SUPERSYMMETRIC GAUGE THEORIES IN NONCOMMUTATIVE GEOMETRY

FIRST STEPS TOWARDS THE MSSM

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ABSTRACT

One of the prime achievements of noncommutative geometry (NCG), a branch of mathematics, is its application to (particle) physics. We delve into the application of NCG to supersymmetry, which has received relatively little attention so far. The results are twofold. First, we present a noncommutative approach to the supersymmetric version of the Einstein-Yang-Mills system. Second, we make some first efforts towards a description of the Minimal Supersymmetric Standard Model (MSSM) by deriving super-QCD, the supersymmetric extension of the theory of quarks and gluons.

Keywords: noncommutative geometry, supersymmetry, MSSM, supersymmetric Einstein Yang Mills model, super-QCD.

Provando e riprovando

—Accademia del Cimento—

In that Empire, the Art of Cartography attained such Perfection that the map of a single Province occupied the entirety of a City, and the map of the Empire, the entirety of a Province. In time, those Unconscionable Maps no longer satisfied, and the Cartographers Guilds struck a Map of the Empire whose size was that of the Empire, and which coincided point for point with it.

—Jorge Luis Borges—
(translation by Andrew Hurley)

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You're trying so carefully to avoid graduating for all these years and in a moment of inattentiveness it sneaks up on you from behind and surprises you: suddenly time has come to make the last efforts of your thesis. Writing these acknowledgments is amongst them.

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1 INTRODUCTION

This thesis is the result of my Master's research in the field of Theoretical High Energy Physics, that I have been conducting under the supervision of Dr. W. van Suijlekom from summer '08 to August '09. It deals with the noncommutative approach to supersymmetry. Noncommutative geometry (NCG) is a vast and modern field of mathematics in which many notions from other mathematical branches meet. In a nutshell, NCG is about extends the correspondence between commutative algebras and spaces to algebras whose elements do not commute, corresponding to some virtual noncommuting space. In the past decades it became clear that noncommutative geometry is well suited for physical applications.

Indeed, by now one of the prime successes of noncommutative geometry from a physical point of view is that it provides an alternative, geometrical, interpretation of the Standard Model (SM). This Standard Model is a theory that describes nature at the smallest scales accessible to man. It does so in terms of elementary particles and their interactions. The accuracy of the SM is astonishing and its predictions got (and get) confirmed in experiments time and time again .[1] For theoretical reasons, however, it is widely recognized that at certain energies beyond those currently available in experiments the predictions made by the Standard Model should become less and less accurate. Its validity has to 'break down' at some point.

This is where supersymmetry (SUSY) comes into play. Supersymmetry describes a certain symmetry between the two types of elementary particles; fermions and bosons. That is, it states that for each fermion appearing in a certain theory there should be a boson that is somehow associated to it. In such a case we call these particles each others *superpartners*. A particularly important example of a supersymmetric theory is the Minimal Supersymmetric Standard Model (MSSM) which states that for every particle that appears in the Standard Model, there should be a —yet unobserved— superpartner.[2] The MSSM is regarded by many as being a promising candidate for accurately describing nature 'beyond the Standard Model'.

The subject of this thesis is to make first efforts to see to what extent noncommutative geometry is capable of giving an alternative description not only for the SM, but for the MSSM as well.

This thesis is divided into several parts. The first is dedicated to set the stage for a number of mathematical notions and concepts (not necessarily typical for noncommutative geometry) that will be needed in the rest of this thesis. I cannot emphasize enough that is not meant as —and by far is— an exhausting coverage of noncommutative geometry; in general, we only introduce here what will be needed later on. Due to the nature of this part, it will unfortunately be rather prosaic; it mainly contains definitions and examples.

1 All particles predicted by the SM have indeed been detected in particle accelerators. Except for one; the infamous Higgs boson.
2 This characterization is not completely right, but it is more or less.

What follows is the actual core of this thesis; the application of noncommutative geometry to supersymmetric field theory. In Part II (Chapters 9 to 11) we consider the Einstein-Yang-Mills model. Using a vocabulary that is (partly) typical for noncommutative geometry, we investigate to what extent this model exhibits supersymmetry.

In Part III we set our first steps on MSSM-grounds: we consider a simplification of the SM by discarding most of its particles and try to 'supersymmetrize' what is left using noncommutative geometry.

In both Part II as III there is a chapter that serves as an interlude; Chapter 9 provides a very sketchy introduction to supersymmetry, whereas Chapter 12 can be regarded as a brief introduction to the Standard Model, but in the approach of noncommutative geometry. Even if your are a physicist that is (very) familiar with the Standard Model, I can advice you to have a look at it.

Part I

PRELIMINARIES

2 $(C^*\text{-})$ALGEBRAS & MODULES

The concepts that appear in this chapter were already widely used before the advent NCG and are consequently certainly not unique for NCG. Despite that, they may be considered as being part of the toolbox of noncommutative geometry. We will briefly discuss what a C^*-algebra is and give some examples. After that, C^*-modules are introduced. At the end of the chapter the notion of Morita equivalence will be mentioned.

2.1 $(C^*\text{-})$ALGEBRAS

▷ **Definition 2.1 [(Unital) algebra].** *An (associative)* algebra $\mathcal{A}$ *is a vector space over some field* F *(in general* $\mathbb{R}$ *or* $\mathbb{C}$*), equipped with a product* $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *that obeys*

$$\begin{aligned} (ab)c &= a(bc) && \textit{(associativity)} \\ a(b+c) &= ab + ac, \quad (a+b)c = ac + bc && \textit{(linearity 1)} \\ f(ab) &= (fa)b = a(fb) && \textit{(linearity 2)} \end{aligned}$$

for all $a, b, c \in \mathcal{A}$, $f \in F$. *In general, an algebra need not have a* unit $1 \in \mathcal{A}$ *obeying* $1a = a1 = a \quad \forall\, a \in \mathcal{A}$. *If is does though, the algebra is called* unital.

Just as for a vector spaces, we can define on a algebras something called a norm:

▷ **Definition 2.2 [Norm].** *A* norm *is a map* $\|.\| : \mathcal{A} \to F$ *that obeys*

$$\begin{aligned} \|a+b\| &\leq \|a\| + \|b\| \quad \textit{(triangle inequality)} \\ \|fa\| &= |f|\|a\| \\ \|a\| &\geq 0 \textit{ and } \|a\| = 0 \Leftrightarrow a = 0 \end{aligned}$$

for all $a, b \in \mathcal{A}$, $f \in F$. *Here* $|.| : F \to \mathbb{R}$ *is a norm on* F.

▷ **Definition 2.3 [Involution, involutive algebra].** *An* involution *on an algebra* $\mathcal{A}$ *over a field* $\mathcal{F}$ *is a map* $^* : \mathcal{A} \to \mathcal{A}$ *(that sends* $a \to a^*$*), such that*

$$\begin{aligned} (ab)^* &= b^* a^*, \\ (a+b)^* &= a^* + b^*, \\ (a^*)^* &= a, \\ (fa)^* &= \bar{f} a^*. \end{aligned}$$

for all $a, b \in \mathcal{A}$, $f \in F$. *We call an algebra on which an involution is defined an* involutive algebra.

▷ **Definition 2.4 [Banach algebra].** *Let* $\mathcal{A}$ *be an associative algebra over the real or complex numbers that is equipped with a norm and that is* complete *with respect to that norm. If additionally we have*

$$\|ab\| \leq \|a\|\|b\| \quad \forall\, a, b \in \mathcal{A},$$

we call $\mathcal{A}$ *a* Banach algebra.

So the definition of a Banach algebra combines the notions 'algebra' and 'norm'. In the same sense we can introduce an involution and call what we end up with a C^*-algebra:

▷ **Definition 2.5 [C^*-algebra].** *A* C^*-algebra *is a Banach algebra* $\mathcal{A}$ *equipped with an involution that obeys*

$$\|a^*a\| = \|a\|^2 \quad \forall\, a \in \mathcal{A}. \tag{2.1}$$

Equation 2.1 is known as the C^*-identity.

Note that from being a Banach algebra it follows that $\|aa^*\| = \|a\|^2$ as well, since

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\| \Rightarrow \|a\| \leq \|a^*\|.$$

By replacing a with a^* we similarly have that $\|a^*\| \leq \|a\|$ and consequently $\|a^*\| = \|a\|$ for any $a \in \mathcal{A}$. Combining this with the C^*-identity yields

$$\|aa^*\| = \|(a^*a)^*\| = \|a^*a\| = \|a\|^2.$$

We will give one of the most elementary examples of a C^*-algebra.

▷ **Example 2.6.** *is the algebra $M_N(\mathbb{C})$ of $N \times N$ matrices with complex coefficients, regarded as operators on $\mathbb{C}^N$. If we define a norm on $M_N(\mathbb{C})$ by*

$$\|T\| := \min\{c \;:\; \|Tz\| \leq c\|z\| \;\forall\, z \in \mathbb{C}^N\|,$$

and let T^* be T after transposing and complex conjugation of its entries, it is seen that $M_N(\mathbb{C})$ is indeed a C^*-algebra.

Another example —and one that will play an important role— is the following.

▷ **Example 2.7.** *Let X be a compact Hausdorff space and let $C(X, \mathbb{C})$ (or just $C(X)$ for short) be the space of continuous functions from X to $\mathbb{C}$. This is not only a vector space over $\mathbb{C}$; we can turn this into an algebra as well by defining a product fg by*

$$(fg)(x) := f(x)g(x) \qquad \forall\, x \in X. \tag{2.2}$$

In addition, we can make it an involutive algebra by defining

$$C(X) \ni f \to f^* \in C(X), \qquad f^*(x) := \overline{f(x)} \qquad \forall x \in X. \tag{2.3}$$

Here the symbol $\overline{z}$ simply indicates the complex conjugate of $z \in \mathbb{C}$. Now we add to this data a norm (the so called sup-norm*) given by*

$$\|f\| := \sup_{x \in X} |f(x)|. \tag{2.4}$$

This can be seen to be a Banach algebra[1]*:*

$$\|fg\| = \sup_{x\in X} |(fg)(x)| = \sup_{x\in X} |f(x)||g(x)| \leq \sup_{x\in X} |f(x)| \sup_{x\in X} |g(x)| = \|f\|\|g\|.$$

In addition, it satisfies the C^*-identity (2.1) as well:

$$\|f^*f\| = \sup_{x\in M} |\overline{f(x)}f(x)| = \sup_{x\in M} |f(x)f(x)| = \|f^2\|.$$

1 The proof of completeness uses topology; we will omit it here but refer to [16, §2.7] instead.

We combine the above two examples.

▹ **Example 2.8.** *Let* $C(X, M_N(\mathbb{C}))$ *be the space of* matrix-valued functions*. A product and involution are given by*

$$(fg)(x) := f(x)g(x) \quad \forall x \in X,$$

and

$$f^*(x) := f(x)^*.$$

That this is also a C^**-algebra follows from the previous examples.*

For future use we can define a concept closely related to an algeba:

▹ **Definition 2.9 [Opposite algebra].** *Suppose* $\mathcal{A}$ *is an algebra over a field* F*. We can construct the* opposite algebra $\mathcal{A}^o$ *of* $\mathcal{A}$ *as follows; take*

$$\mathcal{A}^o := \{a^o, a \in \mathcal{A}\}, \tag{2.5}$$

with addition and scalar multiplication the same as in the case of $\mathcal{A}$*:* $(a+b)^o = a^o + b^o$*,* $(\lambda a)^o = \lambda a^o \ \forall a, b \in \mathcal{A}, \lambda \in F$*. The thing that makes* $\mathcal{A}^o$ *different from* $\mathcal{A}$ *is the multiplication:*

$$(ba)^o := a^o b^o.$$

2.2 MODULES

Another important concept that has a central role in what follows is that of a *module*.[2] It can be regarded as a generalization of the notion of a vector space in the sense that the latter is defined over a field whereas the first can be defined over any *ring*.

▹ **Definition 2.10 [Module].** *Let* $\mathcal{A}$ *be an algebra over a field* F*. A* right module $\mathcal{E}$ *is a vector space over* F *that allows for a right action* $\mathcal{E} \times \mathcal{A} \to \mathcal{E}$*,* $(\eta, a) \to (\eta a)$ *that fulfills*

$$\eta(ab) = (\eta a)b, \tag{2.6a}$$

$$\eta(a+b) = \eta a + \eta b, \tag{2.6b}$$

$$(\eta + \zeta)a = \eta a + \zeta a \tag{2.6c}$$

for all $\eta, \zeta \in \mathcal{E}$ *and* $a, b \in \mathcal{A}$*.*

N.B. As said above, modules are in principle defined in terms of rings. Rings and algebras are related however, which enables us to extend the definition of a module in terms of an algebra by additionally requiring that it is a vector space as well.

We can just as easily define a *left module* when it has a *left action* $(\eta, a) \to a\eta$. In addition we can define a *bimodule* if, for two algebras $\mathcal{A}$ and $\mathcal{B}$, there is a mutually commuting left-$\mathcal{A}$ and right-$\mathcal{B}$ action:

$$(a\eta)b = a(\eta b) \qquad \forall a \in \mathcal{A}, b \in \mathcal{B} \text{ and } \eta \in \mathcal{E}. \tag{2.7}$$

Needless to say, the properties (2.6a) to (2.6c) defining the module vary slightly in those cases.

The swarm of left/right/bimodules can sometimes be confusing. In cases where confusion may possibly occur we write ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$ for a left-$\mathcal{A}$ right-$\mathcal{B}$ bimodule.

2 For a good treatment [1] is recommended.

▷ **Example 2.11.** *Note that any (associative) algebra $\mathcal{A}$ is a module over itself, by using the multiplication on $\mathcal{A}$; just compare the Definitions 2.1 of an algebra to Definition 2.10 of a module.*

Let us expand the previous example a bit:

▷ **Example 2.12.** *Define $\mathcal{A}^n := \mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}$ (n times). Elements of this space are then n-tuples with entries in $\mathcal{A}$: $\{(a_1, a_2, \ldots, a_n) \mid a_1, \ldots a_n \in \mathcal{A}\}$. We make $\mathcal{A}^n$ a $\mathcal{A}$-bimodule by defining*

$$a(a_1, a_2, \ldots, a_n) := (aa_1, aa_2, \ldots, aa_n),$$
$$(a_1, a_2, \ldots, a_n)a := (a_1a, a_2a, \ldots, a_na).$$

The properties (2.6a) to (2.6c) are then easily verified.

If $\mathcal{A}$ is an algebr over $\mathbb{C}$, we can identify this space with $\mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}$ by

$$\mathcal{A}^n \ni \{a_1, \ldots a_n\} \leftrightarrow \sum_i e_i \otimes a_i \in \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}.$$

Here e_i is the i'th (canonical) basis vector of $\mathbb{C}^n$.

▷ **Example 2.13.** *Let X be a compact space. We denote by $\Gamma(X, S)$ the space of* spinor-valued functions; *for any $\psi \in \Gamma(X, S)$ we write $\psi(x) \in S_x$, where S_x is a complex, finite-dimensional vector space. $\Gamma(X, S)$ is seen to be a left $C(X)$-module by the definition*

$$(f\psi)(x) := f(x)\psi(x) \in S_x \quad \forall f \in C(M), \psi \in \Gamma(X, S). \tag{2.8}$$

From the compactness of X we infer that elements of $C(X)$ act as bounded multiplication operators *on $\Gamma(X, S)$.*

Related to the concept of a module is that of a contragredient module.

▷ **Definition 2.14 [Contragredient module].** *Let $\mathcal{A}$ and $\mathcal{B}$ be two involutive algebras and let $\mathcal{E}$ be a left-$\mathcal{A}$ right-$\mathcal{B}$ bimodule. We define the* contragredient module *$\mathcal{E}^o$ of $\mathcal{E}$ as the space*

$$\mathcal{E}^o := \{\overline{\eta}, \eta \in \mathcal{E}\}. \tag{2.9}$$

It has left-$\mathcal{B}$ right-$\mathcal{A}$ action, inherited from $\mathcal{E}$:

$$b\,\overline{\eta}\,a := \overline{a^*\eta b^*}, \tag{2.10}$$

and is thus a right-$\mathcal{A}$ left-$\mathcal{B}$ bimodule.

N.B. Equivalently this makes $\mathcal{E}^o$ a left-$\mathcal{A}^o$ right-$\mathcal{B}^o$ bimodule, where $\mathcal{A}^o$ and $\mathcal{B}^o$ are the opposite algebras of $\mathcal{A}$ and $\mathcal{B}$ respectively (see Definition 2.9); if, for $a^o \in \mathcal{A}^o$ and $\overline{\eta} \in \mathcal{E}^o$, we define $a^o\overline{\eta} := \overline{a^*\eta}$, we easily see that

$$a_1^o a_2^o \overline{\eta} = \overline{a_1^* a_2^* \eta} = \overline{(a_2 a_1)^* \eta} = (a_2 a_1)^o \overline{\eta}, \tag{2.11}$$

in correspondence with the product structure on $\mathcal{A}^o$. The case for $\mathcal{B}$ is similar, $\mathcal{E}^o$ becoming a left-$\mathcal{A}^o$ right-$\mathcal{B}^o$ bimodule.

▷ **Definition 2.15 [Endomorphisms of a module].** *For a right $\mathcal{A}$-module $\mathcal{E}$ we let*

$$\mathrm{End}_{\mathcal{A}}(\mathcal{E}) := \{\phi : \mathcal{E} \to \mathcal{E} \ : \phi(\eta a) = \phi(\eta)a,\ \phi(\eta_1 + \eta_2) = \phi(\eta_1) + \phi(\eta_2) \\ \forall\, a \in \mathcal{A}, \eta, \eta_1, \eta_2 \in \mathcal{E}\} \tag{2.12}$$

*be the $\mathcal{A}$-*linear endomorphisms of $\mathcal{E}$*. This can be given the structure of an algebra, with the product given by composition.*

Given an algebra $\mathcal{A}$ and modules $\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathcal{F}$, we can construct a *tensor product over $\mathcal{A}$* of $\mathcal{E}$ and $\mathcal{F}$, denoted by $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$.[3] For such a tensor product we have the relations

$$\begin{aligned}(\eta + \zeta) \otimes_{\mathcal{A}} \xi &= \eta \otimes_{\mathcal{A}} \xi + \zeta \otimes_{\mathcal{A}} \xi \\ \eta \otimes_{\mathcal{A}} (\xi + \chi) &= \eta \otimes_{\mathcal{A}} \xi + \eta \otimes_{\mathcal{A}} \chi \\ \eta a \otimes_{\mathcal{A}} \xi &= \eta \otimes_{\mathcal{A}} a\xi \qquad \forall a \in \mathcal{A}, \eta, \zeta \in \mathcal{E}, \xi, \chi \in \mathcal{F}.\end{aligned}$$

In cases where it is obvious that we take the tensor product over an algebra instead of over a field, we may drop the reference to the algebra in the notation.

2.3 INNER PRODUCTS & NORMS ON MODULES

The definitions and examples merely had preparatory purposes: just as we enriched algebras with extra structures to yield C^*-algebras, we can make the transition from ordinary modules to something called C^*-modules.

▷ **Definition 2.16 [Pre-C^*-module, Hermitian pairing].** *Suppose $\mathcal{A}$ is a C^*-algebra over $\mathbb{C}$ and suppose $\mathcal{E}$ is a right-$\mathcal{A}$ module. We call $\mathcal{E}$ a* pre-C^*-module *if we define a pairing $(.,.) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$, that behaves nicely with the vector space structure:*

$$(\eta, z_1\zeta + z_2\xi) = z_1(\eta, \zeta) + z_2(\eta, \xi) \qquad \forall z_1, z_2 \in \mathbb{C} \tag{2.13}$$

and on top of that satisfies

$$\begin{aligned}(\eta, \zeta a + \xi b) &= (\eta, \zeta)a + (\eta, \xi)b, \\ (\eta, \zeta)^* &= (\zeta, \eta), \\ (\eta, \eta) &\geq 0, \quad (\eta, \eta) = 0 \Leftrightarrow \eta = 0 \quad \forall a, b \in \mathcal{A}, \eta, \zeta, \xi \in \mathcal{E}.\end{aligned} \tag{2.14}$$

We will generally refer to a pairing as above as a ($\mathcal{A}$-valued) Hermitian pairing.

If on $\mathcal{E}_{\mathcal{A}}$ such a pairing $(.,.)$ is defined, the contragredient module ${}_{\mathcal{A}}\mathcal{E}^o$ naturally carries a pairing $(.,.)'$ as well, by $(\overline{\eta}, \overline{\zeta})' := (\eta, \zeta)$.

Now if you would want to derive a norm from the pairing as just mentioned, you run into trouble: since it takes values in $\mathcal{A}$ instead of in $\mathbb{C}$, a norm like $\|\eta\| := \sqrt{(\eta, \eta)}$ will not work since the square root of an element in $\mathcal{A}$ is in general not defined. We *can* however, take

$$\|.\|_{\mathcal{E}} : \mathcal{E} \to \mathbb{R} \qquad \text{with} \quad \|\eta\|_{\mathcal{E}} := \sqrt{\|(\eta, \eta)\|)_{\mathcal{A}}}, \tag{2.15}$$

where the second norm is that on $\mathcal{A}$. This norm is seen to satisfy all the requirements that have been laid out in Definition 2.2.

3 We will not delve into the precise construction of a tensor product over an algebra, but mention that it is not too different from the construction of a tensor product over a field and refer for further information to [1, § 19].

▷ **Definition 2.17.** *If a pre-C^*-module $\mathcal{E}$ is complete in the norm defined by (2.15), we call it a C^**-module.

▷ **Example 2.18.** *If we look at Examples 2.11 and 2.12 —but now with $\mathcal{A}$ a C^*-algebra— we can define in a simple manner an $\mathcal{A}$-valued Hermitian pairing on $\mathcal{A}$ and $\mathcal{A}^n$ by*

$$(a, b) := a^* b \quad \forall\, a, b \in \mathcal{A}$$

and

$$(\,(a_1, \ldots a_n), (b_1, \ldots, b_n)\,) := \sum_{i=1}^{n} a_i^* b_i \quad \forall\, a_i, b_i \in \mathcal{A}$$

respectively. That these definitions obey the properties of Definition 2.16 is easy to check.

▷ **Example 2.19.** *The $C(X)$-module of spinor-valued functions $\Gamma(X, S)$ (see Example 2.13) can be furnished with a Hermitian pairing*

$$(.,.) : \Gamma(S) \times \Gamma(S) \to C(X), \tag{2.16}$$

by setting

$$(\psi, \chi)(x) := \langle \psi(x), \chi(x) \rangle, \tag{2.17}$$

where with $\langle .,. \rangle : S_x \times S_x \to \mathbb{C}$ the complex inner product on the spinor representation is meant.

Given a C^*-module $\mathcal{E}_{\mathcal{A}}$ with Hermitian pairing $(.,.)$, we can form the set of operators

$$|\eta\rangle\langle\zeta| : \mathcal{E} \to \mathcal{E} \qquad |\eta\rangle\langle\zeta|(\xi) := \eta(\zeta, \xi). \tag{2.18}$$

Since for each $a \in \mathcal{A}$, $|\eta\rangle\langle\zeta|(\xi a) = \eta(\zeta, \xi)a = |\eta\rangle\langle\zeta|(\xi)a$ so that all operators of this form are elements of $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ (see Definition 2.15), their composition being of the form

$$|\eta\rangle\langle\zeta| \circ |\xi\rangle\langle\chi| = |\eta(\zeta, \xi)\rangle\langle\chi| \quad \forall\, \eta.\zeta, \xi, \chi \in \mathcal{E}.$$

Furthermore we can give an adjoint of each of these operators with respect to the pairing $(.,.)$, that is

$$(\eta, |\zeta\rangle\langle\xi|\chi) = (\eta, \zeta)(\xi, \chi) = (\xi(\zeta, \eta), \chi) = (|\xi\rangle\langle\zeta|\eta, \chi) \quad \forall\, \eta, \chi \in \mathcal{E},$$

i.e. $|\zeta\rangle\langle\xi|^* = |\xi\rangle\langle\zeta|$. All finite sums of this kind of terms form an algebra, which we denote by $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$.[4]

2.4 MORITA EQUIVALENCE

Morita equivalence is a relation between two rings (and therefore algebras). Though it is a weaker relation than isomorphism, many of the properties of an object are preserved under Morita equivalence.

For the existence of a Morita equivalence of two C^*-algebras, two definitions can be given.

4 Actually, we take its *closure* with respect to the norm derived from $(.,.)$.

▷ **Definition 2.20 [Morita equivalence 1].** *Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be* Morita equivalent *if there exist C^* bimodules ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{F}_{\mathcal{A}}$ such that*

$$\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F} \simeq \mathcal{A} \qquad \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E} \simeq \mathcal{B} \tag{2.19}$$

▷ **Definition 2.21 [Morita equivalence 2].** *$\mathcal{A}$ and $\mathcal{B}$ are* Morita equivalent *if there exist a full right $\mathcal{A}$ module*[5] *$\mathcal{E}$ such that*

$$\mathrm{End}^0_{\mathcal{A}}(\mathcal{E}) \simeq \mathcal{B}, \tag{2.20}$$

where $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ as defined in the previous section.

That these definitions are in fact equivalent is seen for example in [14, Thm. 4.26].

▷ **Lemma 2.22.** *Any C^*-algebra $\mathcal{A}$ is Morita equivalent to itself.*

Proof. If we take the first definition with $\mathcal{A} = \mathcal{B}$, we immediately see that this is true with $\mathcal{E} = \mathcal{F} = \mathcal{A}$. □

This fact will prove to be of great interest of use, as we will see in Section 6.

5 In this context *full* means that the *closure* of $(\mathcal{E}, \mathcal{E}) := \mathrm{span}\{(\eta, \zeta) : \eta, \zeta \in \mathcal{E}\}$ must be equal to $\mathcal{E}$.

3 ASPECTS OF DIFFERENTIAL GEOMETRY

The purpose of this chapter is mainly to prepare for the next one, in which many of the ideas and concepts introduced here, will be used. Differential geometry is branch of mathematics that deals with (topological) spaces that are not $\mathbb{R}^n$ for some n, but 'locally' look like it. We introduce the notion of a manifold and its tangent and cotangent spaces. From there on, we will work towards the definition of the scalar curvature of a manifold, introducing the concepts of vector fields, covector fields, connections and metrics.

3.1 MANIFOLDS

▷ **Definition 3.1 [Manifold].** *Let M be a Hausdorff space. We call M a* manifold *if there exists an* atlas, *i.e. a family of* charts $\{(U_i, \phi_i)\}$, *where $U_i \subset M$ with $\bigcup_i U_i = M$ and $\phi_i : U_i \to \mathbb{R}^n$ a homeomorphism such that for each two U_i, U_j with $U_i \cap U_j \neq \emptyset$ the* transition map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ *is smooth. Such a manifold is said to be of* dimension n.
Intuitively this says that if we can find a set of patches U_i, each of which is isomorphic to a subset of $\mathbb{R}^n$ for some fixed n, where all of these patches together totally cover M, then we call M a manifold. We say M is *locally isomorphic* to $\mathbb{R}^n$. In general the term *local* is sometimes used if we have a description in terms of some explicitly chosen chart. Often we will —for obvious reasons— denote a chart by $\{(U_i, x_i)\}$.

▷ **Example 3.2.** *A simple, though instructive example of a manifold is the two-sphere with radius 1:*

$$S^2 := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We can describe this space with an atlas consisting of two charts, U_1 and U_2. They are given by

$$U_1 := S^2 - \{(1,0,0)\} \qquad U_2 := S^2 - \{(-1,0,0)\}$$

If we define maps $\phi_{1,2} : U_{1,2} \to \mathbb{R}^2$ by

$$\phi_1(x_1, x_2, x_3) = \frac{(x_2, x_3)}{1 + x_1}, \qquad \phi_S(x_1, x_2, x_3) = \frac{(x_2, x_3)}{1 - x_1}.$$

Because of the specific signs in the numerator, these maps are well defined. We see that $U_1 \cup U_2 = S^2$ and that ϕ_1^{-1} is given by

$$\phi_1^{-1}(y_1, y_2) = \left(\frac{1 - y_1^2 - y_2^2}{1 + y_1^2 + y_2^2}, \frac{2y_1}{1 + y_1^2 + y_2^2}, \frac{2y_2}{1 + y_1^2 + y_2^2} \right).$$

Now $U_1 \cap U_2 = S - \{(1,0,0), (-1,0,0)\}$ so that for one of the two transition functions

$$\phi_{21} := \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

we have

$$\phi_{21}(y_1, y_2) = \left(\frac{y_1}{y_1^2 + y_2^2}, \frac{y_2}{y_1^2 + y_2^2} \right),$$

which is indeed an infinitely differentiable nowhere vanishing function. For the transition function ϕ_{12} we have a similar result, from which we may conclude that S^2 is a manifold, locally isomorphic to $\mathbb{R}^2$.

With a *curve* we mean a continuous map $\gamma : I \to M$ from some real interval I, sweeping out a path in M. We denote the variable that runs through I with t.

We can take two such curves γ_1, γ_2, with respective intervals I_1, I_2. If these curves have the same value p at some point $s \in I_1 \cup I_2$, i.e. $\gamma_1(s) = \gamma_2(s) = p$, and if for some ϕ_i, belonging to a U_i in which the curves $\gamma_1(t_1)$ and $\gamma_2(t_2)$ are contained for all $t_1 \in I_1$ and $t_2 \in I_2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_i \circ \gamma_1(t)\Big|_{t=s} = \frac{\mathrm{d}}{\mathrm{d}t}\phi_i \circ \gamma_2(t)\Big|_{t=s},$$

(i.e. the derivatives of these two curves at the point s coincide), we say that these curves are *tangent* at the point s. Now 'being tangent' is equivalence relation and we can consider the equivalence classes of such tangent curves. We denote the representative —the 'tangent vector'— of an equivalence class containing γ by $\gamma'(s)$.

Without loss of generality, we can suppose $s = 0$ from here on.

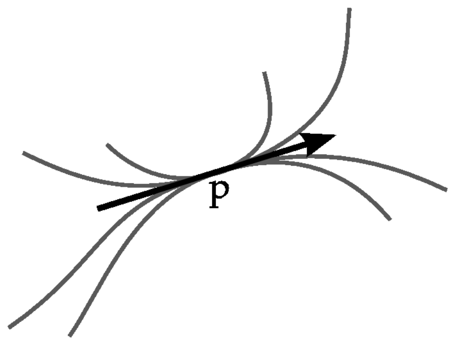

Figure 3.1: A set of tangent curves at a point $p \in M$. The representative of their equivalence class is depicted with an arrow.

We can then use such a tangent vector $\gamma'(0)$ to define a smooth derivation $v : C^\infty(M) \to \mathbb{R}$. We define v by its action on any smooth function $f \in C^\infty(M, \mathbb{R})$ as

$$v(f) := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0}. \tag{3.1}$$

By using the properties of $C^\infty(M, \mathbb{R})$ —that are derived from Example 2.7— it is easily checked that such a v satisfies

$$v(f + g) = v(f) + v(g) \qquad \forall\, f, g \in C^\infty(M, \mathbb{R}) \tag{3.2a}$$
$$v(\lambda f) = \lambda v(f) \qquad \forall\, \lambda \in \mathbb{R} \tag{3.2b}$$
$$v(fg) = f(p)v(g) + v(f)g(p) \qquad \forall\, f, g \in C^\infty(M, \mathbb{R}), \tag{3.2c}$$

(where $p = \gamma(0)$), i.e. v is indeed a derivation.

In fact, we can turn the above properties into a definition:

▷ **Definition 3.3 [Tangent vector].** *A* tangent vector *is a map* $v : C^\infty(M) \to \mathbb{R}$ *that satisfies* (3.2a) *to* (3.2c).

Now as we have seen above, there may be many inequivalent tangent vectors at one point. For their collection, we have:

▷ **Definition 3.4 [Tangent space].** *The* tangent space T_pM *is the space consisting of the equivalence classes of curves tangent to* M *at* p.

This tangent space can be given the structure of a vector space, in the following way. As we have seen above, a tangent vector $\gamma'(0)$ defines a derivation. Suppose we have two such —inequivalent— tangent vectors, $\gamma_1'(0)$ and $\gamma_2'(0)$ giving rise to derivations v_1 and v_2. We can define

$$\begin{aligned} (v_1 + v_2)(f) &:= v_1(f) + v_2(f) \qquad && \forall\, v_1, v_2 \in T_xM \\ (\lambda v)(f) &:= \lambda v(f) && \forall\, \lambda \in \mathbb{R}. \end{aligned}$$

It is then seen that these definitions indeed define a vector space structure.

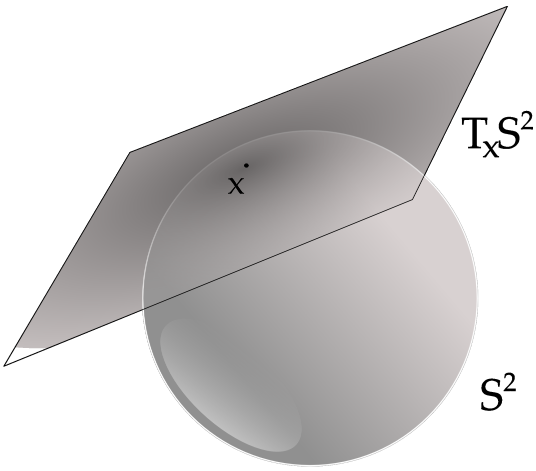

Figure 3.2: The two-sphere S^2, with one of its tangent spaces T_xS^2.

