A}_\lambda^{op}$, and that this is a Fréchet space as well.

With the purpose of extending them, we prove that the functions multiplication, involution and ϕ are Cauchy continuous. By ϕ we mean the function defined in Definition 3.2.

Proposition 4.4. *The following functions, defined for elements of the polynomial noncommutative torus, are Cauchy continuous with respect to the Fréchet topology.*

(i) $(a,b) \mapsto ab \quad (A_\lambda \times A_\lambda \to A_\lambda)$

(ii) $a \mapsto a^* \quad (A_\lambda \to A_\lambda)$

(iii) $\phi : A_\lambda^{op} \to A_\lambda$

Proof of (i). We let $p \in \mathbb{N}$, $a, b \in A_\lambda$ and try to estimate the p^{th} Fréchet norm of ab as follows.

$$\begin{aligned}
\|ab\|_{(p)} &= \sup_{k,l} \left| \sum_{m,n} a_{mn} b_{k-m,l-n} \lambda^{n(k-m)} \right| (|k|+1)^p (|l|+1)^p \\
&\leqslant \sup_{k,l} \sum_{m,n} |a_{mn}||b_{k-m,l-n}|(|k|+1)^p(|l|+1)^p \\
&\leqslant \sum_{m,n} |a_{mn}| \sup_{k,l} |b_{k-m,l-n}|(|k|+1)^p(|l|+1)^p \\
&= \sum_{m,n} |a_{mn}| \sup_{k,l} |b_{kl}|(|k+m|+1)^p(|l+n|+1)^p
\end{aligned}$$

We now use the following estimation, which holds for all $p \in \mathbb{N}, k, m \in \mathbb{Z}$:

$$(|k+m|+1)^p \leqslant 2^p(p+1)(|k|+1)^p(|m|+1)^p. \tag{14}$$

The proof of (14) is immediate when $k+m=0$, and can otherwise be done using $|k+m|+1 \leqslant 2|k+m|$ and the binomial theorem. Using this estimation twice gives

$$\|ab\|_{(p)} \leqslant 2^{2p}(p+1)^2 \sum_{m,n} |a_{mn}|(|m|+1)^p(|n|+1)^p \sup_{k,l} |b_{kl}|(|k|+1)^p(|l|+1)^p\,.$$

We still need to estimate the above series over m and n, for which we use the following lemma. The proof comes afterwards.

Lemma 4.5. *There exists a constant $C > 0$ such that for all p and $a \in \mathcal{A}_\lambda$:*

$$\sum_{m,n} |a_{mn}|(|m|+1)^p(|n|+1)^p \leqslant C \left\|a\right\|_{(p+2)} .$$

Putting this together with our inequality for $\left\|ab\right\|_{(p)}$ gives

$$\left\|ab\right\|_{(p)} \leqslant 2^{2p}(p+1)^2 C \left\|a\right\|_{(p+2)} \left\|b\right\|_{(p)} .$$

We are left with a standard proof to show multiplication is Cauchy continuous. Let (a_i, b_i) be a Cauchy sequence in $A_\lambda \times A_\lambda$. From the product topology follows that both (a_i) and (b_i) are Cauchy in A_λ. They are therefore Cauchy in every Fréchet norm. Now proving that the sequence of products $(a_i b_i)$ is Cauchy in every Fréchet norm is straightforward:

$$\begin{aligned}
\left\|a_i b_i - a_j b_j\right\|_{(p)} &= \left\|a_i b_i - a_i b_j + a_i b_j - a_j b_j\right\|_{(p)} \\
&\leqslant \left\|a_i(b_i - b_j)\right\|_{(p)} + \left\|(a_i - a_j) b_j\right\|_{(p)} \\
&\leqslant 2^{2p}(p+1)^2 C \big(\left\|a_i\right\|_{(p+2)} \left\|b_i - b_j\right\|_{(p)} \\
&\qquad\qquad + \left\|a_i - a_j\right\|_{(p+2)} \left\|b_j\right\|_{(p)} \big) .
\end{aligned}$$

Since (a_i) and (b_j) are bounded in every Fréchet norm, and we have $\left\|b_i - b_j\right\|_{(p)} \to 0$ as well as $\left\|a_i - a_j\right\|_{(p+2)} \to 0$, we get $\left\|a_i b_i - a_j b_j\right\|_{(p)} \to 0$. Therefore $(a_i b_i)$ is Cauchy in every Fréchet norm, and thus Cauchy in A_λ. □

Proof of (ii). This is much simpler, since we have

$$\begin{aligned}
\left\|a^*\right\|_{(p)} &= \sup_{k,l} |\overline{a_{-k,-l}} \lambda^{kl}|(|k|+1)^p(|l|+1)^p \\
&= \sup_{k,l} |a_{-k,-l}|(|-k|+1)^p(|-l|+1)^p \\
&= \left\|a\right\|_{(p)} .
\end{aligned}$$

If (a_i) is Cauchy, then $\left\|a_i^* - a_j^*\right\|_{(p)} = \left\|(a_i - a_j)^*\right\|_{(p)} = \left\|a_i - a_j\right\|_{(p)} \to 0$ and therefore (a_i^*) is Cauchy as well. □

Proof of (iii). By the rules for coefficients (Lemma 3.1) we have

$$\left\|\phi(a^{op})\right\|_{(p)} = \sup_{k,l} |a_{lk}|(|k|+1)^p(|l|+1)^p = \left\|a\right\|_{(p)} .$$

We use the topology on A_λ^{op} inherited from A_λ, so if (a_i^{op}) is Cauchy in A_λ^{op}, then (a_i) is Cauchy in A_λ, and so for all p: $\left\|a_i - a_j\right\|_{(p)} \to 0$. Then also $\left\|\phi(a_i^{op}) - \phi(a_j^{op})\right\|_{(p)} \to 0$, and so $(\phi(a_i^{op}))$ is Cauchy in A_λ. □

We have proven Theorem 4.4 once we are done with the proof of the lemma we postponed.

Proof of Lemma 4.5. We estimate

$$\begin{aligned}
\sum_{m,n} |a_{mn}|(|m|+1)^p(|n|+1)^p &= \sum_{m,n} |a_{mn}| \frac{(|m|+1)^{p+2}(|n|+1)^{p+2}}{(|m|+1)^2(|n|+1)^2} \\
&\leqslant \sum_{m,n} \frac{\sup_{k,l} |a_{kl}|(|k|+1)^{p+2}(|l|+1)^{p+2}}{(|m|+1)^2(|n|+1)^2} \\
&= \|a\|_{(p+2)} \sum_{m,n} \frac{1}{(|m|+1)^2(|n|+1)^2} \\
&= \|a\|_{(p+2)} \left(\sum_m \frac{1}{(|m|+1)^2} \right)^2 \\
&\equiv C \|a\|_{(p+2)} ,
\end{aligned}$$

where we define the constant $C := (\sum 1/(|m|+1)^2)^2 = (\frac{1}{3}\pi^2 - 1)^2$. □

Now that the proof of Theorem 4.4 is complete, we can extend all structure of the involutive algebra A_λ.

Corollary 3. *The smooth noncommutative 2-torus $\mathcal{A}_\lambda$ is a Fréchet algebra.*

Proof. Remember that $\mathcal{A}_\lambda$ is already a Fréchet *space*. It has algebra structure extended from A_λ. Since multiplication on A_λ is jointly continuous, (this was proven in Proposition 4.4(i)) multiplication is also jointly continuous on $\mathcal{A}_\lambda$. □

It follows that $\mathcal{A}_\lambda^{op}$ is a Fréchet algebra as well.

4.3 Extending Results to the Smooth Torus

We can now easily prove the same rules for coefficients we had in Lemma 3.1. Notice that the unit 1 of A_λ is also the unit of $\mathcal{A}_\lambda$, and Lemma 3.1(iv) does not have to be extended.

Lemma 4.6 (Extended rules for coefficients)**.** *For $a, b \in \mathcal{A}_\lambda$ we also have:*

(i) $(ab)_{kl} = \sum_{m,n} a_{mn} b_{k-m,l-n} \lambda^{n(k-m)}$

(ii) $(a^*)_{kl} \;=\; \overline{a_{-k,-l}} \lambda^{kl}$

(iii) $\phi(a^{op})_{kl} \;=\; a_{lk}$

Here we used Notation 2 with subsequently ab, a^* and $\phi(a^{op})$ instead of a.

Proof. (i) We would like to make the same calculations as in the proof of Lemma 3.1(i), but now for $a, b \in \mathcal{A}_\lambda$. In that proof the sums over m, n and k, l are interchanged. This is allowed when

$$\sum_{k,l,m,n} a_{mn} b_{kl} u^{m+k} v^{n+l}$$

converges absolutely with respect to every p-norm. We use (14) twice to find

$$\sum_{klmn} \left\| a_{mn} b_{kl} u^{m+k} v^{n+l} \right\|_{(p)} = \sum_{klmn} |a_{mn} b_{kl}| (|m+k|+1)^p (|n+l|+1)^p$$
$$\leqslant 2^{2p}(p+1)^2 \sum_{m,n} |a_{mn}| (|m|+1)^p (|n|+1)^p \sum_{k,l} |b_{kl}| (|k|+1)^p (|l|+1)^p$$
$$\leqslant 2^{2p}(p+1)^2 C^2 \left\| a \right\|_{(p+2)} \left\| b \right\|_{(p+1)} < \infty \, .$$

We employed Lemma 4.5 in the last line. By absolute convergence we have (with respect to the Frechet topology)

$$\sum_{k,l} \sum_{m,n} a_{mn} b_{kl} u^{m+k} v^{n+l} = \sum_{m,n} \sum_{k,l} a_{mn} b_{kl} u^{m+k} v^{n+l} \, .$$

All other calculations in the proof of Lemma 3.1(i) also hold for $a, b \in \mathcal{A}_\lambda$. By these calculations result (i) follows.

(ii) The function $\mathcal{A}_\lambda \to \mathbb{C}$ defined by $a \mapsto (a^*)_{kl} - \overline{a_{-k,-l}} \lambda^{kl}$ is a composition of continuous functions by Lemma 4.3 and Proposition 4.4, and therefore continuous. Since this function is zero on A_λ, which is a dense subset of $\mathcal{A}_\lambda$, it is also zero on $\mathcal{A}_\lambda$. This gives result (ii).

(iii) The method of part (ii) also applies here, now applied to the function $\mathcal{A}_\lambda \to \mathbb{C}$ defined by $a \mapsto \phi(a^{op})_{kl} - a_{lk}$. Observe $a \mapsto a^{op}$ is continuous by definition of the topology on $\mathcal{A}_\lambda^{op}$. Therefore (again using Lemma 4.3 and Proposition 4.4) we find that the function $a \mapsto \phi(a^{op})_{kl} - a_{lk}$ is continuous. It is zero on a dense subset and therefore zero on all $\mathcal{A}_\lambda$.

□

Proposition 4.7. *The map $\phi : \mathcal{A}_\lambda^{op} \to \mathcal{A}_\lambda$ is an isomorphism of Fréchet algebras. Moreover, $\phi(\phi(a^{op})^{op}) = a$ for all $a \in \mathcal{A}_\lambda$.*

Proof. The map ϕ is defined by continuous extension, which means $\phi(\lim a_i) := \lim \phi(a_i)$. This implies that ϕ is still linear. Since involution is continuous, we find that ϕ is still involutive, for

$$\phi((\lim_i a_i)^*) = \phi(\lim_i a_i^*) = \lim_i \phi(a_i)^* = (\lim_i \phi(a_i))^* = \phi(\lim_i a_i)^* \,.$$

Since multiplication is continuous, a similar argument gives that ϕ is still multiplicative. As we have seen in the proof of 4.4(iii), the function $\phi : A_\lambda \to A_\lambda$ is not only Cauchy continuous, but conserves every Frechet norm. This means that for every p, ϕ is an isometry between normed spaces: $(A_\lambda^{op}, \|.\|_{(p)}) \to (A_\lambda, \|.\|_{(p)})$. Its continuous extension is therefore also an isometry between $(\mathcal{A}_\lambda^{op}, \|.\|_{(p)})$ and $(\mathcal{A}_\lambda, \|.\|_{(p)})$.

Left to show is surjectivity. For this let $\lim_i a_i \in \mathcal{A}_\lambda$ with $a_i \in A_\lambda$. Then for every i we have a $b_i^{op} \in A_\lambda$ such that $\phi(b_i^{op}) = a_i$. We have $\|b_i - b_j\|_{(p)} = \|a_i - a_j\|_{(p)} \to 0$ since ϕ is a linear isometry. Therefore (b_i) is Cauchy, and by completeness of $\mathcal{A}_\lambda$ it converges. Once again by continuous extension we get

$$\phi(\lim_i b_i^{op}) = \lim_i \phi(b_i^{op}) = \lim_i a_i \,,$$

and so ϕ is surjective. It is therefore bijective and preserves the algebraic structure as well as the Fréchet space structure of $\mathcal{A}_\lambda$ and $\mathcal{A}_\lambda^{op}$.

Now for the last claim of the proposition observe that $\phi(\phi(a^{op})^{op}) = a$ holds for all $a \in A_\lambda$. Therefore the map $a \mapsto \phi(\phi(a^{op})^{op}) - a$ is zero on a dense subset of $\mathcal{A}_\lambda$. It is a continuous map since we have just proven ϕ is continuous and $a \mapsto a^{op}$ is continuous by definition. A continuous function that is zero on a dense subset is zero everywhere, therefore we have $\phi(\phi(a^{op})^{op} = a$ for all $a \in \mathcal{A}_\lambda$. □

This result enables us to use Proposition 2.2 with as algebra A the smooth torus; $A = \mathcal{A}_\lambda$. Therefore, we can look at $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ instead of $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda^{op}$. For every $c = \sum a_j \otimes b_j \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$, let (c_{klmn}) denote the coefficient sequence of c, defined by

$$c_{klmn} := \sum_j (a_j)_{kl} (b_j)_{mn} \,.$$

Remember from Section 3.2 that the coefficient sequence of an element in $A_\lambda \otimes A_\lambda$ was in $C_c(\mathbb{Z}^2) \otimes C_c(\mathbb{Z}^2) = C_c(\mathbb{Z}^4)$. We will later on see that the coefficient sequence of an element in $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ is in $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2) \subseteq \mathcal{S}(\mathbb{Z}^4)$. Extending the product $*_\lambda$ from $C_c(\mathbb{Z}^4)$ to $\mathcal{S}(\mathbb{Z}^4)$, we define

$$(c_{klmn}) *_\lambda (d_{klmn}) := \left(\sum_{pqrs} c_{pqrs} d_{k-p,l-q,m-r,n-s} \lambda^{q(k-p)+s(m-r)} \right)_{klmn} \,, \qquad (15)$$

which allows us to generalize Proposition 3.3.

Proposition 4.8. *The function $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda \to \mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2)$ defined by $c \mapsto (c_{klmn})$ is a unital algebra isomorphism, when we equip $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2) \subseteq \mathcal{S}(\mathbb{Z}^4)$ with the product $*_\lambda$.*

Proof. Let $f : \mathcal{A}_\lambda \to \mathcal{S}(\mathbb{Z}^2)$ be defined by $f(a) := (a_{kl})$. Then f is bijective by Theorem 4.2. Define the function $g := f \otimes f : \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda \to \mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2)$, that is,

$$g\Big(\sum_j a_j \otimes b_j\Big) = \sum_j f(a_j) \otimes f(b_j) = \Big(\sum_j (a_j)_{kl}(b_j)_{mn}\Big)_{k,l,m,n\in\mathbb{Z}} \tag{16}$$

where we have used $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2) \subseteq \mathcal{S}(\mathbb{Z}^4)$, for elaboration see Section A.2 in the appendix.

Since $g = f \otimes f$ and f is bijective, g is bijective as well. From (16) we see that g sends $c \in A_\lambda \otimes A_\lambda$ to its coefficient sequence, so $g_{\restriction A_\lambda \otimes A_\lambda}$ is the unital algebra isomorphism from Proposition 3.3. By definition of the norms[5] on $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2)$ we get $\|g(c)\|_{(p)} = \|c\|_{(p)}$, therefore g is Cauchy continuous. Thus g is the continuous extension of $g_{\restriction A_\lambda \otimes A_\lambda}$ which preserves all unital algebra structure. Multiplication is continuous by Proposition 4.4, and so g also preserves all unital algebra structure. □

Notice that we haven't defined an involution on $\mathcal{S}(\mathbb{Z}^4)$, which explains why we have an isomorphism of unital algebras $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ and $\mathcal{S}(\mathbb{Z}^4)$, and not of unital *-algebras. Nevertheless, this result is enough to prove the following theorem, which extends Theorem 3.4.

