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Note

Perturbations and operator trace functions

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Abstract

We study the spectral functional $A \mapsto \operatorname{Tr} f(D+A)$ for a suitable function f, a self-adjoint operator D having compact resolvent, and a certain class of bounded self-adjoint operators A. Such functionals were introduce by Chamseddine and Connes in the context of noncommutative geometry. Motivated by the physical applications of these functionals, we derive a Taylor expansion of them in terms of Gâteaux derivatives. This involves divided differences of f evaluated on the spectrum of D, as well as the matrix coefficients of A in an eigenbasis of D. This generalizes earlier results to infinite dimensions and to any number of derivatives.

Keywords: Noncommutative geometry; Perturbation theory

1. Introduction

The spectral action in noncommutative geometry [4] is given as the trace $\operatorname{Tr} f(D)$ of a suitable function $f(D)$ of an unbounded self-adjoint operator D, which is assumed to have compact resolvent. One is interested in this trace function as D is perturbed to $D+A$ where A is a certain self-adjoint bounded operator. For instance, the so-called inner fluctuations of a spectral triple are of this type; they are central in the applications of noncommutative geometry to high-energy physics [1–3] (cf. also [6]). A natural question that arises is what happens to the trace function when D is perturbed to $D+A$. It is the goal of this paper to address this question.

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We aim for a Taylor expansion of the spectral action by Gâteaux deriving it with respect to A. As we will see, the context of finite-dimensional noncommutative manifolds (*i.e.* spectral triples) allows for a derivation of results previously obtained only for finite-dimensional (matrix) algebras [13]. Our main result is the expansion:

$$S_D[A] = \sum_{n=0}^{\infty} \frac{1}{n} \sum_{i_1,\dots,i_n} A_{i_1 i_2} \cdots A_{i_n i_1} f'[\lambda_{i_1}, \dots, \lambda_{i_n}]$$

where $f'[\lambda_{i_1}, \dots, \lambda_{i_n}]$ is the divided difference of order n of f' (cf. Definition 14 below) evaluated on the spectrum of D, and A_{ij} are the matrix coefficients with respect to an eigenbasis of D.

This paper is organized as follows. First, we recall in Section 2 some results on perturbations of operators, in the setting of noncommutative geometry. Then, we give a precise definition of the spectral action functional in Section 3. In that section, we also recall the definition of divided differences and derive our main result on the Taylor expansion of the spectral action. We end with some conclusions and an appendix recalling a theorem by Getzler and Szenes.

2. Perturbations and spectral triples

Recall that a *spectral triple* consists of an algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, together with a self-adjoint operator D with compact resolvent such that the commutator $[D, a]$ is a bounded operator for all $a \in \mathcal{A}$. The key example is associated to a compact Riemannian spin manifold M:

$$\big(C^\infty(M), L^2(M, S), \partial\!\!\!/\big),$$

where $\partial\!\!\!/$ is a Dirac operator on the spinor bundle $S \to M$. Indeed, $\partial\!\!\!/$ is an elliptic differential operator of degree one and smooth functions satisfy

$$\big\|[\partial\!\!\!/, f]\big\| = \|f\|_{\mathbf{Lip}} < \infty$$

in the Lipschitz norm of f.

In general, a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be of *finite summability* if there exists an $n \geqslant 0$ such that $(1 + D^2)^{-n/2}$ is a traceclass operator on $\mathcal{H}$. Let us start with a basic and well-known result.

Lemma 1. *Let p be a polynomial on $\mathbb{R}$. Then for any $t > 0$ the operator $p(D)e^{-tD^2}$ is traceclass.*

Proof. By finite summability and Hölder's inequality $(1 + D^2)^{-n/2}$ is traceclass for some n. Thus,

$$p(D)e^{-tD^2} = \varphi(D)\big(1 + D^2\big)^{-n/2}$$

with φ defined by functional calculus for the function

$$\varphi(x) = p(x)\big(1 + x^2\big)^{-n/2} e^{-tx^2}.$$

For $t > 0$, this is a bounded function on $\mathbb{R}$ so that $\varphi(D)(1 + D^2)^{-n/2}$ is in the ideal $\mathcal{L}^1(\mathcal{H})$ of traceclass operators as required. □

In particular, this applies to $p(x) = 1$, *i.e.* finite summability implies so-called θ-summability:

$$\mathrm{Tr}\big(e^{-tD^2}\big) < \infty \quad (t \in \mathbb{R}_+). \tag{1}$$

2.1. Fréchet algebra of smooth operators

Given the derivation $\delta(\cdot) = [|D|, \cdot]$ on $\mathcal{B}(\mathcal{H})$, there is a natural structure of a Fréchet algebra on the smooth domain of δ.

