Hopf algebras of Feynman graphs for gauge theories

Walter D. van Suijlekom

July 22, 2009

Radboud University Nijmegen

Contents

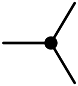
- What Feynman graphs are...
- (Hopf) algebraic structure of renormalization for gauge theories
- Applications: Slavnov–Taylor identities, MHV/BCFW-rules
- Connection between Hopf algebra of graphs and Gerstenhaber algebras.
- Application to Yang–Mills theory

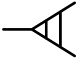
What Feynman graphs are...

Graphs built from a fixed set $\{v_1, \ldots, v_k\}$ of types of vertices (possibly $k = \infty$ [BK]) and a fixed set $\{e_1, \ldots, e_N\}$ of types of edges.

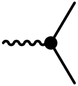
Examples:

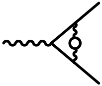
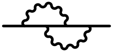
- Scalar ϕ^3-theory:

  vertex:  , edge: .

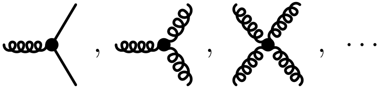
  and one constructs graphs such as  , 

- Electrodynamics:

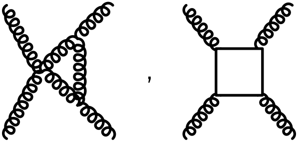
  vertex:  , edges: , .

  and one constructs graphs such as  , 

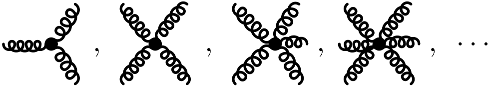
- Yang–Mills theory:

vertices: 

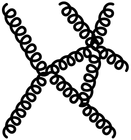
edges: 

and one constructs graphs such as 

- (Gravity):

vertices: 

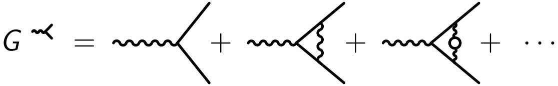
edges: 

and one constructs graphs such as 

Perturbative quantum gauge theory

Idea: probability amplitudes for physical processes are given by expansions in Feynman graphs.

- **Example**: interaction of photon with electron (QED)



- A physicist is interested in numbers, and the **Feynman rules** associate a complex number to a Feynman graph Γ

$$\Gamma \mapsto U(\Gamma) \in \mathbb{C}$$

- However, these numbers are typically infinite... $\rightsquigarrow$ need to **renormalize**

Idea of renormalization

1. **Regularization**: introduce a parameter $z \in \mathbb{C}$ and define new Feynman rules U_z:

$$\Gamma \mapsto U_z(\Gamma) \in \mathbb{C}$$

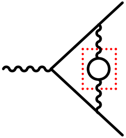
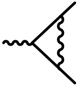
The previous infinity becomes a **pole at** $z = 0$ of the Laurent series expansion in z.

2. **Subtraction**: get rid of the whole pole part of the Laurent series expansion: this gives the renormalized amplitude

$$\Gamma \mapsto R_z(\Gamma) \in \mathbb{C}$$

This applies not only to the Feynman graph Γ, but also to its subgraphs:

For a generic graph Γ: $R_z(\Gamma)$ defined by a recursive procedure

Example: renormalization of  involves  and 

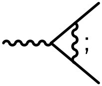
Mathematical structure of renormalization

Group of 'Feynman rules'

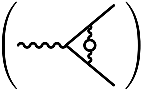
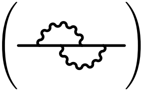
It turns out that the collection of all Feynman rules constitute a group.

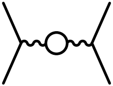
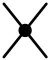
We start by considering the Feynman rules $\Gamma \mapsto U(\Gamma) \in \mathbb{C}$ as characters on the **free commutative algebra H generated by all 1PI Feynman graphs with residue in** $\{v_1, \ldots, v_k\} \cup \{e_1, \ldots, e_N\}$:

- One-particle irreducible graphs:

1PI:  ,  ; and **not 1PI (1PR)**: ,

- Residue of a graph:

res  =  and res  =

Example of a graph not allowed:  since 1PR and residue  $\neq v_i$

Group structure on characters of H

- Unit $\epsilon \in G := \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ is understood as a **co**unit $\epsilon : H \to \mathbb{C}$.
- Multiplication $* : G \times G \to G$ induced by a **co**product $\Delta : H \to H \otimes H$
- Inverse induced by the antipode $S : H \to H$.