The fact that T_pM is a vector space is nice, but for future calculations it is convenient to know a basis of the tangent space. We can construct one as follows: for a chart $\{(U, x)\}$ and $p \in U$ we define $\frac{\partial}{\partial x_i}\big|_p$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) := D_i(f \circ x^{-1})\Big|_{x(p)}. \tag{3.3}$$

for each $f \in C^\infty(M, \mathbb{R})$. Here, D_i is just the i-th derivative of a map from $\mathbb{R}^n$ to $\mathbb{R}$. Looking at definition 3.3, one can see that this map is indeed a vector tangent to p.

▷ **Proposition 3.5 [A basis of T_pM].** *If* (U, x) *is a* n*-dimensional chart of* M *and* $p \in U$*, then*

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \tag{3.4}$$

is a basis for T_pM.

Proof. See [28], Theorem 3.3. □

The above Proposition implies that in this notation any tangent vector X is written as $X = X^i_p \frac{\partial}{\partial x^i}|_p$, where $X^i_p \in \mathbb{R}$ for any of the i's.

Instead of talking about tangent spaces, defined with respect to one point of the manifold, we can also prefer a more 'global' description. We define

$$TM := \coprod_{x \in M} T_x M, \tag{3.5}$$

where $\coprod$ denotes the *disjoint union* of all spaces $T_x M$. We call a smooth function that maps each point p of M to a tangent vector in $T_p M$, a *vector field*. We denote the space of such functions by $\Gamma^\infty(M, TM)$ or $\Gamma^\infty(TM)$. It is a $C^\infty(M)$-module. Its elements are of the form $X = X^i \partial_i, \quad X^i \in C^\infty(M)$. We connect this notation to the 'pointwise' case by setting $X(p) := X^i_p \frac{\partial}{\partial x_i}\Big|_p$.

Closely related to the tangent space is the *cotangent space*:

▷ **Definition 3.6 [Cotangent space].** *The* cotangent space $T^*_p M$ *for some point* $p \in M$ *is defined as the* dual *of the tangent space:*

$$T^*_p M := Hom(T_p M, \mathbb{R})$$

Any $f \in C^\infty(M, \mathbb{R})$ defines a *cotangent vector* $\mathrm{d}f|_p$ at p by

$$\mathrm{d}f(v) := v(f) \tag{3.6}$$

This means in particular that if $\{\frac{\partial}{\partial x^i}\big|_p, i = 1, \ldots, n\}$ is a basis for $T_p M$, we can define a basis $\{\mathrm{d}x^i\big|_p, i = 1, \ldots, n\}$ for $T^*_p M$ which is fixed by

$$\mathrm{d}x^i\Big|_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \frac{\partial}{\partial x^j}\Big|_p x^i = D_j (x^i \circ x^{-1})\Big|_{x(p)} \tag{3.7}$$

i.e. $\mathrm{d}x^i\big|_p \left(\frac{\partial}{\partial x^j}\big|_{x(p)}\right) = \delta^i{}_j$.

As for tangent vectors we can pick a more 'global' approach to cotangent vectors, by defining the $C^\infty(M)$-module of cotangent vector fields or *covector fields* $\Gamma(M, T^*M)$.

3.3 DIFFERENTIAL FORMS & CONNECTIONS

In the previous section we saw that a map $f \in C^\infty(M)$ defined a covector field $\mathrm{d}f$ by $(\mathrm{d}f)(X) := X(f)$ for all $X \in \Gamma(M, TM)$. We can generalize this. For α, β covector fields, define $\alpha \wedge \beta : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to C^\infty(M)$ by

$$\alpha \wedge \beta(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X).$$

This satisfies

$$\begin{aligned} &\alpha_1 \wedge (\alpha_2 + \alpha_3) = \alpha_1 \wedge \alpha_2 + \alpha_1 \wedge \alpha_3, \\ &\alpha_1 \wedge \alpha_2 = -\alpha_2 \wedge \alpha_1. \end{aligned} \tag{3.8}$$

We can take linear combinations of such element and denote the resulting $C^\infty(M)$-module by $\Omega^2(C^\infty(M))$.

Of course we can generalize this to maps $\Gamma(TM) \times ... \times \Gamma(TM) \to C^\infty(M)$ (with k products). We write $\Omega^k(M)$ for the space of all such *differential k-forms*. With respect to a coordinate system $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ we write

$$\Omega^k(C^\infty(M)) := \text{span}\{f_{i_1,\dots i_k}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} : f_{i_1,\dots i_k} \in C^\infty(M)\} \tag{3.9}$$

where $i_1 < \ldots < i_k$. Note that $\Omega^k(C^\infty(M)) = 0$ for $k > n$ because of the antisymmetry property (3.8).

The direct sum of all spaces

$$\Omega(C^\infty(M)) := \bigoplus_{k=0} \Omega^k(C^\infty(M)), \quad \Omega^0(C^\infty(M)) := C^\infty(M), \tag{3.10}$$

thus forms a *graded space.*, where k denotes the *degree* of an element.

We can define a *differential* $\mathrm{d} : \Omega^k(M) \to \Omega(M)^{k+1}$ by

$$d(f_{i_1\dots i_k}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) := \frac{\partial}{\partial x^i}(f_{i_1\dots i_k})\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \tag{3.11}$$

which can be seen to satisfy:

- On $\mathcal{A}^0(M) \equiv C^\infty(M)$, d is defined by (3.6);
- $\mathrm{d}(\alpha_1 + \alpha_2) = \mathrm{d}(\alpha_1) + \mathrm{d}(\alpha_2)$;
- $\mathrm{d}^2 = 0$;
- $\mathrm{d}(\alpha_1 \wedge \alpha_2) = \mathrm{d}(\alpha_1) \wedge \alpha_2 + (-)^{|\alpha_1|}\alpha_1 \wedge \mathrm{d}(\alpha_2)$.

There is another map, the *contraction* $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$, where X is a vector field, which is defined by

$$(\iota_X\alpha)(X_1, \dots, X_{k-1}) := \alpha(X, X_1, \dots, X_{k-1}). \tag{3.12}$$

This map satisfies

- $\iota_X(\alpha_1 + \alpha_2) = \iota_X(\alpha_1) + \iota_X(\alpha_2)$;
- $\iota_X(\alpha_1 \wedge \alpha_2) = \iota_X\alpha_1 \wedge \alpha_2 + (-)^{|\alpha_1|} \wedge \iota_X(\alpha_2)$;
- $\iota_{fX} = f\iota_X \; \forall f \in C^\infty(M)$;
- $\iota_X\iota_Y = -\iota_Y\iota_X$, in particular $\iota_X^2 = 0$.

Suppose we are given a manifold M and for each $p \in M$ a vector space E_p over $\mathbb{R}$, such that we can —in a similar way as for TM and T^*M— construct an object E from that. We again let $\Gamma(M, E)$ be the space of E-valued functions. We then define

$$\Omega^1(M, E) := \Gamma^\infty(M, E) \otimes \Omega^k(C^\infty(M)) \simeq \Gamma^\infty(M, \Lambda^k T^*M \otimes E). \tag{3.13}$$

i.e. the space of functions with values in the E-valued differential 1-forms.

▷ **Definition 3.7 [Connection].** *A* connection ∇ *on such an E as above is a $\mathbb{R}$-linear map*

$$\nabla : \Gamma^\infty(M, E) \to \Gamma^\infty(M, T^*M \otimes E) \tag{3.14}$$

that satisfies

$$\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s \qquad \forall f \in C^\infty(M), s \in \Gamma(M, E). \tag{3.15}$$

By composition a connection ∇ with the contraction ι_X we get an operator

$$\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$$

by

$$\nabla_X := \nabla \circ \iota_X + \iota_X \circ \nabla. \tag{3.16}$$

N.B. In a specific coordinate system $\{\partial_i\}$ we will sometimes write:

$$\nabla_i := \nabla_{\partial_i}.$$

3.4 RIEMANNIAN MANIFOLDS & THE METRIC

A special class of manifolds are ones that carry a *metric*. In this context, we mean with the latter

▷ **Definition 3.8 [Metric, Riemannian manifold].** *we call a map*

$$g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M) \tag{3.17}$$

a metric *when it is $C^\infty(M)$-bilinear and* positive definite *in the sense that $g(X, X) \geq 0 \; \forall \; X \in \Gamma(TM)$ (using the norm defined in Example 2.7). If M possesses such a metric, we call the pair (M, g) a* Riemannian manifold.

A possible way to construct a metric is to associate to each point $p \in M$ a $\mathbb{R}$-bilinear positive definite map

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

The family of maps g_p then forms the metric.

Given a $p \in M$ we can apply to g_p two basis elements of T_p giving rise to the definition

$$g_{ij} := g_p\Big(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\Big), \tag{3.18}$$

from which we can infer that g_p can be written as

$$g_p = g_{ij} \mathrm{d}x^i\big|_p \otimes \mathrm{d}x^j\big|_p,$$

where a sum on i, j is implied.

N.B. A metric g as above induces a pairing $g' : \Gamma(T^*M) \times \Gamma(T^*M) \to C^\infty(M)$ for one-forms (see e.g. [14, § 7,1]). We will refer to this pairing as the metric too and — and write

$$g^{ij} := g'(\mathrm{d}x^i, \mathrm{d}x^j).$$

3.5 THE LEVI-CIVITA CONNECTION & CURVATURE

Given a connection ∇ we can define its *torsion tensor*

$$T(X, Y) := \nabla_X(Y) - \nabla_Y(X) - [X, Y]$$

We call a connection *torsion free* if $T(X, Y) = 0 \; \forall \; X, Y \in \Gamma(TM)$.

If for a Riemannian manifold, a connection ∇ satisfies

$$g(\nabla X, Y) + g(X, \nabla Y) = \mathrm{d}(g(X, Y))$$

we we have written

$$g(X_i \otimes \mathrm{d}x^i, Y) = g(X^i, Y)\mathrm{d}x^i.$$

we call the connection *compatible* with the metric.

▷ **Theorem 3.9.** *For any Riemannian manifold* (M, g) *there is a unique connection* $\nabla^g : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$*, that is both torsion free as compatible with the metric. We call this connection the* Levi-Civita connection*.*

Proof. See [29], Lemma 6.8. □

▷ **Definition 3.10 [Riemann curvature tensor].** *Let* $X, Y \in \Gamma(TM)$*. We define the* Riemann curvature tensor $R(X, Y)$ *by*

$$R(X,Y)Z := (\nabla^g_X \nabla^g_Y - \nabla^g_Y \nabla^g_X - \nabla^g_{[X,Y]})Z, \tag{3.19}$$

for $Z \in \Gamma(TM)$*.*

Properties of $R(X, Y)$ that can be inferred from the definition above are

- $R(X, Y) = -R(Y, X)$;
- $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$;
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y$ (Bianchi identity).

Since for some vector field X the Levi-Civita tensor is an operator on $\Gamma(TM)$, we can get a characterization of ∇^g in terms of *Christoffel symbols* Γ^k_{ij} that are defined by

$$\nabla^g_{\partial_j} \partial_i = \Gamma^k_{ij} \partial_k. \tag{3.20}$$

This expression allows us to translate the properties of the Levi-Civita tensor (torsion-freeness, metric compatibility) into properties of these Christoffel symbols. Without proof we state

$$\Gamma^{\bullet}_{ij} = \Gamma^{\bullet}_{ji} \quad \text{and} \tag{3.21}$$
$$\Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} = \partial_i g_{jk}. \tag{3.22}$$

Additionally, we can use the local expression for the Levi-Civita connection to obtain an expression in components for the Riemann curvature tensor. We define

$$\begin{aligned} R^i_{jkl} \partial_i &:= R(\partial_k, \partial_l)\partial_j = [\nabla^g_{\partial_k} \Gamma^m_{lj} \partial_m - \nabla^g_{\partial_l} \Gamma^m_{kj} \partial_m] \\ &= (\partial_k \Gamma^m_{lj})\partial_m + \Gamma^n_{km} \Gamma^m_{lj} \partial_n - (\partial_l \Gamma^m_{kj})\partial_m - \Gamma^n_{lm} \Gamma^m_{kj} \partial_n, \end{aligned} \tag{3.23}$$

where we have used that $[\partial_i, \partial_j] = 0 \; \forall \, i, j$ and the properties of a connection. Contracting this expression by means of the metric tensor yields

$$R^i_{jkl} = (\partial_k \Gamma^i_{lj}) - (\partial_l \Gamma^i_{kj}) + \Gamma^i_{km} \Gamma^m_{lj} - \Gamma^i_{lm} \Gamma^m_{kj}.$$

From this expression we first obtain the *Ricci tensor* by contracting two indices:

$$R_{jl} := R^k_{jkl}$$

and by contracting again we obtain the *scalar curvature* R :

$$R := g^{jl} R_{jl}. \tag{3.24}$$

Since a given Riemannian manifold produces a unique Levi-Civita connection and the curvature is derived from that, the latter provides useful information about the manifold: it assigns to every point on the manifold a number that is a measure for the amount of curvature on that point.

4 SPIN GEOMETRY

As the name suggests, spin geometry —in contrast to differential geometry— allows us to describe spinor fields, widely used in particle physics. The field of spin geometry is vast and intricate. We therefore by far have the intention of giving a complete introduction, rather we will merely touch upon some of the most important concepts to us, only now and then going into detail a bit. This chapter is mainly based upon [14], (in particular Chapters 5 and 9). For other good and detailed accounts we refer to e.g. [12].

▷ **Definition 4.1 [Clifford algebra].** *Let V be a finite dimensional vector space over $\mathbb{R}$, endowed with a bilinear symmetric map $g : V \times V \to \mathbb{R}$. The* Clifford algebra *$Cl(V, g)$ is the algebra generated by products of elements of V modulo the relation $\{v_1, v_2\} := v_1v_2 + v_2v_1 = 2g(v_1, v_2)$.*

If a set of elements $\{e_i\}$ of V is some basis for V, this characteristic relation can be casted into a form more common in physics:

$$\{e_i, e_j\} = 2g_{ij},$$

Here we have written $g_{ij} := g(e_i, e_j)$.

We can extend the definition of a Clifford algebra to a vector space over the complex numbers by a technique called *complexification*. For a real vector space V we take $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \simeq V \oplus iV$. Then we can extend a given bilinear symmetric map $g : V \times V \to \mathbb{R}$, to $g^{\mathbb{C}} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ by

$$g^{\mathbb{C}}(v_1 + iv_2, v_3 + iv_4) := g(v_1, v_3) - g(v_2, v_4) + i(g(v_2, v_3) + g(v_1, v_4)).$$

Using $g^{\mathbb{C}}$, we can, in a similar fashion as for a real vector space, construct a Clifford algebra $Cl(V^{\mathbb{C}}, g^{\mathbb{C}})$. We will write $\mathbb{C}l(V)$ for short.

We can define an inner product on any Clifford algebra quite easily. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ for V (such that $e_ie_j = -e_je_i$ and $e_i^2 = 1\,\forall\, i, j$), any element u of the Clifford algebra $\mathbb{C}\mathrm{l}(V)$ can be written as $u = \sum_I u_I e_I$, where each I is a strictly ordered set[1] and $u_I \in \mathbb{C}$ for each I. In this notation we define an inner product $\langle\, .\,,\, .\,\rangle : \mathbb{C}\mathrm{l}(V) \times \mathbb{C}\mathrm{l}(V) \to \mathbb{C}$ by

$$\langle\, u\,,\, v\,\rangle := \sum_I \overline{u}_I v_I. \tag{4.1}$$

In that same notation, we can make $\mathbb{C}\mathrm{l}(V)$ involutive, by defining for any $u = \sum_I u_I e_I \in \mathbb{C}\mathrm{l}(V)$:

$$u^* := \sum_I \overline{u}_I e_I^!, \tag{4.2}$$

where with $e_I^!$ we mean the total reversal of the order in which all elements appear.

1 See the beginning of §A.1 for more on this particular notation.

For each Clifford algebra $\mathbb{C}l(V)$ we can define a *Chirality element* γ that is given in terms of an orthonormal basis $\{e_1, \ldots, e_n\}$ of V by

$$\gamma := (-i)^m e_1 e_2 \cdots e_n \tag{4.3}$$

where $n = 2m$ or $n = 2m + 1$ according to whether n is even or odd. This definition guarantees that $\gamma^* = \gamma$ and $\gamma^2 = 1$.

Let us apply the aforementioned concepts. Throughout this chapter (M, g) will denote a compact Riemannian manifold with metric g.

▷ **Example 4.2 [$\mathbb{C}l(T_x^*M)$].** *As we saw in Chapter 3, for each point p on the manifold T_p^*M has the structure of a vector space, spanned by the basis $\mathrm{d}x^\mu|_p$. We can therefore construct a Clifford algebra $\mathbb{C}l(T_p^*M)$ generated by the complexifications of $\mathrm{d}x^\mu|_p$ modulo the Clifford algebra relation featuring the complexification of the symmetric bilinear form using the complexification of $g^{\mu\nu}(p) := g(\mathrm{d}x^\mu|_p, \mathrm{d}x^\nu|_p)$.*

Thus we can associate a complex Clifford algebra to each point of the manifold. We write $\Gamma(M, \mathbb{C}l(T^*M))$ [or just $\Gamma(\mathbb{C}l(T^*M))$] for the space of 'Clifford algebra-valued functions', that is for an element $\alpha \in \Gamma(M, \mathbb{C}l(T^*M))$ we have $\alpha(x) \in \mathbb{C}l(T_x^*M)$ for each $x \in M$.

Note that, as the Clifford algebra itself, $\Gamma(M, \mathbb{C}l(T^*M))$ is an algebra by 'pointwise multiplication': for $\alpha, \beta \in \Gamma(M, \mathbb{C}l(T^*M))$

$$(\alpha\beta)(x) := \alpha(x)\beta(x) \in \mathbb{C}l(T^*M), \tag{4.4}$$

i.e. $\alpha\beta \in \Gamma(M, \mathbb{C}l(T^*M))$ as well. In a similar way we can make $\Gamma(\mathbb{C}l(T^*M))$ involutive [$\alpha^*(x) := \alpha(x)^*$, where $\alpha(x)^*$ is given by (4.2)] and it can be endowed with a Hermitian pairing $(.,.) : \Gamma(\mathbb{C}l(T^*M)) \times \Gamma(\mathbb{C}l(T^*M)) \to C^\infty(M)$ by:

$$(\alpha, \beta)(x) := \langle\, \alpha(x)\, ,\, \beta(x)\, \rangle, \tag{4.5}$$

where $\langle ., . \rangle$ is the inner product (4.1) on the Clifford algebra. Last, we introduce a *chirality element* γ on $\Gamma(\mathbb{C}l(T^*M))$ by

$$\gamma(x) := \gamma, \tag{4.6}$$

where with right hand side we mean the chirality element γ on each Clifford algebra [cf. (4.3)].

We recall the space of spinor-valued functions as introduced in Example 2.13. The coordinate space appearing there is in this context M of course. We will write $\Gamma(S)$ for $\Gamma(M, S)$. So far we have hardly said anything about S, with the exception that for each $x \in M$, S_x has the structure of a complex vector space. We mention here that S_x is a representation of the so called spin^c-group of $\mathbb{C}l(T_x^*M)$ that consists of elements that are a product of an even number of unitary elements of $\mathbb{C}l(T_x^*M)$. If the dimension of M is even, S_x falls apart in two *irreducible representations* of equal dimension —eigenspaces of the chirality element γ with eigenvalues ± 1— that are denoted by S_x^+ and S_x^- respectively.
The dimension can be determined from the dimension of the manifold. We suffice by saying that for a manifold M that locally looks like the Euclidean

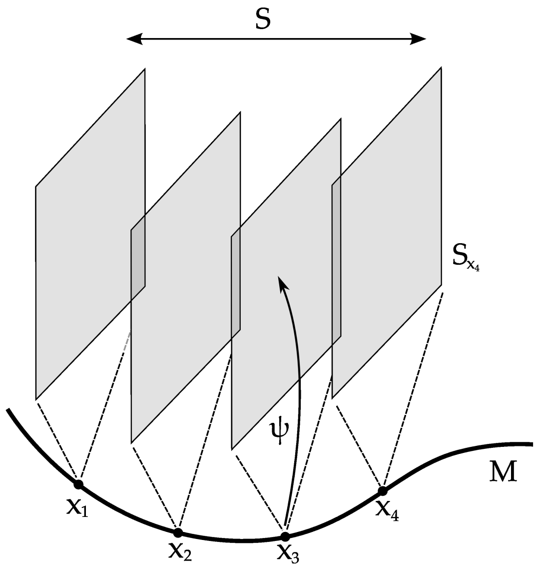

Figure 4.1: The spinor-valued functions ψ maps a point $x \in M$ to an element in the vector space S_x.

space $\mathbb{R}^4$, the spinors will have 4 complex components as well and turn out to be Dirac spinors.[2] We refer to [14, §5.2, 5.3] for details.

If there exists a $C(M)$-linear algebra homomorphism

$$c : \Gamma(\mathbb{C}l(T^*M)) \to \mathrm{End}_{C(M)}(\Gamma(S)) \tag{4.7}$$

(the *spin homomorphism*), the elements $\gamma^\mu := c(\mathrm{d}x^\mu)$ act on the spinor-valued functions $\Gamma(S)$ in the following way:

$$(\gamma^\mu \psi)(x) := \gamma^\mu(x)\psi(x), \tag{4.8}$$

where the right hand side is just matrix multiplication.

Thus $\Gamma(S)$ is not only a left $C(M)$-module, but (via the homomorphism c) a left $\Gamma(\mathbb{C}l(T^*M))$-module as well. These two modules structures are compatible, for elements of the two different algebras commute with each other.

From here on, we will restrict to only the *smooth* elements of $C(M)$, $\Gamma(S)$, $\Gamma(\mathbb{C}\mathrm{l}(T^*M))$.[3] To maintain a concise notation, though, we still write $\Gamma(S)$ and $\Gamma(\mathbb{C}l(T^*M))$ where we in fact mean $\Gamma^\infty(S)$ and $\Gamma^\infty(\mathbb{C}l(T^*M)$ respectively.

▷ **Definition 4.3 [Space of square integrable spinors, $L^2(M, S)$].** *On the space of smooth spinor-valued functions $\Gamma(S)$ we can define an inner product*

$$\langle ., . \rangle : \Gamma(S) \times \Gamma(S) \to \mathbb{C} \tag{4.9}$$

given by

$$\langle\, \psi \,,\, \phi \,\rangle := \int_M (\psi, \phi)(x)\sqrt{g}\, \mathrm{d}^4 x, \tag{4.10}$$

2 This does *not* imply, however, that number of components of the spinor always equals the dimension of the manifold.

3 We must mention that the subalgebra $C^\infty(M)$ of $C(M)$ is *not* a C^*-algebra, since it fails to be complete with respect to norm on $C(M)$. We will ignore this though.

where with $(.,.)$ we denote the $C^\infty(M)$-valued pairing that can be obtained from the one in Example 2.19 by restriction. Upon completing $\Gamma(S)$ with respect to this inner product, we get the Hilbert space $L^2(M,S)$; the space of square integrable spinors*.*

4.1 THE DIRAC OPERATOR

There is a certain type of manifolds that allows the construction of spinor-valued functions, these are called *spin manifolds*. We shall not go into what the requirements are for such a spin manifold to exists. We do mention that a four-dimensional compact Riemannian manifold is indeed a spin manifold as long as it is *orientable*, its *second Stiefel-Whitney class* of S vanishes[4] and one is able to define a bijective antilinear map $C : \Gamma(S) \to \Gamma(S)$ satisfying certain conditions ([14, §9.2]).

▷ **Definition 4.4 [Spin connection].** *On a spin manifold M we can define a unique Hermitian connection (the* spin connection*)*

$$\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S) \simeq \Gamma(T^*M) \otimes_{C^\infty(M)} \Gamma(S)$$

that satisfies the Leibniz-rule

$$\nabla^S(c(\alpha)\psi) = c(\nabla^g\alpha)\psi + c(\alpha)\nabla^S\psi \quad \forall\, \psi \in \Gamma(S), \\ \nu \in \Gamma(\mathbb{C}l(T^*M)), \psi \in \Gamma(S). \tag{4.11}$$

Here ∇^g is the Levi-Civita connection on M, as introduced in Section 3.5.

There is a local expression for the spin connection, it is (see [14, §9.3])

$$\nabla^S = \mathrm{d}x^\mu \otimes (\partial_\mu + \omega_\mu), \quad \text{with } \omega_\mu = \widetilde{\Gamma}^\beta_{\mu\alpha}\gamma^\alpha\gamma_\beta, \tag{4.12}$$

where $\widetilde{\Gamma}^\beta_{\mu\alpha}$ are the Christoffel symbols determined by ∇^g [cf. (3.20)] (but with respect to a different basis than the $\mathrm{d}x^\mu$) and γ^α, γ^β are the *tetrads* as defined in Appendix A.

Using the spin-homomorphism c we can construct another map $\hat{c} : \Gamma(\mathbb{C}l(T^*M)) \otimes \Gamma(S) \to \Gamma(S)$ by defining

$$\hat{c}(\alpha \otimes \psi) := c(\alpha)\psi \quad \forall\, \alpha \in \Gamma(\mathbb{C}l(T^*M)), \psi \in \Gamma(S).$$

Combining $\hat{c}$ and ∇^S we get an operator on $\Gamma(S)$:

▷ **Definition 4.5 [Dirac operator].** *The Dirac operator on $\Gamma(S)$ is given by*

$$\not{\partial}_M := i\hat{c}\circ\nabla^S, \tag{4.13}$$

where we have tacitly used the embedding of $\Gamma(T^*M)$ in $\Gamma(\mathrm{Cl}(T^*M))$.

It is not hard to find a local expression of the Dirac operator, once you know the one for the spin connection:

$$\not{\partial}_M = i\gamma^\mu(\partial_\mu + \omega_\mu), \tag{4.14}$$

where ω_μ is as above.

Adjoining the i in the definition of the Dirac operator has the following result:

4 See e.g. [20], Theorem 1.7

▷ **Theorem 4.6.** *$\not{\partial}_M$ as in (4.13) is a self-adjoint operator on $L^2(M, S)$.*

Proof. See [14, §9.4].[5] □

▷ **Lemma 4.7.** *The commutator $[\not{\partial}_M, a]$ for any $a \in C^\infty(M)$ is a bounded operator on $L^2(M, S)$ that is locally of the form $\gamma^\mu \partial_\mu(a)$.*

Proof. Using the properties of the spin connection, we get $[\not{\partial}_M, a] = c(\mathrm{d}a)$ for the commutator of $\not{\partial}_M$ with a. Since c is $C^\infty(M)$-linear, this becomes for a certain chart

$$[\not{\partial}_M, a] = c(\mathrm{d}x^\mu)(\partial_\mu a) = \gamma^\mu \partial_\mu(a).$$

We then have for the norm

$$\begin{aligned} \|[\not{\partial}_M, a]\|^2 &= \sup_{x\in M} \|c(\mathrm{d}a)(x)\|^2 \\ &= \sup_{x\in M} [(\gamma^\mu \partial_\mu(a), \gamma^\nu \partial_\nu(a)(x)] \qquad (4.15) \\ &= \sup_{x\in M} [(\gamma^\mu, \gamma^\nu)(x)\overline{(\partial_\mu a)(x)}(\partial_\nu a)(x)] \\ &= \sup_{x\in M} |(\partial^\mu a)(x)|^2 = \|\partial^\mu a\|_{C^\infty(M)} \leq \infty, \qquad (4.16) \end{aligned}$$

since $a \in C^\infty(M)$. The pairing $(.,.)$ appearing in the second and third line, is that on $\Gamma(\mathbb{C}\mathrm{l}(T^*M))$ given by (4.5). □

As we shall see, the square of a Dirac operator is a variable that will frequently appear. For $\not{\partial}_M$ we shall calculate it explicitly. It's important to note that because of (4.11) the spin connection ∇^S_μ and gamma matrices do not commute. Instead we find [6]

$$\nabla^S_\mu \gamma^\nu = \nabla^S_\mu c(\mathrm{d}x^\nu) = c(\mathrm{d}x^\nu)\nabla^S_\mu + \Gamma^\nu_{\lambda\mu} c(\mathrm{d}x^\lambda) = \gamma^\nu \nabla^S_\mu + \Gamma^\nu_{\lambda\mu}\gamma^\lambda.$$

Bearing this in mind, we get for the square of $\not{\partial}_M$:

$$\begin{aligned} \not{\partial}_M^2 &= -\gamma^\mu\gamma^\nu \nabla^S_\mu \nabla^S_\nu + \Gamma^\nu_{\lambda\mu}\gamma^\mu\gamma^\lambda\nabla^S_\nu \\ &= -g^{\mu\nu}\nabla^S_\mu\nabla^S_\nu - \tfrac{1}{2}\gamma^\mu\gamma^\nu[\nabla^S_\mu, \nabla^S_\nu] + g^{\mu\lambda}\Gamma^\nu_{\mu\lambda}\nabla^S_\nu \\ &= -g^{\mu\nu}(\nabla^S_\mu\nabla^S_\nu - \Gamma^\lambda_{\mu\nu}\nabla^S_\lambda) - \tfrac{1}{2}\gamma^\mu\gamma^\nu[\nabla^S_\mu, \nabla^S_\nu], \qquad (4.17) \end{aligned}$$

where in the second step we have employed the identity $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

Now we use the local form of the spin connection (4.12) to get for

Now by using the local form of ∇^S_μ and the properties of the Riemann tensor 3.23 one can show (e.g. [14, Thm. 9.16]) that for the second term of (4.17) we have

$$\gamma^\mu\gamma^\nu[\nabla^S_\mu, \nabla^S_\nu] = \frac{1}{2}R, \qquad (4.18)$$

where R is the scalar curvature of M. Thus we arrive at

$$\not{\partial}_M^2 = \Delta + \tfrac{1}{4}R,$$

where $\Delta = -g^{\mu\nu}(\nabla_\mu\nabla_\nu - \Gamma^\lambda_{\mu\nu}\nabla_\lambda)$ is called a *Laplacian*. For future use we write it out locally to yield

$$\not{\partial}_M^2 = -[g^{\mu\nu}\partial_\mu\partial_\nu + (2\omega^\mu - \Gamma^\mu)\partial_\mu + \partial^\mu(\omega_\mu) + \omega^\mu\omega_\mu - \Gamma^\mu\omega_\mu + \tfrac{1}{4}R]. \qquad (4.19)$$

5 There is a subtlety here. Rather than $\not{\partial}_M$ its *closure* $\overline{\not{\partial}_M}$ is self-adjoint, we will write $\not{\partial}_M$ for the latter.

6 Would we be using *flat* gamma-matrices (i.e. $\gamma^a = e^a_\mu\gamma^\mu$) as in Appendix A, this would be equivalent with $\nabla_\nu e^\mu_a := \partial_\nu e^\mu_a + \omega_\nu e^\mu_a - \Gamma^\mu_{\lambda\nu} e^\lambda_a = 0$.

5 SPECTRAL TRIPLES

We have arrived at a point where we can introduce one of the key notions of noncommutative geometry: that of a spectral triple.

▷ **Definition 5.1 [Spectral triple].** *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ an involutive unital algebra and D an operator on $\mathcal{H}$. We call the set $(\mathcal{A}, \mathcal{H}, D)$ a* spectral triple *when*

- *$\mathcal{A}$ has a representation as bounded operators on $\mathcal{H}$;*
- *D is self-adjoint and has compact resolvent;*
- *for all $a \in \mathcal{A}$, the commutator $[D, a]$ is a bounded operator on $\mathcal{H}$.*

Several contents introduced in the previous chapters seamlessly fit into this definition:

▷ **Example 5.2 [Canonical spectral triple].** *In the case of a compact manifold M without boundary, the spin geometry $(C^\infty(M), L^2(M, S), \not{\partial}_M = i\hat{c} \circ \nabla^S)$ that we constructed in Chapter 4 is a particular example of a spectral triple. Most of the demands were already seen to be met: $C^\infty(M)$ is involutive by Example 2.7, which indeed has a representation as bounded operators on $\mathcal{H}$ by Example 2.13. The operator $\not{\partial}_M$ is self-adjoint (as was mentioned in the previous Chapter) and $[\not{\partial}_M, a]$ was indeed seen to be bounded by Lemma 4.7. We will often refer to this spectral triple as the* canonical spectral triple

Besides the canonical example as above, there is another one of particular interest to us:

▷ **Example 5.3.** *When we take the algebra to be a finite direct sum of matrix-algebras, i.e. $\mathcal{A} = \bigoplus_{n_i} M_{n_i}(\mathbb{K})$ —where $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$— let $\mathcal{H}$ be some finite dimensional representation of $\mathcal{A}$ and D a symmetric matrix acting on $\mathcal{H}$, we get a spectral triple. That it indeed is one, is easy to check, since everything is finite-dimensional.*

▷ **Definition 5.4 [Even spectral triple].** *If $\mathcal{H}$ is $\mathbb{Z}_2$ graded, i.e. there exits a unitary operator χ on $\mathcal{H}$ with $\chi^* = \chi$ and $\chi^2 = 1$ that in addition fulfills*

$$\cdot\ \chi D = -D\chi;$$
$$\cdot\ \chi a = a\chi, \quad \forall\, a \in \mathcal{A}, \tag{5.1}$$

we call $(\mathcal{A}, \mathcal{H}, D, \chi)$ an even spectral triple.

▷ **Example 5.5.** *For the spectral triple $(C^\infty(M), L^2(M, S), D)$ we define*

$$\chi := c(\gamma) \tag{5.2}$$

where c is the spin homomorphism *as defined in the previous Chapter and where γ is the chirality element of the $\Gamma(\mathbb{C}\mathrm{l}(M))$. To see that this χ meets the requirements as laid out above, the local expression for $\not{\partial}_M$ and the properties of c have to be used.*
More often than not χ is denoted γ_5 when $\mathcal{H} = L^2(M, S)$.