Proposition 2. *The following define a multiplicative family of semi-norms on $\mathcal{B}(\mathcal{H})$:*

$$\big\|\delta^n(T)\big\| \quad \big(T \in \mathcal{B}(\mathcal{H})\big)$$

indexed by $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. The derivation property of δ yields

$$\big\|\delta^n(T_1T_2)\big\| = \left\| \sum_{k=0}^{n} \binom{n}{k} \delta^k(T_1)\delta^{n-k}(T_2) \right\| \leqslant \sum_{k=0}^{n} \binom{n}{k} \big\|\delta_k(T_1)\big\| \big\|\delta_{n-k}(T_2)\big\|. \qquad \square$$

We will denote

$$\mathcal{B}^n(\mathcal{H}) = \big\{T \in \mathcal{B}(\mathcal{H}):\ \big\|\delta^k(T)\big\| < \infty \text{ for all } k \leqslant n\big\}.$$

Evidently, we have

$$\mathcal{B}^\infty(\mathcal{H}) \subset \cdots \subset \mathcal{B}^2(\mathcal{H}) \subset \mathcal{B}^1(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$$

where by definition $\mathcal{B}^\infty(\mathcal{H}) = \bigcap_{n \in \mathbb{Z}_{\geqslant 0}} \mathcal{B}^n(\mathcal{H})$.

Remark 3. Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called regular if both the algebra $\mathcal{A}$ and $[D, \mathcal{A}]$ are in the smooth domain of δ. This can thus be reformulated as:

the algebra generated by a and $[D, b]$ $(a, b \in \mathcal{A})$ is a subalgebra of $\mathcal{B}^\infty(\mathcal{H})$.

In particular, the $\mathcal{A}$-bimodule of *Connes' differential one-forms* [4, Sect. VI.1],

$$\Omega^1_D(\mathcal{A}) = \left\{ \sum_j a_j[D, b_j] \right\}$$

is a subspace of $\mathcal{B}^\infty(\mathcal{H})$.

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2.2. Perturbations of heat operators

In this subsection, we take a closer look at the heat operator e^{-tD^2} and its perturbations. First, recall that the standard m-simplex is given by an m-tuple $(t_1, \dots, t_m)$ satisfying $0 \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant 1$. Equivalently, it can be given by an $m+1$-tuple $(s_0, s_1, \dots, s_m)$ such that $s_0 + \cdots + s_m = 1$ and $0 \leqslant s_i \leqslant 1$ for any $i = 0, \dots, m$. Indeed, we have $s_0 = t_1$, $s_i = t_{i+1} - t_i$ and $s_m = 1 - t_m$ and, vice versa, $t_k = s_0 + s_1 + \cdots + s_{k-1}$.

For later use, we prove the following bound, which already appeared in a slightly different form in [10].

Proposition 4. *For any $m \geqslant 0$ and $0 \leqslant k \leqslant m+1$ we have the bound*

$$\int_{\Delta_m} d^m s (s_0 \cdots s_{k-1})^{-1/2} \leqslant \frac{\pi^k}{(m-k)!}.$$

Proof. In terms of the parameters t_i for the m-simplex, we have to find an upper bound for

$$\int_0^1 dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \frac{1}{\sqrt{t_1(t_2 - t_1)\cdots(t_k - t_{k-1})}},$$

where $t_{m+1} \equiv 1$. First, note that by a standard substitution

$$\int_0^{t_2} dt_1 \frac{1}{\sqrt{t_1(t_2 - t_1)}} = \pi.$$

For the subsequent integral over t_2:

$$\int_0^{t_3} dt_2 \frac{1}{\sqrt{t_3 - t_2}} \leqslant \int_0^{t_3} dt_2 \frac{1}{\sqrt{t_2(t_3 - t_2)}} = \pi$$

since $t_2 \leqslant 1$. This we can repeat k times, leaving us with the integral

$$\int_0^1 dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_{k+1}} dt_k = \frac{1}{(m-k)!}. \qquad \square$$

Lemma 5. *Let A be a bounded operator and denote $D_A = D + A$. Then*

$$e^{-t(D_A)^2} = e^{-tD^2} - t \int_0^1 ds\, e^{-st(D_A)^2} P(A) e^{-(1-s)tD^2}$$

with $P(A) = DA + AD + A^2$.

