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Perturbation Semigroup for matrix algebras

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Abstract

In this thesis we analyze the perturbation semigroup introduced by A. Chamseddine, A. Connes and W. van Suijlekom, and we try to get a better understanding of its structure. We will concretely determine the perturbation semigroup for all matrix algebras and use some toy models to see the physical use of the perturbation semigroup, in particular to the Standard Model of Particle Physics.

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Chapter 1

Introduction

Geometry is an ancient part of mathematics which can be traced back to the ancient Greeks and further. Euclides and Newton used geometry in their work and Einstein used geometry for his famous theory of gravity. In order to develop his general theorem of relativity he had to use some kind of geometry, more specifically Einstein used Riemannian geometry. In this way, Einstein's theory could describe gravity. The other three fundamental forces, the weak and strong nuclear force and the electromagnetic force, could not yet be described by it despite many efforts (e.g. Kaluza-Klein [1]). Many have tried to generalize Einstein's theory, but it was Alain Connes in the twentieth century that found a generalization that allows for the inclusion of the other forces as well [2]. The result was non-commutative geometry and it generalizes Riemannian geometry. With this generalization it was also possible to describe the Standard Model of Particle Physics at least at the classical level [3].

In 2013 an article was published in which non-commutative geometry was further generalized by the disposal of one of the conditions, namely the first order condition [4]. In this thesis we will take a closer look at a semigroup structure emerging through this generalization and try to apply it to some toy models.

In this first chapter we will give a short introduction of the subject. In the next chapter we will give some definitions needed later, and we will also prove that the perturbation semigroup is in fact a semigroup. After that we will continue with some examples. We will then determine the perturbation semigroup for all matrix algebras. Starting with $\mathbb{C}^N$ we show that

$$\mathrm{Pert}(\mathbb{C}^N) \cong C^{N(N-1)/2}$$

and we will take a look at the embedding of the unitaries in the perturbation semigroup. This explains how the perturbation semigroup is a generalization of the gauge group. We will also determine the perturbation semigroup of $M_N(\mathbb{C})$. For both $\mathbb{C}^N$ and $M_N(\mathbb{C})$ we will first determine the perturbation semigroup for some example before we generalize it. For $M_i(\mathbb{C})$, where $i = 2, 3, 4$, we will first determine the structure that follows from the definition after which we will try to find defining properties. As it will turn out, in general it is too hard to understand the structure of the perturbation semigroup from the definition. Instead we will start by analyzing defining properties and determine its structure that way. If we have done that we will also take a quick look at the invertible elements in the perturbation semigroup and at the way the unitaries in the algebra are embedded in the perturbation semigroup. An other interesting matrix algebra besides $M_N(\mathbb{C})$ is $M_N(\mathbb{R})$ for which we will find that

$$\mathrm{Pert}(M_N(\mathbb{R})) \cong \left(\mathbb{R}^{(N-1)(N+2)/2} \rtimes M_{(N-1)(N+2)/2}(\mathbb{R})\right) \times M_{N(N-1)/2}(\mathbb{R}).$$

We will also take a look at the perturbation semigroup of the quaternions $\mathbb{H}$, which is a real subalgebra of $\mathrm{Pert}(M_2(\mathbb{C}))$, and then we will generalize this to the perturbation semigroup of $M_N(\mathbb{H})$, which is a real subalgebra of $\mathrm{Pert}(M_{2N}(\mathbb{C}))$.

In chapter 4 we will take a look at the general theory of the perturbation semigroup, in the sense that we will look at the perturbation semigroup of the direct sum and of the tensor product. For the perturbation semigroup of a direct sum we will find an explicit expression, while for the perturbation semigroup of the tensor product there does not seem to be such an explicit expression.

After that it will be time to take a look at the perturbation semigroup of the Standard Model of Particle Physics. Before we can determine that perturbation semigroup we will consider the perturbation semigroup of smooth functions on a manifold after which we will replace the smooth functions by the smooth functions with values in a finite dimensional $*$-algebra.

In the last chapter we will take a look at the action of this perturbation semigroup on hermitian matrices. We will apply it to some toy models, some of which have physical meaning. We will take a look at the action of the perturbation semigroup on diagonal matrices, after which we will consider off-diagonal, but still hermitian, matrices. These have some physical application: it will turn out that the famous Higgs field is encoded in one of the results.

In the appendix we will prove a few results on semigroups which we have not proven in the text. For instance, we include the semigroup isomorphism theorems and a result on the invertible elements of a semidirect product.

Chapter 2

Perturbation semigroup

2.1 Definitions

In order for us to be able to do mathematics we need a few definitions.

Definition 2.1. A complex unital algebra is a vector space $\mathcal{A}$ with a bilinear associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit 1 satisfying 1a = a1 = a for all $a \in \mathcal{A}$.

Definition 2.2. An involutive algebra (or $*$-algebra) is a complex algebra $\mathcal{A}$ with a conjugate-linear map $* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$

$$\begin{aligned} (ab)^* &= b^* a^*, \\ (a^*)^* &= a. \end{aligned}$$

We will restrict to involutive unital $*$-algebras and we will simply refer to them as algebras. An example of a $*$-algebra is $\mathbb{C}^N$ with componentwise multiplication. An other example is $M_N(\mathbb{C})$ with matrix multiplication.

Proposition 2.3. *The vector space $\mathbb{C}^N$ with componentwise multiplication is a $*$-algebra, with the action of $*$ given by conjugation. The monoid $M_N(\mathbb{C})$ with matrix multiplication is a $*$-algebra, where $*$ acts as hermitian conjugation.*

Proof. Let $v, w \in \mathbb{C}^N$ then $(vw)_i = (v_i w_i)_i \in \mathbb{C}^N$ since every $v_i, w_i \in \mathbb{C}$, thus $v_i w_i \in \mathbb{C}$. We also know that $\mathbb{C}$ is commutative, thus in general $\mathbb{C}^N$ is commutative. The unit is given by $1 = (1)_i$, the vector with as entries 1. The $*$ is conjugation. So let $v, w \in \mathbb{C}^N$ then

$$(vw)^* = \overline{vw} = \overline{wv} = w^* v^*$$

and

$$(v^*)^* = (\overline{v})^* = \overline{\overline{v}} = v.$$

For $M_N(\mathbb{C})$ we have the usual matrix multiplication. So if $A, B \in M_N(\mathbb{C})$ then $AB \in M_N(\mathbb{C})$. The unit is given by the unit matrix I_N, i.e. by the diagonal matrix I_N with as entries 1 on the diagonal and zeros elsewhere. The $*$ is hermitian conjugation, so for $A, B \in M_N(\mathbb{C})$ we get

$$(AB)^* = (\overline{AB})^{\mathsf{T}} = \overline{B}^{\mathsf{T}} \overline{A}^{\mathsf{T}} = B^* A^*$$

and
$$(A^*)^* = (\overline{A}^{\mathsf{T}})^* = \overline{(\overline{A}^{\mathsf{T}})}^{\mathsf{T}} = (\overline{\overline{A}}^{\mathsf{T}})^{\mathsf{T}} = A.$$
We conclude that both $\mathbb{C}^N$ and $M_N(\mathbb{C})$ are $*$-algebras. $\square$

Central in finite dimensional non-commutative geometry is

Definition 2.4. A finite spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$ of a $*$-algebra $\mathcal{A}$ represented faithfully on a finite-dimensional Hilbert space $\mathcal{H}$, together with a symmetric linear operator $D : \mathcal{H} \to \mathcal{H}$.

The name spectral triple comes from the fact that the geometry of $\mathcal{A}$ is encoded in the spectrum of D. It is useful to allow for finite spectral triples on real algebras, instead of complex ones, as above.

Definition 2.5. A real unital algebra is a vector space $\mathcal{A}$ (over $\mathbb{R}$) with a bilinear associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit 1 satisfying $1a = a1 = a$ for all $a \in \mathcal{A}$.
An involutive algebra (or $*$-algebra) is a real algebra $\mathcal{A}$ together with a real linear map (the involution) $* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$
$$\begin{aligned} (ab)^* &= b^* a^*, \\ (a^*)^* &= a. \end{aligned}$$

Remark 2.6. *Note that both $\mathbb{R}^N$ and $M_N(\mathbb{R})$ are also real unital $*$-algebras, just as in Proposition 2.3. The action of $*$ on $\mathbb{R}^N$ is trivial, while the action of $*$ on $M_N(\mathbb{R})$ is just matrix transposition.*

The difference between this definition and that of a complex $*$-algebra is that the real algebras are closed under multiplication with real numbers only. A particularly interesting example in this context is given by the quaternions, $\mathbb{H}$, which is a real subalgebra of $M_2(\mathbb{C})$, defined by
$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C} \right\}.$$
As one can see, the entries of the matrices are complex, but the algebra itself is real. Indeed, upon multiplying a matrix in $\mathbb{H}$ with a complex number λ, and demanding this to be in $\mathbb{H}$, forces $\lambda = \overline{\lambda}$.

We will also be working with semigroups in this thesis, so we need a few definitions on that.

Definition 2.7. A semigroup S is a set with an associative operation $\circ : S \times S \to S$. If S has a unit it is called a monoid.

Definition 2.8. A group G is a set with an associative operation $\circ : G \times G \to G$, an identity element e such that $ge = g = eg$ for all $g \in G$ and for every $g \in G$ there is an element $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$. We will refer to g^{-1} as the inverse element of g, since g^{-1} is unique for every g.

Remark 2.9. *Every group G is thus in particular a monoid and a semigroup.*

Just as for groups we have homomorphism between semigroups.

Definition 2.10. Let S, T be two semigroups, then the function $\phi : S \to T$ is called a semigroup homomorphism if
$$\phi(xy) = \phi(x)\phi(y)$$
for all $x, y \in S$.

Notation 2.11. *For a semigroup (or a monoid) S we write $S^\times$ for the group of invertible elements in S.*

Associated to any spectral triple is the following group.

Definition 2.12. The group $\mathcal{U}(\mathcal{A})$ is the group of unitaries, i.e.

$$\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = 1 = u^*u\}.$$

2.2 Semigroup of inner perturbations

The starting point of this thesis is [4]. In that paper the theory of non-commutative geometry is enriched by neglecting the first order condition. This gave rise to the inner perturbations as we will now describe. At first the fluctuated metrics were formed with help of $\mathcal{U}(\mathcal{A})$ in the sense that, for $u \in \mathcal{U}(\mathcal{A})$ we had

$$D \mapsto uDu^*,$$

which can be rewritten as

$$D \mapsto uDu^* = D + u[D, u^*].$$

However, since the elements of $\mathcal{U}(\mathcal{A})$ are unitary elements the spectrum of D remains the same. This motivated the use of $\mathcal{A}$ to fluctuate the metric. If we now use $u[D, u^*]$ as a prototype for the new elements we get

$$D \mapsto D + A, \quad A = \sum a_j[D, b_j],$$

where $a_j, b_j \in \mathcal{A}$ and $A = A^*$, as in [5]. In [4] this action was generalized to the action of the so called perturbation semigroup $\mathrm{Pert}(\mathcal{A})$. The elements of this perturbation semigroup are $\sum a_j \otimes b_j^{op} \in \mathrm{Pert}(\mathcal{A})$ and work on D as $\sum a_j D b_j$. Let us work towards a precise definition of the perturbation semigroup.

Definition 2.13. Let $\mathcal{A}$ be an algebra, then the opposite algebra of $\mathcal{A}$ is denoted by $\mathcal{A}^{op}$. It is given by $\mathcal{A}$ as a vector space, with product $a \circ b = ba$.

Definition 2.14. Let $\mathcal{A}$ be an associative algebra with unit then $\Omega^1(\mathcal{A})$ is the space of *one-forms* given by
$$\Omega^1(\mathcal{A}) = \{\sum_i a_i \delta b_i \mid a_i, b_i \in \mathcal{A}\},$$
where
$$\delta : \mathcal{A} \to \Omega^1(\mathcal{A})$$

be a map such that

$$\delta(ab) = (\delta a)b + a\delta b;$$
$$\delta(\alpha a + \beta b) = \alpha\delta a + \beta\delta b;$$
$$\delta(a)^* = -\delta(a^*).$$

Here $a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$.

Remark 2.15. *The condition*

$$\delta(ab) = (\delta a)b + a\delta b$$

implies that $\delta 1 = 0$. Therefore also $\delta\mathbb{C} = 0$.

This $\Omega^1(\mathcal{A})$ is further generalized in [6]. One now has

Lemma 2.16. *The map η is a surjection*

$$\eta : \{\sum a_j \otimes b_j^{op} \in \mathcal{A} \otimes \mathcal{A}^{op} \mid \sum a_j b_j = 1\} \to \Omega^1(\mathcal{A}), \quad \eta(\sum a_j \otimes b_j^{op}) = \sum a_j \delta(b_j)$$

and one has

$$\eta(\sum b_j^* \otimes a_j^{*op}) = (\eta(\sum a_j \otimes b_j^{op}))^*.$$

Proof. Let $\omega = \sum a_i\delta(b_i) \in \Omega^1(\mathcal{A})$, then we can write

$$\omega = (1 - \sum a_i b_i)\delta(1) + \sum a_i\delta(b_i),$$

because we have $\delta(1) = 0$. The preimage of this element is

$$1 \otimes 1^{op} - \sum a_i b_i \otimes 1^{op} + \sum a_i \otimes b_i^{op}.$$

This is normalized since we have

$$1 \cdot 1 - \sum a_i b_i \cdot 1 + \sum a_i b_i = 1.$$

So η is a surjection.

Let $\sum a_i\delta(b_i) \in \Omega^1(\mathcal{A})$, then we see that $\sum a_i \otimes b_i^{op}$ is normalized. We have

$$(\sum a_i\delta(b_i))^* = -(\sum \delta(a_i)b_i)^* = \sum b_i^*\delta(a_j^*),$$

hence

$$\eta\Big(\sum b_j^* \otimes a_j^{*op}\Big) = \Big(\eta\Big(\sum a_j \otimes b_j^{op}\Big)\Big)^*.$$

Note that we have

$$\sum b_j^* a_j^* = (\sum a_j b_j)^* = 1^* = 1$$

so $\sum b_i^* \otimes a_i^{*op}$ is also normalized.

Now suppose that $\omega = \sum x_i\delta(y_i)$, such that $\omega = \omega^*$. We can now write ω as $\omega = \frac{1}{2}\sum a_i\delta(b_i) + \frac{1}{2}\sum b_i^*\delta(a_i^*)$ for given a_i, b_i. Note that ω is in fact self-adjoint. We can rewrite ω as

$$\omega = (1 - \sum a_i b_i)\delta(\frac{1}{2}) + \delta(\frac{1}{2})(1 - \sum b_i^* a_i^*) + \frac{1}{2}\sum a_i\delta(b_i) + \frac{1}{2}\sum b_i^*\delta(a_i^*),$$

since $\delta(\frac{1}{2}) = 0$. The preimage of ω is

$$1 \otimes \frac{1}{2}^{op} - \sum a_i b_i \otimes \frac{1}{2}^{op} + \frac{1}{2} \otimes 1^{op} - \sum \frac{1}{2} \otimes (a_i b_i)^{*op} + \frac{1}{2} \sum a_i \otimes b_i^{op} + \frac{1}{2} \sum b_i^* \otimes a_i^{*op},$$

where we have used that $(b_i^* a_i^*)^{op} = (a_i b_i)^{*op}$. Also note that this preimage is in fact self-adjoint. We see that this preimage is normalized since we have

$$1 \cdot \frac{1}{2} - \sum a_i b_i \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot \sum (a_i b_i)^* + \frac{1}{2} \sum a_i b_i + \frac{1}{2} \sum b_i^* a_i^* = 1.$$

□

We now want to show transitivity of the inner fluctuations, which means that inner fluctuations of inner fluctuations are still inner fluctuations.

Proposition 2.17. *Let $A = \sum a_i \otimes b_i^{op} \in \mathcal{A} \otimes \mathcal{A}^{op}$ be self-adjoint and normalized by $\sum a_i b_i = 1$. Then for $A, A' \in \mathcal{A} \otimes \mathcal{A}^{op}$, both normalized, we have*

$$(D(\eta(A)))(\eta(A')) = D(\eta(A'A)).$$

Here $D' = D(\eta(A))$ stands for the inner fluctuation of D, which means

$$D' = D + \sum a_i [D, b_i].$$

Proof. Let $A = \sum a_i \otimes b_i^{op}$ and $A' = \sum x_s \otimes y_s^{op}$ be normalized and self-adjoint. We have

$$D' = D(\eta(A)) = D + \sum a_i [D, b_i]$$

and in a similar way

$$D'' = D'(\eta(A')) = (D(\eta(A)))(\eta(A')) = D(\eta(A)) + \sum x_s [D(\eta(A)), y_s].$$

Expanding this gives

$$D'' = D + \sum a_i [D, b_i] + \sum x_s [D, y_s] + \sum \sum x_s [a_i [D, b_i], y_s].$$

If we now use

$$x_s [a_i [D, b_i], y_s] = x_s (a_i [D, b_i] y_s - y_s a_i [D, b_i])$$

and we use that

$$\sum \sum x_s y_s a_i [D, b_i] = \sum a_i [D, b_i]$$

we get

$$D'' = D + \sum x_s [D, y_s] + \sum \sum x_s a_i [D, b_i] y_s.$$

However $x_s a_i [D, b_i] y_s$ can be expanded as

$$x_s a_i [D, b_i] y_s = x_s a_i [D, b_i y_s] - x_s a_i b_i [D, y_s]$$

and we know that

$$\sum \sum x_s a_i b_i [D, y_s] = \sum x_s [D, y_s].$$

So we get

$$D'' = D + \sum \sum x_s a_i [D, b_i y_s].$$

If we now use

$$(\sum x_s \otimes y_s^{op})(\sum a_i \otimes b_i^{op}) = \sum x_s a_i \otimes (b_i y_s)^{op}$$

we get the result. □

Thus inner fluctuations of inner fluctuations are still inner fluctuations. The self-adjoint normalized elements of $\mathcal{A}\otimes\mathcal{A}^{op}$ form a semigroup $\mathrm{Pert}(\mathcal{A})$ under multiplication.

Definition 2.18. The perturbation semigroup is given by

$$\mathrm{Pert}(\mathcal{A}) = \Big\{ \sum a_j \otimes b_j^{op} \in \mathcal{A}\otimes\mathcal{A}^{op} \mid \sum a_j b_j = 1, \sum a_j \otimes b_j^{op} = \sum b_j^* \otimes a_j^{*op} \Big\},$$

where the sums are finite and the 1 is the unit in $\mathcal{A}$.

Theorem 2.19. $\mathrm{Pert}(\mathcal{A})$ *is a semigroup and has a unit.*

Proof. To show that $\mathrm{Pert}(\mathcal{A})$ is in fact a semigroup we have to show the operation $\circ : \mathrm{Pert}(\mathcal{A}) \times \mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A})$ is associative, thus for all $a, b, c \in \mathrm{Pert}(\mathcal{A})$ we have $(a\circ b)\circ c = a\circ(b\circ c)$. However, since $\mathcal{A}$ is associative we know that $\mathcal{A}\otimes\mathcal{A}^{op}$ is associative as well. Since every element in the perturbation semigroup comes from $\mathcal{A}\otimes\mathcal{A}^{op}$, we know that the operation $\circ$ is associative. We now need to show that the operation is closed, i.e. that the product of two elements is again in the perturbation semigroup. Again let $\sum_j a_j \otimes \widetilde{a}_j^{op}, \sum_k b_k \otimes \widetilde{b}_k^{op} \in \mathrm{Pert}(\mathcal{A})$, then

$$\begin{array}{rcl} \Big(\sum_j a_j \otimes \widetilde{a}_j^{op}\Big)\Big(\sum_k b_k \otimes \widetilde{b}_k^{op}\Big) & = & \sum_{j,k}(a_j b_k \otimes \widetilde{a}_j^{op}\widetilde{b}_k^{op}), \\ & = & \sum_{j,k}(a_j b_k \otimes (\widetilde{b}_k\widetilde{a}_j)^{op}). \end{array}$$

This has to be both normalized and self-adjoint. This is the case since we have

$$\begin{array}{rcl} \sum_{j,k}(a_j b_k)(\widetilde{b}_k\widetilde{a}_j) & = & \sum_{j,k} a_j(b_k\widetilde{b}_k)\widetilde{a}_j, \\ & = & \sum_j a_j(\sum_k b_k\widetilde{b}_k)\widetilde{a}_j, \\ & = & \sum_j a_j\widetilde{a}_j, \\ & = & 1, \end{array}$$

where we have subsequently used the normalization condition for $\sum_k b_k\otimes\widetilde{b}_k^{op}$ and $\sum_j a_j\otimes \widetilde{a}_j^{op}$. Moreover, self-adjointness follows from

$$\begin{array}{rcl} \sum_{j,k}(\widetilde{b}_k\widetilde{a}_j)^* \otimes (a_j b_k)^{*op} & = & \sum_{j,k} \widetilde{a}_j^*\widetilde{b}_k^* \otimes a_j^{*op} b_k^{*op}, \\ & = & \Big(\sum_j \widetilde{a}_j^* \otimes a_j^{*op}\Big)\Big(\sum_k \widetilde{b}_k^* \otimes b_k^{*op}\Big), \\ & = & \Big(\sum_j a_j \otimes \widetilde{a}_j^{op}\Big)\Big(\sum_k b_k \otimes \widetilde{b}_k^{op}\Big), \\ & = & \sum_{j,k}(a_j b_k) \otimes (\widetilde{b}_k\widetilde{a}_j)^{op}, \end{array}$$

where we have used the self-adjointness of both $\sum_k b_k \otimes \widetilde{b}_k^{op}$ and $\sum_j a_j \otimes \widetilde{a}_j^{op}$.

