

Workshop on Noncommutative Manifolds

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Dirac operator on $SU_q(2)$

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"The Dirac operator on $SU_q(2)$ "

Introduction

- Noncommutative Geometry vs. Quantum Groups
- Construct q -version of spin geometry on $SU(2)$:
 - Homogeneous space:

$$SU(2) = \frac{\text{Spin}(4)}{\text{Spin}(3)} = \frac{SU(2) \times SU(2)}{SU(2)}$$

with $\text{Spin}(3)$ the diagonal $SU(2)$ subgroup of $\text{Spin}(4)$.

Quotient map: $(p, q) \mapsto pq^{-1}$

- Action of $\text{Spin}(4) = SU(2) \times SU(2)$ on $SU(2)$:

$$(p, q) \cdot x = pxq^{-1}$$

Algebraic preliminaries

Let q be a positive real number, $q \neq 1$.

Definition. Define the algebra $\mathcal{A} := \mathcal{A}(SU_q(2))$ of polynomials on $SU_q(2)$ to be the $*$ -algebra generated by a and b , subject to the following relations:

$$\begin{aligned}ba &= qab, & b^*a &= qab^*, & bb^* &= b^*b \\ a^*a + q^2b^*b &= 1, & aa^* + bb^* &= 1.\end{aligned}$$

As a consequence, $a^*b = qba^*$ and $a^*b^* = qb^*a^*$.

Correspondence with [Kl-Schm],[Chakr-Pal],[Con]: $a \leftrightarrow a^*, b \leftrightarrow -b$.

This becomes a *Hopf \ast -algebra* with

- the *coproduct* $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ defined by

$$\Delta a := a \otimes a - q b \otimes b^*,$$

$$\Delta b := b \otimes a^* + a \otimes b,$$

- the *counit* $\varepsilon : \mathcal{A} \rightarrow \mathbb{C}$ defined by $\varepsilon(a) = 1$ and $\varepsilon(b) = 0$,

- the *antipode* $S : \mathcal{A} \rightarrow \mathcal{A}$ defined as an antilinear map by

$$Sa = a^*, \quad Sb = -qb,$$

$$Sb^* = -q^{-1}b^*, \quad Sa^* = a.$$

Definition. The $*$ -algebra $\mathcal{U} := \mathcal{U}_q(\mathfrak{su}(2))$ is generated by elements e, f, k , with k invertible, satisfying the relations

$$ek = qke, \quad kf = qfk, \quad k^2 - k^{-2} = (q - q^{-1})(fe - ef)$$

Correspondence with [Kl-Schm]: $q \leftrightarrow q^{-1}$, or, equivalently: $e \leftrightarrow f$.

Hopf $$ -algebra* structure given by: *coproduct* Δ :

$$\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f,$$

counit $\epsilon(k) = 1, \epsilon(f) = \epsilon(e) = 0$, *antipode* S ,

$$Sk = k^{-1}, \quad Sf = -qf, \quad Se = -q^{-1}e,$$

and *star structure*: $k^* = k, f^* = e$.

Representation theory of $\mathcal{U}_q(su(2))$

The irreducible finite dimensional representations σ_l of $\mathcal{U}_q(su(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, and given by

$$\begin{aligned}\sigma_l(k) |lm\rangle &= q^m |lm\rangle, \\ \sigma_l(f) |lm\rangle &= \sqrt{[l-m][l+m+1]} |l, m+1\rangle, \\ \sigma_l(e) |lm\rangle &= \sqrt{[l-m+1][l+m]} |l, m-1\rangle,\end{aligned}$$

on the irreducible \mathcal{U} -modules $V_l = \text{Span}\{|lm\rangle\}_{m=-l, \dots, l}$.

The brackets denote q -integers: $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, provided $q \neq 1$.

Action of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(SU_q(2))$

Dual pairing of $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \rightarrow \mathbb{C}$ induces left and right action of $h \in \mathcal{U}_q(\mathfrak{su}(2))$ on $x \in \mathcal{A}(SU_q(2))$:

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle \quad x \triangleleft h := \langle h, x_{(1)} \rangle x_{(2)},$$

where we use Sweedler's notation for the coproduct in $\mathcal{A}(SU_q(2))$:

$$\Delta x = x_{(1)} \otimes x_{(2)}, \quad (x \in \mathcal{A})$$

Using the antipode, the right action can be transformed into a left action, which we will denote by $h \cdot x$.

Left regular representation of $\mathcal{A}(SU_q(2))$

We establish the left regular representation of \mathcal{A} as an equivariant representation with respect to two copies of \mathcal{U} acting via \cdot and \triangleright on the left.

Definition. Let λ and ρ be mutually commuting representations of the Hopf algebra \mathcal{U} on a vector space V . A representation π of the algebra \mathcal{A} on V is (λ, ρ) -equivariant if the following compatibility relations hold:

$$\begin{aligned}\lambda(h) \pi(x)\xi &= \pi(h_{(1)} \cdot x) \lambda(h_{(2)})\xi, \\ \rho(h) \pi(x)\xi &= \pi(h_{(1)} \triangleright x) \rho(h_{(2)})\xi,\end{aligned}$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$.

Equivariant representation of $\mathcal{A}(SU_q(2))$

Representation space:

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l$$

Two copies of $\mathcal{U}_q(su(2))$ act via the irreducible representations σ on the first and the second leg of the tensor product, respectively:

$$\lambda(h) = \sigma_l(h) \otimes \text{id}, \quad \rho(h) = \text{id} \otimes \sigma_l(h) \quad \text{on } V_l \otimes V_l.$$

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \dots, l$.