Proof. Note that $e^{-tD_A^2}$ is the unique solution of the Cauchy problem

$$\begin{cases} (d_t + D_A)u(t) = 0, \\ u(0) = 1 \end{cases}$$

with $d_t = d/dt$. Using the fundamental theorem of calculus, we find that

$$d_t\left[e^{-tD^2} - \int_0^t dt'\, e^{-(t-t')D_A^2} P(A) e^{-t'D^2}\right] = -D_A^2\left(e^{-tD^2} - \int_0^t dt'\, e^{-(t-t')D_A^2} P(A) e^{-t'D^2}\right)$$

showing that the bounded operator $e^{-tD^2} - \int_0^t dt'\, e^{-(t-t')D_A^2} P(A) e^{-t'D^2}$ also solves the above Cauchy problem. □

The following estimates were proved in a slightly different form in [10].

Lemma 6. *If the operators A, A_i are bounded, and $\alpha_i \in \{0,1\}$ are such that $\sum_i \alpha_i = k$, then*

$$\left|\int_{\Delta_n} \operatorname{Tr} A_0 |D_A|^{\alpha_0} e^{-s_0 t D_A^2} A_1 |D|^{\alpha_1} e^{-s_1 t D^2} \cdots A_n |D|^{\alpha_n} e^{-s_n t D^2}\, d^n s\right| \leqslant \frac{\|A_0\| \cdots \|A_n\| \operatorname{Tr} e^{-(1-\epsilon)tD^2}}{(n-k)!(\pi^{-2}\epsilon t)^{k/2}}$$

for any $0 < \epsilon < 1$.

Proof. Recall Hölder's inequality:

$$\left|\operatorname{Tr}(T_0 \cdots T_n)\right| \leqslant \|T_0\|_{s_0^{-1}} \cdots \|T_n\|_{s_n^{-1}} \tag{2}$$

when $s_0 + \cdots + s_n = 1$. Also, we estimate for some arbitrary $0 < \epsilon < 1$:

$$\begin{aligned} \left\|A_i e^{-s_i t D^2}\right\|_{s_i^{-1}} &\leqslant \|A_i\| \left(\operatorname{Tr} e^{-tD^2}\right)^{s_i} \leqslant \|A_i\| \left(\operatorname{Tr} e^{-(1-\epsilon)tD^2}\right)^{s_i}, \\ \left\|A_i |D| e^{-s_i t D^2}\right\|_{s_i^{-1}} &\leqslant \|A_i\| \left\||D| e^{-\epsilon s_i t D^2}\right\| \left(\operatorname{Tr} e^{-(1-\epsilon)tD^2}\right)^{s_i} \\ &\leqslant (\epsilon s_i t)^{-1/2} \|A_i\| \left(\operatorname{Tr} e^{-(1-\epsilon)tD^2}\right)^{s_i} \end{aligned}$$

writing $e^{-stD^2} = e^{-\epsilon st D^2} e^{-(1-\epsilon)stD^2}$. We have used Lemma 1 and the fact that

$$\left\|e^{-\epsilon st D^2}\right\| \leqslant 1, \qquad \left\||D| e^{-\epsilon st D^2}\right\| \leqslant \sup_{x \in \mathbb{R}_+} \left\{x e^{-\epsilon st x^2}\right\} = (2e\epsilon st)^{-1/2}.$$

Moreover, Theorem C in [10] (cf. Appendix A) gives

$$\operatorname{Tr} e^{-t(1-\epsilon/2)(D_A)^2} \leqslant e^{(1+2/\epsilon)t\|A\|^2} \operatorname{Tr} e^{-t(1-\epsilon)D^2}. \tag{$*$}$$

This further yields

$$\left\| A_0 |D_A| e^{-s_0 t D_A^2} \right\|_{s_0^{-1}} \leqslant \|A_0\| \left\| |D_A| e^{-\epsilon/2 s_i t D_A^2} \right\| \left(\operatorname{Tr} e^{-(1-\epsilon/2) t D_A^2}\right)^{s_i}$$
$$\leqslant (e\epsilon s_0 t)^{-1/2} e^{(1+2/\epsilon) t \|A\|^2} \|A_0\| \left(\operatorname{Tr} e^{-(1-\epsilon) t D^2}\right)^{s_0}.$$

