Inner perturbations in noncommutative geometry

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Overview

- Spectral geometry
- Matrix algebra and noncommutative geometry
- Semigroup of perturbations
- Examples of perturbation semigroup
- Towards the Standard Model of Particle Physics

References

A. Chamseddine, Alain Connes, WvS. Inner Fluctuations in Noncommutative Geometry without the first order condition. *J. Geom. Phys.* 73 (2013) 222-234 [arXiv:1304.7583]

A. Chamseddine, Alain Connes, WvS. Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

WvS. The spectral model of particle physics, *Nieuw Archief voor Wiskunde*, June 2014.

WvS. *Noncommutative Geometry and Particle Physics*. Mathematical Physics Studies, Springer, July 2014.

and also: `http://www.noncommutativegeometry.nl`

Spectral geometry

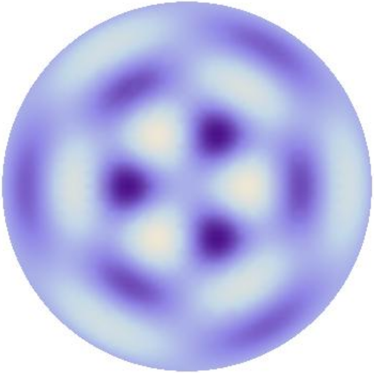
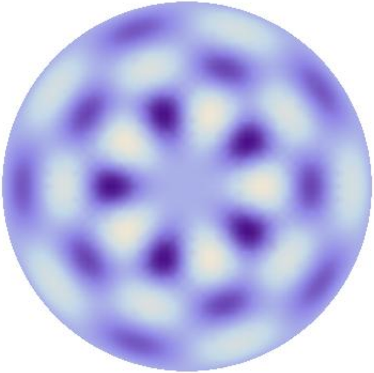
"Can one hear the shape of a drum?" (Kac, 1966)

Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

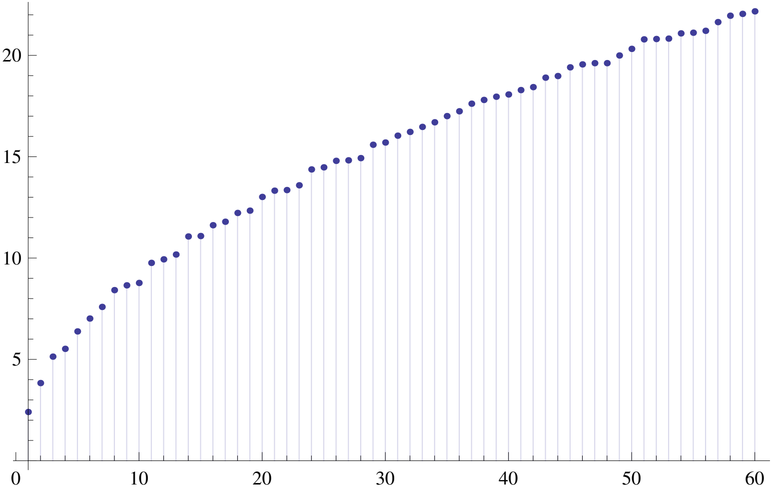
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