Theorem 4.9. *The perturbation semigroup is isomorphic to*

$$\mathrm{Pert}(\mathcal{A}_\lambda) \cong \left\{ (c_{klmn}) \in \mathcal{S}(\mathbb{Z}^2)\otimes\mathcal{S}(\mathbb{Z}^2) \;\middle|\; \begin{array}{l} \sum_{m,n} c_{m,l-n,n,k-m}\lambda^{(k-m)(l-n)} = 1_{kl} \\ c_{klmn} = \overline{c_{-n,-m,-l,-k}}\lambda^{kl+mn} \end{array} \right\}$$

when we equip $\mathcal{S}(\mathbb{Z}^2)\otimes\mathcal{S}(\mathbb{Z}^2)$ with the product $*_\lambda$ as defined in (15).

Proof. The conditions for Proposition 2.2 are satisfied when we take $A = \mathcal{A}_\lambda$ and ϕ defined by continuous extension as before. Therefore we get $\mathrm{Pert}(\mathcal{A}_\lambda) \cong$

$$\left\{ \sum_j a_j \otimes b_j \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda \;\middle|\; \begin{array}{l} \sum_j a_j \phi(b_j^{op}) = 1\,, \\ \sum_j a_j \otimes b_j = \sum_j \phi(b_j^{*op}) \otimes \phi(a_j^{*op}) \end{array} \right\}. \tag{17}$$

Let g be as in Proposition 4.8. When we can show that g maps (17) to the set on the right of Theorem 4.9 we are done.

[5]See Section A.2 for the definition of the norms on $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2)$.

Let $\sum a_j \otimes b_j \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ and let $(c_{klmn}) := g(\sum a_j \otimes b_j)$ be its coefficient sequence. Because our rules for coefficients generalise to the smooth torus by Lemma 4.6, we find that the formulas in front of (10) are still equivalent. Notice that this time ϕ has a more general meaning, and the sums over k, l and m, n can contain an infinite amount of nonzero terms. Since the sum over j is finite we can still interchange it with other sums without problems of convergence. Therefore

$$\sum_j a_j \phi(b_j^{op}) = 1 \iff \sum_{m,n} c_{m,l-n,n,k-m} \lambda^{(k-m)(l-n)} = 1_{kl} \,.$$

In the same way the formulas in front of (11) are still equivalent, therefore

$$\sum_j a_j \otimes b_j = \sum_j \phi(b_j^{*op}) \otimes \phi(a_j^{*op}) \iff c_{klmn} = \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn} \,.$$

Thus the conditions in (17) are equivalent to the conditions on the right side of Theorem 4.9. Therefore $\sum a_j \otimes b_j$ is in (17) if and only if $g(\sum a_j \otimes b_j) = (c_{klmn})$ is in the set on the right side of Theorem 4.9. □

This expression of the perturbation semigroup concludes the section about the smooth noncommutative torus. We defined the smooth noncommutative torus as an extension in the Fréchet topology. This definition took some work, as well as proving that the smooth noncommutative torus is a Fréchet algebra. However, after this effort, extending the results from Section 3 was relatively easy. All rules we had for the polynomial noncommutive torus extended to similar ones for the smooth version. One may say the two algebras are not very different from each other.

5 Noncommutative Torus as Operator Algebra

In this section we will discuss the noncommutative torus (referring to both the polynomial and the smooth version) as an algebra consisting of bounded operators. To be more precise, we construct a faithful representation (see Defintion 2.2) of A on $B(H)$[6], for a Hilbert space H.

A concrete application of this Section is provided in Section 6, where we discuss the smooth noncommutative as part of a spectral triple. As we have

[6]We denote $B(X)$ for the space of bounded operators on X, as explained in the appendix, Section A.2

said in Section 2.1, a spectral triple consists partly of an algebra represented faithfully on a Hilbert space. So in order to define a spectral triple where $\mathcal{A}_\lambda$ is the algebra, we will need a representation. This representation will be based on the representation we give in the first part of this section.

The second part of this section explores some implications of viewing the noncommutative torus as an operator algebra. This point of view turns out to be very powerful, and eventually leads us to find a new expression for the perturbation semigroup in the third part of this section. However, we will be forced to depart from the description in terms of operators on H, and we will have to use operators on another space.

5.1 Representing the Noncommutative Torus

First of all, we will need a Hilbert Space to represent $\mathcal{A}_\lambda$ on. This Hilbert space will be $l^2(\mathbb{Z}^2)$, consisting of all sequences $\xi : \mathbb{Z}^2 \to \mathbb{C}$ such that

$$\sum_{k,l} |\xi(k,l)|^2 < \infty\,.$$

This space has an inner product defined by $\langle \xi, \eta \rangle := \sum_{k,l} \overline{\xi(k,l)}\eta(k,l)$ for all $\xi, \eta \in l^2(\mathbb{Z}^2)$. It is a standard fact from the field of functional analysis that the space $l^2(\mathbb{Z}^2)$ is complete in the induced norm $\|\xi\|_2 := \sqrt{\langle \xi, \xi \rangle}$. It is therefore a Hilbert space. Representing $\mathcal{A}_\lambda$ faithfully on $l^2(\mathbb{Z}^2)$ means that we have a representation $\rho : \mathcal{A}_\lambda \to B(l^2(\mathbb{Z}^2))$, which is injective. To do this, we will need the operators U and V (suggestively alike to u and v) defined for every $\xi \in l^2(\mathbb{Z}^2)$ as

$$\begin{aligned} U(\xi)(k,l) &:= \lambda^{-k}\,\xi(k,l-1)\,, \\ V(\xi)(k,l) &:= \xi(k-1,l)\,. \end{aligned} \tag{18}$$

To see that these operators are bounded, and thus contained in $B(l^2(\mathbb{Z}^2))$, we calculate for $\xi \in l^2(\mathbb{Z}^2)$:

$$\|U\xi\|_2^2 = \sum_{k,l} |\lambda^{-k}\xi(k,l-1)|^2 = \sum_{k,l} |\xi(k,l-1)|^2 = \sum_{k,l} |\xi(k,l)|^2 = \|\xi\|_2^2\,.$$

Similarly $\|V\xi\|_2^2 = \|\xi\|_2^2$. Therefore $\|U\| = \|V\| = 1$ and $U, V \in B(l^2(\mathbb{Z}^2))$.

We first prove that we can represent the polynomial noncommutative torus on $B(l^2(\mathbb{Z}^2))$, and subsequently extend the representation to the smooth noncommutative torus.

Proposition 5.1. *The mapping $u \mapsto U, v \mapsto V$ yields a faithful algebra representation $\rho : A_\lambda \to B(l^2(\mathbb{Z}^2))$.*

Proof. We carry out the following steps:
(i) We show that U is unitary, by observing that it is surjective and proving that it preserves the inner product. Analogously V is unitary.
(ii) We prove that $VU = \lambda UV$.
(iii) We conclude from (i) and (ii) that this map is well-defined and preserves all structure.
(iv) We prove injectivity, which means ρ is faithful.

(i) Since U is simply a pointwise multiplication with a phase $\lambda^{-k} \neq 0$, composed with a translation, it is surjective. To show that it preserves inner products, we take $\xi, \eta \in l^2(\mathbb{Z}^2)$ and compute

$$\begin{aligned} \langle U\xi, U\eta \rangle &= \sum \overline{U\xi(k,l)} U\eta(k,l) \\ &= \sum \lambda^k \overline{\xi(k,l-1)} \lambda^{-k} \eta(k,l-1) \\ &= \sum \overline{\xi(k,l)} \eta(k,l) \\ &= \langle \xi, \eta \rangle . \end{aligned}$$

Therefore $U^*U = 1$ and because U is surjective we have $UU^* = 1$ as well. The proof that V is unitary is analogous (if not simpler, since no phasefactor λ is involved).

(ii) We compute

$$\begin{aligned} VU\xi(k,l) &= U\xi(k-1,l) \\ &= \lambda^{-(k-1)} \xi(k-1,l-1) \\ &= \lambda \lambda^{-k} V\xi(k,l-1) \\ &= \lambda UV\xi(k,l) , \end{aligned}$$

and find $VU = \lambda UV$.

(iii) An arbitrary element $\sum a_{kl} u^k v^l$ of A_λ gets mapped to $\sum a_{kl} U^k V^l$, which has to be in the *-algebra $B(l^2(\mathbb{Z}^2))$ since a *-algebra is closed under multiplication, involution and finite linear combinations. Therefore ρ is well-defined. It is easy to see that ρ is linear. We defined A_λ to be the *-algebra with the relations $uu^* = u^*u = 1, vv^* = v^*v = 1, vu = \lambda uv$, and the elements U and V which generate $\rho(A_\lambda)$ have exactly those relations. Therefore the two algebras A_λ and $\rho(A_\lambda)$ necessarily have the same rules for multiplication and involution (namely, those of Lemma 3.1) and $\rho(1) = 1$. So the mapping $\sum a_{kl} u^k v^l \mapsto \sum a_{kl} U^k V^l$ preserves all *-algebra structure.

(iv) Suppose $\sum a_{kl} U^k V^l = 0$. Then for all $m, n \in \mathbb{Z}$ we look at the Kronecker delta function $\delta_{m,n} \in l^2(\mathbb{Z}^2)$, defined as $\delta_{m,n}(m,n) := 1$ and $\delta_{m,n}(k,l) := 0$ for $(k,l) \neq (m,n)$.

$$0 = \left(\sum a_{kl} U^k V^l \delta_{m,n}\right)(0,0) = \sum a_{kl} \delta_{m,n}(-l,-k) = a_{-n,-m}\,,$$

and therefore $\sum a_{kl} u^k v^l = 0$. So ρ is injective.

□

We now also represent the smooth torus $\mathcal{A}_\lambda$ as a subset of $B(l^2(\mathbb{Z}^4))$. We apply the technique of continuous extension, as we did in Subsection 4.2.

Proposition 5.2. *The faithful representation $\rho : A_\lambda \to B(l^2(\mathbb{Z}^2))$ is Cauchy continuous, and thus extends to a continuous faithful representation $\rho : \mathcal{A}_\lambda \to B(l^2(\mathbb{Z}^2))$.*

Proof. Use that the operator norm on $B(l^2)$ is a norm, as well as satisfies $\|AB\| \leqslant \|A\|\,\|B\|$ (since $B(l^2)$ is a Banach algebra) to find

$$\|\rho(a)\| = \left\| \sum_{k,l} a_{kl} U^k V^l \right\| \leqslant \sum_{k,l} |a_{kl}|\, \|U\|^k \, \|V\|^l = \sum_{k,l} |a_{kl}|\,.$$

Now apply Lemma 4.5 with $p = 0$ to obtain

$$\|\rho(a)\| \leqslant \sum_{k,l} |a_{kl}| \leqslant C\,\|a\|_{(2)}\,.$$

So if (a_m) is Cauchy in A_λ then in particular $\|a_m - a_n\|_{(2)} \to 0$ and so $\|\rho(a_m) - \rho(a_n)\| \to 0$. Therefore ρ sends Cauchy sequences to Cauchy sequences. We can extend it to a unique continuous map on $\mathcal{A}_\lambda$, which we also call ρ. □

To symplify our following formulas, we will make no difference between $a \in \mathcal{A}_\lambda$ and $\rho(a) \in B(l^2(\mathbb{Z}^2))$. In fact, we will only use the small case u and v, and talk about $\mathcal{A}_\lambda$ as if it were an algebra of operators.

5.2 Operators and Coefficients

Recall Notation 2 to identify $a \in \mathcal{A}_\lambda$ with its coefficients $(a_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$. If we see a as an operator, how does its action on $\xi \in l^2(\mathbb{Z}^2)$ depend on its coefficients?

Proposition 5.3. *For $a \in \mathcal{A}_\lambda$, $\xi \in l^2(\mathbb{Z}^2)$ and $p, q \in \mathbb{Z}$ we have*

$$a\xi(p,q) = \sum_{k,l} a_{kl}\lambda^{-pk}\,\xi(p-l, q-k)\,.$$

Proof. Use the definitions of $U = \rho(u)$ and $V = \rho(v)$ in (18) repeatedly to find

$$\begin{aligned} a\xi(m,n) &= \left(\sum a_{kl}u^k v^l\right)\xi(m,n) \\ &= \sum a_{kl}u^k v^l \xi(m,n) \\ &= \sum a_{kl}\lambda^{-mk}\, v^j \xi(m, n-k) \\ &= \sum a_{kl}\lambda^{-mk}\,\xi(m-l, n-k)\,, \end{aligned}$$

where $m, n \in \mathbb{Z}$ and sums are over $k, l \in \mathbb{Z}$. □

In the last Proposition some kind of twisted discrete convolution emerges between the sequences (a_{kl}) and ξ. The main difference with the twisted convolutions we had earlier, as in Lemma 4.6(i), is found in the indices k, l whose places are switched in $\xi(m-l, n-k)$. Note that this can be avoided by altering the notation in (5).

The same method as for $a \in \mathcal{A}_\lambda$ can be applied to $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$. As we have seen in (8), we can write:

$$c = \sum_{k,l,m,n} c_{klmn}\, u^k v^l \otimes u^m v^n\,, \tag{19}$$

where (c_{klmn}) is the coefficient sequence of c. We have just seen that u and v are bounded operators on $l^2(\mathbb{Z}^2)$. Since $(c_{klmn}) \in \mathcal{S}(\mathbb{Z}^4)$ and the tensor product of bounded operators is bounded by the product of their norms, we get by Lemma 4.5 that c is a bounded operator on $l^2(\mathbb{Z}^2) \otimes l^2(\mathbb{Z}^2)$.

Proposition 5.4. *For $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$, $\varphi \in l^2(\mathbb{Z}^4)$ and $p, q, r, s \in \mathbb{Z}$ we have*

$$c\varphi(p,q,r,s) = \sum_{klmn} c_{klmn}\lambda^{-pk-rm}\varphi(p-l, q-k, r-n, s-m)\,.$$

Proof. Let $\xi, \eta \in l^2(\mathbb{Z}^4)$. The action of c on $\xi \otimes \eta$ is given by:

$$\begin{aligned} c(\xi \otimes \eta)(p,q,r,s) &= \sum_{klmn} c_{klmn}(u^k v^l \otimes u^m v^n)(\xi \otimes \eta)(p,q,r,s) \\ &= \sum_{klmn} c_{klmn}(u^k v^l \xi(p,q))(u^m v^n \eta(r,s)) \\ &= \sum_{klmn} c_{klmn}(\lambda^{-pk}\xi(p-l,q-k))(\lambda^{-rm}\eta(r-n,s-m)) \\ &= \sum_{klmn} c_{klmn}\lambda^{-pk-rm}(\xi \otimes \eta)(p-l,q-k,r-n,s-m)\,. \end{aligned}$$

By linearity and continuity of c it follows that for a finite sum $\sum \xi_j \otimes \eta_j$ in $l^2(\mathbb{Z}^2) \otimes l^2(\mathbb{Z}^2)$, the action of c is:

$$\begin{aligned} &c(\sum_j \xi_j \otimes \eta_j)(p,q,r,s) \\ &= \sum_r c(\xi_j \otimes \eta_j)(p,q,r,s) \\ &= \sum_j \sum_{klmn} c_{klmn}\lambda^{-pk-rm}(\xi_j \otimes \eta_j)(p-l,q-k,r-n,s-m) \\ &= \sum_{klmn} c_{klmn}\lambda^{-pk-rm}(\sum_j \xi_j \otimes \eta_j)(p-l,q-k,r-n,s-m)\,. \end{aligned}$$

The interchange of the sums in the last equality is valid because the sum over j is finite. For an arbitrary element $\varphi \in l^2(\mathbb{Z}^4)$ we can write $\varphi = \sum_j \xi_j \otimes \eta_j$.[7] From this the proposition follows. □

5.3 Eigenvectors and Commutation Relations

We aim to find a new expression for the perturbation semigroup of the smooth noncommutative torus in terms of operators. As it will turn out, we can define such an expression in terms of operators on the space $l^\infty(\mathbb{Z}^4)$ of bounded sequences, but not on the space $l^2(\mathbb{Z}^4)$. We will explain later what causes this problem, but let us focus for now on interpreting the elements $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ as operators on $l^\infty(\mathbb{Z}^4)$.