Combining these estimates with (2), we obtain for instance in the case that the first k α_i are nonzero (*i.e.* $\alpha_0 = \cdots = \alpha_{k-1} = 1$):

$$\left|\operatorname{Tr} A_0 |D_A|^{\alpha_0} e^{-s_0 t D_A^2} A_1 |D|^{\alpha_1} e^{-s_1 t D^2} \cdots A_n |D|^{\alpha_n} e^{-s_n t D^2}\right|$$
$$\leqslant \frac{\|A_0\| \cdots \|A_n\|}{s_0 \cdots s_k (\epsilon t)^{k/2}} \operatorname{Tr} e^{-(1-\epsilon) t D^2}$$

making use of the fact that $s_0 + s_1 + \cdots + s_n = 1$. The bounds of Proposition 4 complete the proof. □

Let us introduce the following convenient notation (cf. [10]). If $A_0, \ldots, A_n$ are operators, we define a t-dependent quantity by

$$\langle A_0, \ldots, A_n \rangle_n := t^n \operatorname{Tr} \int_{\Delta_n} A_0 e^{-s_0 t D^2} A_1 e^{-s_1 t D^2} \cdots A_n e^{-s_n t D^2} d^n s. \tag{3}$$

Note the difference in notation with [10], for which the same symbol is used for the supertrace of the same expression, rather than the trace. Also, we are integrating over the 'inflated' n-simplex $t\Delta^n$, yielding the factor t^n. The forms $\langle A_0, \ldots, A_n \rangle$ satisfy, *mutatis mutandis*, the following properties.

Lemma 7. *(See [10].) In each of the following cases, we assume that the operators A_i are such that each term is well defined*:

1. $\langle A_0, \ldots, A_n \rangle_n = \langle A_i, \ldots, A_n, \ldots, A_{i-1} \rangle_n$;
2. $\langle A_0, \ldots, A_n \rangle_n = \sum_{i=0}^n \langle 1, \ldots, A_i, \ldots, A_n, A_0, \ldots, A_{i-1} \rangle_n$;
3. $\sum_{i=0}^n \langle A_0, \ldots, [D, A_i], \ldots, A_n \rangle_n = 0$;
4. $\langle A_0, \ldots, [D^2, A_i], \ldots, A_n \rangle_n = \langle A_0, \ldots, A_{i-1} A_i, , \ldots, A_n \rangle_{n-1} - \langle A_0, \ldots, A_i A_{i+1}, \ldots, A_n \rangle_{n-1}$.

2.3. Gâteaux derivatives

As a preparation for the next section, we recall the notion of Gâteaux derivatives, referring to the excellent treatment [12] for more details.

Definition 8. The *Gâteaux derivative* at $x \in X$ of a map $F : X \to Y$ between locally convex topological vector spaces is defined for $h \in X$ by

$$F'(x)(h) = \lim_{u \to 0} \frac{F(x + uh) - F(x)}{u}.$$

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In general, the map $F'(x)(\cdot)$ is not linear, in contrast with the Fréchet derivative. However, if X and Y are Fréchet spaces, then the Gâteaux derivatives actually defines a linear map $F'(x)(\cdot)$ for any $x \in X$ [12, Theorem 3.2.5]. In this case, higher order derivatives are denoted as F'', F''' *et cetera*, or more conveniently as $F^{(k)}$ for the k-th order derivative. The latter will be understood as a linear bounded operator from $X \times \cdots \times X$ ($k+1$ copies) to Y.

Theorem 9 *(Taylor's formula with integral remainder). For a Gâteaux $k+1$-differentiable map $F: X \to Y$ between Fréchet spaces X and Y it holds for $x, a \in X$ that*

$$F(x) = F(a) + F'(a)(x-a) + \frac{1}{2!}F''(a)(x-a, x-a) + \cdots + \frac{1}{n!}F^{(k)}(a)(x-a, \ldots, x-a) + R_k(x)$$

with integral remainder given by

$$R_k(x) = \frac{1}{k!}\int_0^1 F^{(k+1)}\big(a + t(x-a)\big)\big((1-t)h, \ldots, (1-t)h, h\big)\,dt.$$

3. Trace functionals

In this section, we consider trace functionals of the form $A \mapsto \operatorname{Tr} f(D+A)$. Here D is the self-adjoint operator forming a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and A is a bounded operator. We derive a Taylor expansion of this functional in A. Our main motivation comes from the spectral action principle introduced by Chamseddine and Connes [1,2] and we define accordingly

Definition 10. (See Chamseddine and Connes [2].) The spectral action functional $S_D[A]$ is defined by

$$S_D[A] = \operatorname{Tr} f(D+A) \quad \big(A \in \mathcal{B}(\mathcal{H})\big).$$

The square brackets indicate that $S_D[A]$ is considered as a functional of $A \in \mathcal{B}(\mathcal{H})$.

