Semigroup of inner perturbations in noncommutative geometry

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Overview

- Gauge theory from spectral triples
- Semi-group of inner perturbations, Morita self-equivalence
- Examples: Yang–Mills, SM, Beyond SM

References

A. Chamseddine, Alain Connes, WvS. Inner Fluctuations in Noncommutative Geometry without the first order condition. *J. Geom. Phys.* 73 (2013) 222-234 [arXiv:1304.7583]

A. Chamseddine, Alain Connes, WvS. Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

WvS. The spectral model of particle physics, *Nieuw Archief voor Wiskunde*, December 2014.

WvS. *Noncommutative Geometry and Particle Physics*. Mathematical Physics Studies, Springer, July 2014.

and also: `http://www.waltervansuijlekom.nl`

Spectral triples

Noncommutative Riemannian spin manifolds

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extended to real spectral triple:
  - $J : \mathcal{H} \to \mathcal{H}$ real structure (anti-unitary)

  such that

$$J^2 = \pm 1; \qquad JD = \pm DJ$$

- Action of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$: $a^{\mathrm{op}} = Ja^* J^{-1}$ and

$$[a^{\mathrm{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy first-order condition if

$$[[D, a], b^{\mathrm{op}}] = 0$$

Spectral invariants

$$\mathrm{Tr}\ f(D/\Lambda) + \frac{1}{2}\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \qquad U = uJuJ^{-1}$$

- Gauge group: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$.
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and blue term vanishes if D satisfies first-order condition

Semi-group of inner perturbations

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$$

- For real spectral triples we use the map $\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ sending $T \mapsto T \otimes \hat{T}$ so that

$$D \mapsto \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j$$

Relation to Morita equivalence

Suppose $\mathcal{A} \sim_M \mathcal{B}$.
Can we construct a spectral triple on $\mathcal{B}$ from $(\mathcal{A}, \mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$.
- The self-adjoint operator $(1 \otimes_\nabla D)(\eta \otimes \psi) := \nabla_D(\eta)\psi + \eta \otimes D\psi$ requires a universal connection on $\mathcal{E}$:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

  where ∇_D indicates that $a\delta(b) \in \Omega^1(\mathcal{A})$ is represented as $a[D, b]$.
- Then $(\mathcal{B}, \mathcal{H}', 1 \otimes_\nabla D)$ is a spectral triple [Connes, 1996].

Morita equivalence

with real structure

Again, suppose $\mathcal{A} \sim_M \mathcal{B}$.

- If there is a real structure J on $(\mathcal{A}, \mathcal{H}, D)$, then we define

$$\mathcal{H}' := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}) \otimes_{\mathcal{A}} \overline{\mathcal{E}}$$

with the conjugate (left $\mathcal{A}$-) module $\overline{\mathcal{E}}$ and define analogously the operator $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1$ on $\mathcal{H}'$, where

$$\overline{\nabla} : \overline{\mathcal{E}} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \overline{\mathcal{E}},$$

and we also define

$$J' : \mathcal{H}' \to \mathcal{H}', \qquad \eta \otimes \psi \otimes \overline{\rho} \mapsto \rho \otimes J\psi \otimes \overline{\eta}$$

Proposition (Chamseddine–Connes–vS, 2013)

We have $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1 = 1 \otimes_\nabla (D \otimes_{\overline{\nabla}} 1)$ and the tuple $(\mathcal{B}, \mathcal{H}', (1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1; J')$ is a real spectral triple.

Morita self-equivalence

- If $\mathcal{B} = \mathcal{A}$ and $\mathcal{E} = \mathcal{A}$ we have $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}} \simeq \mathcal{H}$ and $J' \equiv J$.
- The operator D is perturbed to $D' \equiv D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$ where

$$A_{(1)} := \sum_j a_j [D, b_j], \qquad \widetilde{A}_{(1)} := \sum_j \hat{a}_j [D, \hat{b}_j] = \pm J A_{(1)} J^{-1};$$

$$A_{(2)} := \sum_j \hat{a}_j [A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k [[D, b_k], \hat{b}_j]$$

and blue terms vanish if D satisfies first-order condition

- Gauge transformations $D' \mapsto U D' U^*$ implemented by

$$A_{(1)} \mapsto u A_{(1)} u^* + u[D, u^*]$$

$$A_{(2)} \mapsto J u J^{-1} A_{(2)} J u^* J^{-1} + J u J^{-1} [u[D, u^*], J u^* J^{-1}]$$

