

Inner perturbations in noncommutative geometry

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Overview

- Motivation: NCG and HEP
- Noncommutative Riemannian spin manifolds (aka spectral triples)
- Gauge theory from spectral triples: gauge group, semi-group of inner perturbations
- Examples: Yang–Mills, SM, Beyond SM

References

A. Chamseddine, Alain Connes, WvS. Inner Fluctuations in Noncommutative Geometry without the first order condition. *J. Geom. Phys.* 73 (2013) 222-234 [arXiv:1304.7583]

A. Chamseddine, Alain Connes, WvS. Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

WvS. The spectral model of particle physics, *Nieuw Archief voor Wiskunde*, September 2014.

WvS. *Noncommutative Geometry and Particle Physics*. Mathematical Physics Studies, Springer, summer 2014.

and also: <http://www.noncommutativegeometry.nl>

A fermion in a spacetime background

- Spacetime is a (pseudo) Riemannian manifold M :

local coordinates x_μ generate algebra $C^\infty(M)$.

- Propagator is described by Dirac operator D_M .

- Combined they capture the metric:

$$d(x, y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$

NCG and HEP

Replace spacetime by **spacetime** \times **finite (nc) space**: $M \times F$

- F is considered as **internal space** (Kaluza–Klein like)
- F is described by a **noncommutative algebra**, such as $M_3(\mathbb{C})$, just as spacetime is described by **coordinate functions** $x_\mu(p)$.
- ‘Propagation’ of particles in F is described by a **Dirac-type operator** D_F which is actually simply a hermitian matrix.

Finite spaces

- Finite space F , discrete topology

$$F = \quad 1 \bullet \quad 2 \bullet \quad \cdots \cdots \quad N \bullet$$

- Smooth functions on F are given by N -tuples in \mathbb{C}^N , and the corresponding algebra $C^\infty(F)$ corresponds to **diagonal matrices**

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & f(N) \end{pmatrix}$$

- The **finite Dirac operator** is an arbitrary hermitian matrix D_F , giving rise to a distance function on F as

$$d(p, q) = \sup_{f \in C^\infty(F)} \{|f(p) - f(q)| : \|[D_F, f]\| \leq 1\}$$

Example: two-point space

$$F = \{1, 2\}$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \bar{c} \\ c & 0 \end{pmatrix}; \quad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(p, q) = \begin{cases} |c|^{-1} & p \neq q \\ 0 & p = q \end{cases}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F .

- Instead of diagonal matrices, we consider **block diagonal** matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_N \end{pmatrix},$$

where the a_1, a_2, \dots, a_N are square matrices of size n_1, n_2, \dots, n_N .

- Hence we will consider the **matrix algebra**

$$\mathcal{A}_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- A **finite Dirac operator** is still given by a hermitian matrix.

Example: **noncommutative** two-point space

The two-point space can be given a noncommutative structure by considering the **algebra** \mathcal{A}_F of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A **finite Dirac operator** for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \bar{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Spectral triples

Noncommutative Riemannian spin manifolds

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extended to **real** spectral triple:
 - $J: \mathcal{H} \rightarrow \mathcal{H}$ real structure (anti-unitary)

such that

$$J^2 = \pm 1; \quad JD = \pm DJ$$

- **Action of \mathcal{A}^{op}** on \mathcal{H} : $a^{\text{op}} = Ja^*J^{-1}$ and

$$[a^{\text{op}}, b] = 0; \quad a, b \in \mathcal{A}$$

- D is said to satisfy **first-order condition** if

$$[[D, a], b^{\text{op}}] = 0$$

Spectral invariants

$$\text{Tr } f(D/\Lambda) + \frac{1}{2} \langle J\tilde{\psi}, D\tilde{\psi} \rangle$$

- **Invariant** under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \quad U = uJuJ^{-1}$$

- **Gauge group**: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$.
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and **blue** term vanishes if D satisfies **first-order** condition

Semi-group of inner perturbations

$$\text{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\text{op}} \in \mathcal{A} \otimes \mathcal{A}^{\text{op}} \mid \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\text{op}} = \sum_j b_j^* \otimes a_j^{\text{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\text{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\text{op}}$.
- $\text{Pert}(\mathcal{A})$ acts on D :

$$D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$$

- For **real** spectral triples we use the map $\text{Pert}(\mathcal{A}) \rightarrow \text{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ sending $A \mapsto A \otimes \hat{A}$ so that

$$D \mapsto \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j$$

Perturbation semigroup for matrix algebras

Proposition

Let \mathcal{A}_F be the algebra of block diagonal matrices (fixed size). Then the *perturbation semigroup of \mathcal{A}_F* is

$$\text{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \mid \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \bar{B}_j \otimes \bar{A}_j \end{array} \right\}$$

The semigroup law in $\text{Pert}(\mathcal{A}_F)$ is given by the matrix product in $\mathcal{A}_F \otimes \mathcal{A}_F$:

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \quad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

are called **normalization** and **self-adjointness condition**, respectively.

- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j \right) \left(\sum_k A'_k \otimes B'_k \right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $\mathcal{A}_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we can write an arbitrary element of $\text{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

leaving only one free complex parameter so that $\text{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.