Remark 11. Actually, Chamseddine and Connes considered $S_D[A]$ for so-called gauge fields associated to the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. These are self-adjoint elements A in $\Omega^1_D(\mathcal{A})$ which by Remark 3 is a subset of $\mathcal{B}^2(\mathcal{H})$.

For the function f we assume that it is a Laplace–Stieltjes transform:

$$f(x) = \int_{t>0} e^{-tx^2}\,d\mu(t)$$

for which we make the additional:

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Assumption 1. For all $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $0 \leqslant \epsilon < 1$, there exist constants $C_{\alpha\beta\gamma\epsilon}$ such that

$$\int_{t>0} \operatorname{Tr} t^{\alpha} |D|^{\beta} e^{-t(\epsilon D^2 - \beta)} \left| d\mu(t) \right| < C_{\alpha\beta\gamma\epsilon}.$$

In view of Theorem 9, we have the following Taylor expansion (around 0) in $A \in \mathcal{B}^2(\mathcal{H})$ for the spectral action $S_D[A]$:

$$S_D[A] = \sum_{n=0}^{\infty} \frac{1}{n!} S_D^{(n)}(0)(A, \dots, A). \tag{4}$$

Indeed, S_D is Fréchet differentiable on $\mathcal{B}^2(\mathcal{H})$ as the following Proposition establishes.

Proposition 12. *If $n = 0, 1, \dots$ and $A \in \mathcal{B}^2(\mathcal{H})$, then $S_D^{(n)}(0)(A, \dots, A)$ exists and*

$$S_D^{(n)}(0)(A, \dots, A) = n! \sum_{k=0}^{n} (-1)^k \sum_{\varepsilon_1, \dots, \varepsilon_k} \big\langle 1, (1-\varepsilon_1)\{D, A\} + \varepsilon_1 A^2, \dots, (1-\varepsilon_k)\{D, A\} + \varepsilon_k A^2 \big\rangle_k \, d\mu(t),$$

where the sum is over multi-indices $(\varepsilon_1, \dots, \varepsilon_k) \in \{0, 1\}^k$ such that $\sum_{i=1}^{k} (1 + \varepsilon_i) = n$.

Proof. We will prove this by induction on n; the case $n = 0$ being trivial. By definition of the Gâteaux derivative and using Lemma 5,

$$\begin{aligned} S_D^{(n+1)}(0)(A, \dots, A) = n! \sum_{k=0}^{n} \sum_{\varepsilon_1, \dots, \varepsilon_k} \Bigg[& \sum_{i=1}^{k} (-1)^{k+1} \big\langle 1, (1-\varepsilon_1)\{D, A\} + \varepsilon_1 A^2, \dots, \\ & \underset{i}{\{D, A\}}, \dots, (1-\varepsilon_k)\{D, A\} + \varepsilon_k A^2 \big\rangle_{k+1} \\ & + \sum_{i=1}^{k} (-1)^k \big\langle 1, (1-\varepsilon_1)\{D, A\} + \varepsilon_1 A^2, \dots, 2(1-\varepsilon_i) A^2, \dots, \\ & (1-\varepsilon_k)\{D, A\} + \varepsilon_k A^2 \big\rangle_k \Bigg] d\mu(t). \end{aligned}$$

The first sum corresponds to a multi-index $\vec{\varepsilon}\,' = (\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_i, \dots, \varepsilon_k)$, the second sum corresponds to $\vec{\varepsilon}\,' = (\varepsilon_1, \dots, \varepsilon_i + 1, \dots, \varepsilon_k)$ if $\varepsilon_i = 0$, counted with a factor of 2. In both cases, we compute that $\sum_j (1 + \varepsilon'_j) = n + 1$. In other words, the induction step from n to $n+1$ corresponds to inserting in a sequence of 0's and 1's (of, say, length k) either a zero at any of the $k+1$ places, or replace a 0 by a 1 (with the latter counted twice). In order to arrive at the right combinatorial coefficient $(n+1)!$, we have to show that any $\vec{\varepsilon}\,'$ satisfying $\sum_i (1 + \varepsilon'_i) = n + 1$ appears in precisely $n + 1$ ways from $\vec{\varepsilon}$ that satisfy $\sum_i (1 + \varepsilon_i) = n$. If $\vec{\varepsilon}\,'$ has length k, it contains $n + 1 - k$

times 1 as an entry and, consequently, $2k-n-1$ a 0. This gives (with the double counting for the 1's) for the number of possible $\vec{\varepsilon}$:

$$2(n+1-k)+2k-n-1=n+1$$

as claimed. This completes the proof. □

Example 13.