We would like to generalize the statement in Proposition 5.4 for $\varphi \in l^\infty(\mathbb{Z}^4)$. It is easy to see that the right-hand side of the equation in Proposition 5.4

[7] We used that $l^2(\mathbb{Z}^2)\hat{\otimes} l^2(\mathbb{Z}^2) = l^2(\mathbb{Z}^4)$, for an elaboration see Section A.2 in the appendix.

converges absolutely whenever φ is bounded. So when φ is in $l^\infty(\mathbb{Z}^4)\setminus l^2(\mathbb{Z}^4)$ the equation in Proposition 5.4 still has a well-defined meaning. This leads us to the following definition.

Definition 5.1. *Let $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$. Define $\hat{c} : l^\infty(\mathbb{Z}^4) \to \mathbb{C}^{\mathbb{Z}^4}$ by*

$$\hat{c}\varphi(p,q,r,s) := \sum_{klmn} c_{klmn}\lambda^{-pk-rm}\varphi(p-l,q-k,r-n,s-m)\,,$$

where $\mathbb{C}^{\mathbb{Z}^4}$ denotes the space of all sequences on $\mathbb{Z}^4$.

In order to talk about the behaviour of this function $\hat{c}$, we will need information about its norm. Denote by $\|\cdot\|_p$ the usual norm on l^p. For instance, $\|\cdot\|_\infty$ is the supremum norm on l^∞. In this context, we denote the operator norm by $\|\cdot\|$. (See also (29) in the appendix.)

Proposition 5.5. *We have $\hat{c} \in B(l^\infty(\mathbb{Z}^4))$ with $\hat{c}_{\restriction l^2} = c$ and*

$$\|\hat{c}\| = \|(c_{klmn})\|_1\,.$$

Proof. From the similarity between Proposition 5.4 and Definition 5.1 we find $\hat{c}_{\restriction l^2} = c$. It is easy to see that $\hat{c}$ is a linear operator. For $\hat{c}$ to be in $B(l^\infty(\mathbb{Z}^4))$ we still need to show that it maps $l^\infty(\mathbb{Z}^4)$ to $l^\infty(\mathbb{Z}^4)$ and is bounded.
For every $\varphi \in l^\infty$ we have:

$$\begin{aligned}\|\hat{c}\varphi\|_\infty &= \sup_{pqrs}|\sum_{klmn} c_{klmn}\lambda^{-pk-rm}\varphi(p-l,q-k,r-n,s-m)|\\ &\leqslant \sup_{pqrs}\sum_{klmn}|c_{klmn}|\,|\varphi(p-l,q-k,r-n,s-m)|\\ &= \sum_{klmn}|c_{klmn}|\cdot\|\varphi\|_\infty\\ &= \|(c_{klmn})\|_1\cdot\|\varphi\|_\infty\,.\end{aligned}$$

Since $\|(c_{klmn})\|_1 \leqslant C\,\|c\|_{(2)} < \infty$ by Lemma 4.5, we have that $\hat{c}\varphi \in l^\infty$. Moreover, since $\|\hat{c}\varphi\|_\infty \leqslant \|(c_{klmn})\|_1\,\|\varphi\|_\infty$, we immediately get $\hat{c} \in B(l^\infty(\mathbb{Z}^4))$ and $\|\hat{c}\| \leqslant \|(c_{klmn})\|_1$. For the "$\geqslant$" direction, define

$$\varphi_0(k,l,m,n) := \frac{\overline{c_{-l,-k,-n,-m}}}{|c_{-l,-k,-n,-m}|}\,.$$

It directly follows that

$$\|\varphi_0\|_\infty = \sup_{klmn}\frac{|\overline{c_{-l,-k,-n,-m}}|}{|c_{-l,-k,-n,-m}|} = 1\,.$$

In particular $\varphi_0 \in l^\infty$. Finally, we calculate

$$\begin{aligned}
\|\hat{c}\varphi_0\|_\infty &= \sup_{pqrs} \left| \sum_{klmn} c_{klmn} \lambda^{-pk-rm} \frac{\overline{c_{k-q,l-p,m-s,n-r}}}{|c_{k-q,l-p,m-s,n-r}|} \right| \\
&\geqslant \left| \sum_{klmn} c_{klmn} \lambda^{0} \frac{\overline{c_{k,l,m,n}}}{|c_{k,l,m,n}|} \right| \qquad \text{(Take } (p,q,r,s)=0.\text{)} \\
&= \left| \sum_{klmn} |c_{klmn}| \right| \\
&= \|(c_{klmn})\|_1 \, .
\end{aligned}$$

So also $\|\hat{c}\| \geqslant \|(c_{klmn})\|_1$, hence we have equality. □

This leads to the following proposition.

Proposition 5.6. *The map $c \mapsto \hat{c}$ is an injective homomorphism of $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ into $B(l^\infty(\mathbb{Z}^4))$.*

Proof. Let $c, d \in \mathcal{A}_\lambda$ and $\mu \in \mathbb{C}$. The first two parts of Proposition 5.5 give us well-definedness and injectivity, for $\hat{c} = \hat{d} \Rightarrow \hat{c}_{\restriction l^2} = \hat{d}_{\restriction l^2} \Rightarrow c = d$. We have seen in Theorem 4.9 that there is a order-preserving correspondence between c and its coefficient sequence (c_{klmn}). It is in particular linear, therefore $(c + \mu d)_{klmn} = c_{klmn} + \mu d_{klmn}$, so using Definition 5.1 gives $\widehat{(c + \mu d)}\varphi = \hat{c}\varphi + \mu\hat{d}\varphi$. Therefore $\widehat{(c + \mu d)} = \hat{c} + \mu\hat{d}$.

Showing that $\widehat{cd} = \hat{c}\hat{d}$ takes some more work. For this, let $\varphi \in l^\infty$ be arbitrary. We can find a sequence $(\varphi_i)_{i\in\mathbb{N}}$ in l^2 that converges pointwise to φ. If we plug φ_i into Definition 5.1, and use Proposition 5.4 after swapping finite sum and limit, we obtain:

$$\hat{c}\varphi(k,l,m,n) = \lim_i c\varphi_i(k,l,m,n) \, . \tag{20}$$

Because (20) holds for d instead of c, we get $d\varphi_i \to \hat{d}\varphi$ pointwise. Again use (20), now with $d\varphi_i$ instead of φ_i; we find that $cd\varphi_i \to \hat{c}\hat{d}\varphi$ pointwise. Using (20) one last time with cd instead of c gives $cd\varphi_i \to \widehat{cd}\varphi$ pointwise. Since pointwise limits are unique, we obtain $\hat{c}\hat{d} = \widehat{cd}$.

□

Define $C := (\widehat{\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda})$, the image of $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ under the function $c \mapsto \hat{c}$. Evidently $C \subseteq B(l^\infty(\mathbb{Z}^4))$, and Proposition 5.6 directly implies the following.

Corollary 4. *Under the isomorphism $c \mapsto \hat{c}$ we have $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda \cong C$ as unital *-algebras.*

We have a new description of $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$, namely in terms of operators on the space $l^\infty(\mathbb{Z}^4)$. The perturbation semigroup can thus also be seen as a set of operators. The goal for the rest of this section will be to obtain an expression for the perturbation semigroup that is fully in terms of operators. For this we will need to rewrite the normalisation condition and the self-adjointness condition.

We define the vector $\tau \in l^\infty(\mathbb{Z}^4)$ by

$$\tau(p, q, r, s) := \lambda^{-pq} 1_{q+r,p+s} ,$$

which is related to the normalisation condition in the following way.

Lemma 5.7. *Let $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$. The normalisation condition* (10) *is equivalent to*

$$\hat{c}\tau = \tau .$$

Proof. We calculate

$$\begin{aligned}
\hat{c}\tau(p, q, r, s) &= \sum_{klmn} c_{klmn} \lambda^{-pk-rm} \tau(p - l, q - k, r - n, s - m) \\
&= \sum_{klmn} c_{klmn} \lambda^{-pk-rm} \lambda^{-(p-l)(q-k)} 1_{q-k+r-n,p-l+s-m} \\
&= \lambda^{-pq} \sum_{klmn} c_{klmn} \lambda^{-rm+l(q-k)} 1_{q-k+r-n,p-l+s-m} \\
&= \lambda^{-pq} \sum_{m,n} c_{q+r-n,p+s-m,m,n} \lambda^{-rm+(p+s-m)(n-r)} \\
&= \lambda^{-pq} \sum_{m,n} c_{m,p+s-n,n,q+r-m} \lambda^{-rn+(p+s-n)(q-m)} \\
&= \lambda^{-pq} \lambda^{-(p+s)r} \sum_{m,n} c_{m,p+s-n,n,q+r-m} \lambda^{(p+s-n)(q+r-m)} .
\end{aligned}$$

Therefore

$$\begin{aligned}
\hat{c}\tau = \tau \iff& \lambda^{-(p+s)r} \sum_{m,n} c_{m,p+s-n,n,q+r-m} \lambda^{(p+s-n)(q+r-m)} = 1_{q+r,p+s} \\
\iff& \sum_{m,n} c_{m,p+s-n,n,q+r-m} \lambda^{(p+s-n)(q+r-m)} = 1_{q+r,p+s} \\
\iff& \sum_{m,n} c_{m,l-n,n,k-m} \lambda^{(l-n)(k-m)} = 1_{kl} .
\end{aligned}$$

□

The vector τ is the reason we identified $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ with a subset of $B(l^\infty(\mathbb{Z}^4))$, (namely C,) instead of a subset of $B(l^2(\mathbb{Z}^4))$. Since $|\tau(p, q, -q, -p)| = 1$ for all $p, q \in \mathbb{Z}$, the sum over all $|\tau(p, q, r, s)|^2$ is infinite, so $\tau \notin l^2(\mathbb{Z}^4)$. Therefore one cannot prove a similar version of Lemma 5.7 where we interpret $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ as an operator on $l^2(\mathbb{Z}^4)$. We tried to find different formulas to express the normalisation condition in terms of operators on $l^2(\mathbb{Z}^4)$, but none were found as simple as $\hat{c}\tau = \tau$.

We now express the self-adjointness condition in terms of operators on $l^\infty(\mathbb{Z}^4)$, to parallel $\hat{c}\tau = \tau$. Define the operator Ω on l^∞ by:

$$\Omega\varphi(p, q, r, s) := \varphi(r, s, p, q)\,.$$

This will be related to the self-adjointness condition later on. For every $c \in A_\lambda \otimes A_\lambda$ define the element $c^\dagger \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ by its coefficient sequence:

$$c^\dagger_{klmn} := \overline{c_{-l,-k,-n,-m}}\lambda^{kl+mn}\,.$$

This allows us to define the same function $\cdot^\dagger$ on C. Simply set $\hat{c}^\dagger := \widehat{c^\dagger}$ for every $\hat{c} \in C$. We have two results on the behaviour of this function.

Lemma 5.8. *The function $\cdot^\dagger$ is multiplicative and its own inverse, in other words*

$$(cd)^\dagger = c^\dagger d^\dagger \quad \textit{and} \quad c^{\dagger\dagger} = c\,,$$

for all $c, d \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$. Moreover, the above formulas hold for all $c, d \in C$.

Proof. It is sufficient to prove these two assertions for coefficient sequences, by virtue of Proposition 4.8. For $(c_{pqrs}), (d_{pqrs}) \in \mathcal{S}(\mathbb{Z}^4)$ we let $\cdot^\dagger$ work on the twisted convolution product and find

$$\begin{aligned}\big((c_{pqrs}) *_\lambda (d_{pqrs})\big)^\dagger_{klmn} &= \overline{\big((c_{pqrs}) *_\lambda (d_{pqrs})\big)_{-l,-k,-n,-m}}\lambda^{kl+mn} \\ &= \sum_{pqrs} \overline{c_{pqrs}d_{-l-p,-k-q,-n-r,-m-s}}\lambda^{q(l+p)+s(n+r)}\lambda^{kl+mn} \\ &= \sum_{pqrs} \overline{c_{-q,-p,-s,-r}}\,\overline{d_{q-l,p-k,s-n,r-m}}\lambda^{-p(l-q)-r(n-s)+kl+mn}\,.\end{aligned}$$

On the other hand, the twisted convolution product of $(c^\dagger_{pqrs})$ and $(d^\dagger_{pqrs})$ gives

$$\big((c^\dagger_{pqrs}) *_\lambda (d^\dagger_{pqrs})\big)_{klmn} = \sum_{pqrs} c^\dagger_{pqrs} d^\dagger_{k-p,l-q,m-r,n-s}\lambda^{q(k-p)+s(m-r)}\,.$$

If we now use the following identities:

$$\begin{aligned} c^{\dagger}_{pqrs} &= \overline{c_{-q,-p,-s,-r}}\lambda^{pq+rs} \,, \\ d^{\dagger}_{k-p,l-q,m-r,n-s} &= \overline{d_{q-l,p-k,s-n,r-m}}\lambda^{(p-k)(q-l)+(r-m)(s-n)} \,, \end{aligned}$$

it follows, after some thorough inspection of the exponent of λ, that

$$\big((c_{pqrs}) *_{\lambda} (d_{pqrs})\big)^{\dagger} = (c^{\dagger}_{pqrs}) *_{\lambda} (d^{\dagger}_{pqrs}) \,.$$

We thus have multiplicativity, and only need to show that the function $\cdot^{\dagger}$ is its own inverse. Again, we look at the coefficient sequence $(c_{klmn}) \in \mathcal{S}(\mathbb{Z}^4)$ instead of the element $c \in \mathcal{A}_{\lambda} \otimes \mathcal{A}_{\lambda}$. By definition we find

$$c^{\dagger\dagger}_{klmn} = \overline{c^{\dagger}_{-l,-k,-n,-m}}\lambda^{kl+mn} = \overline{\overline{c_{klmn}}\lambda^{kl+mn}}\lambda^{kl+mn} = c_{klmn} \,.$$

Because we defined $\hat{c}^{\dagger} = \widehat{c^{\dagger}}$, it immediately follows that $\hat{c}^{\dagger\dagger} = \widehat{c^{\dagger\dagger}} = \hat{c}$. A similar argument, together with the knowledge that $c \mapsto \hat{c}$ is multiplicative, gives $(\hat{c}\hat{d})^{\dagger} = \hat{c}^{\dagger}\hat{d}^{\dagger}$. Therefore the formulas $(cd)^{\dagger} = c^{\dagger}d^{\dagger}$ and $c^{\dagger\dagger}$ hold for all $c, d \in C$. □

Lemma 5.9. *Let $c \in \mathcal{A}_{\lambda} \otimes \mathcal{A}_{\lambda}$. The self-adjointness condition* (11) *is equivalent to*

$$\hat{c}\Omega = \Omega\hat{c}^{\dagger} \,.$$

Proof. When we let the left-hand side work on an arbitrary $\varphi \in l^{\infty}(\mathbb{Z}^4)$, we get

$$\begin{aligned} \hat{c}\Omega\varphi(p,q,r,s) &= \sum_{klmn} c_{klmn}\lambda^{-pk-rm}\Omega\varphi(p-l,q-k,r-n,s-m) \\ &= \sum_{klmn} c_{klmn}\lambda^{-pk-rm}\varphi(r-n,s-m,p-l,q-k) \,. \end{aligned}$$

When we do the same for the right-hand side, we get

$$\begin{aligned} \Omega\hat{c}^{\dagger}\varphi(p,q,r,s) &= \widehat{c^{\dagger}}\varphi(r,s,p,q) \\ &= \sum_{klmn} c^{\dagger}_{klmn}\lambda^{-rk-pm}\varphi(r-l,s-k,p-n,q-m) \\ &= \sum_{klmn} \overline{c_{-l,-k,-n,-m}}\lambda^{kl+mn-rk-pm}\varphi(r-l,s-k,p-n,q-m) \\ &= \sum_{mnkl} \overline{c_{-n,-m,-l,-k}}\lambda^{mn+kl-rm-pk}\varphi(r-n,s-m,p-l,q-k) \,. \end{aligned}$$

We thus have $\hat{c}\Omega\varphi = \Omega\hat{c}^{\dagger}\varphi$ for all φ if and only if $c_{klmn} = \overline{c_{-n,-m,-l,-k}}\lambda^{kl+mn}$ for all k, l, m, n, and the latter is the self-adjointness condition. □

We finally arrive at our main result of this section. We defined a fixed vector τ and a fixed operator Ω, and showed they can express the two conditions of the perturbation semigroup. We now give a description of the perturbation semigroup as a subset of $B(l^\infty(\mathbb{Z}^4))$.