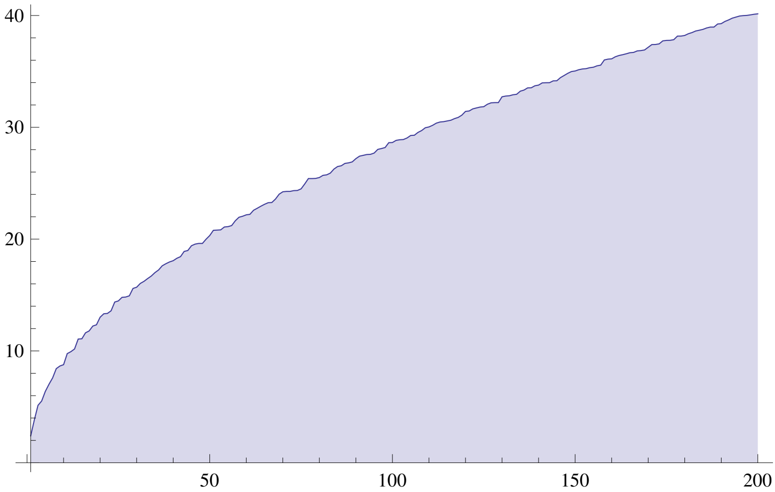
The disc

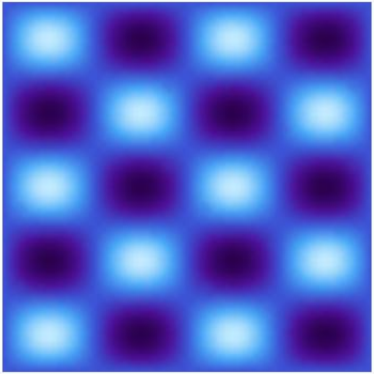
Wave numbers on the disc

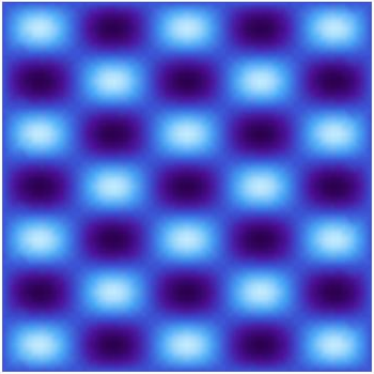


Wave numbers on the disc: high frequencies

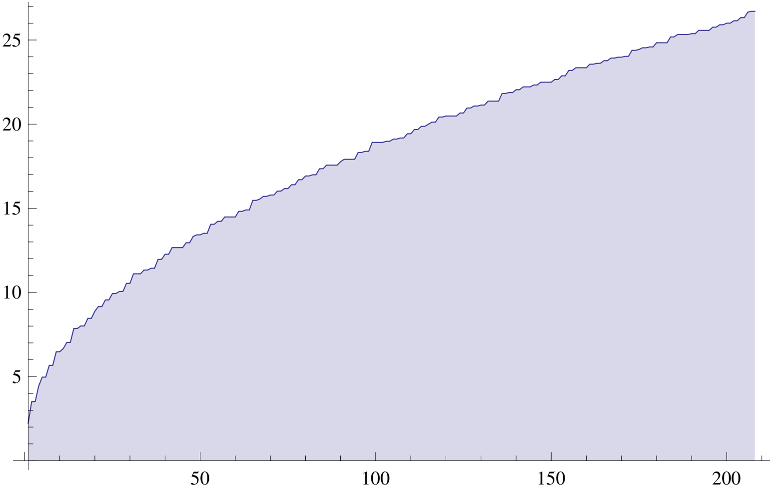


The square



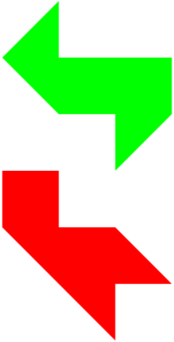


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

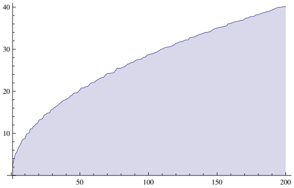
so the answer to Kac's question is no.

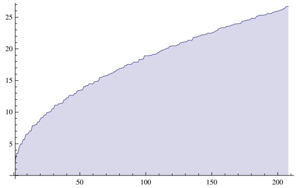
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension n of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_n \mathrm{Vol}(M)}{n(2\pi)^n}\Lambda^n$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):





Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.

The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

  with square $\Delta_{\mathbb{S}^1}$.

- The eigenfunctions of $D_{\mathbb{S}^1}$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt$$

  with eigenvalue $n \in \mathbb{Z}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads
$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$
- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:
$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}\right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

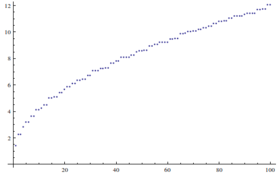
- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



The 4-dimensional torus

- Consider the 4-torus $\mathbb{T}^4$ parametrized by t_1, t_2, t_3, t_4 and the Laplacian is

$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

- The Dirac operator on $\mathbb{T}^4$ is

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$

- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

Spectral action functional

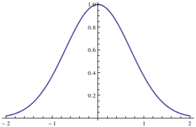
- Reconsider Weyl's estimate, in a smooth version:

$$\mathrm{Tr}\ \chi\left(\frac{D_M}{\Lambda}\right) = \sum_\lambda \chi\left(\frac{\lambda}{\Lambda}\right)$$

  for a smooth cutoff function $\chi : \mathbb{R} \to \mathbb{R}$.