- More generally, $\text{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a **noncommutative example**, $\mathcal{A}_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\text{Pert}(M_2(\mathbb{C}))$ are **4×4 -matrices** satisfying the **normalization** and **self-adjointness** condition. In a suitable basis:

$$\text{Pert}(M_2(\mathbb{C})) = \left\{ \left(\begin{array}{cccc} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{array} \right) \mid \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \dots, x_9 \in \mathbb{R} \end{array} \right\}$$

and one can show that

$$\text{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

- More generally (B.Sc. thesis Niels Neumann),

$$\text{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

Example: noncommutative two-point space

- Consider **noncommutative two-point space** described by $\mathbb{C} \oplus M_2(\mathbb{C})$
- It turns out that

$$\text{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \text{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \text{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F :

$$D_F = \begin{pmatrix} 0 & \bar{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \bar{c}\bar{\phi}_1 & \bar{c}\bar{\phi}_2 \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- The **group of unitary block diagonal matrices** is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \bar{\lambda}u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Example: perturbation semigroup of a manifold

Recall, for any involutive algebra \mathcal{A}

$$\text{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\text{op}} \in \mathcal{A} \otimes \mathcal{A}^{\text{op}} \mid \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\text{op}} = \sum_j b_j^* \otimes a_j^{\text{op}} \end{array} \right\}$$

- We can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two variables, *i.e.* elements in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\text{Pert}(C^\infty(M))$ translate accordingly and yield

$$\text{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \mid \begin{array}{l} f(x, x) = 1 \\ f(x, y) = \overline{f(y, x)} \end{array} \right\}$$

- The action of $\text{Pert}(C^\infty(M))$ on the partial derivatives appearing in a **Dirac operator** D_M is given by

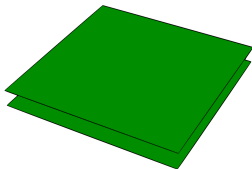
$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x, y) \Big|_{y=x} =: \partial_\mu + A_\mu$$

Applications to particle physics

- As said, combine (4d) Riemannian spin manifold M with finite noncommutative space F :

$$M \times F$$

- F is internal space at each point of M



- Described by matrix-valued functions on M : algebra $C^\infty(M, \mathcal{A}_F)$

Dirac operator on $M \times F$

- Recall that in 4d:

$$D_M = \begin{pmatrix} 0 & D_M^+ \\ D_M^- & 0 \end{pmatrix}.$$

- Dirac operator on $M \times F$ is the combination

$$D_{M \times F} = \begin{pmatrix} D_F & D_M^+ \\ D_M^- & -D_F \end{pmatrix}.$$

- The crucial property of this specific form is that it squares to the sum of the two Laplacians on M and F :

$$D_{M \times F}^2 = D_M^2 + D_F^2$$

Yang–Mills theory

On a 4-dimensional background:

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = D_M$
- $J = C \otimes (\cdot)^*$

Proposition (Chamseddine–Connes, 1996)

- $\text{Tr } f(D)$: *pure gravity (including higher-derivatives)*
- *The perturbations of D are given by hermitian $\gamma^\mu A_\mu$, describing an $\mathfrak{su}(n)$ -gauge field on M .*
- *Gauge group $\mathcal{G}(\mathcal{A}) \simeq C^\infty(M, SU(n))$*
- *The **spectral action** of perturbed Dirac operator is given by*

$$\text{Tr } f(D') \sim (\dots) + \frac{f(0)}{24\pi^2} \int_M \text{Tr } F_{\mu\nu} F^{\mu\nu}$$

Example beyond first-order

$$\mathcal{A}'_F = \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$$

$$\mathcal{H}_F = (\mathbb{C}_R \oplus \mathbb{C}_L) \otimes (\mathbb{C}^2)^\circ \oplus \mathbb{C}^2 \otimes (\mathbb{C}_R^\circ \oplus \mathbb{C}_L^\circ)$$

$$J_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \quad (C : \text{complex conjugation}),$$

$$D_F = \begin{pmatrix} 0 & \bar{c} \otimes 1_2 & \begin{matrix} \bar{d} & 0 \\ 0 & 0 \end{matrix} & 0 \\ c \otimes 1_2 & 0 & 0 & 0 \\ \begin{matrix} d & 0 \\ 0 & 0 \end{matrix} & 0 & 0 & 1_2 \otimes c \\ 0 & 0 & 1_2 \otimes \bar{c} & 0 \end{pmatrix}$$

The **algebra action** of $(\lambda_R, \lambda_L, m) \in \mathcal{A}'_F$ on \mathcal{H}_F is given explicitly by

$$\pi(\lambda_R, \lambda_L, m) = \begin{pmatrix} \lambda_R 1_2 & & & \\ & \lambda_L 1_2 & & \\ & & m & \\ & & & m \end{pmatrix}, \pi^\circ(\lambda_R, \lambda_L, m) = \begin{pmatrix} m^t & & & \\ & m^t & & \\ & & \lambda_R 1_2 & \\ & & & \lambda_L 1_2 \end{pmatrix}.$$