Theorem 5.10. *The perturbation semigroup is isomorphic to*

$$\mathrm{Pert}(\mathcal{A}_\lambda) \cong \left\{ \hat{c} \in C \;\middle|\; \hat{c}\tau = \tau\,,\ \hat{c}\Omega = \Omega\hat{c}^\dagger \right\}\,,$$

for $C \subseteq B(l^\infty(\mathbb{Z}^4))$ a subalgebra, $\tau \in l^\infty(\mathbb{Z}^4)$ and $\Omega \in B(l^\infty(\mathbb{Z}^4))$.

Proof. Recall Theorem 4.9, in which we in particular used the function $g : c \mapsto (c_{klmn})$ to find a new expression for $\mathrm{Pert}(\mathcal{A}_\lambda)$. Apply g^{-1} to this expression (the right-hand side in Theorem 4.9) to find that $\mathrm{Pert}(\mathcal{A}_\lambda)$ is isomorphic to

$$\left\{ c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda \;\middle|\; \begin{array}{l} \sum_{m,n} c_{m,l-n,n,k-m}\lambda^{(k-m)(l-n)} = 1_{kl} \\ c_{klmn} = \overline{c_{-n,-m,-l,-k}}\lambda^{kl+mn} \end{array} \right\}. \tag{21}$$

By Corollary 4, $\mathrm{Pert}(\mathcal{A}_\lambda)$ is isomorphic to the image of (21) under the function $c \mapsto \hat{c}$. Loosely speaking, we replace $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$ by $\hat{c} \in C$. Now by Lemma 5.7 and Lemma 5.9, the two conditions in 21 are equivalent to $\hat{c}\tau = \tau$ and $\hat{c}\Omega = \Omega\hat{c}^\dagger$. From this the theorem follows. □

Compare this result with Proposition 3.9 in [13]. Here Niels Neumann derived a similar expression for the perturbation semigroup of $M_N(\mathbb{C})$, the algebra of $N \times N$-matrices. He expressed $\mathrm{Pert}(M_N(\mathbb{C}))$ as a subset of $M_{N^2}(\mathbb{C})$, which can be seen as the set of bounded operators on $\mathbb{C}^{N^2}$. An important feature of the expression of Theorem 5.10 (just as for the analogous expression in [13]) is the visibility of its structure.

To illustrate this, we will prove that the set on the right-hand side in Theorem 5.10 is indeed a semigroup. Take elements $\hat{c}, \hat{d}$ in this set, we will show that $\hat{c}\hat{d}$ is still in the same set. First of all, since C is a subalgebra, we have $\hat{c}\hat{d} \in C$. Second, since $\hat{c}\tau = \tau$ and $\hat{d}\tau = \tau$:

$$\hat{c}\hat{d}\tau = \hat{c}\tau = \tau\,.$$

Last of all, $\hat{c}\Omega = \Omega\hat{c}^\dagger$, $\hat{d}\Omega = \Omega\hat{d}^\dagger$ and $\cdot^\dagger$ is multiplicative. Therefore

$$\hat{c}\hat{d}\Omega = \hat{c}\Omega\hat{d}^\dagger = \Omega\hat{c}^\dagger\hat{d}^\dagger = \Omega(\hat{c}\hat{d})^\dagger\,.$$

It follows that the expression in Theorem 5.10 is indeed a semigroup, as we already knew.

As we have said, the expressions for $\mathrm{Pert}(M_N(\mathbb{C}))$ and $\mathrm{Pert}(\mathcal{A}_\lambda)$ are similar. It might be tempting to conclude that this is only because of the natural relation between $M_n(\mathbb{C})$ and $\mathcal{A}_\lambda$, which is nicely portrayed in [17]. However, a premature calculation suggests a more general result.

Conjecture 5.1. *For any spectral triple (A, H, D) there is a vector space X and an embedding $\iota : B(H \otimes H) \to B(X)$ with $\iota(A \otimes A^{op}) = C$. Furthermore, there exist $\tau \in X$, $\Omega \in B(X)$ and a multiplicative function $\cdot^\dagger$ on C such that under ι:*

$$\mathrm{Pert}(A) \cong \left\{ c \in C \;\middle|\; c\tau = \tau,\ \Omega c = \Omega c^\dagger \right\} .$$

It is our presumption that this result can be obtained with the use of the weak topology[8] on $H \otimes H$, and defining X as the extension of $H \otimes H$ in this topology. More research on this is needed.

6 The Torus Triple

We will discuss the spectral triple of the noncommutative torus, and will do so in two parts. In the first part of this section we will finally show that the smooth noncommutative torus $\mathcal{A}_\lambda$ is part of a spectral triple. For this we define a Hilbert space $\mathcal{H}$ and an unbounded operator $\mathcal{D}$, (conform with [23],) and then follow Definition 2.1 to prove $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$ is a spectral triple.

We have discussed the interpretation of the perturbation semigroup of a spectral triple in 2.2, and saw that this semigroup gave rise to perturbations on an operator. In the second part of this section we investigate the perturbations originating from the spectral triple of the noncommutative torus.

6.1 The Spectral Triple of the Noncommutative Torus

We define $\mathcal{H} := l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2)$. Notice that the elements of $\mathcal{H}$ can be written like column vectors:

$$\mathcal{H} = \left\{ \begin{pmatrix} \xi \\ \eta \end{pmatrix} \;\middle|\; \xi, \eta \in l^2(\mathbb{Z}^2) \right\} .$$

We define an inner product on $\mathcal{H}$, written as $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, as the sum of the inner products on $l^2(\mathbb{Z}^2)$, that is

$$\langle \xi, \eta \rangle_{\mathcal{H}} = \langle \xi_1, \eta_1 \rangle + \langle \xi_2, \eta_2 \rangle .$$

[8]See [9] for more information on this topology. However, for this thesis one needs only to understand the related ‘weak operator topology’, which will be defined in Section 7.

This inner product makes $\mathcal{H}$ into a Hilbert space.

We want to define an unbounded operator $\mathcal{D}$ in $\mathcal{H}$, but in order to do this we first define an unbounded operator $\mathcal{D}_0$ with the domain

$$\mathrm{dom}(\mathcal{D}_0) := C_c(\mathbb{Z}^2) \oplus C_c(\mathbb{Z}^2)\,.$$

We remark that $\mathrm{dom}(\mathcal{D}_0)$ is dense in $\mathcal{H}$. Let $\xi, \eta \in C_c(\mathbb{Z}^2)$ be finite sequences, then we set

$$\mathcal{D}_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix} (k,l) := \begin{pmatrix} \eta(k,l)(k+il) \\ \xi(k,l)(k-il) \end{pmatrix} \,. \tag{22}$$

Because it simplifies the formulas encountered in spectral theory, we will use so-called "bra-ket notation", introduced by Paul Dirac. (A good introduction is [19].) In bra-ket notation, $|\psi\rangle$ denotes a vector in $\mathcal{H}$, $\langle\psi|\varphi\rangle$ denotes the inner product of vectors $|\psi\rangle$ and $|\varphi\rangle$, $\langle\psi|$ denotes a functional defined by:

$$|\varphi\rangle \mapsto \langle\psi|\varphi\rangle \ .$$

Another concept in bra-ket notation is the so-called matrix element $\langle\psi|a|\varphi\rangle$. This is nothing more than the inner product of $|\psi\rangle$ with $a\,|\varphi\rangle$.

Perhaps more important than the definition of $\mathcal{D}_0$ is its set of eigenvectors and its spectrum.

Lemma 6.1. *An orthonormal basis of eigenvectors of $\mathcal{D}_0$ is given by the vectors*

$$\left|\psi^{\epsilon}_{k,l}\right\rangle := \frac{1}{\sqrt{2\sqrt{k^2+l^2}}} \begin{pmatrix} \sqrt{k+il}\,\delta_{k,l} \\ \epsilon\,\sqrt{k-il}\,\delta_{k,l} \end{pmatrix} \,,$$

which respectively have the eigenvalues

$$\lambda_{k,l,\epsilon} := \epsilon\sqrt{k^2+l^2}\,.$$

Here k and l run over $\mathbb{Z}$ and ϵ runs over $\{\pm 1\}$.

The proof is straightforward, and is omitted.

We stress that $\lambda_{k,l,\epsilon}$ should not be confused with the parameter λ in $\mathcal{A}_\lambda$. The first lambda is a real eigenvalue dependent on k, l and ϵ, the second is a fixed complex number of norm 1.

Proposition 6.2. *The unbounded operator $\mathcal{D}_0$ extends to a closed self-adjoint unbounded operator $\mathcal{D}$.*

Proof. We will show that $\mathcal{D}_0$ is symmetric, by which we mean that for $\xi, \eta \in \operatorname{dom}(\mathcal{D}_0)$ we have $\langle \mathcal{D}_0\xi, \eta\rangle = \langle \xi, \mathcal{D}_0\eta\rangle$. Writing $\xi = (\xi_1, \xi_2)^t$, $\eta = (\eta_1, \eta_2)^t$ gives

$$\begin{aligned}
\langle \mathcal{D}_0\xi, \eta\rangle_{\mathcal{H}} &= \sum_{k,l} \left(\overline{\xi_2(k,l)(k+il)}\eta_1(k,l) + \overline{\xi_1(k,l)(k-il)}\eta_2(k,l)\right) \\
&= \sum_{k,l} \left(\overline{\xi_1(k,l)}\eta_2(k,l)(k+il) + \overline{\xi_2(k,l)}\eta_1(k,l)(k-il)\right) \\
&= \langle \xi, \mathcal{D}_0\eta\rangle_{\mathcal{H}} \, .
\end{aligned}$$

Therefore $(\mathcal{D}_0^*)_{\restriction \operatorname{dom}(\mathcal{D}_0)} = \mathcal{D}_0$. We define $\mathcal{D} := \mathcal{D}_0^*$, and get that $\mathcal{D}$ is an extension of $\mathcal{D}_0$. We can now use Paragraph VIII.2 of [21], which states that a symmetric operator $\mathcal{D}_0$ is called *essentially self-adjoint* when $\mathcal{D}_0^*$ is self-adjoint, and that in this case $\mathcal{D}$ is closed. The corollary in that same paragraph states that $\mathcal{D}_0$ is essentially self-adjoint iff the ranges of $\mathcal{D}_0 \pm i$ are dense.

By Lemma 6.1 the eigenvectors $\left|\psi_{k,l}^{\epsilon}\right\rangle$ form a basis of $\mathcal{H}$. Since they are eigenvectors of $\mathcal{D}_0$, they are also eigenvectors of $\mathcal{D}_0 \pm i$, and therefore contained in their ranges. A linear subspace containing a basis is automatically dense, therefore the ranges of $\mathcal{D}_0 \pm i$ are dense in $\mathcal{H}$. So $\mathcal{D}_0$ is essentially self-adjoint, and it follows that $\mathcal{D}$ is a closed self-adjoint extension of $\mathcal{D}_0$. □

Definition 6.1. *The* ***torus triple*** *is the triple* $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$*, where* $\mathcal{A}_\lambda$ *is the smooth noncommutative 2-torus,* $\mathcal{H} = l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2)$ *and* $\mathcal{D} = \mathcal{D}_0^*$*.*

The main goal of this section is to prove that the torus triple is a spectral triple, following Definition 2.1. First of all we need to represent $\mathcal{A}_\lambda$ on $B(\mathcal{H})$. Thanks to Proposition 5 we already have a representation of $\mathcal{A}_\lambda$ on $l^2(\mathbb{Z}^2)$. Now also interpret $a \in \mathcal{A}_\lambda$ as a bounded operator on $\mathcal{H}$ by the identification

$$a \mapsto \begin{pmatrix} a \\ a \end{pmatrix}, \quad \text{or in other terms:} \quad a\begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} a\xi \\ a\eta \end{pmatrix}. \tag{23}$$

We remark that Proposition 5.3 extends in the following way.

$$\begin{aligned}
a\begin{pmatrix} \xi \\ \eta \end{pmatrix}(p,q) = \begin{pmatrix} a\xi(p,q) \\ a\eta(p,q) \end{pmatrix} &= \begin{pmatrix} \sum_{k,l} a_{kl}\lambda^{-pk}\xi(p-l,q-k) \\ \sum_{k,l} a_{kl}\lambda^{-pk}\eta(p-l,q-k) \end{pmatrix} \\
&= \sum_{k,l} a_{kl}\lambda^{-pk} \begin{pmatrix} \xi \\ \eta \end{pmatrix}(p-l,q-k)
\end{aligned} \tag{24}$$

Denote by $\delta_{k,l} \in l^2(\mathbb{Z}^2)$ the vector defined as $\delta_{k,l}(k,l) = 1$ and $\delta_{k,l}(p,q) = 0$ for $(p,q) \neq (k,l)$.

For $k,l \in \mathbb{Z}$ let $\varphi(k,l) \in [0,2\pi)$ be the phase such that

$$e^{2i\varphi(k,l)} = \frac{k+il}{\sqrt{k^2+l^2}} .$$

This gives a new expression for the eigenvectors $\left|\psi^{\epsilon}_{k,l}\right\rangle$ which is easier to work with.

Corollary 5. *An orthonormal basis of eigenvectors of $\mathcal{D}$ is given by*

$$\left|\psi^{\epsilon}_{k,l}\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi(k,l)}\,\delta_{k,l} \\ \epsilon\, e^{-i\varphi(k,l)}\,\delta_{k,l} \end{pmatrix} ,$$

with eigenvalues $\lambda_{k,l,\epsilon} = \epsilon\sqrt{k^2+l^2}$, where $\epsilon \in \{\pm 1\}$ and $k,l \in \mathbb{Z}$.

Proof. The vectors $\left|\psi^{\epsilon}_{k,l}\right\rangle$ are the same as in Lemma 6.1, since

$$e^{i\varphi(k,l)} = \frac{\sqrt{k+il}}{\sqrt{\sqrt{k^2+l^2}}} \qquad \text{and} \qquad e^{-i\varphi(k,l)} = \frac{\sqrt{k-il}}{\sqrt{\sqrt{k^2+l^2}}} .$$

Therefore they form an orthonormal basis. Since the vectors $\left|\psi^{\epsilon}_{k,l}\right\rangle \in \mathrm{dom}(\mathcal{D}_0)$ are eigenvectors of $\mathcal{D}_0$ and $\mathcal{D}_{\restriction \mathrm{dom}(\mathcal{D}_0)} = \mathcal{D}_0$, they are eigenvectors of $\mathcal{D}$. □

In physics, the fact that $\mathcal{D}$ has a basis of orthonormal eigenvectors would be written like

$$\mathcal{D} = \sum_{\substack{k,l\in\mathbb{Z}, \\ \epsilon=\pm 1}} \lambda_{k,l,\epsilon} \left|\psi^{\epsilon}_{k,l}\right\rangle \left\langle\psi^{\epsilon}_{k,l}\right| . \tag{25}$$

On the right-hand side appears an infinite sum over operators. Usually, one would define the right-hand side to be the limit of the partial sums with respect to the operator norm.[9] However, since $\mathcal{D}$ is not bounded, this sum would not converge. Fortunately there is a topology in which the right-hand side of (25) does converge, called the weak operator topology. In this topology a series of operators $\sum A_k$ converges to an operator A (and we write $A = \sum A_k$) if and only if

$$\langle \psi_m | A | \psi_n \rangle = \sum_k \langle \psi_m | A_k | \psi_m \rangle ,$$

for a certain basis $\{|\psi_m\rangle\}$. For a more direct definition of the weak operator topology see [9]. The identity (25) follows immediately with respect to the weak operator topology.

We can now prove the central theorem of this section.

[9] The operator norm is defined in Section A.2 of the appendix.

Theorem 6.3. *The torus triple* $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$ *is a spectral triple.*

Proof. We check that $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$ satisfies the three conditions in Definition 2.1.