Theorem (Connes–Kreimer, 2000)

There exists a counit, coproduct and antipode on the algebra H of Feynman graphs, turning H into a Hopf algebra (and G a group). The counit is

$$\epsilon(\Gamma) = \begin{cases} 1 & \text{if } \Gamma = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

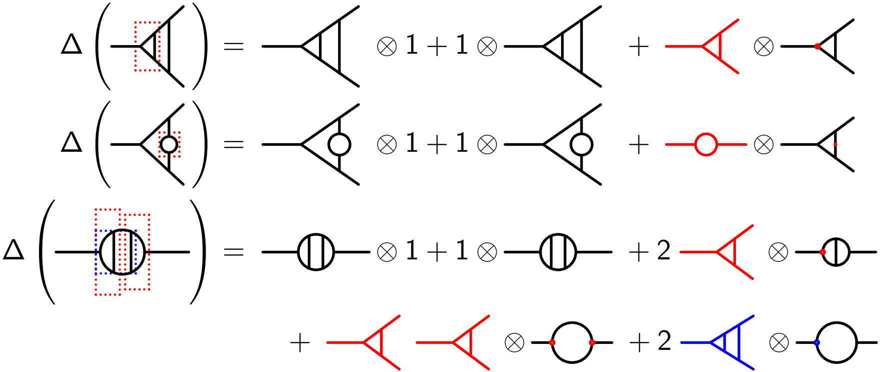
and the coproduct is defined by

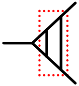
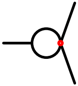
$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over (disjoint unions of) 1PI subgraphs with residue v_i or e_j.

Examples of the coproduct with $v =$  and $e =$

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$



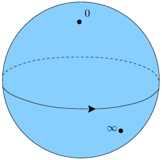
not allowed:  $\rightsquigarrow$ 

Renormalization as a decomposition in G

- The above Hopf algebra H is the algebraic structure underlying the recursive procedure of renormalization.
- In fact, for a character $U_z : H \to \mathbb{C}$, there exists a character $C_z : H \to \mathbb{C}$ ('counterterm') defined for $z \neq 0$ as

$$C_z(\Gamma) = -T\left[U_z(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C_z(\gamma) U_z(\Gamma/\gamma)\right]$$

  with T the projection onto the pole part, so that $R_z = C_z * U_z$ is finite at $z = 0$ [Connes and Kreimer, 2000].
- This decomposition is unique if one requires $C_{z=\infty} = \epsilon$ and can be interpreted as a Birkhoff decomposition in the group $G = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$:



$$U_z = C_z^{-1} * R_z$$

Gauge theories

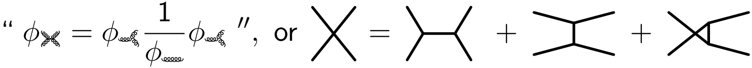
- The physical (renormalized) Green's functions are given by

$$\phi_r(p, \mu, \alpha, \pm, \ldots)\ R_{z=0}(G^r)(p, \mu, \alpha, \pm, \ldots)$$

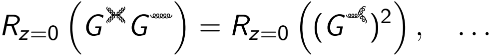
with $r = v_i, e_j$ and ϕ_r the corresponding formfactors (depending on momenta, Lorentz and spinor indices, chiralities *et cetera*) and

$$G^{v_i} = 1 + \sum_{\mathrm{res}(\Gamma)=v_i} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|}, \qquad G^{e_j} = 1 - \sum_{\mathrm{res}(\Gamma)=e_j} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|}$$

- Gauge symmetries imply certain identities between these formfactors, such as in pure Yang–Mills theories:



- For renormalizability of gauge theories it is essential for these identities to hold at any loop order: the Slavnov–Taylor identities for the couplings



- Thus, we first need an expression for the coproduct on the G^r's.

Structure of H

Gradings

- Grading by loop number $l(\Gamma) = h^1(\Gamma)$:

$$H = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^l, \qquad q_l : H \to H^l$$

- Multigrading by number of vertices:

$$d_i(\Gamma) = \#\text{vertices } v_i \text{ in } \Gamma - \delta_{v_i, \mathrm{res}(\Gamma)}$$

with

$$H = \bigoplus_{n_1, \ldots, n_k \in \mathbb{Z}^k} H^{n_1, \ldots, n_k}, \qquad p_{n_1, \ldots, n_k} : H \to H^{n_1, \ldots, n_k}$$

- These are related via $\sum_{i=1}^{k} (\mathrm{val}(v_i) - 2) d_i = 2l$.