Perturbation semi-group and Morita self-equivalences

Proposition (Chamseddine–Connes–vS, 2013)

- *The linear map* $\eta : \mathrm{Pert}(\mathcal{A}) \to \Omega^1(\mathcal{A})$, $\eta(a \otimes b^{\mathrm{op}}) = a\delta(b)$ *is surjective.*
- *If* $\sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathrm{Pert}(\mathcal{A})$ *then the perturbed operator*

$$\sum_j a_j D b_j = D + \sum_j a_j [D, b_j] \equiv D + A_{(1)}$$

and, for real spectral triples:

$$\sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j = D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

Perturbation semigroup for matrix algebras

Proposition

Let $\mathcal{A}_F$ be the algebra of block diagonal matrices (fixed size). Then the perturbation semigroup of $\mathcal{A}_F$ is

$$\mathrm{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\}$$

The semigroup law in $\mathrm{Pert}(\mathcal{A}_F)$ is given by the matrix product in $\mathcal{A}_F \otimes \mathcal{A}_F$:

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \qquad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

are called normalization and self-adjointness condition, respectively.

- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j \right) \left(\sum_k A'_k \otimes B'_k \right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $\mathcal{A}_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.
- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a noncommutative example, $\mathcal{A}_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C}))$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{array} \right\}$$

  and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

- More generally (B.Sc. thesis Niels Neumann),

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

Example: noncommutative two-point space

- Consider noncommutative two-point space described by $\mathbb{C} \oplus M_2(\mathbb{C})$
- It turns out that

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Example: perturbation semigroup of a manifold

Recall, for any involutive algebra $\mathcal{A}$

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- We can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two variables, *i.e.* elements in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\}$$

- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x} =: \partial_\mu + A_\mu$$

Example: perturbation semigroup of an almost-commutative manifold

- For the algebra $C^\infty(M, \mathcal{A}_F)$ we have

  $$\mathrm{Pert}(C^\infty(M, \mathcal{A}_F)) \simeq C^\infty(M \times M - \Delta, \mathcal{A}_F \otimes \mathcal{A}_F^{\mathrm{op}})^{\mathrm{s.a.}} \times C^\infty(M, \mathrm{Pert}(\mathcal{A}_F))$$

  with Δ the diagonal in $M \times M$.
- If $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ then this contains

$$\begin{aligned} & C^\infty(M, \mathbb{C} \otimes \mathbb{H}) \\ \times & C^\infty(M \times M - \Delta, \mathbb{C} \otimes \mathbb{C}^{\mathrm{op}})^{\mathrm{s.a.}} \\ \times & C^\infty(M \times M - \Delta, \mathbb{H} \otimes \mathbb{H}^{\mathrm{op}})^{\mathrm{s.a.}} \\ \times & C^\infty(M \times M - \Delta, M_3(\mathbb{C}) \otimes M_3(\mathbb{C})^{\mathrm{op}})^{\mathrm{s.a.}} \end{aligned}$$

Physical applications: Yang–Mills theory

On a 4-dimensional background:

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = D_M \otimes 1$
- $J = C \otimes (.)^*$

Proposition (Chamseddine-Connes, 1996)

- $\mathrm{Tr}\ f(D)$: *pure gravity (including higher-derivatives)*
- *The perturbations of D are given by hermitian $\gamma^\mu A_\mu$, describing an $\mathfrak{su}(n)$-gauge field on M.*
- *Gauge group $\mathcal{G}(\mathcal{A}) \simeq C^\infty(M, SU(n))$*
- *The spectral action of perturbed Dirac operator is given asymptotically (as $\Lambda \to \infty$) by*

$$\mathrm{Tr}\ f(D'/\Lambda) \sim (\cdots) + \frac{f(0)}{24\pi^2} \int_M \mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu} + \cdots$$

