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★**Noncommutative geometry, quantum fields and motives.**

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Noncommutative geometry has developed rapidly since it was founded by Connes in the early 1980s. Much of the early material was described in his book [*Noncommutative geometry*, Academic Press, San Diego, CA, 1994; MR1303779 (95j:46063)] which formed a landmark for the developments in the field during the last two decades. These involved many applications in both physics and many fields of mathematics, some of which we can follow in the current book under review.

Recall that the basic idea underlying noncommutative geometry is to "trade spaces for algebras", in the wording of the wonderful textbook [J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser Boston, Boston, MA, 2001; MR1789831 (2001h:58038)]. In fact, for commutative (C^*-)algebras, there is a one-to-one correspondence with (Hausdorff topological) spaces, and this motivates one to consider noncommutative algebras as describing virtual noncommutative spaces. Moreover, the whole machinery of differential geometry can be translated into algebraic terms, and more specifically, Riemannian manifolds can be traded for functional analytical data called a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. It consists of an algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, and an unbounded self-adjoint operator D on $\mathcal{H}$ satisfying certain compatibility conditions (cf. Definition 1.120 of the book under review). This notion lies at the heart of many new applications of noncommutative geometry, ever since it was first introduced in Connes' book (there, it was called a K-cycle). For an overview of these applications, we refer to the book by Várilly [*An introduction to noncommutative geometry*, Eur. Math. Soc., Zürich, 2006; MR2239597 (2007e:58011)] (see also the rich list of references therein) or to the lectures in [*An invitation to noncommutative geometry*, World Scientific, Hackensack, NJ, 2008].

One of these new applications of noncommutative geometry to physics dates back to 1995, just after the appearance of Connes' book. In the last chapter of that book the Standard Model of high-energy physics was described using noncommutative geometry, following the work of Connes and J. Lott [Nuclear Phys. B Proc. Suppl. **18B** (1990), 29–47 (1991); MR1128127 (93a:58015)]. The experimentally well-established Standard Model describes all known elementary particles and their interactions. In the joint work of Connes with A. H. Chamseddine [Comm. Math. Phys. **186** (1997), no. 3, 731–750; MR1463819 (99c:58010)] this derivation was enhanced via the introduction of the so-called spectral action principle. This principle allowed them to derive physical Lagrangians of field theories from the spectrum of a self-adjoint operator that is naturally associated to a (noncommutative) spectral triple. In fact, the whole Lagrangian describing the dynamics and interactions of the Standard Model of high-energy physics could be given a purely geometrical description via this principle. Importantly, this naturally includes the Higgs boson, an elementary particle that—at the moment of writing—still awaits detection in particle accelerators. More recently, an enhancement in the noncommutative model was proposed by Connes and M. Marcolli in collaboration with Chamseddine [Adv. Theor. Math. Phys. **11** (2007), no. 6, 991–1089; MR2368941 (2009d:81414)]. This model includes neutrino masses that were observed in the meanwhile, and also resolved the so-called "fermion doubling problem" of [F. Lizzi et al., Phys. Rev. D (3) **55** (1997), no. 10, 6357–6366; MR1454002 (98e:81110)]. The noncommutative geometrical description of the Standard Model of high-energy physics is found in Chapter 1, Sections 9–17 of the current book.

Another application of the methods of noncommutative geometry to physics is in renormalization of perturbative quantum field theories. This started with the work of Connes and D. Kreimer [Comm. Math. Phys. **210** (2000), no. 1, 249–273; MR1748177 (2002f:81070); Comm. Math. Phys. **216** (2001), no. 1, 215–241; MR1810779 (2002f:81071)] on a connection between renormalization and the Riemann–Hilbert problem, which was eventually fully accomplished by Connes and Marcolli [Int. Math. Res. Not. **2004**, no. 76, 4073–4091; MR2109986 (2006b:81173)].

As before in the case of the Standard Model, the complicated intricacies with which theoretical physicists deal are mathematically dissected using techniques from noncommutative geometry. In this case, it is the process of renormalization that is described using Hopf algebras. The central idea is to consider renormalization as the basis for a rich algebraic structure, instead of as an artifact of quantum field theories. For a physicist, renormalization provides a way to obtain meaningful predictions from the typically divergent and ill-defined expressions of perturbative quantum field theory. Through the work of Connes and Kreimer, it turned out that this at first sight suspicious procedure actually has a well-founded mathematical description in terms of commutative Hopf algebras. More recently, in [op. cit.] the authors of the current book explicitly identified the Riemann–Hilbert correspondence that is present in perturbative renormalization. The techniques involved are very much akin to the Tannakian formalism, and include categories of flat vector bundles, differential Galois theory and the motivic Galois group. The book discusses all of this material in Sections 1–8 of the first chapter.