N.B. Connected Hopf algebra: $H^0 = H^{0,\ldots,0} = \mathbb{C}1$.

Structure of H

Hopf subalgebras

- **Question: what is the coproduct on Green's functions?**
- Example: scalar ϕ^3-theory (with one type of vertex $v =$  and one type of edge $e =$ —) :

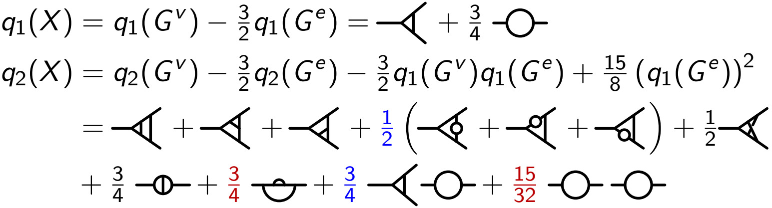
Proposition

The elements $X = G^v (G^e)^{-3/2}$ and G^e generate a Hopf subalgebra in H:

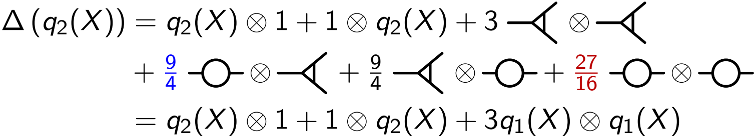
$$\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X), \qquad \Delta(G^e) = \sum_{l=0}^{\infty} G^e X^{2l} \otimes q_l(G^e)$$

Let us check that $\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X)$ at lowest loop order (with $v =$  and $e =$). Recall that

$$X = G^v (G^e)^{-3/2}.$$



and



Structure of H

Hopf subalgebras

- Again: scalar ϕ^3-theory (with one type of vertex $v =$  and one type of edge $e =$ —) :

Proposition

The elements $X = G^v (G^e)^{-3/2}$ and G^e generate a Hopf subalgebra in H:

$$\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X), \qquad \Delta(G^e) = \sum_{l=0}^{\infty} G^e X^{2l} \otimes q_l(G^e)$$

- This is recognized as the Hopf algebra dual to (a subgroup of) the group $\mathbb{C}[[\lambda]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C}, 0)$. Namely, a character ϕ on this Hopf subalgebra defines
  1. An invertible formal power series by $\sum_{l=0}^{\infty} \phi(q_l(G^e))\lambda^l$
  2. A formal diffeomorphism on $\mathbb{C}$ by $\lambda \mapsto \sum_{l=0}^{\infty} \phi(q_l(X))\lambda^{l+1}$.

- In general (vertices $\{v_1, \ldots, v_k\}$ and edges $\{e_1, \ldots, e_N\}$), we define for each vertex v

$$X_v := \left(\frac{G^v}{\prod_i (G^{e_j})^{\mathrm{val}_j(v)/2}} \right)^{1/\mathrm{val}(v)-2}$$

Proposition (vS, 2008)

The coproduct on the Green's functions reads

$$\Delta(G^r) = \sum_{n_1,\ldots,n_k} G^r (X_{v_1})^{n_1(\mathrm{val}(v_i)-2)} \cdots (X_{v_k})^{n_k(\mathrm{val}(v_k)-2)} \otimes p_{n_1,\ldots,n_k}(G^r),$$

- On the elements X_{v_i} we then have

$$\Delta(X_v) = \sum_{n_1,\ldots,n_k} X_v (X_{v_1})^{n_1(\mathrm{val}(v_i)-2)} \cdots (X_{v_k})^{n_k(\mathrm{val}(v_k)-2)} \otimes p_{n_1,\ldots,n_k}(X_v),$$

Thus, X_v and G^e (equivalently, G^v and G^e) for all vertices v and edges e generate a Hopf subalgebra, when restricted to each multidegree...

Structure of H

Hopf subalgebras and ideals

Theorem (vS, 2008)

1. *The elements G^{v_i} and G^{e_j} generate a Hopf subalgebra H' in H with dual group*

$$G := \mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \left(\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]^\times\right)^N \rtimes \overline{\mathbf{Diff}}(\mathbb{C}^k)$$

2. *The ideal $J := \langle X_{v_i} - X_{v_j} \rangle$ in H' is a Hopf ideal, i.e. H'/J is a Hopf algebra with dual group*

$$\mathrm{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset \left(\mathbb{C}[[\lambda]]^\times\right)^N \rtimes \overline{\mathbf{Diff}}(\mathbb{C})$$

- The relations $X_{v_i} = X_{v_j}$ in the quotient Hopf algebra H'/J are called (generalized) Slavnov–Taylor identities for the couplings.