Proposition

The largest subalgebra $\mathcal{A}_F \subset \mathcal{A}'_F \equiv \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$ for which the first-order condition holds (for the above \mathcal{H}_F, D_F and J_F) is given by

$$\mathcal{A}_F = \left\{ \left(\lambda_R, \lambda_L, \begin{pmatrix} \lambda_R & 0 \\ 0 & \mu \end{pmatrix} \right) : (\lambda_R, \lambda_L, \mu) \in \mathbb{C}_R \oplus \mathbb{C}_L \oplus \mathbb{C} \right\}$$

Proposition

The *perturbed Dirac operator* D'_F is parametrized by *three complex scalar fields* ϕ, σ_1, σ_2 :

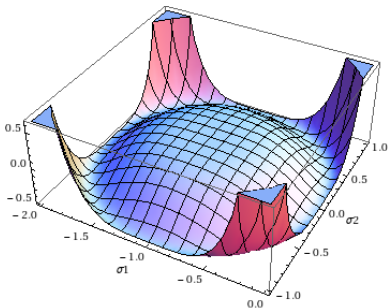
$$D'_F = \begin{pmatrix} 0 & \bar{c}\phi \otimes 1_2 & \bar{d}\bar{v} \cdot \bar{v}^t & 0 \\ c\phi \otimes 1_2 & 0 & 0 & 0 \\ dv \cdot v^t & 0 & 0 & 1_2 \otimes c\phi \\ 0 & 0 & 1_2 \otimes \bar{c}\bar{\phi} & 0 \end{pmatrix}$$

with $v = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$.

Spectral action functional

Spectral action functional gives rise to a scalar potential

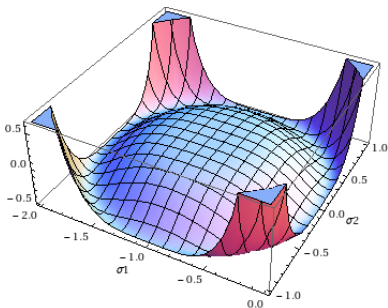
$$V(\phi, \sigma_1, \sigma_2) = -\frac{f_2}{\pi^2} \Lambda^2 (4|c|^2 |\phi|^2 + |d|^2 (|\sigma_1|^2 + |\sigma_2|^2)^2) \\ + \frac{f(0)}{4\pi^2} \left(4|c|^4 |\phi|^4 + 4|c|^2 |d|^2 |\phi|^2 (|\sigma_1|^2 + |\sigma_2|^2)^2 \right. \\ \left. + |d|^4 (|\sigma_1|^2 + |\sigma_2|^2)^4 \right)$$



Spontaneous symmetry breaking to first-order

Proposition

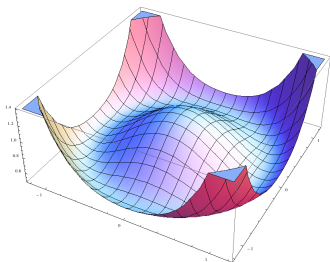
The potential $V(\phi = 0, \sigma_1, \sigma_2)$ has a local minimum at $(\sigma_1, \sigma_2) = (\sqrt{w}, 0)$ with $w = \sqrt{2f_2\Lambda^2/(f(0)|d|^2)}$ and this point spontaneously breaks the symmetry group $\mathcal{U}(\mathcal{A}'_F)$ to $\mathcal{U}(\mathcal{A}_F)$.



“Usual” SSB

After the fields (σ_1, σ_2) have reached their vevs $(\sqrt{w}, 0)$, there is a remaining potential for the ϕ -field:

$$V(\phi) = -\frac{2f_2}{\pi^2}\Lambda^2|c|^2|\phi|^2 + \frac{f(0)}{\pi^2}|c|^4|\phi|^4.$$



Selecting one of the minima of $V(\phi)$ spontaneously breaks the symmetry further from $\mathcal{U}(\mathcal{A}_F) = U(1)_R \times U(1)_L \times U(1)$ to $U(1)_L \times U(1)$, and generates mass terms for the $L - R$ abelian gauge field.

Beyond the Standard Model

One starts with the algebra

$$\mathcal{A}_{\text{PS}} := \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

and an off-diagonal Dirac operator

$$D_F := \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$$

- The largest 'first-order' subalgebra of \mathcal{A}_{PS} is $\mathbb{C} \oplus \mathbb{H}_L \oplus M_3(\mathbb{C})$.
- Symmetry breaking from Pati–Salam $SU(2)_R \times SU(2)_L \times SU(4)$ to Standard Model $U(1) \times SU(2)_L \times SU(3)$.
- Perturbation semigroup of \mathcal{A}_{PS} gives rise to many new scalar fields, including a **real scalar singlet** σ which is coupled to the Higgs sector:

$$V(\sigma, h) = -\frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2) + \frac{1}{24} \lambda_h h^4 + \frac{1}{2} h^2 \sigma^2 + \frac{1}{4} \lambda_\sigma \sigma^4$$

which allows for $m_h = 125.5\text{GeV}$ and $m_\sigma \sim 10^{12}\text{GeV}$.