We claim that the unit in $\mathrm{Pert}(\mathcal{A})$ is given by $1\otimes 1$, since

$$\begin{array}{rcl} (\sum_j a_j \otimes \widetilde{a}_j^{op}) \circ (1\otimes 1) & = & \sum_j (a_j \otimes \widetilde{a}_j^{op}) \circ (1\otimes 1), \\ & = & \sum_j (a_j \cdot 1) \otimes (\widetilde{a}_j^{op} \cdot 1), \\ & = & \sum_j a_j \otimes \widetilde{a}_j^{op}. \end{array}$$

Similarly

$$\begin{array}{rcl} (1\otimes 1) \circ (\sum_j a_j \otimes \widetilde{a}_j^{op}) & = & \sum_j (1\otimes 1) \circ (a_j \otimes \widetilde{a}_j^{op}), \\ & = & \sum_j (1 \cdot a_j) \otimes (1 \cdot \widetilde{a}_j^{op}), \\ & = & \sum_j a_j \otimes \widetilde{a}_j^{op}, \end{array}$$

where we have used that 1 is a unit in both $\mathcal{A}$ and $\mathcal{A}^{op}$. We conclude that $\mathrm{Pert}(\mathcal{A})$ is a semigroup and it is a monoid if $\mathcal{A}$ has a unit. □

Let us now consider how the unitary group is embedded in the perturbation semigroup.

Proposition 2.20. *Let $\mathcal{A}$ be a $*$-algebra, then we have*

$$\begin{aligned} \mathcal{U}(\mathcal{A}) &\to \mathrm{Pert}(\mathcal{A}), \\ u &\mapsto u \otimes u^{*op}. \end{aligned} \tag{2.1}$$

Proof. We need to show that $u \otimes u^{*op}$ is both normalized and self-adjoint. The normalization condition follows by definition, since we have $uu^* = 1$. We also have

$$u \otimes u^{*op} = (u^*)^* \otimes (u^{op})^*,$$

hence the element is self-adjoint, which proves the proposition. □

Chapter 3

Perturbation semigroup for matrix algebras

Now that we have our definitions in place, we want to further investigate the perturbation semigroup by studying some examples.

3.1 Perturbation semigroup $\mathrm{Pert}(\mathbb{C}^N)$

Since the examples are finite dimensional vector spaces, we can work in a basis for $\mathcal{A} = \mathbb{C}^N$, which allows for an explicit form of $\mathrm{Pert}(\mathcal{A})$. For a vector space $\mathbb{C}^N$ (and also $\mathbb{R}^N$) we have

$$\mathcal{A} \cong \mathcal{A}^{op}$$

since it is commutative.

3.1.1 $\mathcal{A} = \mathbb{C}$

First we look at the case $\mathcal{A} = \mathbb{C}$. Now $\sum_j a_j \otimes b_j^{op}$ reduces to $\sum_j a_j b_j$, since the tensor product is linear over $\mathbb{C}$ and $\mathbb{C} \cong \mathbb{C}^{op}$. This $\sum_j a_j b_j$ enters precisely in our normalization condition, thus equals 1. We can therefore conclude that the case $\mathcal{A} = \mathbb{C}$ is trivial

$$\mathrm{Pert}(\mathbb{C}) \cong \{1\}.$$

$\mathcal{U}(\mathbb{C})$ maps onto the perturbation semigroup, following the embedding in (2.1), since we have

$$\lambda \in \mathcal{U}(1) \mapsto \lambda \otimes \overline{\lambda}.$$

However, the tensor product is $\mathbb{C}$-linear, therefore we can bring $\overline{\lambda}$ to the other side, which gives

$$\lambda\overline{\lambda} \otimes 1 = 1 \otimes 1.$$

Thus in this case $\mathcal{U}(1) \to \{1\}$.

3.1.2 $\mathcal{A} = \mathbb{C}^2$

Now let us consider $\mathcal{A} = \mathbb{C}^2$, so that $\mathcal{A} \otimes \mathcal{A}^{op} \cong \mathbb{C}^4$. As a basis for $\mathcal{A}$ we take

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In terms of this basis the product of $\mathbb{C}^2$ behaves much like a Kronecker delta, in the sense that

$$e_i e_j = \begin{cases} 0 & \text{if } i \neq j, \\ e_i & \text{if } i = j. \end{cases}$$

Thus for an element $\sum_{i,j} C_{ij} e_i \otimes e_j^{op}$, with coefficients C_{ij}, one gets that the normalization conditions reads

$$C_{11}e_1e_1 + C_{12}e_1e_2 + C_{21}e_2e_1 + C_{22}e_2e_2 = C_{11}e_1 + C_{22}e_2 = e_1 + e_2,$$

since $e_1 + e_2$ is the identity in $\mathbb{C}^2$. So it follows that $C_{11} = C_{22} = 1$. To determine what restrictions are imposed on C_{21} and C_{12}, we use the self-adjointness condition. We see that

$$\begin{aligned} \sum_{i,j} C_{ij} e_i \otimes e_j^{op} &= \sum_{i,j} C_{ij} e_i \otimes e_j, \\ &= \sum_{i,j} C_{ij}^* e_j^* \otimes e_i^*, \\ &= \sum_{i,j} \overline{C_{ij}} e_j \otimes e_i, \\ &= \sum_{i,j} \overline{C_{ji}} e_i \otimes e_j^{op}, \end{aligned}$$

where we have used that $\mathcal{A} \cong \mathcal{A}^{op}$. So we see that $C_{ij} = \overline{C_{ji}}$, hence $C_{12} = \overline{C_{21}}$.

Upon identifying the basis

$$e_1 \otimes e_1 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_1 \otimes e_2 \rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mathit{etc.}$$

the element

$$\sum_{i,j} C_{ij} e_i \otimes e_j^{op} \in \operatorname{Pert}(\mathbb{C}^2)$$

thus becomes

$$\begin{pmatrix} 1 \\ C_{12} \\ \overline{C_{12}} \\ 1 \end{pmatrix}, \quad C_{12} \in \mathbb{C}.$$

This is isomorphic to $\mathbb{C}$, so we get

$$\operatorname{Pert}(\mathbb{C}^2) \cong \mathbb{C}.$$

For $\mathbb{C}^2$ the unitaries are mapped to $\mathrm{Pert}(\mathbb{C}^2)$ as in (2.1). With the identification

$$\mathcal{U}(\mathbb{C}^2) \cong \mathcal{U}(1) \times \mathcal{U}(1),$$

we have

$$(\lambda, \mu) \in \mathcal{U}(1) \times \mathcal{U}(1) \mapsto (\lambda, \mu) \otimes (\overline{\lambda}, \overline{\mu}).$$

Once again we have a $\mathbb{C}$-linear tensor product, thus we get

$$(\lambda, \mu) \otimes (\overline{\lambda}, \overline{\mu}) = (\lambda, \mu) \otimes \overline{\lambda}(1, \lambda\overline{\mu}) = (1, \overline{\lambda}\mu) \otimes (1, \lambda\overline{\mu}).$$

Note that one only sees the difference between the components, but not the components itself, leaving a $\mathcal{U}(1) \subset \mathrm{Pert}(\mathbb{C}^2)$.

3.1.3 $\mathcal{A} = \mathbb{C}^N$

Now let us take a look at the general case $\mathcal{A} = \mathbb{C}^N$ so that $\mathcal{A} \otimes \mathcal{A}^{op} \cong \mathbb{C}^{N^2}$. As a basis for $\mathcal{A}$ we take

$$e_i = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i$$

the standard basis.

Proposition 3.1. *For any $N \geq 1$ we have*

$$\mathrm{Pert}(\mathbb{C}^N) \cong \mathbb{C}^{N(N-1)/2}$$

with the semigroup structure given by componentwise multiplication.

Proof. Our normalization condition states that $C_{ii} = 1$ for all i. Since

$$\sum_{i,j} C_{ij} e_i e_j$$

reduces to

$$\sum_{i} C_{ii} e_i e_i$$

and $\sum_i e_i e_i$ is the unit in $\mathbb{C}^N$. That this sum reduces to the unit comes from the product $e_i e_j$ which behaves much like a Kronecker product in the sense that

$$e_i e_j = \begin{cases} 0 & \text{if } i \neq j, \\ e_i & \text{if } i = j. \end{cases} \tag{3.1}$$

The self-adjointness condition states that $C_{ij} = \overline{C_{ji}}$ for all i, j, since

$$\begin{aligned} \sum_{i,j} C_{ij} e_i \otimes e_j^{op} &= \sum_{i,j} C_{ij} e_i \otimes e_j, \\ &= \sum_{i,j} C_{ij}^* e_j^* \otimes e_i^*, \\ &= \sum_{i,j} \overline{C_{ij}} e_j \otimes e_i, \\ &= \sum_{i,j} \overline{C_{ji}} e_i \otimes e_j^{op} \end{aligned}$$

So there are N^2 variables, among which N are equal to one, while the others are pairwise conjugated.

Now let $v, w \in \mathrm{Pert}(\mathbb{C}^N)$ and $z = vw$. This is once again in the semigroup, since for the k-th component we have

$$z_k = z_{ij} = v_{ij}w_{ij} = C_{ij}D_{ij}$$

for given i, j and components C_{ij}, D_{ij}, such that $v_{ij} = C_{ij}e_i \otimes e_j$ and similarly for w_{ij} and z_{ij}. Suppose $i = j$ then

$$z_k = z_{ii} = v_{ii}w_{ii} = C_{ii}D_{ii} = 1 * 1 = 1.$$

So z is normalized. For arbitrary i, j we have

$$z_{ij} = v_{ij}w_{ij} = \overline{v_{ji}w_{ji}} = \overline{z_{ji}},$$

hence $z_{ij} = \overline{z_{ji}}$. So z is self-adjoint. Thus the product is indeed the semigroup structure required. □

In general we have the following proposition for the embeddeding of the unitaries in the perturbation semigroup.

Proposition 3.2. *Under the embedding* $\mathcal{U}(\mathbb{C}^N) \hookrightarrow \mathrm{Pert}(\mathbb{C}^N)$ *we have*

$$\mathcal{U}(\mathbb{C}^N) \to \mathcal{U}(1)^{N-1}.$$

Proof. Let

$$(\lambda_1, \ldots, \lambda_N) \in \mathcal{U}(\mathbb{C}^N) \cong \mathcal{U}(1)^N,$$

then the embedding reads

$$\begin{aligned}(\lambda_1, \ldots, \lambda_N) \otimes (\overline{\lambda_1}, \ldots, \overline{\lambda_N}) &= (\lambda_1, \ldots, \lambda_N) \otimes \overline{\lambda_1}(1, \lambda_1\overline{\lambda_2}, \ldots, \lambda_1\overline{\lambda_N}),\\ &= (1, \overline{\lambda_1}\lambda_2, \ldots, \overline{\lambda_1}\lambda_N) \otimes (1, \lambda_1\overline{\lambda_2}, \ldots, \lambda_1\overline{\lambda_N}),\end{aligned}$$

Where we have used that the tensor product is $\mathbb{C}$-linear and that every variable is unitary. We only see the difference between every pair of variables, therefore we get that the unitaries $\mathcal{U}(\mathbb{C}^N)$ are embedded in $\mathrm{Pert}(\mathbb{C}^N)$ as $\mathcal{U}(1)^N \to \mathcal{U}(1)^{N-1}$. □

Note that it does not matter which variable we extract from the vector ($\overline{\lambda_1}$ in the above proof), since $(\overline{\lambda_i}\lambda_j)(\lambda_i\overline{\lambda_k}) = \lambda_j\overline{\lambda_k}$.

3.2 Perturbation semigroup $\mathrm{Pert}(M_N(\mathbb{C}))$

Another interesting case is the perturbation semigroup of $M_N(\mathbb{C})$. The first cases we will consider are $\mathcal{A} = M_2(C), M_3(\mathbb{C})$ and $M_4(\mathbb{C})$ which will then be generalized to $\mathcal{A} = M_N(\mathbb{C})$.

First let us look at $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})^{op}$ in general. One can look at the basis components in order to determine the perturbation semigroup. As basis we take $\{e_{ij}\}$, where e_{ij} is a matrix with a 1 on position (i, j) and zero's everywhere else.

Lemma 3.3. *We have the following identification*

$$M_N(\mathbb{C})^{op} \to M_N(\mathbb{C}),$$
$$A^{op} \mapsto A^\intercal.$$

Proof. Note that the product behaves the same on both sides. Since

$$A^{op}B^{op} = (BA)^{op}$$

and

$$A^\intercal B^\intercal = (BA)^\intercal.$$

Also note that the dimensions of both $*$-algebras are equal. Thus this identification is correct. □

Under this identification we then have

$$e_{ij}^{op} \leftrightarrow e_{ji}.$$

Furthermore, if we multiply two basis matrices we get

$$e_{ij}e_{kl} = \delta_k^j e_{il}. \tag{3.2}$$

We also introduce the notation $C_{ij,kl}$ as the coefficient corresponding to $e_{ij} \otimes e_{kl}^{op}$ in $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})^{op}$. The last thing we need to note is that we can identify elements in the perturbation semigroup with elements in $M_{N^2}(\mathbb{C})$, since we take the tensor product of two $M_N(\mathbb{C})$ matrices.

3.2.1 $\mathcal{A} = M_2(\mathbb{C})$

The first case we look at is $\mathcal{A} = M_2(\mathbb{C})$. Note that we have four basis elements for which the normalization condition becomes

$$\begin{aligned}(C_{11,11} + C_{12,21})e_{11} + (C_{11,12} + C_{12,22})e_{12} + (C_{21,11} + C_{22,21})e_{21}\\ + (C_{21,12} + C_{22,22})e_{22} = e_{11} + e_{22}\end{aligned}$$

Thus, we need to have

$$\begin{aligned}C_{11,11} + C_{12,21} &= 1,\\ C_{11,12} + C_{12,22} &= 0,\\ C_{21,11} + C_{22,21} &= 0,\\ C_{21,12} + C_{22,22} &= 1.\end{aligned}$$

If we combine two arbitrary basis elements in the tensor product we can make the following identification

$$\begin{aligned}M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{op} &\to M_4(\mathbb{C}),\\ e_{ij} \otimes e_{kl}^{op} &\mapsto e_{ij} \otimes e_{lk} = e_{2(i-1)+l,\,2(j-1)+k}.\end{aligned}$$

in terms of the basis elements $e_{ij} \otimes e_{lk}$ and then extend this linearly to all of $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{op}$. For the self-adjointness we get $C_{ij,kl} = \overline{C_{lk,ji}}$, which will be proven in section 3.2.4. Taking the above identification into account and applying these conditions we get for $A \in \mathrm{Pert}(M_2(\mathbb{C}))$

$$A = \begin{pmatrix} x_1 & z_3 & \overline{z_3} & 1-x_1 \\ z_1 & z_2 & \overline{z_5} & -z_1 \\ \overline{z_1} & z_5 & \overline{z_2} & -\overline{z_1} \\ x_2 & z_4 & \overline{z_4} & 1-x_2 \end{pmatrix}; \qquad z_1, \dots z_5 \in \mathbb{C},\ x_1, x_2 \in \mathbb{R}.$$

Surprisingly for two of such matrices, their product once again has this general form. Now let $A \in M_4(\mathbb{C})$ correspond to an element in $\mathrm{Pert}(M_2(\mathbb{C}))$ via the above identification. Then the above form for A can be obtained by demanding

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\widehat{\Omega}\overline{A} = A\widehat{\Omega}, \text{where } \widehat{\Omega} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This $\widehat{\Omega}$ can be rewritten as a block matrix

$$\widehat{\Omega} = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11}^\mathsf{T} & e_{12}^\mathsf{T} \\ e_{21}^\mathsf{T} & e_{22}^\mathsf{T} \end{pmatrix} = \sum e_{ij} \otimes e_{ji}.$$

Especially the last identity is useful, since we see that the eigenvectors of $\widehat{\Omega}$ are given by $e_1 \otimes e_2 \pm e_2 \otimes e_1$, with eigenvalue 1 and -1 depending on the $+$ or $-$ sign, and $e_1 \otimes e_1 \pm e_2 \otimes e_2$, with eigenvalue 1. If we now change to a basis consisting of these eigenvectors we will get

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We also see that in this new base, the vector e_1 is left invariant, i.e. e_1 is an eigenvector of the matrix Ω with eigenvalue 1. This eigenvector is identified with $e_1 \otimes e_1 + e_2 \otimes e_2$ in the original basis and therefore it is also an eigenvector of the matrix A in the perturbation semigroup. Thus we get

$$\Omega e_1 = e_1; \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives

$$\mathrm{Pert}(M_2(\mathbb{C})) \cong \Big\{ A \in M_4(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \Big\},$$

with
$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix}.$$
The invertible elements in the perturbation semigroup are given by the invertible matrices in $M_4(\mathbb{C})$ which fulfill the conditions. Thus the invertible elements are given by
$$\mathrm{Pert}(M_2(\mathbb{C}))^\times \cong \left\{ A \in GL_4(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \right\}.$$

We also want to know how the unitaries are embedded in the perturbation semigroup. For a unitary matrix u we have $uu^* = I_2$. As it turns out there are two possible 2 by 2 unitary matrices, namely
$$\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \text{ and } \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix},$$
where $|a|^2 + |b|^2 = 1$. Using equation (2.1) we can get the general form of these matrices in the perturbation semigroup. Computation gives
$$\begin{pmatrix} |a|^2 & -ab & -\overline{ab} & |b|^2 \\ a\overline{b} & a^2 & -\overline{b}^2 & -a\overline{b} \\ \overline{a}b & -b^2 & \overline{a}^2 & -\overline{a}b \\ |b|^2 & ab & \overline{ab} & |a|^2 \end{pmatrix}$$
and
$$\begin{pmatrix} |a|^2 & ab & \overline{ab} & |b|^2 \\ a\overline{b} & -a^2 & \overline{b}^2 & -a\overline{b} \\ \overline{a}b & b^2 & -\overline{a}^2 & -\overline{a}b \\ |b|^2 & -ab & -\overline{ab} & |a|^2 \end{pmatrix}.$$
We see that these matrices have the same general form as we found for $\mathrm{Pert}(M_2(\mathbb{C}))$. We also see that these matrices are again unitary matrices, since we have
$$\begin{pmatrix} |a|^2 & \mp ab & \mp\overline{ab} & |b|^2 \\ a\overline{b} & \pm a^2 & \mp\overline{b}^2 & -a\overline{b} \\ \overline{a}b & \mp b^2 & \pm\overline{a}^2 & -\overline{a}b \\ |b|^2 & \pm ab & \pm\overline{ab} & |a|^2 \end{pmatrix} \begin{pmatrix} |a|^2 & \overline{a}b & a\overline{b} & |b|^2 \\ \mp\overline{ab} & \pm\overline{a}^2 & \mp\overline{b}^2 & \pm\overline{ab} \\ \mp ab & \mp b^2 & \pm a^2 & \pm ab \\ |b|^2 & -\overline{a}b & -a\overline{b} & |a|^2 \end{pmatrix} =$$
$$\begin{pmatrix} (|a|^2+|b|^2)^2 & 0 & 0 & 0 \\ 0 & (|a|^2+|b|^2)^2 & 0 & 0 \\ 0 & 0 & (|a|^2+|b|^2)^2 & 0 \\ 0 & 0 & 0 & (|a|^2+|b|^2)^2 \end{pmatrix} = I_4,$$
where we have used that $|a|^2 + |b|^2 = 1$.