▷ **Definition 5.6 [Real Spectral Triple].** *Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we can define a* real structure*; an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, that satisfies $J^2 = \epsilon$ and $JD = \epsilon' DJ$, where $\epsilon, \epsilon' \in \{\pm 1\}$. A spectral triple endowed with a real structure, is called a* real spectral triple *and is denoted by $(\mathcal{A}, \mathcal{H}, D, J)$.*

The definitions of an even spectral triple and a real spectral triple can be combined, yielding a *real even spectral triple* $(\mathcal{A}, \mathcal{H}, D, J, \chi)$. We have to specify how χ and J interact, though. To this end we define a third sign $\epsilon'' \in \{\pm 1\}$ with $\gamma J = \epsilon'' J \gamma$.

▷ **Example 5.7.** *If M is a four-dimensional compact Riemannian manifold, we have $\epsilon = -1$, $\epsilon' = \epsilon'' = 1$. When taking $(C^\infty(M), L^2(M, S), \not{\partial}_M)$ as a spectral triple, we can explicitly define J as being related to the charge conjugation matrix C in the following way ([19, § 6.8]):*

$$(J\psi)(x) := C\overline{\psi(x)} \quad \forall x \in M. \tag{5.3}$$

Readers that are confused about the signs, should keep in mind that we are in a Euclidean set up here.

For reference we list different values of ϵ, ϵ' and ϵ'' for all possible even KO-dimensions. We mention that for a Riemannian spin manifold M, the KO-dimension equals the dimension of M.

KO dimension H	$J^2 = \epsilon$	$JD = \epsilon' DJ$	$J\gamma = \epsilon''\gamma J$
0	+	+	+
2	-	+	-
4	-	+	+
6	+	+	-

Table 5.1: The values for $\epsilon, \epsilon', \epsilon''$ defined by the value of J^2 and whether J (anti)commutes with D and γ.

In [8], A. Connes formulated seven axioms that a certain spectral triple must meet in order to be called *a noncommutative (spin) geometry*. Except for one, we will not cover them here, for they will not be needed in this thesis and introducing them would require quite a lot of effort. Instead we refer to [14, Ch. 10], or [33, Ch. 3] for details. There are two things to mention here. One is that for a compact Riemannian manifold M without boundary, the set $(C^\infty(M), L^2(M, S), D, J, \chi)$ is an example of a noncommutative geometry (see [14], chapter 11). Secondly, under suitable circumstances [17], finite spectral triples fulfill these axioms as well.

As announced above, we mention two demands related to the seven axioms.

In addition to $\mathcal{H}$ being a left $\mathcal{A}$-module (meaning that there exists a representation of $\mathcal{A}$ on $\mathcal{H}$), we can make $\mathcal{H}$ a right $\mathcal{A}$-module as well by using J:

$$\psi b := J b^* J^* \psi \qquad \forall\, \psi \in \mathcal{H}, b \in \mathcal{A}, \tag{5.4}$$

where (2.6a) is easily seen to be satisfied:

$$(\psi a) b := (J b^* J^*)(J a^* J^*)\psi = J(ab)^* J^* \psi = \psi(ab).$$

The requirement (2.7) for bimodules then becomes our first demand:

$$[a, Jb^*J^{-1}] = 0. \tag{5.5}$$

We could have stated this demand in terms of a representation of $\mathcal{A} \otimes \mathcal{A}^o$ as well. Recalling definition 2.9 of the opposite algebra, we can define a *opposite representation* $b^o = Jb^*J^{-1}$ on $\mathcal{H}$ (indeed satisfying $a^o b^o = (ba)^o$). Then from the representations of $\mathcal{A}$ and $\mathcal{A}^o$ on $\mathcal{H}$, we construct one of $\mathcal{A} \otimes \mathcal{A}^o$ on $\mathcal{H}$ by $a \otimes b^o \to aJb^*J^{-1}$, which is only seen to be a representation after requiring (5.5).

The second demand is the *order one condition:* in addition to commuting with the representation of $\mathcal{A}$ (5.5), the representation of $\mathcal{A}^o$ must commute with that of $[D, \mathcal{A}]$ as well:

$$[[D, a], b^o] = [[D, a], Jb^*J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}. \tag{5.6}$$

As we will see, when it comes to doing particle physics, the above examples will nut suffice; they simply do not provide enough data to make models that show resemblance with reality. Luckily we can rely on

▷ **Theorem 5.8.** *If* $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ *of and* $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ *are two spectral triples, then*

$$(\mathcal{A}, \mathcal{H}, D) := (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2) \tag{5.7}$$

is one as well.

Proof. The tensor product of a Hilbert space is another, with respect to the inner product

$$\langle \,.\,, \,.\, \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle \,.\,, \,.\, \rangle_{\mathcal{H}_1} \langle \,.\,, \,.\, \rangle_{\mathcal{H}_2} \tag{5.8}$$

The representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is simply defined by $\pi(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ where $\pi_{1,2}$ are the representations on $\mathcal{H}_{1,2}$ respectively. The representation π is easily seen to be bounded using (5.8).

It is immediate that D is self-adjoint (with respect to 5.8), since γ_1 is. Furthermore, by using (5.1), we find

$$[D, a_1 \otimes a_2] = [D_1, a_1] \otimes a_2 + \gamma_1 a_1 \otimes [D_2, a_2] \tag{5.9}$$

which is bounded since the representations $\mathcal{A}_1 \to B(\mathcal{H}_1)$ and $\mathcal{A}_2 \to B(\mathcal{H}_2)$ are. □

In the case of two real spectral triples with isometries J_1 and J_2 respectively, we can make the tensor product a real spectral triple as well by defining a J on the tensor product. In order to (anti)commute with the Dirac operator, this is $J_1 \otimes J_2$, $J_1 \otimes J_2\chi_2$ or $J_1\chi_1 \otimes J_2$, depending on the values of n_1 and n_2 (see [32]).

Note that if we have two spectral triples with KO-dimension n_1 and n_2 respectively, their tensor product is of KO-dimension $n = n_1 + n_2$.

We anticipate a bit to chapter 8, if we state that it is the tensor product of carefully chosen spectral triples with which we can construct realistic physical models.

▷ **Definition 5.9 [Unitarily equivalent spin geometries ([33], §7.1)].** *Two spin geometries* $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ *and* $(\mathcal{A}, \mathcal{H}, D', \gamma', J')$ *are said to be* unitarily equivalent, *if there exists a unitary operator* $U : \mathcal{H} \to \mathcal{H}$ *such that*

- $D' = UDU^*$;
- $J' = UJU^*$;
- $\gamma' = U\gamma U^*$;
- $U\pi(a)U^* = \pi(\sigma(a)) \; \forall \, a \in \mathcal{A}$.

Here, with π *we explicated the representation* $\mathcal{A}$ *must necessarily have on* $\mathcal{H}$ *and* $\sigma : \mathcal{A} \to \mathcal{A}$ *is an* automorphism *of* $\mathcal{A}$.

▷ **Example 5.10.** *If for a certain spin geometry we take* $U = uJuJ^*$, *where* $u \in U(\mathcal{A}) := \{u \in \mathcal{A} : u^*u = uu^* = 1\}$ *a unitary element of* $\mathcal{A}$, *we see that*

- $\gamma' = uJuJ^*\gamma Ju^*J^*u^* = \epsilon'' uJ\gamma J^*u^* = (\epsilon'')^2\gamma = \gamma$, *where we have used that* γ *must commute with elements of the algebra, and gives a* ϵ'' *upon interchanging it with* J;
- $J' = (uJuJ^*)J(Ju^*J^*u^*) = uJuJu^*J^*u^* = uJJu^*J^*uu^* = \epsilon J^* = J$, *where we have used* (5.5);
- *the resulting automorphism is* $\sigma_u(a) = uau^*$: *again using* (5.5) *gives* $\sigma(a) = uJuJ^*aJu^*J^*u^* = uau^*$. *This is an example of what is called an* inner automorphism.

For D *we have:*

$$
\begin{aligned}
D' &= uJuJ^*DJu^*J^*u^* = \epsilon' uJuDu^*J^*u^* = \epsilon' uJu(u^*D + [D, u^*])J^*u^* \\
&= \epsilon' uJDJ^*u^* + \epsilon' uJu[D, u^*]J^*u^* = uDu^* + \epsilon' JJ^*uJu[D, u^*]J^*u^* \\
&= D + u[D, u^*] + \epsilon' Ju[D, u^*]J^* =: D_u
\end{aligned}
$$

where we have made use of (5.5) *and the order one condition* (5.6).

All *inner automorphisms* σ_u for $u \in U(\mathcal{A})$ in fact form a group of which the product is given by composition (with $\sigma_{u_1} \circ \sigma_{u_2} = \sigma_{u_1 u_2}$). The element σ_1 is seen to be the identity, and the inverse of σ_u is σ_{u^*}. We denote the group of inner automorphisms by $Inn(\mathcal{A})$. It will later on become clear that it can be interpreted as the *gauge group*. See e.g. [9, § 9.9], [19, § 9.3] for a more thorough discussion on this.

6 INNER FLUCTUATIONS OF THE DIRAC OPERATOR

There are many interesting problems and questions regarding the concept of a spectral triples. The answer of one of them will have a crucial physical interpretation. We will cover it in the last two sections of this chapter. Before that we cover some preparatory subjects.

6.1 CONNES' ALGEBRA OF DIFFERENTIAL FORMS

▷ **Definition 6.1 [Derivation].** *Let $\mathcal{E}$ be a bimodule over an algebra $\mathcal{A}$. A map* $\mathrm{d} : \mathcal{A} \to \mathcal{E}$ *is called a* derivation *when*

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b \quad \forall\, a, b \in \mathcal{A}.$$

If $\mathcal{A}$ is unital (definition 2.1) it follows that $\mathrm{d}a = \mathrm{d}(1a) = (\mathrm{d}1)a + 1\mathrm{d}a$, i.e. $\mathrm{d}1 = 0$.

▷ **Example 6.2.** *For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, $\mathrm{ad}\, D : \mathcal{A} \to B(\mathcal{H})$, defines a derivation, since*

$$\begin{aligned} \mathrm{ad}\, D(ab)\psi &= [D, ab]\psi = a[D, b]\psi + [D, a]b\psi \\ &= [a\,\mathrm{ad}\, D(b) + \mathrm{ad}\, D(a)b]\psi \quad \forall\, \psi \in \mathcal{H}. \end{aligned} \tag{6.1}$$

We will be exploiting this example right away:

▷ **Definition 6.3 [Connes' algebra of differential forms].** *For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we define* Connes' algebra of differential forms *as*

$$\begin{aligned} &\Omega_D\mathcal{A} := \bigoplus_{k=0} \Omega_D\mathcal{A} \qquad \text{with } \Omega^0_D\mathcal{A} := \mathcal{A} \qquad \text{and} \\ &\Omega^k_D\mathcal{A} := \mathrm{span}\{a_0[D, a_1]\cdots[D, a_k],\ a_0, \ldots, a_k \in \mathcal{A}\}. \end{aligned} \tag{6.2}$$

By the third demand for a spectral triple (cf. Definition 5.1) we see that $\Omega_D\mathcal{A} \subseteq B(\mathcal{H})$, the bounded operators on $\mathcal{H}$.

Note that, as the name suggests, the algebra comes with a product; for two elements $a_0[D, a_1]\cdots[D, a_k] \in \Omega^k_D\mathcal{A}$ and $a_{k+2}[D, a_1]\cdots[D, a_{k+l+1}] \in \Omega^l_D\mathcal{A}$ we have

$$\begin{aligned} &(a_0[D, a_1]\cdots[D, a_k])(a_{k+1}[D, a_{k+2}]\cdots[D, a_{k+l+1}]) \\ &= \sum_{i=1}^{k}(-1)^{k-i}a_0[D, a_1]\cdots[D, (a_ia_{i+1})]\cdots[D, a_{k+1}]\cdots[D, a_{k+l+1}] \\ &\quad + (-1)^k(a_0a_1[D, a_2]\cdots[D, a_{k+l+1}]). \end{aligned}$$

▷ **Example 6.4.** *For the canonical spectral from Example 5.2 triple we have that*

$$\Omega^k_{\partial\!\!\!/_M} C^\infty(M) := \mathrm{span}\{a_0\gamma^\mu\partial_\mu(a_1)\cdots\gamma^\nu\partial_\nu(a_k) : a_0, \ldots a_k \in C^\infty(M)\}, \tag{6.3}$$

since the term ω_μ of D drops out due to the commutator.

We will focus on the second ($\Omega^1_D\mathcal{A}$) and third terms ($\Omega^2_D\mathcal{A}$) of (6.2) though many results can obtained for $\Omega_D\mathcal{A}$in general (e.g. [19, Chapter 7]).

▷ **Lemma 6.5.** *$\Omega^1_D\mathcal{A}$ and $\Omega^2_D\mathcal{A}$ are both $\mathcal{A}$-bimodules.*

Proof. The left modules structure is immediately clear. As for the right $\mathcal{A}$ module structure of $\Omega^1_D\mathcal{A}$ we have

$$\Omega^1\mathcal{A}\times\mathcal{A}\ni(a[D,b],c)\to a[D,b]c=a[D,bc]-ab[D,c]\in\Omega^1_D\mathcal{A}\quad\forall\, a,b,c\in\mathcal{A}.$$

The argument for $\Omega^2_D\mathcal{A}$ is analogous. □

Note that $\Omega^1_D\mathcal{A}$ is involutive, with its involution given by

$$(a[D,b])^*=-[D,b^*]a^*=b^*[D,a^*]-[D,b^*a^*]\in\Omega^1_D\mathcal{A}$$

where we have used that D is self-adjoint. The algebra $\Omega^2_D\mathcal{A}$ is involutive as well as is seen by a slightly longer calculation.

▷ **Theorem 6.6.** *Let $\mathcal{A}=C^\infty(M)$. Taking the algebra $\Omega^1\mathcal{A}$ and $\Omega^2\mathcal{A}$ as constructed in Section 3.3 we have the following isomorphisms:*

$$\Omega^1\mathcal{A}\simeq\Omega^1_D\mathcal{A}\quad\textit{and}$$
$$\Omega^2\mathcal{A}\simeq\Omega^2_D\mathcal{A}/J,\tag{6.4}$$

with $J=\{\sum_i[D,a_i][D,b_i]:a_i,b_i\in\mathcal{A},\sum_i a_i[D,b_i]=0\}$.

Proof. We give a sketch of the proof. The thing to do is define a map $\pi:\Omega^k\mathcal{A}\to\Omega^k_D\mathcal{A}$ that is both an *algebra-homomorphism* [$\pi(\omega_1\omega_2)=\pi(\omega_1)\pi(\omega_2)$ $\forall\,\omega_1,\omega_2\in\Omega_D\mathcal{A}$] and satisfies

$$\pi\circ d=\mathrm{ad}_D\circ\pi.\tag{6.5}$$

This last equation will assure us that if $\pi(\omega)=0$ for some ω, $\pi(\mathrm{d}\omega)=0$ as well.

We take for the map π:

$$\pi(a_0\mathrm{d}a_1\cdots\mathrm{d}a_k):=a_0[D,a_1]\cdots[D,a_k],$$

i.e. $\pi(a_0\mathrm{d}a_1):=a_0[D,a_1]$ on $\Omega^1\mathcal{A}$ and $\pi(a_0\mathrm{d}a_1\mathrm{d}a_2):=a_0[D,a_1][D,a_2]$ on $\Omega^2\mathcal{A}$. On both $\Omega^1\mathcal{A}$ and $\Omega^2\mathcal{A}$, π is easily seen to be an algebra homomorphism. As for the other requirement, we obviously have $\pi\circ da=\mathrm{ad}_D\circ\pi(a)$ on $\Omega^1\mathcal{A}$. For $\Omega^2\mathcal{A}$ however there are in general elements $\sum_i a_i\mathrm{d}b_i$ of $\Omega^1\mathcal{A}$ with

$$\sum_i\pi(a_i\mathrm{d}b_i)=a_i[D,b_i]=0$$

but with $\sum_i[D,a_i][D,b_i]\neq0$. We solve this by simply dividing out all elements that prevent π from satisfying (6.5):

$$J:=\{\sum_i[D,a_i][D,b_i]:a_i,b_i\in\mathcal{A},\sum_i a_i[D,b_i]=0\}.$$

□

Similar isomorphisms can be constructed for any algebra $\mathcal{A}$, part of a spectral triple and for the whole of $\Omega_D\mathcal{A}$ (i.e. for any degree), see for example [19, Ch. 7].

Previously, in Chapter 3 we defined a *connection* in the context of differential geometry. Here we will extend this definition somewhat.

▷ **Definition 6.7.** *A* connection ∇ *over a right $\mathcal{A}$-module $\mathcal{E}$ is a map*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A} \tag{6.6}$$

that obeys a Leibniz rule

$$\nabla(\eta a) = (\nabla \eta) a + \eta \otimes \mathrm{d}a \quad \forall\, a \in \mathcal{A}, \eta \in \mathcal{E}. \tag{6.7}$$

Instead of over a right module we can define a connection over a left $\mathcal{A}$-module $\mathcal{F}$ as well. In that case a connection is defined as a map

$$\nabla : \mathcal{F} \to \Omega^1 \mathcal{A} \otimes \mathcal{F}$$

that obeys

$$\nabla(a\eta) = a(\nabla \eta) + \mathrm{d}a \otimes \eta \quad \forall\, a \in \mathcal{A}, \eta \in \mathcal{F}. \tag{6.8}$$

If on a right $\mathcal{A}$-module $\mathcal{E}$ a Hermitian pairing $(.,.) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ is defined, we can regard connections that are compatible with the pairing.

▷ **Definition 6.8.** *Let $\mathcal{E}$ be a right $\mathcal{A}$-module with a Hermitian pairing $(.,.)$: $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ defined on it. A connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A}$ is called a* Hermitian connections *if it obeys*

$$(\nabla \eta, \zeta)' + (\eta, \nabla \zeta)' = \mathrm{d}(\eta, \zeta) \quad \forall \eta, \zeta \in \mathcal{E}$$

where we have extended the pairing $(.,.)$ to $(.,.)'\mathcal{E} \otimes \Omega^1 \mathcal{A} \times \mathcal{E} \to \Omega^1 \mathcal{A}$ by

$$(\eta \otimes \mathrm{d}a, \zeta)' := (\eta, \zeta)\mathrm{d}a \quad \forall a \in \mathcal{A}, \eta, \zeta \in \mathcal{E} \tag{6.9}$$

As in the case of connections, Hermitian connections can be defined for left modules equally well, for which only a minor modification to (6.9) is required.

6.3 MORITA EQUIVALENCE & INNER FLUCTUATIONS

As hinted in Section 2.4, the notion of Morita equivalence plays a crucial role. In this chapter we will see what that role precisely is. The central question is what the implications are when the algebra of a certain spectral triple is Morita equivalent to some other algebra. We start out by considering a general spectral triple and cover the case of a *real* spectral triple thereafter.

We recall definition 2.21: two C^*-algebras $\mathcal{A}, \mathcal{B}$ being Morita equivalent means that there exists a full right $\mathcal{A}$-module $\mathcal{E}$ such that $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E}) \simeq \mathcal{B}$.[1]

6.3.1 *Inner fluctuations for a spectral triple*

Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, so it meets all the demands as stated in definition 5.1. Suppose in addition that $\mathcal{A}$ is Morita equivalent to another

1 As mentioned in the previous chapter, the restriction $C^\infty(M)$ of $C(M)$ is not a C^*-algebra anymore. The definition of Morita equivalence then requires $\mathcal{E}$ to be something called a *finitely generated projective module*. This does not have any consequences for the results.

algebra $\mathcal{B}$.

The question is whether we can find a second spectral triple $(\mathcal{B}, \mathcal{H}', D')$ that is compatible with the Morita equivalence of $\mathcal{A}$ and $\mathcal{B}$. The relation $\mathcal{B} \simeq \mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ suggests that we take $\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$, where the representation of $\mathcal{B}$ on $\mathcal{H}'$ is simply given by

$$b(\eta \otimes \psi) := b(\eta) \otimes \psi, \quad \forall\, b \in \mathcal{B}. \tag{6.10}$$

This respects the $\mathcal{A}$-linearity of the tensor product. Note that $\mathcal{B}$ is involutive and unital too, as is demanded for the algebra of a spectral triple.

We can endow $\mathcal{H}'$ with an inner product derived from the respective inner products on $\mathcal{E}$ and $\mathcal{H}$:

$$\langle \eta_1 \otimes \psi_1, \eta_2 \otimes \psi_2 \rangle_{\mathcal{H}'} := \langle \psi_1, (\eta_1, \eta_2)\psi_2 \rangle_{\mathcal{H}},$$

where $(.,.)$ is the $\mathcal{A}$-valued pairing on $\mathcal{E}$ (cf. definition 2.16).

The remaining question is what $D' : \mathcal{H}' \to \mathcal{H}'$ looks like; simply taking $D' = \mathrm{id} \otimes D$ will not suffice, for then D' does not respect the $\mathcal{A}$-linearity of the tensor product:

$$D'(\eta a \otimes_{\mathcal{A}} \phi) = (\eta \otimes_{\mathcal{A}} aD\phi) \neq (\eta \otimes_{\mathcal{A}} Da\phi) = D'(\eta \otimes_{\mathcal{A}} a\phi),$$

since the commutator $[D, a]$ does not vanish identically in general.

The way out is to specify a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ as in (6.6) and define $D' : \mathcal{H}' \to \mathcal{H}'$ by

$$D'(\eta \otimes \psi) := \eta \otimes D\psi + \nabla(\eta)\psi,$$

where we use the identification $\Omega^1\mathcal{A} \simeq \Omega^1_D\mathcal{A}$ (cf. Theorem 6.6) and the fact that $\Omega^1_D\mathcal{A}$ has by demand a representation as bounded operators on $\mathcal{H}$. We easily check that this definition *does* respect $\mathcal{A}$-linearity:

$$\begin{aligned} D'(\eta \otimes_{\mathcal{A}} a\psi) - D'(\eta a \otimes_{\mathcal{A}} \psi) &= (\nabla\eta)a\psi + \eta \otimes_{\mathcal{A}} Da\psi - (\nabla\eta)a\psi \\ &\quad - \eta \otimes_{\mathcal{A}} \mathrm{d}a\psi - \eta a \otimes_{\mathcal{A}} D\psi \\ &= \eta \otimes_{\mathcal{A}} [D, a]\psi - \eta \otimes_{\mathcal{A}} \mathrm{d}a\psi = 0, \end{aligned} \tag{6.11}$$

upon using Theorem 6.6 again.

Note that any connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ will do the trick, so actually the Morita equivalence of $\mathcal{A}$ with $\mathcal{B}$ induces a set of spectral triples

$$(\mathcal{B}, \mathcal{H}', D') = (\mathrm{End}^0_{\mathcal{A}}(\mathcal{E}), \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, 1 \otimes D + \nabla \otimes \mathrm{id}) \tag{6.12}$$

with $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$.

As mentioned before, any algebra is in particular Morita equivalent to itself —in which case $\mathcal{E} \simeq \mathcal{A}$— so we can apply the operator D' to $\mathcal{H}' = \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H}$ too:

$$D'(a \otimes \psi) = (a \otimes D + \nabla(a))\psi$$

However, $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{H} \simeq \mathcal{H}$ by the identification $a \otimes_{\mathcal{A}} \psi \leftrightarrow a\psi$. Taking in particular $a = 1$ we get

$$D'\psi = (D + A)\psi \quad A := \nabla(1) \in \Omega^1\mathcal{A},$$

where A is of the form

$$A = \sum_j a_j[D, b_j], \quad a_j, b_j \in \mathcal{A}, \tag{6.13}$$

since any element of $\Omega^1_D\mathcal{A}$ is. Furthermore, from the self-adjointness of D, we infer the demand $A^* = A$.

We call the induced operators $D' = D + A$ the *inner fluctuations* of D.

6.3.2 *Inner fluctuations for a real spectral triple*

When considering a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ (see definition 5.6), we can ask again what the consequences of a Morita equivalence of $\mathcal{A}$ with some $\mathcal{B}$ are. Compared to the previous case there is a further restriction: any induced spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$, for some $J' : \mathcal{H}' \to \mathcal{H}'$ needs to be compatible with the relation $J'D' = \epsilon' D'J'$.

Note that if $\mathcal{A}$ is Morita equivalent with some $\mathcal{B}$, $\mathcal{E}$ is a $\mathcal{B}$-$\mathcal{A}$-module. Then the contragredient module $\mathcal{E}^o$ (see Definition 2.14) naturally is a $\mathcal{A}$-$\mathcal{B}$-module.

We then take for the constituents of the spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$:

- $\mathcal{H}' := \mathcal{E} \otimes_\mathcal{A} \mathcal{H} \otimes_\mathcal{A} \mathcal{E}^o$ where for the second tensor product we use the left $\mathcal{A}$-representation on $\mathcal{H}$ (5.4), i.e.

$$\eta \otimes_\mathcal{A} \psi \otimes_\mathcal{A} a\overline{\zeta} = \eta \otimes_\mathcal{A} \psi a \otimes_\mathcal{A} \overline{\zeta} = \eta \otimes_\mathcal{A} Ja^*J^*\psi \otimes_\mathcal{A} \overline{\zeta}.$$

- Compared to (6.10), the action of $\mathcal{B}$ on $\mathcal{H}'$ is modified slightly to yield

$$b(\eta \otimes \psi \otimes \overline{\zeta}) := b(\eta) \otimes \psi \otimes \overline{\zeta},$$

  easily seen to be respecting $\mathcal{A}$-linearity.

- We define J' by:

$$J'(\eta \otimes \psi \otimes \overline{\zeta}) := \zeta \otimes J\psi \otimes \overline{\eta}.$$

Note that by using J' there is a right representation of $\mathcal{B}$ on $\mathcal{H}'$ as well:

$$\begin{aligned}(\eta \otimes \psi \otimes \overline{\zeta})b &:= J'b^*J'(\eta \otimes \psi \otimes \overline{\zeta}) \\ &= (\eta \otimes \psi \otimes \overline{b^*\zeta}) \\ &= \eta \otimes \psi \otimes \overline{\zeta}b\end{aligned}$$

compatible with (2.10).

Last we have to specify D'. In analogy with the previous situation, the choice $D' = \mathrm{id} \otimes D \otimes \mathrm{id}$ would ruin $\mathcal{A}$-linearity, so again we pick an expression involving connections on both $\mathcal{E}$ and $\mathcal{E}^o$:

$$D'(\eta \otimes \psi \otimes \overline{\zeta}) := \nabla(\eta)\psi \otimes \overline{\zeta} + \eta \otimes D\psi \otimes \overline{\zeta} + \eta \otimes \psi \nabla' \overline{\zeta} \tag{6.14}$$

where $\nabla : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega^1_D\mathcal{A}$ as seen before and $\nabla' : \mathcal{E}^o \to \Omega^1_D\mathcal{A} \otimes_\mathcal{A} \mathcal{E}^o$ satisfying

$$\nabla'(a\overline{\zeta}) = a\nabla'(\overline{\zeta}) + \mathrm{d}a \otimes \overline{\zeta}.$$

Here, the right representation of $\mathcal{A}$ on $\mathcal{H}$ is extended to one of $\Omega^1_D\mathcal{A}$ by

$$\psi \mathrm{d}a := J(\mathrm{d}a)^* J^{-1}\psi = J[D,a]^* J^{-1}\psi. \tag{6.15}$$

Up to now ∇ and ∇' were unrelated, but the demand $D'J' = \epsilon' J'D'$ restricts them. If we write everything out and demand this equality, we see that if $\nabla\eta$ is of the form $\sum_i \eta_i \otimes \omega_{\eta_i}$ for certain $\eta_i \in \mathcal{E}, \omega_{\eta_i} = \Omega^1_D\mathcal{A}$ we have that $\omega_{\overline{\eta}_i} = \epsilon' \omega^*_{\eta_i}$.
As in the previous case we take as a special case that $\mathcal{A}$ is Morita equivalent to itself, leading to $\mathcal{E} \simeq \mathcal{A}, \mathcal{E}^o \simeq \mathcal{A}^o$. Making the identification $\mathcal{A}\otimes_{\mathcal{A}}\mathcal{H}\otimes_{\mathcal{A}}\mathcal{A}^o \simeq \mathcal{H}$ by $a_1 \otimes_{\mathcal{A}} \psi \otimes_{\mathcal{A}} \overline{a_2} \leftrightarrow aJa_2J^{-1}\psi$ and applying D' to $1 \otimes \psi \otimes \overline{1} \in \mathcal{H}'$ yields

$$D'\psi = [D + \nabla(1) + \epsilon' J\nabla(1)^* J^*]\psi,$$

corresponding to inner fluctuations of D of the form

$$D \to D + A + \epsilon' JAJ^{-1}, \quad A = \sum_i a_i[D,b_i],\ a_i, b_i \in \mathcal{A}, \tag{6.16}$$

after imposing the demand of self-adjointness for D.

Note that this particular form relies on the non-commutativity of $\mathcal{A}$. To see this, take $\mathcal{A}$ commutative. This implies that for each $a \in \mathcal{A}$, $a = Ja^*J^*$ (see [19, §6.8]) and let $\Omega^1_D \ni A = \sum_i a_i[D,b_i],\ A^* = A$. Then

$$\begin{aligned} JAJ^* &= \sum_i Ja_iJ^*J[D,b_j]J^* = \epsilon' \sum_i Ja_iJ^*[D,b_j^*] \\ &= \epsilon' \sum_i [D,b_j^*]Ja_iJ^* = \epsilon' \sum_i [D,b_j^*]a_i^* \\ &= -\epsilon' \Big(\sum_i a_i[D,b_j] \Big)^* = -\epsilon' A, \end{aligned}$$

where we successively exploited (5.6) and the self-adjointness of A. Consequently for a Riemannian spin manifold all fluctuations $A + \epsilon JAJ^{-1}$ vanish.

As a last topic in this Chapter, we combine the theory from Section 5, with inner fluctuations.

▷ **Proposition 6.9.** *The noncommutative spin geometries* $(\mathcal{A}, \mathcal{H}, D_A, \gamma, J)$ *and* $(\mathcal{A}, \mathcal{H}, D'_A, \gamma, J)$ *are unitarily equivalent under the unitary transformation of the form* $uJuJ^*$ *for* $u \in U(\mathcal{A})$ *(i.e. 'under the gauge group') when* $D'_A = D_{A^u}$*. Here*

$$A^u := uAu^* + u[D,u^*] \tag{6.17}$$

Proof. As was seen in Example 5.10, taking $U = uJuJ^*$ for $u \in U(\mathcal{A})$ did not have effect on γ and J. For the fluctuated Dirac operator we have

$$\begin{aligned} UD_AU &= U(D + A + \epsilon' JAJ^*)U^* \\ &= D + u[D,u^*] + \epsilon' Ju[D,u^*]J^* + UAU^* + \epsilon' UJAJ^*U^*. \end{aligned}$$

Now for the fourth term we have

$$UAU^* = uJuJ^*\big(\sum_i a_i[D,b_i]\big)Ju^*J^*u^* = \sum_i uJuJ^*a_iJu^*J^*[D,b_i]u^*,$$

where we have used the order one condition (5.6). If we employ (5.5) too, we get

$$UAU^* = \sum_i uJuJ^*Ju^*J^*a_i[D,b_i]u^* = \sum_i ua_i[D,b_i]u^* = uAu^*.$$

With a similar calculation we can obtain $UJAJ^*U^* = eJuAu^*J^*$. Collecting terms, we get

$$UD_AU = D + uAu^* + u[D, u^*] + \epsilon' J(uAu^* + u[D, u^*])J^* = D_{A^u}.$$

□

7 SPECTRAL ACTION

As was introducted in Chapter 5, a spectral triple consists of the data $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}, \mathcal{H}$ a Hilbert space and D a Dirac operator. The latter we can use in

▷ **Definition 7.1.** *For a Dirac operator D, a* cut-off scale *of Λ and some positive, even function f we can define the* spectral action*:*

$$\mathrm{Tr}(f(D_A/\Lambda)). \tag{7.1}$$

Here $D_A = D + A + \epsilon' JAJ^*$ is the Dirac operator with its inner fluctuations.

This action functional was first proposed and used by A. Connes and A.H. Chamseddine ([8], [4]). We shall encounter it numerous times in the remaining part of this thesis, its central role will automatically become clear.

▷ **Lemma 7.2.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and let $U(\mathcal{A})$ be the unitary elements of $\mathcal{A}$ (see Section 5), then the spectral action as defined above is* gauge invariant*:*

$$\mathrm{Tr}(f(D_{A^u}/\Lambda)) = \mathrm{Tr}(f(D_A/\Lambda)) \quad \forall\, u \in U(\mathcal{A}).$$

Proof. As was proved in Proposition 6.9, the gauge transformation $A^u = uAu^* + u[D, u^*]$ of A was such that $D_{A^u} = UD_AU^*$, $U = uJuJ^*$, for some $u \in U(\mathcal{A})$. The result then automatically follows from the properties of the trace. □

The action functional in the above form is in many cases not useful for direct computations. There is however a certain expansion in powers of Λ that can be made in order to simplify calculations. For this we first need to elaborate on the *heat kernel*.

7.1 THE HEAT KERNEL

The theory of this section is taken mainly from [13].

Let V be a bundle of a Riemann manifold (M, g) and $P : C^\infty(V) \to C^\infty(V)$ be an operator of the form

$$P = -(g^{\mu\nu}\partial_\mu\partial_\nu + K^\mu\partial_\mu + L) \tag{7.2}$$

with $K^\mu, L \in \mathrm{End}(V)$.