Proposition. *A (λ, ρ) -equivariant $*$ -representation π of $\mathcal{A}(SU_q(2))$ on V is necessarily given by the left regular representation. Explicitly:*

$$\begin{aligned}\pi(a) |lmn\rangle &= A_{lmn}^+ |l + \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2}\rangle + A_{lmn}^- |l - \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2}\rangle, \\ \pi(b) |lmn\rangle &= B_{lmn}^+ |l + \frac{1}{2}, m + \frac{1}{2}, n - \frac{1}{2}\rangle + B_{lmn}^- |l - \frac{1}{2}, m + \frac{1}{2}, n - \frac{1}{2}\rangle,\end{aligned}$$

where for example the constants A_{lmn}^\pm are given by

$$\begin{aligned}A_{lmn}^+ &= q^{(-2l+m+n-1)/2} \left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]} \right)^{\frac{1}{2}}, \\ A_{lmn}^- &= q^{(2l+m+n+1)/2} \left(\frac{[l-m][l-n]}{[2l][2l+1]} \right)^{\frac{1}{2}}.\end{aligned}$$

Spinor representation

We amplify representation π of \mathcal{A} to the **spinor representation** defined by $\pi' = \pi \otimes \text{id}$ on $V \otimes \mathbb{C}^2$, and set $\rho' = \rho \otimes \text{id}$, but λ' as the tensor product of the representations λ on V and $\sigma_{\frac{1}{2}}$ on $V_{\frac{1}{2}} = \mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}).$$

Proposition. *The representation π' of \mathcal{A} is (λ', ρ') -equivariant.*

Clebsch-Gordan decomposition:

$$V \otimes \mathbb{C}^2 = \left(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l \right) \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j).$$

Basis vectors ($l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$):

$$|j\mu n\uparrow\rangle := C_{j+1,\mu} |j + \frac{1}{2}, \mu - \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle - S_{j+1,\mu} |j + \frac{1}{2}, \mu + \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle,$$

where $\mu = -j, \dots, j$ and $n = -(j + \frac{1}{2}), \dots, j + \frac{1}{2}$

$$|j\mu n\downarrow\rangle := S_{j\mu} |j - \frac{1}{2}, \mu - \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle + C_{j\mu} |j - \frac{1}{2}, \mu + \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle,$$

where $\mu = -j, \dots, j$ and $n = -(j - \frac{1}{2}), \dots, j - \frac{1}{2}$, and the q -Clebsch-Gordan coefficients come from the well-known representation theory of $\mathcal{U}_q(su(2))$:

$$C_{j\mu} := q^{-(j+\mu)/2} \frac{[j - \mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}, \quad S_{j\mu} := q^{(j-\mu)/2} \frac{[j + \mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}.$$

\implies expressions for π' in basis $\{|j\mu n\uparrow\rangle, |j\mu n\downarrow\rangle\}$ contain off-diagonal terms.

Invariant Dirac operator

Proposition. *Any self-adjoint operator on $\mathcal{H} = (V \otimes \mathbb{C}^2)^{\text{cl}}$, that commutes with both actions ρ', λ' of $\mathcal{U}_q(\mathfrak{su}(2))$ is of the form*

$$D|j\mu n\uparrow\rangle = d_j^\uparrow |j\mu n\uparrow\rangle, \quad D|j\mu n\downarrow\rangle = d_j^\downarrow |j\mu n\downarrow\rangle.$$

Restrict form of eigenvalues by imposing **bounded commutator** condition:

$$[D, \pi'(x)] \in \mathcal{B}(\mathcal{H}), \quad (x \in \mathcal{A}).$$

- D with as eigenvalues q -analogues of the classical Dirac operator (like $[j]$) gives unbounded commutators (cf. [Bib-Kul]).

- ‘Classical’ Dirac operator, with D eigenvalues linear in j with opposite signs on the \uparrow and \downarrow -eigenspaces, respectively.

Proposition. *If D has eigenvalues linear in j , the commutators $[D, \pi'(x)]$ ($x \in \mathcal{A}$) are **bounded** operators.*

Relation with [Gos], with **unbounded commutators**?

Difference in definition of spinor space: $\mathbb{C}^2 \otimes V$ (instead of $V \otimes \mathbb{C}^2$).

Define on $\mathbb{C}^2 \otimes V$:

$$\pi'(x) = \text{id} \otimes \pi(x);$$

$$\rho'(h) = \text{id} \otimes \rho(h);$$

$$\lambda'(h) = \sigma_{\frac{1}{2}}(h_{(1)}) \otimes \lambda(h_{(2)}).$$

Let us (naïvely) define the Dirac operator to be diagonal in the $\uparrow - \downarrow$ basis obtained from the Clebsch-Gordan decomposition, with j -linear eigenvalues. This is exactly [Gos]. A computation shows that $[D, \pi'(x)]$ is **unbounded**.

But,..., the abovesly defined representation π' is not (λ', ρ') -equivariant.

In other words, the principle of \mathcal{U} -equivariance does not allow $\mathbb{C}^2 \otimes V$ as a spinor space.

Modular conjugation operator

Definition. *The modular conjugation operator J is the antilinear operator on \mathcal{H} which is defined on the orthonormal spinor basis by*

$$J |j\mu n\uparrow\rangle := i^{2(2j+\mu+n)} |j, -\mu, -n, \uparrow\rangle;$$

$$J |j\mu n\downarrow\rangle := i^{2(2j-\mu-n)} |j, -\mu, -n, \downarrow\rangle.$$

Proposition. *With D as before and J as above, the commutant property and the first-order condition are satisfied up to compact operators:*

$$[\pi'(x), J\pi'(y)J^{-1}] \in \mathcal{K}(\mathcal{H});$$

$$[\pi'(x), [D, J\pi'(y)J^{-1}]] \in \mathcal{K}(\mathcal{H}); \quad (\forall x, y \in \mathcal{A}(SU_q(2)))$$

References

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