We now want to change to a basis consisting of eigenvectors, more precisely the eigenvectors $e_1 \otimes e_1 \pm e_2 \otimes e_2$ and $e_1 \otimes e_2 \pm e_2 \otimes e_1$. We do this with a transformation matrix, which is given by
$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

with inverse

$$M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Note that both M and M^{-1} are unitary matrices since the columns of M are orthogonal. In terms of this new basis we get

$$M^{-1}\begin{pmatrix} |a|^2 & -ab & -\overline{ab} & |b|^2 \\ a\overline{b} & a^2 & -\overline{b}^2 & -a\overline{b} \\ \overline{a}b & -b^2 & \overline{a}^2 & -\overline{a}b \\ |b|^2 & ab & \overline{ab} & |a|^2 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & |a|^2-|b|^2 & -2\Re(ab) & -2i\Im(ab) \\ 0 & 2\Re(a\overline{b}) & \Re(a^2-b^2) & i\Im(a^2-b^2) \\ 0 & 2i\Im(a\overline{b}) & i\Im(a^2+b^2) & \Re(a^2+b^2) \end{pmatrix}$$

and

$$M^{-1}\begin{pmatrix} |a|^2 & ab & \overline{ab} & |b|^2 \\ a\overline{b} & -a^2 & \overline{b}^2 & -a\overline{b} \\ \overline{a}b & b^2 & -\overline{a}^2 & -\overline{a}b \\ |b|^2 & -ab & -\overline{ab} & |a|^2 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & |a|^2-|b|^2 & 2\Re(ab) & 2i\Im(ab) \\ 0 & 2\Re(a\overline{b}) & -\Re(a^2-b^2) & -i\Im(a^2-b^2) \\ 0 & 2i\Im(a\overline{b}) & -i\Im(a^2+b^2) & -\Re(a^2+b^2) \end{pmatrix}.$$

Apart from some minus signs, these matrices are equal. The result is again unitary, since it is the product of three unitary matrices. These resulting matrices can be parametrized by the lower right 3×3 block. So we get

$$\mathcal{U}(M_2(\mathbb{C})) \to \left\{ \begin{pmatrix} |a|^2-|b|^2 & \mp 2\Re(ab) & \mp 2i\Im(ab) \\ 2\Re(a\overline{b}) & \pm\Re(a^2-b^2) & \pm i\Im(a^2-b^2) \\ 2i\Im(a\overline{b}) & \pm i\Im(a^2+b^2) & \pm\Re(a^2+b^2) \end{pmatrix} \middle|\, |a|^2+|b|^2 = 1 \right\}.$$

3.2.2 $\mathcal{A} = M_3(\mathbb{C})$

There is not really a big difference between the cases where $n=2$ versus $n=3$. Instead of four basis elements, we now need nine basis elements, one for every possible position in the matrix. The normalization condition becomes

$$\begin{aligned} &C_{11,11}+C_{12,21}+C_{13,31}=1, && C_{11,12}+C_{12,22}+C_{13,32}=0, && C_{11,13}+C_{12,23}+C_{13,33}=0, \\ &C_{21,11}+C_{22,21}+C_{23,31}=0, && C_{21,12}+C_{22,22}+C_{23,32}=1, && C_{21,13}+C_{22,23}+C_{23,33}=0, \\ &C_{31,11}+C_{32,21}+C_{33,31}=0, && C_{31,12}+C_{32,22}+C_{33,32}=0, && C_{31,13}+C_{32,23}+C_{33,33}=1. \end{aligned}$$

When combining two arbitrary basis elements in the perturbation semigroup we make the following identification

$$\begin{aligned} M_3(\mathbb{C}) \otimes M_3(\mathbb{C})^{op} &\to M_9(\mathbb{C}), \\ e_{ij}\otimes e_{kl}^{op} &\mapsto e_{ij}\otimes e_{lk} = e_{3(i-1)+l,\,3(j-1)+k}, \end{aligned}$$

again in terms of the basis $e_{ij}\otimes e_{lk}$. The self-adjoint condition translates to $C_{ij,kl} = \overline{C_{lk,ji}}$, this will be proven in section 3.2.4. Taking the above identification into account and applying these conditions we get for $A \in \mathrm{Pert}(M_3(\mathbb{C}))$

$$A = \begin{pmatrix} x_1 & z_7 & z_8 & \overline{z_7} & x_2 & \overline{z_9} & \overline{z_8} & z_9 & 1-x_1-x_2 \\ z_1 & z_{10} & z_{11} & z_{12} & z_2 & z_{13} & z_{14} & z_{15} & -z_1-z_2 \\ z_3 & z_{16} & z_{17} & z_{18} & z_4 & z_{19} & z_{20} & z_{21} & -z_3-z_4 \\ \overline{z_1} & \overline{z_{12}} & \overline{z_{14}} & \overline{z_{10}} & \overline{z_2} & \overline{z_{15}} & \overline{z_{11}} & \overline{z_{13}} & -\overline{z_1}-\overline{z_2} \\ x_3 & z_{22} & z_{23} & \overline{z_{22}} & x_4 & \overline{z_{24}} & \overline{z_{23}} & z_{24} & 1-x_3-x_4 \\ z_5 & z_{25} & z_{26} & z_{27} & z_6 & z_{28} & z_{29} & z_{30} & -z_5-z_6 \\ \overline{z_3} & \overline{z_{18}} & \overline{z_{20}} & \overline{z_{16}} & \overline{z_4} & \overline{z_{21}} & \overline{z_{17}} & \overline{z_{19}} & -\overline{z_3}-\overline{z_4} \\ \overline{z_5} & \overline{z_{27}} & \overline{z_{29}} & \overline{z_{25}} & \overline{z_6} & \overline{z_{30}} & \overline{z_{26}} & \overline{z_{28}} & -\overline{z_5}-\overline{z_6} \\ x_5 & z_{31} & z_{32} & \overline{z_{31}} & x_6 & \overline{z_{33}} & \overline{z_{32}} & z_{33} & 1-x_5-x_6 \end{pmatrix};$$

$$z_1, \ldots, z_{33} \in \mathbb{C},\ x_1, \ldots, x_6 \in \mathbb{R}.$$

Let $A \in M_9(\mathbb{C})$ then the above form for A can be obtained by demanding

$$A(e_1 + e_5 + e_9) = (e_1 + e_5 + e_9),$$

$$\widehat{\Omega}\overline{A} = A\widehat{\Omega}, \text{where } \widehat{\Omega} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

such that $A \in \mathrm{Pert}(M_3(\mathbb{C}))$. Furthermore, we can bring $\widehat{\Omega}$ to a more compact form, i.e.

$$\widehat{\Omega} = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} = \begin{pmatrix} e_{11}^{\mathsf{T}} & e_{12}^{\mathsf{T}} & e_{13}^{\mathsf{T}} \\ e_{21}^{\mathsf{T}} & e_{22}^{\mathsf{T}} & e_{23}^{\mathsf{T}} \\ e_{31}^{\mathsf{T}} & e_{32}^{\mathsf{T}} & e_{33}^{\mathsf{T}} \end{pmatrix} = \sum e_{ij} \otimes e_{ji}.$$

The eigenvectors are now given by $e_k \otimes e_l \pm e_l \otimes e_k$ for $l \neq k$, with eigenvalue ± 1. For $l = k$ the eigenvectors are given by $e_k \otimes e_k$ for $k = 1, 2, 3$, all with eigenvalue 1. Note that $e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$ is also an eigenvector with eigenvalue 1. We can diagonalize $\widehat{\Omega}$ with a new basis consisting of these eigenvectors, which will lead to

$$\Omega = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&0&-1 \end{pmatrix}.$$

First note that $\Omega e_1 = e_1$ and if we change basis so that e_1 becomes the vector $e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$ in the previous basis, we see that e_1 is also an eigenvector of a matrix A in the perturbation semigroup. Combining this gives

$$\mathrm{Pert}(M_3(\mathbb{C})) \cong \Big\{ A \in M_9(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \Big\},$$

with

$$\Omega = \begin{pmatrix} I_6 & 0 \\ 0 & -I_3 \end{pmatrix}.$$

The invertible elements of Pert($\mathrm{M}_3(\mathbb{C})$) are given by the invertible matrices in the perturbation semigroup. Thus

$$\mathrm{Pert}(M_3(\mathbb{C}))^\times \cong \Big\{ A \in GL_9(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \Big\}.$$

Just as for $\mathrm{Pert}(M_2(\mathbb{C}))$, we want to know how the unitaries are embedded in $\mathrm{Pert}(M_3(\mathbb{C}))$. However for $N = 2$ we had two possible unitary matrices, if N becomes larger, there are also more possible unitary matrices. The approach is similar. We construct a transformation matrix using the eigenvectors. Note that the columns need to be orthogonal. For the unitary matrix u we use that $\sum_j u_{ij}\overline{u_{kj}} = \delta^i_k$. This way, though a lot of work, we can determine the embedding of the unitaries in $\mathrm{Pert}(M_3(\mathbb{C}))$.

3.2.3 $\mathcal{A} = M_4(\mathbb{C})$

The last case which will be discussed before we take a look at the general case is $\mathcal{A} = M_4(\mathbb{C})$. We do this to get more insight in the structure of the perturbation semigroup. In terms of our coefficients $C_{ij,kl}$ described above, the normalization condition reads

$$\sum_{i=1}^{4} C_{ki,ik} = 1 \text{ for } k \in \{1, \ldots, 4\},$$

$$\sum_{i=1}^{4} C_{ki,il} = 0 \text{ for } k, l \in \{1, \ldots, 4\}, k \neq l.$$

Once again when combining two arbitrary basis elements in the perturbation semigroup we make the following identification

$$M_4(\mathbb{C}) \otimes M_4(\mathbb{C})^{op} \to M_{16}(\mathbb{C}),$$
$$e_{ij} \otimes e^{op}_{kl} \mapsto e_{ij} \otimes e_{lk} = e_{4(i-1)+l,\,4(j-1)+k},$$

in terms of the basis $e_{ij} \otimes e_{lk}$. The self-adjoint condition is now given by $C_{ij,kl} = \overline{C_{lk,ji}}$, which will be proven in section 3.2.4. Taking the identification into account and applying these conditions we get for $A \in \mathrm{Pert}(M_4(\mathbb{C}))$

$$A = \begin{pmatrix}
x_1 & z_{19} & z_{21} & z_{23} & \overline{z_{19}} & x_2 & z_{24} & \overline{z_{22}} & \overline{z_{21}} & \overline{z_{24}} & x_3 & \overline{z_{20}} & \overline{z_{23}} & z_{22} & z_{20} & 1-x_1-x_2-x_3 \\
z_1 & z_{31} & z_{35} & \overline{z_{39}} & z_{65} & z_2 & z_{73} & z_{77} & z_{78} & z_{74} & z_3 & z_{66} & z_{40} & z_{36} & z_{32} & -z_1-z_2-z_3 \\
z_4 & z_{43} & z_{47} & z_{51} & z_{69} & z_5 & z_{81} & z_{87} & z_{88} & z_{82} & z_6 & z_{70} & z_{52} & z_{48} & z_{44} & -z_4-z_5-z_6 \\
z_7 & z_{55} & z_{59} & z_{63} & \overline{z_{57}} & z_8 & z_{85} & \overline{z_{62}} & \overline{z_{61}} & z_{86} & z_9 & \overline{z_{58}} & z_{64} & \overline{z_{60}} & z_{56} & -z_7-z_8-z_9 \\
\overline{z_1} & \overline{z_{65}} & \overline{z_{78}} & \overline{z_{40}} & \overline{z_{31}} & \overline{z_2} & \overline{z_{74}} & \overline{z_{36}} & \overline{z_{35}} & \overline{z_{73}} & \overline{z_3} & \overline{z_{32}} & \overline{z_{39}} & \overline{z_{77}} & \overline{z_{66}} & -\overline{z_1}-\overline{z_2}-\overline{z_3} \\
x_4 & z_{111} & z_{91} & z_{99} & \overline{z_{111}} & x_5 & z_{109} & \overline{z_{92}} & \overline{z_{91}} & \overline{z_{109}} & x_6 & \overline{z_{112}} & \overline{z_{99}} & z_{92} & z_{112} & 1-x_4-x_5-x_6 \\
z_{10} & z_{95} & z_{101} & z_{105} & \overline{z_{97}} & z_{11} & z_{107} & \overline{z_{104}} & \overline{z_{103}} & z_{108} & z_{12} & \overline{z_{98}} & \overline{z_{106}} & z_{102} & z_{96} & -z_{10}-z_{11}-z_{12} \\
z_{13} & \overline{z_{71}} & \overline{z_{90}} & \overline{z_{54}} & \overline{z_{45}} & z_{14} & \overline{z_{84}} & \overline{z_{50}} & \overline{z_{49}} & \overline{z_{83}} & z_{15} & \overline{z_{46}} & \overline{z_{53}} & \overline{z_{89}} & \overline{z_{72}} & -z_{13}-z_{14}-z_{15} \\
\overline{z_4} & \overline{z_{69}} & \overline{z_{88}} & \overline{z_{52}} & \overline{z_{43}} & \overline{z_5} & \overline{z_{82}} & \overline{z_{48}} & \overline{z_{47}} & \overline{z_{81}} & \overline{z_6} & \overline{z_{44}} & \overline{z_{51}} & \overline{z_{87}} & \overline{z_{70}} & -\overline{z_4}-\overline{z_5}-\overline{z_6} \\
\overline{z_{10}} & z_{97} & z_{103} & z_{106} & \overline{z_{95}} & \overline{z_{11}} & \overline{z_{108}} & \overline{z_{102}} & \overline{z_{101}} & \overline{z_{107}} & \overline{z_{12}} & \overline{z_{96}} & \overline{z_{105}} & z_{104} & z_{98} & -\overline{z_{10}}-\overline{z_{11}}-\overline{z_{12}} \\
x_7 & z_{113} & z_{93} & z_{100} & \overline{z_{113}} & x_8 & z_{110} & \overline{z_{94}} & \overline{z_{93}} & \overline{z_{110}} & x_9 & \overline{z_{114}} & \overline{z_{100}} & z_{94} & z_{114} & 1-x_7-x_8-x_9 \\
z_{16} & \overline{z_{67}} & \overline{z_{80}} & \overline{z_{42}} & \overline{z_{33}} & z_{17} & \overline{z_{76}} & \overline{z_{38}} & \overline{z_{37}} & \overline{z_{75}} & z_{18} & \overline{z_{34}} & \overline{z_{41}} & \overline{z_{79}} & \overline{z_{68}} & -z_{16}-z_{17}-z_{18} \\
\overline{z_7} & z_{57} & z_{61} & \overline{z_{64}} & \overline{z_{55}} & \overline{z_8} & \overline{z_{86}} & \overline{z_{60}} & \overline{z_{59}} & \overline{z_{85}} & \overline{z_9} & \overline{z_{56}} & \overline{z_{63}} & \overline{z_{62}} & z_{58} & -\overline{z_7}-\overline{z_8}-\overline{z_9} \\
\overline{z_{13}} & z_{45} & z_{49} & z_{53} & z_{71} & \overline{z_{14}} & z_{83} & z_{89} & z_{90} & z_{84} & \overline{z_{15}} & z_{72} & z_{54} & z_{50} & z_{46} & -\overline{z_{13}}-\overline{z_{14}}-\overline{z_{15}} \\
\overline{z_{16}} & z_{33} & \overline{z_{37}} & z_{41} & z_{67} & \overline{z_{17}} & z_{75} & z_{79} & z_{80} & z_{76} & \overline{z_{18}} & z_{68} & z_{42} & z_{38} & z_{34} & -\overline{z_{16}}-\overline{z_{17}}-\overline{z_{18}} \\
x_{10} & z_{25} & z_{27} & z_{29} & \overline{z_{25}} & x_{11} & z_{30} & \overline{z_{28}} & \overline{z_{27}} & \overline{z_{30}} & x_{12} & \overline{z_{26}} & \overline{z_{29}} & z_{28} & z_{26} & 1-x_{10}-x_{11}-x_{12}
\end{pmatrix},$$

$$z_1, \ldots, z_{114} \in \mathbb{C} \; x_1, \ldots, x_{12} \in \mathbb{R}.$$

Now suppose $A \in M_{16}(\mathbb{C})$ then we obtain the above form by demanding that

$$A(e_1 + e_6 + e_{11} + e_{16}) = (e_1 + e_6 + e_{11} + e_{16})$$

and

$$\widehat{\Omega}\overline{A} = A\widehat{\Omega},$$

where

$$\widehat{\Omega} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

We can rewrite this $\widehat{\Omega}$ as a block matrix

$$\widehat{\Omega} = \begin{pmatrix} e_{11} & e_{21} & e_{31} & e_{41} \\ e_{12} & e_{22} & e_{32} & e_{42} \\ e_{13} & e_{23} & e_{33} & e_{43} \\ e_{14} & e_{24} & e_{34} & e_{44} \end{pmatrix} = \begin{pmatrix} e_{11}^{\mathsf{T}} & e_{12}^{\mathsf{T}} & e_{13}^{\mathsf{T}} & e_{14}^{\mathsf{T}} \\ e_{21}^{\mathsf{T}} & e_{22}^{\mathsf{T}} & e_{23}^{\mathsf{T}} & e_{24}^{\mathsf{T}} \\ e_{31}^{\mathsf{T}} & e_{32}^{\mathsf{T}} & e_{33}^{\mathsf{T}} & e_{34}^{\mathsf{T}} \\ e_{41}^{\mathsf{T}} & e_{42}^{\mathsf{T}} & e_{43}^{\mathsf{T}} & e_{44}^{\mathsf{T}} \end{pmatrix} = \sum e_{ij} \otimes e_{ji}.$$

The eigenvectors are given by $e_l \otimes e_k \pm e_k \otimes e_l$ for $l \neq k$, with eigenvalues 1 and -1 depending on the sign, and $e_i \otimes e_i$ for $i = 1, \ldots, 4$ with eigenvalue 1. We now see that

$$\sum e_i \otimes e_i$$

is also an eigenvector with eigenvalue one. We can diagonalize $\widehat{\Omega}$ with a new basis consisting of these eigenvectors, which will lead to

$$\Omega = \begin{pmatrix} I_{10} & 0 \\ 0 & -I_6 \end{pmatrix}.$$

As we can see e_1 is an eigenvector of Ω and if we identify e_1 in terms of the new basis with the vector $\sum e_i \otimes e_i$ in the old basis, we see that e_1 is also an eigenvector of a matrix A in the perturbation semigroup. Combining this gives

$$\mathrm{Pert}(M_4(\mathbb{C})) \cong \Big\{A \in M_{16}(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega\Big\},$$

with

$$\Omega = \begin{pmatrix} I_{10} & 0 \\ 0 & -I_6 \end{pmatrix}.$$

The invertible elements in this perturbation semigroup are given by

$$\mathrm{Pert}(M_4(\mathbb{C}))^\times \cong \Big\{A \in GL_{16}(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega\Big\}.$$

In a similar way as for $\mathrm{Pert}(M_3(\mathbb{C}))$ we can construct the embedding of the unitaries in $\mathrm{Pert}(M_4(\mathbb{C}))$. We construct a transformation matrix using the eigenvectors. Note that the columns need to be orthogonal. For the unitary matrix u we use that $\sum_j u_{ij}\overline{u_{kj}} = \delta^i_k$. This way, though a lot of work, we can determine the embedding of the unitaries in $\mathrm{Pert}(M_4(\mathbb{C}))$.