▷ **Lemma 7.3.** *Given a P as above, there exist unique ∇, E such that*

$$P = \nabla\nabla^* - E. \tag{7.3}$$

Here, E is given by

$$E = L - g^{\mu\nu}\partial_\nu(\omega'_\mu) - g^{\mu\nu}\omega'_\mu\omega'_\nu + g^{\mu\nu}\omega'_\rho\Gamma^\rho_{\mu\nu} \tag{7.4}$$

$$\omega'_\mu = \frac{1}{2}(g_{\mu\nu}K_\nu + g_{\mu\nu}g^{\rho\sigma}\Gamma^\nu_{\rho\sigma}) \tag{7.5}$$

where ω'_μ is determined by the local form of ∇:

$$\nabla_\mu = \partial_\mu + \omega'_\mu.$$

Proof. See [13], Lemma 4.8.1. □

We introduce the variable $\Omega_{\mu\nu}$ that denotes the curvature of ∇:

$$\Omega_{\mu\nu} := \partial_\mu(\omega'_\nu) - \partial_\nu(\omega'_\mu) - [\omega'_\mu, \omega'_\nu]. \tag{7.6}$$

If P is indeed of the form (7.2) we can make an expansion of the operator e^{-tP} in of powers of t. It is given by

$$\mathrm{Tr}\; e^{-tP} \sim \sum_{n\geq 0} t^{(n-m)/2} \int_M a_n(x,P)\sqrt{g}\mathrm{d}^m x, \tag{7.7}$$

where m is the dimension of M. The coefficients $a_n(x,P)$ are called the Seeley-De Witt coefficients. It follows that $a_n(x,P) = 0$ for odd n.[1] A deep Theorem by Gilkey [13, Ch 4.8] shows that the first three even coefficients of the expansion are given by

$$a_0(x,P) = (4\pi)^{-m/2}\,\mathrm{Tr}(\mathrm{id}); \tag{7.8a}$$

$$a_2(x,P) = (4\pi)^{-m/2}\,\mathrm{Tr}(-R/6\,\mathrm{id} + E); \tag{7.8b}$$

$$\begin{aligned} a_4(x,P) = (4\pi)^{-m/2}\frac{1}{360}\,\mathrm{Tr}\big(&-12R_{;\mu}{}^{\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\ &+ 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RE + 180E^2 + 60E_{;\mu}{}^{\mu} \\ &+ 30\Omega_{\mu\nu}\Omega^{\mu\nu}\big), \end{aligned} \tag{7.8c}$$

where $R_{;\mu}{}^{\mu} := \nabla^\mu\nabla_\mu R$ and the same for E. Also, $\Omega_{\mu\nu}$ and E are given by (7.5) and (7.4) respectively and both depend only on x and P. For notational purposes, we define

$$a_n(P) := \int_M a_n(x,P)\sqrt{g}\mathrm{d}^m x. \tag{7.9}$$

In all cases that we will consider, the manifold will be taken without boundary[2]: $\partial M = \varnothing$. Then the terms $E_{;\mu}{}^{\mu}$, $R_{;\mu}{}^{\mu}$ vanish by an application of Stokes's Theorem on manifolds[3]:

$$\int_M A_{;\mu}{}^{\mu}\sqrt{g}\mathrm{d}^m x = \int_M (\nabla^\mu\nabla_\mu A)\sqrt{g}\mathrm{d}^m x = \int_{\partial M} \epsilon_{\mu a_1\cdots a_m}\nabla^\mu A, \tag{7.10}$$

where, $\epsilon := \sqrt{g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$. Of course, for a manifold without boundary, the right hand side of (7.10) vanishes.

7.2 EXPANSION OF THE ACTION FUNCTIONAL

Now we have the following expansion for the spectral action.

▷ **Theorem 7.4 [Expansion of the spectral action].** *If D is of the form* (7.2)*, the spectral action* (7.1) *can be expanded as*

$$\mathrm{Tr}\, f(D/\Lambda) = \sum_{n\geq 0} \Lambda^{m-n} a_n(D^2)\frac{2f_{m-n}}{\Gamma((m-n)/2)} + a_m(D^2)f(0), \tag{7.11}$$

for certain f_{n-m} depending on the function f. Here m is the dimension of the manifold under consideration.

1 E.g. see [13, Theorem 1.7.6].
2 Equivalently we can demand all functions on the manifold to have compact support, i.e. they 'vanish at infinity'.
3 See e.g. [34], appendix B for more details on this.

Proof. We take a test function $k(u)$, which is the Laplace transform of another:

$$k(u) = \int_0^\infty e^{-su} h(s) \mathrm{d}s.$$

From this we can obtain an even function $f(u)$ by taking the square of the argument:

$$f(u) := k(u^2) = \int_0^\infty e^{-su^2} h(s) \mathrm{d}s.$$

Now we can replace u by an (unbounded) operator D/Λ and take the trace on both sides yielding

$$\mathrm{Tr}\, f(D/\Lambda) = \int_0^\infty \mathrm{Tr}(e^{-sD^2/\Lambda^2}) h(s) \mathrm{d}s.$$

If D^2 is of the form (7.2), we can use (7.7) and this turns into

$$\begin{aligned} \mathrm{Tr}\, f(D/\Lambda) &= \sum_{n \geq 0} \int_0^\infty \left[(s/\Lambda^2)^{(n-m)/2} a_n(D^2) \right] h(s) \mathrm{d}s \\ &= \sum_{n \geq 0} \Lambda^{m-n} a_n(D^2) \int_0^\infty s^{(n-m)/2} h(s) \mathrm{d}s. \end{aligned} \tag{7.12}$$

Now, we can use for $n \neq m$ the Laplace transform of $s^{(n-m)/2}$:

$$s^{(n-m)/2} = \frac{1}{\Gamma((m-n)/2)} \int_0^\infty e^{-sv} v^{(m-n)/2-1} \mathrm{d}v,$$

whereas for $n = m$ we use

$$\int h(s) \mathrm{d}s = \lim_{v \to 0} \int_0^\infty e^{-sv} h(s) \mathrm{d}s = f(0).$$

Inserting these in (7.12), we get:

$$\begin{aligned} \mathrm{Tr}\, f(D/\Lambda) = a_m(D^2) f(0) &+ \sum_{n \geq 0} \Lambda^{m-n} a_n(D^2) \frac{1}{\Gamma((m-n)/2)} \\ &\times \int_0^\infty \left[\int_0^\infty e^{-sv} h(s) \mathrm{d}s \right] v^{(m-n)/2-1} \mathrm{d}v. \end{aligned}$$

We can make a change of variable $w^2 := v$, which gives us

$$\begin{aligned} \mathrm{Tr}\, f(D/\Lambda) &= \sum_{n \geq 0} \Lambda^{m-n} a_n(D^2) \frac{2}{\Gamma((m-n)/2)} \int_0^\infty f(w) w^{m-n-1} \mathrm{d}w \\ &\quad + a_m(D^2) f(0) \\ &= \sum_{n \geq 0} \Lambda^{m-n} a_n(D^2) \frac{2 f_{m-n}}{\Gamma((m-n)/2)} + a_m(D^2) f(0), \end{aligned} \tag{7.13}$$

where we defined

$$f_k := \int_0^\infty f(w) w^{k-1} \mathrm{d}w.$$

□

8 A FIRST APPLICATION: THE EINSTEIN-YANG-MILLS SYSTEM

A first application of the theory presented in the previous chapters is a model that not only outgrows a toy theory but will play a central role in what follows. We obtain the Einstein-Yang-Mills system, after we have constructed a new spectral triple from an existing one by taking the tensor product with an algebra of matrices.

Up to the end of this thesis M will denote a four-dimensional compact Riemannian manifold without boundary. We take our spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to be the tensor product of the canonical spectral triple (see Example 5.7) and the finite spectral triple $(M_N(\mathbb{C}), M_N(\mathbb{C}), 0)$:

$$\begin{aligned} \mathcal{A} &= C^\infty(M) \otimes M_N(\mathbb{C}); \\ \mathcal{H} &= L^2(M,S) \otimes M_N(\mathbb{C}); \\ D &= \partial\!\!\!/_M \otimes \mathrm{id}, \quad \text{with } \partial\!\!\!/_M = i\gamma^\mu \nabla^S_\mu, \end{aligned}$$

where the representation of the complex $N \times N$ matrix $M_N(\mathbb{C})$ on $M_N(\mathbb{C})$ is just by left multiplication)

We make $(\mathcal{A}, \mathcal{H}, D)$ a *real* spectral triple by defining $J : \mathcal{H} \to \mathcal{H}$ by

$$J(\psi \otimes A) := J_M \psi \otimes A^*, \quad \psi \in \mathcal{H}, A \in M_N(\mathbb{C}), \tag{8.1}$$

where J_M is the real structure on $L^2(M,S)$ (see Example 5.7) and A^* is the adjoint of A.

Note that elements of the algebra $\mathcal{A}$ can equally well be interpreted as 'matrix-valued functions', i.e. functions from M to $M_N(\mathbb{C})$ by the identification

$$(a \otimes A)(x) := a(x)A, \quad a \in C^\infty(M), \; A \in M_N(\mathbb{C}).$$

Since $a(x) \in \mathbb{C}$ we have that $C^\infty(M) \otimes M_N(\mathbb{C}) \simeq C^\infty(M, M_N(\mathbb{C}))$.

Keep in mind that an element $\chi(x)$ of $S_x \otimes M_N(\mathbb{C})$ actually has several indices, both spinor (indicated by a, b, etc. and running from $1, \ldots, 4$) as matrix (indicated by i, j, etc. and running from $1, \ldots, N$), so in component form $\chi = \psi \otimes T$ is can pointwise be written like $\chi(x)^a_{ij} = \psi(x)^a \otimes T_{ij}$.

8.1 INNER FLUCTUATIONS

As we shall see later on, the inner fluctuations generated by the Morita equivalence of the algebra $\mathcal{A}$ have an important physical interpretation. To determine these fluctuations, we first need the local expression of the fluctuated Dirac operator. As we have seen in Chapter 6, all inner fluctuations are of the form (6.16), where now

$$\begin{aligned} \sum_j a_j [D, b_j] &= i \sum_j (\tilde{a}_j \otimes \tilde{A}_j)(\gamma^\mu \nabla^S_\mu \otimes \mathrm{id})(\tilde{b}_j \otimes \tilde{B}_j) \\ &= i\gamma^\mu \Big(\sum_j \tilde{a}_j \nabla^S_\mu(\tilde{b}_j) \otimes \tilde{A}_j \tilde{B}_j \Big) =: \gamma^\mu a_\mu \otimes A, \end{aligned}$$

where $\tilde{a}_j, \tilde{b}_j, a_\mu \in C^\infty(M)$ and $\tilde{A}_j, \tilde{B}_j, A \in M_N(\mathbb{C})$.

Since the dimension of the manifold is even, we have for the action of J_M on gamma-matrices ([14, § 5.3]):

$$J_M \gamma^\mu J_M^* = J_M \gamma^\mu J_M^{-1} = -\gamma^\mu, \tag{8.2}$$

where the first equality follows from J_M being an isometry on an even-dimensional manifold. Now for an operator A of the form $A = \gamma^\mu a_\mu \otimes \tilde{A}$ —with $a_\mu \in C^\infty(M)$— we have

$$\begin{aligned} JAJ^*(\psi \otimes T) &= J_M a_\mu \gamma^\mu J_M^* \otimes (\tilde{A}T^*)^* \\ &= -\overline{a_\mu}\gamma^\mu \otimes T\tilde{A}^* \\ &= -a_\mu \gamma^\mu \otimes TA \end{aligned}$$

by the self-adjointness of A and using that $\gamma^{\mu *} = \gamma^\mu$. Adding terms and setting $\mathrm{ad}(A)T := [A, T]$, we have

$$A + JAJ^* = \gamma^\mu a_\mu \otimes \mathrm{ad}(\tilde{A}), \tag{8.3}$$

for the fluctuations of the Dirac operator.

Combining the expression for $\partial\!\!\!/_M$ (4.14) and (8.3), we can state the complete, 'fluctuated' Dirac operator in local coordinates:

$$D_A = ie_a^\mu \gamma^a[(\partial_\mu + \omega_\mu) \otimes \mathrm{id} - ia_\mu \otimes \mathrm{ad}(\tilde{A})] \tag{8.4}$$

So, even though we started out with the trivial Dirac operator on $M_N(\mathbb{C})$, we still get a non-trivial one after allowing for fluctuations!

For some of the calculations that will follow, it is most convenient to interpret the algebra of matrices as follows. We associate with each element $T \in M_N(\mathbb{C})$ a function from M to $M \times M_N(\mathbb{C})$, that assigns to each point x the value (x, T). We then make the identification

$$\Gamma^\infty(M, S) \otimes M_N(\mathbb{C}) \simeq \Gamma^\infty(M, S) \otimes_{C^\infty(M)} \Gamma(M, M \times M_N(\mathbb{C})).$$

With this identification we are allowed to 'pull' the $C^\infty(M)$-function a_μ through the —now $C^\infty(M)$-linear— tensor product resulting in

$$-i\gamma^\mu \otimes a_\mu \,\mathrm{ad}\,\tilde{A} =: \gamma^\mu \otimes \mathbb{A}_\mu, \tag{8.5}$$

where $\mathbb{A} := -ia_\mu \,\mathrm{ad}(\tilde{A})$. The result is that $\mathbb{A}$ is *skew-Hermitian* ($\mathbb{A}^* = -\mathbb{A}$), rather than that it is self-adjoint.

Note that, in order for D_A to respect the $C^\infty(M)$-linearity of the tensor product, ∂_μ must naturally act on its second part as well. We therefore need to write

$$D_A = ie_a^\mu \gamma^a[(\partial_\mu + \omega_\mu) \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\mu + \mathbb{A}_\mu)]. \tag{8.6}$$

Note that this is the same operator as in (8.4) but on $\Gamma^\infty(M, S) \otimes_{C^\infty(M)} \Gamma^\infty(M, M \times M_N(\mathbb{C}))$ instead of $\Gamma^\infty(M, S) \otimes M_N(\mathbb{C})$.

8.1.1 *The symmetries of the fluctuations*

In the previous calculations we already saw —and used— that A was self-adjoint by the demand of the self-adjointness of D_A, it is thus a $\mathfrak{u}(N)$-valued one-form.[1] Now, $U(N)$ is not a simple group but $U(N) \simeq U(1) \times SU(N)$ resulting in $\mathfrak{u}(N) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(N)$ for the corresponding Lie algebras. But since $A + JAJ^{-1}$ is in the adjoint representation [cf. (8.3)] of $U(N)$, the scalar $\mathfrak{u}(1)$-part is irrelevant when taking a Lie bracket like $[A, T]$ and we might therefore just as well discard that part from the outset, keeping only a traceless object. The symmetry group of the fluctuations is therefore effectively $\mathfrak{su}(N)$.

8.2 THE INNER PRODUCT: FERMIONS

We can then define an inner product on $\mathcal{H} = L^2(M, S) \otimes M_N(\mathbb{C})$ by combining the inner product on $L^2(M, S)$ (4.10) with the Hilbert Schmidt inner product on $M_N(\mathbb{C})$:

$$\begin{aligned}\langle\, \psi_1 \otimes T_1\,,\, \psi_2 \otimes T_2\,\rangle &= \langle\, \psi_1\,,\, \psi_2\,\rangle_{L^2(M,S)} \langle\, T_1\,,\, T_2\,\rangle_{M_N(\mathbb{C})} \\ &:= \int_M (\psi_1, \psi_2)(x) \sqrt{g}\, \mathrm{d}^4x\, \mathrm{Tr}(T_1^* T_2),\end{aligned}$$

This inner product will have a very important application in the system under consideration as the next theorem shows.

▷ **Theorem 8.1.** *The fermionic part of the Lagrangian of the $SU(N)$ Einstein-Yang-Mills system is given by the expression*

$$\langle \chi, D_A \chi \rangle, \quad \chi \in \mathcal{H}. \tag{8.7}$$

Proof. Combine the inner product and the formula for the fluctuated Dirac operator to yield

$$\begin{aligned}\langle\, \chi\,,\, D_A\chi\,\rangle &:= \langle\, \psi \otimes T\,,\, i\gamma^\mu[(\partial_\mu + \omega_\mu) \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{A}_\mu]\psi \otimes T\,\rangle \\ &= \sum_{i,j} |T_{ij}|^2 \int_M \overline{\psi(x)} \gamma^\mu [i(\partial_\mu + \omega_\mu)\psi](x) \sqrt{g}\, \mathrm{d}^4x \\ &\qquad - \int_M \overline{\psi(x)} i\gamma^\mu(x)\psi)(x)] \sqrt{g}\, \mathrm{d}^4x\, \mathrm{Tr}(T^*[A_\mu, T]), \\ &= i \int_M \overline{\psi(x)} i\gamma^\mu(x)[t_1(\partial_\mu + \omega_\mu)] + t_2\psi](x) \sqrt{g}\, \mathrm{d}^4x,\end{aligned} \tag{8.8}$$

with $t_1 := \mathrm{Tr}(T^*T)$ and $t_{2\mu} := \mathrm{Tr}(T[A_\mu, T])$. Upon renaming objects, this corresponds to the expressions of the literature for the fermion part of the Lagrangian (compare e.g. [37, § 16.2], but on a curved space). □

Of course the value of this inner product should not depend on whether we think of $\mathcal{H}$ as $L^2(M, S) \otimes M_N(\mathbb{C})$ or as $\Gamma^\infty(S) \otimes_{C^\infty(M)} \Gamma(M, M \times M_N(\mathbb{C}))$. To make this explicit, in the latter case we would have

$$\begin{aligned}\langle\, \chi\,,\, D_A\chi\,\rangle &:= \langle\, \psi \otimes T\,,\, i\gamma^\mu[(\partial_\mu + \omega_\mu) \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\mu + \mathbb{A}_\mu)]\psi \otimes T\,\rangle \\ &= \langle\, \psi\,,\, i\langle T, T\rangle \gamma^\mu(\partial_\mu + \omega_\mu)\psi\,\rangle + \langle\, \psi\,,\, i\langle T, \gamma^\mu(\partial_\mu + \mathbb{A}_\mu)T\rangle\psi\,\rangle \\ &= i \int_M \Big(\overline{\psi(x)} i\gamma^\mu \{ [\mathrm{Tr}(T^*T)(\partial_\mu + \omega_\mu) \\ &\qquad\qquad + [\mathrm{Tr}(T^*[A_\mu, T])\psi](x)\} \Big) \sqrt{g}\, \mathrm{d}^4x,\end{aligned}$$

which clearly equals (8.8) after using the $C^\infty(M)$-linearity of the trace and identifying t_1 and t_2 with the same expressions we did earlier.

1 In the Dirac operator we pulled out factor i, causing the finite part of A_μ to be skew-Hermitian.

8.3.1 *Finding expressions for E and $\Omega_{\mu\nu}$*

The coefficients of the heat kernel expansion involve the operators E — defined by (7.3)— and $\Omega_{\mu\nu}$, the curvature tensor of the connection on the tensor product. So before we can proceed, we need expressions for them.

▷ **Theorem 8.2.** *The operator D_A^2, with D_A given by (8.6), is of the form (7.2).*
Proof. This calculation bears strong resemblance with the one that led to (4.19). Writing out and using taking care of (4.11) gives

$$\begin{aligned}
D_A^2 &= -\gamma^\mu[\nabla^S_\mu \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\mu + \mathbb{A}_\mu)][\gamma^\nu\{\nabla^S_\nu \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\nu + \mathbb{A}_\nu)\}] \\
&= -\gamma^\mu\gamma^\nu[\nabla^S_\mu\nabla^S_\nu \otimes \mathrm{id} \ + \mathrm{id} \otimes (\partial_\mu + \mathbb{A}_\mu)(\partial_\nu + \mathbb{A}_\nu) \\
&\quad + \nabla^S_\nu \otimes (\partial_\mu + \mathbb{A}_\mu) + \nabla^S_\mu \otimes (\partial_\nu + \mathbb{A}_\nu)] \\
&\qquad + \Gamma^\nu_{\lambda\mu}\gamma^\mu\gamma^\lambda[\nabla^S_\nu \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\nu + \mathbb{A}_\nu)] \\
&= -g^{\mu\nu}[\nabla^S_\mu \otimes \mathrm{id} \ + \mathrm{id} \otimes (\partial_\mu + \mathbb{A}_\mu)][\nabla^S_\nu \otimes \mathrm{id} \ + \mathrm{id} \otimes (\partial_\nu + \mathbb{A}_\nu)] \\
&\quad + \Gamma^\nu[\nabla^S_\nu \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\nu + \mathbb{A}_\nu)] \\
&\qquad - \tfrac{1}{2}\gamma^\mu\gamma^\nu([\nabla^S_\mu, \nabla^S_\nu] \otimes \mathrm{id} \ + \mathrm{id} \otimes [\partial_\mu + \mathbb{A}_\mu, \partial_\nu + \mathbb{A}_\nu])
\end{aligned}$$

where we have used the symmetry of the Christoffel symbols ($\Gamma^\bullet_{\mu\nu} = \Gamma^\bullet_{\nu\mu}$) and the definition $\Gamma^\nu := \Gamma^\nu_{\mu\lambda}g^{\mu\lambda}$. Notice the similarities with (4.19). Now for the first of the commutators of the connections we take the result of equation (4.18), while for the second we have

$$[\partial_\mu + \mathbb{A}_\mu, \partial_\nu + \mathbb{A}_\nu] = \partial_\mu(\mathbb{A}_\nu) - \partial_\nu(\mathbb{A}_\mu) + [\mathbb{A}_\mu, \mathbb{A}_\nu] =: \mathbb{F}_{\mu\nu} \tag{8.9}$$

the curvature tensor of $\mathbb{A}_\mu$. Inserting these results as well as the local expression for ∇^S (4.12) and collecting terms of the same order of ∂_μ yields:

$$\begin{aligned}
D_A^2 &= -[g^{\mu\nu}(\partial_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\mu)(\partial_\nu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\nu) \\
&\quad + [(2\omega^\mu - \Gamma^\mu) \otimes \mathrm{id} + 2\,\mathrm{id} \otimes \mathbb{A}^\mu](\partial_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\mu) \\
&\qquad + (\partial^\mu\omega_\mu + \omega^\mu\omega_\mu - \Gamma^\mu\omega_\mu + \tfrac{1}{4}R) \otimes \mathrm{id} + \mathrm{id} \otimes (\partial^\mu\mathbb{A}_\mu + \mathbb{A}^\mu\mathbb{A}_\mu) \\
&\qquad\quad + 2\omega^\mu \otimes \mathbb{A}_\mu - \Gamma^\mu \otimes \mathbb{A}_\mu - \tfrac{1}{2}\gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu}].
\end{aligned}$$

So indeed D_A is of the form (7.2), with

$$\begin{aligned}
K^\mu &= (2\omega^\mu - \Gamma^\mu) \otimes \mathrm{id} + 2\,\mathrm{id} \otimes \mathbb{A}^\mu \\
L &= (\partial^\mu\omega_\mu + \omega^\mu\omega_\mu - \Gamma^\mu\omega_\mu + \tfrac{1}{4}R) \otimes \mathrm{id} + \mathrm{id} \otimes (\partial^\mu\mathbb{A}_\mu + \mathbb{A}^\mu\mathbb{A}_\mu) \\
&\quad + 2\,\omega^\mu \otimes \mathbb{A}_\mu - \Gamma^\mu \otimes \mathbb{A}_\mu - \tfrac{1}{2}\gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu}.
\end{aligned}$$

which proves the theorem. □

With this we can both determine ω'_μ (and consequently $\Omega_{\mu\nu}$) and E uniquely. First of all, ω'_μ is given by (7.5):

$$\omega'_\mu = \frac{1}{2}g_{\mu\nu}(K^\nu + \Gamma^\nu \otimes \mathrm{id}) = \omega_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{A}_\mu. \tag{8.10}$$

With this we find for E [cf. (7.4)]:

$$\begin{aligned}
E &= L - g^{\mu\nu}\partial_\nu\omega'_\mu - g^{\mu\nu}\omega'_\mu\omega'_\nu + g^{\mu\nu}\omega'_\rho\Gamma^\rho_{\mu\nu} \\
&= [\partial_\mu(\omega^\mu) + \omega^\mu\omega_\mu - \Gamma^\mu\omega_\mu + \tfrac{1}{4}R] \otimes \mathrm{id} + 2\,\omega_\mu \otimes \mathbb{A}^\mu \\
&\qquad \mathrm{id} \otimes (\partial^\mu\mathbb{A}_\mu + \mathbb{A}^\mu\mathbb{A}_\mu) + \tfrac{1}{2}\gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu} - \Gamma^\nu[\mathrm{id} \otimes \mathbb{A}_\nu] \\
&\qquad\quad - [\partial^\mu\omega_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\mu\mathbb{A}^\mu] - [\omega^\mu\omega_\mu \otimes \mathrm{id} + 2\,\omega^\mu \otimes \mathbb{A}_\mu \\
&\qquad\qquad + \mathrm{id} \otimes \mathbb{A}^\mu\mathbb{A}_\mu] + \Gamma^\mu[\omega_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{A}_\mu] \\
&= \tfrac{1}{4}R \otimes \mathrm{id} - \tfrac{1}{2}\gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu}.
\end{aligned}$$

Next, from (7.5) we have

$$\Omega_{\mu\nu} := \partial_\mu(\omega'_\nu) - \partial_\nu(\omega'_\mu) + [\omega'_\mu, \omega'_\nu],$$

where $\partial_\mu = \partial_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\mu$ is understood. Inserting ω'_μ from (8.10) yields

$$\begin{aligned}\Omega_{\mu\nu} &= (\partial_\mu\omega_\nu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\mu\mathbb{A}_\nu) - (\partial_\nu\omega_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\nu\mathbb{A}_\mu) \\ &\quad + [\omega_\mu \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{A}_\mu, \omega_\nu \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{A}_\nu] \\ &= (\partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]) \otimes \mathrm{id} \\ &\quad + \mathrm{id} \otimes (\partial_\mu\mathbb{A}_\nu - \partial_\nu\mathbb{A}_\mu + [\mathbb{A}_\mu, \mathbb{A}_\nu]) \\ &= \frac{1}{4}R^{ab}_{\mu\nu}\gamma_{ab} \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{F}_{\mu\nu},\end{aligned}$$

where the derivation of the $R^{ab}_{\mu\nu}\gamma_{ab}$ is taken from (4.18).

8.3.2 *The spectral action*

We have shown that the fluctuated Dirac operator D_A meets the demands needed to apply the theory of § 7.1, using equation (7.11). The dimension of the manifold under consideration is 4, so we get:

$$\mathrm{Tr}\, f(D_A/\Lambda) \sim 2f_4\Lambda^4 a_0(D_A^2) + 2f_2\Lambda^2 a_2(D_A^2) + f(0)a_4(D_A^2) + \mathcal{O}(\Lambda^{-2}). \quad (8.11)$$

In order to determine the first terms of this this expansion we need to obtain the exact expressions of the Seeley-De Witt coefficients $a_0(D_A^2), a_2(D_A^2)$ and $a_4(D_A^2)$ from the heat kernel expansion[2]. In order to do so, we need to translate equations 7.8a, 7.8b and 7.8c into the current situation. For that, we use $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$, where the trace of A is taken over spinor space. For the traces over the space $M_N(\mathbb{C})$ —denoted by Tr_N if otherwise confusion could arise— we have

$$\mathrm{Tr}_N(\mathrm{id}) = \dim M_N(\mathbb{C}) = N^2.$$

Now for the first coefficient $a_0(D_A^2)$ we immediately find

$$a_0(D_A^2) = (4\pi)^{-2}\int_M \mathrm{Tr}(\mathrm{id})\,\mathrm{Tr}_N(\mathrm{id})\sqrt{g}\mathrm{d}^4x = \frac{N^2}{4\pi^2}\int_M \sqrt{g}\,\mathrm{d}^4x. \quad (8.12)$$

Next for $a_2(D_A^2)$ we have two terms. The first is

$$\mathrm{Tr}\left(-\frac{R}{6} \otimes \mathrm{id}\right) = -\frac{4N^2}{6}R, \quad (8.13)$$

where the 4 comes from the trace of the identity in spinor space and the N^2 from that of $M_N(\mathbb{C})$. The second term is

$$\mathrm{Tr}(E) = \big(N^2R - \sum_{\mu<\nu}\mathrm{Tr}(\gamma^\mu\gamma^\nu)\,\mathrm{Tr}(\mathbb{F}_{\mu\nu})\big) = N^2R, \quad (8.14)$$

since $g_{\mu\nu}$ is symmetric in its indices, but $\mathbb{F}^{\mu\nu}$ is antisymmetric. Combining (8.13) and (8.14), we obtain

$$a_2(D_A^2) = \frac{N^2}{(4\pi)^2}\int_M\left(-\frac{4}{6}R + R\right)\sqrt{g}\,\mathrm{d}^4x = \frac{N^2}{48\pi^2}\int_M R\sqrt{g}\,\mathrm{d}^4x. \quad (8.15)$$

2 Recall that $a_n(D_A^2)$ is $a_n(x, D_A^2)$ after integration.

Next for $a_4(D_A^2)$ we have quite a few terms and will deal with them one by one.

The first four terms of (7.8c) yield us no surprises:

$$\operatorname{Tr}(-12R_{;\mu}{}^{\mu}) = -48N^2R_{;\mu}{}^{\mu}, \tag{8.16}$$

$$\operatorname{Tr}(5R^2) = 20N^2R^2, \tag{8.17}$$

$$\operatorname{Tr}(-2R_{\mu\nu}R^{\mu\nu}) = -8N^2R_{\mu\nu}R^{\mu\nu}, \tag{8.18}$$

$$\operatorname{Tr}(2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) = 8N^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{8.19}$$

Next, we have

$$\begin{aligned}\operatorname{Tr}(RE) &= R\Big[\frac{1}{4}R\operatorname{Tr}(\mathrm{id})\operatorname{Tr}_N(\mathrm{id}) - \sum_{\mu<\nu}\operatorname{Tr}(\gamma^\mu\gamma^\nu)\operatorname{Tr}(\mathbb{F}_{\mu\nu})\Big]\\ &= N^2R^2,\end{aligned} \tag{8.20}$$

where we exploited the antisymmetry of $\mathbb{F}_{\mu\nu}$ again. For the square of E we obtain

$$\begin{aligned}\operatorname{Tr}(E^2) &= \frac{1}{16}R^2\operatorname{Tr}(\mathrm{id})\operatorname{Tr}_N(\mathrm{id}) - \frac{2}{4}\sum_{\mu<\nu}R\operatorname{Tr}(\gamma^\mu\gamma^\nu)\operatorname{Tr}(\mathbb{F}_{\mu\nu})\\ &\quad + \sum_{\mu<\nu,\rho<\sigma}\operatorname{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\operatorname{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma})\\ &= \frac{N^2}{4}R^2 + 4\sum_{\mu<\nu,\rho<\sigma}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\operatorname{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma})\\ &= \frac{N^2}{4}R^2 - 2\operatorname{Tr}(\mathbb{F}^{\mu\nu}\mathbb{F}_{\mu\nu}).\end{aligned} \tag{8.21}$$

Here, changing to a summation over all indices gave a difference of a factor four, and using the antisymmetry of the field tensor resulted in factor minus two again, and the vanishing of the term containing $\mathbb{F}_\mu{}^\mu$. The next term is

$$\operatorname{Tr}(E_{;\mu}{}^{\mu}) = \big(N^2R_{;\mu}{}^{\mu} - \sum_{\rho<\sigma}g^{\rho\sigma}\operatorname{Tr}(\mathbb{F}_{\rho\sigma;\mu}{}^{\mu})\big) = N^2R_{;\mu}{}^{\mu}, \tag{8.22}$$

by antisymmetry of $\mathbb{F}_{\mu\nu}$. And finally, for the last term we have:

$$\begin{aligned}\operatorname{Tr}(\Omega_{\mu\nu}\Omega^{\mu\nu}) &= \frac{N^2}{16}R^{ab}_{\mu\nu}R^{cd\mu\nu}\operatorname{Tr}(\gamma_{ab}\gamma_{cd}) + \frac{1}{2}R^{ab}_{\mu\nu}\operatorname{Tr}(\gamma_{ab})\operatorname{Tr}(\mathbb{F}^{\mu\nu})\\ &\quad + 4\operatorname{Tr}(F_{\mu\nu}\mathbb{F}^{\mu\nu})\\ &= -\frac{N^2}{2}R_{ab\mu\nu}R^{ab\mu\nu} + 4\operatorname{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}),\end{aligned} \tag{8.23}$$

where in the second step we used (A.8) and have exploited the property $R_{ijkl} = -R_{jikl}$.