Example beyond first-order

$$\mathcal{A}'_F = \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$$

$$\mathcal{H}_F = (\mathbb{C}_R \oplus \mathbb{C}_L) \otimes (\mathbb{C}^2)^\circ \oplus \mathbb{C}^2 \otimes (\mathbb{C}_R^\circ \oplus \mathbb{C}_L^\circ)$$

$$J_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \qquad (C : \text{complex conjugation}),$$

$$D_F = \begin{pmatrix} 0 & \overline{c} \otimes 1_2 & \begin{smallmatrix} \overline{d} & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ c \otimes 1_2 & 0 & 0 & \\ \begin{smallmatrix} d & 0 \\ 0 & 0 \end{smallmatrix} & 0 & 0 & 1_2 \otimes c \\ 0 & 0 & 1_2 \otimes \overline{c} & 0 \end{pmatrix}$$

The algebra action of $(\lambda_R, \lambda_L, m) \in \mathcal{A}'_F$ on $\mathcal{H}_F$ is given explicitly by

$$\pi(\lambda_R, \lambda_L, m) = \begin{pmatrix} \lambda_R 1_2 & & & \\ & \lambda_L 1_2 & & \\ & & m & \\ & & & m \end{pmatrix}, \pi^\circ(\lambda_R, \lambda_L, m) = \begin{pmatrix} m^t & & & \\ & m^t & & \\ & & \lambda_R 1_2 & \\ & & & \lambda_L 1_2 \end{pmatrix}.$$

Proposition

The largest subalgebra $\mathcal{A}_F \subset \mathcal{A}'_F \equiv \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$ *for which the first-order condition holds (for the above* $\mathcal{H}_F, D_F$ *and* J_F*) is given by*

$$\mathcal{A}_F = \left\{ \left(\lambda_R, \lambda_L, \begin{pmatrix} \lambda_R & 0 \\ 0 & \mu \end{pmatrix} \right) : (\lambda_R, \lambda_L, \mu) \in \mathbb{C}_R \oplus \mathbb{C}_L \oplus \mathbb{C} \right\}$$

Proposition

The perturbed Dirac operator D'_F *is parametrized by three complex scalar fields* ϕ, σ_1, σ_2*:*

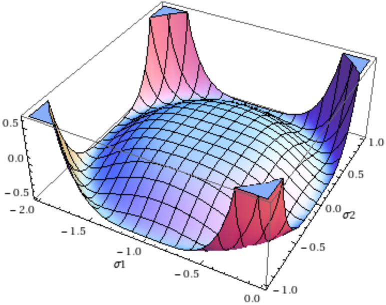
$$D'_F = \begin{pmatrix} 0 & \overline{c}\overline{\phi}\otimes 1_2 & \overline{d}\overline{v}\cdot\overline{v}^t & 0 \\ c\phi\otimes 1_2 & 0 & 0 & \\ dv\cdot v^t & 0 & 0 & 1_2\otimes c\phi \\ 0 & 0 & 1_2\otimes\overline{c}\overline{\phi} & 0 \end{pmatrix}$$

with $v = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$.

Spectral action functional

Spectral action functional gives rise to a scalar potential

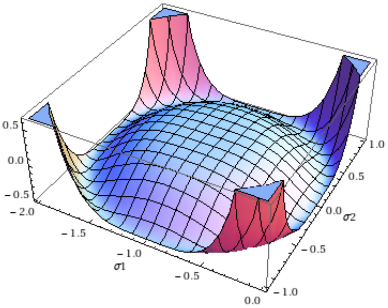
$$V(\phi,\sigma_1,\sigma_2) = -\frac{f_2}{\pi^2}\Lambda^2\left(4|c|^2|\phi|^2 + |d|^2(|\sigma_1|^2+|\sigma_2|^2)^2\right) + \frac{f_0}{4\pi^2}\Big(4|c|^4|\phi|^4 + 4|c|^2|d|^2|\phi|^2(|\sigma_1|^2+|\sigma_2|^2)^2 + |d|^4(|\sigma_1|^2+|\sigma_2|^2)^4\Big)$$



Spontaneous symmetry breaking to first-order

Proposition

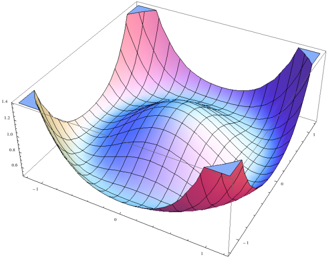
The potential $V(\phi = 0, \sigma_1, \sigma_2)$ has a local minimum at $(\sigma_1, \sigma_2) = (\sqrt{w}, 0)$ with $w = \sqrt{2f_2\Lambda^2/(f_0|d|^2)}$ and this point spontaneously breaks the symmetry group $\mathcal{U}(\mathcal{A}'_F)$ to $\mathcal{U}(\mathcal{A}_F)$.