There have also been many cross-fertilizations of noncommutative geometry with other fields within mathematics. This applies in particular to number theory, and many of these are actually treated in the other three chapters of the book. An intriguing question is what noncommutative geometry has to say about the Riemann zeta function and its zeros, and this question is addressed in the second chapter. The work of Connes [Selecta Math. (N.S.) **5** (1999), no. 1, 29–106; MR1694895 (2000i:11133)] just before the turn of the second millennium actually aims and arrives at a spectral realization of the zeros of the Riemann zeta function. Again, this is highly inspired by physics: Riemann's estimate for counting the zeros of the zeta function is obtained via a counting of energy levels in the absorption spectrum of a quantum mechanical system. This is described in the first part of Chapter 2.

A study of the terms that are present in this estimate leads naturally to a noncommutative space: it is the quotient of the adèles $\mathbb{A}_{\mathbb{Q}}$ by the action of $\mathbb{Q}^*$ by multiplication. This is naturally understood as the configuration space of the physical system. Since it is not a Hausdorff topological space, it is better described by a noncommutative crossed product algebra. The main result of this chapter is Theorem 2.47 giving the spectral realization of the critical zeros of the Riemann zeta function.

The above adèle class space forms a natural connection point with the remaining two chapters of the book. In Chapter 3, a geometrical description is given of the same quotient space in terms of $\mathbb{Q}$-lattices, and once again a prominent role is played by quantum mechanics. In Chapter 4, on the other hand, the quotient space is described in terms of Artin motives and their noncommutative generalizations: endomotives. These chapters are based on the work of the two authors and their collaborations with N. Ramachandran [Selecta Math. (N.S.) **11** (2005), no. 3-4, 325–347; MR2215258 (2007a:11078); in *Operator Algebras: The Abel Symposium 2004*, 15–59, Springer, Berlin, 2006; MR2265042 (2007i:11080)] and with C. Consani [Adv. Math. **214** (2007), no. 2, 761–831; MR2349719 (2009f:58014)].

As said, many modern developments in noncommutative geometry were initiated just after the appearance of the book by Connes, and so it is in this case. In the work of J.-B. Bost and Connes [Selecta Math. (N.S.) **1** (1995), no. 3, 411–457; MR1366621 (96m:46112)] a quantum statistical mechanical system was constructed that exhibited the Riemann ζ-function as its partition function, and that "realized the explicit class field theory of $\mathbb{Q}$" (p. 476). In general, a quantum statistical mechanical system involves a C^*-dynamical system (A, σ) consisting of a C^*-algebra A and a time evolution σ for which one studies certain equilibrium states. The latter are by definition linear functionals on A that satisfy a compatibility (KMS) condition (Definition 3.6) with respect to the time evolution, which depends on a parameter β: the inverse temperature.

For the Bost–Connes system, the symmetry group of the system was found to be $\widehat{\mathbb{Z}}^*$, obtained as the group of idèle classes of $\mathbb{Q}$ modulo the connected component of the identity. At the critical temperature given by the pole of the partition function $\zeta(\beta)$, this symmetry is spontaneously broken. A set of generators could be found for the cyclotomic extension of $\mathbb{Q}$ as the image of a certain "rational subalgebra" under the KMS states of the system at zero temperature. The Galois group of the cyclotomic extension of $\mathbb{Q}$ intertwines the above $\widehat{\mathbb{Z}}^*$-action.

More recently, in [*Frontiers in number theory, physics, and geometry. I*, 269–347, Springer, Berlin, 2006; MR2261099 (2007k:58010)] the authors gave a geometrical interpretation of the above Bost–Connes system as a 1-dimensional $\mathbb{Q}$-lattice, and described an analog of the system for 2-dimensional $\mathbb{Q}$-lattices. The partition function of the corresponding quantum statistical mechanical system is now the product $\zeta(\beta)\zeta(\beta-1)$, and exhibits two phase transitions. In contrast to the previous abelian extension of $\mathbb{Q}$, the system describes the Galois theory of the field of modular functions.

The class field theory of an imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ also fits into this program. In the previously cited work of the two authors with Ramachandran this case was studied in terms of 1-dimensional $\mathbb{K}$-lattices yielding a quantum statistical mechanical system with the Dedekind zeta function as a partition function.

The quantum statistical mechanical systems for these three cases are nicely combined in Chapter 3 of the book under review, with Sections 3–4 considering the explicit class field theory of $\mathbb{Q}$, Sections 5–7 the nonabelian case related to the Galois theory of the modular field, and finally Section 8 the explicit class field theory of imaginary quadratic fields.