The only thing left, is putting all pieces together. Combining equations (8.16), to (8.23) —but with the right coefficients (see [7.8c])— we get

$$\begin{aligned} a_4(D_A^2) = & \frac{1}{16\pi^2}\frac{1}{360}\int_M \Bigg[(60-48)N^2 R_{;\mu}{}^{\mu} + \left(\frac{180}{4}+20-60\right)N^2R^2 \\ & - 8N^2 R_{\mu\nu}R^{\mu\nu} + \left(8-\frac{30}{2}\right)N^2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & + (30\cdot 4 - 180\cdot 2)\,\mathrm{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})\Bigg] \\ = & \frac{1}{16\pi^2}\frac{N^2}{360}\int_M \Big[12 R_{;\mu}{}^{\mu} + 5R^2 - 8R_{\mu\nu}R^{\mu\nu} \\ & - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big]\sqrt{g}\,\mathrm{d}^4x - \frac{1}{24\pi^2}\int_M \mathrm{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})\sqrt{g}\,\mathrm{d}^4x. \end{aligned} \tag{8.24}$$

8.3.3 *Final expression*

Inserting the results from the previous section into (8.11) we have

$$\begin{aligned} \mathrm{Tr}\, f(D_A/\Lambda) & \sim 2f_4\Lambda^2 a_0(D_A^2) + 2f_2\Lambda^2 a_2(D_A^2) + f(0)a_4(D_A^2) + \mathcal{O}(\Lambda^{-2}) \\ & = \frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu},A)\sqrt{g}\,\mathrm{d}^4x + \mathcal{O}(\Lambda^{-2}), \end{aligned} \tag{8.25}$$

where

$$\begin{aligned} \mathcal{L}(g_{\mu\nu},A) = 2f_4\Lambda^4N^2 + \frac{N^2}{6}f_2\Lambda^2 R + f(0)\frac{N^2}{1440}\Big[5R^2 - 8R_{\mu\nu}R^{\mu\nu} \\ - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big] - \frac{f(0)}{6}\,\mathrm{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}) \end{aligned} \tag{8.26}$$

where we have discarded boundary terms $R_{;\mu}{}^{\mu}, E_{;\mu}{}^{\mu}$ since the M is without boundary.

To summarize, we have constructed a spectral triple from the canonical one and the finite spectral triple $(M_N(\mathbb{C}), M_N(\mathbb{C}), 0)$, fluctuated the Dirac operator on the former and saw that a component appeared on the latter. That component turned out to have a $\mathfrak{su}(N)$-symmetry. We then took the spectral action $\mathrm{Tr}\, f(D_A/\Lambda)$ and made an expansion in powers of the mass-scale Λ, according to Section 7.2 and found three terms, one is the Einstein-Hilbert action of General Relativity and another is the Lagrangian of the $SU(N)$-Yang-Mills field. Since the term $\langle\psi, D_A\psi\rangle$ accounts for the fermionic propagator and interactions with the gauge field, the sum

$$\langle\psi, D_A\psi\rangle + \mathrm{Tr}\, f(D_A/\Lambda) \tag{8.27}$$

gives the full action of the Einstein Yang-Mills system plus terms $\mathcal{O}(\Lambda^{-2})$.

Part II

NONCOMMUTATIVE GEOMETRY & SUPERSYMMETRY

9 INTRODUCTION TO SUPERSYMMETRY

It is fairly pointless to discuss supersymmetry (SUSY) —as we will be doing in the following chapters— without having at least a certain grasp of what is meant with it. In this chapter we will present some basic notions of supersymmetry, mostly in the parlance of physics. We underline that it is by no means an exhausting coverage of the field. Good —though not very rigorous— introductions are [21], [23] and [2]. A more mathematical approach can be found in [11].

9.1 EXTENSIONS OF THE POINCARE GROUP

As is well known, all isometries —rotations, boosts and translations— of Minkowski space $\mathbb{R}^{3,1}$ (with metric g) are described by the Poincaré group. The Lie algebra corresponding to the 10-dimensional Poincaré group is generated by $J_{\mu\nu}$ $(= -J_{\nu\mu})$ [boosts and rotations] and P_μ $(\mu, \nu = 1, \ldots, 4)$ [translations] satisfying

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[J_{\mu\nu}, P_\rho] &= g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu \\
[J_{\mu\nu}, J_{\rho\sigma}] &= g_{\mu\sigma} M_{\nu\rho} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho} + g_{\nu\rho} M_{\mu\sigma}.
\end{aligned}$$

The question was raised whether extensions of the Poincaré algebra exist to incorporate a possible symmetry that would prove to be valuable for physics. In [7] Coleman and Mandula proved that —given certain conditions— these are all the symmetries of the S-matrix.

Several years later however, Haag et al. [15] showed that extending the Poincaré algebra *can* possibly lead to new physics, if instead of a Lie algebra one uses a

▷ **Definition 9.1 [Lie superalgebra].** A Lie superalgebra $\mathcal{G}$ *is a vector space (over* $\mathbb{R}$ *or* $\mathbb{C}$*) supplied with a map* $[.,.] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ *satisfying*

$$[A, B] = -(-)^{|A||B|}[B, A], \quad \forall\, A, B \in \mathcal{G} \tag{9.1}$$

and

$$\begin{aligned}
&(-)^{|A||C|}[A, [B, C]] \;+\; (-)^{|C||B|}[C, [A, B]] \\
&\qquad +\; (-)^{|B||A|}[B, [C, A]] = 0 \quad \forall\, A, B, C \in \mathcal{G}.
\end{aligned} \tag{9.2}$$

Here $|.| : \mathcal{G} \to \{0, 1\}$ *denotes the* order *of an element, thus making* $\mathcal{G}$ $\mathbb{Z}_2$*-graded.*
This generalizes a Lie algebra, where all elements are considered to be of order 0.

It is then possible to extend the Poincaré algebra with a set of variables Q_a^i and $\overline{Q}_a^i$ $(i = 1, \ldots, N, a = 1, 2)$ of order 1, transforming in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group respectively. This extended algebra is called the *supersymmetry algebra*.

Troughout this thesis we will be considering the case $N = 1$ only.

Adding Q and $\overline{Q}$, a number of new commutators have to be determined.

▷ **Lemma 9.2.** *We have the following results for the extra commutators that appear due to adding Q and $\overline{Q}$.*

$$[P_{ab}, Q_c] = [P_{ab}, \overline{Q}_c] = 0, \tag{9.3a}$$
$$[Q_a, J_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu}Q)_a, \quad [\overline{Q}_a, J_{\mu\nu}] = \tfrac{1}{2}(\overline{\sigma}_{\mu\nu}\overline{Q})_a \tag{9.3b}$$
$$[Q_a, \overline{Q}_b] = -2iP_{\alpha\beta} \tag{9.3c}$$
$$[Q_a, Q_b] = [\overline{Q}_a, \overline{Q}_b] = 0, \tag{9.3d}$$

where we have written $P_{\alpha\beta} = (\sigma^\mu)_{\alpha\beta}P_\mu$, $\sigma^\mu = (1, \overline{\sigma})$ being the Pauli matrices.
Proof. See [36, Ch. 2]. □

▷ **Lemma 9.3.** *On any representation $|m, s, s_3\rangle$ (with mass m, spin s) of the supersymmetry algebra we have*

$$\overline{Q}_2|m, s, s_3\rangle \sim |m, s, s_3 + \tfrac{1}{2}\rangle \qquad \overline{Q}_1|m, s, s_3\rangle \sim |m, s, s_3 - \tfrac{1}{2}\rangle. \tag{9.4}$$

Proof. See [35, Ch. 2]. □
N.B. a similar theorem can be stated (e.g.[21]) for massless representations of the Poincaré algebra.

The above lemma suggests that supersymmetry is a relation between fermions (with half integer spin) and bosons (with integer spin). This will indeed turn out to be the case.

9.2 A FIRST PEEK AT SUSY: THE WESS-ZUMINO MODEL

Now how does this relate to field theory?
▷ **Definition 9.4 [Supersymmetry transformation].** *For a constant, two component spinor ϵ, we define (cf. [35, pg. 21]) a* supersymmetry transformation *on a field ϕ as*

$$\delta_\epsilon\phi(x) := [(\epsilon Q) + (\epsilon^*\overline{Q})]\phi(x). \tag{9.5}$$

N.B. It is important not to think of δ as some map, rather we associate with some field a new one $\delta_\epsilon\phi(x)$.

If we define such a $\delta_\epsilon\phi_i(x)$ for each of the fields $\phi_1, \ldots, \phi_n$ appearing in the action of a certain system, we can talk about whether or not this action is invariant under supersymmetry. For this we define:

$$\delta S[\phi_1, \ldots, \phi_n] := \frac{\mathrm{d}}{\mathrm{d}t} S[\phi_1 + t\delta_\epsilon\phi_1, \ldots, \phi_n + t\delta_\epsilon\phi_n]\Big|_{t=0} \tag{9.6}$$

If then $\delta S[\phi_1, \ldots, \phi_n] = 0$ we call the system to be *supersymmetry invariant*.

▷ **Example 9.5 [Wess-Zumino].** *The action of a system containing a free Weyl fermion ψ and scalar ϕ, is given by*

$$S[\phi, \phi^*, \chi, \overline{\chi}] = -\int \left[|\partial_\mu\phi(x)|^2 + i\psi(x)^*\overline{\sigma}^\mu\partial_\mu\psi(x)\right]\mathrm{d}^4x.$$

One can define $\delta_{\epsilon,\epsilon^*}\phi(x)$ and $\delta_{\epsilon,\epsilon^*}\psi(x)$ by

$$\delta_\epsilon[\phi(x)] := \epsilon \cdot \chi(x). \tag{9.7}$$

(implying that $\overline{Q}\phi(x) = 0$) and

$$\delta_{\epsilon^*}[\psi(x)] := -i[\sigma^\mu\epsilon^*]\partial_\mu\phi(x),. \tag{9.8}$$

respectively, and show that the action is invariant under (9.7) and (9.8) (see [23, ch. 3]). Fields such as ϕ and ψ are called each others superpartners*.*

Actually, in proving the invariance of the Wess-Zumino model, the equation of motion for ψ has to be used; it only holds *on shell*. We can make this work off shell though as well by introducing a complex scalar field D (an 'auxiliary field') appearing in the Lagrangian through $\mathcal{L}_D = |D(x)|^2$. Modifying (9.7) and (9.8) slightly to contain D, supersymmetry is seen to be holding both on shell as off shell. Furthermore we can check that (9.3c) holds by applying the maps (9.7) and (9.8) twice.
Notice that this is a pointwise approach: at each separate point of Minkowski space the bosonic and fermionic variables are linked to each other trough δ.

The example above is a nice illustration of a requisite in order for a system to exhibit supersymmetry at all:

> The total number of fermionic and bosonic degrees of freedom is the same.

Upon applying the equations of motion for ψ in the Wess-Zumino model, the number of degrees of freedom (two) match, but off shell we have four real fermionic degrees of freedom but only two bosonic. Adding two off shell bosonic degrees of freedom (the equations of motion for D are $D = 0$, thus D does not contribute to on shell degrees of freedom), mended this problem.

N.B. In many of the more advanced treatments of supersymmetry (e.g. [35]), ordinary space is extended to yield a *superspace* $(x^\mu, \theta, \overline{\theta})$ (where θ and $\overline{\theta}$ are two-component Grassmann variables). The particle content of a certain model is then described in terms of *superfields* (fields depending on all coordinates of superspace). The action is recovered by integrating potentials that are functions of these superfields over superspace by means of a *Berezin integral*. Though this kind of machinery can be convenient, we will not adopt it here.

9.3 SUPERSYMMETRIC GAUGE THEORIES

Many —if not all— of the theories describing nature at its most fundamental level are gauge theories. How do supersymmetry and gauge symmetry relate?

Since sypersymmetry links particles of different spin and gauge transformations only have an effect on the gauge ('internal') group of the particles, we require both transformations to be compatible. This boils down to requiring that:

> A particle and its superpartner are in the same representation of the gauge group.

Two pivotal examples of supersymmetric gauge theories are

▷ **Example 9.6 [Super Yang-Mills theory].** *The Yang-Mills model —the model discussed in Chapter 8 but on a flat manifold— exhibits ($N = 1$) supersymmetry under suitable conditions (cf. [23, ch. 3.3]).*

▷ **Example 9.7 [MSSM].** *If we consider the Standard Model (SM, see Chapter 12) —certainly a gauge theory— we can add a superpartner for each type of* *particle that appears. The theory that describes the SM-particles and their —yet unobserved— superpartners is called the* Minimally Supersymmetric Standard Model (MSSM)[1]*. The adjective 'minimally' is justified by the fact that the MSSM is the supersymmetric extension of the SM with the smallest number of additional superpartners: it is an example of* $N = 1$ *supersymmetry.*

The list of the MSSM's merits is quite impressive. See [6, ch. 1] for a short overview.

We will be encountering both examples again in the succeeding chapters.

1 The situation is actually somewhat more involved; in addition to all Standard Model particles having a superpartner, the MSSM requires an extra two Higgs doublets instead of only one.

10 PREPARING THE EINSTEIN-YANG-MILLS SYSTEM FOR SUPERSYMMETRY

Having considered the Einstein-Yang-Mills system in Chapter 8 and some basic notions of supersymmetry in Chapter 9, we would like to combine these to obtain a realization of supersymmetry for the Einstein-Yang-Mills system in the framework of noncommutative geometry.

We immediately stumble upon two problems. The first is that bosonic and fermionic fields are not in the same representation of the gauge group, as is required for supersymmetry. Indeed, in Section 8.1 we were able to reduce the degrees of freedom for the bosons by requiring self-adjointness and unimodularity, and found them to be in the adjoint representation of $\mathfrak{su}(N)$. The fermions, on the other hand, are still of the form $\psi \otimes A$, with $A \in M_N(\mathbb{C})$.

Secondly a spinor $\psi(x) \in S_x$ has eight real degrees of freedom —four components with complex coefficients— whereas the continuous part of the gauge potential has only four: A_μ, $\mu = 1, \ldots, 4$.

These two problems are summarized in table 10.1. We will solve them one by one in the subsequent sections.

	Currently		Needed	
	Continuous	Finite	Continuous	Finite
Bosons	4	$N^2 - 1$	4	$N^2 - 1$
Fermions	8	$2N^2$	4	$N^2 - 1$

Table 10.1: The number of real degrees of freedom for fermions and bosons for both the continuous and finite parts of the Hilbert space. The current situation is on the left, whereas the situation needed in order to obtain supersymmetry is given on the right.

10.1 MAJORANA & WEYL FERMIONS

As is well known, the basic fermionic constituents of many supersymmetric theories are Majorana fermions, particles that are their own antiparticle. One would therefore need to restrict to those $\psi \in \mathcal{H}$ for which we can impose the Majorana condition: $J\psi = \psi$, thereby reducing the degrees of freedom. However, in the Euclidean set up, we have $J^2 = -1$ (see table 5.1) with which only $\psi(x) = 0$ is compatible. Indeed (massless) Majorana fermions do not exist in a 4 dimensional Euclidean space, as was pointed out by Schwinger already in 1959 [26].

Different but similar solutions of this problem were given by Van Nieuwenhuizen and Waldron [30] and Nicolai [24]. To obtain a Lagrangian in Euclidean space, whose Green functions are analytic continuations of the Minkowskian counterpart, [30] proposes a scheme that boils down to doubling the spinor degrees of freedom compared to Minkowski space. That is, the kinetic term of a fermion Lagrangian still comprises of two Grassmann variables ($\overline{\psi}$ and ψ), but the number of complex numbers is doubled. This is achieved by relaxing the (Minkowskian) *reality constraint* $\overline{\psi} := \psi^\dagger\gamma^0$ —thus regarding $\overline{\psi}$ and ψ as two *different* variables.[1]

Instead of working with Majorana fermions in Euclidean space, Nieuwenhuizen and Waldron employ the fact that in Minkowski space any Majorana fermion can be written as a Weyl fermion. They then take a Minkowskian Lagrangian containing Weyl fermions, relax the constraint $\overline{\psi} := \psi^\dagger\gamma^0$ and Wick rotate the Lagragian, thus obtaining the Euclidean counterpart of a Minkowskian Lagrangian containing Majorana fermions.

So our goal is converted to finding an inner product that is suited for Weyl spinors. To this end —and to accommodate a mass term for the neutrino—, Chamseddine, Connes and Marcolli propose in [5] (but see also [9, §16.2]), a different inner product compared to (8.7):

$$\langle\, J\psi_1\,,\, D_A\psi_2\,\rangle. \tag{10.1}$$

The pleasant property of this inner product is that it allows for restricting spinors to eigenspaces of γ, thus obtaining Weyl spinors[2]. Indeed, for $\psi_{1,2} \in \mathcal{H}^+ = \{\psi \in \mathcal{H}\,;\, \gamma\psi = \psi\}$ and $(\mathcal{A}, \mathcal{H}, D)$ of KO-dimension 2, we have

$$\langle\, J\psi_1\,,\, D\psi_2\,\rangle = \langle\, J\psi_1\,,\, D\gamma\psi_2\,\rangle = -\langle\, \gamma J\psi_1\,,\, D\psi_2\,\rangle = \langle\, J\gamma\psi_1\,,\, D\psi_2\,\rangle.$$

Here, we have used in the second step that $D\gamma = -\gamma D$ and $\gamma^* = \gamma$ and that $J\gamma = -\gamma J$ in the third step. In our case (KO-dimension 4; $J\gamma = \gamma J$) however, this calculation would yield the disastrous result $\langle J\psi_1, D\psi_2\rangle = -\langle J\psi_1, D\psi_2\rangle$ and will thus not allow a reduction to Weyl spinors.

To resolve this, we retain the inner product (8.7) —with no J occurring— and employ the ideas of [30] as illustrated above, inserting two *different* spinor fields into the inner product. Then we *can* restrict the spinors to eigenspaces of γ, albeit different ones. That is, for $\psi \in \mathcal{H}^+$ we have

$$\langle\, \chi\,,\, D\psi\,\rangle = \langle\, \chi\,,\, D\gamma\psi\,\rangle = -\langle\, \chi\,,\, \gamma D\psi\,\rangle = -\langle\, \gamma\chi\,,\, D\psi\,\rangle$$

which again equals $\langle \chi, D\psi\rangle$ provided $\chi \in \mathcal{H}^-$, i.e. $\gamma\chi = -\chi$.

In conclusion, we are able to reduce fermion degrees of freedom by the right amount, by taking

$$S_f[\mathbb{A}, \psi, \chi] := \langle\, \chi\,,\, D_A\psi\,\rangle \quad \psi \in \mathcal{H}^+, \chi \in \mathcal{H}^-, \tag{10.2}$$

instead of (8.7) or (10.1) as the fermionic part of the action. The resulting action is then in fact the same as that of [30, eq. 47].

Notice that the above arguments hold for fluctuated Dirac operators equally well, since $D_A\gamma = -\gamma D_A$ if $A \in \Omega^1_D\mathcal{A}$, as can easily be seen from (8.3).

Even though it might appear that we have doubled the degrees of freedom by relaxing the reality constraint as above, the path integral is insensitive for this step; the result should be the same regardless of whether we integrate over $\overline{\psi}$ and ψ or over $\overline{\chi}$ and ψ. (See the discussion at the end of [31] for some details.)

1 A direct consequence of this is the loss of the Hermiticity of the Lagrangian, but see [31], pg. 2 for a short discussion on this.

2 This was in fact the very reason for Chamseddine, Connes and Marcolli to work in a spectral triple of KO-dimension 2.

We still have to reduce the finite part of the fermions from $M_N(\mathbb{C})$ to $\mathfrak{su}(N)$, to yield what —in analogy with the bosonic case— we can call *unimodular fermions*.

We will do this in two steps; first from $M_N(\mathbb{C})$ to $\mathfrak{u}(N)$ and second from $\mathfrak{u}(N)$ to $\mathfrak{su}(N)$.

For the first part we simply use the fact that the $M_N(\mathbb{C})$ is the complexification of $\mathfrak{u}(N)$:

$$M_N(\mathbb{C}) \simeq \mathfrak{u}(N) \oplus i\mathfrak{u}(N)$$
or
$$M_N(\mathbb{C}) \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(N).$$

For the Hilbert space this implies that

$$\mathcal{H} = L^2(M, S) \otimes_{\mathbb{C}} M_N(\mathbb{C}) \simeq L^2(M, S) \otimes_{\mathbb{R}} \mathfrak{u}(N).$$

Since for any $u \in \mathfrak{u}(N)$ we have $u = u^a S_a, u^a \in \mathbb{R}, \forall\, a = 1, \ldots, N^2$ with respect to a basis S_a of $\mathfrak{u}(N)$, we can write $\psi(x) \otimes u \in S_x \otimes_{\mathbb{R}} \mathfrak{u}(N)$ in components:

$$\psi(x)_i \otimes u = \psi_i^a \otimes S_a, \quad \text{with } \psi_i^a := \psi(x)_i u^a, i = 1, \ldots 4,$$

in components.

Note that the adjoint action of the bosons leaves this reduction invariant. This is seen by applying $\mathbb{A}(x) = \gamma^\mu A_\mu^a \otimes \mathrm{ad}_{T_a}$ to $\psi(x) = \psi^b \otimes S_b$ for which the finite part is then given by $[T_a, S_b]$. For the latter we have $[T_a, S_b]^* = (S^b)^*(T^a)^* - (T^a)^*(S^b)^* = -[T^a, S^b]$. We may conclude that $\mathbb{A}\psi \in L^2(M, S) \otimes_{\mathbb{R}} \mathfrak{u}(N)$ whenever $\psi \in L^2(M, S) \otimes_{\mathbb{R}} \mathfrak{u}(N)$.

What is left now, is the reduction from $\mathfrak{u}(N)$ to $\mathfrak{su}(N)$.

We do this by splitting any fermion into two parts as was done earlier for the bosons in Chapter 8:

$$\widetilde{\psi} = \mathrm{Tr}\,\widetilde{\psi} + \psi, \quad \widetilde{\psi} \in L^2(M, S) \otimes_{\mathbb{R}} \mathfrak{u}(N),$$

with $\mathrm{Tr}\,\widetilde{\psi} \in L^2(M, S) \otimes \mathfrak{u}(1)$ and $\psi \in L^2(M, S) \otimes \mathfrak{su}(N)$. Inserting these expressions into the inner product, we get

$$\begin{aligned} \langle\, \widetilde{\chi}\,,\, D_A \widetilde{\psi}\,\rangle = {} & \langle\, \mathrm{Tr}\,\widetilde{\chi}\,,\, D_A \,\mathrm{Tr}\,\widetilde{\psi}\,\rangle + \langle\, \chi\,,\, D_A \,\mathrm{Tr}\,\widetilde{\psi}\,\rangle \\ & + \langle\, \mathrm{Tr}\,\widetilde{\chi}\,,\, D_A \psi\,\rangle + \langle\, \chi\,,\, D_A \psi\,\rangle. \end{aligned}$$

Now the first term on the right hand side is reduced to $\langle \mathrm{Tr}\,\widetilde{\chi}, D\,\mathrm{Tr}\,\widetilde{\psi}\rangle$ since the finite part gives us terms like $[u, s]$ with $u \in \mathfrak{u}(1)$ and $s \in \mathfrak{su}(N)$, that of course vanish. That same argument can be applied to reduce the second term to $\langle \chi, D\,\mathrm{Tr}\,\widetilde{\psi}\rangle$ which then vanishes for the inner product on the finite part gives only terms like $\mathrm{Tr}(su) = u\,\mathrm{Tr}\,s = 0$. Finally the third term on the right hand side vanishes too, while the inner product on the finite part yields only terms like $\mathrm{Tr}(u[s_1, s_2]) = u\,\mathrm{Tr}[s_1, s_2]$, where $s_1, s_2 \in \mathfrak{su}(N)$. We then have obtained that

$$\langle\, \widetilde{\chi}\,,\, D_A \widetilde{\psi}\,\rangle = \langle\, \mathrm{Tr}\,\widetilde{\chi}\,,\, D\,\mathrm{Tr}\,\widetilde{\psi}\,\rangle + \langle\, \chi\,,\, D_A \psi\,\rangle, \tag{10.3}$$

i.e. the two different parts decouple and the trace-part lacks all gauge interactions; it describes a totally free fermion. We therefore discard it from the theory, thereby retaining the unimodular part of the fermion.

Combining the two schemes presented in the preceding two sections equalize the bosonic and fermionic degrees of freedom, thereby putting them in the same gauge representation, thus permitting supersymmety.

11 SUPERSYMMETRY IN THE NON-COMMUTATIVE EINSTEIN-YANG-MILLS SYSTEM

The preparations being done in the previous chapter, the Einstein-Yang-Mills system is at least suited for supersymmetry. What is left, is actually showing that the system is supersymmetric.

We still consider the spectral triple as defined in the beginning of Chapter 8, but in addition we take the manifold (M, g) such that all Christoffel symbols (and thus the scalar curvature) vanish everywhere. The Dirac operator can then locally be written as[1] $D_A = i\gamma^\mu(\partial_\mu \otimes + \otimes (\partial_\mu + \mathbb{A}_\mu)) =: i\gamma^\mu D_\mu$. The action corresponding to this system can directly be read off from the curved case as considered in Chapter 8:

$$S[\psi, \chi, \mathbb{A}] = \int_M \mathcal{L}(\psi, \chi, \mathbb{A})(x)\mathrm{d}^4x + \mathcal{O}(\Lambda^{-2}) \tag{11.1}$$

with Lagrangian

$$\mathcal{L}(\psi, \chi, \mathbb{A}) = \mathrm{Tr}_F(\chi, D_A\psi) - \frac{f(0)}{24\pi^2}\mathrm{Tr}(\mathbb{F}^2), \quad \text{with } \psi \in \mathcal{H}^+, \chi \in \mathcal{H}^-, \tag{11.2}$$

where $\mathbb{F} = \gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu}$. Here with $(.\,,.)$ we mean the Hermitian pairing $(.\,,.) : \Gamma^\infty(S) \times \Gamma^\infty(S) \to C^\infty(M)$ and by Tr_F the trace of the *finite part*.

For the constituents of this Lagrangian we have, after performing the reduction as discussed in the previous chapter:

$$\mathbb{A} = A^a \otimes \mathrm{ad}_{T_a} \in B(L^2(M, S)) \otimes \mathrm{ad}\,\mathfrak{su}(N)$$
$$\psi = \psi^a \otimes T_a \in L^2(M, S)^+ \otimes \mathfrak{su}(N),$$

where the plus-sign means that ψ is of positive chirality. In order to see whether this system exhibits supersymmetry, we would like to define[2]

$$\delta\mathbb{A} \in B(\mathcal{H}) \quad \text{and} \quad \delta\psi, \delta\chi \in \mathcal{H} \tag{11.3}$$

—where the expressions for $\delta\mathbb{A}$, $\delta\psi$ and $\delta\chi$ contain their respective superpartners— under which the action in (11.1) is invariant, i.e. $\delta S[\psi, \chi, \mathbb{A}]$ as defined in (9.6) vanishes.

BOSONIC TERMS

Let $\epsilon_{+/-}$ be two constant spinors [i.e. $(\partial_\mu\epsilon_{+/-})(x) = 0 \; \forall\, x \in M$] of positive/negative chirality.

▷ **Definition 11.1.** *We define* $\delta\mathbb{A} \in B(\mathcal{H})$ *by*

$$\delta\mathbb{A} := c_1\gamma^\mu \otimes \mathrm{ad}(\epsilon_-, \gamma_\mu\psi) + c_2\gamma^\mu \otimes \mathrm{ad}(\chi, \gamma_\mu\epsilon_+), \tag{11.4}$$

where $(\epsilon_-, \gamma_\mu\psi) = (\epsilon_-, \gamma_\mu\psi^a)T_a$ *is understood and* $\mathrm{ad} : \mathfrak{su}(N) \to \mathrm{ad}\,\mathfrak{su}(N)$ *is simply defined by* $\mathrm{ad}(T_a) := \mathrm{ad}_{T_a}$.

1 Notice the difference with (8.4).
2 Even though the symbols are the same, no confusion is likely to arise.

The values of $c_{1,2} \in \mathbb{R}$ are yet to be determined from the subsequent calculations.

Since the bosonic part of the action functional only depends on $\mathbb{A}$, we have

$$\delta S_B[\mathbb{A}] := \frac{\mathrm{d}}{\mathrm{d}t} S_B[\mathbb{A} + t\delta\mathbb{A}]\Big|_{t=0}, \tag{11.5}$$

where t is some real parameter.

Taking the bosonic part of (11.2), this means that

$$\begin{aligned}
\delta \int_M \mathrm{Tr}\, \mathbb{F}^2 \mathrm{d}^4x &= \frac{\mathrm{d}}{\mathrm{d}t} \int_M \mathrm{Tr}(\mathrm{d}(\mathbb{A} + t\delta\mathbb{A}) + [\mathbb{A} + t\delta\mathbb{A}, \mathbb{A} + t\delta\mathbb{A}]_F)^2 \mathrm{d}^4x\Big|_{t=0} \\
&= \int_M \Big[2\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}[(\mathrm{d}\mathbb{A} + [\mathbb{A}, \mathbb{A}]_F)(\mathrm{d}t\delta\mathbb{A} + [t\delta\mathbb{A}, \mathbb{A}]_F \\
&\qquad + [\mathbb{A} + t\delta\mathbb{A}]_F) + \mathcal{O}(t^2)]\Big|_{t=0} \Big] \mathrm{d}^4x \\
&= \int_M 2\, \mathrm{Tr}\left[\mathbb{F}(\mathrm{d}\delta\mathbb{A} + [\delta\mathbb{A}, \mathbb{A}]_F + [\mathbb{A} + \delta\mathbb{A}]_F) \right] \mathrm{d}^4x.
\end{aligned} \tag{11.6}$$

Three unknown terms occur in (11.6): $\mathrm{d}\delta\mathbb{A}$, $[\delta\mathbb{A}, \mathbb{A}]_F$ and $[\mathbb{A} + \delta\mathbb{A}]_F$. For the first we have

▷ **Lemma 11.2.** *With $\delta\mathbb{A}$ given by Definition 11.1 we have*

$$\mathrm{d}\delta\mathbb{A} = 2\sigma^{\mu\nu} \otimes [c_1\, \mathrm{ad}(\epsilon_-, \partial_\mu\gamma_\nu\psi) - c_2\, \mathrm{ad}(\partial_\mu\gamma_\nu\chi, \epsilon_+)]. \tag{11.7}$$

with $\sigma^{\mu\nu} := \frac{1}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$.

Proof. The proof can be split up into two parts. First we have that

$$\begin{aligned}
\mathrm{d}\mathbb{A} &= c(\mathrm{d}_M \sum_i a_i \mathrm{d}_M b_i) \otimes \mathrm{ad}_{T_i} \\
&= c(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)\partial_\mu(a_i\partial_\nu b_i) \otimes \mathrm{ad}_{T_i} = 2\sigma^{\mu\nu} \otimes \partial_\mu\mathbb{A}_\nu.
\end{aligned} \tag{11.8}$$

[e.g. (3.11)]. Applying Definition 11.1 to this expression, we simply get

$$\begin{aligned}
\mathrm{d}\delta\mathbb{A} &= 2\sigma^{\mu\nu} \otimes \partial_\mu\delta\mathbb{A}_\nu \\
&= 2\sigma^{\mu\nu} \otimes [c_1\, \mathrm{ad}\, \partial_\mu(\epsilon_-, \gamma_\nu\psi) + c_2\, \mathrm{ad}\, \partial_\mu(\chi, \gamma_\nu\epsilon_+)].
\end{aligned}$$

Secondly, the spin connection ∇^S is Hermitian, i.e.

$$(\mathrm{d}\psi, \chi) + (\psi, \mathrm{d}\chi) = \mathrm{d}(\psi, \chi) \quad \forall\, \psi, \chi \in L^2(M, S), \tag{11.9}$$

on a flat manifold. We can extend the pairing $(.\,,.) : \Gamma(S) \times \Gamma(S) \to C(M)$ to another $(.,.)' : \Gamma(S) \times \Gamma(S) \otimes \Omega^1(C(M)) \to \Omega^1(C^\infty(M))$ by

$$(a, b \otimes \mathrm{d}c)' := (a, b)\mathrm{d}c.$$

Using this pairing in combination with (11.9) and then applying the map c [cf. (4.7)] we get

$$\gamma^\mu(\partial_\mu\psi, \chi) + \gamma^\mu(\psi, \partial_\mu\chi) = \gamma^\mu\partial_\mu(\psi, \chi). \tag{11.10}$$

Now applying this to the case with one of the spinors constant, we get the desired result. □

Next, for the other part of the field strength we have

▷ **Lemma 11.3.** *With $\delta\mathbb{A}$ given by Definition 11.1 we have*

$$[\delta\mathbb{A}, \mathbb{A}]_F + [\mathbb{A}, \delta\mathbb{A}]_F = 2\sigma^{\mu\nu} \otimes \big(c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\mu \mathbb{A}_\nu, \psi) + c_2 \,\mathrm{ad}(\gamma_\mu \mathbb{A}_\nu \chi, \epsilon_+)\big). \tag{11.11}$$

Proof. A simple calculation shows that

$$\begin{aligned}[\delta\mathbb{A}, \mathbb{A}]_F + [\mathbb{A}, \delta\mathbb{A}]_F &= \gamma^\mu\gamma^\nu \otimes ([\delta\mathbb{A}_\mu, \mathbb{A}_\nu] + [\mathbb{A}_\mu, \delta\mathbb{A}_\nu]) \\ &= \gamma^\mu\gamma^\nu \otimes \big([c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\mu\psi) + c_2 \,\mathrm{ad}(\chi, \gamma_\mu\epsilon_+), \mathbb{A}_\nu] \\ &\qquad + [\mathbb{A}_\mu, c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\nu\psi) + c_2 \,\mathrm{ad}(\chi, \gamma_\nu\epsilon_+)]\big) \\ &= \gamma^\mu\gamma^\nu \otimes \big(- \mathrm{ad}[A_\nu, c_1(\epsilon_-, \gamma_\mu\psi) + c_2(\chi, \gamma_\mu\epsilon_+)] \\ &\qquad + \mathrm{ad}[A_\mu, c_1(\epsilon_-, \gamma_\nu\psi) + c_2(\chi, \gamma_\nu\epsilon_+)]\big),\end{aligned}$$

where we have used the fact that $\mathrm{ad} : \mathfrak{su}(N) \to \mathrm{ad}\,\mathfrak{su}(N)$ is a Lie algebra homomorphism and that the Lie bracket is antisymmetric. Then by ordering terms by coefficients and using that $\epsilon_{+/-}$ lack a finite part and that the gauge potential is self-adjoint (i.e. $\overline{A^a_\mu} = A^a_\mu$), this yields

$$[\delta\mathbb{A}, \mathbb{A}]_F + [\mathbb{A}, \delta\mathbb{A}]_F = 2\sigma^{\mu\nu} \otimes \big[c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\nu \mathbb{A}_\mu, \psi) + c_2 \,\mathrm{ad}(\gamma_\nu \mathbb{A}_\mu \chi, \epsilon_+)\big].$$

□

We are now ready to determine the full expression for $\delta S_B[\mathbb{A}]$:

▷ **Proposition 11.4.** *The map 11.1 results in*

$$\delta S_B[\mathbb{A}] = -\frac{4f(0)N}{3\pi^2}[c_1\langle\epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu\psi\rangle + c_2\langle F^{\mu\nu}\gamma_\mu D_\nu\chi, \epsilon_+\rangle], \tag{11.12}$$

for the bosonic part of the Einstein Yang-Mills action. Here $\langle .,. \rangle$ denotes the inner product on $\mathcal{H}$ and we have written $D_\mu := \partial_\mu + \mathbb{A}_\mu$.