$$\begin{aligned}
S_D^{(1)}(0)(A) &= \int \big(-\langle 1,\{D,A\}\rangle_1\big)\,d\mu(t),\\
S_D^{(2)}(0)(A,A) &= 2\int \big(-\langle 1,A^2\rangle_1+\langle 1,\{D,A\},\{D,A\}\rangle_2\big)\,d\mu(t),\\
S_D^{(3)}(0)(A,A,A) &= 3!\int \big(\langle 1,A^2,\{D,A\}\rangle_2+\langle 1,\{D,A\},A^2\rangle_2\\
&\quad -\langle 1,\{D,A\},\{D,A\},\{D,A\}\rangle_3\big)\,d\mu(t).
\end{aligned}$$

3.1. Divided differences

Recall the definition of and some basic results on divided differences.

Definition 14. Let $g:\mathbb{R}\to\mathbb{R}$ and $x_0,x_1,\dots,x_n$ be distinct points on $\mathbb{R}$. The divided difference of order n is defined by the recursive relations

$$\begin{aligned}
g[x_0] &= g(x_0),\\
g[x_0,x_1,\dots,x_n] &= \frac{g[x_1,\dots,x_n]-g[x_0,x_1,\dots,x_{n-1}]}{x_n-x_0}.
\end{aligned}$$

On coinciding points we extend this definition as the usual derivative:

$$g[x_0,\dots,x,\dots,x,\dots,x_n]:=\lim_{u\to 0} g[x_0,\dots,x+u,\dots,x,\dots,x_n].$$

Finally, as a shorthand notation, we write for an index set $I=\{i_1,\dots,i_n\}$:

$$g[x_I]=g[x_{i_1},\dots,x_{i_n}].$$

Also note the following useful representation due to Hermite [14].

Proposition 15. *For any* $x_0,\dots,x_n\in\mathbb{R}$:

$$f[x_0,x_1,\dots,x_n]=\int\limits_{\Delta_n} f^{(n)}(s_0x_0+s_1x_1+\cdots+s_nx_n)\,d^n s.$$

As an easy consequence, we derive

$$\sum_{i=0}^{n} f[x_0, \dots, x_i, x_i, \dots, x_n] = f'[x_0, x_1, \dots, x_n].$$

Proposition 16. *For any $x_1, \dots, x_n \in \mathbb{R}$ we have for $f(x) = g(x^2)$:*

$$f[x_0, \dots, x_n] = \sum_I \left(\prod_{\{i-1,i\} \subset I} (x_i + x_{i+1}) \right) g\left[x_I^2\right]$$

where the sum is over all ordered index sets $I = \{0 = i_0 < i_1 < \cdots < i_k = n\}$ such that $i_j - i_{j-1} \leqslant 2$ for all $1 \leqslant j \leqslant k$ (i.e. there are no gaps in I of length greater than 1).

Proof. This follows from the chain rule for divided difference: if $f = g \circ \varphi$, then [9]

$$f[x_0, \dots, x_n] = \sum_{k=1}^{n} \sum_{0=i_0<i_1<\cdots<i_k=n} g\left[\varphi(x_{i_0}), \dots, \varphi(x_{i_k})\right] \prod_{j=0}^{k-1} \varphi[x_{i_j}, \dots, x_{i_{j+1}}].$$

For $\varphi(x) = x^2$ we have $\varphi[x, y] = x + y$, $\varphi[x, y, z] = 1$ and all higher divided differences are zero. Thus, if $i_{j+1} - i_j > 2$ then $\varphi[x_{i_j}, \dots, x_{i_{j+1}}] = 0$. In the remaining cases one has

$$\varphi[x_{i_j}, \dots, x_{i_{j+1}}] = \begin{cases} x_{i_j} + x_{i_{j+1}} & \text{if } i_{j+1} - i_j = 1, \\ 1 & \text{if } i_{j+1} - i_j = 2, \end{cases}$$

and this selects in the above summation precisely the index sets I. □

Example 17. For the first few terms, we have

$$\begin{aligned} f[x_0, x_1] &= (x_0 + x_1) g\left[x_0^2, x_1^2\right], \\ f[x_0, x_1, x_2] &= (x_0 + x_1)(x_1 + x_2) g\left[x_0^2, x_1^2, x_2^2\right] + g\left[x_0^2, x_2^2\right], \\ f[x_0, x_1, x_2, x_3] &= (x_0 + x_1)(x_1 + x_2)(x_2 + x_3) g\left[x_0^2, x_1^2, x_2^2, x_3^2\right] \\ &\quad + (x_2 + x_3) g\left[x_0^2, x_2^2, x_3^2\right] + (x_0 + x_1) g\left[x_0^2, x_1^2, x_3^2\right]. \end{aligned}$$

