Gauge theories and noncommutative manifolds

(work in progress)

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NCG and gauge theories

Goal:

Understand the nature of gauge theories appearing in noncommutative geometry

Or, how does a noncommutative manifold give rise to a gauge theory?
(At least) two possibilities:

1. Noncommutative algebra A describes (virtual) noncommutative background manifold on which to do gauge theory
2. Noncommutative algebra A describes gauge theory on a background manifold:

   "noncommutative $*$-algebras have non-trivial inner automorphisms"

which seem incompatible...

Examples of first (gauge theory on nc space)

- Yang–Mills theory on noncommutative torus: moduli space of minima [Connes–Rieffel 1987]
- Instantons on noncommutative 4-manifolds [Nekrasov–Schwarz 1999, ..., Brain–Landi–vS]
- Noncommutative field theories in physics [DFR 1995, SW 1999, ..., Grosse, Wulkenhaar, Steinacker, ...]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}$$

Mathematical setup:

- Module M over $*$-algebra A
- Connections $\nabla : M \to M \otimes_A \Omega^1(A)$ (gauge fields)
- Group $\mathcal{U}(\mathrm{End}_A(M))$ of unitary endomorphisms (gauge group)

N.B.: In general, $\mathcal{U}(\mathrm{End}_A(M))$ does not contain $\mathcal{U}(A)$.

Examples of second (internal gauge theory)

- Noncommutative manifold describing (ordinary) Yang–Mills theory [Chamseddine–Connes 1996, Boeijink–vS 2009]

$$\mathrm{Inn}(M_N(\mathbb{C})) \simeq \mathrm{PSU}(N)$$

- Noncommutative description of Standard Model of elementary particles [Chamseddine–Connes–Marcolli 2006]

$$\mathrm{Inn}(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) \simeq G_{SM}$$

Mathematical setup:

- K-cycles (representative of K-homology class) aka spectral triple on A
- 'Inner fluctuations' of K-cycle (gauge fields)
- Group $\mathrm{Inn}(A)$ of inner automorphisms (gauge group)

Propose a common framework

- Algebra A and identify its center $A_0 \simeq C(X)$; X is the background manifold.
- Find Hilbert bundle V over X
- A acts on $\Gamma(V)$ by $*$-endomorphisms (fiberwise)

$$\Longrightarrow \qquad \boxed{\mathcal{U}(A) \hookrightarrow \mathcal{U}(\Gamma \operatorname{End} V)}$$

and similarly for (spin, Riemannian) differential structure given by K-cycle.
Naturally, formulated in KK-theory [Kasparov, ... , BJ, Kucerovsky, Mesland]
Evidence:

- Example of Yang–Mills [Boeijink-vS 2009], Standard Model
- Need for 'inner gauge transformations' in search for instanton moduli space on nc spaces [Brain–Landi–vS]
- Emergent geometry from matrix models [Steinacker et al.]
- Explicit examples $\mathbb{T}^2_\theta$, $\mathbb{T}^2 \times \mathbb{T}^2_\theta$, $\mathbb{S}^3_\theta$

Noncommutative manifolds

Basic device: a spectral triple, or, equivalently, an unbounded K-cycle $(\mathcal{H}, D)$ for a (unital) C^*-algebra A ($\|a^*a\| = \|a\|^2$):

- algebra A of bounded operators on
- a Hilbert space $\mathcal{H}$,
- a self-adjoint operator D with compact resolvent and such that the commutator $[D, a]$ is bounded for all $a \in \mathcal{A} \subset A$ (dense subalgebra).

Example

Let M be a compact Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M) \subset C(M) = A$
- $\mathcal{H} = L^2(\mathcal{S})$, square integrable spinors
- $D = \partial\!\!\!/$, Dirac operator

Then D has compact resolvent because $\partial\!\!\!/$ elliptic self-adjoint.
Also $[D, f]$ bounded for $f \in C^\infty(M)$ with $\|[D, f]\| = \|f\|_{\mathrm{Lip}}$.