Proof. We use the results of the previous two lemmas by inserting (11.7) and (11.11) into (11.6) to get

$$\begin{aligned}\delta \int_M \mathrm{Tr}\,\mathbb{F}^2 \mathrm{d}^4x &= 2 \int_M \Big[\mathrm{Tr}\,\big[\mathbb{F}(\mathrm{d}\delta\mathbb{A} + [\delta\mathbb{A}, \mathbb{A}] + [\mathbb{A} + \delta\mathbb{A}]\big) \\ &= 4 \int_M \Big[\mathrm{Tr}(\gamma^\mu\gamma^\nu\sigma^{\alpha\beta})\,\mathrm{Tr}\,\big[\mathbb{F}_{\mu\nu}\big(c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\beta\partial_\alpha\psi) \\ &\qquad + c_2 \,\mathrm{ad}(\gamma_\beta\partial_\alpha\chi, \epsilon_+) + c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\beta\mathbb{A}_\alpha, \psi) \\ &\qquad + c_2 \,\mathrm{ad}(\gamma_\beta\mathbb{A}_\alpha\chi, \epsilon_+)\big)\big]\Big] \mathrm{d}^4x.\end{aligned} \tag{11.13}$$

Employing the identity (A.7) we get

$$2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu\sigma^{\alpha\beta}) = 8[\delta^{\mu\beta}\delta^{\nu\alpha} - \delta^{\mu\alpha}\delta^{\nu\beta}], \tag{11.14}$$

so that (11.13) turns into

$$\begin{aligned}&32 \int_M \Big[\mathrm{Tr}[\mathbb{F}^{\mu\nu}(c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\mu\partial_\nu\psi) + c_2 \,\mathrm{ad}(\gamma_\mu\partial_\nu\chi, \epsilon_+) \\ &\qquad + \big(c_1 \,\mathrm{ad}(\epsilon_-, \gamma_\mu\mathbb{A}_\nu, \psi) + c_2 \,\mathrm{ad}(\gamma_\mu\mathbb{A}_\nu\chi, \epsilon_+)\big]\Big] \mathrm{d}^4x \\ &= 32N \int_M \Big[\mathrm{Tr}[F^{\mu\nu}(c_1(\epsilon_-, \gamma_\mu\partial_\nu\psi) + c_2(\gamma_\mu\partial_\nu\chi, \epsilon_+) \\ &\qquad + \big(c_1(\epsilon_-, \gamma_\mu\mathbb{A}_\nu, \psi) + c_2(\gamma_\mu\mathbb{A}_\nu\chi, \epsilon_+)\big]\Big] \mathrm{d}^4x \\ &= 32N \int_M \Big[\mathrm{Tr}[(c_1(\epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu\psi) + c_2(F^{\mu\nu}\gamma_\mu D_\nu\chi, \epsilon_+)]\Big] \mathrm{d}^4x \\ &= 32N[c_1\langle\epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu\psi\rangle + c_2\langle F^{\mu\nu}\gamma_\mu D_\nu\chi, \epsilon_+\rangle]\end{aligned} \tag{11.15}$$

where we have written $D_\mu = \partial_\mu + \mathbb{A}_\mu$, used the antisymmetry of $\mathrm{F}_{\mu\nu}$ and that for $\mathfrak{su}(N)$ the inner product is a scalar multiple of the killing form:

$$\mathrm{Tr}[\mathrm{ad}\, T_1\, \mathrm{ad}\, T_2] = N\, \mathrm{Tr}[T_1 T_2] \quad \forall\, T_1, T_2 \in \mathfrak{su}(N). \tag{11.16}$$

Collecting terms, pulling $F_{\mu\nu}$ into the pairing and multiplying with the correct factor from the Lagrangian, gives the desired result. □

FERMIONIC TERMS

Now for the fermionic part of the action we can perform similar calculations. To do so we must first define $\delta\psi$ and $\delta\chi$.

▷ **Definition 11.5.** *For $\psi, \in \mathcal{H}^+, \chi \in \mathcal{H}^-$ we define $\delta\psi \in \mathcal{H}^+, \delta\chi \in \mathcal{H}^-$ by*

$$\delta\psi := c_3 F\epsilon_+ = c_3 \gamma^\mu \gamma^\nu F^a_{\mu\nu} \epsilon_+ \otimes T_a \qquad \textit{and} \qquad \delta\chi := c_4 F\epsilon_-. \tag{11.17}$$

Here again, $c_{3,4} \in \mathbb{R}$ are yet to be determined.

For the fermionic part of the action we then have the following proposition.

▷ **Proposition 11.6.** *With the definitions given above, we have for the fermionic part of the action*

$$\delta S_F[\psi, \chi, \mathbb{A}] = -2c_4 \langle \epsilon_-, F_{\mu\nu}\gamma^\mu D^\nu \psi\rangle - 2c_3 \langle F_{\mu\nu}\gamma^\mu D^\nu \chi, \epsilon_+\rangle. \tag{11.18}$$

Proof. We take the same approach as in the bosonic case:

$$\begin{aligned} \delta S_F[\psi, \chi, \mathbb{A}] &= \frac{\mathrm{d}}{\mathrm{d}t} S_F[\psi + t\delta\psi, \chi + t\delta\chi, \mathbb{A} + t\delta\mathbb{A}]\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \langle \chi + t\delta\chi, D_{A+t\delta A}(\psi + t\delta\psi)\rangle\Big|_{t=0} \\ &= c_4 \langle F\epsilon_-, D_A\psi\rangle + \langle \chi, \delta\mathbb{A}\psi\rangle + c_3\langle \chi, D_A F\epsilon_+\rangle. \end{aligned} \tag{11.19}$$

Let us look at the terms on the right hand side one by one. Writing out F, using the self-adjointness of D_A and using the identity (A.9) we have for the first

$$(F\epsilon_-, D_A\psi) = i([\delta^{\sigma\mu}\gamma^\nu - \delta^{\sigma\nu}\gamma^\mu + \delta^{\mu\nu}\gamma^\sigma - i\epsilon^{\sigma\mu\nu\lambda}\gamma^5\gamma_\lambda] D_\sigma F_{\mu\nu}\epsilon_-, \psi). \tag{11.20}$$

The first two terms of (11.20) add up by the antisymmetry of $F_{\mu\nu}$, whereas the third term vanishes for that very reason. The fourth term is in fact the celebrated Bianchi identity (e.g. [10, §18.3]):

$$[D_\mu F_{\nu\sigma} + D_\sigma F_{\mu\nu} + D_\nu F_{\sigma\mu}](x) = 0 \quad \forall\, x \in M.$$

We are thus left with:

$$c_4 \langle F\epsilon_-, D_A\psi\rangle = 2c_4 i(\gamma^\nu D^\mu F_{\mu\nu}\epsilon_-, \psi)$$

By exactly the same reasoning we can rewrite the third term of (11.19). Now we are still left with the second term of (11.19), which yields for each point $x \in M$:

$$\mathrm{Tr}_F(\chi, \delta\mathbb{A}\psi)(x) = f_{abc}(\chi^a, \gamma^\mu\psi^c)[c_1(\epsilon_-, \gamma_\mu\psi) + c_2(\chi, \gamma_\mu\epsilon_+)](x). \tag{11.21}$$

Both terms are seen to vanish separately using the antisymmetry of f_{abc} and a Fierz transformation (see Appendix A.1 for all the details). Adding the results for the first and third terms of (11.19) yields the expression:

$$\begin{aligned} \delta S_F[\chi, \psi, \mathbb{A}] &= 2c_4 i\langle \gamma^\nu D^\mu F_{\mu\nu}\epsilon_-, \psi\rangle + 2c_3 i\langle \chi, \gamma^\nu D^\mu F_{\mu\nu}\epsilon_+\rangle \\ &= -2c_4 i\langle \epsilon_-, F_{\mu\nu}\gamma^\mu D^\nu\psi\rangle - 2c_3 i\langle F_{\mu\nu}\gamma^\mu D^\nu\chi, \epsilon_+\rangle. \end{aligned}$$

□

Then adding the separate results for the fermions and bosons, we get:

▷ **Theorem 11.7.** *The action given by* (11.1) *is a supersymmetry invariant, given* (11.7) *and* (11.17)*, provided that* $3\pi^2 c_4 = 2if(0)Nc_1$ *and* $3\pi^2 c_3 = 2if(0)Nc_2$*.*

Proof. With Propositions 11.4 and 11.6, this boils down to nothing more than comparing the expressions (11.12) and (11.18) to conclude that

$$\begin{aligned}
&\delta S_F[\chi, \psi, \mathbb{A}] + \delta S_B[\mathbb{A}] \\
&= -\left(\frac{4f(0)N}{3\pi^2}c_1 + 2ic_4\right)\langle \epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu \psi\rangle \\
&\quad - \left(\frac{4f(0)N}{3\pi^2}c_2 + 2ic_3\right)\langle F^{\mu\nu}\gamma_\mu D_\nu \chi, \epsilon_+\rangle = 0
\end{aligned} \tag{11.22}$$

iff $2f(0)Nc_1 = -3i\pi^2 c_4$ and $2f(0)Nc_2 = -3i\pi^2 c_3$. □

11.1 SUPERSYMMETRY UP TO THE PLANCK SCALE

In the previous chapter we have demonstrated that the action of the Yang-Mills system was invariant under supersymmetry. Despite that the signature of the manifold was Euclidean instead of Minkowskian —in contrast to practically all other treatments— resulted in several difficulties, the methods and results were essentially not that different from the canonical approach of the subject, apart from some notational matters perhaps. The point where the results of the noncommutative and canonical approaches to field theory differ most, is that the spectral action of the former contributes much more to the action than only (11.1). In terms of a series in Λ, we only took the terms $\mathcal{O}(\Lambda^0)$ [i.e. $a_4(D_A^2)$] into account in the previous Chapter. But we have absolutely no guaranty that the spectral action as a whole is supersymmetric! There's no *a priori* reason that the terms on all orders of Λ —together with the inner product— are invariant (given (11.4) and (11.7)) —or that their sum is at least.

The consequence of additional terms generated by (11.4), (11.7) and/or (11.17) would be to alter these expressions in order to preserve supersymmetry, if that is possible. But let's not panic too soon and get to work.

Since all fermionic contributions to the Lagrangian are of order Λ^0 (originating from the inner product), we can focus our attention solely on the spectral action. As future input for the latter we have

▷ **Proposition 11.8.** *The square of the operator* $D_{A+t\delta A}$ *with* δA *given by*

$$\delta A = \gamma^\mu \otimes [c_1(\epsilon_-, \gamma_\mu \psi) + c_2(\chi, \gamma_\mu \epsilon_+)], \quad \psi \in \mathcal{H}^+, \chi \in \mathcal{H}^- \tag{11.23}$$

(with c_1*,* c_2 *determined in the previous section) is of the form*

$$D^2_{A+t\delta A} = -[g_{\nu\nu}\partial^\mu\partial^\nu + K^\mu\partial_\mu + L] \tag{11.24}$$

for certain $K^\mu, L \in \mathrm{End}(\mathcal{H})$*.*

Proof. We will explicitly calculate

$$\begin{aligned}
D^2_{A+t\delta A} &= [D + A + t\delta A + J(A + t\delta A)J^*]^2 \\
&= D_A^2 + i\{D_A, \delta\mathbb{A}\}t - (\delta\mathbb{A})^2 t^2,
\end{aligned} \tag{11.25}$$

where $\delta A + J\delta A J = i\delta \mathbb{A}$ was already determined to yield

$$\gamma^\mu \otimes \mathrm{ad}[(\epsilon_-, \gamma_\mu \psi) + (\chi, \gamma_\mu \epsilon_+)]. \tag{11.26}$$

For the rest of this chapter we only take the first term of $\delta\mathbb{A}$ (featuring ψ) into account and remark that the arguments and results for the other term are analogous. We first focus on the second term on the right hand side of (11.25). Since the manifold under consideration is flat, we get

$$\begin{aligned}\{D_A, \delta\mathbb{A}\} &= [i\gamma^\nu(\partial_\nu \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\nu + \mathbb{A}_\nu))][c_1\gamma^\mu \otimes \mathrm{ad}(\epsilon_-, \gamma_\mu\psi)] \\ &\quad + [ic_1\gamma^\mu \otimes \mathrm{ad}(\epsilon_-, \gamma_\mu\psi)][\gamma^\nu(\partial_\nu \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\nu + \mathbb{A}_\nu))] \\ &= ic_1\gamma^\nu\gamma^\mu \otimes \mathrm{ad}(\epsilon_-, \gamma_\mu(\partial_\nu + \mathbb{A}_\nu)\psi) + 2ic_1 \otimes \mathrm{ad}(\epsilon_-, \gamma^\mu\psi)\,\partial_\mu \\ &\quad + 2ic_1 \otimes \mathrm{ad}(\epsilon_-, \gamma^\nu\psi)\,\mathbb{A}_\nu.\end{aligned}$$

Plugging this expression into (11.25), we receive

$$\begin{aligned}K^\mu \to K^{\mu\prime} &= K^\mu + 2c_1 \otimes \mathrm{ad}(\epsilon_-, \gamma^\mu\psi)t, \\ L \to L' &= L + c_1\gamma^\nu\gamma^\mu \otimes \mathrm{ad}(\epsilon_-, \gamma_\mu(\partial_\nu + \mathbb{A}_\nu)\psi)t \\ &\qquad + 2c_1 \otimes \mathrm{ad}(\epsilon_-, \gamma^\nu\psi)\mathbb{A}_\nu t + \mathcal{O}(t^2),\end{aligned} \tag{11.27}$$

compared to the values of K and L for only D_A. □

This result is useful, for it allows us to perform a the heat kernel expansion, analogous to (7.7). After done that, we possess the correct machinery to see to what extent each of the coefficients $a_n(D_A^2)$ (for $n = 0, 2, 4$) is invariant under supersymmetry. Since the spectral action is just part of the total action, we can adopt a definition for the supersymmetry transformation for the former, quite similar to (11.5).

▷ **Definition 11.9.** *The supersymmetry transformation for the spectral action is defined as*

$$\delta \operatorname{Tr} f(D_A/\Lambda) := \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr} f(D_{A+t\delta A}/\Lambda)\Big|_{t=0}.$$

We can again make a Laplace transformation of the spectral action and formally expand the resulting exponential in a Laurent series in Λ:

$$\begin{aligned}\delta \operatorname{Tr} f(D_A/\Lambda) &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr} e^{-s(D_{A+t\delta A}/\Lambda)^2} h(s)\mathrm{d}s\Big|_{t=0} \\ &\sim \sum_n \frac{\mathrm{d}}{\mathrm{d}t} a_n(D_{A+t\delta A})\Big|_{t=0} \Lambda^{4-n} \int_0^\infty s^n h(s)\mathrm{d}s,\end{aligned}$$

where in the last step we have used the heat kernel expansion (7.7), which is justified by the theorem above. The objects that are of particular interest are thus

$$\frac{\mathrm{d}}{\mathrm{d}t} a_n(D^2_{A+t\delta A})\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} a_n(D_A^2 + \{\delta\mathbb{A}, D_A\}t + \mathcal{O}(t^2))\Big|_{t=0}, \tag{11.28}$$

the first of which are given by (7.8a), (7.8b) and (7.8c). The E and $\Omega_{\mu\nu}$ appearing in these formulas are of course different than before, but still related to K^μ and L (as given above) in the same way; by (7.4) and (7.5).

Short calculations —similar to those in Section 8.3.1— show that the changes of K^μ to $K^{\mu\prime}$ and L to L' have the following effect on the variables E and $\Omega_{\mu\nu}$:

$$\begin{aligned} E \to E' &= E + c_1\gamma^\mu\gamma^\nu \otimes \mathrm{ad}(\epsilon_-, \gamma_\nu D_\mu\psi)t \\ &\qquad - c_1 \mathrm{id} \otimes \mathrm{ad}(\epsilon_-, \gamma^\mu D_\mu \psi)t + \mathcal{O}(t^2) \\ \Omega_{\mu\nu} \to \Omega'_{\mu\nu} &= \Omega_{\mu\nu} + ic_1 \mathrm{id} \otimes [\mathrm{ad}(\epsilon_-, \gamma_\nu D_\mu \psi) \\ &\qquad - c_1\, \mathrm{ad}(\epsilon_-, \gamma_\mu D_\nu\psi)]t + \mathcal{O}(t^2). \end{aligned}$$

Having found these particular expressions, we are ready to determine (11.28) for $n = 0, 2, 4$.

▷ **Theorem 11.10.** *Under the supersymmetry map from Definition 11.1 we have for the first three coefficients of the heat kernel expansion:*

$$\frac{\mathrm{d}}{\mathrm{d}t} a_0(D^2_{A+t\delta A})\Big|_{t=0} = 0 \tag{11.29a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a_2(D^2_{A+t\delta A})\Big|_{t=0} = 0 \tag{11.29b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a_4(D^2_{A+t\delta A})\Big|_{t=0} = \frac{c_1}{6\pi^2}\langle \epsilon_-, F^{\mu\nu}\gamma_\nu D_\mu\psi\rangle, \tag{11.29c}$$

where with $\langle .,.\rangle$ the inner product on $\mathcal{H}$ is meant.

Proof. The first coefficient $a_0(D_A^2)$ is trivial: the identity does not transform under supersymmetry. For the second coefficient, there is only one contribution [see (7.8b)];

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}(E')\Big|_{t=0} &= c_1\,\mathrm{Tr}[\gamma^\mu\gamma^\nu \otimes \mathrm{ad}(\epsilon_-, \gamma_\nu D_\mu\psi) - \mathrm{id} \otimes \mathrm{ad}(\epsilon_-, \gamma^\mu D_\mu\psi)] \\ &= 0. \end{aligned}$$

The results for the third coefficient (7.8c) should of course be no different than (11.12), but it might be instructive to calculate it explicitly anyway:

$$\delta a_4(D_A^2) = \frac{1}{192\pi^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M \mathrm{Tr}\left(6E'^2 + \Omega'_{\mu\nu}\Omega'^{\mu\nu}\right)\mathrm{d}^4x\Big|_{t=0}. \tag{11.30}$$

We use that $E = -\frac{1}{2}\gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu}$ and get for the first term of (11.30):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M \mathrm{Tr}\,E'^2\mathrm{d}^4x\Big|_{t=0} &= -2\frac{1}{2}c_1\int_M \Big[\mathrm{Tr}\left[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\lambda\sigma}\,\mathrm{ad}(\epsilon_-, \gamma_\nu D_\mu\psi)\right. \\ &\qquad \left. - \gamma^\mu\gamma^\nu \otimes \mathbb{F}_{\mu\nu}\,\mathrm{ad}(\epsilon_-, \gamma^\lambda D_\lambda\psi)\right]\Big]\mathrm{d}^4x \\ &= -4c_1\left(\delta^{\lambda\nu}\delta^{\sigma\mu} - \delta^{\lambda\mu}\delta^{\nu\sigma}\right) \\ &\qquad \times \int_M \mathrm{Tr}[\mathbb{F}_{\lambda\sigma}\,\mathrm{ad}(\epsilon_-, \gamma_\nu D_\mu\psi)]\mathrm{d}^4x \\ &= 8c_1\int_M \mathrm{Tr}\left[\mathbb{F}^{\mu\nu}\,\mathrm{ad}(\epsilon_-, \gamma_\nu D_\mu\psi)\right]\mathrm{d}^4x \\ &= -8Nc_1\langle\epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu\psi\rangle, \end{aligned}$$

where at various points we have used that $\mathbb{F}$ is antisymmetric. For the second term in (11.30) we have with $\Omega_{\mu\nu} = 1 \otimes \mathbb{F}_{\mu\nu}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M \mathrm{Tr}\,\Omega'^2_{\mu\nu}\mathrm{d}^4x\Big|_{t=0} &= 2c_1\int_M \mathrm{Tr}\left[1 \otimes \mathbb{F}_{\mu\nu}][\mathrm{id} \otimes \mathrm{ad}(\epsilon_-, \gamma_{[\nu}D_{\mu]}\psi)]\right]\mathrm{d}^4x \\ &= 16c_1\int_M \mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\,\mathrm{ad}(\epsilon_-, \gamma^\nu D^\mu\psi)\right]\mathrm{d}^4x \\ &= -16Nc_1\langle\epsilon_-, F_{\mu\nu}\gamma^\mu D^\nu\psi\rangle. \end{aligned}$$

We thus get for (11.30):

$$\begin{aligned}\delta a_4(D_A^2) &= -\frac{c_1}{192\pi^2}(48+16)N\langle\epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu\psi\rangle \\ &= -\frac{c_1}{3\pi^2}\langle\epsilon_-, F^{\mu\nu}\gamma_\mu D_\nu\psi\rangle.\end{aligned}$$

□

So the theorem shows that both $a_0(D_A^2)$ (proportional to Λ^4) and $a_2(D_A^2)$ (proportional to Λ^2) are supersymmetry invariants, and the result for $a_4(D_A^2)$ is as expected. That means that for at least all positive powers of Λ the inner product and the spectral action together are supersymmetry invariant. These results are not only not that bad, they are in fact all we could have hoped for: we can let Λ run up to the Planck scale without getting into trouble.

Note that we have not considered terms $a_n(D_A^2)$ for $n \geq 6$, not only because they are less interesting from a physical point of view but also for the pragmatic reason that their specific expressions in terms of E and $\Omega_{\mu\nu}$ are enormous at best (e.g. [13, pg 327]).

Though these results are encouraging, it is still somewhat unsatisfactory that we had to resort to a heat kernel expansion; a question whether or not the *full* spectral action is supersymmetry invariant remains to be answered. As was noted [3] by A. Chamseddine, noncommutative geometry treats bosons (spectral action) and fermions (inner product) on such a different footing, that it seems unlikely that noncommutative field theory is a truly supersymmetric theory. Hence, any attempt (such as [27]) that combines both the inner product and the spectral action into a single expression is well worth studying from the perspective of supersymmetry.

Part III

ON NONCOMMUTATIVE GEOMETRY & THE MSSM

12 THE STANDARD MODEL FROM NONCOMMUTATIVE GEOMETRY

The list of merits of noncommutative geometry is long, the derivation of the full Standard Model (SM) Lagrangian being the most prominent by far from a physical point of view. In this chapter we will very briefly sketch how this Lagrangian is derived from noncommutative geometry. The main goal is to touch upon a number of the steps leading to the Standard Model and to put the next chapter into perspective (the MSSM is after all an extension of the SM), rather than providing a detailed and thorough mathematical treatment. For details and a complete review we refer the reader to [9, Ch. 12 – 17], on which this chapter is primarily based.

As in the case of the Einstein-Yang-Mills model, a spectral triple is defined as the tensor product of the canonical spectral triple on a Riemannian spin manifold and a suitably chosen finite spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$. The former is dealt with extensively in the Preliminaries. We will focus primarily on the latter, only now and then referring to the spinor part.

12.1 THE ALGEBRA

The starting point is defining the algebra:

$$\mathcal{A}_{LR} := \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \tag{12.1}$$

where $\mathbb{H}$ denotes the algebra of the *quaternions*, which can be represented by 2×2 matrices:

$$\mathbb{H} := \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

(The subscripts L and R have no specific meaning yet.) The algebra is involutive by the involution on its components. That is, for an element $(\lambda, q_L, q_R, m) \in \mathcal{A}_{LR}$ involution is defined by

$$(\lambda, q_L, q_R, m)^* = (\overline{\lambda}, q_L^\dagger, q_R^\dagger, m^*) \in \mathcal{A}_{LR}.$$

Here $q^\dagger$ denotes the involution on quaternions:

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}^\dagger := \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}$$

12.2 THE REPRESENTATIONS OF THE ALGEBRA

We will characterize the various representations of $\mathcal{A}_{LR}$ by their dimension, written in boldface: $\mathbf{3}$ denotes the three dimensional representation of $M_3(\mathbb{C})$ and so forth. In addition, a subscript L or R is used to indicate whether $\mathbb{H}_L$ or $\mathbb{H}_R$ acts on it. We denote by $\mathcal{E}^o$ the contragredient (see Definition 2.14) of a module $\mathcal{E}$: $\mathbf{3}^o$ is the contragredient of $\mathbf{3}$.

We then construct the finite Hilbert space as follows. Let the representation $\mathcal{E}$ of $\mathcal{A}_{LR}$ be given by

$$\mathcal{E} := \mathbf{2}_L \otimes \mathbf{1}^o \oplus \mathbf{2}_R \otimes \mathbf{1}^o \oplus \mathbf{2}_L \otimes \mathbf{3}_o \oplus \mathbf{2}_R \otimes \mathbf{3}^o, \tag{12.2}$$

we take the direct sum with its opposite representation and take three copies of it:

$$\mathcal{H}_F := (\mathcal{E} \oplus \mathcal{E}^o)^{\oplus 3}. \tag{12.3}$$

We can also write $\mathcal{H}_F = \mathcal{H}_f \oplus \mathcal{H}_{\overline{f}}$ where $\mathcal{H}_f = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E}$ and $\mathcal{H}_{\overline{f}} = \mathcal{E}^o \oplus \mathcal{E}^o \oplus \mathcal{E}^o$. This notation will become more clear later on.

The number of copies is put in by hand, but the construction of $\mathcal{E} \oplus \mathcal{E}^o$ is by no means ad hoc, though, as presented here, it might so. We refer to [9, Ch. 13] for the details.

We make the finite spectral triple *real* and *even* by defining an isometry J_F and grading operator respectively. The first is given by

$$J_F : \mathcal{E} \oplus \mathcal{E}^o \to \mathcal{E} \oplus \mathcal{E}^o, \qquad J_F(f, f') := (\overline{f'}, \overline{f}), \tag{12.4}$$

that is, J_F both interchanges and conjugates the components of the different elements. The grading on the other hand is defined with respect to the isometry

$$\gamma_F := c - J_F c J_F \qquad c = (0, 1, -1, 0). \tag{12.5}$$

This has the property that

$$\gamma_F f_L = f_L \qquad \text{and} \qquad \gamma_F f_R = -f_R$$

for left and right handed particles respectively. (From the representations as above we can directly see that the first term of (12.5) gives a sign to left and right handed elements, whereas the second term does the same on $\mathcal{E}^o$.)

In addition it is easy to check that

$$J_F^2 = 1 \qquad \text{and} \qquad J_F \gamma_F = -\gamma_F J_F,$$

implying that we are dealing with a finite spectral triple of KO-dimension 6 (cf. table 5.1).[1]

Now with a bit of foresight the compatibility between the algebra and the Dirac operator is investigated. It follows that if the finite Dirac operator will have components that map from $\mathcal{H}_f$ to $\mathcal{H}_{\overline{f}}$, i.e. it has *off diagonal* components[2] we are in fact restricted to a subalgebra of $\mathcal{A}_{LR}$ instead. The algebra $\mathcal{A}_F$ of maximal dimension that allows for off diagonal Dirac operators is seen so be of the form

$$\mathcal{A}_F := \{(\lambda, q, \lambda, m) : \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})\}.$$

We continue with this algebra.

1 This has the effect that the tensor product of the two spectral triple is of KO-dimension $(4+6) \mod 8 = 2$; see the end of Chapter 5.
2 As we will see, this is something we would want.

These representations (12.2) have an interpretation of course. If we write $|\uparrow\rangle_{L,R}$ and $|\downarrow\rangle_{L,R}$ for the two basisvectors of $\mathbb{H}_{L,R}$, we can define (the finite part of) the left and right handed *up-quarks* $u_{L,R}$ by

$$u_L := |\uparrow\rangle_L \otimes \mathbf{3}^o \subset \mathbf{2}_L \otimes \mathbf{3}^o, \qquad \text{and} \qquad u_R := |\uparrow\rangle_R \otimes \mathbf{3}^o, \tag{12.6}$$

respectively. The left and right handed *down-quarks* are then given by

$$d_L := |\downarrow\rangle_L \otimes \mathbf{3}^o, \qquad \text{and} \qquad d_R := |\downarrow\rangle_R \otimes \mathbf{3}^o.$$

In a similar fashion we can take

$$\nu_{L,R} := |\uparrow\rangle_{L,R} \otimes \mathbf{1}^o \qquad \text{and} \qquad e_{L,R} := |\downarrow\rangle_{L,R} \otimes \mathbf{1}^o$$

for the (left/right handed) *neutrino* and *electron* respectively. Together they form the *leptons*. The antiparticles of these particles are then elements of the opposite representation $\mathcal{E}^o$. This is in accordance with the definition of J_F, sending particles to antiparticles and vice versa. The different copies of course represent the three *generations* of particles.

In the notation as introduced above, we can define the precise representations of the algebra. On the quarks we have, for $a = (\lambda, q, \lambda, m) \in \mathcal{A}_F$

$$a \begin{pmatrix} u_L \\ d_L \end{pmatrix} := q^t \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad a \begin{pmatrix} u_R \\ d_R \end{pmatrix} := q^t(\lambda) \begin{pmatrix} u_R \\ d_R \end{pmatrix},$$

where q^t denotes the transpose of q and with $q(\lambda)$ we mean the embedding of the scalar λ in a quaternion $q(\lambda)$:

$$q(\lambda) := \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}.$$

Note that the quaternions act only on the $\mathbf{2}$-part, leaving the $\mathbf{3}^o$-part unaffected.

On the leptons we define the representation in the same way:

$$a \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} := q^t \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad a \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} := q^t(\lambda) \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}.$$

We have a representation of the algebra on the antiparticles as well. For any antilepton[3] $\bar{l}$ it is given by

$$a\,\bar{l} := \lambda\,\bar{l},$$

acting on the $\mathbf{1}$ part. For an antiquark $\overline{q}$ on the other hand, we have

$$a\,\overline{q} := m\,\overline{q}, \tag{12.7}$$

acting on the $\mathbf{3}$ of course.

The isometry J_F defined above then has the effect that the particles not only have a left representation but also a right (or opposite) representation by

$$\pi^o(m) := J_F \pi(m^*) J_F^*.$$

3 We only use this particular notation here to emphasize that we are dealing with an antiparticle.

For example, the representation (12.7) on the antiquarks causes a right representation on the quarks by

$$\pi^o(m)f := J_F\pi(m^*)J_F f = m^t f, \qquad f \in \mathbf{2}_{L,R} \otimes \mathbf{3}^o, \tag{12.8}$$

which can be written as fm as well.

N.B. The identification of the various standard model particles is further backed up if one considers

$$SU(\mathcal{A}_F) := \{u \in \mathcal{A}_F : uu^* = u^*u = 1, \det(u) = 1\}.$$

Not only do you find

$$SU(\mathcal{A}_F) \sim U(1) \times SU(2) \times SU(3)$$

but, using the adjoint action $u(u^*)^o$ of all elements $u \in U(1) < SU(\mathcal{A}_F)$, you get precisely the correct hypercharges for the fermions ([9, § 13.3]).

12.3 THE DIRAC OPERATORS & THEIR INNER FLUCTUATIONS

As we mentioned in chapter 5, a Dirac operator on the tensor product of the canonical and finite spectral triple is of the form:

$$D_A = \not{\partial}_M \otimes \mathrm{id} + \gamma^5 \otimes D_F.$$

For the first part we take the canonical Dirac operator $\not{\partial}_M = \hat{c} \circ \nabla^S$ which was seen to locally equal $i\gamma^\mu(\partial_\mu + \omega_\mu)$. For the Dirac operator on the finite spectral triple we introduce a 3×3 matrix Υ_u that mixes between generations (in this case between the up, charm and top quarks). In a similar fashion Υ_e, Υ_ν and Υ_d can be introduced. Then D_F is taken to be

$$D_F := \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}, \tag{12.9}$$

with S given by

$$S := S_l \oplus (S_q \otimes \mathrm{id}_3), \tag{12.10}$$

where

$$S_l : \mathbf{2}_R \otimes \mathbf{1}^o \oplus \mathbf{2}_L \otimes \mathbf{1}^o \to \mathbf{2}_R \otimes \mathbf{1}^o \oplus \mathbf{2}_L \otimes \mathbf{1}^o$$

is of the form

$$S_l := \begin{pmatrix} 0 & 0 & \Upsilon_\nu^* & 0 \\ 0 & 0 & 0 & \Upsilon_e^* \\ \Upsilon_\nu & 0 & 0 & 0 \\ 0 & \Upsilon_e & 0 & 0 \end{pmatrix} \tag{12.11}$$

on the basis (ν_R, e_R, ν_L, e_L). The map

$$S_q \otimes \mathrm{id}_3 : \mathbf{2}_R \otimes \mathbf{3}^o \oplus \mathbf{2}_L \otimes \mathbf{3}^o \to \mathbf{2}_R \otimes \mathbf{3}^o \oplus \mathbf{2}_L \otimes \mathbf{3}^o$$

acts trivially on $\mathbf{3}^o$ and S_q is given by the same expression as (12.11) but with $\Upsilon_\nu \to \Upsilon_u$ and $\Upsilon_e \to \Upsilon_d$.