3.2.4 $\mathcal{A} = M_N(\mathbb{C})$

With these examples in mind we now proceed and determine $\mathrm{Pert}(M_N(\mathbb{C}))$. First note that the matrices in the perturbation semigroup $\mathrm{Pert}(M_N(\mathbb{C}))$ will be elements of $M_{N^2}(\mathbb{C})$. For the normalization condition we have the following proposition

Proposition 3.4. *For $i = 1, \ldots, N$ the normalization condition is equivalent to*

$$\sum_j C_{ij,ji} = 1$$

and for $i, l = 1, \ldots, N$ and $i \neq l$ it is equivalent to

$$\sum_j C_{ij,jl} = 0.$$

Proof. An element $\sum a_j \otimes b_j^{op} \in M_N(\mathbb{C}) \otimes M_N(\mathbb{C})^{op}$ has to be normalized, which means

$$\sum a_j b_j = 1,$$

where the 1 is the unit in $M_N(\mathbb{C})$. Therefore, in terms of our basis e_{ij} and coefficients $C_{ij,kl}$, we get

$$\sum C_{ij,kl} e_{ij} e_{kl} = \sum e_{ii},$$

where the right hand side is the unit 1. Using (3.2) we see that we can rewrite it as

$$\sum C_{ij,kl} e_{il} \delta_k^j = \sum e_{ii}.$$

For $i = l$ the coefficients on the LHS need to equal 1, hence

$$\sum_j C_{ij,ji} = 1.$$

For $i \neq l$ the coefficients on the RHS equal zero, so

$$\sum_j C_{ij,jl} = 0.$$

□

Remark 3.5. *Note that this proposition, and hence the normalization condition, implies that $\sum_i e_i \otimes e_i$ is an eigenvector for such a matrix A in the perturbation semigroup with eigenvalue* 1*.*

While in the previous section we determined the general form of a matrix in the semigroup after which we found two defining properties, this time we do it the other way around. We now determine a matrix $\widehat{\Omega}$ which will give the general form of matrices in the perturbation semigroup, but first let us look at the following lemma

Lemma 3.6. *For $A = \sum C_{ij,kl} e_{ij} \otimes e_{kl}^{op}$ the self-adjointness condition is equivalent to demanding*

$$C_{ij,kl} = \overline{C_{lk,ji}}.$$

Proof. Let $A = A^*$, then we have

$$\begin{array}{rcl} A & = & \sum C_{ij,kl} e_{ij} \otimes e_{kl}^{op} \\ & = & \sum \overline{C_{ij,kl}} e_{kl}^{*} \otimes e_{ij}^{*op} \\ & = & \sum \overline{C_{ij,kl}} e_{lk} \otimes e_{ji}^{op} \end{array}.$$

If we now relabel the last expression, we get

$$\sum \overline{C_{lk,ji}} e_{ij} \otimes e_{kl}^{op},$$

hence

$$C_{ij,kl} = \overline{C_{lk,ji}}$$

□

We now have the following proposition.

Proposition 3.7. *Let $A = \sum C_{ij,kl} e_{ij} \otimes e_{kl}^{op}$. Then $C_{ij,kl} = \overline{C_{lk,ji}}$ if and only if $\widehat{\Omega}\overline{A} = A\widehat{\Omega}$ with $\widehat{\Omega} = \sum e_{ij} \otimes e_{ij}^{op} \in M_N(\mathbb{C}) \otimes M_N(\mathbb{C})^{op}$.*

Proof. We can write $\widehat{\Omega}$ as $\widehat{\Omega} = \sum \delta_m^r \delta_n^s e_{mn} \otimes e_{rs}^{op}$. Starting with the right hand side of the equation we get

$$\begin{aligned} A\widehat{\Omega} &= (\sum C_{ij,kl} e_{ij} \otimes e_{kl}^{op})(\sum \delta_m^r \delta_n^s e_{mn} \otimes e_{rs}^{op}) \\ &= \sum C_{ij,kl} \delta_m^r \delta_n^s e_{ij} e_{mn} \otimes (e_{rs} e_{kl})^{op} \\ &= \sum C_{ij,kl} \delta_m^r \delta_n^s \delta_j^m \delta_s^k e_{in} \otimes e_{rl}^{op} \\ &= \sum C_{ij,kl} \delta_j^r \delta_n^k e_{in} \otimes e_{rl}^{op} \\ &= \sum C_{ij,kl} e_{ik} \otimes e_{jl}^{op}. \end{aligned}$$

The left hand side of the equation reads

$$\begin{aligned} \widehat{\Omega}\overline{A} &= (\sum \delta_m^r \delta_n^s e_{mn} \otimes e_{rs}^{op})(\sum \overline{C_{ij,kl}} e_{ij} \otimes e_{kl}^{op}) \\ &= \sum \overline{C_{ij,kl}} \delta_m^r \delta_n^s e_{mn} e_{ij} \otimes (e_{kl} e_{rs})^{op} \\ &= \sum \overline{C_{ij,kl}} \delta_m^r \delta_n^s \delta_i^n e_{mj} \otimes \delta_r^l e_{ks}^{op} \\ &= \sum \overline{C_{ij,kl}} \delta_m^l \delta_i^s e_{mj} \otimes e_{ks}^{op} \\ &= \sum \overline{C_{ij,kl}} e_{lj} \otimes e_{ki}^{op} \\ &= \sum \overline{C_{lk,ji}} e_{ik} \otimes e_{jl}^{op}. \end{aligned}$$

Thus we have $C_{ij,kl} = \overline{C_{lk,ji}}$ if and only if $\widehat{\Omega}\overline{A} = A\widehat{\Omega}$. □

We now make the following identification

$$\begin{aligned} &M_N(\mathbb{C}) \otimes M_N(\mathbb{C})^{op} \to M_{N^2}(\mathbb{C}), \\ &\qquad C_{ij,kl} e_{ij} \otimes e_{kl}^{op} \mapsto C_{ij,kl} e_{ij} \otimes e_{lk} = C_{ij,kl} e_{N(i-1)+l,\, N(j-1)+k}, \end{aligned}$$

after which we can bring $\widehat{\Omega}$ into a more appealing form as a block matrix. So we get

$$\widehat{\Omega} = \sum e_{ij} \otimes e_{ji}.$$

Lemma 3.8. *The eigenvectors of $\widehat{\Omega}$ are given by $e_k \otimes e_l \pm e_l \otimes e_k$ with eigenvalue 1 and -1 ($k \neq l$).*

Proof. First suppose $k \neq l$, then we have

$$\begin{aligned} \widehat{\Omega}(e_k \otimes e_l \pm e_l \otimes e_k) &= (\sum e_{ij} \otimes e_{ji})(e_k \otimes e_l \pm e_l \otimes e_k), \\ &= \sum e_{ij} e_k \otimes e_{ji} e_l \pm \sum e_{ij} e_l \otimes e_{ji} e_k, \\ &= \sum e_i \delta_k^j \otimes e_j \delta_l^i \pm \sum e_i \delta_l^j \otimes e_j \delta_k^i, \\ &= e_l \otimes e_k \pm e_k \otimes e_l, \\ &= \pm(e_k \otimes e_l \pm e_l \otimes e_k). \end{aligned}$$

Thus $e_k \otimes e_l \pm e_l \otimes e_k$ is indeed an eigenvector with eigenvalue 1 or -1 depending on the sign.

Now suppose $k = l$. We now need to show that $e_k \otimes e_k$ is an eigenvector of $\widehat{\Omega}$ for all k. This is indeed the case, since we have

$$\begin{aligned} \widehat{\Omega}(e_k \otimes e_k) &= (\sum e_{ij} \otimes e_{ji})(e_k \otimes e_k), \\ &= \sum e_{ij} e_k \otimes e_{ji} e_k, \\ &= \sum e_i \delta_k^j \otimes e_j \delta_k^i, \\ &= e_k \otimes e_k, \end{aligned}$$

Therefore $e_k \otimes e_k$ is an eigenvector with eigenvalue 1. Note that we now have all the eigenvectors since there are $N(N+1)$ eigenvectors $e_k \otimes e_l + e_l \otimes e_k$, however interchanging l and k will not change the eigenvector, so essentially we have $N(N+1)/2$ eigenvectors of this form. We also have $N(N-1)$ eigenvectors $e_k \otimes e_l - e_l \otimes e_k$ (for $l \neq k$), but once again interchanging k and l will give a minus sign and therefore we essentially have $N(N-1)/2$ eigenvectors of this form. If we now add these we see that we have $N(N+1)/2 + N(N-1)/2 = N^2$ eigenvectors, hence the lemma is proven. □

Since $e_k \otimes e_k$ is an eigenvector for all k and the eigenvalue is 1 for all these vectors, we see that their sum must be an eigenvector with eigenvalue 1 as well, i.e. we have

$$\widehat{\Omega}(\sum_i e_i \otimes e_i) = \sum_i e_i \otimes e_i.$$

We change to a basis consisting of eigenvectors, where we take $\sum e_i \otimes e_i$ in terms of the old basis to be identified with e_1 in the new basis. This will give us

$$\Omega = \begin{pmatrix} I_{N(N+1)/2} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix}. \tag{3.3}$$

Note that the number of minus ones and plus ones match the number of eigenvalues. As we have seen before $\sum e_i \otimes e_i$ in terms of our old basis, is an eigenvector of a matrix A in the perturbation semigroup. In the new basis e_1 is thus an eigenvector of such a matrix A. Combining the above results gives the following theorem

Theorem 3.9. *We have*

$$\mathrm{Pert}(M_N(\mathbb{C})) \cong \left\{ A \in M_{N^2}(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \right\} \tag{3.4}$$

where

$$\Omega = \begin{pmatrix} I_{N(N+1)/2} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix}.$$

The semigroup structure is given by matrix multiplication.

Proof. We have already seen that

$$M_N(\mathbb{C}) \otimes M_N(\mathbb{C})^{op} \cong M_N(\mathbb{C}) \otimes M_N(\mathbb{C}) = M_{N^2}(\mathbb{C}),$$

hence the perturbation semigroup consists of matrices in $M_{N^2}(\mathbb{C})$. As seen in the above propositions the normalization condition and the self-adjointness can be translated into two defining properties. Let $A \in \mathrm{Pert}(M_N(\mathbb{C}))$ then

$$Ae_1 = e_1$$

and

$$\Omega\overline{A} = A\Omega,$$

where

$$\Omega = \begin{pmatrix} I_{N(N+1)/2} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix}.$$

Thus the elements in the perturbation semigroup are matrices $A \in M_{N^2}(\mathbb{C})$ which fulfill two conditions, hence we get

$$\mathrm{Pert}(M_N(\mathbb{C})) \cong \left\{ A \in M_{N^2}(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \right\}.$$

We now need to prove that for $A, B \in \mathrm{Pert}(M_N(\mathbb{C}))$ also $AB \in \mathrm{Pert}(M_N(\mathbb{C}))$. So suppose $A, B \in \mathrm{Pert}(M_N(\mathbb{C}))$, then

$$(AB)e_1 = A(Be_1) = Ae_1 = e_1$$

and.

$$\Omega(\overline{AB}) = \Omega\overline{A}\overline{B} = A\Omega\overline{B} = AB\Omega = (AB)\Omega.$$

Hence $AB \in \mathrm{Pert}(M_N(\mathbb{C}))$. Therefore the perturbation semigroup is closed under the operation. $\square$

Similarly the invertible elements in the perturbation semigroup are given by

$$\mathrm{Pert}(M_N(\mathbb{C}))^\times \cong \left\{ A \in GL_{N^2}(\mathbb{C}) \mid Ae_1 = e_1,\ \Omega\overline{A} = A\Omega \right\}.$$

Now let us take a closer look at $\mathrm{Pert}(M_N(\mathbb{C}))$ again. Let $A \in M_{N^2}(\mathbb{C})$ with $Ae_1 = e_1$, then we get that

$$A = \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix}.$$

Here v is a row vector of length $N^2 - 1$, while $B \in M_{N^2-1}(\mathbb{C})$. However, we also know that $\Omega\overline{A} = A\Omega$. If we set

$$\Omega' = \begin{pmatrix} I_{N(N+1)/2-1} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix} = \begin{pmatrix} I_{(N+2)(N-1)/2} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix},$$

then the condition that $\Omega\overline{A} = A\Omega$ can be rewritten as

$$\Omega'\overline{B} = B\Omega',$$

and

$$\overline{v} = v\Omega'.$$

Note that the equality $\overline{v} = v\Omega'$ does indeed hold. If we work this out we see that

$$v = \begin{pmatrix} v_1 & iv_2 \end{pmatrix},$$

where v_1 and v_2 are both real row vectors of resp. length $(N-1)(N+2)/2$ and $N(N-1)/2$. We also see that

$$B = \begin{pmatrix} B_1 & iB_2 \\ iB_3 & B_4 \end{pmatrix},$$

where $B_1, \ldots, B_4$ are all real matrices. The dimensions of these matrices are $(N-1)(N+2)/2 \times (N-1)(N+2)/2$, $N(N-1)/2 \times (N-1)(N+2)/2$, $(N-1)(N+2)/2 \times N(N-1)/2$, $N(N-1)/2 \times N(N-1)/2$, resp.

Now define

$$\begin{aligned} V &= \left\{ v \in \mathbb{C}^{N^2-1} \mid \overline{v} = v\Omega' \right\}, \\ S &= \left\{ A \in M_{N^2-1}(\mathbb{C}) \mid \Omega'\overline{A} = A\Omega' \right\}. \end{aligned}$$

We can now construct the semidirect product of V and S which gives

Lemma 3.10. *Let V, S be as above, then $V \rtimes S$ is in fact a semigroup defined by*

$$(v, A) \cdot (v', A') = (v' + vA', AA').$$

Proof. First note that $vA' \in V$ since

$$\begin{aligned} vA'\Omega' &= v\Omega'\overline{A'}, \\ &= \overline{v}\overline{A'}, \\ &= \overline{vA'}. \end{aligned}$$

Of course $AA' \in S$ since S is a semigroup. Therefore this semigroup is closed under the operation. The unit of this semigroup is given by

$$1 = (0, I),$$

since

$$(v, A) \cdot (0, I) = (0 + vI, AI) = (v, A).$$

Thus it is in fact a semigroup with the above operation. □

With this lemma we get

Theorem 3.11. *For V and S as above we can write*

$$\mathrm{Pert}(M_N(\mathbb{C})) \cong V \rtimes S.$$

Proof. Let $A, A' \in \mathrm{Pert}(M_N(\mathbb{C}))$, then we have

$$A = \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix}$$

and

$$A' = \begin{pmatrix} 1 & v' \\ 0 & B' \end{pmatrix}$$

for suitable $v, v' \in V$ and $B, B' \in S$. If we now multiply A and A' we get

$$AA' = \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & v' \\ 0 & B' \end{pmatrix} = \begin{pmatrix} 1 & v' + vB \\ 0 & BB' \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & v' + vB \\ 0 & BB' \end{pmatrix}$$

can be parameterized by the second column and that column equals the semidirect product defined in the previous lemma. So we get

$$\mathrm{Pert}(M_N(\mathbb{C})) \cong V \rtimes S.$$

□

First consider the following lemma

Lemma 3.12. *Let V be a vectorspace and G a group which acts on V, then $V \rtimes G$ is a group, with the operation*

$$(v, A) \cdot (v', A') = (v' + vA', AA').$$

Proof. That the group is closed under the operation and that there is a unit has already been proven in the previous proposition. All that is left is to prove that there is an inverse element. In fact we have

$$(v, A)^{-1} = (-vA^{-1}, A^{-1}),$$

because

$$(v, A) \cdot (-vA^{-1}, A^{-1}) = (v + (-vA^{-1})A, AA^{-1}) = (v - v, I) = 1.$$

We know that A^{-1} exists, since G is a group. □

We can now look at the invertible elements in the perturbation semigroup, but first let

$$T = S^\times = \left\{A \in GL_{N^2-1}(\mathbb{C}) \mid \Omega'\overline{A} = A\Omega'\right\}.$$

Then we have

Proposition 3.13. *For V a vector space and S a semigroup working on V we have*

$$(V \rtimes S)^\times = V \rtimes S^\times.$$

This follows from

$$0 \neq \det(A) = \begin{vmatrix} 1 & v \\ 0 & B \end{vmatrix} = \det(B),$$

where $A \in \mathrm{Pert}(M_N(\mathbb{C}))$. This gives

Proposition 3.14. *Let*

$$V = \left\{v \in \mathbb{C}^{N^2-1} \mid \overline{v} = v\Omega'\right\}$$

and

$$T = \left\{A \in GL_{N^2-1}(\mathbb{C}) \mid \Omega'\overline{A} = A\Omega'\right\}$$

then we have

$$\mathrm{Pert}(M_N(\mathbb{C}))^\times \cong V \rtimes T.$$

In general one can find the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{C}))$ using a transformation matrix consisting of eigenvectors. However, these eigenvectors need to be orthogonal. Note that we can use Gramm-Schmidt orthogonalization in order to construct these eigenvectors. We only need to orthogonalize the vectors with eigenvalue 1 and -1 seperately, since the eigenvectors with eigenvalue 1 are orthogonal to the eigenvectors with eigenvalue -1. If we look at the eigenvectors with eigenvalue 1 (or -1) we see that every linear combination of them is also an eigenvector with eigenvalue 1 (or -1). For the unitary matrix u we use that $\sum u_{ij}\overline{u_{kj}} = \delta^i_k$. This way, though a lot of work, we can determine the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{C}))$.

3.3 Perturbation semigroup of real matrix algebras

Now that we have the semigroup $\mathrm{Pert}(M_N(\mathbb{C}))$ we can take a look at the perturbation semigroup of a real matrix algebra, to wit $\mathrm{Pert}(M_N(\mathbb{R}))$ and $\mathrm{Pert}(M_N(\mathbb{H}))$.

3.3.1 $\mathcal{A} = M_N(\mathbb{R})$

We can start with the results we obtained for $\mathrm{Pert}(M_N(\mathbb{C}))$. In the calculations we did use that fact the the entries were complex, in the sense that $C_{ij,kl} = \overline{C_{lk,ji}}$. We can however use the same reasoning for real entries and neglect complex conjugation, so

$$\mathrm{Pert}(M_N(\mathbb{R})) \cong \Big\{ A \in M_{N^2}(\mathbb{R}) \mid Ae_1 = e_1,\ \Omega A = A\Omega \Big\},$$

where

$$\Omega = \begin{pmatrix} I_{N(N+1)/2} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix}.$$

The invertible elements in the perturbation semigroup are then given by

$$\mathrm{Pert}(M_N(\mathbb{R}))^\times = \Big\{ A \in GL_{N^2}(\mathbb{R}) \mid Ae_1 = e_1,\ \Omega A = A\Omega \Big\}.$$

If we now take a closer look at $\mathrm{Pert}(M_N(\mathbb{R}))$ we see that we get

$$\begin{pmatrix} 1 & v_1 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix}.$$

Note that this equals the general form we have seen for $\mathrm{Pert}(M_N(\mathbb{C}))$, however the complex parts now equal zero. If we now, once again, multiply two of such matrices, say A and A', we see that

$$AA' = \begin{pmatrix} 1 & v_1 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix} \begin{pmatrix} 1 & w_1 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{pmatrix} = \begin{pmatrix} 1 & w_1 + v_1 B_2 & 0 \\ 0 & B_1 C_1 & 0 \\ 0 & 0 & B_2 C_2 \end{pmatrix}.$$

The upper left 2×2-block looks exactly like the matrix we got in the previous section. While the lower right entry is just simple matrix multiplication. Using our knowledge about a semidirect product gives us

Theorem 3.15. *We have*

$$\mathrm{Pert}(M_N(\mathbb{R})) \cong \Big(\mathbb{R}^{(N-1)(N+2)/2} \rtimes M_{(N-1)(N+2)/2}(\mathbb{R}) \Big) \times M_{N(N-1)/2}(\mathbb{R}).$$

Proof. The first part is just the application of Lemma 3.10 with $V = \mathbb{R}^{(N-1)(N+2)/2}$ and $S = M_{(N-1)(N+2)/2}(\mathbb{R})$. Note that we do not have to impose extra conditions on V and S with respect to Ω, since the matrix got this form from Ω. The second part is the lower right entry, which behaves as simple matrix multiplication. □

Taking a closer look at the invertible elements we see that matrices $A \in \mathrm{Pert}(M_N(\mathbb{R}))$ have a rather interesting form

$$A = \begin{pmatrix} 1 & v & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix},$$

where v is a $1\times(N-1)(N+2)/2$-matrix, C a $(N-1)(N+2)/2\times(N-1)(N+2)/2$-matrix and D a $N(N-1)/2 \times N(N-1)/2$-matrix. Note that this is the same matrix as in the $M_N(\mathbb{C})$ case, but the complex entries now equal zero. If A is invertible then so are C and D because we have

$$0 \neq \det(A) = 1 \cdot \begin{vmatrix} C & 0 \\ 0 & D \end{vmatrix} = \det(C)\det(D).$$

If we now multiply two matrices in the perturbation semigroup, say A, B, we see that

$$AB = \begin{pmatrix} 1 & v & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix} = \begin{pmatrix} 1 & w & 0 \\ 0 & E & 0 \\ 0 & 0 & F \end{pmatrix} = \begin{pmatrix} 1 & w+vD & 0 \\ 0 & CE & 0 \\ 0 & 0 & DF \end{pmatrix}.$$

Here the second column behaves similar as in the complex-case, while the last column is just matrix multiplication. Note that the only condition we have on the vector and the matrices is that the entries are real. So if we use the result about $V \rtimes G$ from the previous section,

$$(v, A) \cdot (v', A') = (v' + vA', AA')$$

with

$$V = \mathbb{R}^{(N-1)(N+2)/2},$$
$$G = GL_{(N-1)(N+2)/2}(\mathbb{R}),$$

we get

$$\mathrm{Pert}(M_N(\mathbb{R}))^\times \cong \left(\mathbb{R}^{(N-1)(N+2)/2} \rtimes GL_{(N-1)(N+2)/2}(\mathbb{R})\right) \times GL_{N(N-1)/2}(\mathbb{R})$$

In a similar way as for $\mathrm{Pert}(M_N(\mathbb{C}))$ we can find the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{R}))$. There is however one difference, namely that for the unitary matrix we have $\sum u_{ij}u_{ki} = \delta^i_k$. This way we can determine the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{R}))$. Note that the moment we will find the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{C}))$, we have also found the embedding in $\mathrm{Pert}(M_N(\mathbb{R}))$, since we can neglect the complex terms.