3.2. Taylor expansion of the spectral action

We fix a complete set of eigenvectors $\{\psi_n\}_n$ of D with respective eigenvalue $\lambda_n \in \mathbb{R}$, forming an orthonormal basis for $\mathcal{H}$. We also denote $A_{mn} := (\psi_m, A\psi_n)$ so that $\sum_{m,n} A_{mn} |\psi_m)(\psi_n|$ converges to A in the weak operator topology.

Theorem 18. *If f satisfies Assumption 1 and $A \in \mathcal{B}^2(\mathcal{H})$, then*

$$S_D^{(n)}(0)(A, \dots, A) = n! \sum_{i_1,\dots,i_n} A_{i_n i_1} A_{i_1 i_2} \cdots A_{i_{n-1} i_n} f[\lambda_{i_p}, \lambda_{i_1}, \dots, \lambda_{i_n}].$$

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A similar result was obtained in finite dimensions in [13].

Proof. Proposition 12 gives us an expression for $S_D^{(n)}$ in terms of the brackets $\langle\cdots\rangle$. We compute for these:

$$\begin{aligned}
&(-1)^k\big\langle 1, (1-\varepsilon_1)\{D,A\}+\varepsilon_1 A^2, \dots, (1-\varepsilon_k)\{D,A\}+\varepsilon_k A^2\big\rangle_k\, d\mu(t)\\
&\quad= (-1)^k \sum_{i_0=i_k, i_1,\dots,i_k} \int_{\Delta_k} \left(\prod_{j=1}^k \big((1-\varepsilon_j)(\lambda_{i_{j-1}}-\lambda_{i_j})A + \varepsilon_j A^2\big)_{i_{j-1}i_j} \right)\\
&\qquad \times e^{-(s_0 t\lambda_{i_0}^2+\cdots+s_k t\lambda_{i_k}^2)}\, d^k s\, d\mu(t)\\
&\quad= \sum_{i_0=i_k, i_1,\dots,i_k} \left(\prod_{j=1}^k \big((1-\varepsilon_j)(\lambda_{i_{j-1}}-\lambda_{i_j})A + \varepsilon_j A^2\big)_{i_{j-1}i_j} \right) g\big[\lambda_{i_0}^2,\dots,\lambda_{i_k}^2\big].
\end{aligned}$$

Glancing back at Proposition 16 we are finished if we establish a one-to-one relation between the order index sets $I=\{0=i_0<i_1<\cdots<i_k=n\}$ such that $i_{j-1}-i_j\leqslant 2$ for all $1\leqslant j\leqslant k$ and the multi-indices $(\varepsilon_1,\dots,\varepsilon_k)\in\{0,1\}^k$ such that $\sum_{i=1}^k(1+\varepsilon_i)=n$. If I is such an index set, we define a multi-index:

$$\varepsilon_j = \begin{cases} 0 & \text{if } \{i_j-1, i_j\}\subset I,\\ 1 & \text{otherwise.}\end{cases}$$

Indeed, then $i_j = i_{j-1}+1+\varepsilon_j$ so that

$$\sum_{i=1}^k (1+\varepsilon_i) = i_0 + \sum_{i=1}^k (1+\varepsilon_i) = i_k = n.$$

It is now clear that, vice-versa, if ε is as above, we define $I=\{0=i_0<i_1<\cdots<i_k=n\}$ by $i_j=i_{j-1}+1+\varepsilon_j$ and starting with $i_0=0$. □

Corollary 19. *If $n\geqslant 0$ and $A\in\mathcal{B}^2(\mathcal{A})$, then*

$$S_D^{(n)}(0)(A,\dots,A) = (n-1)! \sum_{i_1,\dots,i_n} A_{i_1i_2}\cdots A_{i_ni_1} f'[\lambda_{i_1},\dots,\lambda_{i_n}].$$

Consequently,

$$S_D[A] = \sum_{n=0}^\infty \frac{1}{n} \sum_{i_1,\dots,i_n} A_{i_1i_2}\cdots A_{i_ni_1} f'[\lambda_{i_1},\dots,\lambda_{i_n}].$$

An interesting consequence is the following, which was obtained recently at first order for bounded operators [11].