For the map T —the off diagonal part of the Dirac operator we discussed earlier— a fifth 3×3 matrix Υ_R is introduced, after which the map

$$T : |\uparrow\rangle_R \otimes \mathbf{1}^o \to \mathbf{1} \otimes |\uparrow\rangle_R^o,$$

connecting the neutrinos with the antineutrino, is given by $T(\nu_R) := \Upsilon_R \overline{\nu_R}$ on the right handed neutrinos and $T = 0$ everywhere else.

Again this might seem kind of ad hoc, but in fact Connes and Marcolli make a classification of Dirac operators. It turns out that —given the algebra $\mathcal{A}_F$— any Dirac operator that meets all requirements such as the order one condition (5.6) is of this form.

With the representations of the algebra being defined, the inner fluctuations of these two Dirac operators can be determined.

For the canonical Dirac operator we can define for $a_i, a_i' \in \mathcal{A}$ where $\mathcal{A} := C^\infty(M, \mathcal{A}_F)$, i.e. $a_i(x) = (\lambda_i, q_i, m_i), a_i'(x) = (\lambda_i', q_i', m_i') \in \mathcal{A}_F$

$$\begin{aligned}\Omega^1_{\partial\!\!\!/_M}\mathcal{A} \ni A^{(1,0)}(x) &:= \sum_i a_i[\partial\!\!\!/_M, a_i'](x) \\ &= \sum_i \left(\lambda_i \partial\!\!\!/_M(\lambda_i'), q_i \partial\!\!\!/_M(q_i'), m_i \partial\!\!\!/_M(m_i')\right),\end{aligned} \tag{12.12}$$

acting diagonally on generations. Here we have introduced the notation $A^{(1,0)}$ to distinguish the fluctuations coming from the canonical Dirac operator from those coming from the finite one ($A^{(0,1)}$).

By requiring self-adjointness for the fluctuated Dirac operator and furthermore demanding that the third component of (12.12) has vanishing trace, it can be shown ([9], Proposition 1.207) that these inner fluctuations can be parametrized by three gauge fields (Λ, Q, V), having $U(1)$, $SU(2)$ and $SU(3)$ symmetry respectively.

Then there is the finite Dirac operator D_F. Given a set of elements

$$\begin{aligned}a_i(x) &= \left(\lambda_i, \begin{pmatrix}\alpha & \beta \\ -\bar{b} & \bar{\alpha}\end{pmatrix}_i, \begin{pmatrix}\lambda & 0 \\ 0 & \bar{\lambda}\end{pmatrix}_i, m_i\right), \\ a_i'(x) &= \left(\lambda_i', \begin{pmatrix}\alpha' & \beta' \\ -\bar{b}' & \bar{\alpha}'\end{pmatrix}_i, \begin{pmatrix}\lambda' & 0 \\ 0 & \bar{\lambda}'\end{pmatrix}_i, m_i'\right),\end{aligned}$$

it can be shown that its inner fluctuations on $\mathcal{H}_f$ are of the form

$$\sum_i a_i[\gamma^5 \otimes D_F, a_i'](x)\Big|_{\mathcal{H}_f} = \gamma^5 \otimes \left(A_q^{(0,1)} + A_l^{(0,1)}\right) \tag{12.13}$$

acting on quarks and leptons respectively, with

$$A_q^{(0,1)} = \begin{pmatrix}0 & X \\ X' & 0\end{pmatrix} \otimes \mathrm{id}_3,$$

where id_3 denotes the identity on the color sector. Here X and X' are given by

$$X = \begin{pmatrix}\Upsilon_u^*\phi_1 & \Upsilon_u^*\overline{\phi_2} \\ -\Upsilon_d^*\phi_2 & \Upsilon_d^*\overline{\phi_1}\end{pmatrix}, \qquad X' = \begin{pmatrix}\Upsilon_u\phi_1' & \Upsilon_d\overline{\phi_2'} \\ -\Upsilon_u\overline{\phi_2'} & \Upsilon_d\overline{\phi_1'}\end{pmatrix}$$

respectively. That is, as far as the quark sector is concerned, the inner fluctuations of D_F act trivially on the *color* sector, but mix between generations. For the leptonic part of the fluctuations, we have

$$A_l^{(0,1)} = \begin{pmatrix} 0 & Y \\ Y' & 0 \end{pmatrix},$$

with Y and Y' the same as for the quark sector but with the replacements $\Upsilon_u \to \Upsilon_\nu$ and $\Upsilon_d \to \Upsilon_e$.

Here the fields $\phi = \phi_1 + j\phi_2, \phi' = \phi'_1 + j\phi'_2 \in C^\infty(M, \mathbb{H})$ are quaternion-valued fields, defined in terms $a_i(x)$ and $a'_i(x)$:

$$\phi_1 := \sum_i \lambda_i(\alpha'_i - \lambda'_i), \qquad \phi_2 := \sum_i \lambda_i\beta'_i$$
$$\phi'_1 := \sum_i \alpha_i(\lambda'_i - \alpha'_i) + \beta_i\overline{\beta_i}, \qquad \phi'_2 := \sum_i \beta_i(\overline{\lambda'_i} - \overline{\alpha'_i}) - \alpha_i\beta'_i.$$

Note that, in contrast to S, the map T does not generate any inner fluctuations.

The astonishing part is that having defined all the contents of the spectral triple completely, everything else comes out naturally! From the data of the spectral triple we can define the fluctuations of the Dirac operator(s). Having defined these, the inner product and spectral action

$$\langle J\psi\,, D_A\,\psi\rangle + \mathrm{Tr}(f(D_A/\Lambda)) \tag{12.14}$$

(where for the latter a heat kernel expansion is used) together produce the action functional. Of course, this is no picknick: performing the actual calculations requires care and patience, and a lot of problems and subtleties pop up along the way, but in the end you come up with an action: the full action of the Standard model, including the Higgs mechanism —the Higgs field being described by the different $\phi's$— neutrino mass and everything coupled to gravity!

13 SUPER-QCD

As a first step on the long road towards the MSSM, we consider a somewhat simplified version of QCD —the theory of quarks and gluons. For that we will be regarding only one of three generations of particles and we leave all leptons and weak gauge bosons out. Consequently we in fact have only one quark which is described by $\mathbf{3}^o$ instead of $\mathbf{2}_{L/R}\otimes\mathbf{3}^o$ (see (12.6)). Nevertheless, this model retains many of the characteristic features of QCD itself. It is this simplified version of QCD that we try to make supersymmetric by adding the superpartners of the particles involved.

13.1 THE SET UP

If we want any chance of finding supersymmetry, we need to enlarge the finite part of the Hilbert space such that it contains not only the quarks and antiquarks, but the gluinos[1] —the supersymmetric partners of the gluons and therefore fermions— as well. To this end, we define the finite spectral triple as follows.

▹ **Definition 13.1.** *Let the finite real spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F, J_F)$ be given by*

- $\mathcal{A}_F := M_3(\mathbb{C})$;
- $\mathcal{H}_F := \mathbf{3} \oplus M_3(\mathbb{C}) \oplus \mathbf{3}^o$, *i.e. we add the algebra* as a vector space *to the Hilbert space. Note that we have interchanged the the $M_3(\mathbb{C})$ representations of quarks and antiquarks compared to the previous Chapter [cf. (12.7) and (12.8)]. This has no physical consequences though. Troughout this chapter we will use that as a vector space $M_3(\mathbb{C}) \simeq \mathbb{C}^9$;*
- *D_F is given by*

$$D_F := \begin{pmatrix} 0 & d & 0 \\ d^* & 0 & e^* \\ 0 & e & 0 \end{pmatrix}, \tag{13.1}$$

with $d : M_3(\mathbb{C}) \to \mathbf{3}$ and $e : M_3(\mathbb{C}) \to \mathbf{3}^o$ (both linear) yet to be defined. Note that D_F is self-adjoint by construction.

- *J_F is defined by*

$$J_F(q_1, g, q_2) := (\overline{q_2}, g^*, \overline{q_1}), \tag{13.2}$$

that is, for $\mathbf{3}$ and $\mathbf{3}^o$ J_F acts the same as in the SM [see (12.4)] and for $M_3(\mathbb{C})$ it is the same as in the Einstein-Yang-Mills model (8.1) —with $N = 3$.

Note that we leave the algebra untouched; the gauge group should for the superpartners be $SU(\mathcal{A}_F) \simeq SU(3)$ too.

For the representation of $\mathcal{A}_F$ on $\mathcal{H}_F$ we take

$$\pi = \pi_1 \oplus \pi_2 \oplus \pi_1' \tag{13.3}$$

1 We will postpone the (partial) justification of this terminology until §14.4.

where

$$\pi_1(m)q := mq \qquad \forall m \in M_3(\mathbb{C}), q \in \mathbf{3},$$

$$\pi_2(m)g := mg \qquad \forall\, m \in M_3(\mathbb{C}), g \in M_3(\mathbb{C})$$

and $\pi_1' = 1$.

As with J, the representation of the algebra is as expected: on $\mathbf{3}$ and $\mathbf{3}^o$ it is —up to interchanging $\mathbf{3}$ and $\mathbf{3}^o$— the same as in the Standard Model ([9, §13.2]), on $M_3(\mathbb{C})$ it is the same as in the EYM model .

▷ **Lemma 13.2.** *With the representation as above, we have for the* opposite representation

$$\pi^o = \pi_1^o \oplus \pi_2^o \oplus {\pi_1'}^o, \tag{13.4}$$

with

$$\begin{aligned} \pi_2^o(m)g &= gm, \qquad \forall\, m \in M_3(\mathbb{C}), g \in M_3(\mathbb{C}) \\ {\pi_1'}^o(m)q &= m^t q, \qquad \forall\, m \in M_3(\mathbb{C}), q \in \mathbf{3}^o, \end{aligned} \tag{13.5}$$

and ${\pi_1}^o = 1$.

Proof. This follows directly from the definition $\pi^o(m) = J_F \pi(m^*) J_F^*$. □

13.2 INNER FLUCTUATIONS

Everything being properly defined now, the only objects that can still be varied are d and e, appearing in D_F. We are in fact restricted even further, as is seen by the next lemma.

▷ **Lemma 13.3.** *With J_F as in (13.2) the requirement $D_F J_F = J_F D_F$ uniquely determines e in terms of d:*

$$\overline{e(g)} = d(g^*) \qquad \forall\, g \in M_3(\mathbb{C}). \tag{13.6}$$

Proof. Applying $D_F J_F$ and $J_F D_F$ on an element (q_1, g, q_2), gives —amongst others— the equality (13.6), which indeed fixes e completely. □

So actually, specifying d fixes D_F completely.

▷ **Definition 13.4.** *Let $d : M_3(\mathbb{C}) \to \mathbf{3}^o$ be such that*

$$d(g) = gv \qquad \forall\, g \in M_3(\mathbb{C}). \tag{13.7}$$

Here, v is a fixed 3-tuple which depends on d.

This definition for d corresponds to $d^*(q) = q\overline{v}^t$ [i.e. $(q\overline{v}^t)_{ij} = q_i \overline{v}_j$] for the adjoint of d:

$$\begin{aligned} \langle d(g), q \rangle_{\mathbf{3}} &= \sum_i (\overline{gv})_i q_i = \sum_{i,j} (g^*)_{ji} \overline{v}_j q_i \\ &= \mathrm{Tr}(g^* q \overline{v}^t) = \langle g, q\overline{v}^t \rangle \quad \forall\, g \in M_3(\mathbb{C}), q \in \mathbf{3}^o. \end{aligned} \tag{13.8}$$

Here v is the same as in (13.7). Note that for any two 3-tuples v_1, v_2 and a matrix m we have $m(v_1 v_2^t) = (mv_1)v_2^t$ and $(v_1 v_2^t)m = v_1(m^t v_2)^t$.

By Lemma 13.3 this gives for the map e:

$$\begin{aligned} e(g) &= g^t \overline{v} \qquad \forall g \in M_3(\mathbb{C}), \\ e^*(q) &= vq^t \qquad \forall q \in \mathbf{3}. \end{aligned}$$

where again v is the same as in (13.7).

An argument that this D_F is bona fide is provided by the next lemma.

▷ **Lemma 13.5.** *Given that $\pi_1' = \pi_1^o = 1$, the finite Dirac operator with d and e as above satisfies the order one condition (5.6).*

Proof. Writing out (5.6) and using $\pi_1' = \pi_1^o = 1$ gives the simultaneous demands:

$$\begin{aligned} d\pi_2(m) &= \pi_1(m)d, & d\pi_2(m)\pi_2^o(n) &= \pi_1(n)d\pi_2^o(m), \\ d^*\pi_1(m) &= \pi_2(m)d^*, & \pi_2^o(m)d^*\pi_1(n) &= \pi_2^o(n)\pi_2(m)d^*, \\ e^*\pi_1^{o\prime}(m) &= \pi_2^o(m)e^*, & \pi_2(m)e^*\pi_1^{o\prime}(n) &= \pi_2^o(n)\pi_2(m)e^*, \\ e\pi_2^o(m) &= \pi_1^{o\prime}(m)e, & e\pi_2(m)\pi_2^o(n) &= \pi_1^{o\prime}(n)e\pi_2(m), \end{aligned}$$

which are easily seen to be met for the given representations and maps d and e. □

▷ **Theorem 13.6.** *The inner fluctuations $A^{(0,1)}$ and $J_F(A^{(0,1)})^*J_F^*$ of D_F,[2] with d and e as above, are of the form*

$$A^{(0,1)} = \sum_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \pi_2(m_i)(1-\pi_2(n_i))\circ e^* \\ 0 & e\circ(\pi_2(n_i)-1) & 0 \end{pmatrix} \tag{13.9}$$

and

$$J_F A^{(0,1)} J_F^* = \sum_i \begin{pmatrix} 0 & d\circ\pi_2^o(m_i)(1-\pi_2^o(n_i)) & 0 \\ (\pi_2^o(n_i)-1)\circ d^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13.10}$$

with $n_i, m_i \in M_3(\mathbb{C})$.

Proof. We have $A^{(0,1)} := \sum_i \pi(m_i)[D_F, \pi(n_i)]$ [cf. (6.13)] which, applied to an element $(q_1, g, q_2) \in \mathcal{H}_F$, gives

$$\begin{aligned} &\sum_i \pi(m_i)[D_F, \pi(n_i)](q_1, g, q_2) \\ &= \sum_i \big(m_i[d(n_i g) - n_i d(g)], m_i[d^*(n_i q_1) - n_i d^*(q_1)] \\ &\qquad + m_i e^*(q_2) - m_i n_i e^*(q_2), e(n_i g) - e(g)\big) \\ &= \sum_i \big(0, m_i[1-n_i]e^*(q_2), e((n_i - 1)g)\big) \end{aligned} \tag{13.11}$$

corresponding to (13.9). Note that the specific form of d causes its inner fluctuations to vanish.

For the other part, $J_F(A^{(0,1)})^*J_F^*$, we have

$$\begin{aligned} \sum_i J_F(\pi(m_i)[D_F, \pi(n_j)])^* J_F^* &= -\sum_i J_F[D_F, \pi(n_i^*)]\pi(m_i^*)J_F \\ &= -\sum_i [D_F, \pi^o(n_i)]\pi^o(m_i), \end{aligned}$$

2 We use the notation $A^{(0,1)}$ to distinguish the inner fluctuations of D_F from those of the Dirac operator on the spinor part.

where we have used that $D_F J_F = J_F D_F$. We therefore get

$$\begin{aligned}
&J_F(A^{(0,1)})^* J_F(q_1, g, q_2)\\
&= \sum_i \big(- d(gm_i n_i) - d(gm_i), -d^*(q) + d^*(q_1)n_i \\
&\qquad - e^*(n_i^t m_i^t) + e^*(m_i^t q_2)n_i, -e(gm_i n_i) + n_i^t e(gm_i)\big)\\
&= \sum_i \big(d(gm_i(1-n_i)), d^*(q_1)(n_i - 1), 0\big). \qquad (13.12)
\end{aligned}$$

This time, it's the inner fluctuations of e that vanish. □

These expressions are not that insightful, though. The next lemma sheds some light on the nature of the fluctuations. In analogy with $A^{(0,1)}$ we introduce the notation $D^{(0,1)} := D_F + A^{(0,1)} + J_F(A^{(0,1)})^* J_F^*$ to make a distinction between the finite Dirac operator and the (fluctuated) Dirac operator $D^{(1,0)}$ on the spinor part.

▷ **Lemma 13.7.** *Requiring self-adjointness for $A^{(0,1)}$ gives*

$$D^{(0,1)} = g_3 \begin{pmatrix} 0 & A_{\widetilde{q}} & 0 \\ A_{\widetilde{q}}^* & 0 & B_{\overline{\widetilde{q}}}^* \\ 0 & B_{\overline{\widetilde{q}}} & 0 \end{pmatrix} \qquad (13.13)$$

with

$$\begin{aligned}
A_{\widetilde{q}}^*(q) &= q\overline{\widetilde{q}}^t, & A_{\widetilde{q}}(g) &= g\widetilde{q},\\
B_{\overline{\widetilde{q}}}^*(q) &= \widetilde{q}q^t, & B_{\overline{\widetilde{q}}}(g) &= g^t\overline{\widetilde{q}}.
\end{aligned}$$

for some $\widetilde{q} \in \mathbb{C}^3$. Here, g_3 is the QCD-coupling constant.

Proof. Requiring (13.9) and its counterpart (13.10) to be self-adjoint yields the demand

$$\sum_i n_i^* - 1 = \sum_i m_i(1 - n_i), \qquad (13.14)$$

for the elements of the algebra.

If we add the expressions (13.11) and (13.12) for $A^{(0,1)}$ and $J_F A^{(0,1)} J_F^*$ respectively to that of D_F, we get

$$\begin{aligned}
&D^{(0,1)}(q_1, g, q_2)\\
&= \sum_i \big(d(g(1 + m_i - m_i n_i)), d^*(q_1)n_i + (1 + m_i - m_i n_i)e^*(q_2), e(n_i g)\big)\\
&= \sum_i \big(g(1 + m_i - m_i n_i)v, q_1(\overline{n_i^* v})^t + [(1 + m_i - m_i n_i)v]q_2^t, g^t\overline{n_i^* v}\big)\\
&= g_3\big(g\widetilde{q}, q_1\overline{\widetilde{q}}^t + \widetilde{q}q_2^t, g^t\overline{\widetilde{q}}\big) \qquad (13.15)
\end{aligned}$$

after having defined

$$\widetilde{q} := g_3^{-1}\sum_i [1 + m_i(1 - n_i)]v = g_3^{-1}\sum_i n_i^* v. \qquad (13.16)$$

It is clear that (13.15) corresponds to (13.13). □

We identify $\widetilde{q}$ and $\overline{\widetilde{q}}$ as the *squark* and *anti squark* respectively.

N.B. We will occasionally write $D^{(0,1)}_{\widetilde{q}}$ instead of $D^{(0,1)}$ to indicate that/how the Dirac operator depends on $\widetilde{q}$.

Note that if we take the tensor product of the spectral triple as in Definition 13.1 with the canonical one, a squark becomes in fact an element of $C^\infty(M, \mathbb{C}^3)$, the calculations above only holding on each separate $x \in M$.

There is one more thing to check. The demand of self-adjointness for the inner fluctuations of the finite Dirac operator put constraints on the elements of the algebra, as was the case for those of the canonical Dirac operator. Are these constraints compatible? The answer is positive, as was pointed out by A. Connes, A.H. Chamseddine and M. Marcolli [5, §3.5.3] and their argument can be seen to be applicable equally well to our case.

13.3 THE LAGRANGIAN

Having found an expression for $D^{(0,1)}$, we are in a position to determine all additional terms in the Lagrangian that result from adding a finite Dirac operator and enlarging the Hilbert space with $M_3(\mathbb{C})$. The full spectral triple is (as in the models discusses previously) taken to be the tensor product of the canonical and finite spectral triples. In contrast to the previous chapters, we may take the spin manifold M to be curved again, with scalar curvature R.

We recall that the full Dirac operator —including its inner fluctuations— of the tensor product of the canonical and finite spectral triples, is given by:

$$D_A = D^{(1,0)} + \gamma^5 \otimes D^{(0,1)}, \tag{13.17}$$

where we use the notations $D^{(1,0)}$ and $D^{(0,1)}$ to designate the canonical and finite Dirac operators respectively.

13.3.1 *The inner product*

Due to adding $\gamma^5 \otimes D^{(0,1)}$ and the gluinos to the theory, we have two additional terms in the fermionic Lagrangian. On the one hand we have for $\psi_g \in L^2(M, S) \otimes M_3(\mathbb{C})$

$$\langle \psi_g, D^{(1,0)} \psi_g \rangle = g_3 \langle \psi_g, i\gamma^\mu [(\partial_\mu + \omega_\mu) \otimes \mathrm{id} + \mathrm{id} \otimes (\partial_\mu + (\mathbb{A}_\mu)] \psi_g \rangle, \tag{13.18}$$

with the gauge field $\mathbb{A}_\mu$ in the adjoint representation, as in the Einstein-Yang-Mills model. On the other hand we have an interaction due to $\gamma^5 \otimes D^{(0,1)}$. This gives (for the finite part only):

$$\begin{aligned} &\langle (q_1, g, q_2), D^{(0,1)}(q_1, g, q_2) \rangle \\ &= g_3 \big[\langle q_1, g\widetilde{q} \rangle_{\mathbf{3}^o} + \langle g, q_1 \overline{\widetilde{q}}^t \rangle_{M_3(\mathbb{C})} + \langle g, \widetilde{q} q_2^t \rangle_{M_3(\mathbb{C})} + \langle q_2, g^t \overline{\widetilde{q}} \rangle_{\mathbf{3}} \big] \\ &= g_3 \big[\langle q_1, g\widetilde{q} \rangle_{\mathbf{3}^o} + \langle g\widetilde{q}, q_1 \rangle_{\mathbf{3}^o} + \langle g^t \overline{\widetilde{q}}, q_2 \rangle_{\mathbf{3}} + \langle q_2, g^t \overline{\widetilde{q}} \rangle_{\mathbf{3}} \big], \end{aligned} \tag{13.19}$$

after having employed a calculation similar to (13.8). We will spend more time on these terms again in the last part of this chapter.

13.3.2 *Spectral action*

Then there are contributions to the Lagrangian coming from the spectral action (7.1). We will first prove some lemmas that will be of later use.

▷ **Lemma 13.8.** *For the square of D_A as in (13.17) we have*

$$\operatorname{Tr} D_A^2 = \operatorname{Tr}(D^{(1,0)})^2 + \operatorname{Tr}(D^{(0,1)})^2 \tag{13.20}$$

with

$$\operatorname{Tr}(D^{(0,1)})^2 = 12 g_3^2 |\widetilde{q}|^2$$

Proof. In writing out the square of D_A we find an additional term $\gamma^5[D^{(1,0)}, 1 \otimes D^{(0,1)}]$ that vanishes upon taking the trace. For the square of the finite part we find

$$\operatorname{Tr}(D^{(0,1)})^2 = 2g_3^2 \operatorname{Tr}[A_{\widetilde{q}}^* A_{\widetilde{q}}] + 2g_3^2 \operatorname{Tr}[B_{\overline{\widetilde{q}}}^* B_{\overline{\widetilde{q}}}].$$

If we apply this first operator on the right hand side on a quark q we get:

$$[A_{\widetilde{q}} A_{\widetilde{q}}^* q]_i = [(q^t \overline{\widetilde{q}})\widetilde{q}]_i = \sum_j (\overline{\widetilde{q}} \widetilde{q}^t)_{jj} q_i,$$

i.e. $A_{\widetilde{q}} A_{\widetilde{q}}^* = \operatorname{diag} |\widetilde{q}|^2$. With a similar calculation for $B_{\overline{\widetilde{q}}}$, we arrive at the result. □

▷ **Lemma 13.9.** *For the fourth power of the finite Dirac operator $D^{(0,1)}$ we have*

$$\operatorname{Tr}(D^{(0,1)})^4 = 16 g_3^4 |\widetilde{q}|^4. \tag{13.21}$$

Proof. The calculation bears strong resemblance with the previous lemma, the main difference lies in additional cross terms. If we write out the fourth power of (13.13) we get

$$\operatorname{Tr}(D^{(0,1)})^4 = 2g_3^4 \operatorname{Tr}(A_{\widetilde{q}} A_{\widetilde{q}}^*)^2 + 4g_3^4 \operatorname{Tr}[B_{\overline{\widetilde{q}}} A_{\widetilde{q}}^* A_{\widetilde{q}} B_{\overline{\widetilde{q}}}^*] + 2g_3^4 \operatorname{Tr}(B_{\overline{\widetilde{q}}} B_{\overline{\widetilde{q}}}^*)^2.$$

For the first term on the right hand side of this expression we have

$$[A_{\widetilde{q}} A_{\widetilde{q}}^* A_{\widetilde{q}} A_{\widetilde{q}}^*(q)]_i = \operatorname{diag} |\widetilde{q}|^4 q_i,$$

whereas for the second we find

$$[B_{\overline{\widetilde{q}}} A_{\widetilde{q}}^* A_{\widetilde{q}} B_{\overline{\widetilde{q}}}^*]_{ij} = |\widetilde{q}|^2 \overline{\widetilde{q}}_i \widetilde{q}_j.$$

Adding these two expressions to the one involving $B_{\overline{\widetilde{q}}}$, yields

$$\operatorname{Tr}(D^{(0,1)})^4 = 2g_3^4 \operatorname{Tr} \operatorname{diag} |\widetilde{q}|^4 + 4g_3^4 |\widetilde{q}|^4 + 2g_3^4 \operatorname{Tr} \operatorname{diag} |\widetilde{q}|^4 = 16 g_3^4 |\widetilde{q}|^4.$$

□

To proceed, we will —as in Chapter 8— make an expansion in powers of D_A^2. We first determine $E' \in \operatorname{End}(\mathcal{H})$, defined by

$$D_A^2 = \nabla^* \nabla - E', \tag{13.22}$$

where $\nabla = \nabla^S + g_3 \mathbb{A}$. Adding a finite Dirac operator is easily seen to have the following effect on E and $\Omega_{\mu\nu}$

$$\begin{aligned} -E \to -E' &= -E - i\gamma^5 \gamma^\mu [D_\mu, 1 \otimes D^{(0,1)}] + 1 \otimes {D^{(0,1)}}^2 \\ \Omega_{\mu\nu} &\to \Omega_{\mu\nu} \end{aligned} \tag{13.23}$$

compared to $E = \frac{1}{4} R \otimes \operatorname{id} - \sum_{\mu<\nu} \gamma^\mu \gamma^\nu \otimes \mathbb{F}_{\mu\nu}$ prior to adding squarks and gluinos. The minus sign giving rise to the commutator comes from interchanging γ^μ and γ^5. The term ω_μ then drops from the expression, leaving the commutator of $D_\mu := \partial_\mu + g_3 \mathbb{A}_\mu$ with $D^{(0,1)}$.

▷ **Theorem 13.10.** *Adding the finite Dirac operator $\gamma^5 \otimes D^{(0,1)}$, has the following effect (at the orders of Λ^2 and Λ^0) on the bosonic part of the action:*

$$S_B \to S_B + \int_M \Big[-\frac{6f_2}{\pi^2} g_3^2 \Lambda^2 |\widetilde{q}(x)|^2 + g_3^2 \frac{f(0)}{4\pi^2} (8 g_3^2 |\widetilde{q}(x)|^4 + 6 |D_\mu \widetilde{q}(x)|^2 - 3 R g_3^2 |\widetilde{q}(x)|^2) \Big] \sqrt{g}\, \mathrm{d}^4 x. \tag{13.24}$$

Proof. From (7.8b) wee see that the contributions to the Lagrangian of $\mathcal{O}(\Lambda^2)$ come from $\mathrm{Tr}(E')$. Since the trace of second term of (13.23) vanishes, we are left with

$$\mathrm{Tr}(E') = \mathrm{Tr}(E) - 4\,\mathrm{Tr}(D^{(0,1)})^2 = \mathrm{Tr}(E) - 48 g_3^2 |\widetilde{q}|^2,$$

by virtue of Lemma 13.8.

Since $\Omega_{\mu\nu}$ is unaltered, all extra terms we have on $\mathcal{O}(\Lambda^0)$ result from $\mathrm{Tr}(RE')$ and $\mathrm{Tr}(E'^2)$ [see (7.8c)]. For the first we have

$$\mathrm{Tr}(RE') = \mathrm{Tr}(RE) - 4R\,\mathrm{Tr}(D^{(0,1)})^2 = \mathrm{Tr}(RE) - 48 g_3^2 R |\widetilde{q}|^2, \tag{13.25}$$

whereas the second gives

$$\begin{aligned}\mathrm{Tr}(E'^2) &= \mathrm{Tr}(E^2) + \mathrm{Tr}(i\gamma^5\gamma^\mu[D_\mu, 1 \otimes D^{(0,1)}])^2 + \mathrm{Tr}[1 \otimes (D^{(0,1)})^2]^2 \\ &\quad - \frac{1}{2}\,\mathrm{Tr}[R \otimes (D^{(0,1)})^2] \\ &= \mathrm{Tr}(E^2) + 4\,\mathrm{Tr}([D_\mu, D^{(0,1)}][D^\mu, D^{(0,1)}]) + 4\,\mathrm{Tr}[(D^{(0,1)})^4] \\ &\quad - 2R\,\mathrm{Tr}(D^{(0,1)})^2, \end{aligned} \tag{13.26}$$

where in the first step we have used that terms of the Clifford algebra proportional to $\gamma^\mu\gamma^\nu$ ($\mu < \nu$), $\gamma^5\gamma^\mu$ and 1 are orthogonal, and we consequently only retain the squares of the terms in (13.23) plus one cross-term. Now for the last term of (13.26) we can use Lemma 13.9, whereas for the second we use that D_μ acts as follows on the different particles:

$$\begin{aligned} D_\mu q &= (\partial_\mu + g_3 A_\mu) q && \text{on quarks} \\ D_\mu g &= (\partial_\mu + g_3 \mathbb{A}_\mu) g && \text{on gluinos} \\ D_\mu \overline{q} &= (\partial_\mu + g_3 \overline{A}_\mu) \overline{q} && \text{on antiquarks.} \end{aligned}$$

Thus we get[3]

$$\begin{aligned} &[D_\mu, D^{(0,1)}](q_1, g, q_2) \\ &= \big((\partial_\mu + A_\mu)(g\widetilde{q}) - [(\partial_\mu + \mathrm{ad}\, A_\mu)g]\widetilde{q}, (\partial_\mu + \mathrm{ad}\, A_\mu)(q_1 \overline{\widetilde{q}}^t) \\ &\qquad - [(\partial_\mu + A_\mu)q_1]\overline{\widetilde{q}}^t + (\partial_\mu + \mathrm{ad}\, A_\mu)(\widetilde{q} q_2^t) - \widetilde{q}[(\partial_\mu + \overline{A}_\mu)q_2]^t, \\ &\qquad\quad (\partial_\mu + \overline{A}_\mu)(g^t \overline{\widetilde{q}}) - [(\partial_\mu + \mathrm{ad}\, A_\mu)g]^t \overline{\widetilde{q}}\big) \\ &= \big(g(\partial_\mu + A_\mu)\widetilde{q}, q_1[\partial_\mu \overline{\widetilde{q}}]^t - (q_1 \overline{\widetilde{q}}^t)A_\mu + [(\partial_\mu + A_\mu)\widetilde{q}]q_2^t, g^t(\partial_\mu - A_\mu^t)\overline{\widetilde{q}}\big) \\ &= \big(g(\partial_\mu + A_\mu)\widetilde{q}, q_1[\overline{(\partial_\mu + A_\mu)\widetilde{q}}]^t + [(\partial_\mu + A_\mu)\widetilde{q}]q_2^t, g^t\overline{(\partial_\mu + A_\mu)\widetilde{q}}\big), \end{aligned}$$

where we have frequently used that $A_\mu^* = -A_\mu$.