3.3.2 $\mathcal{A} = \mathbb{H}$

We want to determine the perturbation semigroup of the quaternions, notated by $\mathbb{H}$. First, recall that

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}.$$

For matrices in $\mathbb{H}$ we have the following lemma

Lemma 3.16. *For $A \in M_2(\mathbb{C})$ to be in $\mathbb{H}$ we have $\widehat{J}\overline{A} = A\widehat{J}$ where*

$$\widehat{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{12} - e_{21}.$$

Proof. For $A \in M_2(\mathbb{C})$ we can write

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. The RHS reads

$$\widehat{J}\overline{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix} = \begin{pmatrix} \overline{\gamma} & \overline{\delta} \\ -\overline{\alpha} & -\overline{\beta} \end{pmatrix},$$

while the LHS reads

$$A\widehat{J} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\beta & \alpha \\ -\delta & \gamma \end{pmatrix}.$$

Therefore $\alpha = \overline{\delta}$ and $\beta = -\overline{\gamma}$. Hence

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbb{H}.$$

□

Since the quaternions form a real subalgebra of $M_2(\mathbb{C})$ we can start by looking at the matrices in $\mathrm{Pert}(M_2(\mathbb{C}))$. Recall that the general form of the matrices was given by

$$A = \begin{pmatrix} x_1 & z_2 & \overline{z_2} & 1 - x_1 \\ z_1 & z_4 & \overline{z_5} & -z_1 \\ \overline{z_1} & z_5 & \overline{z_4} & -\overline{z_1} \\ x_2 & z_3 & \overline{z_3} & 1 - x_2 \end{pmatrix},$$

$$z_i \in \mathbb{C}, \text{ for } i = 1, \dots, 5, \ x_1, x_2 \in \mathbb{R}.$$

We now have to impose our condition regarding $\widehat{J}$ in order to get a matrix in $\mathrm{Pert}(\mathbb{H})$. In the tensor product $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{op}$ this $\widehat{J}$ behaves almost the same way. Now for $A \otimes B^{op} \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{op}$ to be in $\mathbb{H} \otimes \mathbb{H}^{op}$ we have

$$(\widehat{J} \otimes \widehat{J}^{op})\overline{(A \otimes B^{op})} = (\widehat{J} \otimes \widehat{J}^{op})(\overline{A} \otimes \overline{B}^{op}) = (A \otimes B^{op})(\widehat{J} \otimes \widehat{J}^{op}).$$

Once again using the identification

$$\begin{aligned} M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{op} &\to M_4(\mathbb{C}), \\ e_{ij} \otimes e_{kl}^{op} &\mapsto e_{ij} \otimes e_{lk} \end{aligned}$$

we get

$$\widehat{J} \otimes \widehat{J}^{op} \mapsto \widetilde{J} = (e_{12} - e_{21}) \otimes (e_{12} - e_{21})^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

So for a matrix $A \in \mathrm{Pert}(\mathbb{H})$ we need to have

$$\widetilde{J}\overline{A} = A\widetilde{J}.$$

The matrix A in the perturbation semigroup can now be written as

$$A = \begin{pmatrix} x & z_2 & \overline{z_2} & 1-x \\ z_1 & z_3 & z_4 & -z_1 \\ \overline{z_1} & \overline{z_4} & \overline{z_3} & -\overline{z_1} \\ 1-x & -z_2 & -\overline{z_2} & x \end{pmatrix},$$

where $x \in \mathbb{R}, z_1, z_2, z_3, z_4 \in \mathbb{C}$. Since this is the same form as for $A \in \mathrm{Pert}(M_2(\mathbb{C}))$ it follows that we have a similar commutation relation for this A with $\widehat{\Omega}$, namely $\widehat{\Omega}\overline{A} = A\widehat{\Omega}$. Once again we can diagonalize $\widehat{\Omega}$ to

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This new basis consists of eigenvectors, which are

$$e_1 \otimes e_2 \pm e_2 \otimes e_1,$$

$$e_1 \otimes e_1 \pm e_2 \otimes e_2.$$

Recall that these are indeed eigenvectors of $\widehat{\Omega}$. Let us now write $\widetilde{J} = \widehat{J} \otimes \widehat{J}^{op}$ in terms of this new basis. We see that

$$\begin{aligned} \Big((e_{12} - e_{21}) \otimes (e_{21} - e_{12})\Big)(e_1 \otimes e_2 \pm e_2 \otimes e_1) &= \pm(e_1 \otimes e_2 \pm e_2 \otimes e_1), \\ \Big((e_{12} - e_{21}) \otimes (e_{21} - e_{12})\Big)(e_1 \otimes e_1 \pm e_2 \otimes e_2) &= \mp(e_1 \otimes e_1 \pm e_2 \otimes e_2). \end{aligned}$$

Since the first eigenvector of Ω was $e_1 \otimes e_1 + e_2 \otimes e_2$ we retrieve the following expression for $\widetilde{J}$ in terms of the new basis

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We see that e_1 in this new basis, $e_1 \otimes e_1 + e_2 \otimes e_2$ in the old, is an eigenvector of J. With this we can find the general expression for $\mathrm{Pert}(\mathbb{H})$.

Proposition 3.17. *We have*

$$\mathrm{Pert}(\mathbb{H}) \cong \Big\{A \in M_4(\mathbb{C}) \mid Ae_1 = e_1, \Omega\overline{A} = A\Omega, J\overline{A} = AJ\Big\}.$$

Proof. Since we started with the perturbation semigroup $\mathrm{Pert}(M_2(\mathbb{C}))$, we only have to show that imposing the condition $J\overline{A} = AJ$ gives $\mathrm{Pert}(\mathbb{H})$. But we know that for a general 2×2-matrix B, we can retrieve the general form of a quaternion by demanding that $\widehat{J}\overline{B} = B\widehat{J}$. We also saw that a similar relation holds in the tensor product of two such matrices, this time with $\widetilde{J}$. If we now choose the matrices A such that $A \in \mathrm{Pert}(M_2(\mathbb{C}))$ the same reasoning holds and thus the matrix has the same commutation relation with $\widetilde{J}$. Changing to the new basis then gives J and this gives the result. □

Since Ω and J have the same commutation relation with $A \in \mathrm{Pert}(\mathbb{H})$, also the sum and difference of Ω and J must have this commutation relation with A. If we now set

$$\Upsilon = (\Omega - J)/2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma = (\Omega + J)/2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we get

$$\mathrm{Pert}(\mathbb{H}) \cong \left\{ A \in M_4(\mathbb{C}) \mid Ae_1 = e_1, \Upsilon\overline{A} = A\Upsilon, \Gamma\overline{A} = A\Gamma \right\}.$$

Now let $A \in \mathrm{Pert}(\mathbb{H})$ then by $Ae_1 = e_1$ we get

$$A = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 & x_6 \\ 0 & x_7 & x_8 & x_9 \\ 0 & x_{10} & x_{11} & x_{12} \end{pmatrix}.$$

Now $\Gamma\overline{A} = A\Gamma$ gives

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 & x_6 \\ 0 & x_7 & x_8 & x_9 \end{pmatrix}$$

and $\Upsilon\overline{A} = A\Upsilon$ then gives

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & iy_1 \\ 0 & x_3 & x_4 & iy_2 \\ 0 & iy_3 & iy_4 & x_5 \end{pmatrix},$$

where $x_1, \ldots, x_5, y_1, \ldots, y_4 \in \mathbb{R}$. If we now have two of such matrices, say A and B we see that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & CD \end{pmatrix}$$

for given C, D. Thus AB is again in the perturbation semigroup. Note that we can parameterize A with a given 3×3-matrix. Now define

$$\Gamma' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

so Γ without the first row and column. Then we have

Proposition 3.18. *The perturbation semigroup for $\mathbb{H}$ is given by*

$$\mathrm{Pert}(\mathbb{H}) \cong \{A \in M_3(\mathbb{C}) \mid \Gamma'\overline{A} = A\Gamma'\},$$

which is a monoid.

Proof. We got a general form for $A \in \mathrm{Pert}(\mathbb{H})$ which can be parameterized by a 3×3-matrix which has a commutation relation with Γ'. This gives the expression $\mathrm{Pert}(\mathbb{H})$.

We also have $I_3 \in \mathrm{Pert}(\mathbb{H})$, since

$$\Gamma'\overline{I_3} = \Gamma' I_3 = \Gamma' = I_3\Gamma'.$$

Also for $A, B \in \mathrm{Pert}(\mathbb{H})$ we have

$$\Gamma'\overline{(AB)} = (\Gamma'\overline{A})\overline{B} = A\Gamma'\overline{B} = AB\Gamma' = (AB)\Gamma',$$

hence $AB \in \mathrm{Pert}(\mathbb{H})$, which proves the proposition. $\square$

For $A \in \mathrm{Pert}(\mathbb{H})$, A invertible, we have $A^{-1} \in \mathrm{Pert}(\mathbb{H})$, since

$$\Gamma'\overline{A^{-1}} = \Gamma'\overline{A}^{-1} = (\overline{A}\Gamma'^{-1})^{-1} = (\overline{A}\Gamma')^{-1} = (\Gamma' A)^{-1} = A^{-1}\Gamma'^{-1} = A^{-1}\Gamma'.$$

Corollary 3.19. *The invertible elements are given by*

$$\mathrm{Pert}(\mathbb{H})^{\times} \cong \{A \in GL_3(\mathbb{C}) \mid \Gamma'\overline{A} = A\Gamma'\}.$$

In order to find the embedding of the unitaries in $\mathrm{Pert}(\mathbb{H})$ we can use the result we have found for $\mathrm{Pert}(M_2(\mathbb{C}))$. The unitary 2 by 2 matrices are given by

$$\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \text{ and } \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix},$$

where $|a|^2 + |b|^2 = 1$. We see that the first matrix is in fact an element of $\mathbb{H}$. The second matrix is not an element of $\mathbb{H}$, because in order for it to be, we need to have $a = 0$ and $b = 0$. So we only need to look at

$$\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}.$$

The computation has already been done in section 3.2.1. We get the general form

$$\begin{pmatrix} |a|^2 & -ab & -\overline{ab} & |b|^2 \\ a\overline{b} & a^2 & -\overline{b}^2 & -a\overline{b} \\ \overline{a}b & -b^2 & \overline{a}^2 & -\overline{a}b \\ |b|^2 & ab & \overline{ab} & |a|^2 \end{pmatrix},$$

which can be brought to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & |a|^2 - |b|^2 & -2\Re(ab) & -2i\Im(ab) \\ 0 & 2\Re(a\overline{b}) & \Re(a^2 - b^2) & i\Im(a^2 - b^2) \\ 0 & 2i\Im(a\overline{b}) & i\Im(a^2 + b^2) & \Re(a^2 + b^2) \end{pmatrix}$$

by a basis transformation. The embedding is then given by

$$\mathcal{U}(\mathbb{H}) \to \left\{ \begin{pmatrix} |a|^2 - |b|^2 & -2\Re(ab) & -2i\Im(ab) \\ 2\Re(a\overline{b}) & \Re(a^2 - b^2) & i\Im(a^2 - b^2) \\ 2i\Im(a\overline{b}) & i\Im(a^2 + b^2) & \Re(a^2 + b^2) \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\}.$$

3.3.3 $\mathcal{A} = M_N(\mathbb{H})$

We now want to determine the perturbation semigroup for $\mathcal{A} = M_N(\mathbb{H})$. For a matrix in $\mathrm{Pert}(M_N(\mathbb{H}))$ we have a matrix similar to $\widetilde{J}$ that we had for $\mathrm{Pert}(\mathbb{H})$, we call this matrix $\widetilde{L}$. We had $\widetilde{J}\overline{A} = A\widetilde{J}$, this becomes $\widetilde{L}\overline{B} = B\widetilde{L}$, for $B \in \mathrm{Pert}(M_N(\mathbb{H}))$.

Lemma 3.20. *We can retrieve* $\mathrm{Pert}(M_N(\mathbb{H}))$ *from* $\mathrm{Pert}(M_{2N}(\mathbb{C}))$ *by imposing*

$$\widetilde{L}\overline{A} = A\widetilde{L}$$

on $A \in \mathrm{Pert}(M_{2N}(\mathbb{C}))$. *Here* $\widetilde{L} = I_{N^2} \otimes \widetilde{J}$.

Proof. Let $A \in M_{4N^2}(\mathbb{C})$ and let $\widetilde{L} = I_{N^2} \otimes \widetilde{J}$. If we now think of A as a $N^2 \times N^2$-matrix, with as entries 4 by 4 matrices, we can write $A = \sum e_{ij} \otimes A_{ij}$, with e_{ij} the standard basis for $M_N(\mathbb{C})$ and $A_{ij} \in M_4(\mathbb{C})$.

Now we compute both $\widetilde{L}\overline{A}$ and $A\widetilde{L}$. Computing $\widetilde{L}\overline{A}$ gives

$$\widetilde{L}\overline{A} = (I_{N^2} \otimes \widetilde{J})(\sum e_{ij} \otimes \overline{A_{ij}}) = \sum I_{N^2} e_{ij} \otimes \widetilde{J}\overline{A_{ij}} = \sum e_{ij} \otimes \widetilde{J}\overline{A_{ij}},$$

while $A\widetilde{L}$ gives

$$A\widetilde{L} = (\sum e_{ij} \otimes A_{ij})(I_{N^2} \otimes \widetilde{J}) = \sum e_{ij} I_{N^2} \otimes A_{ij}\widetilde{J} = \sum e_{ij} \otimes A_{ij}\widetilde{J}.$$

Thus we get

$$\widetilde{L}\overline{A} = A\widetilde{L} \Leftrightarrow \widetilde{J}\overline{A_{ij}} = A_{ij}\widetilde{J} \Leftrightarrow A_{ij} \in \mathbb{H} \otimes \mathbb{H}^{op},$$

where we have used equation (3.5).

So by imposing $\widetilde{L}\overline{A} = A\widetilde{L}$, for $A \in \mathrm{Pert}(M_{2N}(\mathbb{C}))$, we retrieve $\mathrm{Pert}(M_N(\mathbb{H}))$ from $\mathrm{Pert}(M_{2N}(\mathbb{C}))$. $\square$

We now want to diagonalize $\widetilde{L}$ just as we did with $\widetilde{J}$ for $\mathbb{H}$. In order to do so we will prove that $\widehat{\Omega}$ and $\widetilde{L}$ commute. First let us define the basis $\{e_{i,j} \otimes f_{\alpha\beta}\}_{i,j=1,\ldots,N,\alpha,\beta=1,2}$ for both $\widetilde{L}$ and $\widehat{\Omega}$. So $f_{\alpha\beta}$ is the standard basis for $M_2(\mathbb{C})$, while $e_{i,j}$ is the standard basis for $M_N(\mathbb{C})$. In terms of this basis we can write

$$\widetilde{L} = \Big(\sum e_{kk} \otimes (f_{12} - f_{21})\Big) \otimes \Big(\sum e_{kk} \otimes (f_{21} - f_{12})\Big)$$

and

$$\widehat{\Omega} = \sum (e_{ij} \otimes f_{\alpha\beta}) \otimes (e_{ji} \otimes f_{\beta\alpha}).$$

Note that f_{12}, f_{21} are the basis vectors of J. We now have the following proposition

Proposition 3.21. *Let* $\widetilde{L}, \widehat{\Omega}$ *be as above, then we have*

$$\widetilde{L}\widehat{\Omega} = \widehat{\Omega}\widetilde{L}.$$

Proof. By multiplying $\widetilde{L}$ and $\widehat{\Omega}$ we can show that the LHS reads

$$\begin{aligned}
\widetilde{L}\widehat{\Omega} &= \Big((\sum e_{kk} \otimes (f_{12} - f_{21})) \otimes (\sum e_{kk} \otimes (f_{21} - f_{12}))\Big)\Big(\sum (e_{ij} \otimes f_{\alpha\beta}) \otimes (e_{ji} \otimes f_{\beta\alpha})\Big), \\
&= \sum \Big(\sum e_{kk} e_{ij} \otimes (f_{12} - f_{21}) f_{\alpha\beta}\Big) \otimes \Big(\sum e_{kk} e_{ji} \otimes (f_{21} - f_{12}) f_{\beta\alpha}\Big), \\
&= \sum \Big(e_{ij} \otimes (f_{1\beta}\delta^2_\alpha - f_{2\beta}\delta^1_\alpha)\Big) \otimes \Big(e_{ji} \otimes (f_{2\alpha}\delta^1_\beta - f_{1\alpha}\delta^2_\beta)\Big).
\end{aligned}$$

If we now just consider the $f_{\alpha\beta}$ part of the equation we get

$$\sum(f_{1\beta}\delta^2_\alpha - f_{2\beta}\delta^1_\alpha) \otimes (f_{2\alpha}\delta^1_\beta - f_{1\alpha}\delta^2_\beta) = f_{11} \otimes f_{22} + f_{22} \otimes f_{11} - f_{12} \otimes f_{21} - f_{21} \otimes f_{12}.$$

The RHS however reads

$$\begin{aligned}
\widehat{\Omega}\widetilde{L} &= \Big(\sum(e_{ij} \otimes f_{\alpha\beta}) \otimes (e_{ji} \otimes f_{\beta\alpha})\Big)\Big((\sum e_{kk} \otimes (f_{12} - f_{21})) \otimes (\sum e_{kk} \otimes (f_{21} - f_{12}))\Big), \\
&= \sum\Big(\sum e_{ij}e_{kk} \otimes f_{\alpha\beta}(f_{12} - f_{21})\Big) \otimes \Big(\sum e_{ji}e_{kk} \otimes f_{\beta\alpha}(f_{21} - f_{12})\Big), \\
&= \sum\Big(e_{ij} \otimes (f_{\alpha 2}\delta^1_\beta - f_{\alpha 1}\delta^2_\beta)\Big) \otimes \Big(e_{ji} \otimes (f_{\beta 1}\delta^2_\alpha - f_{\beta 2}\delta^1_\alpha)\Big).
\end{aligned}$$

If we now once again consider the $f_{\alpha\beta}$ part of the equation we get

$$\sum(f_{\alpha 2}\delta^1_\beta - f_{\alpha 1}\delta^2_\beta) \otimes (f_{\beta 1}\delta^2_\alpha - f_{\beta 2}\delta^1_\alpha) = -f_{12} \otimes f_{21} - f_{21} \otimes f_{12} + f_{11} \otimes f_{22} + f_{22} \otimes f_{11}.$$

Hence the two terms are equal, thus the two matrices commute. $\square$

Corollary 3.22. *$\widehat{\Omega}$ and $\widetilde{L}$ have a common set of eigenvectors.*

We know that $\widetilde{L}$ has the same amount of eigenvectors with eigenvalue 1 as -1. Diagonalizing then gives us

$$\Omega = \begin{pmatrix} I_{N(2N+1)} & 0 \\ 0 & -I_{N(2N-1)} \end{pmatrix}$$

and

$$L = \begin{pmatrix} -I_N & 0 & 0 \\ 0 & I_{2N^2} & 0 \\ 0 & 0 & -I_{N(2N-1)} \end{pmatrix},$$

such that for $A \in \mathrm{Pert}(M_N(\mathbb{H})) : \Omega\overline{A} = A\Omega$ and $L\overline{A} = AL$. We thus get

$$\mathrm{Pert}(M_N(\mathbb{H})) \cong \Big\{A \in M_{4N^2}(\mathbb{C}) \mid Ae_1 = e_1, \Omega\overline{A} = A\Omega, L\overline{A} = AL\Big\}.$$

As we have shown Ω and L commute, so every linear combination of these two must satisfy a similar commutation relation with A. Now let us define

$$\Psi = (\Omega - L)/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{N-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Theta = (\Omega + L)/2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{2N^2} & 0 \\ 0 & 0 & -I_{N(2N-1)} \end{pmatrix},$$

also let

$$\Theta' = \begin{pmatrix} I_{2N^2} & 0 \\ 0 & -I_{N(2N-1)} \end{pmatrix}.$$

The reason why we choose this particular notation will become clear in the following theorem, but first notice that we now have

$$\mathrm{Pert}(M_N(\mathbb{H})) \cong \Big\{A \in M_{4N^2}(\mathbb{C}) \mid Ae_1 = e_1, \Psi\overline{A} = A\Psi, \Theta\overline{A} = A\Theta\Big\}.$$