Noncommutative vector bundles

Consider a C^*-algebra B. Then, a Hilbert B-module is a right B-module E with a sesquilinear B-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to B$ such that

$$\langle \eta, \xi b \rangle = \langle \eta, \xi \rangle b$$

and $\langle \eta, \eta \rangle \geq 0$ with equality if and only if $\eta = 0$; and E is complete in the norm $\eta \to \| \langle \eta, \eta \rangle \|_B^{1/2}$

Example

If $B = \mathbb{C}$ then E is a Hilbert space.

Example

B is itself a B-module, with $\langle b, b' \rangle = b^* b'$.

Example

$B = C(M)$ for a Hausdorff topological space M. Then every Hilbert module is isomorphic to the sections $\Gamma(M, V)$ of a Hilbert bundle V [Takahashi].

KK-theory

These two extremes of (abstract) elliptic 1st-order differential operators and (noncommutative) ‘Hilbert bundles’ are combined in (unbounded) KK-theory.

Definition

Let A and B be C^*-algebras. An (odd) unbounded KK-cycle is a triple (E, D) where

- *E is a Hilbert-B-module, with $\langle \cdot, \cdot \rangle : E \times E \to B$,*
- *$A \subset \mathrm{End}_B(E)$, and*
- *D a self-adjoint regular operator in E such that $(1 + D^2)^{-1}$ extends to an element of $\mathrm{End}^0_B(E)$ for all $a \in A$,*

and the set $\mathcal{A}$ of all $a \in A$ s.t. $[D, a] \in \mathrm{End}_B(E)$ is dense in A.

- The set of all unbounded KK-cycles is denoted by $\Psi(A, B)$; if A is separable, there is a surjective map $\Psi(A, B) \to KK(A, B)$.
- A spectral triple ${}_A(\mathcal{H}, D)$ is an element in $\Psi(A, \mathbb{C})$ (K-homology).
- A Hilbert B-module E is the element $(B, 0)$ in $\Psi(\mathbb{C}, B)$ (K-theory).

Examples

So, with C^*-algebras A, B, we look for

$$A \to E \rightleftharpoons B$$

and a B-linear operator D (plus conditions).

Example ($\Psi(A, B)$)

A is Morita equivalent to B iff $A \simeq \mathrm{End}_B(E)$ for a finitely generated right B module. In this case, E is a Hilbert B-module on which A acts by bounded endomorphisms. Thus, $(E, 0)$ defines a KK-cycle in $\Psi(A, B)$.

Example ($\Psi(C(M), C(Y))$)

Families of (spin) manifolds indexed by a compact space: fibration $M \to Y$ with fibers manifolds.

Morita equivalence

Suppose $A \sim_M B$. Can we construct a spectral triple on B from ${}_A(\mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$.
- Definition of operator $D'(\eta, \psi) := \nabla(\eta)\psi + \eta \otimes D\psi$ requires a (compatible) connection on $\mathcal{E}$:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$$

  w.r.t. the derivation $d := [D, .]$ and Connes' differential one-forms are

$$\Omega^1_D(\mathcal{A}) = \left\{ \sum_j a_j [D, b_j] : a_j, b_j \in \mathcal{A} \right\}$$

- Then ${}_B(\mathcal{H}', D')$ is a spectral triple [Connes, 1996]. In fact, this is a manifestation of the unbounded internal Kasparov product:

$$\Psi(B, A) \times \Psi(A, \mathbb{C}) \to \Psi(B, \mathbb{C})$$

  with $(\mathcal{E}, 0)$ representing a class in $\Psi(B, A)$.