- For simplicity, restrict to a Gaussian function

$$\chi(x) = e^{-x^2}$$

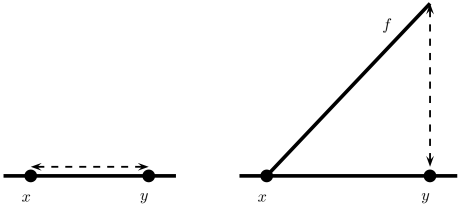


  so that we can use heat asymptotics: $\mathrm{Tr}\ e^{-D_M^2/\Lambda^2} \sim \frac{\mathrm{Vol}(M)\Lambda^n}{(4\pi)^{n/2}}$

Hearing the shape of a drum

- As said, the geometry of M is not fully determined by spectrum of D_M.
- This can be improved by considering besides D_M also the algebra $C^\infty(M)$ of smooth functions on M, with pointwise product and addition
- In fact, the distance function on M is equal to

$$d(x,y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \text{ gradient } f \leq 1 \}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$. For example, on the circle we have $[D_{\mathbb{S}^1}, f] = -i \frac{df}{dt}$

Finite spaces

- Finite space F, discrete topology

$$F = \quad {}_1\bullet \qquad {}_2\bullet \qquad \cdots\cdots \qquad {}_N\bullet$$

- Smooth functions on F are given by N-tuples in $\mathbb{C}^N$, and the corresponding algebra $C^\infty(F)$ corresponds to diagonal matrices

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix}$$

- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(p,q) = \sup_{f \in C^\infty(F)} \{ |f(p) - f(q)| : \|[D_F, f]\| \leq 1 \}$$

Example: two-point space

$$F = {}_1\bullet \qquad {}_2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(p,q) = \begin{cases} |c|^{-1} & p \neq q \\ 0 & p = q \end{cases}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F.

- Instead of diagonal matrices, we consider block diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \dots a_N$ are square matrices of size $n_1, n_2, \dots, n_N$.

- Hence we will consider the matrix algebra

$$\mathcal{A}_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- A finite Dirac operator is still given by a hermitian matrix.

Example: **noncommutative** two-point space

The two-point space can be given a noncommutative structure by considering the algebra $\mathcal{A}_F$ of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Spectral triples

Noncommutative Riemannian spin manifolds

More generally, we consider

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by a unital associative $*$-algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D in $\mathcal{H}$ such that the resolvent $(i + D)^{-1}$ is a compact operator and $[D, a]$ is bounded for each $a \in \mathcal{A}$.

- Spectral action functional:

$$\mathrm{Tr}\ \chi(D/\Lambda)$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto uDu^*$$

Semigroup of inner perturbations

Extend this to more general perturbations:

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$$

- Spectral action functional:

$$\mathrm{Tr}\ \chi(D+\omega) = \sum_{n=0}^{\infty} \frac{1}{2\pi i n} \mathrm{Tr}\ \oint \chi'(z) \omega (z-D)^{-1} \cdots \omega (z-D)^{-1}$$

Perturbation semigroup for matrix algebras

Proposition

Let $\mathcal{A}_F$ be the algebra of block diagonal matrices (fixed size). Then the perturbation semigroup of $\mathcal{A}_F$ is

$$\mathrm{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\}$$

The semigroup law in $\mathrm{Pert}(\mathcal{A}_F)$ is given by the matrix product in $\mathcal{A}_F \otimes \mathcal{A}_F$:

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \qquad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

are called normalization and self-adjointness condition, respectively.

- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j \right) \left(\sum_k A'_k \otimes B'_k \right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $\mathcal{A}_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

  leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.
- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a noncommutative example, $\mathcal{A}_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C}))$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \dots x_9 \in \mathbb{R} \end{array} \right\}$$

  and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

- More generally (B.Sc. thesis Niels Neumann),

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

Example: noncommutative two-point space

- Consider noncommutative two-point space described by $\mathbb{C} \oplus M_2(\mathbb{C})$
- It turns out that

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Physicists call ϕ_1 and ϕ_2 the Higgs field.
- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Example: perturbation semigroup of a manifold

Recall, for any involutive algebra $\mathcal{A}$

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- We can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two variables, *i.e.* elements in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\}$$

- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

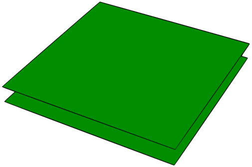
$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x} =: \partial_\mu + A_\mu$$

Applications to particle physics

- Combine (4d) Riemannian spin manifold M with finite noncommutative space F:

$$M \times F$$

- F is internal space at each point of M



- Described by matrix-valued functions on M: algebra $C^\infty(M, \mathcal{A}_F)$

Dirac operator on $M \times F$

- Recall the form of D_M:

$$D_M = \begin{pmatrix} 0 & D_M^+ \\ D_M^- & 0 \end{pmatrix}.$$

- Dirac operator on $M \times F$ is the combination

$$D_{M\times F} = \begin{pmatrix} D_F & D_M^+ \\ D_M^- & -D_F \end{pmatrix}.$$

- The crucial property of this specific form is that it squares to the sum of the two Laplacians on M and F:

$$D_{M\times F}^2 = D_M^2 + D_F^2$$

- Using this, we can expand:

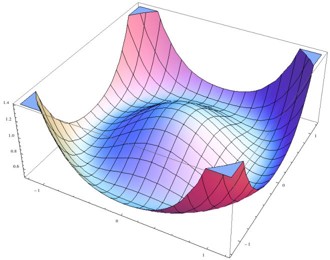
$$\mathrm{Tr}\; e^{-D_{M\times F}^2/\Lambda^2} = \frac{\mathrm{Vol}(M)\Lambda^4}{(4\pi)^2}\mathrm{Tr}\;\left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4}\right) + \mathcal{O}(\Lambda^{-1}).$$

The Higgs mechanism

We apply this to the noncommutative two-point space described before

- Algebra $\mathcal{A}_F = \mathbb{C} \oplus M_2(\mathbb{C})$
- Perturbation of Dirac operator D_F parametrized by ϕ_1, ϕ_2.
- Potential for the perturbed Dirac operator is

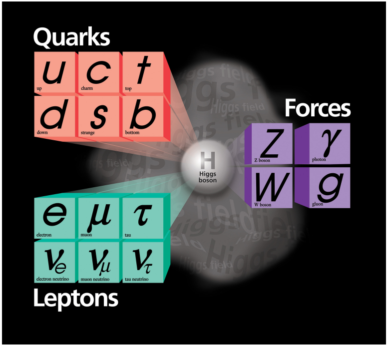
$$V(\phi) = -2\Lambda^2(|\phi_1|^2 + |\phi_2|^2) + (|\phi_1|^2 + |\phi_2|^2)^2$$



- Minimum breaks symmetry spontaneously, giving mass to Higgs boson (125.5 GeV, corresponding to $10^{-18}m$).

The spectral Standard Model

- The full Standard Model is based on the algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
- The finite Dirac operator is given by a 96×96-dimensional hermitian matrix, containing masses for the leptons and quarks.
- This allows for a derivation of the particle content of the Standard Model from pure geometry



- The spectral action functional describes their dynamics and interactions

Summary

- Noncommutative geometry allows for a derivation of the Standard Model of particle physics
- Perturbations of D form a semigroup, unitary elements form a subgroup of this semigroup.
- Matrix algebras give rise to matrix semigroups, perturbing the Dirac operator to yield physical fields (electromagnetic, Higgs, etc.).
- Spectral action functional gives dynamics and interactions.
- At the classical level... still awaiting a rigorous quantization