Application I: ST identities and renormalization

We now apply the above results to (Yang–Mills) gauge theories (eg. QCD).

- Suppose we are given a gauge invariant regularization scheme, such that

$$U_z(X_{v_i}) = U_z(X_{v_j}); \qquad (\forall i, j = 1, \ldots, k).$$

  or, more explicitly, in QCD: Slavnov–Taylor identities for the couplings,

$$U_z\left(\frac{G^{\text{4-gluon vertex}}}{(G^{\text{gluon propagator}})^2}\right) = U_z\left(\left(\frac{(G^{\text{3-gluon vertex}})}{(G^{\text{gluon propagator}})^{3/2}}\right)^2\right) \text{ i.e. } U_z(G^{\text{4-gluon vertex}} G^{\text{gluon propagator}}) = (U_z(G^{\text{3-gluon vertex}}))^2.$$

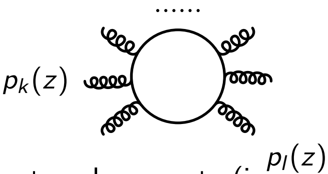
- Thus, the map U_z is a character on the quotient H'/J and there exists a decomposition $U_z = C_z^{-1} * R_z$ as characters on H'/J.
- Consequently, both C_z and R_z vanish on J, in other words

$$C_z(X_{v_i}) = C_z(X_{v_j}); \qquad R_z(X_{v_i}) = R_z(X_{v_j})$$

  In particular, the **Slavnov–Taylor identities for the couplings are satisfied by the renormalized Feynman rules**.
- **N.B.** This is a completely (Hopf) algebraic proof of this physical fact.

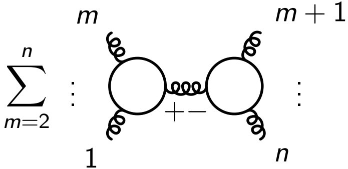
(Potential) Application II: MHV/BCFW rules [KvS]

Consider the following n-valence tree-level gluon amplitude in QCD:



$$=: A_n^{\text{tree}}$$

- Shift two external momenta (i.e. p_k, p_l) by a complex amount z.
- One can show that $A(z)$ is meromorphic in z with simple poles;
- These poles appear in the propagators that 'split $A_n(z)$ into two pieces':



implying $A_n = \sum_{m=2}^{n} A_{m+1}^{\text{tree}} \frac{1}{P_m^2} A_{n-m+1}^{\text{tree}}$

- At higher loop order, this would become: $A_n = \sum_{m=2}^{n} A_{m+1} \frac{1}{A_2} A_{n-m+1}$; compare to our Hopf ideal which is generated (in this case) by

$$\boxed{G^{(m+1)} \frac{1}{G^{(2)}} G^{(n-m+1)} = G^{(n)}}$$

General setup

Comodule Gerstenhaber algebras

We now establish a connection between the Hopf algebra of renormalization and a Gerstenhaber structure in the context of gauge theories, with the goal of **explaining the origin of the Hopf ideals J**.

- In general, we assign to each vertex v_i a parameter λ_i for $i = 1, \ldots, k$.
- To each edge e_j we assign a field ϕ_j and a corresponding antifield $\phi_j^\ddagger$ for $j = 1, \ldots, N$ (with certain degrees); the anti-bracket is defined by

$$\left(\phi_i(x), \phi_j^\ddagger(y)\right) = \delta_{ij}\delta(x - y).$$

  and zero on other generators.
- This makes the following a **Gerstenhaber algebra**:

$$\boxed{A := \mathcal{F}([\phi_1, \phi_1^\ddagger, \ldots, \phi_N, \phi_N^\ddagger]) \otimes \mathbb{C}[[\lambda_1, \cdots, \lambda_k]]}$$

  i.e. **a graded algebra with a Lie bracket of degree 1**.

The algebra $A = \mathcal{F}([\phi_1, \phi_1^\ddagger, \ldots, \phi_N, \phi_N^\ddagger]) \otimes \mathbb{C}[[\lambda_1, \cdots, \lambda_k]]$ consists of $\mathbb{C}[[\lambda_1, \cdots, \lambda_k]]$-linear functionals in the fields.

Proposition (vS, 2008)

The algebra A is a Gerstenhaber comodule algebra over H'. In other words, there exists a map $\rho : A \to A \otimes H'$ compatible with the coproduct on H' and respecting the bracket in A.