Morita self-equivalences as gauge fields

- In the case $\mathcal{B} = \mathcal{A}$ and $\mathcal{E} = \mathcal{A}$ we have of course $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \simeq \mathcal{H}$.
- However, the operator D is perturbed to $D_\omega := D + \omega$ with $\omega^* = \omega \in \Omega^1_D(\mathcal{A})$ the connection one-form (gauge field) in $\nabla = d + \omega$. These are the so-called inner fluctuations.
- The (gauge) group $\mathcal{U}(A)$ of unitary elements in A acts on $\mathcal{H}$.
- This induces an action of $\mathcal{U}(\mathcal{A})$ on the connection one-form A, since $D_\omega \mapsto u D_\omega u^*$ implies

$$\omega \mapsto u \omega u^* + u[D, u^*]$$

- Spectral action principle:

$$\mathrm{Tr}\, f(D_\omega/\Lambda)$$

Example: $PSU(N)$ Yang–Mills theory

Conventionally described by a $PSU(N)$-principal bundle $P \to M$ with a connection; the gauge group is $\Gamma(\mathrm{Ad}P) = \Gamma(P \times_{PSU(N)} PSU(N))$. Physics described by Yang–Mills action

$$\int_M \mathrm{Tr}_{\mathfrak{su}(N)} F \wedge *F.$$

The noncommutative approach:

Starting with a Riemannian spin manifold $(L^2(M, \mathcal{S}), \partial\!\!\!/) \in \Psi(C(M), \mathbb{C})$, we generate a spectral triple (+reality) for a Morita equivalent algebra:

$$(L^2((\mathrm{End}\, V) \otimes \mathcal{S}), 1 \otimes \partial\!\!\!/ + \nabla \otimes 1) \in \Psi(\Gamma(\mathrm{End}(V), \mathbb{C})$$

with $V \to M$ a vector bundle equipped with a connection ∇.

Proposition (Chamseddine-Connes (1996), Boeijink-vS (2009))

Given $(L^2((\text{End } V) \otimes \mathcal{S}), 1 \otimes \partial\!\!\!/ + \nabla \otimes 1) \in \Psi(\Gamma(\text{End}(V), \mathbb{C})$ *as above. Then:*

- *There exists a PSU(N)-p.b. P such that* $P \times_{PSU(n)} M_n(\mathbb{C}) = \text{End } V$.
- *The inner fluctuations* ω *of this spectral triple correspond 1-1 to connections on P.*
- *The gauge group* $\text{Inn}(A) \simeq \Gamma(\text{Ad}P)$
- *The spectral action is*

$$\text{Tr}\, f(D_\omega/\Lambda) - \text{Tr}\, f(D/\Lambda) = \int_M \text{Tr}_{\mathfrak{su}(N)} F_\omega \wedge *F_\omega + \mathcal{O}(\Lambda^{-1})$$

Conversely, every PSU(N)-gauge theory on M can be obtained in this way.

Relation to KK-theory

The ingredients for the above construction are:

$$(\Gamma(\mathrm{End}\, V), 0) \in \Psi(\Gamma(\mathrm{End}\, V), C(M))$$
$$(L^2(M, S), \partial\!\!\!/) \in \Psi(C(M), \mathbb{C})$$

so that the spectral triple

$$(L^2((\mathrm{End}\, V) \otimes \mathcal{S}), \partial\!\!\!/_V) \in \Psi(\Gamma(\mathrm{End}\, V), \mathbb{C})$$

is obtained as their (unbounded) internal Kasparov product:

$$\Psi(\Gamma(\mathrm{End}\, V), C(M)) \times \Psi(C(M), \mathbb{C}) \to \Psi(\Gamma(\mathrm{End}\, V), \mathbb{C}).$$

Moreover, the inner fluctuations of $(L^2((\mathrm{End}\, V) \otimes \mathcal{S}), \partial\!\!\!/_V)$ correspond to connections on End V, entering in the Kasparov internal product.