Although the explicit spectral realization of the zeros of the zeta function was based on the noncommutative space of adèle classes, the full strength of noncommutative geometry was not used. In the work of the authors with Consani cited above, the proper noncommutative description was given in cohomological terms. The proper theory for this is cyclic cohomology, which is one of the main tools in noncommutative geometry, introduced by Connes already in the 1980s. The basic objects in this work are endomotives, which are noncommutative analogues of Artin motives. This description allows a very natural assignment of an algebraic and an analytic (C^*-algebraic) data, similar to the rational subalgebras constructed in the Bost–Connes system and their generalizations. In fact, there is an endomotivic description of the Bost–Connes system: it is explicitly constructed by means of endomorphisms of the multiplicative group $\mathbb{G}_m$ with ground field $\mathbb{Q}$.

In general, an endomotive is defined over a ground field $\mathbb{K}$ as a projective system of Artin motives with the action of a semigroup by endomorphisms. They come in two flavours: algebraic and analytic, and the latter carries an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Moreover, there is a natural notion of measure on an analytic endomotive, which yields a state on the corresponding C^*-algebra A. This then paves the way for a quantum statistical mechanical description of the endomotive, with the time evolution σ arising naturally from Tomita's modular theory. This description will be used to construct a noncommutative analogue of the action of the Frobenius on l-adic cohomology.

The authors' use of vocabulary from physics to characterize the mathematical processes involved is quite original. First, one considers "cooling", by which is meant the process of extracting classical points from the noncommutative system (A, σ). Although the high temperature states are more interesting from the functional analytical point of view (being type III states), the low temperature states (of type I_∞) reveal the arithmetic structure with a more interesting symmetry group. The next step is referred to as "distillation": the derivation of a cyclic module from the above cooling procedure. As a replacement of the aforementioned l-adic cohomology, one considers the cyclic homology of this distilled cyclic module. The homology group carries an action of $\mathbb{R}_+^*$ which is then thought of as a "Frobenius in characteristic zero". For the Bost–Connes system, the combination of this action with the natural Galois action on cyclic homology is related to the spectral realization of zeros of the Riemann zeta function yielding a cohomological version of the Weil explicit formula as a trace formula.

This construction has been extended to global fields along the same lines involving cooling, distillation, leading to a trace formula for the action of the idèle class group on the cyclic cohomology group of the distilled module. This "recovers the Weil explicit formula and reformulates the Riemann Hypothesis as a positivity problem" (p. 651). In addition, it is the starting point for a dictionary between Weil's proof for the Riemann Hypothesis for function fields and the noncommutative geometry of the adèle class space. This material is covered in the last chapter of the book, and sets off a discussion of what a good setup for a theory of quantum gravity would be. This builds on practically the rest of the book, exploring similarities between spontaneous symmetry breaking in the Standard Model (cf. Section 1.9.2) and the interactions between quantum statistical mechanics and number theory described in the other chapters of the book. An extension of this spontaneous symmetry breaking is conceivable "to the full gravitational sector, in which the geometry of space-time emerges through a symmetry breaking mechanism and a cooling process" (p. xxi).

As illustrated by the above summary, this book touches on many different areas in mathematics such as number theory, functional analysis, arithmetic geometry, algebraic geometry and of course noncommutative geometry, as well as in physics involving elementary particle physics, perturbative quantum field theory, quantum statistical mechanics and quantum gravity. Moreover, it mixes them in an unexpected manner, introducing for instance methods from physics in number theory. It would be impossible to include introductions to all of these fields in a book as the present one, but the authors manage very well in filtering and presenting the central ideas whilst including a rich and precise list of references to the literature. For example, the conceptually rather difficult subject of renormalization in perturbative quantum field theories is presented in a very transparent manner in Sections 2–5 of the first chapter. The same applies for instance to the presentation in Section 3.6, giving an excellent and concise overview on the field of modular functions. However, it must be noted that reading the book in the usual order is not necessarily the best way: some of the used concepts find their detailed explanation only later in the book. Nevertheless, the many cross-references within and between the chapters help the reader to find his/her way in this book and explore the many interconnections between the apparently so different subjects.

Although the book discusses many of the developments in noncommutative geometry during the last decade and a half, it does certainly not form a closed chapter. In fact, at present one already witnesses many new developments, continuing on the material in the book. For instance, an endomotivic description of the "field" with one element has appeared in [A. Connes, C. Consani and M. Marcolli, "Fun with $\mathbb{F}_1$", preprint, arxiv.org/abs/0806.2401] whereas the CKM-matrix of the Standard Model (see Section 1.9.3) served as an invariant in Connes' reconstruction theorem [A. Connes, "On the spectral characterization of manifolds", preprint; per revr.]. This already illustrates the power of the present research book, and will undoubtedly serve as an inspiration to the formidable mathematical question on the structure of the following two spaces: spacetime and the space of primes. *Walter D. van Suijlekom*