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Corollary 20. *If $n \geqslant 0$ and $A \in \mathcal{B}^2(\mathcal{A})$ and if f' has compact support, then*

$$S_D^{(n)}(0)(A, \dots, A) = \frac{(n-1)!}{2\pi i} \operatorname{Tr} \oint f'(z) A (z-D)^{-1} \cdots A (z-D)^{-1}.$$

The contour integral encloses the intersection of the spectrum of D with supp f'.

Proof. This follows directly from Cauchy's formula for divided differences (cf. [8, Ch. I.1])

$$g[x_0, \dots, x_n] = \frac{1}{2\pi i} \oint \frac{g(z)}{(z-x_0)\cdots(z-x_n)} dz$$

with the contour enclosing the points x_i. □

4. Outlook

We have obtained a Taylor expansion for the spectral action in noncommutative geometry. As such, it is natural to consider its quadratic part as the starting point for a free quantum field theory. Expectedly, this involves the usual nuances of a gauge theory such as gauge fixing, Gribov ambiguities, *et cetera*. Under the assumption of vanishing tadpole

$$S_D^{(1)}(A) = 0 \quad \bigl(A \in \Omega^1(\mathcal{A})\bigr),$$

also exploited in [5], one indeed encounters a degeneracy in the quadratic part. In fact, in this case $S_D^{(2)}(A, [D, a]) = 0$ for all $a \in \mathcal{A}$. This vanishing on pure gauge fields will be considered in more detail elsewhere. Once this issue has been dealt with, the higher derivatives of the spectral action account for interactions, allowing for a development of a perturbative quantization of the spectral action.

Another application of the present work is to matrix models, as our Taylor expansion is very similar to Lagrangians encountered in matrix models. In fact, if the spectral triple is $(M_N(\mathbb{C}), \mathbb{C}^N, D)$ with D a symmetric $N \times N$-matrix, then the spectral action is exactly the hermitian one-matrix model (cf. [7]). An honest infinite-dimensional example might be provided by the spectral triples that are involved in Moyal deformations (see [15] and references therein). It would be interesting to apply the above results and develop a quantum theory for these models.

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Appendix A. A theorem by Getzler and Szenes

In [10] Getzler and Szenes proof the following theorem. For completeness, we repeat it here (specified to our finitely-summable case).

Theorem 21 *(Getzler–Szenes). Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a finitely-summable spectral triple and* V *a self-adjoint bounded operator on* $\mathcal{H}$*. Then* $(\mathcal{A}, \mathcal{H}, D_V)$ *with* $D_V = D + V$ *is a finitely-summable spectral triple, and*

$$\operatorname{Tr} e^{-(1-\epsilon/2)t(D_V)^2} \leqslant e^{(1+2/\epsilon)t\|V\|^2} \operatorname{Tr} e^{-(1-\epsilon)tD^2}$$

for any $0 < \epsilon < 1$ *and* $t > 0$.

Proof. This follows from the fact that for two positive self-adjoint operator A and B we have

$$\operatorname{Tr} e^{-A-B} \leqslant \operatorname{Tr} e^{-A}. \tag{5}$$

Indeed, let

$$A = (1-\epsilon)tD^2,$$
$$B = \epsilon t D^2/2 + (1-\epsilon/2)t\big(DV + VD + V^2\big) + (1+2/\epsilon)t\|V\|^2,$$

so that $A + B = (1-\epsilon/2)(D+V)^2 + (1+2/\epsilon)\|V\|^2$. Obviously, A is positive. To see that B is positive, we use the fact that

$$0 \leqslant \epsilon t D^2/2 + 2tV^2/\epsilon + t(DV + VD),$$

which is just positivity of $(\sqrt{\epsilon t/2}D + \sqrt{2t/\epsilon}V)^2$. Combining this with $V^2 \leqslant \|V^2\|$ and multiplying by the positive number $(1-\epsilon/2)$ we obtain

$$0 \leqslant (1-\epsilon/2)\big(\epsilon t D^2/2 + 2t\|V\|^2/\epsilon + t(DV + VD)\big) = B - \epsilon^2/4tD^2 - (1-\epsilon/2)tV^2,$$

ensuring positivity of B. Eq. (5) then implies

$$\operatorname{Tr} e^{-(1-\epsilon/2)t(D^2+DV+VD+V^2)} e^{-(1+2/\epsilon)t\|V\|^2} \leqslant \operatorname{Tr} e^{-(1-\epsilon)tD^2}$$

as desired. □

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