Factorizing spectral triples (work in progress)

Main question: **what are the conditions on ${}_A(\mathcal{H}, D)$ to factorize as**

$$ {}_A(\mathcal{H}, D) \simeq {}_A(E, T) \otimes_{A_0} (\mathcal{H}_0, D_0) $$

with A_0 (contained in) the center of A, and

- E is a Hilbert A_0-module such that $\mathcal{H} \simeq E \hat{\otimes}_{A_0} \mathcal{H}_0$
- There is an action of $a \in A$ on E, such that $a \sim a \otimes \mathrm{id}$.
- $D \sim T \otimes 1 + \nabla \otimes 1 + 1 \otimes D_0$ with ∇ a connection on E.

In other words, we look for conditions on when the following Kasparov product

$$ \Psi(A, A_0) \times \Psi(A_0, \mathbb{C}) \to \Psi(A, \mathbb{C}). $$

has an inverse.

If this holds, then ${}_{A_0}(\mathcal{H}_0, D_0)$ describes a background spin manifold M [Connes, 2008], and E consists of sections of a Hilbert bundle V on M [Takahashi, 1979]. Moreover

- The inner fluctuations of $(\mathcal{H}, \pi, D)$ are parametrized by connections on V and elements in $\Gamma(\mathrm{End}\, V)$.
- The gauge group $\mathcal{U}(A)$ acts on E by unitary endomorphisms.

Examples: models in particle physics

Consider for a finite dimensional algebra A_F, acting on a vector space $\mathcal{H}_F$, and a linear operator D_F on $\mathcal{H}_F$:

$$(L^2((M \times \mathcal{H}_F) \otimes \mathcal{S}), D_F \otimes 1 + 1 \otimes \partial\!\!\!/) \in \Psi(C(M, A_F), \mathbb{C})$$

It factorizes in terms of

$$(L^2(M, \mathcal{H}_F), D_F) \in \Psi(C(M, \mathcal{A}_F), C(M))$$
$$(L^2(M, \mathcal{S}), \partial\!\!\!/) \in (\Psi(C(M), \mathbb{C})$$

Examples:

- Yang–Mills theory
- Full Standard model including Higgs mechanism [CCM]

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad \mathcal{H}_F = \mathbb{C}^{96}, \qquad D_F = \ldots$$

- Supersymmetric models [van den Broek], eg.

$$A_F = M_3(\mathbb{C}), \qquad \mathcal{H}_F = \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \mathbb{C}^3, \qquad D_F = \ldots$$

- ...

Example: Noncommutative torus

Explicitly, A_θ (aka $C(\mathbb{T}^2_\theta)$) is the C^*-algebra generated by U, V, U^*, V^* such that

$$U^*U = 1 = UU^*; \qquad V^*V = 1 = VV^*; \qquad UV = e^{2\pi i\theta} VU.$$

With θ irrational, the center of A_θ is trivial, $A_{\theta,0} \simeq \mathbb{C}$.

- The natural automorphic action of $\mathbb{T}^2$ on A_θ allows to define the Hilbert space $\mathcal{H}_\theta$ of square-integrable (over $\mathbb{T}^2$) 'functions' on $\mathbb{T}^2_\theta$.
- With infinitesimal generators δ_1 and δ_2 for this action of $\mathbb{T}^2$, we define a 'Dirac operator' on $\mathcal{H}_\theta \otimes \mathbb{C}^2$ by

$$\partial\!\!\!/ = \sum_k i\gamma^k \delta_k = \begin{pmatrix} 0 & \delta_1 - i\delta_2 \\ -\delta_1 - i\delta_2 & 0 \end{pmatrix}$$

Then:

- $(\mathcal{H}_\theta \otimes \mathbb{C}^2, \partial\!\!\!/)$ defines an element in $\Psi(A_\theta, \mathbb{C})$, which factorizes trivially with background manifold **a point** (i.e. $(\mathbb{C}, 0) \in \Psi(\mathbb{C}, \mathbb{C})$).
- The Hilbert bundle consists of the single fiber $\mathcal{H}_\theta \otimes \mathbb{C}^2$ on which A_θ acts by bounded operators.