Consequently, there is an action of $G \subset \left(\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]^\times\right)^N \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^k)$ on A.

For instance, we have

$$\rho : \phi_j \longmapsto \sum_{n_1 \cdots n_k} \phi_j \lambda_1^{n_1} \cdots \lambda_k^{n_k} \otimes p_{n_1 \cdots n_k}\left((G^{e_j})^{1/2} \right) \qquad \text{(invertible series)}$$

$$\rho : \lambda_i \longmapsto \sum_{n_1 \cdots n_k} \lambda_i \lambda_1^{n_1} \cdots \lambda_k^{n_k} \otimes p_{n_1 \cdots n_k}\left((X_v)^{\mathrm{val}(v_i)-2} \right) \qquad \text{(formal diffeos),}$$

where we recall $X_{v_i} = \left(\frac{G^{v_i}}{\prod_j \left(G^{e_j}\right)^{\mathrm{val}_j(v_i)/2}} \right)^{1/\mathrm{val}(v_i)-2} \in H'$.

Master equation

- Next, one considers an element $S \in A$ (the *action*) of the following form
$$\boxed{S = \sum_r \int \phi_r m(r)}$$
with $m(r)$ monomials in the fields that interact/propagate at r and ϕ_r are the formfactors. For a vertex: $\phi_{v_i} = \lambda_i$ (modulo momenta, ...).
- The ideal $I = \langle (S, S) \rangle$ implements the 'master equation' $(S, S) = 0$ and
$$I = \langle p_\alpha(\lambda_1, \ldots, \lambda_k) \rangle_\alpha, \qquad p_\alpha \text{ polynomials}$$
- A theory (defined by S) is called simple if $I = \langle \lambda_i - \lambda^{\mathrm{val}(v_i)-2} \rangle_i$ with $\lambda := \lambda_j$ corresponding to some fixed valence 3 vertex v_j.

Proposition (vS, 2008)

If S defines a simple theory, then the subgroup $G^I \subset G$ leaving I invariant is dual to H'/J (with $J = \langle X_{v_i} - X_{v_j} \rangle_{i,j}$), i.e.
$$G^I \simeq \mathrm{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset (\mathbb{C}[[\lambda]]^\times)^N \rtimes \overline{\mathrm{Diff}}(\mathbb{C}).$$

Renormalization group

- The renormalization group appears as a subgroup of $\mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ when one realizes that the regularized Feynman rules actually depend on a mass scale μ. In fact, there is a decomposition

$$U_{\mu,z} = C_z^{-1} * R_{\mu,z}$$

- The effective coupling constants depent on μ and are defined via the action of $R_{\mu,z}$:

$$\lambda_i(\mu) = R_{\mu,z=0}\lambda_i$$

  The beta-function is then defined as

$$\beta(\lambda_i(\mu)) = \mu \frac{d}{d\mu}\lambda_i(\mu).$$

Proposition (vS,2008)

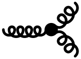
For a simple theory, all β-functions are expressed in terms of $\beta(\lambda)$ for the fundamental coupling constant λ:

$$\beta(\lambda_i) = \beta(\lambda^{\mathrm{val}(v_i)-2}).$$

Application to Yang–Mills theory

- The action is (essentially) the Yang–Mills action for a connection one-form ω (with simple gauge group):

$$S = \|F(\omega)\|^2 + \cdots = \int F^a_{\mu\nu} F^{\mu\nu}_a + \cdots \,, \qquad \text{with } F = d\omega + \tfrac{1}{2}\omega \wedge \omega.$$

- Feynman graphs are built from $v_3 =$  and $v_4 =$ 
- The corresponding Hopf algebra H coacts on the Gerstenhaber algebra A of $\mathbb{C}[[\lambda_3, \lambda_4]]$-linear functionals in $\omega, \omega^\ddagger, \ldots$, and

$$\boxed{\mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \overline{\mathsf{Diff}}(\mathbb{C}^2)}$$

- Gauge symmetry $\rightsquigarrow$ master equation $(S, S) = 0 \iff \boxed{\lambda_4 - \lambda_3^2 = 0}$
- The subgroup G^I of $\mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^2)$ that leaves this equation invariant is dual to the Hopf algebra H'/J with

$$J = \langle X_{\text{[3-gluon vertex]}} - X_{\text{[4-gluon vertex]}} \rangle$$

so that $G^I \subset \mathbb{C}[[\lambda]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C})$ identifying $\lambda_4 = \lambda_3^2 \equiv \lambda^2$.