(i) By Proposition 5.2 we have a faithful representation $\rho : \mathcal{A}_\lambda \to B(l^2(\mathbb{Z}^2))$. Together with the identification from (23) this gives that $\mathcal{A}_\lambda$ is faithfully represented on $\mathcal{H}$.

(ii) Let $\mu \in \mathbb{C} \setminus \mathbb{R}$ be a number. We will show the resolvent $(\mathcal{D} + \mu)^{-1}$ is compact. Recall that we can decompose $\mathcal{D}$ according to (25). Since $-\mu \notin \mathbb{R}$ is not an eigenvalue, we can define a diagonal operator E by

$$E := \sum_{k,l,\epsilon} (\lambda_{k,l,\epsilon} + \mu)^{-1} \left|\psi^\epsilon_{k,l}\right\rangle \left\langle\psi^\epsilon_{k,l}\right| .$$

It easily follows that E is the inverse of $\mathcal{D} + \mu$. By Lemma 3.3.5 of [9], $E = (\mathcal{D} + \mu)^{-1}$ is compact iff its eigenvalues $(\lambda_{k,l,\epsilon} + \mu)^{-1}$ vanish at infinity. That is, iff for any $\delta > 0$ the set

$$\Lambda_\delta := \left\{(k, l, \epsilon) \;\middle|\; k, l \in \mathbb{Z},\, \epsilon = \pm 1,\, |(\lambda_{k,l,\epsilon} + \mu)^{-1}| \geqslant \delta\right\}$$

is finite. Suppose $(k, l, \epsilon) \in \Lambda_\delta$, then we must have $|\lambda_{k,l,\epsilon} + \mu| \leqslant 1/\delta$. Together with the definition of $\lambda_{k,l,\epsilon}$ and the triangle inequality, this gives $\sqrt{k^2 + l^2} \leqslant 1/\delta + |\mu|$. Therefore $|k|$ and $|l|$ are both bounded by $1/\delta + |\mu|$. Because of this, and since ϵ takes just two values, $(k, l, \epsilon) \in \Lambda_\delta$ takes only a finite amount of values. We conclude that the values $(\lambda_{k,l,\epsilon} + \mu)^{-1}$ vanish at infinity, and thus $(\mathcal{D} + \mu)^{-1}$ is compact.

(iii) Let $a \in \mathcal{A}_\lambda$, we need to show that $[\mathcal{D}, a]$ is bounded. We compute $[\mathcal{D}, a]$ by calculating $\mathcal{D}a$ and $a\mathcal{D}$ separately. The action of $\mathcal{D}a$ on $(\xi, \eta)^t \in l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2)$ is

$$\begin{aligned}
\mathcal{D}a \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p, q) &= \mathcal{D} \begin{pmatrix} a\xi \\ a\eta \end{pmatrix} (p, q) \\
&= \begin{pmatrix} a\eta(p, q)(p + iq) \\ a\xi(p, q)(p - iq) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k,l} a_{kl} \lambda^{-pk} \eta(p - l, q - k)(p + iq) \\ \sum_{k,l} a_{kl} \lambda^{-pk} \xi(p - l, q - k)(p - iq) \end{pmatrix} \\
&= \sum_{k,l} a_{kl} \lambda^{-pk} \begin{pmatrix} \eta(p - l, q - k)(p + iq) \\ \xi(p - l, q - k)(p - iq) \end{pmatrix} .
\end{aligned}$$

We compute the action of $a\mathcal{D}$ as

$$a\mathcal{D}\begin{pmatrix}\xi\\ \eta\end{pmatrix}(p,q) = \sum_{k,l} a_{kl}\lambda^{-pk}\mathcal{D}\begin{pmatrix}\xi\\ \eta\end{pmatrix}(p-l,q-k)$$

$$= \sum_{k,l} a_{kl}\lambda^{-pk}\begin{pmatrix}\eta(p-l,q-k)(p-l+i(q-k))\\ \xi(p-l,q-k)(p-l-i(q-k))\end{pmatrix}.$$

Putting the actions of $\mathcal{D}a$ and $a\mathcal{D}$ together gives

$$[\mathcal{D},a]\begin{pmatrix}\xi\\ \eta\end{pmatrix}(p,q) = \sum_{k,l} a_{kl}\lambda^{-pk}\begin{pmatrix}\eta(p-l,q-k)(l+ik)\\ \xi(p-l,q-k)(l-ik)\end{pmatrix}. \qquad (26)$$

This holds for all $a \in \mathcal{A}_\lambda$, so in particular for $u^k v^l \in \mathcal{A}_\lambda$:

$$[\mathcal{D},u^k v^l]\begin{pmatrix}\xi\\ \eta\end{pmatrix}(p,q) = \lambda^{-pk}\begin{pmatrix}\eta(p-l,q-k)(l+ik)\\ \xi(p-l,q-k)(l-ik)\end{pmatrix}.$$

Use that $\|(\xi,\eta)^t\|^2_{\mathcal{H}} = \|\xi\|_2^2 + \|\eta\|_2^2 = \sum(|\xi(k,l)|^2 + |\eta(k,l)|^2)$ to find

$$\begin{aligned}\left\|[\mathcal{D},u^k v^l]\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\|^2_{\mathcal{H}} &= \sum_{p,q}\Big(|\eta(p-l,q-k)(l+ik)|^2 \\ &\qquad + |\xi(p-l,q-k)(l-ik)|^2\Big) \\ &\leqslant (l^2+k^2)\sum_{p,q}\left(|\xi(p,q)|^2 + |\eta(p,q)|^2\right) \\ &= (k^2+l^2)\left\|\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\|^2_{\mathcal{H}}.\end{aligned}$$

And therefore, in the operator norm on $B(\mathcal{H})$ we have $\left\|[\mathcal{D},u^k v^l]\right\| \leqslant k^2 + l^2 < (|k|+1)^2(|l|+1)^2$. With this knowledge we can estimate the operator norm of $[\mathcal{D},a]$ by

$$\begin{aligned}\|[\mathcal{D},a]\| &= \left\|\sum_{k,l} a_{kl}[\mathcal{D},u^k v^l]\right\| \\ &\leqslant \sum_{k,l}|a_{kl}|\left\|[\mathcal{D},u^k v^l]\right\| \\ &\leqslant \sum_{k,l}|a_{kl}|(|k|+1)^2(|l|+1)^2 \\ &\leqslant C\|a\|_{(4)},\end{aligned}$$

using Lemma 4.5. Since $a \in \mathcal{A}_\lambda$, in particular the fourth Fréchet norm $\|a\|_{(4)}$ is a finite number. Therefore $[\mathcal{D},a]$ is bounded.

From (i), (ii) and (iii) it follows that $(\mathcal{A}_\lambda,\mathcal{H},\mathcal{D})$ is a spectral triple. □

6.2 Perturbing $\mathcal{D}$

As discussed in 2.2, we learned from [11] is that the perturbation semigroup of an algebra A of a spectral triple (A, H, D) describes perturbations of D. We now discuss what happens when we apply this technique to the noncommutative torus. The operator $\mathcal{D}$ from the torus triple can be perturbed by elements $\sum_j a_j \otimes b_j^{op} \in \mathrm{Pert}(\mathcal{A}_\lambda)$ in the following way:

$$\mathcal{D} \mapsto \mathcal{D}' := \sum_j a_j \mathcal{D} b_j \,.$$

We call $\mathcal{D}'$ the perturbed operator. We also write $\mathcal{D}' = \mathcal{D}'(\sum a_j \otimes b_j^{op})$ to stress that the perturbed operator is dependent on a semigroup element. For now let $\sum a_j \otimes b_j^{op}$ be fixed. It follows easily from the normalisation condition that

$$\mathcal{D}' = \mathcal{D} + A$$

where A, called the perturbation of $\mathcal{D}$, is defined as

$$A := \sum_j a_j [\mathcal{D}, b_j] \,.$$

To have a better understanding of the operator A, we relate it to the coefficient sequences that also occur in Theorem 4.9.

Theorem 6.4. *Write $c := \sum_j a_j \otimes \phi(b_j^{op}) \in (id \otimes \phi)(\mathrm{Pert}(\mathcal{A}_\lambda))$, and let (c_{klmn}) be the coefficient sequence of c. Then the perturbation of $\mathcal{D}$ can be written as*

$$A \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p, q) = \sum_{klmn} c_{klmn} \lambda^{-pk-pn+ln} \begin{pmatrix} \eta(p-l-m, q-k-n)(m+in) \\ \xi(p-l-m, q-k-n)(m-in) \end{pmatrix} .$$

Proof. Substituting b for a and replacing k, l with m, n in (26) gives

$$[\mathcal{D}, b] \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p, q) = \sum_{m,n} b_{mn} \lambda^{-pm} \begin{pmatrix} \eta(p-n, q-m)(n+im) \\ \xi(p-n, q-m)(n-im) \end{pmatrix} .$$

Therefore

$$\begin{aligned} a[\mathcal{D}, b] \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p, q) &= \sum_{k,l} a_{kl} \lambda^{-pk} [\mathcal{D}, b] \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p-l, q-k) \\ &= \sum_{klmn} a_{kl} b_{mn} \lambda^{-pk-(p-l)m} \begin{pmatrix} \eta(p-l-n, q-k-m)(n+im) \\ \xi(p-l-n, q-k-m)(n-im) \end{pmatrix} \\ &= \sum_{klmn} a_{kl} b_{nm} \lambda^{-pk-(p-l)n} \begin{pmatrix} \eta(p-l-m, q-k-n)(m+in) \\ \xi(p-l-m, q-k-n)(m-in) \end{pmatrix} . \end{aligned}$$

Now when we use $c_{klmn} = \sum_j (a_j)_{kl}\phi(b_j^{op})_{mn} = \sum_j (a_j)_{kl}(b_j)_{nm}$, and we switch the (finite) sum over j with the sum over k, l, m, n, we get

$$A \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p,q) = \sum_j a_j[\mathcal{D}, b_j] \begin{pmatrix} \xi \\ \eta \end{pmatrix} (p,q)$$
$$= \sum_{klmn} c_{klmn} \lambda^{-pk-pn+ln} \begin{pmatrix} \eta(p-l-m, q-k-n)(m+in) \\ \xi(p-l-m, q-k-n)(m-in) \end{pmatrix} .$$

□

An important consequence of Theorem 6.4 is that $A = \sum a_j[\mathcal{D}, b_j]$ is uniquely defined by the semigroup element $\sum a_j \otimes b_j^{op}$. If $\sum a'_j \otimes b'_j = \sum a_j \otimes b_j$ then the coefficient sequence (c'_{klmn}) of $\sum a'_j \otimes b'_j$ is equal to the coefficient sequence (c_{klmn}) of $\sum a_j \otimes b_j$. Therefore they induce the same A, in other words $\sum a'_j[\mathcal{D}, b'_j] = \sum a_j[\mathcal{D}, b_j]$. This is used in the following lemma.

Lemma 6.5. *The perturbation operator A is self-adjoint.*

Proof. To avoid confusion, denote $A \equiv A(c)$ for the perturbation operator of an element $c \in \mathrm{Pert}(\mathcal{A}_\lambda)$. By the discussion above, $c = d$ implies $A(c) = A(d)$. From the self-adjointness condition and Lemma 5.9 (together with the fact that $\Omega^2 = 1$) we get $A(c) = A(\Omega c^\dagger \Omega)$ and therefore also $\mathcal{D}'(c) = \mathcal{D}'(\Omega c^\dagger \Omega)$. Write $c = \sum_j a_j \otimes \phi(b_j^{op})$. Then $\mathcal{D}'(c) = \sum_j a_j \mathcal{D} b_j$ and

$$\mathcal{D}'(\Omega c^\dagger \Omega) = \sum_j b_j^* \mathcal{D} a_j^* = \sum_j b_j^* \mathcal{D}^* a_j^* = \big(\sum_j a_j \mathcal{D} b_j\big)^* = \mathcal{D}'(c)^* .$$

And therefore $\mathcal{D}'(c) = \mathcal{D}'(c)^*$. We now have $\mathcal{D}'(c) = \mathcal{D} + A(c)$ where $\mathcal{D}'(c)$ and $\mathcal{D}$ are self-adjoint. Therefore also $A(c)$ is self-adjoint. □

As before, we fix $\sum a_j \otimes b_j^{op} \in \mathrm{Pert}(\mathcal{A}_\lambda)$ and write (c_{klmn}) for its coefficient sequence. We can now write the perturbation operator in a more elegant form than we did in Theorem 6.4.

Theorem 6.6. *There exists $E \in \mathcal{A}_\lambda$ such that*

$$A = \begin{pmatrix} 0 & E \\ E^* & 0 \end{pmatrix} .$$

Proof. From Theorem 6.4 follows immediately that $A = \begin{pmatrix} 0 & E \\ E^* & 0 \end{pmatrix}$ for some $E \in B(l^2(\mathbb{Z}^2))$, but what is this E?

$$\begin{aligned} E\xi(p,q) &= \sum_{klmn} c_{klmn} \lambda^{-p(k+n)+ln} \xi(p-l-m, q-k-n)(m-in) \\ &= \sum_{klmn} c_{k-n,l-m,m,n} \lambda^{-pk+(l-m)n} \xi(p-l, q-k)(m-in) \\ &= \sum_{k,l} \left(\sum_{m,n} c_{k-n,l-m,m,n} \lambda^{(l-m)n} (m-in) \right) \lambda^{-pk} \xi(p-l, q-k) \end{aligned}$$

This gives the formula from Proposition 5.3, with E instead of $a \in \mathcal{A}_\lambda$, when we define

$$E_{kl} := \sum_{m,n} c_{k-n,l-m,m,n} \lambda^{(l-m)n} (m-in)\,. \tag{27}$$

Thanks to Theorem 4.2 we only need to show that $(E_{kl}) \in \mathcal{A}_\lambda$, since this implies $\sum E_{kl} u^k v^l \in \mathcal{A}_\lambda$ and since $(\sum E_{kl} u^k v^l)\xi(p,q) = E\xi(p,q)$ as we have just shown, we must have $E = \sum E_{kl} u^k v^l \in \mathcal{A}_\lambda$. Using the triangle inequality gives us

$$\begin{aligned} &\sup_{k,l} |E_{kl}|(|k|+1)^p(|l|+1)^p \\ &\qquad \leqslant \sup_{k,l} \sum_{m,n} |c_{k-n,l-m,m,n}||m-in|(|k|+1)^p(|l|+1)^p \\ &\qquad \leqslant \sum_{m,n} \sup_{k,l} |c_{k-n,l-m,m,n}||m-in|(|k|+1)^p(|l|+1)^p \\ &\qquad = \sum_{m,n} \sup_{k,l} |c_{klmn}||m-in|(|k+n|+1)^p(|l+m|+1)^p\,. \end{aligned}$$

We use (14) and $|m-in| \leqslant |m|+|n| < (|m|+1)(|n|+1)$ to find that the above expression is smaller than or equal to

$$\begin{aligned} &\sum_{m,n} \sup_{k,l} |c_{klmn}| 2^{2p}(p+1)^2(|k|+1)^p(|l|+1)^p(|m|+1)^{p+1}(|l|+1)^{p+1} \\ &\leqslant \sum_{m,n} \frac{1}{(|m|+1)^2(|n|+1)^2} 2^{2p}(p+1)^2 \left\|c\right\|_{(p+3)} < \infty\,. \end{aligned}$$

This holds for every $p \in \mathbb{N}$, and therefore we have proven $(E_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$, which means we are done. □

We now have a self-adjoint element $E \in \mathcal{A}_\lambda$ which characterises the perturbation operator A. There are two things to be said about E, just looking at its definition in (27).

First of all we can write the coefficients of E in a more familiar way by substituting $m \mapsto l - n$ and $n \mapsto k - m$. This gives

$$E_{kl} = \sum_{m,n} c_{m,n,l-n,k-m} \lambda^{n(k-m)} ((l-n) - i(k-m)) \,,$$

which looks very much like the normalization condition from Theorem 3.4. In fact, only the factor $(l-n)-i(k-m)$ prevents us from concluding that $E = 1$.

Second of all we can consider the perturbations originating from the polynomial torus, and in that case find that E is also an element of the polynomial torus.