Example: Torus $\times$ noncommutative torus

Consider the algebra $C(\mathbb{T}^2, A_\theta) \simeq C(\mathbb{T}^2) \otimes A_\theta$, acting on the Hilbert space $L^2(\mathbb{T}^2, \mathcal{H}_\theta) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Then, together with the operator

$$\not{\partial}\psi(x) = (\gamma^k \otimes 1)(\partial_k \psi)(x) + (\gamma \otimes i\gamma^k)\delta_k(\psi(x))$$

with $\gamma = i\gamma^1\gamma^2$ this forms a spectral triple on $C(\mathbb{T}^2, A_\theta)$ (deforming the spin geometry of $\mathbb{T}^4$).

Proposition

- *$C(\mathbb{T}^2, \mathcal{H}_\theta)$ is a Hilbert module over $C(\mathbb{T}^2)$.*
- *The C^*-algebra $C(\mathbb{T}^2, A_\theta)$ acts by bounded endomorphisms on $C(\mathbb{T}^2, \mathcal{H}_\theta)$*
- *The Hilbert space $L^2(\mathbb{T}^2, \mathcal{H}_\theta)$ is isomorphic to $C(\mathbb{T}^2, \mathcal{H}_\theta) \otimes_{C(\mathbb{T}^2)} L^2(\mathbb{T}^2)$.*

Moreover, with the above Dirac operator, we have a factorization:

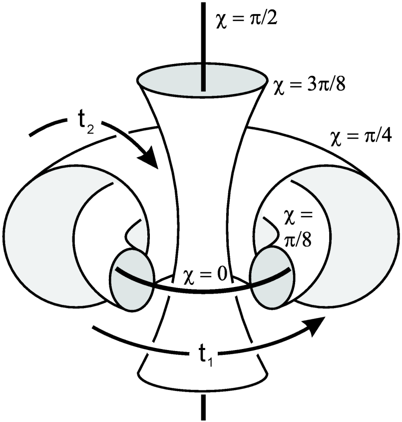
$$(L^2(\mathbb{T}^2, \mathcal{H}_\theta) \otimes \mathbb{C}^4, \not{\partial}) \simeq (C(\mathbb{T}^2, \mathcal{H}_\theta \otimes \mathbb{C}^2), i\gamma^k\delta_k) \otimes_{C(\mathbb{T}^2)} (L^2(\mathbb{T}^2) \otimes \mathbb{C}^2, i\gamma^k\partial_k).$$

The noncommutative 3-sphere

Consider the three-sphere $\mathbb{S}^3$ parametrized by toroidal coordinates

$$\alpha = e^{it_1} \cos \chi; \qquad \beta = e^{it_2} \sin \chi$$

where $0 \leq t_i < 2\pi$ parametrize a 2-torus and $0 \leq \chi \leq \pi/2$.



We will deform the algebra $C(\mathbb{S}^3)$ along this torus action, by 'inserting' the noncommutative torus A_θ:

$$C(\mathbb{S}^3_\theta) := C(\mathbb{S}^3 \times_{\mathbb{T}^2} A_\theta)$$

Isospectral deformation of $\mathbb{S}^3$

More precisely,

$$C(\mathbb{S}^3_\theta) = C(\mathbb{S}^3 \times_{\mathbb{T}^2} A_\theta) := \left\{ f \in C(\mathbb{S}^3, A_\theta) : f(t \cdot x) = t \cdot f(x) \forall t \in \mathbb{T}^2 \right\}$$

which is generated as a C^*-algebra by α, β with relations

$$\alpha \times_\theta \beta = e^{2\pi i \theta} \beta \times_\theta \alpha; \quad \alpha \times_\theta \beta^* = e^{-2\pi i \theta} \beta^* \times_\theta \alpha; \qquad \alpha \times_\theta \alpha^* + \beta \times_\theta \beta^* = 1.$$