Theorem 3.23. *We have*

$$\mathrm{Pert}(M_N(\mathbb{H})) \cong \Big(\mathbb{R}^{N-1} \rtimes M_{N-1}(\mathbb{R})\Big) \times T,$$

where

$$T = \Big\{A \in M_{4N^2-N}(\mathbb{C}) \mid \Theta'\overline{A} = A\Theta'\Big\}.$$

Proof. Let us start with a matrix $A \in M_{4N^2}$ and let us write A in the following suggestive form

$$A = \begin{pmatrix} a & v_1 & v_2 & v_3 \\ b & B_{11} & B_{12} & B_{13} \\ c & B_{21} & B_{22} & B_{23} \\ d & B_{31} & B_{32} & B_{33} \end{pmatrix},$$

where $a, b, c, d \in \mathbb{C}$, $B_{11} \in M_{N-1}(\mathbb{C})$, $B_{22} \in M_{2N^2}(\mathbb{C})$ and $B_{33} \in M_{N(2N-1)}(\mathbb{C})$. The other block matrices B_{ij} and the vectors v_i are chosen in a similar suitable way. Then the first condition $Ae_1 = e_1$ implies that $a = 1$ and $b = c = d = 0$. The second condition $\Psi\overline{A} = A\Psi$ gives

$$\begin{pmatrix} 1 & \overline{v_1} & \overline{v_2} & \overline{v_3} \\ 0 & \overline{B_{11}} & \overline{B_{12}} & \overline{B_{13}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & v_1 & 0 & 0 \\ 0 & B_{11} & 0 & 0 \\ 0 & B_{21} & 0 & 0 \\ 0 & B_{31} & 0 & 0 \end{pmatrix},$$

hence all $v_2, v_3, B_{12}, B_{13}, B_{21}, B_{31}$ equal zero, while v_1 is a real vector and B_{11} is a real matrix. The third and final condition, $\Theta\overline{A} = A\Theta$, implies

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{B_{22}} & \overline{B_{23}} \\ 0 & -\overline{B_{32}} & -\overline{B_{33}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{22} & -B_{23} \\ 0 & B_{32} & -B_{33} \end{pmatrix}.$$

So we see that B_{22}, B_{33} are real, while B_{23}, B_{32} are pure complex. Thus, we have obtained the following general form

$$A = \begin{pmatrix} 1 & v_1 & 0 \\ 0 & B_{11} & 0 \\ 0 & 0 & C \end{pmatrix},$$

where $C \in T$. As we have seen multiple times before the upper left 2×2 block can be parameterized by a semidirect product, namely $\mathbb{R}^{N-1} \rtimes M_{N-1}(\mathbb{R})$, while the lower right entry has no further condition, but to be in T. This gives

$$\mathrm{Pert}(M_N(\mathbb{H})) \cong \Big(\mathbb{R}^{N-1} \rtimes M_{N-1}(\mathbb{R})\Big) \times T,$$

which proves the theorem. □

Note that this result is in accordance with the perturbation semigroup for $\mathbb{H}$, since the semidirect product vanishes for $N = 1$.

For the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{H}))$ we follow the same steps as we would take for $\mathrm{Pert}(M_N(\mathbb{C}))$. We construct a transformation matrix (this can be the same as for $\mathrm{Pert}(M_{2N}(\mathbb{C}))$. The columns of this transformation matrix (the eigenvectors of a matrix A) need to be orthogonal, which can be established by Gramm-Schmidt orthogonlization. For the unitary matrix u we use that $\sum u_{ij}\overline{u_{kj}} = \delta^i_k$. However, we also demand that $u \in M_N(\mathbb{H})$. This way we can determine the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{H}))$. Note that we can get the embedding of the unitaries in $\mathrm{Pert}(M_N(\mathbb{H}))$ from $\mathrm{Pert}(M_{2N}(\mathbb{H}))$ by imposing a commutation relation with L.

Chapter 4

General cases

When we know the perturbation semigroup of two $*$-algebras, we would also like to know the perturbation semigroup of their direct sum and the perturbation semigroup of their tensor product. This is important in order to reach our final goal, namely to determine the perturbation semigroup of the Standard Model of Particle Physics, which uses both of these structures.

4.1 Perturbation semigroup of direct sum

If the perturbation semigroup of two $*$-algebras is known, we would like to construct the perturbation semigroup of their direct sum. We have the following result.

Theorem 4.1. *Let $\mathcal{A}, \mathcal{B}$ be $*$-algebras, then we have*

$$\mathrm{Pert}(\mathcal{A} \oplus \mathcal{B}) \cong \mathrm{Pert}(\mathcal{A}) \times \mathrm{Pert}(\mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B}^{op} \oplus \mathcal{B} \otimes \mathcal{A}^{op})^{s.a.} \tag{4.1}$$

where s.a. stands for the self-adjoint elements as we have for the perturbation semigroup.

The self-adjoint elements have the general form $\sum a_i \otimes b_i^{op} + b_i^* \otimes a_i^{*op}$, which is in fact self-adjoint. Note that for $\mathcal{A}, \mathcal{B}$ matrix algebras the last term reduces to $\mathcal{A} \otimes \mathcal{B}$.

Proof. We have by definition

$$\mathrm{Pert}(\mathcal{A} \oplus \mathcal{B}) = \left\{ \sum a_j \otimes b_j^{op} \in (\mathcal{A} \oplus \mathcal{B}) \otimes (\mathcal{A} \oplus \mathcal{B})^{op} \,\middle|\, \begin{array}{l} \sum a_j b_j = 1 \\ \sum a_j \otimes b_j^{op} = \sum b_j^* \otimes a_j^{*op} \end{array} \right\}.$$

Now define

$$\begin{array}{rll} \phi: & \mathrm{Pert}(\mathcal{A} \oplus \mathcal{B}) & \to \mathrm{Pert}(\mathcal{A}) \times \mathrm{Pert}(\mathcal{B}), \\ & \sum (a_j, b_j) \otimes (\widetilde{a}_j^{op}, \widetilde{b}_j^{op}) & \mapsto (\sum a_j \otimes \widetilde{a}_j^{op}, \sum b_j \otimes \widetilde{b}_j^{op}). \end{array}$$

Note that ϕ is surjective, since $\phi(\sum (a_j, 1) \otimes (\widetilde{a}_j^{op}, 1))$ will give $\mathrm{Pert}(\mathcal{A})$. The same goes for $\mathrm{Pert}(\mathcal{B})$ with $\phi(\sum (1, b_j) \otimes (1, \widetilde{b}_j^{op}))$.

Note that elements which are normalized, are mapped to normalized elements, since for $\sum (a_j, b_j) \otimes (\widetilde{a}_j^{op}, \widetilde{b}_j^{op}) \in (\mathcal{A} \oplus \mathcal{B}) \otimes (\mathcal{A} \oplus \mathcal{B})^{op}$ one gets

$$\begin{aligned} \sum (a_j, b_j)(\widetilde{a}_j, \widetilde{b}_j) &= \sum (a_j \widetilde{a}_j, b_j \widetilde{b}_j), \\ &= \sum (a_j \widetilde{a}_j, 0) + (0, b_j \widetilde{b}_j), \\ &= (\sum a_j \widetilde{a}_j, 0) + (0, \sum b_j \widetilde{b}_j), \\ &= 1 \equiv (1, 1). \end{aligned}$$

For the self-adjointness the same reasoning holds, it is preserved by ϕ. Now let us consider the kernel of ϕ. The kernel appears to consists of elements of the form $\sum(a_j,0)\otimes(0,\widetilde{b_j}^{op})$ and $\sum(0,b_j)\otimes(\widetilde{a}_j^{op},0)$. However these elements alone are not self-adjoint. Combining them gives self-adjoint elements in the kernel. This gives

$$\sum(0,b_j)\otimes(a_j^{op},0)+(a_j^*,0)\otimes(0,b_j^{*op})$$

for the elements in the kernel. But one quickly sees that

$$\sum(0,b_j)\otimes(a_j^{op},0)+(a_j^*,0)\otimes(0,b_j^{*op})\in\mathcal{B}\otimes\mathcal{A}^{op}\oplus\mathcal{A}\otimes\mathcal{B}^{op}.$$

However, as we just saw, not every element of $\mathcal{B}\otimes\mathcal{A}^{op}\oplus\mathcal{A}\otimes\mathcal{B}^{op}$ is in $\ker(\phi)$, only the self-adjoint elements are. Hence

$$\ker(\phi)\cong(\mathcal{B}\otimes\mathcal{A}^{op}\oplus\mathcal{A}\otimes\mathcal{B}^{op})^{s.a},$$

the self-adjoint elements. By the first isomorphism theorem for semigroups we now know that

$$\mathrm{Pert}(\mathcal{A}\oplus\mathcal{B})/\ker(\phi)\cong\mathrm{Im}(\phi)=\mathrm{Pert}(\mathcal{A})\times\mathrm{Pert}(\mathcal{B}).$$

However, we also know that

$$(\mathcal{A}\oplus\mathcal{B})\oplus(\mathcal{A}\oplus\mathcal{B})^{op}\cong\mathcal{A}\otimes\mathcal{B}^{op}\oplus\mathcal{B}\otimes\mathcal{A}^{op}\oplus\mathcal{A}\otimes\mathcal{A}^{op}\oplus\mathcal{B}\otimes\mathcal{B}^{op}.$$

For the perturbation semigroup we need to impose a normalization condition and a self-adjoint condition on the above expression. We see that, with the self-adjoint condition, the first two terms are precisely $\ker(\phi)$. We already saw that

$$\mathrm{Pert}(\mathcal{A}\oplus\mathcal{B})/\ker(\phi)\cong\mathrm{Pert}(\mathcal{A})\times\mathrm{Pert}(\mathcal{B}),$$

hence this equals

$$\mathrm{Pert}(\mathcal{A}\oplus\mathcal{B})\cong\mathrm{Pert}(\mathcal{A})\times\mathrm{Pert}(\mathcal{B})\times(\mathcal{B}\otimes\mathcal{A}^{op}\oplus\mathcal{A}\otimes\mathcal{B}^{op})^{s.a}.$$

□

The self-adjoint part will reduce to a simple tensor product between $\mathcal{A}$ and $\mathcal{B}$ in the cases we will consider.

Example 4.2. *Let $\mathbb{C}^N$ and $\mathbb{C}^M$ be two $*$-algebras, then $\mathrm{Pert}(\mathbb{C}^{N+M})\cong\mathbb{C}^{(N+M)(N+M-1)/2}$ and $\mathrm{Pert}(\mathbb{C}^{N+M})\cong\mathrm{Pert}(\mathbb{C}^N\oplus\mathbb{C}^M)$. We also see that*

$$\mathrm{Pert}(\mathbb{C}^N)\times\mathrm{Pert}(\mathbb{C}^M)\times(\mathbb{C}^N\otimes\mathbb{C}^M)\cong\mathbb{C}^{N(N-1)/2}\times\mathbb{C}^{M(M-1)/2}\times\mathbb{C}^{NM}\cong\mathbb{C}^{(N+M)(N+M-1)/2}.$$

Hence, we have

$$\mathrm{Pert}(\mathbb{C}^N\oplus\mathbb{C}^M)\cong\mathrm{Pert}(\mathbb{C}^N)\times\mathrm{Pert}(\mathbb{C}^M)\times\mathbb{C}^N\otimes\mathbb{C}^M,$$

or

$$\mathbb{C}^{(N+M)(N+M-1)}\cong\mathbb{C}^{N(N-1)/2}\times\mathbb{C}^{M(M-1)/2}\times\mathbb{C}^N\otimes\mathbb{C}^M.$$

4.2 Perturbation semigroup of tensor product

Just as for the perturbation semigroup of the direct sum of two $*$-algebras, we would like to get an explicit expression for the perturbation semigroup of the tensor product of two $*$-algebras. This is however more difficult, as we will now explain.

Let us take a look at $\mathrm{Pert}(\mathcal{A} \otimes \mathcal{B})$, we know that the elements of $\mathrm{Pert}(\mathcal{A} \otimes \mathcal{B})$ have the general form

$$\sum (a_j \otimes b_j) \otimes (\tilde{a}_j^{op} \otimes \tilde{b_j}^{op}) \in (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{A} \otimes \mathcal{B})^{op},$$

such that

$$\sum (a_j \otimes b_j)(\tilde{a}_j^{op} \otimes \tilde{b_j}^{op}) = 1$$

and

$$\sum (a_j \otimes b_j) \otimes (\tilde{a}_j^{op} \otimes \tilde{b_j}^{op}) = \sum (\tilde{a}_j^{*} \otimes \tilde{b_j}^{*}) \otimes (a_j^{*op} \otimes b_j^{*op}).$$

By the isomorphism

$$(\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{A}^{op} \otimes \mathcal{B}^{op}) \cong (\mathcal{A} \otimes \mathcal{A}^{op}) \otimes (\mathcal{B} \otimes \mathcal{B}^{op}),$$

we can modify the terms in the self-adjointness condition to get

$$\sum (a_j \otimes b_j) \otimes (\tilde{a}_j^{op} \otimes \tilde{b_j}^{op}) \cong \sum (a_j \otimes \tilde{a}^{op})_j \otimes (b_j \otimes \tilde{b_j}^{op})$$

and

$$\sum (\tilde{a}_j^{*} \otimes \tilde{b_j}^{*}) \otimes (a_j^{*op} \otimes b_j^{*op}) \cong \sum (\tilde{a}_j^{*} \otimes a_j^{*op}) \otimes (\tilde{b_j}^{*} \otimes b_j^{*op}).$$

Thus

$$\sum (a_j \otimes \tilde{a}_j^{op}) = \sum (\tilde{a}_j^{*} \otimes a_j^{*op})$$

and

$$\sum (b_j \otimes \tilde{b_j}^{op}) = \sum (\tilde{b_j}^{*} \otimes b_j^{*op}).$$

So this would give the impression that the perturbation semigroup of a tensor product splits in the two separate perturbation semigroups plus an extra term. However, let us now consider the normalization condition. This can be rewritten as

$$\sum (a_j \tilde{a}_j^{op} \otimes b_j \tilde{b_j}^{op}) = 1.$$

An easy conclusion could be that both $\sum a_j \tilde{a}_j^{op}$ and $\sum b_j \tilde{b_j}^{op}$ need to equal 1. However, the tensor product is linear over $\mathbb{C}$. Therefore it is possible that $\sum a_j \tilde{a}_j^{op} = \lambda$, while $\sum b_j \tilde{b_j}^{op} = \lambda^{-1}$ for $\lambda \neq 0$, which still gives

$$\sum (a_j \tilde{a}_j^{op} \otimes b_j \tilde{b_j}^{op}) = 1.$$

Hence, it is not possible to say that both sums need to be normalized, thus the perturbation semigroup can not split in two separate perturbation semigroup.

Let us now take a look at an element $\sum (a_j \otimes b_j) \otimes (\tilde{a}_j^{op} \otimes \tilde{b_j}^{op})$, such that $\sum a_j \tilde{a}_j = 0$. Then it is possible that $\sum (a_j \otimes b_j) \otimes (\tilde{a}_j^{op} \otimes \tilde{b_j}^{op}) = 0$, while $\sum b_j \otimes \tilde{b_j}^{op} \neq 0$. So this way we have an element which equals zero in $\mathrm{Pert}(\mathcal{A} \otimes \mathcal{B})$, but is non-zero in $\mathrm{Pert}(\mathcal{B})$.

Chapter 5

Perturbation semigroup of the Standard Model

We now want to determine the perturbation semigroup of the Standard Model of particle physics, that is [3]

$$\mathrm{Pert}\Big(C^\infty\Big(M, \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\Big)\Big), \tag{5.1}$$

where M is a manifold. With $C^\infty(M, \mathcal{A})$ we mean the smooth functions over M which takes values in the $*$-algebra $\mathcal{A}$. Before we can determine the perturbation semigroup of the Standard Model, we need to know what $\mathrm{Pert}(C^\infty(M))$ and $\mathrm{Pert}(C^\infty(M, \mathcal{A}))$ are.

5.1 Perturbation semigroup $\mathrm{Pert}(C^\infty(M))$

Let us first take a look at the definition of our perturbation semigroup and what it means for $C^\infty(M)$. The definition reads

$$\mathrm{Pert}(C^\infty(M)) = \left\{ \sum g_j \otimes h_j^{op} \in C^\infty(M) \otimes C^\infty(M)^{op} \middle| \begin{array}{l} \sum g_j h_j = 1 \\ \sum g_j \otimes h_j^{op} = \sum h_j^* \otimes g_j^{*op} \end{array} \right\}.$$

However, we know that $C^\infty(M) \cong C^\infty(M)^{op}$, thus we get elements in $C^\infty(M) \otimes C^\infty(M)$. A corollary from the Arzelà-Ascoli theorem [7][8] states that $C^\infty(M \times M)$ is dense in $C^\infty(M) \otimes C^\infty(M)$. Therefore we will look at functions in two variables instead. We get

$$\mathrm{Pert}(C^\infty(M)) = \Big\{ f(x,y) \in C^\infty(M \times M) \mid \text{normalized and self-adjoint} \Big\}.$$

We now need to write the normalization and self-adjointness in terms of these new functions $f(x,y)$. We have

$$\begin{array}{ll} C^\infty(M) \otimes C^\infty(M) & \to C^\infty(M \times M) \\ \sum g_j \otimes h_j & \mapsto f. \end{array}$$

So g_j is the first component of f, while h_j is the second. Hence we can write

$$\sum g_j \otimes h_j^{op} = f(x,y)$$

and

$$\sum h_j^* \otimes g_j^{*op} = f(y,x)^*.$$

We also know that $f^* = \overline{f}$. So the self-adjointness condition becomes

$$f(x,y) = \overline{f(y,x)}$$

in terms of the new functions.

If we take a similar look at the functions $\sum g_j \otimes h_j^{op}$ for the normalization condition, thus g_j as the first component and h_j as the second, we see that only the function values $f(x,x)$ need to be normalized. This gives

$$\mathrm{Pert}(C^\infty(M)) = \Big\{ f \in C^\infty(M \times M) \mid f(x,x) = 1, f(x,y) = \overline{f(y,x)} \Big\}.$$

5.2 Perturbation semigroup $\mathrm{Pert}(C^\infty(M, \mathcal{A}))$

In the previous section we considered smooth functions over a manifold M which took values in $\mathbb{C}$. Now we have smooth functions over a manifold M, but it takes values in a finite dimensional matrix algebra $\mathcal{A}$.

Proposition 5.1. *Let M be a manifold and $\mathcal{A}$ a matrix algebra, then*

$$C^\infty(M, \mathcal{A}) \cong C^\infty(M) \otimes \mathcal{A}.$$

In general this can be proven for nuclear Frechet spaces [9]. However for us it is enough if we look at matrix algebras $\mathcal{A}$.

Proof. Let $\{a_i\}_i$ be a basis for the vector space $\mathcal{A}$ and let $x \in M$ then for $f \in C^\infty(M, \mathcal{A})$ we can say

$$f(x) = a = \sum c_i a_i,$$

for $a, a_i \in \mathcal{A}$ and coefficients c_i. We now want to construct a function $h \in C^\infty(M) \otimes \mathcal{A}$ such that we have equality. Let $g_i \in C^\infty(M), a_i \in \mathcal{A}$ such that we have $\sum g_i(x) \otimes a_i$. Now define $g_i(x) = c_i$, where c_i is taken as above. Then we have equality, since the element $f(x) = a = \sum c_i a_i$ can be written as $\sum c_i a_i \otimes 1 = \sum c_i \otimes a_i = \sum g_i(x) \otimes a_i$ by $\mathbb{C}$-linearity. So we get $h(x) = \sum g_i(x) \otimes a_i$. ☐

Remark 5.2. *For matrix algebras the LHS consists of functions which map to matrices, while the RHS consists of matrices with as entries functions from M to $\mathbb{R}$ or $\mathbb{C}$.*

Corollary 5.3. *For M a manifold and $\mathcal{A}, \mathcal{B}$ matrix algebras we have*

$$C^\infty(M, \mathcal{A} \oplus \mathcal{B}) \cong C^\infty(M, \mathcal{A}) \oplus C^\infty(M, \mathcal{B}).$$

We can now look at the perturbation semigroup for $C^\infty(M, \mathcal{A})$. Let us first introduce the semigroup homomorphism μ defined by

$$\begin{aligned} \mu : \mathcal{A} \otimes \mathcal{A}^{op} &\to \mathcal{A}, \\ a \otimes b^{op} &\mapsto ab. \end{aligned} \tag{5.2}$$

We then have

Theorem 5.4. *Let $\mathcal{A}$ be a $*$-algebra and M a manifold, then*

$$\mathrm{Pert}(C^\infty(M \times M, \mathcal{A} \otimes \mathcal{A}^{op})) \cong C^\infty(M, \mathrm{Pert}(\mathcal{A})) \times C^\infty(M \times M - \Delta, \mathcal{A} \otimes \mathcal{A}^{op})^{s.a.}.$$

With Δ we mean the diagonal of M and with s.a. we mean the self-adjoint elements again in terms of our perturbation semigroup.