Corollary 6. *Let E be as in Theorem 6.6 If $c \in Pert(A_\lambda) \subset Pert(\mathcal{A}_\lambda)$, then $E \in A_\lambda$.*

Proof. We have seen that in the polynomial case $(c_{klmn}) \in C_c(\mathbb{Z}^4)$. Let $N \in \mathbb{N}$ be such that $c_{klmn} \neq 0$ only if $|k|, |l|, |m|, |n| \leqslant N$. Now assume $|k| > 2N$. Then we have either $|m| > N$ or $|k - m| > N$. Therefore $c_{m,n,l-n,k-m} = 0$ for all $m, n \in \mathbb{Z}$, and thus $E_{kl} = 0$. By an analogous argument $|l| > 2N$ also implies $E_{kl} = 0$. Therefore $(E_{kl}) \in C_c(\mathbb{Z}^2)$ and we may conclude $E \in A_\lambda$. □

Therefore Theorem 6.6 still holds when we replace $\mathcal{A}_\lambda$ with A_λ. This illustrates the analogy between the polynomial noncommutative torus and the smooth noncommutative torus, and concludes all fundamental results we will prove for these two algebras. In the next section we take a step back, and look at more general spectral triples.

7 Outlook

Let us briefly recall what we have done up until now. In Section 2, we introduced some basic notions of noncommutative geometry. We stated what a spectral triple was, and defined the perturbation semigroup for a general unital algebra A. We saw that, when A belongs to a spectral triple (A, H, D), the perturbation semigroup of A gives rise to perturbations of D.

In Section 3 up to Section 6 we exclusively dealt with the algebra A_λ and its completion $\mathcal{A}_\lambda$. We calculated two new expressions for $\text{Pert}(\mathcal{A}_\lambda)$.

In Section 6 we proved that $\mathcal{A}_\lambda$ belongs to a spectral triple $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$, and subsequently showed which perturbations of $\mathcal{D}$ are obtained from $\mathrm{Pert}(\mathcal{A}_\lambda)$. Importantly, we made much use of the basis $\{u^k v^l | k, l \in \mathbb{Z}\}$. Another fact we learned on the way is that the self-adjoint operator $\mathcal{D}$ has a basis of eigenvectors.

This is where this section picks up. The first result in this section, Lemma 7.1, states that for every spectral triple (A, H, D), the operator D has a basis of eigenvectors. It is for this reason we look at more general spectral triples instead of the torus triple only.

7.1 A Result for Real Spectral Triples

In this section we will express H into a basis of eigenvectors of D. This basis induces a basis for $B(H)$ and consequently a basis for $B(H) \otimes B(H)$. Seeing A as a set of operators on H, we can express A in terms of basis coefficients, and similarly for elements of $A \otimes A^{op}$. Concluding the first part of this section we express $\mathrm{Pert}(A)$ in terms of basis coefficients related to the operator D. In the second part of this section we will look once more at the torus triple, and see how the general expression for $\mathrm{Pert}(A)$ relates to some of the results we had so far.

Lemma 7.1. *Let (A, H, D) be a spectral triple. Then H has a countable basis of eigenvectors of D.*

Proof. Let $\mu \notin \mathbb{R}$ be a complex number. By definition of a spectral triple the bounded operator $(D+\mu)^{-1}$ is compact. Since $D = D^*$, we have

$$(D+\mu)^*(D+\mu) = D^2 + \mu D + \overline{\mu} D + |\mu|^2 = (D+\mu)(D+\mu)^* .$$

Thus $D+\mu$ is normal and therefore also $(D+\mu)^{-1}$ is normal. We can deploy the spectral theorem for compact normal operators on $(D+\mu)^{-1}$. We find that there exists an orthonormal basis $\{|\psi_k\rangle \, | k \in I\}$ for H consisting of eigenvectors of $(D+\mu)^{-1}$ with eigenvalues λ_k such that

$$(D+\mu)^{-1} = \sum_{k \in I} \lambda_k \, |\psi_k\rangle \, \langle\psi_k| \ ,$$

and $\{k \in I | \lambda_k \neq 0\}$ is countable. This is proven in [22]: Theorem 2 and Corollary 4 in Chapter 14. Let us assume $\lambda_k = 0$ for a certain $k \in I$. Then

$$|\psi_k\rangle = (D+\mu)(D+\mu)^{-1} \, |\psi_k\rangle = (D+\mu)0 = 0 \, ,$$

which contradicts $|\psi_k\rangle$ being an eigenvector. Therefore $I = \{k \in I | \lambda_k \neq 0\}$ and I is countable. Define the operator $E := \sum_{k\in I}(\frac{1}{\lambda_k} - \mu) |\psi_k\rangle \langle\psi_k|$, which by orthonormality has $|\psi_k\rangle$ as its eigenvectors. Since these eigenvectors form a basis, it is easy to check that $E + \mu$ is the inverse of $(D + \mu)^{-1}$. Therefore $D = E$. We conclude that $\{\psi_k | k \in I\}$ is a countable basis for H consisting of eigenvectors of D. □

Since in general we do not have $A^{op} \cong A$, it is difficult to repeat the procedure of Section 3.2 to rewrite Pert(A). Therefore we introduce some additional information about (A, H, D), by defining what is called a 'real structure'. This notion was introduced by Alain Connes in 1995, in [5], in order to more elegantly describe the standard model. We will restate two of the definitions given by Connes, and use only some crucial parts of them.

Definition 7.1. *A spectral triple (A, H, D) is said to be* ***even*** *if it comes with an operator γ in H such that $\gamma^* = \gamma$, $\gamma^2 = 1$, $D\gamma = -\gamma D$ and for all $a \in A$ also $\gamma a = a\gamma$.*

This precise definition is not very important, suffice it to say it is normal for a spectral triple to be even. For instance, the torus triple $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$ together with the operator $\gamma : \mathcal{H} \to \mathcal{H}$ defined by

$$\gamma \begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} \xi \\ -\eta \end{pmatrix}$$

is an even spectral triple, as is easy to check.
This allows us to define the notion of a 'real spectral triple', introduced by Connes in [5].

Definition 7.2. *Let (A, H, D) be an even spectral triple. A* ***real structure*** *J is an antilinear isometry*[10] *on H such that*

(i) $JD = DJ$, $J^2 = \epsilon$, $J\gamma = \epsilon'\gamma J$ for $\epsilon, \epsilon' \in \{\pm 1\}$.

(ii) For all $a, b \in A$ the operators a and $[D, a]$ commute with JbJ^*.

If a spectral triple (A, H, D) has a real structure J, we call (A, H, D, J) a ***real spectral triple***[11].

Again, most of the details will not be important, we will simply use key parts of this definition. It is not far-fetched to restrict ourselves to real spectral

[10] That is, $J(\xi + \mu\eta) = J\xi + \overline{\mu}J\eta$ and $\|J\xi\| = \|\xi\|$ for all $\xi, \eta \in H, \mu \in \mathbb{C}$.
[11] We ignore the obvious fact that (A, H, D, J) is not a triple but a quadruple.

triples. The papers [10] and [11] that lay the ground for this thesis also concern real spectral triples, and one needs a real structure to define spin (as explained in [5]).

For the rest of this section, we fix a real spectral triple (A, H, D, J). By part (i) of Definition 2.1 we can faithfully represent A on $B(H)$. For simplicity we will take $A \subseteq B(H)$, in effect identifying A with its image under the faithful representation. The eigenvectors of D from Lemma 7.1 will be denoted by $\{|\psi_k\rangle\}_{k\in I}$, where I is a countable index set. One could take $I = \mathbb{N}$ for simplicity or $I = \mathbb{Z}$ to match the notation of the previous sections. We will need one more notation to state our main result.

Definition 7.3. *For $a \in \mathcal{A}_\lambda$, $k, l \in I$, define the matrix element a^{kl} as*

$$a^{kl} := \langle\psi_k|a|\psi_l\rangle \ .$$

Matrix elements are in particular useful when we are working with the weak operator topology on $B(H)$, explained in section 6. We observe for $a \in \mathcal{A}_\lambda$ that

$$\langle\psi_m|a|\psi_n\rangle = a^{mn} = \sum_{k,l\in I} a^{kl} \langle\psi_m|\psi_k\rangle \langle\psi_l|\psi_n\rangle \ .$$

Therefore we have, in the weak operator topology, the formula

$$a = \sum_{k,l\in I} a^{kl} |\psi_k\rangle \langle\psi_l| \ .$$

Note the resemblance with (5). Since elements of A are associated with a 'sequence' $(a^{kl})_{k,l\in I}$, we can also associate elements of $A \otimes A^{op}$ with corresponding 'sequences', as follows.

Definition 7.4. *To an element $c = \sum_j a_j \otimes b_j^{op} \in A \otimes A^{op}$ we associate the complex numbers*

$$c^{klmn} := \sum_j a_j^{kl} b_j^{mn} ,$$

for all $k, l, m, n \in I$.

With this definition in place we can state the main theorem of this section.

Theorem 7.2. *We have the equality*

$$\mathrm{Pert}(A) = \left\{ c \in A \otimes A^{op} \;\middle|\; \sum_l c^{klln} = \delta_k(n) \, , \; c^{klmn} = \overline{c^{nmlk}} \right\} .$$

Proof. Write $c = \sum_j a_j \otimes b_j^{op}$. We are going to show that the normalisation condition on c is equivalent to $\sum_l c^{klln} = \delta_k(n)$ and that the self adjointness condition is equivalent to $c^{klmn} = \overline{c^{nmlk}}$. First off is the normalisation condition, for which we have

$$\sum_j a_j b_j = \sum_j \left(\sum_{k,l} a_j^{kl} \left|\psi_k\right\rangle \left\langle\psi_l\right| \right) \left(\sum_{m,n} b_j^{mn} \left|\psi_m\right\rangle \left\langle\psi_n\right| \right) .$$

We now want to change the order of summation. This is allowed since the sum over j is finite. We find

$$\begin{aligned} \sum_j a_j b_j &= \sum_{klmn} \sum_j a_j^{kl} \left|\psi_k\right\rangle \left\langle\psi_l\right| b_j^{mn} \left|\psi_m\right\rangle \left\langle\psi_n\right| \\ &= \sum_{klmn} \sum_j a_j^{kl} b_j^{mn} \left|\psi_k\right\rangle \delta_l(m) \left\langle\psi_n\right| \\ &= \sum_{k,l,n} \sum_j a_j^{kl} b_j^{ln} \left|\psi_k\right\rangle \left\langle\psi_n\right| \\ &= \sum_{k,l,n} c^{klln} \left|\psi_k\right\rangle \left\langle\psi_n\right| . \end{aligned}$$

And therefore the normalisation condition becomes, by what is known as the completeness relation in physics and spectral decomposition in mathematics,

$$\begin{aligned} \sum_j a_j b_j = 1 &\iff \sum_{k,l,n} c^{klln} \left|\psi_k\right\rangle \left\langle\psi_n\right| = \sum_k \left|\psi_k\right\rangle \left\langle\psi_k\right| \\ &\iff \sum_l c^{klln} = \delta_k(n) . \end{aligned}$$

The self-adjointness condition will prove to be more difficult. Define the function $\phi : A^{op} \to A$ by

$$\phi(a^{op}) := J a^* J^* .$$

We could go on to prove that ϕ is an isomorphism, and from there follow the path taken in Section 3.3, but we leave that trail for the reader. As instead, we only use injectivity of ϕ, which follows from:

$$J a^* J^* = J b^* J^* \;\Rightarrow\; J^* J a^* J^* J = J^* J b^* J^* J \;\Rightarrow\; a^* = b^* \;\Rightarrow\; a = b .$$

The fact that ϕ is injective yields the equivalence

$$\sum_j a_j \otimes b_j = \sum_j b_j^* \otimes a_j^{*op} \iff \sum_j a_j \otimes J b_j^* J^* = \sum_j b_j^* \otimes J a_j J^* .$$

We can express the right-most condition in terms of the coefficients defined in Definition 7.4. In fact, we have

$$\begin{aligned} Ja^* J^* &= J\Big(\sum_{k,l} a^{kl} \, |\psi_k\rangle \langle\psi_l|\Big)^* J^* = J \sum_{k,l} \overline{a^{kl}} \, |\psi_l\rangle \langle\psi_k| \, J^* \\ &= \sum_{kl} a^{kl} J \, |\psi_l\rangle \langle\psi_k| \, J^* , \end{aligned}$$

and this in turn implies

$$\begin{aligned} \sum_j a_j \otimes J b_j^* J^* &= \sum_j \sum_{klmn} a_j^{kl} b_j^{mn} \, |\psi_k\rangle \langle\psi_l| \otimes J \, |\psi_m\rangle \langle\psi_n| \, J^* \\ &= \sum_{klmn} c^{klmn} \, |\psi_k\rangle \langle\psi_l| \otimes J \, |\psi_m\rangle \langle\psi_n| \, J^* , \end{aligned}$$

where we swapped the order of the sums. Remember that the sums over j are finite, so this is allowed. In the same way

$$\begin{aligned} \sum_j b_j^* \otimes a_j^{*op} &= \sum_j \sum_{klmn} \overline{b_j^{kl}} \, \overline{a_j^{mn}} (|\psi_k\rangle \langle\psi_l|)^* \otimes J(|\psi_m\rangle \langle\psi_n|)^* J^* \\ &= \sum_{klmn} \overline{\sum_j a_j^{mn} b_j^{kl}} \, |\psi_l\rangle \langle\psi_k| \otimes J \, |\psi_n\rangle \langle\psi_m| \, J^* \\ &= \sum_{klmn} \overline{c^{nmlk}} \, |\psi_k\rangle \langle\psi_l| \otimes J \, |\psi_m\rangle \langle\psi_n| \, J^* . \end{aligned}$$

Therefore

$$\begin{aligned} \sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{*op} \iff & \sum_{klmn} c^{klmn} \, |\psi_k\rangle \langle\psi_l| \otimes J \, |\psi_m\rangle \langle\psi_n| \, J^* \\ & = \sum_{klmn} \overline{c^{nmlk}} \, |\psi_k\rangle \langle\psi_l| \otimes J \, |\psi_m\rangle \langle\psi_n| \, J^* \\ \iff & c^{klmn} = \overline{c^{nmlk}} . \end{aligned}$$

This completes the proof.

□

We have ended up with a new expression for the perturbation semigroup. In line with the goal of studying perturbations of the form $D' := \sum a_j D b_j$, the expression of Theorem 7.2 relates the perturbation semigroup to the operator D. How this relation brings us closer to our goal of studying D' is illuminated by a simple calculation, which shows that D' can be written as

$$D' = \sum_{k,l,n \in I} c^{klln} \lambda_l \, |\psi_k\rangle \langle\psi_n| \, , \tag{28}$$

where λ_l is the eigenvalue of $|\psi_l\rangle$. This expression shows the dependance of D' on the eigenvalues of D, and therefore on the spectrum of D, which makes this expression useful for studying spectra. Most importantly, (28) shows how D' depends on the coefficients c^{klmn}. The behaviour of the action $D \mapsto D'$ is thus captured by the constraints on the coefficients c^{klmn} as written in Theorem 7.2.

We have discussed real spectral triples in general, which made our results very abstract. This abstractness can be taken away by looking at an example of a real spectral triple.

7.2 Example: the Noncommutative Torus

In the final part of this thesis we discuss how the the general result for real spectral triples, Theorem 7.2, can be applied in the specific case of the noncommutative torus.

We have seen that the torus triple, denoted by $(\mathcal{A}_\lambda, \mathcal{H}, \mathcal{D})$, is a spectral triple. This triple can be given a real structure, as is done in [5]. (For a construction of four inequivalent real structures on the noncommutative torus, see [23].) Therefore the result of Theorem 7.2 is valid for $A = \mathcal{A}_\lambda$. Since in the resulting description of $\mathrm{Pert}(\mathcal{A}_\lambda)$ the real structure J does not appear, we will not define an explicit real structure for the torus triple.