Proposition (Connes–Landi, 2000)

Let $(L^2(\mathbb{S}^3) \otimes \mathbb{C}^2, \not{\partial}) \in \Psi(C(\mathbb{S}^3, \mathbb{C})$ *be the* ($\mathbb{T}^2$*-equivariant) spin geometry on* $\mathbb{S}^3$ *for the round metric.*
Then $(L^2(\mathbb{S}^3 \times_{\mathbb{T}^2} \mathcal{H}_\theta) \otimes \mathbb{C}^2, \not{\partial})$ *defines an element in* $\Psi(C(\mathbb{S}^3_\theta), \mathbb{C})$.

Q: Does this spectral triple factorize?
Note that the center of $C(\mathbb{S}^3_\theta)$ consists of $\mathbb{T}^2$-invariant elements, and is generated by $\alpha\alpha^*$ and $\beta\beta^*$, *i.e.* $C(\mathbb{S}^3_\theta)_0 \simeq C[0, \pi/2]$.

Proposition (vS)

Let $\Gamma(\mathbb{S}^3_\theta) := C(\mathbb{S}^3 \times_{\mathbb{T}^2} \mathcal{H}_\theta)$. *Then,*

- $\Gamma(\mathbb{S}^3_\theta)$ *is a Hilbert module over* $C(\mathbb{S}^3_\theta)^{\mathbb{T}^2} \simeq C[0,\pi/2]$.
- *The* C^**-algebra* $C(\mathbb{S}^3_\theta)$ *acts by bounded endomorphisms on* $\Gamma(\mathbb{S}^3_\theta)$.
- *The Hilbert space* $L^2(\mathbb{S}^3_\theta)$ *is isomorphic to* $\Gamma(\mathbb{S}^3_\theta)\widehat{\otimes}_{C[0,\pi/2]}L^2[0,\pi/2]$.

Consider the Dirac operator $\not{\partial}$ on $L^2(\mathbb{S}^3) \otimes \mathbb{C}^2$, which we can write explicitly

$$\not{\partial} = \cos\chi^{-1}\gamma^1\partial_{t_1} + \sin\chi^{-1}\gamma^2\partial_{t_2} + \gamma^3(\partial_\chi + f(\chi))$$

On the Hilbert bundle $\mathbb{S}^3 \times_{\mathbb{T}^2} \mathcal{H}_\theta \otimes \mathbb{C}^2$, we introduce the fiberwise operator

$$D = \cos\chi^{-1}\gamma^1\delta_1 + \sin\chi^{-1}\gamma^2\delta_2$$

Conjecture

The noncommutative spin geometry on $\mathbb{S}^3_\theta$ *factorizes:*

$$(L^2(\mathbb{S}^3 \times_{\mathbb{T}^2} \mathcal{H}_\theta) \otimes \mathbb{C}^2, \not{\partial}) \simeq (\Gamma(\mathbb{S}^3_\theta), D) \otimes_{C[0,\pi/2]} (L^2[0,\pi/2], \partial_\chi).$$

i.e. $\not{\partial} \sim D \otimes 1 + \gamma^3 \otimes_\nabla \partial_\chi$

Outlook

- Noncommutative geometry is capable of describing (geometrically!) gauge theories such as Yang–Mills theory, the Standard Model incl. Higgs, SUSY, GUTs, *et cetera*, all coupled to gravity.
- However, this is at the level of classical Lagrangians.
- Ultimate goal is to (nc geometrically) understand quantization of these theories: the above factorization of noncommutative manifolds is a first step: factorizing in a gauge part and a background manifold part might allow for a (background dependent) quantization of the gauge theory.
- Other applications are in the study of instanton moduli spaces on nc spaces, where it turns out that $\mathrm{Inn}(A)$ is required as part of the gauge group [Brain–Landi–vS].