3 For notation's sake we omit the coupling parameter g_3 in this calculation, and put it back in the result.

This means that we have

$$[D_\mu, D^{(0,1)}_{\widetilde{q}}](q_1, g, q_2) = D^{(0,1)}_{(\partial_\mu + g_3 A_\mu)\widetilde{q}}(q_1, g, q_2)$$

with which the second term of (13.26) becomes

$$\begin{aligned}\operatorname{Tr}([D_\mu, D^{(0,1)}][D^\mu, D^{(0,1)}]) &= \operatorname{Tr} D^{(0,1)}_{(\partial_\mu + g_3 A_\mu)\widetilde{q}} D^{(0,1)}_{(\partial_\mu + g_3 A_\mu)\widetilde{q}} \\ &= 12 g_3^2 |(\partial_\mu + g_3 A_\mu)\widetilde{q}|^2. \end{aligned} \tag{13.27}$$

Taking the expansion of the spectral action (7.11), with the coefficients taken from (7.8b) and (7.8c) we get the following extra extra contributions:

$$\begin{aligned}&\mathcal{O}(\Lambda^2) : -2f_2 \frac{1}{(4\pi)^2} 4 \operatorname{Tr}(D^{(0,1)})^2 = -\frac{6}{\pi^2} f_2 g_3^2 |\widetilde{q}|^2, \\ &\mathcal{O}(\Lambda^0) : f(0) \frac{1}{(4\pi)^2} \frac{1}{360} \big[-60(-48 g_3^2 R |\widetilde{q}|^2) + 180(4 \cdot 12 |(\partial_\mu + g_3 A_\mu)\widetilde{q}|^2 \\ &\qquad\qquad + 64 |\widetilde{q}|^4 - 24 R |\widetilde{q}|^2)\big]\end{aligned}$$

with which we arrive at (13.24). □

13.4 THE GAUGE GROUP

Finally we cover a question most important to supersymmetry:

▷ **Theorem 13.11.** *The squarks and the quarks are in the same representation of the gauge group, as are the gluinos and gluons.*

Proof. The thing to check is how the quarks, squarks, gluinos and gluons transform under the gauge group

$$\mathcal{U}(M_3(\mathbb{C})) = \{u \in M_3(\mathbb{C}) : u^* u = u u^* = 1\}. \tag{13.28}$$

First we have a look at how the (s)fermions transform (see Example 5.10):

$$u J u J^* (q_1, g, q_2) = \pi(u) \pi^o(u^*)(q_1, g, q_2) = (u q_1, u g u^*, \overline{u} q_2),$$

i.e.

$q \to uq$	for quarks,
$g \to ugu^*$	for gluinos, and
$q' \to \overline{u}q' = \overline{u}q'$	for antiquarks.

Next, we look at how $D^{(0,1)}$ transforms. We therefore apply[4]

$$u J u J^* D^{(0,1)} u^* (J u J^*)^* = \pi(u) \pi^o(u^*) D^{(0,1)} \pi(u)^* \pi^o(u)$$

to an element (q_1, g, q_2). This results in

$$\begin{aligned}& u J_F u J_F^* D^{(0,1)} u^* J_F u^* J_F^* (q_1, g, q_2) \\ &= \pi(u) \pi^o(u^*) D^{(0,1)} (u^* q_1, u^* g u, u^t q_2) \\ &= \pi(u) \pi^o(u^*) \big((u^* g u) \widetilde{q}, (u^* q_1) \overline{\widetilde{q}}^t + \widetilde{q} (u^t q_2)^t, (u^* g u)^t \overline{\widetilde{q}} \big) \\ &= \big(u (u^* g u) \widetilde{q}, u (u^* q_1) \overline{\widetilde{q}}^t u^* + u \widetilde{q} (u^t q_2)^t u^*, \overline{u} (u^* g u)^t \overline{\widetilde{q}} \big) \\ &= \big(g u \widetilde{q}, q_1 (\overline{u \widetilde{q}})^t + u \widetilde{q} q_2^t, g^t \overline{u \widetilde{q}} \big)\end{aligned}$$

4 Again we will omit the coupling constant g_3 for a moment.

which corresponds to applying $D^{(0,1)}$ but with

$$\widetilde{q} \to u\widetilde{q},$$

indeed the same as with the quarks.

Last, we check how the gluons transform. We review the finite part of $D^{(1,0)}$ applied to a gluino[5]:

$$\begin{aligned}
\pi(u)\pi^o(u^*)(\partial_\mu + \mathbb{A}_\mu)\pi(u^*)\pi^o(u)g &= \pi(u)\pi^o(u^*)(\partial_\mu(u^*gu) + [A_\mu, u^*gu]) \\
&= u\partial_\mu(u^*)g + \partial_\mu g + g(\partial_\mu u)u^* \\
&\qquad + u[A_\mu, u^*gu]u^* \\
&= \partial_\mu g + \mathrm{ad}(uA_\mu u^* + u[\partial_\mu, u^*])g \\
&= (\partial_\mu + \mathbb{A}^u_\mu)g,
\end{aligned}$$

with A^u as in (6.17). This transformation is indeed the same as for gluinos. □

13.5 RESULTS & DISCUSSION

The upshot of all of this is that we have added the superpartners of the QCD-particles (squarks and gluinos) to the theory, in conformity to the 'paradigm' of NCG: fermions are elements of the Hilbert space, whereas scalars come from a finite Dirac operator. The way to achieve this was by

1. adding the algebra of the spectral triple to the Hilbert space;
2. defining maps connecting the 'old' part of the Hilbert space to the part just added.

The freedom to choosing these maps was seen to be very little. On top of that, this construction led to the fact that these superpartners are in the right representation of the gauge group.

At this point there is only one big question left: 'Is this model right?'. To what extent does it describe the supersymmetrization of the QCD — super-QCD? Well, we found a Lagrangian with

· a kinetic term for the squarks [the fourth term of (13.24)], with the latter *minimally coupled* to the gluons. This gives a squark-squark-gluon and squark-squark-gluon-gluon coupling;

· a squark-potential $\sim |\widetilde{q}|^4$ [the third term of (13.24)], resulting in a squark self-interaction. As in the case of the Higgs-Lagrangian this term is of opposite sign compared to the kinetic term;

· a squark mass term $\sim |\widetilde{q}|^2$ [the second term of (13.24)] on the order of Λ^2 is appearing;

· a gluino-gluino-gluon interaction that follows from the inner product;

· a gluino-quark-squark coupling that is provided by the inner product (13.18);

· a term that couples the squarks to gravity.

5 Applying it to a quark or antiquark would give the very same result.

Since these are precisely the same terms that are predicted by the MSSM, it appears that we indeed obtained a description of super-QCD, using noncommutative geometry.[6] At the end of this chapter we will compare these results extensively with results as found in the literature.

Note that though we have added scalars to a theory by means of introducing a finite Dirac operator, the way how is not quite the same as in the treatment of the Higgs-model by Connes, Marcolli and Chamseddine ([5], [9, §15.6 & 16.1]). While in the latter case the finite Dirac operator provides the coupling constants/mixing matrices, here it (partly) provides the fields $\widetilde{q}$, $\overline{\widetilde{q}}$ themselves as well. This is due to the fact that the finite Dirac operator in [9] maps right to left handed particles and vice versa[7], whereas in our case it maps from the space of the squarks to that of the gluinos, both being of different dimension. Another difference with the Higgs-mechanism lies in the fact that in [9] there is an extra contribution $\sim \Lambda^0|\widetilde{q}|^2$ compared to our model.

Note that for supersymmetry at least the number of degrees of freedom need to be the same. For that, the finite part of the gluinos has to be reduced from $M_3(\mathbb{C})$ to $\mathfrak{su}(3)$ —a problem that was dealt with in chapter 10.

In order to compare the results with that of the MSSM[8], we need to spell things out. We first switch to flat Euclidean space by taking $\omega_\mu = 0$ and $R = 0$. For each of the interactions that appear we will at the same time make the switch from the current notation to the one more common in physics and translate the (relevant pieces of the) Lagrangian as found in the literature to this context set up we have used. We assume the degrees of freedom of the gluinos already to be reduced so that we may write $g = g^a T_a$ (the T_a being the eight $\mathfrak{su}(3)$-generators) for their finite part.

- *Squark-quark-gluino*

  We take the finite part of this interaction from (13.19) and write $\chi \otimes g \in L^2(M,S) \otimes \mathfrak{su}(3)$ for a gluino and $\psi \otimes q_1 \in L^2(M,S) \otimes \mathbf{3}^o$ for a quark. We leave out the antisquarks for a moment. We then have:

$$\begin{aligned}
&((\psi \otimes q_1, \chi \otimes g), \gamma^5 \otimes D^{(0,1)}(\psi \otimes q_1, \chi \otimes g))(x) \\
&= g_3((\psi \otimes q_1, \chi \otimes g), (\gamma^5\psi \otimes g\widetilde{q}, \gamma^5\chi \otimes q_1\overline{\widetilde{q}}^t))(x) \\
&= g_3(\psi \otimes q^i e_i, \gamma^5\psi \otimes g^a (T_a)^j{}_k \widetilde{q}^k e_j)(x) \\
&\qquad + g_3\langle \chi \otimes g^a (T_a)^j{}_k \widetilde{q}^k e_j, \gamma^5\chi \otimes q_1^i e_i\rangle(x) \\
&= g_3(\psi^i_{q_1}, \gamma^5\psi^a_g)(x)(T_a)_{ik}(\widetilde{q}(x))^k + g_3(\chi^a_g, \gamma^5\chi^i_{q_1})(x)(\overline{T_a})_{ik}(\overline{\widetilde{q}}(x))^k \\
&= g_3(T_a)_{ik}\left[\langle\psi^i_{q_1}, \gamma^5\psi^a_g\rangle(x)(\widetilde{q}(x))^k - (\chi^a_g, \gamma^5\chi^k_{q_1})(x)(\overline{\widetilde{q}}(x))^i\right]
\end{aligned}$$

  where along the way we switched notations: $\psi \otimes q_1 = \psi^i_{q_1} \otimes e_i$ and $\psi \otimes g = \psi^a_g \otimes T_a$ and similar expressions for χ.

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6 This is besides the simplifications we made to the model, as was mentioned in the introduction of this chapter.

7 The distinction between left and right handed particles is not present in this model, since —lacking a weak sector— this would introduce a property not present in the (MS)SM.

8 Kraml [18] and Chung et al. [6] provide lengthy expositions on the MSSM. In the latter, the various MSSM-interactions are listed in the appendix.

We compare our result to that of Chung ([6], (C.82)):

$$- \sqrt{2} g_3 T^a_{jk} \sum_{u,d} \bigg(G \overline{\widetilde{g}}^a P_L q^k_I \widetilde{q}^{j\,*}_\alpha (\Gamma^{SCKM}_{qL})^*_{I\alpha} + G^{-1} \overline{q}^j_I P_R \widetilde{g}^a \widetilde{q}^k_\alpha (\Gamma^{SCKM}_{qL})_{I\alpha} \\ - G^{-1} \overline{\widetilde{g}}^a P_R q^k_I \widetilde{q}^{j\,*}_\alpha (\Gamma^{SCKM}_{qL})^*_{I\alpha} - G \overline{q}^j_I P_L \widetilde{g}^a \widetilde{q}^k_\alpha (\Gamma^{SCKM}_{qL})_{I\alpha} \bigg).$$

In order to compare we have to delete the matrices Γ^{SCKM} which rotate between generations:

$$- \sqrt{2} g_3 T^a_{jk} \bigg(G \overline{\widetilde{g}}^a P_L q^k \widetilde{q}^{j\,*} + G^{-1} \overline{q}^j P_R \widetilde{g}^a \widetilde{q}^k \\ - G^{-1} \overline{\widetilde{g}}^a P_R q^k \widetilde{q}^{j\,*} - G \overline{q}^j P_L \widetilde{g}^a \widetilde{q}^k \bigg).$$

Using the notation introduced in [6], appendix C.2, this becomes:

$$\sqrt{2} g_3 T^a_{jk} \big(\overline{q}^j \gamma^5 \widetilde{g}^a \widetilde{q}^k - \overline{\gamma^5 \widetilde{g}}^a q^k \widetilde{q}^{j\,*} \big).$$

- *Gluon-gluino-gluino*

  We get a gluon-gluino-gluino interaction from (13.18):

$$\mathcal{L}_{g\widetilde{g}\widetilde{g}} = (\psi_g, i g_3 \gamma^\mu \mathbb{A}_\mu \psi_g)(x) = i g_3 (\psi^c_g, \gamma^\mu A^a_\mu \psi^b_g)(x) \langle T_c, [T_a, T_b] \rangle \\ = i g_3 f_{abc} \overline{\psi_g}^b (x)\, \gamma^\mu\, \psi^c_g(x)\, A^a_\mu(x)$$

  In Kraml ([18]), §1.6.7, we find for the gluon-gluino-gluino term:

$$\mathcal{L}_{g\widetilde{g}\widetilde{g}} = \frac{i g_s}{2}\, f_{abc}\, G^a_\mu\, \overline{\widetilde{g}}^b\, \gamma^\mu\, \widetilde{g}^c.$$

  Both terms are seen to be equal upon identifying $\widetilde{g}^c = \psi^c_g$.

- *Squark-squark-gluon*

  From (13.27) we can extract a squark-squark-gluon term, that looks like[9]:

$$- g_3^2 (g_3 A_\mu \widetilde{q})_i (\partial^\mu \overline{\widetilde{q}})^i - g_3^2 (\partial_\mu \widetilde{q})^i (g_3 \overline{A_\mu \widetilde{q}})_i \\ = - g_3^3 A^a_\mu (T_a)_{ij} \widetilde{q}^j \partial^\mu (\overline{\widetilde{q}})^i - g_3^3 \partial_\mu (\widetilde{q})^i A^a_\mu (\overline{T_a})_{ij} \overline{\widetilde{q}}^j \\ = g_3^3 A^a_\mu (T_a)_{ij} [\overline{\widetilde{q}}^i (\partial_\mu \widetilde{q})^j - g_3^3 (\partial^\mu \overline{\widetilde{q}})^i \widetilde{q}^j]$$

  Kraml ([18], §1.6.2) gives the following expression for this interaction:

$$\mathcal{L}_{\widetilde{q}\widetilde{q}g} = i g_s\, T^a_{rs}\, \delta_{ij} G^a_\mu\, \widetilde{q}^*_{jr} \overleftrightarrow{\partial^\mu}\, \widetilde{q}_{is} = i g_s\, T^a_{rs} G^a_\mu [\widetilde{q}^*_r \partial^\mu (\widetilde{q}_s) - \partial^\mu (\widetilde{q}^*_r) \widetilde{q}_s],$$

  after writing out the differential. These two forms differ two was: we have an extra minus sign and we have the a g_3^2 extra. We will come back to this.[10]

9 We will omit the coordinates '(x)' to reduce notational clutter.

10 The i appearing here is probably harmless since it might be due to the convention in physics to take T^a_{ij} selfadjoint, whereas in our case the T^a_{ij} are skew-Hermitian. The connection between these two conventions is exactely an i.

- *Squark-squark-gluon-gluon*

  Equation 13.27 does not only provide us a squark-squark-gluon interaction, but a squark-squark-gluon-gluon term as well:

$$g_3^2(g_3 A^\mu \widetilde{q})^i (g_3 \overline{A_\mu \widetilde{q}})_i = g_3^4 A^{\mu\, a} A_\mu^b (T_a \widetilde{q})_i (\overline{T_b \widetilde{q}})_i$$
$$= -g_3^4 A^{\mu\, a} A_\mu^b (T_b T_a)_{ij} \overline{\widetilde{q}}_i \widetilde{q}_j .$$

  Since the gluons are in the defining / fundamental representation here, we can use the identity

$$T_b T_a = \frac{1}{6}\delta_{ab}\mathrm{id}_3 + \frac{1}{2}(i f_{bac} + d_{bac})T^c,$$

  resulting in

$$\mathcal{L}_{\widetilde{q}\widetilde{q}gg} = -\frac{1}{6} A^\mu{}_a A_\mu{}^a \overline{\widetilde{q}}_i \widetilde{q}^i - \frac{1}{2} d_{abc}\, A^{\mu\, a} A_\mu^b (T^c)_{ij} \overline{\widetilde{q}}_i \widetilde{q}_j$$

  where the term with f_{abc} vanishes since $A^{\mu\, a} A_\mu^b$ is symmetric upon interchanging a and b.

  Kraml provides us the following expression (§1.6.8) for this interaction:

$$\mathcal{L}_{\widetilde{q}\widetilde{q}gg} \quad = \quad \frac{1}{2}\, g_s^2\, (\frac{1}{3}\delta_{ij}\, \delta_{ab} + d_{abc}\, (T^c)_{ij})\, G_\mu^a\, G^{b\mu}\, \widetilde{q}_j^*\, \widetilde{q}_i,$$

  where two indices are interchanged compared to our result. However, since both [6] (C.93) and [22] (§2.2.3) do provide the same expression that we have, we are inclined to adhere to the latter two.

  We note that, as in the case of the squark-squark-gluon interaction, our answer is off by a factor of g_3^2 and we have an additional minus sign.

- *Four-squark*

  There is a squark self-interaction

$$g_3^4 |\widetilde{q}(x)|^4 = g_3^4 \widetilde{q}(x)_i \overline{\widetilde{q}}(x)^i \widetilde{q}(x)_j \overline{\widetilde{q}}(x)^j,$$

  originating from the third term of (13.24).

  Kraml, §1.6.9 gives for this interaction:

$$\mathcal{L}_{\widetilde{q}\widetilde{q}\widetilde{q}\widetilde{q}} = -\frac{1}{2}\, g_s^2\, T^a_{rs} T^a_{tu} \big(\widetilde{q}^{a*}_{L,r}\, \widetilde{q}^a_{L,s} - \widetilde{q}^{a*}_{R,r}\, \widetilde{q}^a_{R,s}\big) \big(\widetilde{q}^{b*}_{L,t}\, \widetilde{q}^b_{L,u} - \widetilde{q}^{b*}_{R,t}\, \widetilde{q}^b_{R,u}\big).$$

  Here, the generators T^a represent a rotation in flavor space, not present in our work. Setting them all to the identity, we get:

$$\mathcal{L}_{\widetilde{q}\widetilde{q}\widetilde{q}\widetilde{q}} = -\frac{1}{2}\, g_s^2\, \big(\widetilde{q}_i^*\, \widetilde{q}_i\big) \big(\widetilde{q}_i^*\, \widetilde{q}_i\big).$$

  Again, we have a factor of g_3^2 extra compared to the literature, and our result is of opposite sign compared to Kraml.

To summarize, all results are in perfect agreement with the literature, in the sense that all interactions are present and their form is precisely the same. In three terms that we compared however, we were off by two powers of the coupling constants and a sign. However, it are precisely these 'erroneous' terms of the Lagrangian that are accompanied by a factor $f(0)$, in which we can absorb this excess of coupling constants. The minus sign is unresolved still, since f has to be a positive function. There is one other unresolved issue: the constants appearing in our results do not in all cases match those of the literature. One possible solution to this problem is to absorb a constant in the definition of our squark (13.16). It is unclear whether that can solve it.

14 SUMMARY & OUTLOOK

This thesis started out with the definition of an algebra and concluded a large number of pages later with the Lagrangian of super-QCD. Much has happened in between. First, we considered the Einstein-Yang-Mills system, where we were able to identify the fermions and bosons that appear as each others superpartners (Chapter 11). This was not possible, though, before we adapted certain techniques that allowed us to work with Weyl spinors in an Euclidean set up of KO-dimension 4. This, together with making the finite part of the fermions *unimodular*, was required for reducing the number of fermionic degrees of freedom in order to equate them with the bosonic ones (Chapter 10).

In Section 11.1 we casted the calculations in a form more appropriate for NCG, and showed that —in a heat kernel expansion of the spectral action in terms of the *cut off* parameter Λ— the EYM-system is not only supersymmetric on $\mathcal{O}(\Lambda^0)$ but in fact for all positive powers of Λ.

After that, we turned our attention to another system; a one-quark version of Quantum chromodynamics (QCD) in a curved spacetime. We showed that in the noncommutative approach of field theory this part of the Standard Model could be extended to yield super-QCD. With the latter we mean a 'supersymmetrized' model in which the superpartners of the QCD-particles ('squarks' and 'gluinos') are incorporated as well. These superpartners were seen to be in the same representation of the gauge group as the original particles. On top of that, the resulting super-QCD Lagrangian, after making a translation to the flat Euclidean space, was seen to match the Lagrangian of the canonical approach to the MSSM perfectly.[1]

The equivalence of these results is without a doubt promising. Despite that, we are far from there yet: the MSSM is much more than super-QCD as QCD is only a part of the Standard Model. What would be the most logical way to proceed from here? At first sight this would be to extend the model of Chapter 13, for example to yield three quarks (one for each particle generation), but this is to a certain extent straightforward. Adding a second quark by adjoining the weak sector is not that convenient, for the specific representations of $\mathcal{A}_F$ require you to add much more than that in order to maintain an equal number of fermionic and bosonic degrees of freedom.[2] Besides, we think that there is a more informative extension.

The electroweak theory (containing only the leptons, the photon and the weak gauge bosons)[3] is probably the most logical candidate. As a next step, the distinctive MSSM Higgs sector might even be recovered.

1 With the exception that a few constants differed somewhat.
2 With this we mean that adding a weak sector would require to add $\mathcal{H}$ to the algebra. But then you automatically get the weak gauge bosons which requires you to add their superpartners as well, etc.
3 In that case we would take $\mathcal{A}_F = \{(\lambda, q, \lambda) : \lambda \in \mathbb{C}, q \in \mathbb{H}\}$ and $\mathcal{H}_f = \mathbf{2}_L \otimes \mathbf{1}^o \oplus \mathbf{2}_R \otimes \mathbf{1}^o$.

A last intriguing aspect of the noncommutative description of super-QCD that beggs for further reseach is the featured squark mass term; in our opinion this bears so much resemblance to that of the Higgs particle that it might not be a mere coincidence.

All in all, we can say that the signals for a possible noncommutative derivation of the MSSM are definitely there, but much more has to be done for the full MSSM to be realized!

Part IV

APPENDIX

A

GAMMA MATRICES & VIERBEINS — OR TERADS

In order to simplify some calculations we introduce the *vierbeins* (see e.g. [34], § 3.4b); matrix-valued functions that transform a certain metric into a flat one, and that are thus defined by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}. \tag{A.1}$$

This implies that

$$\gamma_a := e^\mu_a \gamma_\mu \tag{A.2}$$

satisfies $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. We define

$$\gamma_{ab} := \frac{1}{2}[\gamma_a, \gamma_b], \tag{A.3}$$

subject to the identity ([25], page 40)[1]:

$$[\gamma_{ab}, \gamma_{cd}] = \delta_{cb}\gamma_{ad} - \delta_{ca}\gamma_{bd} - \delta_{db}\gamma_{ac} + \delta_{da}\gamma_{bc}. \tag{A.4}$$

Later on, we will need to take traces of combinations of *flat* gamma matrices. For this, we have

$$\mathrm{Tr}(\gamma_a\gamma_b) = \tfrac{1}{2}\,\mathrm{Tr}\{\gamma_a, \gamma_b\} = 4\delta_{ab} \tag{A.5}$$

and

$$\begin{aligned}\mathrm{Tr}(\gamma_{ab}\gamma_{cd}) = \tfrac{1}{4}[&\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d) - \mathrm{Tr}(\gamma_b\gamma_a\gamma_c\gamma_d) - \mathrm{Tr}(\gamma_a\gamma_b\gamma_d\gamma_c)\\ &+ \mathrm{Tr}(\gamma_b\gamma_a\gamma_d\gamma_c)].\end{aligned} \tag{A.6}$$

For one of these terms, we get by repeatedly interchanging elements:

$$\begin{aligned}\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d) &= 2\delta_{ab}\,\mathrm{Tr}(\gamma_c\gamma_d) - 2\delta_{ac}\,\mathrm{Tr}(\gamma_b\gamma_d) + 2\delta_{ad}\,\mathrm{Tr}(\gamma_b\gamma_c)\\ &\quad - \mathrm{Tr}(\gamma_b\gamma_c\gamma_d\gamma_a)\\ &= 8\delta_{ab}\delta_{cd} - 8\delta_{ac}\delta_{bd} + 8\delta_{ad}\delta_{bc} - \mathrm{Tr}(\gamma_b\gamma_c\gamma_d\gamma_a),\end{aligned}$$

where we have used (A.5), so that

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d) = 4[\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}]. \tag{A.7}$$

Adding the contributions from the different terms of (A.6), then gives

$$\mathrm{Tr}(\gamma_{ab}\gamma_{cd}) = 4[\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}]. \tag{A.8}$$

Furthermore we have (without proof):

$$\gamma^\mu\gamma^\nu\gamma^\sigma = g^{\mu\nu}\gamma^\sigma - g^{\sigma\nu}\gamma^\mu + g^{\mu\sigma}\gamma^\nu - i\epsilon^{\mu\nu\sigma\lambda}\gamma^5\gamma_\lambda \tag{A.9}$$

1 Diligent readers though may earn my respect by writing all terms out and repeatedly using the anticommutator of two gamma matrices.

▷ **Definition A.1 [Orthonormal Clifford basis].** *Let $Cl(V)$ be the Clifford algebra over a n-dimensional vector space V. Then $\gamma_K := \gamma_{k_1} \cdots \gamma_{k_r}$ for all strictly ordered sets $K = \{k_1 < \ldots < k_r\} \subseteq \{1, \ldots, n\}$ form a basis for $Cl(V)$. If γ_K is as above, we denote with γ^K the element $\gamma^{k_1} \cdots \gamma^{k_r}$. The basis spanned by the γ_K is said to be orthonormal if $\operatorname{Tr} \gamma_K \gamma_L = n n_K \delta_{KL}\ \forall\ K, L$. Here $n_K := (-1)^{r(r-1)/2}$, where r denotes the cardinality of the set K and with δ_{KL} we mean*

$$\delta_{KL} = \begin{cases} 1 & \text{if } K = L \\ 0 & \text{else} \end{cases} . \tag{A.10}$$

▷ **Example A.2.** *Take $V = \mathbb{R}^4$ en let $Cl(4,0)$ be the Euclidean Clifford algebra [i.e. with signature $(+ + + +)$]. Its basis are the sixteen matrices*

$$\begin{array}{lrl} 1 & & \\ \gamma_\mu & & (4 \text{ elements}) \\ \gamma_\mu \gamma_\nu & \mu < \nu & (6 \text{ elements}), \\ \gamma_\mu \gamma_\nu \gamma_\lambda & \mu < \nu < \lambda & (4 \text{ elements}) \\ \gamma_1 \gamma_2 \gamma_3 \gamma_4 =: \gamma_5. & & \end{array}$$

We can identify

$$\begin{array}{ll} \gamma_1\gamma_2\gamma_3 = \gamma_4\gamma_5 & \gamma_1\gamma_3\gamma_4 = \gamma_2\gamma_5 \\ \gamma_1\gamma_2\gamma_4 = -\gamma_3\gamma_5 & \gamma_2\gamma_3\gamma_4 = -\gamma_1\gamma_5, \end{array} \tag{A.11}$$

establishing a connection with the basis most commonly used by physicists.

▷ **Lemma A.3 [Completeness relation].** *If the basis of the clifford algebra is orthonormal, it satisfies the following completeness relation:*

$$\frac{1}{n} \sum_L n_L (\gamma^L)_d{}^c (\gamma_L)_a{}^b = \delta_a{}^c \delta_d{}^b. \tag{A.12}$$

Proof. Since the γ_K form a basis, we can write any element Γ of the Clifford algebra as

$$\Gamma = \sum_K m^K \gamma_K \quad m_K \in \mathbb{C}, \tag{A.13}$$

where the sum runs over all (strictly ordered) sets. By multiplying both sides with γ^L and taking the trace we find the expression for the coefficient m^L to be:

$$m^L = \frac{1}{n} n_L \operatorname{Tr} \Gamma \gamma^L,$$

Applying this result in particular to $\Gamma = \gamma_K$, and writing matrix indices explicitly, (A.13) yields

$$(\gamma_K)_a{}^b = \frac{1}{n} \sum_L n_L (\gamma_K)_c{}^d (\gamma^L)_d{}^c (\gamma_L)_a{}^b,$$

for which (A.12) is required. □

▷ **Theorem A.4 [(Generalized) Fierz identity].** *If for any two strictly ordered sets K, L there exists a third strictly ordered set M and $c \in \mathbb{N}$ such that $\gamma_K \gamma_L = c\,\gamma_M$, we have the four-spinor identity*

$$\langle \psi_1, \gamma^K \psi_2 \rangle \langle \psi_3, \gamma_K \psi_4 \rangle = -\frac{1}{n} \sum_L C_{KL} \langle \psi_3, \gamma^L \psi_2 \rangle \langle \psi_1, \gamma_L \psi_4 \rangle$$
$$C_{KL} \in \mathbb{N}, \tag{A.14}$$

for any $\psi_1, \ldots, \psi_4$ in the n-dimensional spin representation of the Clifford algebra. Here we denote by $\langle ., . \rangle$ the inner product on the spinor representation.

Proof. We start by multiplying the completeness relation (A.12) with $(\gamma_K)_c{}^e (\gamma^K)_b{}^f$ yielding

$$(\gamma_K)_a{}^e (\gamma^K)_d{}^f = \frac{1}{n} \sum_L (\gamma_L \gamma_K)_d{}^e (\gamma^L \gamma^K)_a{}^f,$$

or

$$(\gamma_K)_a{}^e (\gamma^K)_d{}^f = \frac{1}{n} \sum_M C_{KM} (\gamma_M)_d{}^e (\gamma^M)_a{}^f, \tag{A.15}$$

by the assumtion made. Here we have accomodated the proportionality constants in a matrix C_{KL}. Now we have to contract the above expression with the four spinors $\overline{\psi_1^a}, \psi_{2e}, \overline{\psi_3^d}$ and ψ_{4f}. But, remembering that they are Grassmann variables — i.e. their components anticommute— we get one minus sign on the left hand side of (A.15) from interchanging ψ_1 and ψ_3. Hence we we arrive at the result. □

Now how do we compute the constants C_{KL}? Just multiply (A.15) again by $(\gamma_L)_e{}^d (\gamma^L)_f{}^a$, yielding:

$$\mathrm{Tr}(\gamma^K \gamma^L \gamma_K \gamma_L) = \frac{1}{n} \sum_M C_{KM}\, \mathrm{Tr}(\gamma^M \gamma_L)\, \mathrm{Tr}(\gamma_M \gamma^L). \tag{A.16}$$

On the other hand, we have

$$\gamma^K \gamma^L \gamma_K = f_{KL} \gamma^L \quad f_{KL} \in \mathbb{N} \ \text{ (no sum over } L) \tag{A.17}$$

using the anticommutator repeatedly[2]. Putting (A.17) into (A.16) we get:

$$f_{KL}\, \mathrm{Tr}(\gamma^L \gamma_L) = n \sum_M C_{KM} \delta_L^M \delta_M^L$$

or

$$C_{KL} = n_L f_{KL}, \tag{A.18}$$

since

$$\mathrm{Tr}(\gamma^L \gamma_L) = (-1)^{r(r-1)/2} n,$$

by orthonornality.

2 For example: $\gamma^\mu \gamma^\lambda \gamma_\mu = (2 - \dim V)\gamma^\lambda \quad \forall\, \lambda \in \{1, 2, \cdots, \dim V\}$.

▷ **Corrolary A.1 [Fierz identity].** *We work out one example of particular interest to us. Consider again* $Cl(4,0)$ *(*$n=4$*) with the basis as in Example A.2. As can readily be checked, this basis satisfies the requirement for theorem A.4. The spinors we will contract with, are the four Weyl spinors:* $\chi, \epsilon_- \in \mathcal{S}^-, \psi_1, \psi_2 \in \mathcal{S}^+$*. We start with determining the numbers* $f_{1r}, r = 0, \ldots, 4$ *defined by* $\gamma^\mu \gamma_L \gamma_\mu = f_{1r}\gamma_L$ *(see above) where* r *is the cardinality of* L*. We find the recursive relation*

$$\begin{aligned}
\gamma^\mu 1 \gamma_\mu &= n \cdot 1 \equiv f_{10} 1 \\
\gamma^\mu \gamma_\nu \gamma_\mu &= 2\gamma_\nu - \gamma^\mu \gamma_\mu \gamma_\nu = (2 - f_{10})\gamma^\nu \equiv f_{11}\gamma^\nu \\
&\cdots \\
\gamma^\mu (\gamma^{\nu_1} \cdots \gamma^{\nu_n})\gamma_\mu &= [2(-1)^{n-1} - f_{1(n-1)}]\gamma^{\nu_1} \cdots \gamma^{\nu_n} \\
&\equiv f_{1n}\gamma^{\nu_1} \cdots \gamma^{\nu_n} \qquad (n \leq 4)
\end{aligned}$$

which gives

$$f_{10} = 4, \quad f_{11} = -2, \quad f_{12} = 0, \quad f_{13} = 2, \quad f_{14} = -4$$

and consequently, using (A.18)

$$C_{10} = 4, \quad C_{11} = -2, \quad C_{12} = 0, \quad C_{13} = -2, \quad C_{14} = -4.$$

Now applying (A.14) *yields*

$$\begin{aligned}
\langle \chi, \gamma^\mu \psi_1 \rangle \langle \epsilon_-, \gamma_\mu \psi_2 \rangle = &-\frac{1}{4} C_{11} \langle \epsilon_-, \gamma^\mu \psi_1 \rangle \langle \chi, \gamma_\mu \psi_2 \rangle \\
&- \frac{1}{4} C_{13} \langle \epsilon_-, \gamma^\mu \gamma^\nu \gamma^\lambda \psi_1 \rangle \langle \chi, \gamma_\mu \gamma_\nu \gamma_\lambda \psi_2 \rangle
\end{aligned}$$

since only terms with an odd number of γ*-matices survive due to the different chirality of the spinors. Identifing the terms with three* γ*-matrices with* $\pm\gamma^\mu\gamma_5$ *as in* (A.11)*, we get*

$$\begin{aligned}
\langle \chi, \gamma^\mu \psi_1 \rangle \langle \epsilon_-, \gamma_\mu \psi_2 \rangle &= \frac{1}{2} \langle \epsilon_-, \gamma^\mu \psi_1 \rangle \langle \chi, \gamma_\mu \psi_2 \rangle \\
&\quad + \frac{1}{2} \langle \epsilon_-, \gamma^\mu \psi_1 \rangle \langle \chi, \gamma_\mu \psi_2 \rangle \\
&= \langle \epsilon_-, \gamma^\mu \psi_1 \rangle \langle \chi, \gamma_\mu \psi_2 \rangle. \qquad \text{(A.19)}
\end{aligned}$$

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COLOPHON

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