How can the notation of 7.1 be understood for the special case of the noncommutative torus? First of all, the basis of eigenvectors of the general operator D was denoted by $\{|\psi_k\rangle \, | k \in I\}$. Now we already introduced the eigenvectors of $\mathcal{D}$, but they where denoted $\left|\psi^\epsilon_{k,l}\right\rangle$. This notational difference is overcome by using multi-indices $\mathbf{k} = (k_1, k_2, \epsilon_k)$. Let $I := \mathbb{Z} \times \mathbb{Z} \times \{\pm 1\}$ be the set of multi-indices and define for $\mathbf{k} \in I$ the vectors

$$|\psi_{\mathbf{k}}\rangle := \left|\psi^{\epsilon_k}_{k_1,k_2}\right\rangle \, .$$

It is in this notation that the results of Section 7.1 can be directly applied. We only need to trade italic letters for their bold counterparts. For instance, Definition 7.3 determines that the matrix elements of $a \in \mathcal{A}_\lambda$ are written as $a^{\mathbf{kl}} = \langle\psi_{\mathbf{k}}|a|\psi_{\mathbf{l}}\rangle$.

We will now write these matrix elements in terms of more familiar numbers.

Proposition 7.3. *For $\mathbf{k}, \mathbf{l} \in I$, the matrix elements of $a \in \mathcal{A}_\lambda$ are given by:*

$$a^{\mathbf{kl}} = \begin{cases} a_{k_2-l_2,k_1-l_1} \lambda^{k_1(l_2-k_2)} \cos(\varphi_{\mathbf{l}} - \varphi_{\mathbf{k}}) & \text{if } \epsilon_k = \epsilon_l \\ a_{k_2-l_2,k_1-l_1} \lambda^{k_1(l_2-k_2)} \sin(\varphi_{\mathbf{l}} - \varphi_{\mathbf{k}}) i & \text{if } \epsilon_k \neq \epsilon_l \end{cases}$$

where we write $\varphi_{\mathbf{k}} := \varphi(k_1, k_2)$, which does not depend on ϵ_k.

Proof. We first calculate the action of a on an eigenvector $|\psi_{\mathbf{l}}\rangle = \left|\psi^{\epsilon_l}_{l_1,l_2}\right\rangle$ of $\mathcal{D}$. Using (24) and the fact that $\delta_{l_1,l_2}(p-n, q-m) = \delta_{l_1+n,l_2+m}(p,q)$, we find

$$\begin{aligned} a\,|\psi_{\mathbf{l}}\rangle\,(p,q) &= \sum_{m,n} a_{mn} \lambda^{-pm} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi_{\mathbf{l}}} \delta_{l_1,l_2} \\ \epsilon_l e^{-i\varphi_{\mathbf{l}}} \delta_{l_1,l_2} \end{pmatrix} (p-n, q-m) \\ &= \sum_{m,n} a_{mn} \lambda^{-(l_1+n)m} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi_{\mathbf{l}}} \delta_{l_1+n,l_2+m)} \\ \epsilon_l e^{-i\varphi_{\mathbf{l}}} \delta_{l_1+n,l_2+m} \end{pmatrix} (p,q)\,. \end{aligned}$$

We can calculate a matrix element of a by taking the inner product of $|\psi_{\mathbf{k}}\rangle = \left|\psi^{\epsilon_k}_{k_1,k_2}\right\rangle$ with the above formula. We use that the inner product is continuous and linear to find

$$\begin{aligned} \langle\psi_{\mathbf{k}}|a|\psi_{\mathbf{l}}\rangle &= \sum_{m,n} a_{mn} \lambda^{-(l_1+n)m} \frac{1}{\sqrt{2}} \langle\psi_{\mathbf{k}}| \begin{pmatrix} e^{i\varphi_{\mathbf{l}}} \delta_{l_1+n,l_2+m} \\ \epsilon_l e^{-i\varphi_{\mathbf{l}}} \delta_{l_1+n,l_2+m} \end{pmatrix} \\ &= \sum_{m,n} a_{mn} \lambda^{-(l_1+n)m} \frac{1}{2} \left\langle \begin{pmatrix} e^{i\varphi_{\mathbf{k}}} \delta_{k_1,k_2} \\ \epsilon_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \delta_{k_1,k_2} \end{pmatrix}, \begin{pmatrix} e^{i\varphi_{\mathbf{l}}} \delta_{l_1+n,l_2+m} \\ \epsilon_l e^{-i\varphi_{\mathbf{l}}} \delta_{l_1+n,l_2+m} \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \sum_{m,n} a_{mn} \lambda^{-(l_1+n)m} \frac{1}{2} (e^{i(\varphi_{\mathbf{l}}-\varphi_{\mathbf{k}})} + \epsilon_k \epsilon_l e^{-i(\varphi_{\mathbf{l}}-\varphi_{\mathbf{k}})}) \langle \delta_{k_1,k_2}, \delta_{l_1+n,l_2+m} \rangle \\ &= a_{k_2-l_2,k_1-l_1} \lambda^{-k_1(k_2-l_2)} \frac{e^{i(\varphi_{\mathbf{l}}-\varphi_{\mathbf{k}})} + \epsilon_k \epsilon_l e^{-i(\varphi_{\mathbf{l}}-\varphi_{\mathbf{k}})}}{2}\,. \end{aligned}$$

If $\epsilon_k = \epsilon_l$ then $\epsilon_k \epsilon_l = 1$ and therefore $(e^{i\varphi} + \epsilon_k \epsilon_l e^{-i\varphi})/2 = \cos(\varphi)$. If $\epsilon_k \neq \epsilon_l$ then $\epsilon_k \epsilon_l = -1$ and $(e^{i\varphi} + \epsilon_k \epsilon_l e^{-i\varphi})/2 = \sin(\varphi) i$. Using this for $\varphi = \varphi_{\mathbf{l}} - \varphi_{\mathbf{k}}$ gives the proposition. ☐

We have expressed the coefficients $a^{\mathbf{kl}}$ in the coefficients a_{kl} with which we worked throughout this thesis. We can view this as a basis transformation relating two bases of $\mathcal{A}_\lambda$ (in fact $\{|\psi_{\mathbf{k}}\rangle \langle\psi_{\mathbf{l}}|\}$ is a basis of $B(H)$ and $\mathcal{A}_\lambda \subseteq B(H)$).

This allows us to easily relate the corresponding 'bases' of $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda^{op}$. Let $c \in \mathcal{A}_\lambda \otimes \mathcal{A}_\lambda^{op}$ be fixed, and write $c = \sum_j a_j \otimes b_j^{op}$. A little care is needed. In the identification of $\mathcal{A}_\lambda^{op}$ with $\mathcal{A}_\lambda$ we made in Section 3, we switched the places of u and v. In that section we defined a coefficient sequence (c_{klmn}) of elements of $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda$, which were obtained using the isomorphism $(id \otimes \phi)$. Therefore c corresponds to (c_{klmn}) in Theorem 4.9 when we define

$$c_{klmn} := \sum_j (a_j)_{kl} \phi(b_j^{op})_{mn} = \sum_j (a_j)_{kl} (b_j)_{nm} \,.$$

Notice the switch of n and m in the right-most expression. With the definition above, c is in $\mathrm{Pert}(\mathcal{A}_\lambda)$ if and only if (c_{klmn}) is in the set on the right side of Theorem 4.9. That is, if and only if it satisfies

$$\sum_{m,n} c_{m,l-n,n,k-m} \lambda^{(k-m)(l-n)} = 1_{kl} \quad \text{and} \quad c_{klmn} = \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn} \,.$$

Since these conditions are equivalent to respectively the normalisation condition and self-adjointness condition on c, we should get that, for instance, $c_{klmn} = \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn}$ is equivalent to $c^{\mathbf{klmn}} = \overline{c^{\mathbf{nmlk}}}$. We will explicitly show this equivalence, to illustrate the applicability of Theorem 7.2.

Proposition 7.4. *We have*

$$c^{\mathbf{klmn}} = \overline{c^{\mathbf{nmlk}}} \iff c_{klmn} = \overline{c_{-n,-m,-l,-k}} \lambda^{kl+mn} \,.$$

Proof. We have $c^{\mathbf{klmn}} = \sum_j a_j^{\mathbf{kl}} b_j^{\mathbf{mn}}$ by definition, and as we have proven in Proposition 7.3 we can express

$$\begin{aligned} a_j^{\mathbf{kl}} &= (a_j)_{k_2-l_2,k_1-l_1} \lambda^{k_1(l_2-k_2)} \frac{1}{2} (e^{i(\varphi_{\mathbf{l}}-\varphi_{\mathbf{k}})} + \epsilon_k \epsilon_l e^{-i(\varphi_{\mathbf{l}}-\varphi_{\mathbf{k}})}) \\ &\equiv (a_j)_{k_2-l_2,k_1-l_1} \lambda^{k_1(l_2-k_2)} z(\mathbf{k},\mathbf{l}) \,, \end{aligned}$$

and similarly for $b_j^{\mathbf{mn}}$. Here we implicitly defined a function $z : I \times I \to \mathbb{C}$ for notational brevity. The above results imply

$$\begin{aligned} c^{\mathbf{klmn}} &= \sum_j (a_j)_{k_2-l_2,k_1-l_1} (b_j)_{m_2-n_2,m_1-n_1} \lambda^{k_1(l_2-k_2)+m_1(n_2-m_2)} z(\mathbf{k},\mathbf{l}) z(\mathbf{m},\mathbf{n}) \\ &= c_{k_2-l_2,k_1-l_1,m_1-n_1,m_2-n_2} \lambda^{k_1(l_2-k_2)+m_1(n_2-m_2)} z(\mathbf{k},\mathbf{l}) z(\mathbf{m},\mathbf{n}) \,. \end{aligned}$$

It is easy to check that $\overline{z(\mathbf{k},\mathbf{l})} = z(\mathbf{l},\mathbf{k})$. This gives

$$\begin{aligned} \overline{c^{\mathbf{nmlk}}} &= \overline{c_{n_2-m_2,n_1-m_1,l_1-k_1,l_2-k_2}} \lambda^{-n_1(m_2-n_2)-l_1(k_2-l_2)} \overline{z(\mathbf{n},\mathbf{m}) z(\mathbf{l},\mathbf{k})} \\ &= \overline{c_{n_2-m_2,n_1-m_1,l_1-k_1,l_2-k_2}} \lambda^{(k_2-l_2)(k_1-l_1)+(m_1-n_1)(m_2-n_2)} \\ &\quad \times \lambda^{k_1(l_2-k_2)+m_1(n_2-m_2)} z(\mathbf{m},\mathbf{n}) z(\mathbf{k},\mathbf{l}) \,. \end{aligned}$$

Our final result is captured in the equivalences

$$
\begin{aligned}
c^{\mathbf{klmn}} = \overline{c^{\mathbf{nmlk}}} \iff\ & c_{k_2-l_2,k_1-l_1,m_1-n_1,m_2-n_2} \\
& = \overline{c_{n_2-m_2,n_1-m_1,l_1-k_1,l_2-k_2}}\lambda^{(k_2-l_2)(k_1-l_1)+(m_1-n_1)(m_2-n_2)} \\
\iff\ & c_{klmn} = \overline{c_{-n,-m,-l,-k}}\lambda^{kl+mn} .
\end{aligned}
$$

□

The above proposition shows that the expression for the self-adjointness condition we derived in Section 3.3 (a result obtained by specifically investigating the noncommutative torus) can also be obtained from a general result. This illustrates the power of a general approach. Formulas derived for all spectral triples, can afterwards be applied to a special case. This road is not necessarily easier, but may prove more fruitful. For instance, Theorem 7.2 could be applied to any real spectral triple, including those with direct applications to physics.

Real Progress

In retrospect, the noncommutative torus forms a great introduction to the field of noncommutative geometry. It is nontrivial in many aspects, but is still easy enough to grasp. It is diverse, as it comes in different versions, namely the polynomial version, the smooth version and the C*-algebra. These versions are ordered according to complexity just as they are ordered according to size. It seems that most authors prefer to introduce the C*-algebra first. This thesis depicts the advantages of starting with the polynomial version.

As good an example the noncommutative torus is, the expressions encountered in this thesis where often long and inelegant. One has to resort to multi-indices when things get too difficult, but at the same time keep track of the indices in the exponent of λ. The perturbation semigroup of the noncommutative torus allows for an interesting topic of research, but real progress is made when considering general spectral triples.

A Appendix

A.1 Algebras

An algebra is a space whose elements one may add and multiply. What we call an algebra is by others sometimes named an associative algebra over $\mathbb{C}$. We drop the word associative for brevity, and since all our scalars will be in $\mathbb{C}$, we drop 'over $\mathbb{C}$' as well.

Definition A.1. *An* ***algebra*** *is a vector space A together with a bilinear operation $\cdot : A \times A \to A$ such that*

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(ii) $(a + b) \cdot c = a \cdot c + b \cdot c$

(iii) $a \cdot (b + c) = a \cdot b + a \cdot c$

(iv) $\mu(a \cdot b) = (\mu a) \cdot b = a \cdot (\mu b)$

for all elements $a, b \in A$ and scalars $\mu \in \mathbb{C}$. Write $ab := a \cdot b$.

There exist special versions of algebras, so-called unital algebras and involutive algebras.

Definition A.2. *Let A be an algebra, then A is called*

- *a* ***unital algebra*** *if it has a unit $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.*
- *an* ***involutive algebra****, or sometimes a* ****-algebra****, if it has an anti-linear[12] function $^* : A \to A$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$.*

In this thesis we are often dealing with algebras that have all of the above properties, and these are called unital *-algebras. The reader is encouraged to check that $1^* = 1$ in any unital *-algebra.

We also introduce the concept of a homomorphism between algebras. A function is a homomorphism if it preserves all structure.

Definition A.3. *Let A, B be algebras, and $f : A \to B$ a function.*

- *We call f an* ***algebra homomorphism*** *if it is linear and multiplicative.*

[12]That is, $(a + \mu b)^* = a^* + \overline{\mu} b^*$.

- *Let A and B be unital (algebras). f is a* ***unital algebra homomorphism*** *if it is an algebra homomorphism and* $f(1) = 1$.
- *Let A and B be involutive (algebras). f is a* ***involutive algebra homomorphism*** *if it is an algebra homomorphism and* $f(a^*) = f(a)^*$ *for all* $a \in A$.

Instead of 'algebra homomorphism' or 'homomorphism of algebras', we often just say 'homomorphism' and likewise for the special cases above. We will often deal with homomorphisms of unital *-algebras, in which case the homomorphism satisfies all the above properties. A homomorphism of any kind is called an isomorphism (of that same kind) if it is bijective.

A semigroup is a set with considerably less structure than an algebra.

Definition A.4. *A semigroup is a set* S *with a multiplication* $\cdot : S \times S \to S$ *such that for all* $a, b, c \in S$ *we have*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c\,.$$

We immediately see that any algebra is a semigroup, and any subset of an algebra is a semigroup if it is closed under multiplication. For a function $f : S \to T$ to be a homomorphism of semigroups (where S and T are semigroups) it needs to preserve the semigroup structure, which is multiplication. Therefore we say f is a semigroup homomorphism if and only if it is multiplicative. In this thesis we sometimes implicitly use that an isomorphism of algebras is an isomorphism of semigroups.

A.2 Norms and Tensor Products

Definition A.5. *Let* X *be a Banach space. Denote by* $B(X)$ *the vector space of all bounded operators* $c : X \to X$.

Bounded means that the norm of c, defined as

$$\|c\| = \sup\left\{\|cx\| \mid x \in X\,, \|x\| \leqslant 1\right\}\,, \tag{29}$$

is finite. This norm makes $B(X)$ into a Banach space.

The norm of a tensor product can be defined in a bunch of different ways, see for instance [24]. We will define it in the following way, for it makes our proofs a bit lighter. If X, Y are a Banach spaces, and $\sum_j a_j \otimes b_j \in X \otimes Y$, then:

$$\left\| \sum_j a_j \otimes b_j \right\| := \sup_j \|a_j\| \, \|b_j\| \ . \tag{30}$$

It is easy to check that this is a norm on $X \otimes Y$. Even easier to check, yet important to notice, is that $\|a \otimes b\| = \|a\| \, \|b\|$. One calls a norm with this property a "cross norm". Much can be said about cross norms, for which we refer to [24]. Denote by $X \hat{\otimes} Y$ the completion of $X \otimes Y$ in the just mentioned norm.