"Usual" SSB

After the fields (σ_1, σ_2) have reached their vevs $(\sqrt{w}, 0)$, there is a remaining potential for the ϕ-field:

$$V(\phi) = -\frac{2f_2}{\pi^2}\Lambda^2|c|^2|\phi|^2 + \frac{f_0}{\pi^2}|c|^4|\phi|^4.$$



Selecting one of the minima of $V(\phi)$ spontaneously breaks the symmetry further from $\mathcal{U}(\mathcal{A}_F) = U(1)_R \times U(1)_L \times U(1)$ to $U(1)_L \times U(1)$, and generates mass terms for the $L - R$ abelian gauge field.

Beyond the Standard Model

One starts with the algebra

$$\mathcal{A}_{\mathrm{PS}} := \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

and an off-diagonal Dirac operator

$$D_F := \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The largest 'first-order' subalgebra of $\mathcal{A}_{\mathrm{PS}}$ is $\mathbb{C} \oplus \mathbb{H}_L \oplus M_3(\mathbb{C})$.
- Symmetry breaking from Pati–Salam $SU(2)_R \times SU(2)_L \times SU(4)$ to Standard Model $U(1) \times SU(2)_L \times SU(3)$.

Beyond the Standard Model

- Perturbation semigroup of $\mathcal{A}_{\mathrm{PS}}$ gives rise to many new scalar fields

$$\phi_{\dot{a}}^{b}: \quad (2_R, \overline{2}_L, 1), \qquad \Delta_{\dot{a}J}: \quad (2_R, 1_L, 4), \qquad \Sigma_I^J: \quad (1_R, 1_L, 1+15)$$

- The truncation to the Standard Model + real scalar singlet is given via

$$\phi_{\dot{a}}^{b} = \delta_{\dot{a}}^{\dot{1}} \epsilon^{bc} H_c, \qquad \Delta_{\dot{a}I} = \delta_{\dot{a}}^{\dot{1}} \delta_I^1 \sqrt{\sigma}.$$

with spontaneous symmetry breaking:

$$g_R W_{\mu R}^3 = g_1 B_\mu, \qquad W_{\mu R}^{\pm} = 0$$
$$\sqrt{\frac{3}{2}} g V_\mu^{15} = -g_1 B_\mu \qquad (V_\mu)_1^i = 0$$

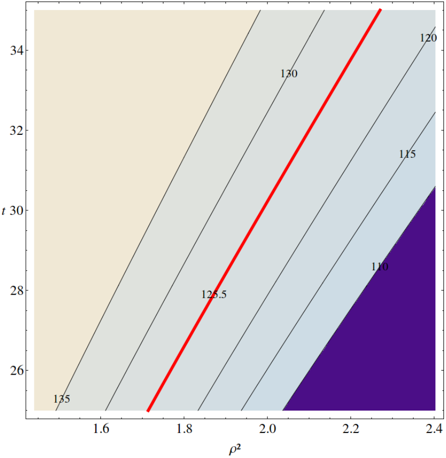
Here V_μ^{15} is the $SU(4)$ gauge field corresponding to the $B-L$ generator

$$\lambda^{15} = \frac{1}{\sqrt{6}} \mathrm{diag}\,(3, -1, -1, -1)$$

- The real scalar singlet σ is coupled to the Higgs sector:

$$V(\sigma, h) = -\frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2) + \frac{1}{24}\lambda_h h^4 + \frac{1}{2} h^2 \sigma^2 + \frac{1}{4}\lambda_\sigma \sigma^4$$

- Instead of the notorious Higgs mass prediction from the nc Standard Model, this real scalar singlet gives a Higgs mass varying with $\rho = m_{\text{top}}/m_\nu$ and the unification scale $t = \log(\Lambda_{\text{GUT}}/M_Z)$



- This allows for $m_h = 125.5\text{GeV}$ and $m_\sigma \sim 10^{12}\text{GeV}$.