Proof. First we make the identification

$$\begin{gathered} C^\infty(M) \otimes C^\infty(M) \to C^\infty(M \times M) \\ \sum g_j \otimes h_j \mapsto f. \end{gathered}$$

So we only need to consider functions in two variables which map to $\mathcal{A} \otimes \mathcal{A}^{op}$. The self-adjoint condition behaves the same way as it did for $\mathrm{Pert}(C^\infty(M))$. However, we now have functions that map to an element in $\mathcal{A} \otimes \mathcal{A}^{op}$. Therefore, instead of complex conjugation we now have

$$f(x, y) = f(y, x)^*,$$

where the $*$ is the $*$ from $\mathcal{A}$.

For $\mathrm{Pert}(C^\infty(M))$ we saw that only the elements $f(x, x)$ had to be normalized. For $\mathrm{Pert}(C^\infty(M, \mathcal{A}))$ we have

$$f(x, x) = \sum a_i \otimes b_i^{op}$$

for given $a_i, b_i \in \mathcal{A}$. We now use the semigroup homomorphism μ to define the normalization condition. We have

$$\mu(f(x, x)) = \mu(\sum a_i \otimes b_i^{op}) = \sum a_i b_i = 1.$$

Note that we have used that μ is a semigroup homomorphism.

As we have seen all the elements need to be self-adjoint, while only the elements $f(x, x)$ need to be normalized. Now split the manifold $M \times M$ in $M \times M - \Delta$ and Δ, where Δ stands for the diagonal. So again for every element we have the self-adjointness condition, while we only have the normalization condition for elements on the diagonal. So the elements of $C^\infty(M \times M - \Delta, \mathcal{A} \otimes \mathcal{A}^{op})$ only need to be self-adjoint, while the elements of $C^\infty(\Delta, \mathcal{A} \otimes \mathcal{A}^{op})$ need to be both self-adjoint and normalized. However, the elements of $\mathcal{A} \otimes \mathcal{A}^{op}$ which are normalized and self-adjoint are precisely the elements in $\mathrm{Pert}(\mathcal{A})$. If we also use that $\Delta \cong M$ we get

$$\mathrm{Pert}(C^\infty(M, \mathcal{A})) \cong C^\infty(M, \mathrm{Pert}(\mathcal{A})) \times C^\infty(M \times M - \Delta, \mathcal{A} \otimes \mathcal{A}^{op})^{s.a.}.$$

□

5.3 Perturbation semigroup of the Standard Model

With all the examples we have worked out and the general theory on perturbation semigroups, we can now construct the perturbation semigroup of the Standard Model of Particle Physics. Recall that we can write the Standard Model of Particle Physics as

$$\mathcal{A} = C^\infty(M) \otimes \Big(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\Big).$$

Here $\mathbb{C}$ describes the photon γ, $\mathbb{H}$ describes the bosons for the weak nuclear force, namely Z and $W^{\pm}$, while $M_3(\mathbb{C})$ describes the gluons [5]. Note that the quarks and leptons, which are fermions, are not described by this expression. Instead they are described by the Hilbert space $\mathcal{H}$. For the Standard Model we have $\mathcal{H} = \mathbb{C}^{96}$. The reason we get 96 and not something else can be found in the Standard Model of Particle Physics by looking at the elementary particles. We see that we have leptons and quarks, for every quark and lepton we have two types. There is also an antiparticle for every quark and for every lepton, while only quarks can have a color. Every particle can also be either right handed or left handed. We also know that for both leptons and quarks there are three generations. So we get

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 3}_{Leptons} + \underbrace{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}_{Quarks} = 24 + 72 = 96.$$

We can now determine the perturbation semigroup of the Standard Model of Particle Physics. In order to do so we use the results we have acquired in the previous sections, which then gives us

$$\begin{aligned}
&\mathrm{Pert}\left(C^\infty(M) \otimes \left(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\right)\right) \cong \\
&\quad C^\infty(M, \mathbb{C}) \times C^\infty(M, \mathrm{Pert}(\mathbb{H})) \times C^\infty(M, \mathrm{Pert}(M_3(\mathbb{C}))) \times C^\infty(M \times M - \Delta, \mathbb{C})^{s.a.} \times \\
&\quad C^\infty(M \times M - \Delta, \mathbb{H} \otimes \mathbb{H}^{op})^{s.a.} \times C^\infty(M \times M - \Delta, M_3(\mathbb{C}) \otimes M_3(\mathbb{C})^{op})^{s.a.} \times \\
&\qquad\qquad \Big(C^\infty(M, \mathbb{C}) \otimes C^\infty(M, \mathbb{H}) \oplus C^\infty(M, \mathbb{C}) \otimes C^\infty(M, M_3(\mathbb{C})) \oplus \\
&\qquad\qquad C^\infty(M, \mathbb{H}) \otimes C^\infty(M, \mathbb{C}) \oplus C^\infty(M, \mathbb{H}) \otimes C^\infty(M, M_3(\mathbb{C})) \oplus \\
&\qquad\qquad C^\infty(M, M_3(\mathbb{C})) \otimes C^\infty(M, \mathbb{H}) \oplus C^\infty(M, M_3(\mathbb{C})) \otimes C^\infty(M, \mathbb{C})\Big)^{s.a.}.
\end{aligned} \tag{5.3}$$

Note that we have used that $\mathbb{C} \cong \mathbb{C} \otimes \mathbb{C}^{op}$.

Chapter 6

Action of the perturbation semigroup

Now that we have determined the perturbation semigroup for several cases, we would like to obtain some physics from them. Since the application of the perturbation semigroup to the full Standard Model of Particle Physics would go beyond the scope of this text, we will restrict ourselves to some toy models. In non-commutative geometry one has finite spectral triples and in particular one has a hermitian matrix D and this matrix is precisely what we will look at. We will consider the action of the perturbation semigroup on the hermitian matrix D. Let $\sum a_j \otimes b_j^{op} \in \mathrm{Pert}(\mathcal{A})$, then this element acts on D by

$$D \mapsto \sum a_j D b_j.$$

6.1 Diagonal hermitian matrices

Let us first consider the case where $\mathcal{A} = \mathbb{C}^2$ and $\mathcal{H} = \mathbb{C}^2$. Also let D be a diagonal hermitian matrix, such that D has the following form

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where $a, b \in \mathbb{R}$ and $a \neq b$. If $a = b$, then D is just the identity matrix multiplied by a constant in which case the action does nothing with D. We now have the spectral triple

$$\left(\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right),$$

let us first look at the action of $\mathrm{Pert}(\mathbb{C}^2)$. For convenience we rewrite the basis elements. Let $e_{ii} := e_i$ thus the matrix with a one at position (i, i) and zeros everywhere else. Note that the normalization condition and self-adjointness condition still hold, since $e_{ii} e_{jj} = \delta^i_j e_{ii}$. The coefficients C_{ij} now become $C_{ii,jj}$.

Example 6.1. *The action is given by*

$$\sum C_{ii,jj} e_{ii} D e_{jj}.$$

We know that $C_{11,11} = C_{22,22} = 1$ *and* $C_{11,22} = \overline{C_{22,11}}$*, so we get as action*

$$e_{11}De_{11} + e_{22}De_{22} + ze_{11}De_{22} + \overline{z}e_{22}De_{11} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + 0 + 0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = D.$$

So the cross terms $ze_{11}De_{22}$ *and* $\overline{z}e_{22}De_{11}$ *vanishes. This is expected from equation* (3.1)*. We know that* $D = ae_{11} + be_{22}$*, so the only non vanishing terms are* $e_{11}De_{11}$ *and* $e_{22}De_{22}$*. So the action of* $\mathrm{Pert}(\mathbb{C}^2)$ *on* D *is trivial.*

Now let us consider the same hermitian matrix, but now with the perturbation semigroup $\mathrm{Pert}(M_2(\mathbb{C}))$. As we have already seen matrices A in the perturbation semigroup have the general form

$$A = \begin{pmatrix} x_1 & z_4 & \overline{z_4} & 1-x_1 \\ z_1 & z_2 & \overline{z_3} & -z_1 \\ \overline{z_1} & z_3 & \overline{z_2} & -\overline{z_1} \\ 1-x_2 & z_5 & \overline{z_5} & x_2 \end{pmatrix}; \qquad z_1, \ldots z_5 \in \mathbb{C},\ x_1, x_2 \in \mathbb{R}.$$

We can rewrite this matrix A as

$$\begin{aligned} A = {} & \tfrac{1}{2}e_{11} \otimes \begin{pmatrix} x_1 & z_4 \\ z_1 & z_2 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} x_1 & \overline{z_1} \\ \overline{z_4} & \overline{z_2} \end{pmatrix} \otimes e_{11} \\ & + \tfrac{1}{2}e_{12} \otimes \begin{pmatrix} \overline{z_4} & 1-x_1 \\ \overline{z_3} & -z_1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} z_4 & z_3 \\ 1-x_1 & -\overline{z_1} \end{pmatrix} \otimes e_{21} \\ & + \tfrac{1}{2}e_{21} \otimes \begin{pmatrix} \overline{z_1} & \overline{z_3} \\ 1-x_2 & z_5 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} z_1 & 1-x_2 \\ z_3 & \overline{z_5} \end{pmatrix} \otimes e_{12} \\ & + \tfrac{1}{2}e_{22} \otimes \begin{pmatrix} \overline{z_2} & -\overline{z_1} \\ \overline{z_5} & x_2 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} z_2 & z_5 \\ -z_1 & x_2 \end{pmatrix} \otimes e_{22}. \end{aligned}$$

This seems like a lot of unnecessary work, but as we will see in the next example it makes live easier. Since we did nothing, but rewrite A as a sum of Kronecker products we see that in this form A is still self-adjoint. First note that for $\mathrm{Pert}(M_2(\mathbb{C}))$ we have the spectral triple

$$\left(M_2(\mathbb{C}), \mathbb{C}^2, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right).$$

Example 6.2. *Let us now look at the action of the perturbation semigroup of* $M_2(\mathbb{C})$ *on*

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

The action

$$D \mapsto \sum a_j D b_j$$

is now given by

$$\begin{aligned} & \tfrac{1}{2}\begin{pmatrix} ax_1 & az_4 \\ 0 & 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} ax_1 & 0 \\ a\overline{z_4} & 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} b\overline{z_3} & -bz_1 \\ 0 & 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} bz_3 & 0 \\ -b\overline{z_1} & 0 \end{pmatrix} \\ + & \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ a\overline{z_1} & a\overline{z_3} \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & az_1 \\ 0 & az_3 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ b\overline{z_5} & bx_2 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & bz_5 \\ 0 & bx_2 \end{pmatrix}. \end{aligned}$$

Which sums up to

$$\begin{pmatrix} ax_1 + \frac{b}{2}(\overline{z_3} + z_3) & \frac{1}{2}(az_4 - bz_1 + az_1 + bz_5) \\ \frac{1}{2}(a\overline{z_4} - b\overline{z_1} + a\overline{z_1} + b\overline{z_5}) & \frac{1}{2}(a\overline{z_3} + az_3) + bx_2 \end{pmatrix}$$

which can be rewritten as

$$\begin{pmatrix} ax_1 + b\Re(z_3) & \frac{1}{2}(az_4 - bz_1 + az_1 + bz_5) \\ \frac{1}{2}\overline{(az_4 - bz_1 + az_1 + bz_5)} & bx_2 + a\Re(z_3) \end{pmatrix}.$$

Note that the result is in fact still hermitian as it should.

Now we want to consider a diagonal hermitian matrix D in dimension N, such that $D = \sum\limits_{k=1}^{N} \lambda_k e_{kk}$. We consider the action of $\mathrm{Pert}(\mathbb{C}^N)$ on this D. Let the basis element of $\mathrm{Pert}(\mathbb{C}^N)$ be given by e_{ii} which can be identified with e_i just as we did above. The coefficients C_{ij} translate to $C_{ii,jj}$. Then we have the following theorem.

Theorem 6.3. *Let $D = \sum\limits_{k=1}^{N} \lambda_k e_{kk}$ be a hermitian matrix, then the action*

$$D \mapsto \sum a_j D b_j$$

of the perturbation semigroup $\mathrm{Pert}(\mathbb{C}^N)$ *is trivial.*

Proof. The elements in the perturbation semigroup are given by $\sum C_{ii,jj} e_{ii} \otimes e_{jj}^{op}$. The action is then given by

$$D \mapsto \sum C_{ii,jj} e_{ii} D e_{jj}.$$

If we work this out we get

$$\begin{aligned} \sum C_{ii,jj} e_{ii}\Big(\sum_{k=1}^{N} \lambda_k e_{kk}\Big) e_{jj} &= \textstyle\sum_{i,j,k} C_{ii,jj} \lambda_k e_{ii} e_{kk} e_{jj}, \\ &= \textstyle\sum_{i,j,k} C_{ii,jj} \lambda_k e_{ij} \delta_k^i \delta_j^k, \\ &= \textstyle\sum_{i,j} C_{ii,jj} \lambda_i e_{ij} \delta_j^i, \\ &= \textstyle\sum_{i} C_{ii,ii} \lambda_i e_{ii}. \end{aligned}$$

We also know that from the normalization condition follows that $C_{ii,jj} = 1$ if $i = j$. Hence

$$D \mapsto \sum_i \lambda_i e_{ii} = D,$$

so $\mathrm{Pert}(\mathbb{C}^N)$ acts on D as

$$D \mapsto D.$$

$\square$

Note that in the above theorem we had the spectral triple

$$\Big(\mathbb{C}^N, \mathbb{C}^N, \sum_{k=1}^{N} \lambda_k e_{kk}\Big).$$

6.2 Off-diagonal hermitian matrices

In this section we want to take a look at off-diagonal hermitian matrices D. First let us take a look at

$$D = \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}$$

and at the action of $\mathrm{Pert}(\mathbb{C}^2)$ and $\mathrm{Pert}(M_2(\mathbb{C}))$ on this D. Let us take the same basis vectors as in the previous section for $\mathrm{Pert}(\mathbb{C}^2)$. Note that the spectral triples for these examples are given by

$$\left(\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}\right)$$

and

$$\left(M_2(\mathbb{C}), \mathbb{C}^2, \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}\right).$$

Example 6.4. *Let D be as above and let*

$$\sum C_{ii,jj} e_{ii} \otimes e_{jj}^{op} \in \mathrm{Pert}(\mathbb{C}^2)$$

be the elements which act on D, then the action of the perturbation semigroup on D is given by

$$D \mapsto \sum C_{ii,jj} e_{ii} D e_{jj}.$$

If we work this out numerically we see that

$$\begin{aligned} \sum C_{ii,jj} e_{ii} D e_{jj} &= \sum C_{ii,jj} e_{ii} (c e_{12} + \overline{c} e_{21}) e_{jj}, \\ &= \sum C_{ii,jj} (c e_{ij} \delta_1^i \delta_j^2 + \overline{c} e_{ij} \delta_2^i \delta_j^1), \\ &= C_{11,22} c e_{12} + C_{22,11} \overline{c} e_{21}. \end{aligned}$$

We also know that by self-adjointness $C_{11,22} = \overline{C_{22,11}}$, so let $C_{11,22} = \phi$ then

$$D \mapsto \begin{pmatrix} 0 & c\phi \\ \overline{c\phi} & 0 \end{pmatrix}.$$

In a similar way we can look at the action of $\mathrm{Pert}(M_2(\mathbb{C}))$ on this D. Recall from the previous section that the elements $A \in \mathrm{Pert}(M_2(\mathbb{C}))$ were given by

$$\begin{aligned} A = \; & \tfrac{1}{2} e_{11} \otimes \begin{pmatrix} x_1 & z_4 \\ z_1 & z_2 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} x_1 & \overline{z_1} \\ \overline{z_4} & \overline{z_2} \end{pmatrix} \otimes e_{11} \\ & + \tfrac{1}{2} e_{12} \otimes \begin{pmatrix} \overline{z_4} & 1 - x_1 \\ \overline{z_3} & -z_1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} z_4 & z_3 \\ 1 - x_1 & -\overline{z_1} \end{pmatrix} \otimes e_{21} \\ & + \tfrac{1}{2} e_{21} \otimes \begin{pmatrix} \overline{z_1} & \overline{z_3} \\ 1 - x_2 & z_5 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} z_1 & 1 - x_2 \\ z_3 & \overline{z_5} \end{pmatrix} \otimes e_{12} \\ & + \tfrac{1}{2} e_{22} \otimes \begin{pmatrix} \overline{z_2} & -\overline{z_1} \\ \overline{z_5} & x_2 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} z_2 & z_5 \\ -z_1 & x_2 \end{pmatrix} \otimes e_{22}. \end{aligned}$$

Example 6.5. *The action of* $\mathrm{Pert}(M_2(\mathbb{C}))$ *on*

$$D = \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}$$

is given by

$$\begin{aligned} D \mapsto \quad & c\tfrac{1}{2}\begin{pmatrix} z_1 & z_2 \\ 0 & 0 \end{pmatrix} + \overline{c}\tfrac{1}{2}\begin{pmatrix} \overline{z_1} & 0 \\ \overline{z_2} & 0 \end{pmatrix} + \overline{c}\tfrac{1}{2}\begin{pmatrix} \overline{z_4} & 1-x_1 \\ 0 & 0 \end{pmatrix} + c\tfrac{1}{2}\begin{pmatrix} z_4 & 0 \\ 1-x_1 & 0 \end{pmatrix} \\ & + c\tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 1-x_2 & z_5 \end{pmatrix} + \overline{c}\tfrac{1}{2}\begin{pmatrix} 0 & 1-x_2 \\ 0 & \overline{z_5} \end{pmatrix} + \overline{c}\tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ \overline{z_2} & -\overline{z_1} \end{pmatrix} + c\tfrac{1}{2}\begin{pmatrix} 0 & z_2 \\ 0 & -z_1 \end{pmatrix}. \end{aligned}$$

Upon summing this we get

$$\begin{aligned} D \mapsto \quad & \tfrac{1}{2}\begin{pmatrix} cz_1 + \overline{cz_1} + \overline{cz_4} + cz_4 & cz_2 + \overline{c} - \overline{c}x_1 + \overline{c} - \overline{c}x_2 + cz_2 \\ \overline{cz_2} + c - cx_1 + c - cx_2 + \overline{z_2} & cz_5 + \overline{cz_5} - \overline{cz_1} - cz_1 \end{pmatrix}, \\ & = \begin{pmatrix} \Re(cz_1 + cz_4) & cz_2 + \overline{c} - \frac{\overline{c}}{2}(x_1 + x_2) \\ \overline{cz_2} + c - \frac{c}{2}(x_1 + x_2) & \Re(cz_5 - cz_1) \end{pmatrix}. \end{aligned}$$

The result is a perturbed matrix D, which is still hermitian.