Proposition A.1. *We have the following isometric isomorphism:*

$$l^2(\mathbb{Z}^2) \hat{\otimes} l^2(\mathbb{Z}^2) \cong l^2(\mathbb{Z}^4) \, .$$

Proof. We will define an isomorphism Γ on finite sequences. Denote by C_c the subset of l^2 of sequences with a finite number of nonzero elements. We will prove Γ is an isometry in the l^2-norm, which means it can be extended to l^2. Lastly, we will show Γ is an isometry in every Fréchet norm, which means it can be extended to $\mathcal{S}$. Define:

$$\Gamma : C_c(\mathbb{Z}^2) \otimes C_c(\mathbb{Z}^2) \to C_c(\mathbb{Z}^4)$$
$$\Gamma(\sum_r \xi_r \otimes \eta_r)(k, l, m, n) := \sum_r \xi_r(k, l) \eta_r(m, n) \, .$$

To show this map is surjective, take a generic $\varphi \in C_c(\mathbb{Z}^4)$. Use a bijection $f : \mathbb{Z}^2 \to \mathbb{Z}$ to define

$$\xi_{f(m,n)}(k, l) := \varphi(k, l, m, n)$$
$$\eta_r(m, n) := \begin{cases} 1 & \text{if } r = f(m, n) \\ 0 & \text{otherwise.} \end{cases}$$

To see that $\xi_r, \eta_r \in C_c(\mathbb{Z}^2)$, we remark that ξ_r has less nonzero terms than φ (which has only finitely many), and η_r has only one nonzero term.
Moreover, we have

$$\sum_r \xi_r(k, l) \eta_r(m, n) = \xi_{f(m,n)}(k, l) = \varphi(k, l, m, n) \, ,$$

hence

$$\Gamma(\sum_r \xi_r \otimes \eta_r) = \varphi \, .$$

Isometric in l^2-norm:

$$\begin{aligned}
\left\| \Gamma(\sum_r \xi_r \otimes \eta_r) \right\|_2^2 &= \sum_{k,l,m,n} |\sum_r \xi_r(k,l)\eta_r(m,n)|^2 \\
&= \sum_{k,l,m,n} \overline{\sum_r \xi_r(k,l)\eta_r(m,n)} \sum_s \xi_s(k,l)\eta_s(m,n) \\
&= \sum_{r,s} \sum_{k,l} \overline{\xi_r(k,l)}\xi_s(k,l) \sum_{m,n} \overline{\eta_r(m,n)}\eta_s(m,n) \\
&= \sum_{r,s} \langle \xi_r, \xi_s \rangle \langle \eta_r, \eta_s \rangle \\
&= \sum_{r,s} \langle \xi_r \otimes \eta_r \,,\, \xi_s \otimes \eta_s \rangle \\
&= \langle \sum_r \xi_r \otimes \eta_r \,,\, \sum_r \xi_r \otimes \eta_r \rangle \\
&= \left\| \sum_r \xi_r \otimes \eta_r \right\|_2^2 .
\end{aligned}$$

□

We would like to repeat the same procedure for the Schwarz space $\mathcal{S}$ instead of the space l^2 of absolute summable sequences. The space $\mathcal{S}(\mathbb{Z}^2)$ is not a Banach space, and therefore the norm on the tensor product becomes more subtle. Define:

$$\left\| \sum_j a_j \otimes b_j \right\|_{(p)} := \sup_{m \in \mathbb{Z}^4} \left| \sum_j (a_j)_{m_1,m_2} (b_j)_{m_3,m_4} \right| |m|_+^p$$

where we introduced the notation

$$|(m_1, m_2, m_3, m_4)|_+ := (|m_1| + 1)(|m_2| + 1)(|m_3| + 1)(|m_4| + 1) .$$

Corollary 7. *We have the following isometric isomorphism:*

$$\mathcal{S}(\mathbb{Z}^2) \hat{\otimes} \mathcal{S}(\mathbb{Z}^2) \cong \mathcal{S}(\mathbb{Z}^4) .$$

Proof. Thanks to the facts already proven about Γ on finite sequences, we only need to show Γ is isometric in the p^{th} Fréchet norm:

$$\left\| \Gamma(\sum_r \xi_r \otimes \eta_r) \right\|_{(p)}^2 = \sup_m |\sum_r \xi_r(m_1, m_2)\eta_r(m_3, m_4)||m|_{+1}^p$$
$$= \left\| \sum_r \xi_r \otimes \eta_r \right\|_{(p)}^2 .$$

The first equality is the definition of the regular norms on $\mathcal{S}(\mathbb{Z}^4)$, the second is our definition of the norms $\|\cdot\|_{(p)}$ on $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2)$. □

Using Γ we can identify the algebraic tensor product of Schwarz spaces, $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2)$, with a subset of $\mathcal{S}(\mathbb{Z}^4)$. We write $\mathcal{S}(\mathbb{Z}^2) \otimes \mathcal{S}(\mathbb{Z}^2) \subseteq \mathcal{S}(\mathbb{Z}^4)$.

A.3 Basis of the Noncommutative Torus

In this section we present a proof of a fact which is often taken for granted. Nevertheless, it is of fundamental importance for the noncommutative torus and this thesis in particular.

Proposition A.2. *The set* $\{u^k v^l | k, l \in \mathbb{Z}\}$ *is a basis for* A_λ.

This consists of two claims:

(i) $\operatorname{span}\{u^k v^l | k, l \in \mathbb{Z}\} = A_\lambda$.

(ii) The set is linearly independent, i.e. $\sum_{k,l} a_{kl} u^k v^l = 0$ implies $a_{kl} = 0$.

Proof of (i). Since A_λ has no smaller unital *-subalgebra such that $u, v \in A_\lambda$, and u, v and 1 are basis vectors themselves, it suffices to show that $\operatorname{span}\{u^k v^l | k, l \in \mathbb{Z}\}$ is closed under the *-algebra operations. Closure under addition and scalar multiplication follows immediately. Left to check is closure under multiplication and involution. For that purpose, let a, b be elements of the span, so that we can write

$$a = \sum_{k,l} a_{kl} u^k v^l , \qquad b = \sum_{k,l} b_{kl} u^k v^l ,$$

for finite sequences (a_{kl}) and (b_{kl}). Even though we may not use the *results* of Lemma 3.1 yet, we can make exactly the same calculations made in the proof to find:

$$a^* = \sum_{k,l} \overline{a_{-k,-l}} \lambda^{kl} u^k v^l , \tag{31}$$

$$ab = \sum_{k,l} \left(\sum_{m,n} a_{mn} b_{k-m,l-n} \lambda^{n(k-m)} \right) u^k v^l . \tag{32}$$

First we show that a^* is in the span. Since (a_{kl}) is a finite sequence, there exists K such that $a_{kl} = 0$ whenever $|k| > K$ or $|l| > K$. With the same K we have $\overline{a_{-k,-l}}\lambda^{kl} = 0$ whenever $|\text{-}k| > K$ or $|\text{-}l| > K$. Therefore $(\overline{a_{-k,-l}}\lambda^{kl})_{k,l\in\mathbb{Z}}$ is a finite sequence, so $a^* \in \operatorname{span}\{u^k v^l | k, l \in \mathbb{Z}\}$.

Showing that ab is in the span takes a little more effort. We can find $K \in \mathbb{N}$ such that $a_{kl} = b_{kl} = 0$ whenever $|k| > K$ or $|l| > K$. Now let $|k| > 2K$, and $m, n \in \mathbb{Z}$. Then either $|m| > K$ or $|k - m| > K$. So either $a_{mn} = 0$ or $b_{k-m,l-n} = 0$. So for all $m, n \in \mathbb{Z}$ we have $a_{mn}b_{k-m,l-n} = 0$, and therefore $\sum_{m,n} a_{mn}b_{k-m,l-n}\lambda^{n(k-m)} = 0$. The same holds whenever $|l| > 2K$, so it follows that $(\sum_{m,n} a_{mn}b_{k-m,l-n}\lambda^{n(k-m)})_{k,l\in\mathbb{Z}}$ is a finite sequence. Therefore $ab \in \operatorname{span}\{u^k v^l | k, l \in \mathbb{Z}\}$. □

Proof of (ii). By definition of A_λ we may write

$$A_\lambda = \mathbb{C}[u, u^*, v, v^*]/I\ ,$$

where I is the two-sided ideal generated by the polynomials

$$uu^* - 1\,; \quad u^*u - 1\,; \quad vv^* - 1\,; \quad v^*v - 1\,; \quad vu - \lambda uv\,.$$

In the remainder of the proof we will write '$\equiv$' for equality in A_λ and '$=$' only for equality in $\mathbb{C}[u, u^*, v, v^*]$.

For convenience we use the notation $w_1 = u, w_2 = u^*, w_3 = v, w_4 = v^*$. The polynomial ring $\mathbb{C}[u, u^*, v, v^*] = \mathbb{C}[w_1, w_2, w_3, w_4]$ is spanned by monomials x in the variables w_j, which can uniquely be written as

$$x = \prod_{k=1}^{K} w_{j_k}\ , \quad \text{where } K \in \mathbb{N},\ j_k \in \{1, 2, 3, 4\}\,.$$

Denote by X the collection of monomials. For $x \in X$ written as above we can define the 'degrees' of x as

$$\deg_u(x) := \sum_{k=1}^{K} (\delta_{j_k,1} - \delta_{j_k,2})\,, \qquad \deg_v(x) := \sum_{k=1}^{K} (\delta_{j_k,3} - \delta_{j_k,4})$$
$$\deg(x) := (\deg_u(x), \deg_v(x))\,. \tag{33}$$

Loosely speaking, $\deg_u(x)$ gives the amount of u in x minus the amount of u^*. We also want a measure for the ordering of the w_j in x. Again loosely speaking, how many v's (minus v^*) are before the u's (minus u^*) in x? This is captured in the phase of x, defined by:

$$\operatorname{phase}(x) := \sum_{k=1}^{K} \sum_{l=1}^{k-1} (\delta_{j_k,1} - \delta_{j_k,2})(\delta_{j_l,3} - \delta_{j_l,4})\,.$$

We can now define a linear function $\varphi : \mathbb{C}[u, u^*, v, v^*] \to \mathbb{C}$ by

$$\varphi(x) := \lambda^{\mathrm{phase}(x)} \delta_{\deg(x),0}$$

which is better recognized by the following rule:

$$\varphi(\sum_{k,l} a_{kl} u^k v^l) = a_{00} \, . \tag{34}$$

Lemma A.3. *We have* $\varphi_{\restriction I} = 0$.

We postpone the proof. This means that for $a, b \in \mathbb{C}[u, u^*, v, v^*]$ with $a \equiv b$ we have $\varphi(a) = \varphi(b)$. So we can view φ as the corresponding function on equivalence classes, and write $\varphi : A_\lambda \to \mathbb{C}$.

Lemma A.4. *Let* $a = \sum_{k,l} a_{kl} u^k v^l$*, then* $\varphi(a^* a) = \sum_{k,l} |a_{kl}|^2$.

This will also be proven after we finish the proof of Proposition A.2. Write $a = \sum_{k,l} a_{kl} u^k v^l$. Then $a \equiv 0$ implies $a^* a \equiv 0$ and thus $\varphi(a^* a) = \varphi(0) = 0$, so by Lemma A.4:

$$\sum_{k,l} |a_{kl}|^2 = 0 \, ,$$

from which follows that $a_{kl} = 0$ for all $k, l \in \mathbb{Z}$. □

Now we present the proofs of the two lemmas we just used.

Proof of Lemma A.3. Because of linearity of φ it suffices to show $\varphi(xhy) = 0$ for the elements h that generate I and monomials $x, y \in X$. We will only show $\varphi(x(u^* u - 1)y) = 0$ and $\varphi(x(vu - \lambda uv)y) = 0$. The claims for the other generators of I follow analogously. Let $x, y \in X$ be arbitrary, and write

$$x = \prod_{k=1}^{K} w_{j_k} \, , \qquad y = \prod_{k=1}^{K'} w_{j'_k} \, .$$

From (33) easily follows:

$$\deg(xy) = \deg(x) + \deg(y) \, , \tag{35}$$

which is what you would expect from a degree. This gives

$$\begin{aligned} \deg(xu^* uy) &= \deg(x) + \deg(u^*) + \deg(u) + \deg(y) \\ &= \deg(x) + (-1, 0) + (1, 0) + \deg(y) \\ &= \deg(xy) \, . \end{aligned} \tag{36}$$

From (35) similarly follows $\deg(xvuy) = \deg(xuvy)$. Before we try to do the same thing for the phase, a little more care is needed. First we define

$$j''_k := \begin{cases} j_k & \text{for } 1 \leqslant k \leqslant K \\ j'_{k-K} & \text{for } K < k \leqslant K + K' \,. \end{cases}$$

This allows us to write

$$xy = \prod_{k=1}^{K+K'} w_{j''_k} \,,$$

which gives

$$\begin{aligned}
\mathrm{phase}(xy) &= \sum_{k=1}^{K} \sum_{l=1}^{k-1} (\delta_{j''_k,1} - \delta_{j''_k,2})(\delta_{j''_l,3} - \delta_{j''_l,4}) \\
&\quad + \sum_{k=1}^{K'} \sum_{l=1}^{K+k-1} (\delta_{j''_{K+k},1} - \delta_{j''_{K+k},2})(\delta_{j''_l,3} - \delta_{j''_l,4}) \\
&= \mathrm{phase}(x) + \sum_{k=1}^{K'} \sum_{l=1}^{K+k-1} (\delta_{j'_k,1} - \delta_{j'_k,2})(\delta_{j''_l,3} - \delta_{j''_l,4}) \\
&= \mathrm{phase}(x) + \sum_{k=1}^{K'} \sum_{l=1}^{K} (\delta_{j'_k,1} - \delta_{j'_k,2})(\delta_{j_l,3} - \delta_{j_l,4})) \\
&\quad + \sum_{k=1}^{K'} \sum_{l=1}^{k-1} (\delta_{j'_k,1} - \delta_{j'_k,2})(\delta_{j''_{K+l},3} - \delta_{j''_{K+l},4}) \\
&= \mathrm{phase}(x) + \deg_v(x) \deg_u(y) \\
&\quad + \sum_{k=1}^{K'} \sum_{l=1}^{k-1} (\delta_{j'_k,1} - \delta_{j'_k,2})(\delta_{j'_l,3} - \delta_{j'_l,4}) \\
&= \mathrm{phase}(x) + \deg_v(x) \deg_u(y) + \mathrm{phase}(y) \,. \qquad (37)
\end{aligned}$$

We now have, using (35) and (37) repeatedly,

$$\begin{aligned}
\mathrm{phase}(xu^*uy) &= \mathrm{phase}(x) + \deg_v(x) \deg_u(u^*uy) + \mathrm{phase}(u^*u) \\
&\quad + \deg_v(u^*u) \deg_u(y) + \mathrm{phase}(y) \\
&= \mathrm{phase}(x) + \deg_v(x) \deg_u(y) + \mathrm{phase}(y) \\
&= \mathrm{phase}(xy) \,.
\end{aligned}$$

From (37) we also find that phase($xvuy$) = phase($xuvy$) + 1. Therefore

$$\begin{aligned}\varphi(xu^*uy) &= \lambda^{\text{phase}(xu^*uy)}\delta_{\deg(xu^*uy)=0} \\ &= \lambda^{\text{phase}(xy)}\delta_{\deg(xy)=0} \\ &= \varphi(xy)\,,\end{aligned}$$

thus $\varphi(x(u^*u-1)y) = 0$. Furthermore

$$\begin{aligned}\varphi(xvuy) &= \lambda^{\text{phase}(xvuy)}\delta_{\deg(xvuy)=0} \\ &= \lambda^{\text{phase}(xuvy)+1}\delta_{\deg(xuvy)=0} \\ &= \lambda\varphi(xuvy)\,,\end{aligned}$$

thus $\varphi(x(vu-\lambda uv)y) = 0$. This completes the proof.

□

Proof of Lemma A.3. As we had before in (31), $a^* \equiv \sum_{k,l} \overline{a_{-k,-l}}\lambda^{kl}u^k v^l$. When we combine this with (32), we find

$$a^*a \equiv \sum_{k,l}\left(\sum_{m,n}\overline{a_{-m,-n}}\lambda^{mn}a_{k-m,l-n}\lambda^{n(k-m)}\right)u^k v^l\,.$$

Combining this with (34) gives:

$$\begin{aligned}\varphi(a^*a) &= \sum_{m,n}\overline{a_{-m,-n}}a_{-m,-n}\lambda^{mn-nm} \\ &= \sum_{k,l}|a_{kl}|^2\,.\end{aligned}$$

□

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