- Thus, in H'/J the identities $X_{\text{[3-gluon vertex]}} = X_{\text{[4-gluon vertex]}}$ hold, or, explicitly

$$\boxed{G^{\text{[4-gluon vertex]}} = \frac{(G^{\text{[3-gluon vertex]}})^2}{G^{\text{[gluon propagator]}}}}$$

- As said, their appearance as generators of a Hopf ideal proves that the ST-identities are compatible with renormalization, so that if U_z satisfies them, it follows that R_z, C_z do so as well.
- Also **the beta functions for λ_3 and λ_4 are the same** so that for the effective couplings: $\lambda_4(\mu) = \lambda_3(\mu)^2$ for all μ and similarly for the quark and ghost couplings.

Application to Yang–Mills theory

- Fields: $A, \omega, \overline{\omega}, h$.
- The gauge-fixed Yang–Mills action S_0 plus BRST-sources (anti-fields) is

$$S = \int_M \mathrm{tr}\Big[- dA * dA - \lambda_3 dA * [A, A] - \frac{1}{4}\lambda_4 [A, A] * [A, A]$$
$$-A * dh + d\overline{\omega} * d\omega + \frac{1}{2}\xi h * h + \lambda_2 d\overline{\omega} * [A, \omega]$$
$$+(sA) * A^{\ddagger} + (s\omega) * \omega^{\ddagger} + (s\overline{\omega}) * \overline{\omega}^{\ddagger} + (sh) * h^{\ddagger}.\Big]$$

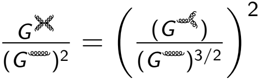
with

$$sA = -d\omega - \lambda_5 [A, \omega], \quad s\omega = -\frac{1}{2}\lambda_6 [\omega, \omega], \quad s\overline{\omega} = -h, \quad sh = 0$$

- The master equation (S, S) implies
  - $s^2 = 0$ which happens iff $\lambda_5 = \lambda_6$
  - $s(S_0) = 0$ which happens iff $\lambda_5 = \lambda_2 = \lambda_3$ and $\lambda_3^2 = \lambda_4$.

- In the quotient by $I = \langle (S,S) \rangle$ we then have in terms of λ:

$$S = \int_M \mathrm{tr} \Big[- dA * dA - \lambda dA * [A,A] - \frac{1}{4}\lambda [A,A] * [A,A]$$
$$-A * dh + d\overline{\omega} * d\omega + \frac{1}{2}\xi h * h + \lambda d\overline{\omega} * [A,\omega]$$
$$+(sA) * A^{\ddagger} + (s\omega) * \omega^{\ddagger} + (s\overline{\omega}) * \overline{\omega}^{\ddagger} + (sh) * h^{\ddagger}. \Big]$$

  with $sA = -d\omega - \lambda[A,\omega], \quad s\omega = -\frac{1}{2}\lambda[\omega,\omega], \quad s\overline{\omega} = -h, \quad sh = 0$
- The subgroup G^I of $\mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_2, \ldots, \lambda_6]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C}^2)$ that leaves this equation invariant is dual to the Hopf algebra H'/J with $J = \langle X_{v_i} - X_{v_j} \rangle$; so that $G^I \subset \mathbb{C}[[\lambda]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C})$ identifying $\lambda_2^2 = \lambda_3^2 = \lambda_4 = \lambda_5^2 = \lambda_6^2 \equiv \lambda^2$.
- In H'/J we have:  , ...
- As said, their appearance as generators of a Hopf ideal proves that the ST-identities are compatible with renormalization
- Also **the beta functions for $\lambda_2, \ldots, \lambda_6$ are the same** so that $\lambda_2^2(\mu) = \lambda_3^2(\mu) = \lambda_4(\mu) = \lambda_5^2(\mu) = \lambda_6^2(\mu) \equiv \lambda^2(\mu)$.

References

- D. Kreimer and WvS. Recursive relations in the core Hopf algebra. *Nucl. Phys.* **B**820 (2009) 682–693.
- WvS. The Hopf algebra of Feynman graphs in QED. *Lett. Math. Phys.* 77 (2006) 265–281.
- WvS. Renormalization of gauge fields: A Hopf algebra approach. *Commun. Math. Phys.* 276 (2007) 773-798.
- WvS. The structure of renormalization Hopf algebras for gauge theories I: Representing Feynman graphs on BV-algebras, *Commun. Math. Phys.* 290 (2009) 291–319.

Cf. http://www.math.ru.nl/∼waltervs