Now let us take a look at a more physics related example. Let

$$D = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and consider the action of $\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$. Note that the spectral triple is now given by

$$\left(\mathbb{C} \oplus M_2(\mathbb{C}), \mathbb{C}^3, \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Example 6.6. *We know by equation* (4.1) *that*

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \cong \mathrm{Pert}(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C})) \times M_2(\mathbb{C}) \otimes \mathbb{C},$$

since $\mathrm{Pert}(\mathbb{C}) \cong \{1\}$ *is trivial, its action will not do anything with D. We also know that* $M_2(\mathbb{C}) \otimes \mathbb{C} \cong M_2(\mathbb{C})$, *hence*

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \cong \mathrm{Pert}(M_2(\mathbb{C})) \times M_2(\mathbb{C}).$$

Thus we only need to know the action of $\mathrm{Pert}(M_2(\mathbb{C}))$ *and the action of* $M_2(\mathbb{C})$ *on D.*

The perturbation semigroup $\mathrm{Pert}(M_2(\mathbb{C}))$ *is embedded in* $\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ *as*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix}$$

where
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = A \otimes B \in \mathrm{Pert}(M_2(\mathbb{C})).$$
The way we have written the matrix A in the previous examples can be used in this example as well. However, where we first had a tensor product of 2×2-matrices, we now have a tensor product of 3×3-matrices
$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$
where A, B are such 2×2-matrices. So we get
$$\begin{aligned} A \otimes B = \; & \tfrac{1}{2} e_{22} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_1 & z_4 \\ 0 & z_1 & z_2 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_1 & \overline{z_1} \\ 0 & \overline{z_4} & \overline{z_2} \end{pmatrix} \otimes e_{22} \\ & + \tfrac{1}{2} e_{23} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{z_4} & 1 - x_1 \\ 0 & \overline{z_3} & -z_1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_4 & z_3 \\ 0 & 1 - x_1 & -\overline{z_1} \end{pmatrix} \otimes e_{32} \\ & + \tfrac{1}{2} e_{32} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{z_1} & \overline{z_3} \\ 0 & 1 - x_2 & z_5 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_1 & 1 - x_2 \\ 0 & z_3 & \overline{z_5} \end{pmatrix} \otimes e_{23} \\ & + \tfrac{1}{2} e_{33} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{z_2} & -\overline{z_1} \\ 0 & \overline{z_5} & x_2 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_2 & z_5 \\ 0 & -z_1 & x_2 \end{pmatrix} \otimes e_{33}. \end{aligned}$$
Note that both the first row and the first column are zero for every matrix. Hence, if we look at the action
$$D \mapsto \sum a_j D b_j,$$
we see that the multiplication Db_j annihilates the first column of D, while the multiplication $a_j D$ annihilates of the first row of D. Thus the action of $\mathrm{Pert}(M_2(\mathbb{C}))$ on D is trivial.

Now consider the term $M_2(\mathbb{C})$, recall that we had $(M_2(\mathbb{C}) \otimes \mathbb{C} \oplus \mathbb{C} \otimes M_2(\mathbb{C}))^{s.a.}$ as in equation (4.1). Note that $M_2(\mathbb{C}) \otimes \mathbb{C}$ is embedded in $\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ by
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix} \otimes \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
In a similar way we get an expression for the term $\mathbb{C} \otimes M_2(\mathbb{C})$. Since the tensor product is $\mathbb{C}-$linear we can set $\lambda = 1$. By self-adjointness we now get
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \phi_1 & \phi_3 \\ 0 & \phi_2 & \phi_4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\phi_1} & \overline{\phi_2} \\ 0 & \overline{\phi_3} & \overline{\phi_4} \end{pmatrix}.$$

The action on D is now given by

$$D \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \phi_1 & \phi_3 \\ 0 & \phi_2 & \phi_4 \end{pmatrix} \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\phi_1} & \overline{\phi_2} \\ 0 & \overline{\phi_3} & \overline{\phi_4} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}.$$

Remark 6.7. *It turns out that the two fields ϕ_1 and ϕ_2 parameterize the famous Higgs field in physics [10][11].*

Remark 6.8. *Note that this was in fact the action of* $\mathrm{Pert}(\mathbb{C} \oplus \mathbb{H})$ *as in the Standard Model. Again* $\mathrm{Pert}(\mathbb{H})$ *acts trivial and for* $\mathbb{H}$ *we have $\phi_3 = -\overline{\phi_2}$ and $\phi_4 = \overline{\phi_1}$. However, ϕ_3, ϕ_4 do not matter for the fluctuated D, hence we get the same result*

$$D = \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}.$$

Remark 6.9. *Note that this example appears in equation* (5.3) *as the term*

$$\Big(C^\infty(M, \mathbb{C}) \otimes C^\infty(M, \mathbb{H}) \oplus C^\infty(M, \mathbb{H}) \otimes C^\infty(M, \mathbb{C})\Big)^{s.a.}.$$

The action of $\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ can be generalized to the action of $\mathrm{Pert}(\mathbb{C} \oplus M_N(\mathbb{C}))$, but first let

$$D = \overline{c}e_{1i} + ce_{i1}$$

for $i = 1, \ldots, N+1$, such that the spectral triple is given by

$$\Big(\mathbb{C} \oplus M_N(\mathbb{C}), \mathbb{C}^{N+1}, \overline{c}e_{1i} + ce_{i1}\Big).$$

Example 6.10. *We know by equation* (4.1) *that*

$$\mathrm{Pert}(\mathbb{C} \oplus M_N(\mathbb{C})) \cong \mathrm{Pert}(\mathbb{C}) \times \mathrm{Pert}(M_N(\mathbb{C})) \times M_N(\mathbb{C}) \otimes \mathbb{C}.$$

Since $\mathrm{Pert}(\mathbb{C})$ *is trivial and we have the identification $M_N(\mathbb{C}) \otimes \mathbb{C} \cong M_N(\mathbb{C})$ we get*

$$\mathrm{Pert}(\mathbb{C} \oplus M_N(\mathbb{C})) \cong \mathrm{Pert}(M_N(\mathbb{C})) \times M_N(\mathbb{C}).$$

Let us first look at the action of $\mathrm{Pert}(M_N(\mathbb{C}))$. *This perturbation semigroup is embedded in* $\mathrm{Pert}(\mathbb{C} \oplus M_N(\mathbb{C}))$ *by*

$$F_A = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

where $A \in M_N(\mathbb{C})$. Note that $(F_A)_{ij} = 0$ if $i = 1$ or $j = 1$. Now let $\sum A_i \otimes B_i^{op} \in$ Pert$(M_N(\mathbb{C}))$ *then $\sum F_{A_i} \otimes F_{B_i^{op}} \in$* Pert$(\mathbb{C} \oplus M_N(\mathbb{C}))$*. The action on D is now given by*

$$\begin{aligned} D \mapsto \; & \sum F_{A_i} D F_{B_i^{op}} \\ &= \sum F_{A_i}(\bar{c} e_{1j} + c e_{j1}) F_{B_i^{op}} \\ &= \sum \left(\bar{c} \sum (F_{A_i})_{kl} e_{1j} F_{B_i^{op}} + c \sum F_{A_i} e_{j1} (F_{B_i^{op}})_{mp} \right) \\ &= \sum \left(\bar{c} \sum (F_{A_i})_{kj} \delta_1^l F_{B_i^{op}} + c \sum F_{A_i} (F_{B_i^{op}})_{jp} \delta_m^j \right) \\ &= 0. \end{aligned}$$

So Pert$(M_N(\mathbb{C}))$ *acts trivial on D.*

We now only need to consider the action of $M_N(\mathbb{C})$ on D. In a similar way as for $M_2(\mathbb{C})$, the elements of $M_N(\mathbb{C})$ are embedded in Pert$(\mathbb{C} \oplus M_N(\mathbb{C}))$ *as*

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & a_{11} & \dots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{N1} & \dots & a_{NN} \end{pmatrix} \otimes \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \oplus \begin{pmatrix} \overline{\lambda} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \overline{a_{11}} & \dots & \overline{a_{N1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \overline{a_{1N}} & \dots & \overline{a_{NN}} \end{pmatrix}.$$

However without loss of generality we can set $\lambda = 1$, since we have a $\mathbb{C}$-linear tensor product. The action is now given by

$$D \mapsto B(\bar{c} e_{1j} + c e_{j1}) e_{11} + e_{11}(\bar{c} e_{1j} + c e_{j1}) B^*,$$

where

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \phi_{11} & \dots & \phi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \phi_{N1} & \dots & \phi_{NN} \end{pmatrix}.$$

If we work this out we get

$$\begin{aligned} D \mapsto & B c e_{j1} + \bar{c} e_{1j} B^* \\ &= \sum_k c e_{k1} \phi_{kj} + \sum_k \bar{c} e_{1k} \overline{\phi_{kj}}, \end{aligned}$$

where we have used that $\phi_{ij} = \overline{\phi_{ji}}$. For $j = 1$ we get

$$\begin{pmatrix} 2\Re(c) & 0 \\ 0 & 0 \end{pmatrix},$$

while for $j = 2, \dots, N+1$ we get

$$\begin{pmatrix} 0 & \bar{c}\overline{\phi_{1,j-1}} & \dots & \bar{c}\overline{\phi_{N,j-1}} \\ c\phi_{1,j-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c\phi_{N,j-1} & 0 & \dots & 0 \end{pmatrix}.$$

This last example is a generalization of a part of the standard model (namely the part $\mathbb{C} \oplus M_3(\mathbb{C})$) and its action on an off-diagonal hermitian operator D. Note that one is not restricted to the forms we have considered for D. The only restriction we have on D is that D is self-adjoint.

Appendix A

Other results for semigroups

In order to get a better understanding of semigroups and monoids in general we try to generalize some theorems which hold for groups to theorems which hold for semigroups. Note that this in indeed a generalization, since every group is in particular a semigroup.

A.1 Definitions

Let us first start with some definitions.

Definition A.1. A semigroup S is a set with an associative operation $\circ : S \times S \to S$. If S has a unit it is called a monoid.

Definition A.2. A group G is a set with an associative operation $\circ : G \times G \to G$, an identity element e such that $ge = g = eg$ for all $g \in G$ and for every $g \in G$ there is an element $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$. We will refer to g^{-1} as the inverse element of g, since g^{-1} is unique for every g.

Remark A.3. *Every group G is thus in particular a monoid and a semigroup.*

Just as for groups we have homomorphism between semigroups.

Definition A.4. Let S, T be two semigroups, then the function $\phi : S \to T$ is called a semigroup homomorphism if

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in S$.

Notation A.5. *For a semigroup (or a monoid) S we write $S^\times$ for the group of invertible elements in S.*

It is also possible that we have a semigroup S and a subset T of this semigroup. Some special subsets are

Definition A.6. Let S be a semigroup and $T \subset S$, we say that

i T is a subsemigroup of S, if $st \in T$ for all $s, t \in T$;

ii T is a submonoid of S, if T is a subsemigroup of S and $e_s \in T$;

iii T is a subgroup of S ($T \leq S$), if T is a submonoid of S and T is a group.

Definition A.7. Let S be a semigroup and T a subset of S. We say that T is a normal subsemigroup, or T is normal, $\Leftrightarrow sT = Ts$ for all $s \in S$.

Remark A.8. *If S is a group, the definition can be reformulated as: T is normal $\Leftrightarrow T = sTs^{-1}$ for all $s \in S$.*

A.2 Generalized isomorphism theorems

A semigroup homomorphism which is bijective is called a semigroup isomorphism. For group isomorphisms we have three well known theorems, namely

Theorem A.9. *(First group isomorphism theorem) Let G, H be groups and $\varphi : G \to H$ a group homomorphism, then we have*

i $\ker\varphi \triangleleft G$, *i.e.* $\ker\varphi$ *is a normal subgroup of G,*

ii $\operatorname{Im}\varphi \leq H$, *i.e.* $\operatorname{Im}\varphi$ *is a subgroup of H,*

iii $\operatorname{Im}\varphi \cong G/\ker\varphi$.

Corollary A.10. *If φ is surjective we get*

$$H \cong G/\ker\varphi.$$

Theorem A.11. *(Second group isomorphism theorem) Let G be a group, $H \leq G$ and $N \triangleleft G$, then*

i $HN \leq G$,

ii $H \cap N \triangleleft H$,

iii $(HN)/N \cong H/(H \cap N)$.

Theorem A.12. *(Third group isomorphism theorem) Let G be a group, $N, M \triangleleft G$ such that $M \subseteq N \subseteq G$, then*

i $N/M \triangleleft G/M$,

ii $(G/M)/(N/M) \cong G/N$.

However in general these theorems do not apply for semigroups, since the notion of quotients is not defined for semigroups. Furthermore is the notion of a kernel not defined for semigroups, since

$$\ker\varphi = \{g \in G \mid \varphi(g) = e_H\}$$

and semigroups do not need to have an identity element. We need to modify the theorems in order to be able to generalize them. The solution has been found in congruence relations.

Definition A.13. A relation $\sim$ is called a congruence relation, or congruence, if it is an equivalence relation and if $x \sim y, z \sim w$ then $xz \sim yw$.

We can now form a new definition for the kernel of a homomorphism.

Definition A.14. Let $\varphi : G \to H$ be a homomorphism between two semigroups, then define the congruence

$$(x, y) \in \ker \varphi \Leftrightarrow \varphi(x) = \varphi(y).$$

Note that this indeed is a generalization of the definition of the kernel for a group homomorphism, since $\varphi(e_G) = e_H$, and if $\varphi(x) = \varphi(y)$ then $\varphi(xy^{-1}) = \varphi(e_G)$ and $xy^{-1} \in \ker \phi$.

Theorem A.15 (First isomorphism theorem)**.** *Let $\varphi : S \to T$ be a semigroup homomorphism then $\ker \varphi$ is a congruence, $\operatorname{Im} \varphi$ is a subsemigroup of T and $S/\ker \varphi \cong \operatorname{Im} \varphi$.*

We follow the proof as in [12].

Proof. First we prove that $\ker \varphi$ is a congruence. That $\ker \varphi$ is a equivalence relation is trivial. Now suppose $x \sim y, z \sim w$ then we now that $\varphi(x) = \varphi(y)$ and $\varphi(z) = \varphi(w)$. But then we get

$$\varphi(xz) = \varphi(x)\varphi(z) = \varphi(y)\varphi(w) = \varphi(yw),$$

so

$$xz \sim yw.$$

Thus $\ker \varphi$ is a congruence. Also $\operatorname{Im} \varphi$ is a subsemigroup of T, because if $\varphi(s), \varphi(t) \in \operatorname{Im} \varphi$ then $\varphi(s)\varphi(t) = \varphi(st) \in \operatorname{Im} \varphi$ since $st \in S$. Now define $\psi : S/\ker \varphi \to T, [x] \mapsto \varphi(x)$, this is a semigroup homomorpishm, since

$$\psi([x][y]) = \psi([xy]) = \varphi(xy) = \varphi(x)\varphi(y) = \psi([x])\psi([y]).$$

Furthermore is ψ injective. Suppose $\psi([x]) = \psi([y])$ then we have $\varphi(x) = \varphi(y)$, so $[x] = [y]$. And we have that ψ is onto $\operatorname{Im} \varphi$, because if $t \in \operatorname{Im} \varphi$ then there is a $s \in S$ such that $\varphi(s) = t$. By construction this means $t = \psi([s])$ and since $\operatorname{Im} \varphi \subseteq T$ we get the result

$$S/\ker \varphi \cong \operatorname{Im} \varphi$$

□

We also have a generalization for the second and third isomorphism theorems. For the proofs we roughly follow [13].

Theorem A.16 (Second isomorphism theorem)**.** *Let S be a semigroup, T a subsemigroup and ρ a congruence, then $\varrho := \rho \cap (T \times T)$ is a congruence, $\widetilde{T} := \cup_{x \in T}[x]$ is a subsemigroup of S and $T/\varrho \cong \widetilde{T}/\rho$*

Proof. Let $\phi : S \to S/\rho$ be the natural quotient homomorphism and let $\phi|_T$ be the restriction of ϕ to T. Then we see that $\operatorname{Im}(\phi|_T) \cong \widetilde{T}/\rho$. Since the image of T is T modulo the congruence ρ. We also have that the kernel is equal to ϱ, so therefore ϱ is a congruence. Applying the first isomorphism theorem gives us $T/\varrho = T/ker(\phi|_T) \cong Im(\phi|_T) = \widetilde{T}/\rho$ □

Theorem A.17 (Third isomorphism theorem)**.** *Let S be a semigroup, ϱ, ϖ congruence relations on S such that $\varrho \subseteq \varpi$, then ϖ/ϱ is a congruence relation on S/ϱ and $(S/\varpi) \cong (S/\varrho)/(\varpi/\varrho)$.*

Proof. Let $\varphi : S \to S/\varpi$ be a semigroup homomorphism, then is the kernel given by ϖ. However since $\varrho \subseteq \varpi$ we get that $\varphi' : S/\varrho \to S/\varpi$, which maps congruence classes to congruence classes, is also a semigroup homomorphism. But the kernel of this map is ϖ/ϱ, and every kernel is a congruence relation and vice versa, so ϖ/ϱ is a congruence. If we now apply the first isomorphism theorem and use the fact that φ' is onto, we get $(S/\varpi) = \mathrm{Im}(\varphi') \cong (S/\varrho)/\mathrm{Ker}(\varphi') = (S/\varrho)/(\varpi/\varrho)$, which gives the result. □

A.3 General theory on semigroups

A.3.1 Morphisms between semigroups

Let $\phi : S \to T$ be a semigroup homomorphism. We now want to construct a function, say $\widetilde{\phi} : S^\times \to T^\times$, between the invertible elements.

Proposition A.18. *Let $\phi : S \to T$ be a semigroup homomorphism, then $\widetilde{\phi} : S^\times \to T^\times$, defined by $\widetilde{\phi} := \phi|_{S^\times}$ is a group homomorphism.*

Proof. If S has no identity we are done, since then $S^\times = \emptyset$. So suppose S has an identity and $S^\times$ is not trivial then we see that the invertible elements of S has to be mapped on the invertible elements in T. This follows from

$$e_T = \phi(e_S) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) \Rightarrow \phi(x^{-1}) = \phi(x)^{-1},$$

where $x \in S^\times$ arbitrary. Note that the above ϕ can be replaced by $\widetilde{\phi}$ since $x \in S^\times$. Thus $\widetilde{\phi} := \phi|_{S^\times} : S^\times \to T^\times$ is a group homomorphism □

A.3.2 Semidirect product

Let us start with the definition of a semidirect product.

Definition A.19. Let S be a semigroup and T a semigroup which works on S. Then we define the semidirect product $S \rtimes T$ by

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 \varphi_{t_1}(s_2), t_1 t_2)$$

with $\varphi : T \to \mathrm{Aut}(S)$.

If we have a semidirect product between a vector space and a semigroup, which works on the vector space, we saw that the invertible elements were the invertible elements of the semigroup which worked on the vector space. So

$$(V \rtimes S)^\times = V \rtimes S^\times$$

where S is a semigroup and V is a vector space on which S acts. But if we have two semigroups instead of a vector space and a semigroup, then what are the invertible elements?

Lemma A.20. *Let S be a semigroups and T a semigroup which works on S then we have*

$$(S \rtimes T)^{\times} = S^{\times} \rtimes T^{\times} \tag{A.1}$$

Proof. That we have the term $T^{\times}$ is clear, since the second component of the semidirect product reads $t_1 t_2$. So for this to be the unit, we need to have $t_2 = t_1^{-1}$.

Now suppose that $s_1 \notin S^{\times}$, but that $s_1 \varphi_{t_1}(s_2) = e_S$. We then see that $\varphi_{t_1}(s_2)$ is the inverse of s_1. Also $\varphi_{t_1}(s_2) \in S$, since $\varphi : T \to \mathrm{Aut}(S)$ and $s_2 \in S$ by definition. Then we see that $s_1 \in S^{\times}$ since $(s_1)^{-1} = \varphi_{t_1}(s_2)$. Of course is φ_{t_1} invertible, since it is an automorphism and therefore also $s_2 \in S^{\times}$, which gives (A.1). $\square$

Bibliography

[1] J.M. Overduin and P.S. Wesson. Kaluza-klein gravity. *Phys.Rept.*, 283:303–380, 1997.

[2] A. Connes. Gravity coupled with matter and the foundation of non-commutative geometry. *Comm. Math. Phys.*, 182(1):155–176, 1996.

[3] A.H. Chamseddine, A. Connes, and M. Marcolli. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.*, 11(6):991–1089, 2007.

[4] A.H. Chamseddine, A. Connes, and W.D. van Suijlekom. Inner fluctuations in noncommutative geometry without the first order condition. *J. Geom. Phys.*, 73:222–234, 2013.

[5] A.H. Chamseddine and A. Connes. The spectral action principle. *Commun.Math.Phys.*, 186:731–750, 1997.

[6] G. Landi. *An introduction to noncommutative spaces and their geometries*, volume 51 of *Lecture Notes in Physics. New Series m: Monographs.* Springer-Verlag, Berlin, 1997.

[7] G. Ascoli. Le curve limiti di una varietà data di curve. *Accad. Dei Lincei Memorie della Cl. Sci. Fis. Mat. Nat.*, 1884.

[8] C. Arzelà. Sulle funzioni di linee. *Mem. Accad. Sci. Ist. Bologna Cl. Sci. Fis. Mat.*, 1895.

[9] D. Vogt. Lectures on frechet spaces. Lecture notes.

[10] A. Connes and M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives.* American Mathematical Society, 2007.

[11] A.H. Chamseddine, A. Connes, and W.D. van Suijlekom. Beyond the spectral standard model: Emergence of pati-salam unification. *JHEP*, 1311:132, 2013.

[12] P.A. Grillet. *Commutative Semigroups.* Advances in Mathematics - Kluwer Academic Publishers. Springer, 2001.

[13] M. Lawson. *Finite Automata.* Chapman and Hall/CRC, 2003.